CHAPTER 2

POSITIVE FORMS AND REPRESENTATIONS

One of the most important concepts of this book is that of a representation. Given an involutive Banach algebra A, it would be a difficult task to establish the existence of representations of A directly. However, we shall set up a correspondence between representations of A and positive forms on A, and in particular between irreducible representations and pure positive forms. Moreover, the classical tools of functional analysis, namely the Hahn-Banach and Krein-Milman theorems, enable us to prove the existence of positive forms and indeed of pure positive forms as well. This is the basic idea of this chapter.

In the various results which follow, the involutive algebras being studied are subjected to a variety of conditions. At a first reading, however, it may be assumed that we are concerned exclusively with C^*-algebras.

2.1. Positive forms

2.1.1. DEFINITION. Let A be an involutive algebra. A linear form f on A is said to be *positive* if $f(x^*x) \geqslant 0$ for each $x \in A$. If A is a normed involutive algebra, a state of A is a continuous positive linear form f on A such that $\|f\| = 1$.

Let A be an involutive algebra and f and g linear forms on A. We say that f dominates g and we write $f \geqslant g$ or $g \leqslant f$ if $f - g$ is positive. This defines a preorder in the dual of A which is compatible with the real vector space structure. If A is a C^*-algebra this relation is in fact a partial ordering, for if $f \geqslant 0$ and $f \leqslant 0$, then f vanishes on A^+, therefore on the set of hermitian elements of A (1.6.5) and hence $f = 0$.

Let Ω be a locally compact space and A the C^*-algebra of continuous complex-valued functions on Ω which vanish at infinity. A continuous linear form on A is simply a bounded measure μ on Ω, and to say that this linear form is positive is exactly the same as saying that the measure μ is positive.

2.1.2. Let A be an involutive algebra and f a positive form on A. For $x, y \in A$ put $(x \mid y) = f(y^*x)$. This scalar product is linear in x, anti-linear in y and $(x \mid x) \geqslant 0$ for each x. A is thus endowed with a pre-Hilbert space structure.

In particular, we have

$$(1) \qquad f(y^*x) = \overline{f(x^*y)} \qquad (x \in A, y \in A),$$

$$(2) \qquad |f(y^*x)|^2 \leqslant f(x^*x)f(y^*y) \quad (x \in A, y \in A).$$

If A is unital, we deduce, putting $y = 1$ in (1) and (2), that

$$(3) \qquad f(x^*) = \overline{f(x)},$$

$$(4) \qquad |f(x)|^2 \leqslant f(1)f(x^*x).$$

Let H be the Hausdorff pre-Hilbert space constructed canonically from the pre-Hilbert space A, so that $H = A/N$ where N is the set of those $x \in A$ for which $f(x^*x) = 0$. By (2), N is equally the set of $x \in A$ such that $f(yx) = 0$ for all $y \in A$, so that N is a *left ideal* of A.

[Putting $(x \mid y) = f(xy^*)$ we would obtain another pre-Hilbert space structure on A with analogous properties.]

2.1.3. Lemma. *Let A be a unital Banach algebra, x' an element of A with $\|x'\| \leqslant 1$ and $x = 1 + x'$. The series*

$$1 + \frac{1}{2}x' + \frac{1}{2!}\cdot\frac{1}{2}\cdot\left(\frac{1}{2} - 1\right)x'^2 + \cdots + \frac{1}{n!}\cdot\frac{1}{2}\left(\frac{1}{2} - 1\right)\cdots\left(\frac{1}{2} - n + 1\right)x'^n + \cdots$$

converges to an element y of A such that $y^2 = x$. If A has an isometric involution and if x is hermitian, then so is y.

The series

$$1 + \frac{1}{2}\|x'\| + \frac{1}{2!}\left|\frac{1}{2}\left(\frac{1}{2} - 1\right)\right| \cdot \|x'\|^2 + \cdots + \frac{1}{n!}\left|\frac{1}{2}\left(\frac{1}{2} - 1\right)\cdots \left(\frac{1}{2} - n + 1\right)\right| \cdot \|x'\|^n + \cdots$$

is convergent from which it follows that the element y exists. If we compute y^2, we obtain a power series in x' whose coefficients we recognize from the classical situation where $A = \mathbf{C}$; indeed, we have

$y^2 = 1 + x' = x$. If A has an isometric involution and if x is hermitian, y is seen to be a limit of hermitian elements and is therefore hermitian because the involution is continuous.

2.1.4. PROPOSITION. *Let A be an involutive Banach algebra having an identity 1 such that $\|1\| = 1$. If f is a positive linear form on A, then f is continuous and $\|f\| = f(1)$.*

If $x \in A$ is hermitian and $\|x\| \leq 1$, then $1 - x$ can be written in the form y^*y (lemma 2.1.3), from which it follows that $f(1-x) \geq 0$ and so $f(x) \leq f(1)$. If $x' \in A$ and $\|x'\| \leq 1$, then $\|x'^*x'\| \leq 1$, and hence, using 2.1.2, formula (4),

$$|f(x')|^2 \leq f(1)f(x'^*x') \leq f(1)^2.$$

This shows that f is continuous and that $\|f\| \leq f(1)$. It is plain that $f(1) \leq \|f\|$, and hence $\|f\| = f(1)$.

2.1.5. PROPOSITION. *Let A be an involutive Banach algebra having an approximate identity* (B 29), *$\tilde{A}$ the involutive algebra obtained from A by adjoining an identity to it, and f a continuous positive linear form on A. Then:*

(i) *$f(x^*) = \overline{f(x)}$ for every $x \in A$.*
$|f(x)|^2 \leq \|f\| \cdot f(x^*x)$ for every $x \in A$.

(ii) *$|f(y^*xy)| \leq \|x\| f(y^*y)$ for all $x, y \in A$.*

(iii) *$\|f\| = \sup_{x \in A, \|x\| \leq 1} f(x^*x)$.*

(iv) *If $(x_i)_{i \in I}$ is a family of elements of A indexed by a directed set, such that $\|x_i\| \leq 1$ and $f(x_i) \to \|f\|$, then $f(x_i^*x_i) \to \|f\|$.*

(v) *If $(u_j)_{j \in J}$ is an approximate identity for A, then $f(u_j) \to \|f\|$, and $f(u_j^*u_j) \to \|f\|$.*

(vi) *f can be uniquely extended to a positive form $\tilde{f}$ on $\tilde{A}$ such that $\tilde{f}(1) = \|f\|$. Every positive form on $\tilde{A}$ which extends f dominates $\tilde{f}$.*

(vii) *In the notation of* (iv), *we have $x_i \to 1$ for the pre-Hilbert space structure defined on $\tilde{A}$ by $\tilde{f}$, so that A is dense in $\tilde{A}$ for this structure.*

We have, for each $x \in A$,

$$f(x^*) = \lim f(x^*u_j) = \lim \overline{f(u_j^*x)} = \lim \overline{f((x^*u_j)^*)} = \overline{f(x^{**})} = \overline{f(x)},$$

$$|f(x)|^2 = \lim |f(xu_j)|^2 \leq f(x^*x) \varlimsup f(u_j^*u_j) \leq \|f\| f(x^*x).$$

This proves (i) and it is plain that (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). We prove (vi). The uniqueness of $\tilde{f}$ is immediate, so we just have to establish the existence of $\tilde{f}$. For each element $(\lambda, x) = \lambda + x$ of $\tilde{A}$ ($\lambda \in \mathbf{C}$, $x \in A$) put $\tilde{f}(\lambda + x) =$

$\lambda\|f\| + f(x)$. Then $\tilde{f}$ is a linear form on $\tilde{A}$ extending f and using (i), we have

$$\begin{aligned}\tilde{f}((\lambda + x)^*(\lambda + x)) &= f(x^*x + \bar{\lambda}x + \lambda x^*) + |\lambda|^2\|f\| \\ &= f(x^*x) + 2 \operatorname{Re} \bar{\lambda}f(x) + |\lambda|^2\|f\| \\ &\geq f(x^*x) - 2|\lambda| \cdot \|f\|^{1/2} f(x^*x)^{1/2} + |\lambda|^2\|f\| \\ &= [f(x^*x)^{1/2} - |\lambda| \cdot \|f\|^{1/2}]^2 \geq 0,\end{aligned}$$

and so $\tilde{f}$ is positive. Clearly, $\tilde{f}(1) = \|f\|$. Now let g be any positive form on $\tilde{A}$ which extends f. $\tilde{A}$ can be given an involutive Banach algebra structure extending that of A and such that $\|1\| = 1$. By (2.1.4), we have $g(1) = \|g\| \geq \|f\|$, from which we obtain

$$\begin{aligned}g((\lambda + x)^*(\lambda + x)) &= f(x^*x + \bar{\lambda}x + \lambda x^*) + |\lambda|^2 g(1) \\ &\geq f(x^*x + \bar{\lambda}x + \lambda x^*) + |\lambda|^2\|f\| \\ &= \tilde{f}((\lambda + x)^*(\lambda + x)),\end{aligned}$$

so that $g \geq \tilde{f}$, and (vi) is proved. If $y \in A$, the form $x \to \tilde{f}(y^*xy)$ on $\tilde{A}$ is positive, because $\tilde{f}(y^*(x^*x)y) = \tilde{f}((xy)^*(xy)) \geq 0$; by (2.1.4), its norm is $f(y^*y)$ and (ii) now follows.

In the notation of (iv), we have

$$\begin{aligned}\tilde{f}((x_i - 1)^*(x_i - 1)) &= f(x_i^*x_i) - \overline{f(x_i)} - f(x_i) + \|f\| \\ &\to \|f\| - \|f\| - \|f\| + \|f\| = 0,\end{aligned}$$

from which (vii) follows.

We have $(u_i \mid u_j) = f(u_j^*u_j) \leq \|f\|$. For every $x \in A$, $(u_i, x) = f(x^*u_i) \to f(x^*) = (1 \mid x)$; since A is dense in $\tilde{A}$ (cf. (vii)), we see that u_i converges weakly to 1. Hence $f(u_i) = (u_i \mid 1) \to (1, 1) = \tilde{f}(1) = \|f\|$. From (iv), $f(u_j^*u_j) \to \|f\|$.

2.1.6. In the notation of 2.1.5, $\tilde{f}$ is called the canonical extension of f to $\tilde{A}$.

COROLLARY. *Let f, g be continuous positive forms on A and $\tilde{f}$, $\tilde{g}$ their canonical extension to $\tilde{A}$. Then*

$$\|f + g\| = \|f\| + \|g\|, \qquad (f + g)^{\sim} = \tilde{f} + \tilde{g}.$$

The first relation follows from 2.1.5 (v), while the second follows from the first.

It follows that the set of states of A is a convex subset of the dual of A.

2.1.7: We retain the notation of 2.1.5. Now let Q be the set of continuous positive forms on A and $\tilde{Q}$ the set of positive forms on $\tilde{A}$ such that $g(1) = \|g \mid A\|$. Then $f \to \tilde{f}$ is a bijection of Q onto $\tilde{Q}$ which preserves both the additive and order structures. If a positive form on $\tilde{A}$ is dominated by an element of $\tilde{Q}$, it must itself belong to $\tilde{Q}$; in fact let $g = g_1 + g_2$ where $g \in \tilde{Q}$ and g_1, g_2 are positive forms on $\tilde{A}$, and let f, f_1, f_2 be the restrictions of g, g_1, g_2 to A. We have

$$g_1 + g_2 = g = \tilde{f} = \tilde{f}_1 + \tilde{f}_2$$

and

$$g_1(1) \geq \tilde{f}_1(1), \qquad g_2(1) \geq \tilde{f}_2(1),$$

from which it follows that

$$g_1(1) = \tilde{f}_1(1), \qquad g_2(1) = \tilde{f}_2(1);$$

and thus $g_1 \in \tilde{Q}$, $g_2 \in \tilde{Q}$.

2.1.8. Let A be a C^*-algebra and f a positive form on A. Then f is continuous. Indeed, let $(x_1, x_2, \ldots)$ be a sequence of elements of A^+ such that $\|x_i\| \leq 1$. We show that the numbers $f(x_i)$ form a bounded set. For each sequence $(\lambda_1, \lambda_2, \ldots)$ of non-negative real numbers satisfying $\Sigma_{i=1}^{\infty} \lambda_i < +\infty$, the series $\Sigma_{i=1}^{\infty} \lambda_i x_i$ converges to an element x of A. For every integer $n \geq 1$, we have $\Sigma_{i=n+1}^{\infty} \lambda_i x_i \geq 0$ by 1.6.1, and so

$$\sum_{i=1}^{n} \lambda_i f(x_i) = f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq f(x).$$

Hence $\Sigma_{i=1}^{\infty} \lambda_i f(x_i) < +\infty$. Since this holds for any sequence (λ_i) of non-negative reals with $\Sigma_{i=1}^{\infty} \lambda_i < +\infty$, it follows that the set of the numbers $f(x_i)$ is bounded. It now follows from this that

$$M = \sup_{x \in A^+, \|x\| \leq 1} f(x) < +\infty.$$

If x is hermitian and of norm ≤ 1, we have

$$|f(x)| \leq |f(x^+)| + |f(x^-)| \leq 2M,$$

and if x is any element of norm ≤ 1, then

$$|f(x)| \leq \left| f\left(\frac{1}{2}(x + x^*)\right)\right| + \left| f\left(\frac{1}{2i}(x - x^*)\right)\right| \leq 4M.$$

Hence f is continuous.

In the sequel, positive forms on C^*-algebras will be mentioned over and over again, and the reader should never forget that these forms are *automatically* continuous.

2.1.9. PROPOSITION. *Let A be a unital C^*-algebra and f a continuous linear form on A. For f to be positive, it is necessary and sufficient that $\|f\| = f(1)$.*

If $f \geq 0$, then $\|f\| = f(1)$ (2.1.4). Now suppose $\|f\| = f(1)$. Let $x \in A^+$ and suppose that $f(x) \ngeq 0$. (We can, of course, assume that $f(1) = 1$.) There is a closed disc $|z - z_0| \leq \rho$ in $\mathbf{C}$ which contains $\operatorname{Sp} x$ but not $f(x)$. Now the spectrum of the normal element $x - z_0$ is contained in the disc $|z| \leq \rho$, from which it follows that $\|x - z_0\| \leq \rho$. Hence

$$|f(x) - z_0\| = |f(x) - z_0 f(1)| = |f(x - z_0)| \leq \|f\| \cdot \|x - z_0\| \geq \rho,$$

which is a contradiction. (An alternative proof is based on B 28.)

The work "state" is borrowed from physics.

References: [174], [618], [619], [638], [1097], [1101], [1323], [1455]. The result of 2.1.8 can be generalized to involutive Banach algebras with an approximate identity [1755].

2.2. Representations

2.2.1. DEFINITION. Let A be an involutive algebra and H a Hilbert space. A ***representation*** of A in H is a morphism of the involutive algebra A into the involutive algebra $\mathscr{L}(H)$.

In other words, a representation of A in H is a map π of A into $\mathscr{L}(H)$ such that

$$\pi(x + y) = \pi(x) + \pi(y), \qquad \pi(\lambda x) = \lambda\pi(x),$$

$$\pi(xy) = \pi(x)\pi(y), \qquad \pi(x^*) = \pi(x)^*$$

for $x, y \in A$, $\lambda \in \mathbf{C}$.

The (Hilbert) dimension of H is called the dimension of π and is denoted by $\dim \pi$. The space H is called the space of π and is denoted by H_π.

Two representations π and π' of A in H and H' are said to be *equivalent*, and we write $\pi \simeq \pi'$ if there is an isomorphism U of the Hilbert space H onto the Hilbert space H' which transforms $\pi(x)$ into $\pi'(x)$ for each $x \in A$. Hence the definition of a *class of representations*.

(For convenience, we will often not distinguish between representations and classes of representations.)

2.2.2. An *intertwining operator* for π and π' is any continuous linear operator $T\colon H \to H'$ such that $T\pi(x) = \pi'(x)T$ for any $x \in A$. The operator U of the previous definition is an intertwining operator. The set of all intertwining operators for π and π' is a vector space whose dimension is called the *intertwining number* of π and π', and is the same as the intertwining number of π' and π. For let $T\colon H \to H'$ be an intertwining operator for π and π'. Then $T^*\colon H' \to H$ is an intertwining operator for π' and π, because

$$T^*\pi'(x) = (\pi'(x^*)T)^* = (T\pi(x^*))^* = \pi(x)T^*.$$

It follows from this, that

$$T^*T\pi(x) = T^*\pi'(x)T = \pi(x)T^*T,$$

and thus $|T| = (T^*T)^{1/2}$ commutes with $\pi(A)$. Let $T = U|T|$ be the polar decomposition of T. We have, for each $x \in A$,

$$U\pi(x)|T| = U|T|\pi(x) = T\pi(x) = \pi'(x)T = \pi'(x)U|T|. \tag{1}$$

If $\operatorname{Ker} T = 0$, then $|T|(H)$ is dense in H and (1) implies

$$U\pi(x) = \pi'(x)U. \tag{2}$$

If moreover $\overline{T(H)} = H'$, i.e. if $\operatorname{Ker} T^* = 0$, then U is an isomorphism of H onto H', and (2) shows that π and π' are equivalent.

2.2.3. Let $(\pi_i)_{i \in I}$ be a family of representations of A in Hilbert spaces H_i. Let H be the Hilbert sum of the H_i. If the set of numbers $\|\pi_i(x)\|$ is bounded for each $x \in A$ (which is the case, by 1.3.7, if A is an involutive Banach algebra), we can construct the continuous linear operator $\pi(x)$ in H which induces $\pi_i(x)$ in each H_i. Then $x \to \pi(x)$ is a representation of A in H, known as the Hilbert sum of the π_i and denoted by $\bigoplus_{i \in I} \pi_i$, or $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n$ in the case of a (finite) family $(\pi_1, \pi_2, \ldots, \pi_n)$ of representations.

If $(\pi_i)_{i \in I}$ is a family of representations of A each of which is equal to some representation π, and if $\operatorname{Card} I = c$, the representation $\bigoplus \pi_i$ is denoted by $c\pi$. Every representation equivalent to a representation of this type is called a multiple of π.

2.2.4. Let π be a representation of A in H. If a closed vector subspace K of H is invariant under $\pi(A)$, the restrictions of the $\pi(x)$'s to K define a *subrepresentation* π' of A in K denoted by π_K or π_E if $E = P_K$. In this situation $H \ominus K$ is also invariant under $\pi(A)$, since if $\xi \in K$ and $\eta \in H \ominus K$, we have that $\pi(x^*)\xi \in K$ for each $x \in A$, and therefore

$$(\pi(x)\eta \mid \xi) = (\eta \mid \pi(x)^*\xi) = (\eta \mid \pi(x^*)\xi) = 0,$$

so that $\pi(x)\eta \in H \ominus K$. Hence P_K commutes with $\pi(A)$. If π'' is the subrepresentation of π defined by $H \ominus K$, then $\pi = \pi' \oplus \pi''$.

Let ρ, ρ' be two representations of A. If ρ' is equivalent to a subrepresentation of ρ, we say that ρ' is *contained* in ρ or that ρ contains ρ', and we write $\rho' \leqslant \rho$ or $\rho \geqslant \rho'$.

2.2.5. Let π be a representation of A in H and let $\xi \in H$. The closure of $\pi(A)\xi$ is a closed subspace of H, invariant under $\pi(A)$. If this subspace is equal to H, we say that ξ is a *cyclic vector* for π.

2.2.6. PROPOSITION. *Let π be a representation of A in H, and let K be the closed subspace of H generated by the $\pi(x)\xi$ ($x \in A, \xi \in H$). Let K' be the closed subspace of H consisting of those $\xi \in H$ such that $\pi(x)\xi = 0$ for every $x \in A$. Then K and K' are invariant under $\pi(A)$ and $\pi(A)'$ [the commutant of $\pi(A)$ in $\mathscr{L}(H)$], are mutually orthogonal and their direct sum is equal to H.*

Clearly, K and K' are invariant under $\pi(A)$. Any operator that commutes with $\pi(x)$ leaves the kernel and range of $\pi(x)$ invariant so that K and K' are invariant under $\pi(A)'$. Let $\xi \in H$. We have

$$\begin{aligned} \xi \in K' &\Leftrightarrow (\pi(A)\xi \mid H) = 0 \Leftrightarrow (\xi \mid \pi(A)H) = 0 \\ &\Leftrightarrow \xi \in H \ominus K, \end{aligned}$$

hence $K' = H \ominus K$.

K is called the *essential subspace* of π. π is said to be *non-degenerate* if $K = H$. The above argument shows that every representation of A can be uniquely expressed as the direct sum of a trivial representation and a non-degenerate representation.

2.2.7. It is plain that a direct sum of non-degenerate representations is non-degenerate and that a representation that admits a cyclic vector is non-degenerate. Conversely:

PROPOSITION. *Every non-degenerate representation of A is a direct sum of representations that admit cyclic vectors.*

Let π be a non-degenerate representation of A in H with $H \neq 0$. It is enough to show that there is a sub-representation of π in a non-trivial space which admits a cyclic vector (for the result then follows from an application of Zorn's lemma). Let ξ be a non-zero element of H. The closure K of $\pi(A)\xi$ is non-trivial and invariant under π. Let $L = H \ominus K$, and put $\xi = \xi_1 + \xi_2$ with $\xi_1 \in K$, $\xi_2 \in L$. If $x \in A$, then $\pi(x)\xi_1 \in K$, $\pi(x)\xi_2 \in L$ and $\pi(x)\xi_1 + \pi(x)\xi_2 = \pi(x)\xi \in K$, so that $\pi(x)\xi_2 = 0$; consequently $\pi(x)\xi = \pi(x)\xi_1$. Thus K is the closure of $\pi(A)\xi_1$.

2.2.8. Let π be a representation of A in H and let $\bar{H}$ be the Hilbert space conjugate to H ($\bar{H} = H$, but when passing from H to $\bar{H}$, multiplication by $\lambda \in \mathbf{C}$ is replaced by multiplication by $\bar{\lambda}$ and scalar products are changed into their complex conjugates). Let A^0 be the reversed involutive algebra of A. It is easily checked that the map $x \to \pi(x^*)$ is a representation of A^0 in $\bar{H}$, which is denoted by $\bar{\pi}^0$.

2.2.9. Let $\tilde{A}$ be the involutive algebra obtained from A by adjoining an identity and let π be a representation of A in H. Then π has a unique extension (known as the canonical extension) to a representation $\tilde{\pi}$ of $\tilde{A}$ in H such that $\tilde{\pi}(1) = 1$.

2.2.10. Let A be an involutive Banach algebra and π a representation of A in H. Recall (1.3.7) that $\|\pi(x)\| \leq \|x\|$ for every $x \in A$. If A has an approximate identity (u_i) and if π is nondegenerate, then $\pi(u_i)$ tends strongly to 1. Indeed, for each $y \in A$ and each $\xi \in H$, $\pi(u_i)(\pi(y)\xi) = \pi(u_i y)\xi$ tends strongly to $\pi(y)\xi$ since

$$\|\pi(u_i y) - \pi(y)\| \leq \|u_i y - y\| \to 0;$$

now, the $\pi(y)\xi$ constitute a total subset of H and moreover $\|\pi(u_i)\| \leq \|u_i\| \leq 1$, so that $\pi(u_i)\eta$ tends strongly to η for each $\eta \in H$.

References: [618], [1101], [1323].

2.3. Topologically irreducible representations

2.3.1. PROPOSITION. *Let A be an involutive algebra, H a Hilbert space and π a representation of A in H. Then the following conditions are equivalent:*

(i) *The only closed subspaces of H invariant under $\pi(A)$ are* 0 *and* H;
(ii) *The commutant of $\pi(A)$ in $\mathscr{L}(H)$ is just the set of scalar operators*;
(iii) *Every non-zero vector of H is a cyclic vector for π, or π is 1-dimensional.*

(i)$\Rightarrow$(iii): suppose condition (i) is satisfied. Let $\xi \in H$, $\xi \neq 0$. If $\pi(A)\xi$ is not dense in H, then $\pi(A)\xi = 0$ by (i). Thus $\mathbf{C}\xi$ is invariant under $\pi(A)$, and so $H = \mathbf{C}\xi$ and π is 1-dimensional.

(iii)$\Rightarrow$(i): suppose condition (iii) is satisfied. Let $K \neq 0$ be a closed subspace of H invariant under $\pi(A)$. We must show that $K = H$, which is plain if $\dim \pi = 1$. Now suppose that every non-zero vector of H is a cyclic vector for π, and let $\xi \in K$, $\xi \neq 0$. Then $\pi(A)\xi \subset K$ and $\overline{\pi(A)\xi} = H$, so that $K = H$.

(ii)$\Rightarrow$(i): suppose condition (ii) is satisfied. Let K be a closed subspace of H invariant under $\pi(A)$. P_K then commutes with $\pi(A)$ (2.2.4), and is therefore a scalar operator so that either $P_K = 0$ or $P_K = 1$, i.e. $K = 0$ or H.

(i)$\Rightarrow$(ii): suppose condition (i) is satisfied. Let T be an element of $\mathscr{L}(H)$ that commutes with $\pi(A)$; we show that T is scalar. Since $T + T^*$ and $T - T^*$ commute with $\pi(A)$, we need only consider the case of hermitian T. The spectral projections of T then commute with $\pi(A)$ and are all therefore equal to either 0 or 1, by (i). Hence T is scalar.

2.3.2. Definition. Let A be an involutive algebra, H a Hilbert space and π a representation of A in H. π is said to be *topologically irreducible* if $H \neq 0$ and if π satisfies the equivalent conditions of 2.3.1.

Such a representation is either trivial, 1-dimensional or else non-trivial and non-degenerate. We denote by $\hat{A}$ the set of classes of non-trivial topologically irreducible representations of A.

Remember that irreducibility of π in the algebraic sense means that the only subspaces of H invariant under $\pi(A)$ are $\{0\}$ and H. If $\dim H = +\infty$, this is a condition far more restrictive than topological irreducibility. We shall presently see (2.8.4), however, that if A is a C^*-algebra then the two notions are equivalent.

A finite-dimensional representation of A is a direct sum of topologically irreducible representations (2.3.5). We shall see (8.5.2) that this result also holds to a certain extent for infinite-dimensional representations, albeit in a rather more subtle form, which is one of the reasons why we shall be particularly concerned with topologically irreducible representations in what follows. A fundamental problem associated with any given involutive algebra A is that of determining, up to equivalence, all the topologically irreducible representations of A. It should be noted that there may well not exist any non-zero topologically irreducible representations (or even any non-zero representations at all) of A; we shall nevertheless see (2.7.3) that every C^*-algebra has "enough" topologically irreducible representations.

2.3.3. Let A be a separable normed involutive algebra and π a topologically irreducible representation of A in a Hilbert space H. Then H is separable: this is obvious if π is zero 1-dimensional; otherwise, let ξ be a non-zero vector of H; if (x_n) is a dense sequence in A, then the $\pi(x_n)\xi$ are dense in H. The same argument shows that, for any normed involutive algebra B, the dimensions of the topologically irreducible representations of B are bounded above by some fixed cardinal.

2.3.4. Let A be an involutive algebra and π, π' two topologically irreducible representations of A with intertwining number n. If $n=0$, then π and π' are not equivalent. If $n>0$, let $T: H_\pi \to H_{\pi'}$ be a non-zero intertwining operator. By 2.2.2, T^*T and TT^* are non-zero scalar operators and π and π' are equivalent.

2.3.5. Let A be an involutive algebra. We study the finite-dimensional representations of A, and while what we say here can be regarded as a special case of certain theorems which we shall meet later on, or equally as a special case of some purely algebraic results, we think it better, for the convenience of the reader, to present a direct treatment here and now.

Let π be a finite-dimensional representation of A. Then

$$\pi = \pi_1 \oplus \cdots \oplus \pi_n,$$

where the π_i are irreducible. This is obvious if $\dim \pi = 0$ with of course, $n=0$. Now suppose that $\dim \pi = q$ and that the assertion has been proved for $\dim \pi < q$. If π is irreducible, there is nothing to prove. Otherwise, $\pi = \pi' \oplus \pi''$ where $\dim \pi' < q$, $\dim \pi'' < q$ and we merely have to make use of the inductive hypothesis. The decomposition $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ is not unique; for example, the representation $\lambda \to \lambda \cdot 1$ of $\mathbf{C}$ in $\mathbf{C}^n$ for $n>1$ admits infinitely many different decompositions into 1-dimensional representations. We shall, nevertheless, establish a uniqueness result. Let ρ_1 and ρ_2 be two irreducible subrepresentations of π, and P_1 and P_2 the projections of H_π onto H_{ρ_1} and H_{ρ_2}. These projections commute with $\pi(A)$, and so the restriction of P_2 to H_{ρ_1} is an intertwining operator for ρ_1 and ρ_2. Hence, unless H_{ρ_1} and H_{ρ_2} are orthogonal, $\rho_1 \cong \rho_2$ (2.3.4). This proves that every irreducible subrepresentation of π is equivalent to one of the π_i's, and thus, rearranging the π_i's, we see that $\pi = \nu_1 \oplus \cdots \oplus \nu_m$, where each ν_i is a multiple $p_i\nu_i'$ of an irreducible representation ν_i' and the ν_i''s are mutually inequivalent. If ρ is an irreducible sub-representation of π, the above discussion shows that H_ρ is orthogonal to all but one of the H_{ν_i}'s and so H_ρ is contained in one of the H_{ν_i}'s. This proves that every subspace H_{ν_i} is uniquely determined, namely, it is the subspace of H_π generated by the spaces of the subrepresentations of π equivalent to ν_i'.

Thus, in the decomposition $\pi = p_1\nu_1' \oplus \cdots \oplus p_m\nu_m'$ of $\pi(\nu_1', \ldots, \nu_m'$ irreducible and inequivalent), the integers p_i and the classes of the ν_i' are uniquely determined, just as are the spaces of the $p_i\nu_i'$.

References: [1101], [1323].

2.4. Positive forms and representations

2.4.1. PROPOSITION. *Let A be an involutive algebra.*

(i) *If π is a representation of A in H and $\xi \in H$, then $x \to (\pi(x)\xi \mid \xi)$ is a positive form on A.*

(ii) *Let π and π' be representations of A in H and H', and let ξ (resp. ξ') be a cyclic vector for π (resp. π'). If $(\pi(x)\xi \mid \xi) = (\pi'(x)\xi' \mid \xi')$ for every $x \in A$, there is a unique isomorphism of H onto H' mapping π to π' and ξ to ξ'.*

We have

$$(\pi(x^*x)\xi \mid \xi) = (\pi(x)^*\pi(x)\xi \mid \xi) = \|\pi(x)\xi\|^2 \geq 0,$$

which gives (i). Now suppose the conditions of (ii) are satisfied. For any $x, y \in A$ we have

$$(\pi(x)\xi \mid \pi(y)\xi) = (\pi(y^*x)\xi \mid \xi) = (\pi'(y^*x)\xi' \mid \xi') = (\pi'(x)\xi' \mid \pi'(y)\xi').$$

Since the $\pi(x)\xi$ (resp. $\pi'(x)\xi'$) are dense in H (resp. H') it follows that there is an isomorphism U of H onto H' such that $U(\pi(x)\xi) = \pi'(x)\xi'$ for any $x \in A$. We show that U transforms π into π', i.e. that $U\pi(x) = \pi'(x)U$ for each $x \in A$; for every $y \in A$, we have

$$\begin{aligned}(U\pi(x))(\pi(y)\xi) &= U\pi(xy)\xi = \pi'(xy)\xi' \\ &= \pi'(x)(\pi'(y)\xi') = (\pi'(x)U)(\pi(y)\xi).\end{aligned}$$

Since the $\pi(y)\xi$ are dense in H, it follows that $U\pi(x) = \pi'(x)U$. Moreover, for each $x \in A$,

$$(\xi' \mid \pi'(x)\xi') = (\xi \mid \pi(x)\xi) = (U\xi \mid U\pi(x)\xi) = (U\xi \mid \pi'(x)\xi'),$$

which implies that $\xi' = U\xi$. Finally, the uniqueness of U is immediate since the values that U takes on the (dense) set of the $\pi(A)\xi$ are predetermined.

2.4.2. In the above notation, the form $x \to (\pi(x)\xi \mid \xi)$ on A is called the *form defined by π and ξ*. If π is fixed while ξ varies in H, we obtain the *forms associated with π*. If S is a set of representations of A, the forms associated with S are just the forms associated with the various elements of S.

Let H be a Hilbert space, B an involutive subalgebra of $\mathscr{L}(H)$ and ξ an element of H. We denote by ω_ξ the positive form on B defined by the identical representation of B and ξ, i.e. the form $x \to (x\xi \mid \xi)$. A positive form on B is said to be a *vector* form if it is equal to ω_ξ for a suitable choice of ξ in H.

2.4.3. Let A be an involutive Banach algebra with an approximate identity (u_i), π a non-degenerate representation of A in H, ξ an element of H and f the positive form defined by π and ξ. Then $\|f\| = (\xi \mid \xi)$. In fact, by 2.1.5 (v),

$$\|f\| = \lim f(u_i) = \lim(\pi(u_i)\xi \mid \xi),$$

and $\pi(u_i)$ tends strongly to 1 (2.2.10). It follows that if $\tilde{A}$ is the involutive algebra obtained by adjoining an identity to A, and $\tilde{f}$ and $\tilde{\pi}$ are the canonical extensions of f and π to $\tilde{A}$, then

$$\tilde{f}(x) = (\tilde{\pi}(x)\xi \mid \xi) \quad \text{for each } x \in \tilde{A}.$$

In particular, if π is the identical representation, assumed to be non-degenerate, of a sub-C^*-algebra A of $\mathscr{L}(H)$ which does not contain 1, then the canonical extension to $\tilde{A} = A + \mathbf{C} \cdot 1$ of $\omega_\xi \mid A$ is $\omega_\xi \mid \tilde{A}$.

2.4.4. Proposition. *Let A be an involutive Banach algebra with an approximate identity, $\tilde{A}$ the involutive algebra obtained by adjoining an identity to A, f a continuous positive form on A, $\tilde{f}$ its canonical extension to $\tilde{A}$, N the left ideal of $\tilde{A}$ consisting of those $x \in \tilde{A}$ such that $\tilde{f}(x^*x) = 0$, A'_f the (Hausdorff) pre-Hilbert space $\tilde{A}/N$ and A_f the Hilbert space which is the completion of A'_f. For each $x \in \tilde{A}$, let $\pi'(x)$ be the operator* *in $\tilde{A}/N$ obtained from left multiplication by x in $\tilde{A}$ by passage to the quotient. Let ξ be the canonical image of 1 in A'_f.*

(i) *Each $\pi'(x)$ has a unique extension to a continuous linear operator $\pi(x)$ in A_f.*

(ii) *The map $x \to \pi(x)(x \in A)$ is a representation of A in A_f.*

(iii) *ξ is a cyclic vector for $\pi(A)$.*

(iv) *$f(x) = (\pi(x)\xi \mid \xi)$ for each $x \in A$.*

By 2.1.5 (ii), we have, for $x, y \in \tilde{A}$,

$$(\pi'(x)\pi'(y)\xi \mid \pi'(x)\pi'(y)\xi) = \tilde{f}(y^*x^*xy) \leq \|x^*x\|\tilde{f}(y^*y)$$
$$= \|x^*x\|(\pi'(y)\xi \mid \pi'(y)\xi),$$

from which (i) follows. It is plain that π' and then π also, are representations in the sense of the algebra structure. For x, y, $z \in \tilde{A}$, we have

$$(\pi(x)\pi(y)\xi \mid \pi(z)\xi) = \tilde{f}(z^*(xy)) = \tilde{f}((x^*z)^*y) = (\pi(y)\xi \mid \pi(x^*)\pi(z)\xi),$$

which implies that $\pi(x)^* = \pi(x^*)$, and (ii) is proved. The set $\pi(A)\xi$ is the canonical image of A in A'_f and is therefore dense in A_f by 2.1.5 (vii); this proves (iii). Finally, we have for each $x \in \tilde{A}$,

$$(\pi(x)\xi \mid \xi) = \tilde{f}(1^*x1) = \tilde{f}(x).$$

We say that the representation π and the vector ξ are *defined by f*, and we denote them by π_f and ξ_f.

2.4.5. With the above notation, let M be the left ideal of A consisting of those $x \in A$ such that $f(x^*x) = 0$. The canonical image of A in A_f can be identified with A/M, and is moreover dense in A_f. A_f can thus be defined as the completion of the Hausdorff pre-Hilbert space A/M, and for each $x \in A$, $\pi(x)$ can be defined as the continuous extension of the operator in A/M of left multiplication by x. This avoids the introduction of $\tilde{A}$ and $\tilde{f}$, although, in this approach, the definition of ξ would be less simple.

2.4.6. Let A be an involutive Banach algebra with an approximate identity, f a continuous positive form on A, and π and ξ the representation and vector defined by f. Then by 2.4.4 (iv) f is precisely the positive form defined by π and ξ.

Conversely, starting with a representation π of A in a Hilbert space and a cyclic vector ξ for π, let f be the positive form defined by π and ξ, which is continuous as π is. Let π' and ξ' be the representation and vector defined by f. Then

$$(\pi(x)\xi \mid \xi) = f(x) = (\pi'(x)\xi' \mid \xi') \quad \text{for each } x \in A,$$

and ξ' is a cyclic vector for π'. By 2.4.1 (ii), there is a unique isomorphism of H_π onto $H_{\pi'}$ mapping π to π' and ξ to ξ'.

In particular, let π be a non-trivial topologically irreducible representation of A. Every non-zero vector of H_π is a cyclic vector for π, and so π is defined up to equivalence by any non-zero form associated with π.

2.4.7. PROPOSITION. *Let A be an involutive Banach algebra with an approximate identity, and π a representation of A. For each $x \in A$,*

$$\|\pi(x)\|^2 = \sup f(x^*x),$$

where f varies over the set of positive forms associated with π such that $\|f\| \leq 1$.

Thanks to 2.2.6, we need only consider the case of non-degenerate π. By 2.4.3, the positive forms associated with π of norm ≤ 1 are just the forms $\omega_\xi \circ \pi$ where $\xi \in H$, $\|\xi\| \leq 1$. Thus

$$\begin{aligned} \|\pi(x)\|^2 &= \sup_{\|\xi\| \leq 1} (\pi(x)\xi \mid \pi(x)\xi) \\ &= \sup_{\|\xi\| \leq 1} (\omega_\xi \circ \pi)(x^*x). \end{aligned}$$

2.4.8. PROPOSITION. *Let A be an involutive Banach algebra with an approximate identity, f a continuous positive form on A and π and ξ the representation and the vector defined by f.*

(i) *If $x_0 \in A$, the form $x \to f(x_0^* x x_0)$ is associated with π.*

(ii) *If f' is a positive from associated with π, f' is the limit (in the norm topology) of forms $x \to f(x_0^* x x_0)$, where $x_0 \in A$.*

If $x_0 \in A$, then

$$f(x_0^* x x_0) = (\pi(x_0^* x x_0)\xi \mid \xi) = (\pi(x)\pi(x_0)\xi \mid \pi(x_0)\xi),$$

and we have (i).

Now let $\xi' \in H_\pi$ and $f' = \omega_{\xi'} \circ \pi$. For every $\epsilon > 0$ there is an $x_0 \in A$ such that $\|\pi(x_0)\xi - \xi'\| \leq \epsilon$, and so for each $x \in A$,

$$\begin{aligned}|f'(x) - f(x_0^* x x_0)| &= |(\pi(x)\xi' \mid \xi') - (\pi(x)\pi(x_0)\xi \mid \pi(x_0)\xi)| \\ &\leq \|\pi(x)\xi'\| \cdot \|\xi' - \pi(x_0)\xi\| \\ &\quad + \|\pi(x)\xi' - \pi(x)\pi(x_0)\xi\| \cdot \|\pi(x_0)\xi\| \\ &\leq \|x\| \cdot \|\xi'\| \cdot \epsilon + \|x\| \cdot \epsilon(\|\xi'\| + \epsilon) \\ &= \|x\|(2\epsilon\|\xi'\| + \epsilon^2),\end{aligned}$$

and $2\epsilon\|\xi'\| + \epsilon^2$ is arbitrarily small.

2.4.9. Let A be a C^*-algebra, I a closed two-sided ideal of A, B the C^*-algebra A/I and ω: $A \to B$ the canonical morphism. If f is a positive form on A such that $f(I) = 0$, then f defines a positive form g on B by passage to the quotient. For each $x \in A$, we have

$$(\pi_g(\omega(x))\xi_g \mid \xi_g) = g(\omega(x)) = f(x),$$

so that $\pi_g \circ \omega$ and ξ_g may be identified with π_f and ξ_f respectively (2.4.1).

PROPOSITION. *Let A be a C^*-algebra, f a positive form on A and I a closed two-sided ideal of A. Then $f(I) = 0$ if and only if $\pi_f(I) = 0$.*

$\operatorname{Ker} \pi_f \subset \operatorname{Ker} f$, so that $\pi_f(I) = 0$ implies $f(I) = 0$. If $f(I) = 0$, then $\pi_f \simeq \pi_g \circ \omega$ in our previous notation, so that $\pi_f(I) = 0$.

2.4.10. COROLLARY. $\operatorname{Ker} \pi_f$ *is the largest closed two-sided ideal of A contained in* $\operatorname{Ker} f$.

2.4.11. COROLLARY. *Let A be a C^*-algebra and f and g positive forms on A. Then the following conditions are equivalent*:

(i) $\operatorname{Ker} \pi_f \subset \operatorname{Ker} \pi_g$.

(ii) *g vanishes on* $\operatorname{Ker} \pi_f$.

References: [618], [619], [638], [1097], [1101], [1323], [1455].

2.5. Pure forms and irreducible representations

We know how to associate representations with positive forms, and we now settle the question of when this procedure leads to irreducible representations.

2.5.1. PROPOSITION. *Let A be an involutive algebra, π a representation of A in H, ξ an element of H and f the positive form defined on A by π and ξ.*

(i) *If T is an hermitian operator on H which commutes with $\pi(A)$ and satisfies $0 \leq T \leq I$, the form*

$$x \to (\pi(x)T\xi \mid T\xi) = (\pi(x)\xi \mid T^2\xi)$$

on A is a positive form f_T which is dominated by f.

(ii) *If ξ is a cyclic vector for π, the map $T \to f_T$ is injective.*

(iii) *If A is an involutive Banach algebra with an approximate identity, every continuous positive form on A which is dominated by f can be written f_T for some T.*

Since $f_T = \omega_{T\xi} \circ \pi$, $f_T \geq 0$. If $x \in A$,

$$f_T(x^*x) = (\pi(x^*x)T\xi \mid T\xi) = \|\pi(x)T\xi\|^2$$
$$= \|T\pi(x)\xi\|^2 \leq \|\pi(x)\xi\|^2 = f(x^*x),$$

so that $f_T \leq f$, and (i) is proved.

If $f_T = f_{T'}$, then

$$(\pi(x)\xi \mid T^2\xi) = (\pi(x)\xi \mid T'^2\xi)$$

for any $x \in A$, so that $T^2\xi = T'^2\xi$ if ξ is a cyclic vector for $\pi(A)$. Since ξ is then a separating vector for the commutant of $\pi(A)$ (A 14) it follows that $T^2 = T'^2$, and hence $T = T'$ as $T \geq 0$, $T' \geq 0$. This proves (ii).

Let g be a positive form on A which is dominated by f. For $x, y \in A$

$$|g(y^*x)|^2 \leq g(x^*x)g(y^*y) \leq f(x^*x)f(y^*y) = \|\pi(x)\xi\|^2 \cdot \|\pi(y)\xi\|^2.$$

Hence the relation

$$(\pi(x)\xi \mid \pi(y)\xi) = g(y^*x),$$

defines a unique continuous sesquilinear form on the subspace $\pi(A)\xi$ of H which is clearly positive and hermitian. There then exists an hermitian operator T_0 on $X = \overline{\pi(A)\xi}$ such that $0 \leq T_0 \leq I$ and

$$g(y^*x) = (\pi(x)\xi \mid T_0\pi(y)\xi).$$

For x, y, $z \in A$, we have

$$(\pi(y)\xi \mid T_0\pi(z)\pi(x)\xi) = g((zx)^*y) = g(x^*(z^*y))$$
$$= (\pi(z^*y)\xi \mid T_0\pi(x)\xi) = (\pi(z)^*\pi(y)\xi \mid T_0\pi(x)\xi)$$
$$= (\pi(y)\xi \mid \pi(z)T_0\pi(x)\xi),$$

from which it follows that $T_0\pi(z) = \pi(z)T_0$ on X. Moreover, X is invariant under $\pi(A)$ and so P_X commutes with $\pi(A)$. Hence T_0P_X is an hermitian operator in H, lying between 0 and I, which commutes with the $\pi(z)$'s ($z \in A$). Let T be its positive square root, which again commutes with the $\pi(z)$'s. Then $0 \leqslant T \leqslant 1$, and

$$g(y^*x) = (\pi(x)\xi \mid T^2\pi(y)\xi) = (\pi(x)T\xi \mid \pi(y)T\xi)$$
$$= (\pi(y^*x)T\xi \mid T\xi) = f_T(y^*x).$$

Finally, suppose that A is an involutive Banach algebra with an approximate identity (u_i) and that g is continuous. Then

$$g(y^*) = \lim g(y^*u_i), \qquad f_T(y^*) = \lim f_T(y^*u_i),$$

and thus $g = f_T$.

2.5.2. Definition. Let A be a normed involutive algebra and f a continuous positive form on A. f is said to be *pure* if $f \neq 0$ and if every continuous positive form on A which is dominated by f is of the form $\lambda f (0 \leqslant \lambda \leqslant 1)$. We denote by $P(A)$ the set of pure states of A.

Let Ω be a locally compact space and A the C^*-algebra of continuous complex-valued functions on Ω that vanish at infinity. The pure positive forms on A may be identified with the positive measures on Ω whose support consists of a single point, i.e. with measures of the form $f \to \lambda f(\omega)$ where $\lambda > 0$ and ω is a fixed point of Ω: it follows from this that the pure states of a commutative C^*-algebra are just the characters of the algebra.

2.5.3. Let A be an involutive Banach algebra with an approximate identity, $\tilde{A}$ the involutive Banach algebra obtained by adjoining an identity to A, f a continuous positive form on A and $\tilde{f}$ its canonical extension to $\tilde{A}$. Then f is pure if and only if $\tilde{f}$ is pure. Indeed, to begin with, the conditions $f = 0$ and $\tilde{f} = 0$ are equivalent. Moreover, as g runs through the set of continuous positive forms on $\tilde{A}$ dominated by f, $\tilde{g}$ runs through the set of continuous positive forms on A dominated by $\tilde{f}$ (2.1.7). Finally, for g to be of the form λf where $0 \leqslant \lambda \leqslant 1$ it is necessary and sufficient that $\tilde{g} = \lambda\tilde{f}$.

2.5.4. Proposition. *Let A be an involutive Banach algebra with an approximate identity, f a continuous positive form on A and π the representation of A defined by f. Then π is non-trivial and topologically irreducible if and only if f is pure.*

Let ξ be the vector of H_π defined by f.

Suppose f is pure, and let E be a projection in H_π which commutes with $\pi(A)$. The form $x \to (\pi(x)E\xi \mid E\xi)$ on A is continuous and positive and is dominated by f [2.5.1 (i)], and is therefore equal to λf for some $0 \leqslant \lambda \leqslant 1$. Thus

$$(\pi(x)E\xi \mid E\xi) = (\pi(x)\lambda^{1/2}\xi \mid \lambda^{1/2}\xi) \quad \text{for each } x \in A.$$

By 2.5.1 (ii), $E = \lambda^{1/2} \cdot 1$ and so $E = 0$ or 1. Furthermore, there exist $x \in A$ such that $f(x) \neq 0$ and hence such that $(\pi(x)\xi \mid \xi) \neq 0$, which shows that π is non-trivial and topologically irreducible.

Now suppose π is non-trivial and topologically irreducible. There exist $x \in A$ such that $(\pi(x)\xi \mid \xi) \neq 0$ and hence $f \neq 0$. Let g be a continuous positive form on A dominated by f. By 2.5.1 (iii), there is a hermitian operator $T \in \pi(A)'$ such that $0 \leqslant T \leqslant I$ and $g(x) = (\pi(x)T\xi \mid T\xi)$ for every $x \in A$. Since π is topologically irreducible, $T = \mu \cdot 1$ with $0 \leqslant \mu \leqslant 1$ and so $g = \mu^2 f$. Hence f is pure, and the proposition is proved.

Proposition 2.5.4 allows us to define a canonical map

$$P(A) \to \hat{A}.$$

This map is surjective by 2.4.6 and 2.5.4, and the inverse image in $P(A)$ of $\pi \in \hat{A}$ is the set of states associated with π (on this subject, cf. 2.5.7).

2.5.5. Proposition. *Let A be an involutive Banach algebra with an approximate identity and B the set of continuous positive forms on A of norm* $\leqslant 1$.

(i) *B is convex and compact in the weak*-topology* $\sigma(A', A)$ *of the dual A' of A.*

(ii) *The extreme points of P consist of* 0 *and the pure states.*

(iii) *B is the weak*-closed convex hull of* 0 *and the set of pure states.*

B is a weak*-closed convex subset of the unit ball of A'. This ball is weak*-compact and so we have (i).

We next show that 0 is an extreme point of B. If $f \in B$ and $-f \in B$, then $f(x^*x) = 0$ for each $x \in A$, so that $|f(x)|^2 \leqslant \|f\| f(x^*x) = 0$ (2.1.5); hence $f = 0$.

Now let f be a pure state, and suppose $f = \lambda f_1 + (1-\lambda) f_2$ with $0 < \lambda < 1$, $f_1, f_2 \in B$. Then λf_1 is dominated by f so that $\lambda f_1 = \mu f$ with $0 \leqslant \mu \leqslant 1$. Since

$$1 = \|f\| = \lambda \|f_1\| + (1-\lambda)\|f_2\|$$

and $\|f_1\|$, $\|f_2\| \leq 1$ we must have $\|f_1\| = \|f_2\| = 1$, and therefore $\lambda = \mu$ and $f_1 = f = f_2$. We have thus proved that f is an extreme point of B.

Conversely, let f be a non-zero extreme point of B. Clearly $\|f\| = 1$. Now let $f = f_1 + f_2$ with f_1, f_2 continuous positive and non-zero. Put $\|f_1\| = \lambda$, so that $\|f_2\| = 1 - \lambda$, and let $g_1 = \lambda^{-1} f_1$, $g_2 = (1-\lambda)^{-1} f_2$. Then $f = \lambda g_1 + (1-\lambda) g_2$, with g_1, $g_2 \in B$. Since f is extreme, $f = g_1 = g_2$. Hence $f_1 = \lambda f$, $f_2 = (1-\lambda) f$ from which it follows that f is a pure state. This proves (ii).

Finally (iii) follows from (i) while (ii) follows from the Krein–Milman theorem.

2.5.6. We retain the above notation and assume in addition that A is unital. B is then the set of positive forms f on A such that $f(1) \leq 1$, and the set $E(A)$ of states of A is the set of positive forms f on A such that $f(1) = 1$. It follows that $E(A)$ is convex and weak*-compact, and that the set of extreme points of $E(A)$ is the set of those extreme points of B which belong to $E(A)$, i.e. the set $P(A)$. $E(A)$ is therefore the closed convex hull of $P(A)$.

2.5.7. PROPOSITION. *Let A be an involutive algebra, π a non-trivial topologically irreducible representation of A, ξ_1 and ξ_2 two vectors of H_π and f_1 and f_2 the positive forms defined by (π, ξ_1) and (π, ξ_2). Then $f_1 = f_2$ if and only if there is a complex number λ of absolute value* 1 *such that $\xi_2 = \lambda \xi_1$.*

If $\xi_2 = \lambda \xi_1$ with $|\lambda| = 1$, then clearly $f_1 = f_2$. On the other hand, suppose $f_1 = f_2$. Since ξ_1 and ξ_2 are cyclic vectors for π (2.3.1), there is an automorphism U of H commuting with the $\pi(x)$'s such that $U\xi_1 = \xi_2$ (2.4.1 (ii)). Now U is a scalar operator (2.3.1), and so $\xi_2 = \lambda \xi_1$ with $|\lambda| = 1$.

In particular, the canonical map $P(A) \to \hat{A}$ is bijective if and only if every topologically irreducible representation of A is one-dimensional. When A is a C^*-algebra, theorem 2.7.3 implies that this condition is fulfilled if and only if A is commutative.

References: [618], [619], [638], [1097], [1101], [1323], [1455].

2.6. Existence of representations of C^*-algebras

2.6.1. THEOREM. *Any C^*-algebra A has an isometric representation on a Hilbert space.*

Let A_h be the real Banach space consisting of the hermitian elements of A, and let x be a non-zero element of A. By (1.6.1), $-x^*x \notin A^+$, and since A^+ is a closed convex cone (1.6.1), there is a continuous linear form f_x on A_h such that $f_x(y) \geqslant 0$ if $y \in A^+$ and $f_x(-x^*x) < 0$ (B 5). Identifying f_x with an hermitian form on A, we see that f_x is a positive form on A and $f_x(x^*x) > 0$, so that the representation π_x defined by f_x following 2.4.4 satisfies $\pi_x(x) \neq 0$. Let the representation π be the direct sum of the π_x's for $x \in A$, $x \neq 0$. Then π is injective and therefore isometric (1.3.7 and 1.8.1) and the theorem is proved.

Thus, as we previously asserted, the closed involutive sub-algebras of $\mathscr{L}(H)$ for H a Hilbert space are indeed the most general examples of C^*-algebras.

2.6.2. Proposition. *Let A be a C^*-algebra and let $x \in A$. Then the following conditions are equivalent:*

(i) $x \geqslant 0$.

(ii) *The operator $\pi(x)$ is $\geqslant 0$ for every representation π of A.*

(iii) *$f(x) \geqslant 0$ for every positive form f on A.*

(i) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (ii): let π be a representation of A and $\xi \in H_\pi$; the form $y \to (\pi(y)\xi \mid \xi)$ on A is positive, and so $(\pi(x)\xi \mid \xi) \geqslant 0$ if (iii) holds. The operator $\pi(x)$ is thus $\geqslant 0$.

(ii) $\Rightarrow$ (i): let π be an isometric representation of A (2.6.1); if the operator $\pi(x)$ is $\geqslant 0$, then $\pi(x)$ is positive in $\pi(A)$ (1.6.5) and so x is positive in A.

2.6.3. Corollary. *Let A be a C^*-algebra, A_h the set of hermitian elements of A, B the set of positive forms on A of norm $\leqslant 1$ and $P \subseteq B$ the set of pure states of A, where B and P are endowed with the weak*-topology. Let $\mathscr{C}(B)$, $\mathscr{C}(P)$ be the sets of continuous real-valued functions on B and P respectively, and for each $x \in A_h$ let F_x be the continuous real-valued function $f \to \langle f, x\rangle$ on B and G_x its restriction to P. Then the map $x \to F_x$ (resp. $x \to G_x$) is an isometric isomorphism of the ordered Banach space A_h onto a subspace of the ordered Banach space $\mathscr{C}(B)$ [resp. $\mathscr{C}(P)$].*

It is clear that the maps are linear and that

$$x \geqslant 0 \Rightarrow F_x \geqslant 0 \Rightarrow G_x \geqslant 0.$$

Suppose conversely that $G_x \geqslant 0$, i.e. that $f(x) \geqslant 0$ for each $f \in P$. By

2.5.5, $f(x) \geqslant 0$ for each $f \in B$, and so $x \geqslant 0$ (2.6.2). Now suppose A is represented in a Hilbert space H (2.6.1) and let $x \in A_h$ with $\|x\| = 1$. There exist $\xi \in H$ of norm 1 with $|(x\xi \mid \xi)|$ arbitrarily close to 1, hence there exist $f \in B$ with $|f(x)|$ arbitrarily close to 1 and therefore also $f \in P$ with $|f(x)|$ arbitrarily close to 1. Consequently, $1 \leqslant \|G_x\| \leqslant F_x\| \leqslant 1$ and the corollary follows.

When A is commutative, the map $x \to G_x$ is simply the Gelfand isomorphism restricted to A_h. An alternative generalization of the Gelfand isomorphism to the non-commutative case, more satisfactory in some ways, will be studied later on (10.5.4).

2.6.4. Corollary. *Let A be a C^*-algebra and g a continuous hermitian linear form on A. Then there exist positive forms f, f' on A such that $g = f - f'$, $\|g\| = \|f\| + \|f'\|$.*

We keep the notation of 2.6.3, and identify A_h with a subspace of $\mathscr{C}(B)$. By the Hahn-Banach theorem, g can be extended to a linear form μ on $\mathscr{C}(B)$ with $\|\mu\| = \|g\|$ (cf. 1.2.6). The measure μ on B can be written $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are positive measures such that $\|\mu\| = \|\mu^+\| + \|\mu^-\|$. Let $f = \mu^+ \mid A_h$, $f' = \mu^- \mid A_h$. Then $g = f - f'$ and

$$\|g\| \leqslant \|f\| + \|f'\| \leqslant \|\mu^+\| + \|\mu^-\| = \|\mu\| = \|g\|,$$

so that

$$\|g\| = \|f\| + \|f'\|.$$

We shall see later (12.3.4) that the decomposition of 2.6.4 is unique.

References: [618], [619], [681], [848], [1097], [1101], [1323], [1455], [1589].

2.7. The enveloping C^*-algebra of an involutive Banach algebra

2.7.1. Proposition. *Let A be an involutive Banach algebra with an approximate identity. Let R be the set of representations of A, R' the set of topologically irreducible representations of A, B the set of continuous positive forms of norm $\leqslant 1$ on A, and P the set of pure states of A. For each $x \in A$, we have*

$$(1) \qquad \sup_{\pi \in R} \|\pi(x)\| = \sup_{\pi \in R'} \|\pi(x)\| = \sup_{f \in B} f(x^*x)^{1/2} = \sup_{f \in P} f(x^*x)^{1/2},$$

and denoting the common value of these four expressions by $\|x\|'$, we *have* $\|x\|' \leqslant \|x\|$. *The map* $x \to \|x\|'$ *is a seminorm on* A *such that*

$$\|xy\|' \leqslant \|x\|'\|y\|', \qquad \|x^*\|' = \|x\|', \qquad \|x^*x\|' = \|x\|'^2$$

for any $x, y \in A$.

We denote the four numbers in the order in which they appear in (1) by a, b, c, d.

$d \leqslant b$: let $f \in P$; then f is associated with a representation $\pi \in R'$ (2.5.4) and $f(x^*x) \leqslant \|\pi(x)\|^2$ (2.4.7);

$b \leqslant a$: obvious;

$a \leqslant c$: this follows from 2.4.7;

$c \leqslant d$: let $f \in B$; by 2.5.5, $f(x^*x)$ lies in the closed interval whose left-hand end-point is 0 and whose right-hand end-point is $\sup_{g \in P} g(x^*x)$, i.e. $f(x^*x) \leqslant \sup_{g \in P} g(x^*x)$.

By 1.3.7, $\|\pi(x)\| \leqslant \|x\|$ for every $\pi \in R$, so that $\|x\|' \leqslant \|x\|$. Moreover, each function $x \to \|\pi(x)\|$ is a seminorm on A, hence $x \to \|x\|'$ is a seminorm on A. We have

$$\|\pi(x^*)\| = \|\pi(x)\| \quad \text{and} \quad \|\pi(x^*x)\| = \|\pi(x)\|^2$$

for each $\pi \in R$, and so

$$\|x^*\|' = \|x\| \quad \text{and} \quad \|x^*x\|' = \|x\|'^2.$$

Finally, if $x, y \in A$, we have for each $\pi \in R$,

$$\|\pi(xy)\| \leqslant \|\pi(x)\| \cdot \|\pi(y)\| \leqslant \|x\|' \cdot \|y\|', \quad \text{hence } \|xy\|' \leqslant \|x\|' \cdot \|y\|'.$$

2.7.2. Let I be the set of $x \in A$ such that $\|x\|' = 0$, which is a closed self-adjoint two-sided ideal of A. The map $x \to \|x\|'$ defines a norm on the quotient A/I. Endowed with this norm, A/I satisfies all the C^*-algebra axioms except that A/I is not complete in general. The completion B of A/I is a C^*-algebra called the *enveloping* C^**-algebra of* A. The canonical map of A into B is a norm-reducing involutive algebra morphism whose image is dense in B.

2.7.3. When A is a C^*-algebra, $\|x\|' = \|x\|$ by 2.6.1, and A may be identified with its enveloping C^*-algebra. In this case, proposition 2.7.1 shows the existence of "sufficiently many" topologically irreducible representations, as we had already claimed, and considerably strengthens theorem 2.6.1:

THEOREM. *For any C^*-algebra A, there is a family (π_i) of topologically irreducible representations of A such that $\|x\| = \sup_i \|\pi_i(x)\|$ for every $x \in A$.*

2.7.4. PROPOSITION. *Let A be an involutive Banach algebra with an approximate identity, B the enveloping C^*-algebra of A and τ the canonical map of A into B.*

(i) *If π is a representation of A, there is exactly one representation ρ of B such that $\pi = \rho \circ \tau$, and $\rho(B)$ is the C^*-algebra generated by $\pi(A)$.*

(ii) *The map $\pi \to \rho$ is a bijection of the set of representations of A onto the set of representations of B.*

(iii) *ρ is non-degenerate (resp. topologically irreducible) if and only if π is non-degenerate (resp. topologically irreducible).*

Let π be a representation of A. In the notation of 2.7.2, π vanishes on I, and defines, by passing to the quotient, a representation π' of A/I such that $\|\pi'(z)\| \leq \|z\|$ for each $z \in A/I$, where $\|\ \|$ denotes the norm of B. π' therefore extends to a representation ρ of B which satisfies $\pi = \rho \circ \tau$. The uniqueness statement of (i) follows from the fact that $\tau(A)$ is dense in B. This also implies that $\pi(A)$ is dense in $\rho(B)$ in the operator-norm topology. Since $\rho(B)$ is a C^*-algebra (1.8.3), it is the C^*-algebra generated by $\pi(A)$. Statements (ii) and (iii) are immediate.

Proposition 2.7.4 shows that B is the solution of a universal problem. It also shows that in the majority of questions concerning representations of involutive Banach algebras with an approximate identity, it is enough to deal only with the C^*-algebra case.

2.7.5. PROPOSITION. *With A, B, τ as in 2.7.4:*

(i) *If f is a continuous positive form on A, there is exactly one positive form g on B such that $f = g \circ \tau$. Moreover $\|g\| = \|f\|$.*

(ii) *The map $f \to g$ is a bijection of the set of continuous positive forms on A onto the set of positive forms on B.*

(iii) *Let M be a bounded set of continuous positve forms on A. Then the map $f \to g$, restricted to M, is bicontinuous for the weak*-topologies $\sigma(A', A)$, $\sigma(B', B)$, where A', B' denote the duals of A and B respectively.*

Let f be a continuous positive form on A. Then for each $x \in A$ we have, using 2.1.5 (i),

$$|f(x)| \leq \|f\|^{1/2} f(x^*x)^{1/2} \leq \|f\|^{1/2} \|f\|^{1/2} \|x\|',$$

so that there is a continuous linear form g on B such that $f = g \circ \tau$ and $\|g\| \leq \|f\|$. If $y \in B$ there is a sequence (x_n) in A such that $\tau(x_n) \to y$ which implies that

$$g(y^*y) = \lim f(x_n^* x_n) \geq 0,$$

and so g is positive. If $x \in A$ and $\|x\| \leq 1$, then

$$|f(x)| = |g(\tau(x))| \leq \|g\| \cdot \|\tau(x)\| \leq \|g\| \cdot \|x\| \leq \|g\|, \quad \text{and hence } \|f\| \leq \|g\|.$$

The uniqueness of g follows from the fact that $\tau(A)$ is dense in B. We have now proved (i), and (ii) is immediate. Let $M \subset A'$ be a set of continuous positive forms on A and $N \subset B'$ its image under the map $f \to g$. The map $f \to g$ of M onto N is plainly bicontinuous for the weak*-topologies $\sigma(A', A)$, $\sigma(B', \tau(A))$. If M is bounded, N is bounded by (i), so that $\sigma(B', \tau(A))$ and $\sigma(B', B)$ coincide on N as $\tau(A)$ is dense in B.

2.7.6. Let A be an involutive Banach algebra with an approximate identity, and let Q be the set of continuous positive forms on A. For each $f \in Q$, let π_f be the representation defined by f. Then $\pi = \bigoplus_{f \in Q} \pi_f$ is called the *universal representation* of A, and is non-degenerate. By (2.2.7) and (2.4.6), every non-degenerate representation of A is a direct sum of representations of the form π_f, so that $\|x\|' = \|\pi(x)\|$ for every $x \in A$. Hence if B is the enveloping C^*-algebra of A, the representation of B corresponding to π is an isomorphism of B onto $\overline{\pi(A)}$.

References: [582], [619], [638], [1097], [1101], [1323], [1455].

2.8. A theorem on transitivity

2.8.1. LEMMA. *Let H be a Hilbert space, $(\xi_1, \ldots, \xi_n)$ an orthonormal system in H, and $\zeta_1, \ldots, \zeta_n$ vectors in H of norm $\leq r$. Then there is a $b \in \mathscr{L}(H)$ of norm $\leq (2n)^{1/2} r$ such that $b\xi_1 = \zeta_1, \ldots, b\xi_n = \zeta_n$. If there is an hermitian element h of $\mathscr{L}(H)$ such that $h\xi_1 = \zeta_1, \ldots h\xi_n = \zeta_n$, b can also be chosen to be hermitian.*

Let K be the subspace of H generated by $\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_n$. We shall define b as an operator which leaves K invariant and which vanishes on $H \ominus K$. This reduces us to the case where $K = H$. Now let $(\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_m)$ be an orthonormal basis for H, and let b be the operator in H whose matrix with respect to $(\xi_1, \ldots, \xi_m)$ is as follows:

(a) the first n columns are the coordinates of $\zeta_1, \dots, \zeta_n$ with respect to the ξ_i;

(b) if h exists, the first n rows are the complex conjugates of the first n columns (this is possible since the existence of h implies that the square matrix comprising the first n rows and the first n columns is hermitian).

(c) all the entries not already defined are zero.

We have

$$b\xi_1 = \zeta_1, \dots, b\xi_n = \zeta_n$$

and

$$\|b\|^2 \leq \mathrm{Tr}(b^*b) \leq 2(\|\zeta_1\|^2 + \cdots + \|\zeta_n\|^2) \leq 2nr^2.$$

If h exists, b is hermitian by construction.

2.8.2. LEMMA. *Let $H_1, \dots, H_p$ be Hilbert spaces, $H = H_1 \oplus \cdots \oplus H_p$, and A a C^*-algebra of operators on H which is strongly dense in the von Neumann algebra $\mathscr{L}(H_1) \times \cdots \times \mathscr{L}(H_p)$. Let $(\xi_1, \dots, \xi_n)$ be an orthonormal system in H and $\eta_1, \dots, \eta_n$ be vectors in H of norm $\leq r$, where we assume that for $i = 1, \dots, n$, ξ_i and η_i both lie in the same subspace $H_{j(i)}$. Then there exists a $b \in A$ of norm $\leq 3rn^{1/2}$ such that $b\xi_1 = \eta_1, \dots, b\xi_n = \eta_n$. If there is an hermitian element h of $\mathscr{L}(H)$ such that $h\xi_1 = \eta_1, \dots, h\xi_n = \eta_n$, then b can in addition be chosen to be an hermitian element of A.*

By Kaplansky's density theorem, the unit ball of A (respectively of the hermitian part of A) is strongly dense in the unit ball of $\mathscr{L}(H_1) \times \cdots \times \mathscr{L}(H_p)$ (respectively of the hermitian part of $\mathscr{L}(H_1) \times \cdots \times \mathscr{L}(H_p)$).

Now by 2.8.1 applied to the spaces $H_1, \dots, H_p$ in turn, there is a $y_0 \in \mathscr{L}(H_1) \times \cdots \times \mathscr{L}(H_p)$ such that

$$y_0\xi_1 = \eta_1, \dots, y_0\xi_n = \eta_n, \qquad \|y_0\| \leq (2n)^{1/2}r,$$

and then by the preceding remark there is $x_0 \in A$ such that

$$\|x_0\xi_1 - y_0\xi_1\| \leq \tfrac{1}{2}r, \dots, \|x_0\xi_n - y_0\xi_n\| \leq \tfrac{1}{2}r, \qquad \|x_0\| \leq (2n)^{1/2}r.$$

Next, again by 2.8.1, there exists $y_1 \in \mathscr{L}(H_1) \times \cdots \times \mathscr{L}(H_p)$ such that

$$y_1\xi_1 = \eta_1 - x_0\xi_1, \dots, y_1\xi_n = \eta_n - x_0\xi_n, \qquad \|y_1\| \leq \tfrac{1}{2}(2n)^{1/2}r$$

and then $x_1 \in A$ such that

$$\|x_1\xi_1 - y_1\xi_1\| \leq \tfrac{1}{4}r, \dots, \|x_1\xi_n - y_1\xi_n\| \leq \tfrac{1}{4}r, \qquad \|x_1\| \leq \tfrac{1}{2}(2n)^{1/2}r.$$

By induction, we construct sequences of elements $y_k \in \mathscr{L}(H_1)\times\cdots\times\mathscr{L}(H_p)$ and $x_k \in A$ such that

$$y_k\xi_i = \eta_i - x_0\xi_i - \cdots - x_{k-1}\xi_i, \qquad \|y_k\| \leq \frac{1}{2^k}(2n)^{1/2}r,$$

$$\|x_k\xi_i - y_k\xi_i\| \leq \frac{1}{2^{k+1}}r, \qquad \|x_k\| \leq \frac{1}{2^k}(2n)^{1/2}r.$$

(If h exists, the y_k and x_k can be chosen to be hermitian.) $x_0 + x_1 + \cdots$ then converges in norm to an element $b \in A$ such that

$$b\xi_1 = \eta_1, \ldots, b\xi_n = \eta_n,$$

$$\|b\| \leq (1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots)(2n)^{1/2}r \leq 3rn^{1/2}.$$

2.8.3. THEOREM. *Let A be a C^*-algebra, $\tilde{A}$ the C^*-algebra obtained by adjoining an identity to A, $\pi_1, \ldots, \pi_p$ representations of A in Hilbert spaces $H_1, \ldots, H_p$, and $\tilde{\pi}_1, \ldots, \tilde{\pi}_p$ their canonical extensions to $\tilde{A}$, where the π_j are assumed to be topologically irreducible, non-zero and mutually inequivalent.*

(i) *Let $T_1 \in \mathscr{L}(H_1), \ldots, T_p \in \mathscr{L}(H_p)$ and $K_1, \ldots, K_p$ be finite-dimensional subspaces of $H_1, \ldots, H_p$ respectively. Then there exists $x \in A$ such that*

$$\pi_j(x) \mid K_j = T_j \mid K_j \quad \textit{for } j = 1, \ldots, p.$$

(ii) *Let $T_1 \in \mathscr{L}(H_1), \ldots, T_p \in \mathscr{L}(H_p)$ be hermitian operators and $K_1, \ldots, K_p$ finite-dimensional subspaces of $H_1, \ldots, H_p$. Then there is an hermitian element x of A such that*

$$\pi_j(x) \mid K_j = T_j \mid K_j \quad \textit{for } j = 1, \ldots, p.$$

(iii) *Let $T_1 \in \mathscr{L}(H_1), \ldots, T_p \in \mathscr{L}(H_p)$ be unitary operators and $K_1, \ldots, K_p$ finite-dimensional subspaces of $H_1, \ldots, H_p$. Then there is a unitary element x of $\tilde{A}$ such that*

$$\tilde{\pi}_j(x) \mid K_j = T_j \mid K_j \quad \textit{for } j = 1, \ldots, p.$$

Let $H = H_1 \oplus \cdots \oplus H_p$, $\pi = \pi_1 \oplus \cdots \oplus \pi_p$. By 1.8.3, $\pi(A)$ is a sub-C^*-algebra of $\mathscr{L}(H)$ which commutes with the projections $E_j = P_{H_j}$. Now if B is the von Neumann algebra generated by $\pi(A)$, we have

$$\pi(A) \subset \mathscr{L}(H_1)\times\cdots\times\mathscr{L}(H_p), \quad \text{and hence } B \subset \mathscr{L}(H_1)\times\cdots\times\mathscr{L}(H_p).$$

Since each π_j is non-degenerate, π is non-degenerate and so $\pi(A)$ is strongly dense in B. Since π_j is topologically irreducible, E_j is a minimal projection of $\pi(A)' = B'$ and the von Neumann algebra induced on H_j by B is $\mathscr{L}(H_j)$. Let $F_j \in B \cap B'$ be the central support of E_j (A 10), and let j and k be distinct indices. If F_j and F_k are not orthogonal, there exist non-zero projections E'_j, E'_k of B' dominated by E_j and E_k respectively and equivalent relative to B' (A 44); since E_j, E_k are minimal projections of B', we have $E'_j = E_j$, $E'_k = E_k$, and so there is a partial isometry of B' with E_j and E_k as initial and final projections respectively; but then π_j and π_k are equivalent which is a contradiction. Hence the F_j's are mutually orthogonal. Now $F_j \geqslant E_j$ for each j, and so $E_j = F_j \in B$. Since $B_{E_j} = \mathscr{L}(H_j)$, we see that $B \supseteq \mathscr{L}(H_1) \times \cdots \times \mathscr{L}(H_p)$ and finally $B = \mathscr{L}(H_1) \times \cdots \mathscr{L}(H_p)$.

Let $T_1 \in \mathscr{L}(H_1), \ldots, T_p \in \mathscr{L}(H_p)$ and $K_1, \ldots, K_p$ be finite-dimensional subspaces of $H_1, \ldots, H_p$. By 2.8.2, there is an $x \in A$ such that $\pi_j(x) \mid K_j = T_j \mid K_j$ for $j = 1, \ldots, p$. If the T_j's are hermitian, x can be chosen so that $\pi(x)$ is hermitian (2.8.2), and as $\pi(x) = \pi(\frac{1}{2}(x + x^*))$, x can itself be chosen to be hermitian. Now suppose the T_j's are unitary. For $j = 1, \ldots, p$ there is a finite-dimensional subspace $K'_j \supseteq K_j$ of H_j and a unitary operator $T'_j \in \mathscr{L}(H_j)$ which leaves K'_j invariant and is such that $T'_j \mid K_j = T_j \mid K_j$; there then exists an hermitian operator $T''_j \in \mathscr{L}(H_j)$, again leaving K'_j invariant, such that $\exp(\mathrm{i}T''_j) \mid K'_j = T'_j \mid K'_j$. By the foregoing work, there is an hermitian element y of A such that $\pi(y) \mid K'_j = T''_j \mid K'_j$ for each j. Then $x = \exp(\mathrm{i}y)$ is a unitary element of $\tilde{A}$ and $\tilde{\pi}_j(x) \mid K_j = T_j \mid K_j$ for each j.

2.8.4. Corollary. *Every topologically irreducible representation of a C^*-algebra is algebraically irreducible.*

We have merely to apply theorem 2.8.3 with $p = 1$ and $\dim K_1 = 1$. Thus we will henceforth speak of irreducible representations of a C^*-algebra without further qualification.

2.8.5. Corollary. *Let A be a C^*-algebra, f a pure positive form on A and N the left ideal of those $x \in A$ such that $f(x^*x) = 0$. Then A/N, with the scalar product derived from $f(y^*x)$ is complete and therefore coincides with the space of the representation defined by f.*

Indeed, let π be this representation, topologically irreducible by (2.5.4). From the construction of π, A/N is a subspace of H_π invariant under π. Now $A/N \neq 0$ and so $A/N = H_\pi$ (2.8.4).

2.8.6. COROLLARY. *Let A be a C^*-algebra, $\tilde{A}$ the C^*-algebra obtained by adjoining an identity to A, and f_1 and f_2 two pure states of A. Then f_1 and f_2 define equivalent representations π_1 and π_2 if and only if there is a unitary element u* of *$\tilde{A}$ such that $f_2(x) = f_1(u^*xu)$ for each $x \in A$.*

Let ξ_1 be the vector of H_π defined by f_1. The states of A that define representations equivalent to π_1 are just the states associated with π_1 (2.4.6), i.e. the forms $x \to (\pi_1(x)\xi \mid \xi)$ where ξ is a unit vector of H_{π_1} (2.4.3). Now the unit vectors of H_{π_1} are just the vectors $\tilde{\pi}_1(u)\xi_1$ where u is unitary in $\tilde{A}$ (2.8.3) and $\tilde{\pi}_1$ is the canonical extension of π_1 to $\tilde{A}$. Finally,

$$(\pi_1(x)\tilde{\pi}_1(u)\xi_1 \mid \tilde{\pi}_1(u)\xi_1) = (\pi_1(u^*xu)\xi_1 \mid \xi_1) = f_1(u^*xu).$$

References: [633], [849], [1323].

2.9. Ideals in C^*-algebras

2.9.1. PROPOSITION. *Let A be a C^*-algebra, f a positive form on A, M its kernel and N the set of $x \in A$ such that $f(x^*x) = 0$. Then $M \supseteq N + N^*$, and if f is pure $M = N + N^*$.*

Since $|f(x)|^2 \leq \|f\| f(x^*x)$, $N \subseteq M$, hence $N^* \subseteq M^* = M$ and so $N + N^* \subseteq M$. Now suppose f is pure, and let $\tilde{A}$ be the C^*-algebra obtained by adjoining an identity to A, and $\tilde{f}$ the canonical extension of f to $\tilde{A}$. Let π and ξ be the irreducible representation and vector defined by f. Let $b \in M$ and let η be its canonical image in H_π. By the construction of π and ξ, $(\eta \mid \xi) = \tilde{f}(1^* \cdot b) = 0$, and hence there is an hermitian operator in H_π which maps ξ to zero and leaves η fixed. By 2.8.3, there is an hermitian element a of A such that $\pi(a)\xi = 0$ and $\pi(a)\eta = \eta$, i.e. such that $a \in N$ and $b - ab \in N$. Thus

$$b^* = (b - ab)^* + b^*a \in N^* + N,$$

and so $M = M^* \subseteq N^* + N$.

2.9.2. LEMMA. *Let A be a unital C^*-algebra, L a closed left ideal of A and $x \in A^+$. If, for every $\epsilon > 0$, there is a positive element u_ϵ of L such that $x \leq u_\epsilon + \epsilon$, then $x \in L$.*

If $t = u_\epsilon^{1/2}$ then t_ϵ is a limit of polynomials in u_ϵ with no constant term and therefore belongs to L. Now

$$\begin{aligned}\|x^{1/2}(t_\epsilon+\sqrt{\epsilon})^{-1}t_\epsilon - x^{1/2}\|^2 &= \|x^{1/2}\sqrt{\epsilon}(t_\epsilon+\sqrt{\epsilon})^{-1}\|^2\\ &= \epsilon\|(t_\epsilon+\sqrt{\epsilon})^{*-1}(x^{1/2})^* x^{1/2}(t_\epsilon+\sqrt{\epsilon})^{-1}\|\\ &= \epsilon\|(t_\epsilon+\sqrt{\epsilon})^{-1}x(t_\epsilon+\sqrt{\epsilon})^{-1}\|\end{aligned}$$

and since $0\leqslant x\leqslant t_\epsilon^2+\epsilon$, we have (1.6.8):

$$\begin{aligned}0&\leqslant(t_\epsilon+\sqrt{\epsilon})^{-1}\times(t_\epsilon+\sqrt{\epsilon})^{-1}\leqslant(t_\epsilon+\sqrt{\epsilon})^{-1}(t_\epsilon^2+\epsilon)(t_\epsilon+\sqrt{\epsilon})^{-1}\\ &=(t_\epsilon^2+\epsilon)(t_\epsilon^2+2\sqrt{\epsilon}t_\epsilon+\epsilon)^{-1}\leqslant 1,\end{aligned}$$

so that

$$\|x^{1/2}(t_\epsilon+\sqrt{\epsilon})^{-1}t_\epsilon - x^{1/2}\|^2\leqslant\epsilon.$$

Therefore $x^{1/2}(t_\epsilon+\sqrt{\epsilon})^{-1}t_\epsilon\in L$, hence $x^{1/2}\in L$ as L is closed and so $x\in L$.

2.9.3. LEMMA. *Let A be a C^*-algebra and L a closed left ideal of A. Then A is the closed left ideal generated by $L\cap A^+$.*

This follows immediately from 1.7.3.

2.9.4. LEMMA. *Let A be a C^*-algebra and L, L' two closed left ideals of A such that $L\subseteq L'$. Suppose every positive form on A that vanishes on L also vanishes on L'. Then $L=L'$.*

We may assume A to be unital. Let $a\in L'\cap A^+$ and let ϵ be >0. The set S_ϵ of positive forms f on A such that $f(1)=\|f\|=1$ and $f(a)\geqslant\epsilon$ is weak*-compact. If $f\in S_\epsilon$, f is not identically zero on L' and therefore nor on L, so that there exists an $x_f\in L$ such that $f(x_f)\neq 0$; consequently there is a weak*-neighbourhood U_f of f in S_ϵ such that $g(x_f)\neq 0$ if $g\in U_f$, and, by the compactness of S_ϵ, there are a finite open covering $(U_i)_{1\leqslant i\leqslant m}$ of S_ϵ and elements $a_1,\ldots,a_m$ of L such that

$$0<|f(a_i)|^2\leqslant f(a_i^*a_i)\quad\text{for } f\in U_i.$$

Hence $f(a_1^*a_1+\cdots+a_m^*a_m)>0$ for every $f\in S_\epsilon$, and multiplying the a_i by sufficiently large scalars, we have $f(a_1^*a_1+\cdots+a_m^*a_m)\geqslant f(a)$ for every $f\in S_\epsilon$. Thus $f(a_1^*a_1+\cdots+a_m^*a_m+\epsilon-a)\geqslant 0$ for every positive form f, from which it follows that $a\leqslant a_1^*a_1+\cdots+a_m^*a_m+\epsilon$ (2.6.2). Hence $a\in L$ (2.9.2). L and L' thus contain the same positive elements and are therefore equal (2.9.3).

2.9.5. THEOREM. *Let A be a C^*-algebra.*

(i) *Let f be a positive form on A and N_f the left ideal of those $x \in A$ such that $f(x^*x) = 0$. Then N_f is maximal regular if and only if f is pure.*

(ii) *The map $f \to N_f$ is a bijection from the set of pure states onto the set of maximal regular left ideals.*

(iii) *Every closed left ideal of A is the intersection of the maximal regular left ideals containing it.*

(a) Let f be a positive form on A. By 2.8.4 and 2.8.5, the canonical representation π of A in A/N_f is non-trivial and algebraically irreducible. There exists $\xi \in H_\pi = A/N_f$ such that N_f is the set of $x \in A$ for which $\pi(x)\xi = 0$ and thus N_f is a maximal regular left ideal of A.

(b) Let L be a closed left ideal of A, and let S be the set of positive forms on A of norm ≤ 1 which vanish on L. If $f \in S$, then $N_f \supset L$, because for each $x \in L$, $x^*x \in L$ and so $f(x^*x) = 0$. By 2.9.4 we have $L = \bigcap_{f \in S} N_f$. Moreover S is convex and weak*-compact and is therefore the weak*-closed convex hull of its set S' of extreme points. It follows that if $x \in A$ satisfies $f(x^*x) = 0$ for every $f \in S'$, then also $f(x^*x) = 0$ for each $f \in S$, so that $\bigcap_{f \in S} N_f = \bigcap_{f \in S'} N_f$. Finally, if $f \in S$ can be written in the form $f_1 + f_2$, for positive forms f_1, f_2, we have, for each $x \in L$,

$$0 \leq f_1(x^*x) + f_2(x^*x) = f(x^*x) = 0,$$

and so $f_1(x^*x) = f_2(x^*x) = 0$, hence $f_1(x) = f_2(x) = 0$ and hence $f_1 \in S$ and $f_2 \in S$. This shows that the non-zero elements of S' are pure forms, and by virtue of (a), the equality $L = \bigcap_{f \in S'} N_f$ proves (iii).

(c) We have just seen that if L is maximal regular, and thus closed (B 1), then L is of the form N_f with f pure. Hence the map described in (ii) is surjective. If f and f' are pure states such that $N_f = N_{f'}$, proposition 2.9.1 shows that f and f' have the same kernel, so that $f = \lambda f'$ for some $\lambda \geq 0$. Since $\|f\| = \|f'\| = 1$, we have $f = f'$, which proves (ii).

(d) Finally, let f be a positive form such that N_f is maximal regular. Then $N_f = N_{f'}$ for some pure f', and the kernel of f', which is $N_{f'} + N_{f'}^* = N_f + N_f^*$, is contained in the kernel of f (2.9.1). Hence f and f' are proportional and so f is pure. This, together with (a), proves (i).

2.9.6. Recall that two representations π_1, π_2 of an algebra in vector spaces E_1, E_2 are said to be algebraically equivalent if there is an isomorphism of E_1 onto E_2 mapping π_1 to π_2.

COROLLARY. *Let A be a C^*-algebra.*

(i) *Every algebraically irreducible representation of the (non-involutive) algebra A in a complex vector space is algebraically equivalent to a representation of the C^*-algebra A (in a Hilbert space).*

(ii) *Let π, π' be two irreducible representations of the C^*-algebra A in Hilbert spaces H, H'. If π and π' are algebraically equivalent, then π and π' are equivalent in the sense of* 2.2.1.

If π is a non-zero algebraically irreducible representation of A in a complex vector space, there is a maximal regular left ideal L of A such that π can be identified with the regular representation of A in A/L. There then exists (2.9.5) a pure positive form f on A such that $L = N_f$, and A/L can be given a Hilbert space structure such that π is a representation of the C^*-algebra A in this Hilbert space. Hence (i).

Now let π, π' be two non-zero irreducible representations of the C^*-algebra A in the Hilbert spaces H, H'. Let ξ, ξ' be unit vectors in H, H' respectively and L, L' be the sets of $x \in A$ such that $\pi(x)\xi = 0$ and $\pi'(x)\xi' = 0$ respectively. Put

$$f(x) = (\pi(x)\xi \mid \xi), \qquad f'(x) = (\pi'(x)\xi' \mid \xi').$$

Then L and L' are the sets of $x \in A$ such that $f(x^*x) = 0$ and $f'(x^*x) = 0$ respectively. If π and π' are algebraically equivalent, ξ and ξ' can be chosen so that $L = L'$. Then $f = f'$ (2.9.5), so that $\pi \simeq \pi'$ (2.4.1) and (ii) follows.

It thus makes sense to speak henceforth of classes of irreducible representations of a C^*-algebra, without specifying whether the equivalence is purely algebraic or unitary.

2.9.7. Recall that a two-sided ideal of an algebra A is said to be primitive if it is the kernel of a non-zero algebraically irreducible representation of A in a vector space. We denote by Prim(A) the set of all primitive two-sided ideals of A.

THEOREM. *Let A be a C^*-algebra.*

(i) *The primitive two-sided ideals of A are just the kernels of the non-zero topologically irreducible representations of A in a Hilbert space.*

(ii) *Every closed two-sided ideal of A is the intersection of the primitive two-sided ideals containing it.*

(i) follows from 2.8.4 and 2.9.6 (i).

Let I be a closed two-sided ideal of A, so that A/I is a C^*-algebra (1.8.2). There therefore exists a family (π_i) of non-zero irreducible representations of A, vanishing on I, such that the intersection of the kernels, $\operatorname{Ker}\pi_i$, is exactly I (2.7.3). Moreover, for each i, $\operatorname{Ker}\pi_i$ is primitive.

Theorem 2.9.7 defines a canonical mapping

$$\hat{A} \to \operatorname{Prim}(A)$$

which is surjective. We shall encounter (4.3.7) a great variety of cases where this map is in fact bijective.

References: [604], [849], [893], [1323], [1456].

2.10. Extension of representations of C^*-algebras

2.10.1. Lemma. *Let A be a unital C^*-algebra, and B a self-adjoint subspace of A such that $1 \in B$. Let F be the set of linear forms g on B such that $g(x^*) = \overline{g(x)}$ for $x \in B$, $g(x) \geq 0$ for $x \in B \cap A^+$ and $g(1) = 1$. Then*

(i) *Every element of F can be extended to a state of A.*

(ii) *Every extreme point of F can be extended to a pure state of A.*

(iii) *Let g be an extreme point of F. If there is exactly one pure state f of A extending g, then f is the only state of A which extends g, and, for every hermitian element $x \in A$, we have*

$$f(x) = \sup_{y = y^* \in B,\, y \leq x} g(y).$$

Let A_h, B_h be the sets of hermitian elements in A, B respectively. Let $g \in F$ and let g' be its restriction to B_h which is real-valued and ≥ 0 on $B \cap A^+$. The element 1 of B_h is an interior point of the cone A^+, and so (B 6) g' can be extended to a real linear form f' on A_h which is ≥ 0 on A^+. Then f' extends to a hermitian linear form f on A, and f is a state of A. The forms g and $f \mid B$ coincide on B_h, and therefore on $B_h + iB_h = B$. This proves (i).

Now let g be an extreme point of F, and let K be the set of states of A that extend g. K is non-empty by (i), and is clearly convex and weak*-compact. We can therefore take f to be an extreme point of K and we shall see that f is then a pure state. Indeed, suppose f_1, f_2 are states of A such that $f = \frac{1}{2}(f_1 + f_2)$; then $g = \frac{1}{2}((f_1 \mid B) + (f_2 \mid B))$ and as g is extreme in F, we have $f_1 \mid B = f_2 \mid B = g$. Hence f_1, $f_2 \in K$ and as f is extreme in K, $f = f_1 = f_2$. This proves (ii).

Now let g be an extreme point of F and suppose there is exactly one pure state f of A which extends g. By the above argument, f is the only extreme point of K and so $K = \{f\}$. Let $x \in A_h$, $x \notin B$ and put

$$\alpha = \sup_{y \in B_h, y \leq x} g(y).$$

There is exactly one linear form g_1 on $B + \mathbf{C}x$ extending g such that $g_1(x) = \alpha$. Clearly, $g_1(y^*) = \overline{g_1(y)}$ for $y \in B + \mathbf{C}x$, and if $y + \lambda x \geq 0$ ($y \in B_h$, $\lambda \in \mathbf{R}$) then $g_1(y + \lambda x) \geq 0$: this is obvious if $\lambda = 0$; if $\lambda > 0$, we have $-\lambda^{-1}y \leq x$, so that $g(-\lambda^{-1}y) \leq \alpha$, and $\lambda\alpha + g(y) \geq 0$, while if $\lambda < 0$, $-\lambda^{-1}y \geq x$, and so $g(-\lambda^{-1}y) \geq \alpha$ and again $\lambda\alpha + g(y) \geq 0$. By the above, g_1 extends to a state of A which must coincide with f. Thus $f(x) = g_1(x) = \alpha$.

2.10.2. PROPOSITION. *Let A be a C^*-algebra, B a sub-C^*-algebra of A and ρ a representation of B in a Hilbert space K. Then there is a Hilbert space H containing K as a subspace such that the norm of H induces the original norm on K, and a representation π of A in H such that $\rho(x) = \pi(x) \mid K$ for each $x \in B$. If ρ is irreducible, π can be chosen to be irreducible.*

Let $\tilde{A}$ be the C^*-algebra obtained by adjoining an identity to A. Let $\tilde{B}$ be the sub-C^*-algebra $B + \mathbf{C} \cdot 1$ of $\tilde{A}$. The representation ρ has a unique extension to a representation $\tilde{\rho}$ of $\tilde{B}$ such that $\tilde{\rho}(1) = 1$, and $\tilde{\rho}$ is irreducible if and only if ρ is irreducible. We can thus confine attention to the case where A is unital and $1 \in B$. Furthermore, we can assume by 2.2.6 and 2.2.7 that ρ admits a cyclic vector ξ. If we put $g(x) = (\rho(x)\xi \mid \xi)$ for $x \in B$, then g is a state of B which is pure if ρ is irreducible. Let f be a state of A extending g, pure if ρ is irreducible (2.10.1). Let $\pi = \pi_f$, $\eta = \xi_f$ and let $H_0 = \overline{\pi(B)\eta} \subset H_\pi$. For each $x \in B$, let $\pi'(x)$ be the restriction of $\pi(x)$ to H_0. Then π' is a representation of B in H_0, η is a cyclic vector for π' and for each $x \in B$, we have

$$(\pi'(x)\eta \mid \eta) = f(x) = g(x) = (\rho(x)\xi \mid \xi).$$

There thus exists an isomorphism from K onto H_0 which maps ξ to η and ρ to π' (2.4.1), and so we may identify K with H_0 and ρ with π'. Finally, if ρ is irreducible, f is a pure state by construction, and so π is irreducible.

2.10.3. LEMMA. *Let A be a C^*-algebra, I a closed two-sided ideal of A, π a representation of A in H, and H' a closed (vector) subspace of H invariant under $\pi(I)$. For each $x \in I$, let $\rho(x) = \pi(x) \mid H'$, and suppose* *that the representation ρ of I in H' is non-degenerate. Then*

(i) *H' is invariant under $\pi(A)$, and if π' denotes the subrepresentation of π defined by H', then*

(ii) *$\pi'(I)$ is strongly dense in $\pi'(A)$.*

Let (u_λ) be an approximate identity in the C^*-algebra I. Since ρ is non-degenerate, $\rho(u_\lambda)$ tends strongly to 1 (2.2.10). If $x \in A$, $xu_\lambda \in I$, hence for each $\xi \in H'$,

$$\pi(x)\rho(u_\lambda)\xi = \pi(x)\pi(u_\lambda)\xi = \pi(xu_\lambda)\xi = \rho(xu_\lambda)\xi \in H',$$

so that, in the limit, $\pi(x)\xi \in H'$. This proves (i). Moreover, we see that $\pi'(x)$ is the strong limit of $\pi'(xu_\lambda)$ from which (ii) follows.

2.10.4. PROPOSITION. *Let A be a C^*-algebra, I a closed two-sided ideal of A and ρ a non-degenerate representation of I in H. Then*

(i) *There is exactly one representation π of A in H which extends ρ.*

(ii) *ρ (I) is strongly dense in $\pi(A)$.*

By (2.10.2) there is a Hilbert space $H' \supset H$ and a representation π_1 of A in H' such that $\rho(x) = \pi_1(x) \mid H$ for each $x \in I$. By 2.10.3, H is invariant under $\pi_1(A)$, and taking for π the subrepresentation of π_1 defined by H, we have the existence statement of (i) while (ii) follows from 2.10.3(ii). Finally let ν be another representation of A in H which extends ρ, and let (u_λ) be an approximate identity of I. For each $x \in A$, $\rho(xu_\lambda)$ tends strongly to $\pi(x)$ and to $\nu(x)$, hence $\pi(x) = \nu(x)$.

References: [590], [630], [1101], [1455].

2.11. Passage to an ideal and to a quotient algebra

Let A be a C^*-algebra. We shall see that any closed two-sided ideal of A leads to a decomposition of the sets $P(A)$, $\hat{A}$ and $\mathrm{Prim}(A)$.

2.11.1. LEMMA. *Let A be a C^*-algebra, I a closed two-sided ideal of A, π a representation of A in a Hilbert space H, K the essential subspace of $\pi \mid I$ and B the strong closure of $\pi(A)$. Then*

(i) *$P_K \in B \cap B'$.*

(ii) *The subrepresentation of π defined by $H \ominus K$ vanishes on I.*

(iii) *If π' is the subrepresentation of π defined by K, then $\pi'(I)$ is strongly dense in $\pi'(A)$.*

The operator P_K is in the strong closure of $\pi(I)$, so that $P_K \in B$. By 2.10.3, K is invariant under $\pi(A)$, and so $P_K \in B'$. Statement (ii) is obvious and (iii) follows from 2.10.3.

2.11.2. PROPOSITION. *Let A be a C^*-algebra, I a closed two-sided ideal of A and $\hat{A}_I$ and $\hat{A}^I$ the sets of $\pi \in \hat{A}$ such that $\pi(I) = 0$ and $\pi(I) \neq 0$ respectively. Then*

(i) *for each $\pi \in \hat{A}_I$, let π' be the quotient representation of π of A/I. Then $\pi \to \pi'$ is a bijection from $\hat{A}_I$ onto $(A/I)^\wedge$.*

(ii) *The map $\pi \to \pi \mid I$ is a bijection from $\hat{A}^I$ onto $\hat{I}$.*

Statement (i) is clear, and we pass to (ii). Let $\pi \in \hat{A}^I$. In the notation of 2.11.1, we have $K \neq \{0\}$ and so $K = H$ as π is irreducible. 2.11.1(iii) then shows that $\pi \mid I$ is irreducible (cf. also 2.11.3), and the map $\pi \to \pi \mid I$ of $\hat{A}^I$ into $\hat{I}$ is bijective by 2.10.4.

2.11.3. LEMMA. *Let A be an algebra over a (commutative) field, and π an irreducible representation of A in a vector space E. Then*

(i) *If I is a two-sided ideal of A and $\pi(I) \neq 0$, then $\pi \mid I$ is irreducible.*

(ii) *If I_1, I_2 are two-sided ideals of A with $\pi(I_1) \neq 0$ and $\pi(I_2) \neq 0$, then $\pi(I_1 \cdot I_2) \neq 0$.*

The set of elements of E on which $\pi(I)$ vanishes is invariant under $\pi(A)$ and is not the whole of E, and therefore consists of the zero vector alone. Hence if ξ is any non-zero element of E, then $\pi(I)\xi \neq 0$, and since $\pi(I)\xi$ is invariant under $\pi(A)$, we have $\pi(I)\xi = E$, which proves (i). Moreover, the preceding work shows that $\pi(I_2)E = E$ and $\pi(I_1)\,\pi(I_2)E = E$, hence $\pi(I_1 \cdot I_2) \neq 0$.

2.11.4. LEMMA. *Let I_1, I_2 be two-sided ideals of the algebra A and I a primitive ideal of A. If $I \supseteq I_1 \cdot I_2$ (in particular if $I \supseteq I_1 \cap I_2$), then either $I \supseteq I_1$ or $I \supseteq I_2$.*

Suppose $I \not\supseteq I_1$ and $I \not\supseteq I_2$. By 2.11.3(ii) applied to an irreducible representation π of A with kernel I, we have $\pi(I_1 \cdot I_2) \neq 0$ and hence $I \not\supseteq I_1 \cdot I_2$.

2.11.5. PROPOSITION. *Let A be a C^*-algebra, I a closed two-sided ideal of A and* $\mathrm{Prim}_I(A)$ *and* $\mathrm{Prim}^I(A)$ *the sets of primitive two-sided ideals of A containing I and not containing I respectively. Then*

(i) *The map $J \to J/I$ is a bijection of* $\mathrm{Prim}_I(A)$ *onto* $\mathrm{Prim}(A/I)$.

(ii) *The map $J \to J \cap I$ is a bijection of* $\mathrm{Prim}^I(A)$ *onto* $\mathrm{Prim}(I)$.

Statement (i) is obvious, so we prove (ii). If $J \in \mathrm{Prim}^I(A)$, then J is the kernel of an irreducible representation π of A such that $\pi(I) \neq 0$, hence $\pi \mid I \in \hat{I}$ (2.11.2) and $J \cap I = \mathrm{Ker}(\pi \mid I) \in \mathrm{Prim}(I)$. If $J' \in \mathrm{Prim}(I)$, there exists $\pi' \in \hat{I}$ such that $J' = \mathrm{Ker}\ \pi'$ and then there is $\pi \in \hat{A}$ such that $\pi' = \pi \mid I$ (2.10.4), from which it follows that $J' = (\mathrm{Ker}\ \pi) \cap I$ and the map described in (ii) is surjective. Let $J_1, J_2 \in \mathrm{Prim}^I(A)$ and suppose that $J_1 \cap I = J_2 \cap I$; then $J_2 \supseteq J_1 \cap I$ and $J_2 \not\supseteq I$, so that $J_2 \supseteq J_1$ (2.11.4) and similarly $J_1 \supseteq J_2$. Thus $J_1 = J_2$ and the map given in (ii) is injective.

2.11.6. Before obtaining a result analogous to propositions 2.11.2 and 2.11.5 for pure states, we establish several properties of general positive forms.

Let A be a C^*-algebra, I a closed two-sided ideal of A and $\omega: A \to A/I$ the canonical morphism. Since $\omega(A^+) = (A/I)^+$, the map $k \to k \circ \omega$ is a bijection of the set of positive forms on A/I onto the set of positive forms on A that vanish on I. It follows from general properties of Banach spaces, that this bijection is norm-preserving and is bicontinuous for the weak*-topologies.

2.11.7. Proposition. *Let A be a C^*-algebra, I a closed two-sided ideal of A and f a positive form on A. Then*

(i) *There exists a unique decomposition $f = f_1 + f_2$ where f_1, f_2 are positive forms on A such that $\|f_1\| = \|f_1 \mid I\|$ and $f_2(I) = 0$.*

(ii) *The pair (π_f, ξ_f) may be identified with $(\pi_{f_1} \oplus \pi_{f_2}, \xi_{f_1} + \xi_{f_2})$; π_{f_2} vanishes on I and $\pi_{f_1} \mid I$ is non-degenerate.*

Put $\pi_f = \pi$. Let K_1 be the essential subspace of $\pi \mid I$, $K_2 = H_\pi \ominus K_1$, $\xi_1 = P_{K_1}\xi_f$, $\xi_2 = P_{K_2}\xi_f$, and π_1 and π_2 the subrepresentations of π defined by K_1 and K_2 [2.11.1(i)]. Then ξ_i is a cyclic vector for π_i, and so if $f_i = \omega_{\xi_i} \circ \pi_i$, (π_i, ξ_i) may be identified with (π_{f_i}, ξ_{f_i}) (2.4.6). $\pi_2(I) = 0$, and so $f_2(I) = 0$. The representations π_1 and $\pi_1 \mid I$ are non-degenerate, and so

$$\|f_1 \mid I\| = \|\xi_1\|^2 = \|f_1\|,$$

by 2.4.3. The relation $\xi_f = \xi_1 + \xi_2$ implies $f = f_1 + f_2$. In particular, $f_2 = 0$ if $\|f\| = \|f \mid I\|$.

It remains to prove the uniqueness statement in (i), so let f_1', f_2' be positive forms on A such that $f = f_1' + f_2'$, $\|f_1'\| = \|f_1' \mid I\|$ and $f_2'(I) = 0$. The above argument, applied to f_1' instead of f, shows that $\pi_{f_1'} \mid I$ is non-degenerate; thus if (u_λ) is an approximate identity of I we have $f_1'(x) =$

$\lim f_1'(xu_\lambda)$ for every $x \in A$ and similarly $f_1(x) = \lim f_1(xu_\lambda)$. f_1 and f_1' therefore coincide on I since $f_2(I) = f_2'(I) = 0$ and thus $f_1' = f_1$ and $f_2' = f_2$.

2.11.8. PROPOSITION. *Let A be a C^*-algebra, I a closed two-sided ideal of A, and $P_I(A)$ and $P^I(A)$ the sets of pure states of A which vanish on I and which do not vanish on I respectively. Then*

(i) *If, for each $f \in P_I(A)$, f' denotes the positive form on A/I obtained from f by passing to the quotient, then $f \to f'$ is a bijection of $P_I(A)$ onto $P(A/I)$.*

(ii) *The map $f \to f|I$ is a bijection of $P^I(A)$ onto $P(I)$.*

Let $f \in P_I(A)$, and let ω denote the canonical morphism of A onto A/I. If f' dominates a positive form g' on A/I, then f dominates $g = g' \circ \omega$ and so $g = \lambda f$ with $0 \leq \lambda \leq 1$, $g' = \lambda f'$ from which it follows that $f' \in P(A/I)$. It is clear that the map $f \to f'$ is injective, so we have only to show that it is surjective. Thus, let $h \in P(A/I)$ and $f = h \circ \omega$. If f dominates a positive form g, then g vanishes on I^+ and therefore on I, and so g can be written $k \circ \omega$ where k is a positive form dominated by h. Hence $k = \lambda h$, with $0 \leq \lambda \leq 1$, so that $g = \lambda f$ and $f \in P_I(A)$. Since $h = f'$, we have indeed shown that the map $f \to f'$ is surjective.

Now let $f \in P^I(A)$. In the decomposition $f = f_1 + f_2$ of 2.11.7, f_1 and f_2 are proportional to f. Now f_2 vanishes on I and f does not vanish on I so that $f_2 = 0$ and $f = f_1$. Hence $\|f|I\| = \|f\| = 1$, from which it follows that $f|I$ is a state of I. By 2.10.3 and 2.11.7(ii), $\pi_{f|I}$ is irreducible and so $f|I \in P(I)$. Let $f, f' \in P^I(A)$; if $f|I = f'|I$, there is an isomorphism of H_{π_f} onto $H_{\pi_{f'}}$, which maps $\xi_{f|I}$ (i.e. ξ_f) into $\xi_{f'|I}$ (i.e. $\xi_{f'}$) and $\pi_{f|I}$ into $\pi_{f'|I}$, and therefore π_f into $\pi_{f'}$ (2.10.4). Thus $f = f'$, and the map $f \to f|I$ of (ii) is injective. Finally, if $g \in P(I)$, the irreducible representation π_g of I can be extended to an irreducible representation π of A (2.10.4). Since $\|\xi_g\| = 1$, $f = \omega_{\xi_g} \circ \pi$ is a pure state of A extending g, and thus $f \in P^I(A)$ with $f|I = g$, which shows that the map $f \to f|I$ of (ii) is surjective.

2.11.9. The six bijections of propositions 2.11.2, 2.11.5 and 2.11.8 are said to be canonical.

To sum up, we have the following diagram of canonical maps:

$$\begin{array}{ccccc} P(A/I) & \rightarrow & P(A) & \leftarrow & P(I) \\ \downarrow & & \downarrow & & \downarrow \\ (A/I)^\wedge & \rightarrow & \hat{A} & \leftarrow & \hat{I} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Prim}(A/I) & \rightarrow & \mathrm{Prim}(A) & \leftarrow & \mathrm{Prim}(I) \end{array}$$

The vertical arrows represent surjective maps, and on each line, the two horizontal arrows represent injections of complementary subsets into the middle set. It is easily checked that this diagram is commutative.

References: [508], [819], [896], [1456].

2.12. Addenda

2.12.1. Let A be a unital C^*-algebra and f, g pure states of A such that $\|f-g\|<2$. Then $\pi_f \simeq \pi_g$. [633].

2.12.2. Let A be a unital C^*-algebra, $E(A)$ the set of states of A, and $f \in E(A)$. Then the following conditions are equivalent:

(i) for every neighbourhood U of f in $E(A)$, there is a $\delta > 0$ and $a \in A$ with $0 \leqslant a \leqslant 1$, $f(a)=1$ and $g(a)<1-\delta$ for $g \in E(A)-U$;

(ii) for every closed G_δ set S of $E(A)$ containing f, there is an $a \in A$ such that $\|a\|=|f(a)|$ and S contains the set of $g \in E(A)$ for which $\|a\|=|g(a)|$;

(iii) f is a pure state. [631].

2.12.3. Let A be a C^*-algebra and K_n the set of states f of A such that $\dim \pi_f = n$ $(<+\infty)$. Then $P(A) \cap K_n$ is an open subset of K_n. [638].

2.12.4. Let A be a von Neumann algebra, $\mathscr{Z}$ its centre and Ω the spectrum of $\mathscr{Z}$. For each $\omega \in \Omega$, let I_ω be the norm-closed two-sided ideal of A generated by Ker ω, and for each $T \in A$, let T_ω be the canonical image of T in A/I_ω. Then

(a) If f is a pure state of A, then $f \mid \mathscr{Z}$ is an element ω of Ω and f vanishes on I_ω.

(b) For every $T \in A$, the function $\omega \to \|T_\omega\|$ is continuous on Ω.

(c) In A/I_ω, the product of two non-zero two-sided ideals is non-zero. [630].

2.12.5. Let A be an involutive unital Banach algebra. Suppose that for every continuous hermitian linear form f on A, there exist two positive forms g, h such that $f=g-h$ and $\|f\|=\|g\|+\|h\|$. Then A is a C^*-algebra. [681].

2.12.6. Let A_1, A_2 be unital C^*-algebras, and $\overline{P(A_i)}$ the weak-closure of $P(A_i)$ in the dual of A_i. Let $\mathscr{L}_i$ be the set of functions $f \to f(x)$ on $\overline{P(A_i)}$ $(x \in A_i)$. If there is a homeomorphism of $\overline{P(A_1)}$ onto $\overline{P(A_2)}$ which maps $\mathscr{L}_1$ into $\mathscr{L}_2$, then there is a bijective linear isometry ρ: $A_1 \to A_2$ such that $\rho(x^*) = \rho(x)^*$ for each $x \in A_1$ and $\rho(x^n) = \rho(x)^n$ for each hermitian x in A_1 (cf. 1.9.10). [839].

2.12.7. Let A be a unital C^*-algebra, $\overline{P(A)}$ the weak*-closure of $P(A)$ in the dual of A and $\mathscr{L}$ the set of functions $f \to f(x)$ on $\overline{P(A)}$ $(x \in A)$. If the product of two elements of $\mathscr{L}$ is again in $\mathscr{L}$, A is commutative. [839].

2.12.8. Let A be a unital C^*-algebra, f a state of A and N the set of $x \in A$ such that $f(x^*x) = 0$. If $\operatorname{Ker} f = N + N^*$, then f is pure. [849].

*2.12.9. For a unital C^*-algebra A to be isomorphic to a von Neumann algebra, it is necessary and sufficient that it satisfies the following conditions:

(i) every increasing family of hermitian elements of A that is bounded above has a least upper bound;

(ii) for every non-zero $x \in A^+$, there is a state f of A with $f(x) \neq 0$, such that for every increasing family (y_i) in A^+ with least upper bound $y \in A^+$, $f(y) = \sup f(y_i)$. [849], [1590], [1591].

*2.12.10. Let A be a unital C^*-algebra and X the compact space $\overline{P(A)}$. For each $x \in A$, let ϕ_x be the function $f \to f(x)$ on $\overline{P(A)}$ and let $\mathscr{L}$ be the set of all the ϕ_x's for $x \in A$. Each state g of A goes over, via $x \to \phi_x$, to a positive linear form on $\mathscr{L}$, and this form can be extended, in infinitely many ways in general, to a positive measure on X; the collection of those subsets of X which are of measure zero for all these measures will be denoted by N_g.

Now let π be a representation of A, and denote by N_π the intersection of the $N_{h \circ \pi}$ as h varies through the set of normal states in the weak*-closure of $\pi(A)$.

Let H be a separable Hilbert space, A a sub-C^*-algebra of $\mathscr{L}(H)$ containing 1, B the weak closure of A and π a representation of A in a separable Hilbert space. Then π can be extended to an ultraweakly continuous representation of B in H_π if and only if $N_\pi \supseteq N_\iota$, where ι denotes the identical representation of A. [848].

2.12.11. Let A be a C^*-algebra and G a topological group. For each $g \in G$, let ζ_g be an automorphism of A. Suppose that $g \to \zeta_g$ is a representation of G and that $\zeta_g(x)$ is a continuous function of g for each $x \in A$. A state f of A is said to be stationary for ζ if $f(\zeta_g(x)) = f(x)$ for each $x \in A$ and $g \in G$. Suppose f satisfies this. Then there exists a unique continuous unitary representation ρ of G in H_{π_f} such that: (1)

$\pi_f(\zeta_g(x)) = \rho(g)\pi_f(x)\rho(g^{-1})$ for each $x \in A$ and $g \in G$; (2) if η denotes the canonical map of A into H_{π_f}, then $\eta(\zeta_g(x)) = \rho(g)\eta(x)$ for each $x \in A$ and $g \in G$. [1460].

2.12.12. Let A be a unital C^*-algebra, x a normal element of A and $\lambda \in \operatorname{Sp} x$. Then there is an irreducible representation of A and a non-zero ξ in H_π such that $\pi(x)\xi = \lambda\xi$. (Apply 2.10.2 to the sub-C^*-algebra B of A generated by 1 and x and to the character of B which takes the value λ at x.) [1455].

2.12.13. Let A be a non-unital C^*-algebra. Then 0 is in the weak*-closure of $P(A)$. [Let $x_1, \ldots, x_n \in A^+$ and $\epsilon > 0$; we have to construct an $f \in P(A)$ such that $f(x_1), \ldots, f(x_n) \leq \epsilon$; replacing $x_1, \ldots, x_n$ by $x_1 + \cdots + x_n$ we can assume $n = 1$ and using 2.5.5, it is enough to construct a state g of A such that $g(x_1) \leq \epsilon$; realizing A in a Hilbert space H by means of a non-degenerate representation, x_1 is not invertible in $A + \mathbf{C} \cdot 1$ and therefore not invertible in $\mathscr{L}(H)$; hence there is a unit vector ξ of H such that $(x_1\xi \mid \xi) \leq \epsilon$: take $g = \omega_\xi$.] This was communicated to me by J. Glimm.

2.12.14. Let A be a C^*-algebra, I and J closed two-sided ideals of A and h a state of A that vanishes on $I \cap J$. Then $h = \lambda f + \mu g$ where $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu = 1$ and where f, g are states of A which vanish on I and J respectively. (It is enough to consider the case where $I \cap J = 0$. Then $\pi_h \mid I$ and $\pi_h \mid J$ have orthogonal essential subspaces. Apply 2.11.7.) [1456].

2.12.15. Let A and B be C^*-algebras, π, π' injective representations of A and ρ, ρ' injective representations of B. Let D and D' be the C^*-algebras of operators on $H_\pi \otimes H_\rho$ and $H_{\pi'} \otimes H_{\rho'}$ respectively generated by the $\pi(x) \otimes \rho(y)$ and $\pi'(x) \otimes \rho'(y)$ respectively where $x \in A$, $y \in B$. There is a unique isomorphism of D onto D' which maps $\pi(x) \otimes \rho(y)$ into $\pi'(x) \otimes \rho'(y)$ for any $x \in A$, $y \in B$. Thus D, regarded as an abstract C^*-algebra, depends only on A and B and is called the C^*-tensor product of A and B. [692], [701], [1613], [1614], [1620], [1728], [1729], [1731], [1847].

*2.12.16. (a) Let A and B be C^*-algebras and μ a linear map from A into B. μ is said to be positive if $\mu(A^+) \subset B^+$. Now let $A^{(n)}$ be the C^*-algebra of $(n \times n)$ matrices with entries in A. Applying μ to each element of such a matrix we obtain a map $\mu^{(n)}$ from $A^{(n)}$ into $B^{(n)}$, and μ

is said to be completely positive if $\mu^{(n)}$ is positive for each n. In general, complete positivity is a more restrictive condition than positivity, although the two notions are equivalent if A is commutative.

(b) Let A be a unital C^*-algebra, H a Hilbert space and μ a linear map from A into $\mathscr{L}(H)$ such that $\mu(1) = I$. Then there is a Hilbert space K containing H as a subspace and a morphism ρ from A into $\mathscr{L}(K)$ such that $\mu(x) = P_H\rho(x) \mid H$ for each $x \in A$ if and only if μ is completely positive. [1506].

2.12.17. Let A be a C^*-algebra, A' the Banach dual of A and A_h and A'_h the hermitian parts of A and A', which are partially ordered vector spaces. Then the following conditions are equivalent: (1) A_h is a lattice; (2) A'_h is a lattice; (3) A is commutative. [606].

2.12.18. Let A be a C^*-algebra, f a pure state of A and N_f the set of $x \in A$ such that $f(x^*x) = 0$, so that A/N_f endowed with the scalar product derived from $f(y^*x)$ is a Hilbert space. Then the norm on this space is equally the quotient norm of the Banach space A/N_f. [1612].

*2.12.19. Let A be a unital C^*-algebra and f a positive form on A.

(a) Let E_f be the set of linear forms on A of the type $x \to f(x_0xx_0^*)$ where $x_0 \in A$. Let F_f be the closure (in the norm topology) of E_f in the dual A' of A. Then if g is a positive form on A, the following conditions are equivalent: (i) $g \in F_f$; (ii) $\pi_g \leq \pi_f$; (iii) there is a ξ in the space of π_f such that $g(x) = (\pi_f(x)\xi \mid \xi)$ for each $x \in A$.

(b) If f is pure, then $F_f = E_f$.

(c) Let E'_f be the set of positive forms on A dominated by a multiple λf of f (where $\lambda \geq 0$). Let F'_f be the closure (in the norm topology) of E'_f in A'. Then F'_f is the set of forms

$$x \to (\pi_f(x)\eta \mid \eta) \text{ on } A, \quad \text{where } \eta \in \overline{\pi_f(A)'\xi_f} \text{ [852].}$$

2.12.20. Let A be an involutive Banach algebra with an approximate identity and f a continuous positive form on A. Let $L^2(f)$ denote the set of $g \in A'$ such that

$$\sup_{x \in A,\, f(x^*x) \neq 0} |g(x)| f(x^*x)^{-1/2} < +\infty.$$

Let H_f be the space of π_f and η_f the canonical map of A into H_f. Then taking the transpose ζ_f of η_f, we have a map of H_f into A' such that $\langle \zeta_f(\xi), x \rangle = (\eta_f(x) \mid \xi)$ for any $\xi \in H_f$ and $x \in A$. The map ζ_f is a conjugate linear bijection of H_f onto $L^2(f)$ which maps ξ_f into f. For each $x \in A$, let

$\pi'(x)$ be the transpose of left-multiplication by x in A, which is a continuous linear operator on A'. Then ζ_f maps $\pi_f(x)$ into $\pi'(x^*) \mid L^2(f)$. Finally, ζ_f is isometric when $L^2(f)$ is endowed with the norm

$$\|g\|_{L^2(f)} = \sup_{x \in A,\, f(x^*x) \neq 0} |g(x)| f(x^*x)^{-1/2}$$

2.12.21. If A is a C^*-algebra in which the only nilpotent element is 0, then A is commutative. [Let $x, y \in A$ with $x = x^*$ and f, g be continuous functions on $\mathrm{Sp}' x$ with $fg = 0$; then $(f(x)yg(x))^2 = 0$, hence $f(x)yg(x) = 0$ and so $f(\pi(x))Tg(\pi(x)) = 0$ for every irreducible representation π of A and every $T \in \mathscr{L}(H_\pi)$, therefore $\pi(A)$ is scalar.] (I. Kaplansky, unpublished; communicated to me by R. V. Kadison.)