CLOSED GEODESICS
ON
ORBIFOLDS

CLOSED GEODESICS ON COMPACT DEVELOPABLE ORBIFOLDS

By

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

McMaster University

DEGREE:	DOCTOR OF PHILOSOPHY, 2011
UNIVERSITY:	McMaster University, Hamilton, Ontario
DEPARTMENT:	Mathematics and Statistics
TITLE:	Closed geodesics on compact developable orbifolds
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PAGES:	xiv, 153

MCMASTER UNIVERSITY
DEPARTMENT OF
MATHEMATICS AND STATISTICS

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Dated: June 2011

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To Miruna

Abstract

Existence of closed geodesics on compact manifolds was first proved by Lyusternik and Fet in [44] using Morse theory, and the corresponding problem for orbifolds was studied by Guruprasad and Haefliger in [33], who proved existence of a closed geodesic of positive length in numerous cases.

In this thesis, we develop an alternative approach to the problem of existence of closed geodesics on compact orbifolds by studying the geometry of group actions. We give an independent and elementary proof that recovers and extends the results in [33] for developable orbifolds. We show that every compact orbifold of dimension 2, 3, 5 or 7 admits a closed geodesic of positive length, and we give an inductive argument that reduces the existence problem to the case of a compact developable orbifold of even dimension whose singular locus is zero-dimensional and whose orbifold fundamental group is infinite torsion and of odd exponent.

Stronger results are obtained under curvature assumptions. For instance, one can show that infinite torsion groups do not act geometrically on simply connected manifolds of nonpositive or nonnegative curvature, and we apply this to prove existence of closed geodesics for compact orbifolds of nonpositive or nonnegative curvature.

In the general case, the problem of existence of closed geodesics on compact orbifolds is seen to be intimately related to the group-

theoretic question of finite presentability of infinite torsion groups, and we explore these and other properties of the orbifold fundamental group in the last chapter.

Acknowledgements

I am deeply honoured to take this opportunity and thank my advisor, Hans Boden, for his crucial advice and guidance, and for his unabated encouragement. I am grateful beyond words to have had in Hans such an excellent mentor and incredible friend who shared so much of his time, wisdom and passion for math.

I am equally honoured to thank Ian Hambleton and Andy Nicas for their continuous support, instruction and inspiration throughout my time at McMaster. This chapter of my life would not have been possible without their extraordinary influence.

My gratitude also goes to the Department of Mathematics, and especially to the geometry and topology group at McMaster, for their welcoming environment and the financial support they provided throughout my studies.

Most of all I thank my daughter and my wife for their enduring patience, love and support. Many thanks to my sister for encouraging me from afar, and of course to my parents for making everything possible.

I dedicate this thesis to Miruna, my daughter, whose very smile has always been the vital inspiration when challenges seemed insurmountable.

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Introduction

An orbifold is perhaps the simplest case of a singular space; it is a topological space which is locally modeled on quotients of open subsets in Euclidean space by linear actions of finite groups. Each point of an orbifold carries additional data, that of a finite isotropy group, and the orbifold structure encodes this information. Orbifolds were first introduced in the 1950's, by Satake [57], [58], in the context of Riemannian geometry under the name of V-manifolds. Later, in the 1970's, Thurston [66] rediscovered and renamed them in his studies of hyperbolic structures of 3-manifolds. During this period, orbifolds were regarded as a natural generalization of manifolds to singular spaces. Early results on orbifolds were motivated by an attempt to adapt and generalize all notions and techniques from the classical theory of manifolds to this more general setting.

Orbifolds arise naturally throughout geometry and one reason for the interest in orbifolds is that they exhibit geometric properties similar to manifolds. The specific problem that we take up in this thesis has to do with the existence of geodesics on compact Riemannian orbifolds. Every compact manifold contains a closed geodesic [41], and a beautiful and elegant proof of this fact is presented in [47], where Morse theory is applied to the energy functional on the loop space to prove existence of closed geodesics on compact manifolds. The corresponding problem for orbifolds was studied by Guruprasad and Haefliger in [33], which adapts the Morse theoretic approach to the orbifold setting and proves the existence of closed geodesics on compact orbifolds whenever (1) the orbifold is not developable, or (2) the orbifold is developable and its orbifold fundamental group is finite or contains an element of infinite order. Despite this progress, the general problem of existence of closed geodesics on compact orbifolds remains open.

In this thesis we investigate the problem of existence of closed geodesics on compact orbifolds and employ a geometric approach that recovers and extends the results of [33] for the developable case. Our results are established by using the geometry of the universal cover and properties of the orbifold fundamental group.

For instance, we show in Theorem 4.3 that a compact developable orbifold $\mathcal{Q}$ admits a closed geodesic of positive length whenever the orbifold fundamental group contains a hyperbolic element, or equivalently whenever $\mathcal{Q}$ admits an intermediate manifold cover. Because infinite order elements in $\pi_1^{orb}(\mathcal{Q})$ are always hyperbolic, this result implies, but is strictly stronger than, the result in [33] for developable orbifolds. This same result is proved by Alexandrino and Javaloyes in [6, Theorem 2.17], and both proofs are elementary and based on [8, Lemma 6.5].

On the level of the orbifold fundamental group, our first existence result reduces the problem to compact orbifolds $\mathcal{Q}$ with $\pi_1^{orb}(\mathcal{Q})$ an infinite group of finite exponent and with finitely many conjugacy classes (cf. Propositions 5.2, 5.3, 5.4). We further reduce the existence problem by noticing that in a general orbifold, the singular locus contains preferred components which form a totally geodesic suborbifold whose singular locus is zero-dimensional. This observation allows us to establish existence of closed geodesics in many situations, as summarized in Proposition 4.7. In particular we obtain existence for closed geodesics on all compact orbifolds of dimension 2, 3, 5 or 7 (cf. Remark 4.2 and Corollary 4.10).

An interesting question related to all of this is whether an infinite torsion group can act geometrically on a simply connected complete Riemannian manifold M (cf. Question 4.6). While we are not able to rule out such actions in the general case, we are able to show that such actions cannot occur if M is assumed to carry a metric of nonpositive or nonnegative sectional curvature (see Proposition 4.13 and Proposition 4.15). As a consequence we conclude that any compact orbifold of nonpositive or nonnegative curvature admits a closed geodesic of positive length (see Corollary 4.14 and Corollary 4.16).

The outline of the thesis is as follows. In the first chapter we collect some standard background material on metric spaces and groups actions on metric spaces and smooth manifolds.

The second chapter introduces orbifolds and the necessary notions for understanding their geometry. We review the notion of a tangent space to an orbifold and give the definition of a Riemannian metric on an orbifold; we define orbifold paths and describe the orbifold fundamental group, a true invariant of the orbifold structure. Along the way, we particularize these notions for the case of developable orbifolds and we conclude the chapter with a series of examples.

In the third chapter we take a closer look at the natural stratification by orbit type of a Riemannian orbifold. We notice that the orbifold decomposes into pieces which are totally geodesic, and use this decomposition to show the existence of closed geodesics of positive length which are contained in the singular locus. This result allows us to reduce the problem of existence of closed geodesics to the particular case of a compact orbifold with only zero dimensional singular locus.

In the fourth chapter we present elementary proofs for the existence of closed geodesics for a large class of compact developable orbifolds and we recover as a particular case the results in [33]. We also show that all 3, 5 and 7-dimensional orbifolds admit closed geodesics of positive length. Furthermore, we reduce the existence problem to a very specific case: that of an even-dimensional compact developable orbifold with finitely many orbifold points. Further we show that one can assume that the orbifold fundamental group is infinite and contains only elliptic isometries. In particular it is torsion and has odd exponent. More existence results are obtained under additional geometric conditions on the universal cover of the orbifold. For example, we show that if the universal cover admits a metric of nonpositive or nonnegative sectional curvature, then the orbifold fundamental group cannot be infinite torsion.

In the last chapter, we summarize properties of the orbifold fundamental group where existence of closed geodesics does not follow by the methods developed previously. The orbifold fundamental group Γ exhibits two seemingly incompatible properties. On the one hand, Γ must be finitely presented, while on the other hand, it can be assumed to be an infinite group of bounded exponent. While examples of infinite torsion groups which are finitely generated and even of finite exponent are known to exist, there are no examples known to be finitely presentable. Finitely generated infinite torsion groups are the object of an important problem in group theory, known as the Burnside problem, and the last chapter presents a brief summary on the questions of Burnside and others, along with several other group-theoretic questions motivated by the problem of existence of closed geodesics on orbifolds.

The chapters are reasonably self-contained, and readers familiar with the basic material of group actions on metric spaces and orbifolds may prefer to begin with chapters 3 and 4, which contain the new results, and to refer back to the earlier chapters as necessary.

Chapter 1

Background

This chapter contains the basic definitions and results on metric spaces and actions by discrete groups. Of particular interest is the case of a Riemannian manifold, and in the last section, we develop the theory in the general context of a smooth manifold and group acting smoothly.

1.1 Metric Spaces

This section is concerned with metric spaces. The presentation is mainly descriptive and we chose not to insist on all the aspects of the theory, but rather to briefly include only the definitions and properties that we will use in this work. For this reason we chose not to label the results or the definitions and also not to include proofs. The material contained in this section is standard and there are many excellent references. With little exceptions, our notations and presentation follow mainly the ones in [11].

Basic Definitions and Facts

A *metric space* is a pair (X, d), where X is a space and d is a *metric* (or *distance function*) on X. That is, d is a real-valued function $d : X \times X \to \mathbb{R}$ satisfying the following two axioms for all $x, y, z \in X$:

i. $d(x, y) = 0$ if and only if $x = y$

ii. $d(x, z) \leq d(x, y) + d(z, y)$.

Aa a direct consequence of the two axioms d is also non-negative ($d(x, y) \geq 0$ for all x, y) and symmetric ($d(x, y) = d(y, x)$ for all x, y).

If (X, d) is a metric space and $x \in X$ we define *open ball of radius* $r > 0$ *about* x as the set $B(x, r) := \{y \in X \mid d(x, y) < r\}$ and the *closed ball of radius* $r > 0$ *about* x to be $\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}$.

Every metric space (X, d) has a natural topology induced by the metric: the topology generated by the set of open balls $B(x, r)$. A metric space is called *proper* if every closed ball $\overline{B}(x, r)$ is compact in this topology.

The *diameter* of a metric space (X, d) is the value in $[0, \infty) \cup \{\infty\}$ given by $\operatorname{diam}(X) := \sup\{d(x, y) \mid x, y \in X\}$.

Suppose (X, d) is a metric space and let $C \subset X$ be a subset of X. For $\varepsilon > 0$ we denote by

$$V_\varepsilon(C) = \{x \in X \mid d(x, y) < \varepsilon \text{ for some } y \in C\}$$

the ε-*neighbourhood* of C in X. Given two sets $C_1, C_2 \subset X$ the *Hausdorff distance* between them is defined by

$$d_H(C_1, C_2) = \inf\{\varepsilon \mid C_1 \subseteq V_\varepsilon(C_2) \text{ and } C_2 \subseteq V_\varepsilon(C_1)\}.$$

A sequence (x_n) of points in X is said to converge to $x \in X$ if and only if for every $r > 0$ there exists a natural number $N = N_r$ such that $d(x_n, x) < r$ for all $n \geq N$. Clearly the metric convergence defined above is equivalent to the convergence in the induced topology. Note also that the properties of d forces the limits to be unique, thus every metric space is Hausdorff. A metric space is said to be *complete* if every Cauchy sequence is convergent.

Isometries

An *isometry* between two metric spaces (X, d) and (X', d') is a bijection $f : X \to X'$ such that $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Every isometry is a homeomorphism and the composition of two isometries is again an isometry. The set of all isometries from a metric space (X, d) to itself, together with the composition of maps, form a group denoted $\mathrm{Isom}(X)$. It is a subgroup of $\mathrm{Homeo}(X)$, the group of homeomorphisms of X.

If γ is an isometry of (X, d) and $x \in X$, it is often convenient to denote the point $\gamma(x) \in X$ by $\gamma.x$ (or γx) and to refer to it as the translate of x by γ. The *displacement function* of γ is the function $d_\gamma : X \to \mathbb{R}_+$ defined by $d_\gamma(x) = d(x, \gamma x)$. The *translation length* of γ is the number $|\gamma| := \inf\{d_\gamma(x) \mid x \in X\}$.

The *minimal set* of γ, denoted $\mathrm{Min}(\gamma) := \{x \in X \mid d_\gamma(x) = |\gamma|\}$, is the set of points where the displacement function attains this infimum. An isometry γ is called *semi-simple* if $\mathrm{Min}(\gamma)$ is non-empty.

We can use the displacement function to classify the isometries of a metric space. An isometry γ of X is called:

(i) *elliptic* if γ has a fixed point (i.e. there exists $x \in X$ so that $\gamma x = x$),

(ii) *hyperbolic* if its displacement d_γ attains a strictly positive minimum (i.e. there exists $x_0 \in X$ such that $0 < d_\gamma(x_0) = d(x_0, \gamma x_0) \leq d(x, \gamma x)$ for all $x \in X$),

(iii) *parabolic* if d_γ does not attain a minimum, i.e. $\mathrm{Min}(\gamma)$ is empty.

Each isometry of a metric space belongs to one of the above disjoint classes and an isometry is semi-simple if and only if it is elliptic or hyperbolic. Moreover, if two isometries are conjugate in $\mathrm{Isom}(X)$, then they have the same translation length and belong to the same class.

Geodesics

Let (X, d) be a metric space.

A *geodesic path* joining two points x and y in X is a map $c : [0, a] \to X$ such that $c(0) = x$, $c(a) = y$ and $d(c(t), c(s)) = |t - s|$ for all $t, s \in [0, a]$. In particular, $d(x, y) = a$ which implies that geodesics minimize distance globally. The image $c([0, a]) \subset X$ is called a *geodesic segment* with endpoints x and y.

A *local geodesic* in X is a map c from an interval $I \subseteq \mathbb{R}$ to X with the property that for every $t \in I$ there exists a neighbourhood $J \subseteq I$ of t such that $d(c(s'), c(s'')) = |s' - s''|$ for all $s', s'' \in J$.

A *geodesic line* in X is an isometric embedding of the real line $\mathbb{R}$ in X, i.e. a map $\ell : \mathbb{R} \to X$ such that $d(\ell(t), \ell(s)) = |t - s|$ for all $t, s \in \mathbb{R}$.

A metric space (X, d) is said to be a *geodesic space* if every two points in X can be connected through a geodesic, and is said to be *uniquely geodesic* if there is exactly one geodesic between any two points in X.

A subset C of a metric space is said to be *convex* if every pair of points $x, y \in C$ can be connected by a geodesic in X and the image of every such geodesic is contained in C.

A *geodesic ray* in X is a map $r : [0, \infty) \to X$ such that $d(r(t), r(s)) = |t - s|$ for all $t, s \geq 0$. The point $r(0)$ is called the *origin* of the ray r. Given two geodesic rays $r, r' : [0, \infty) \to X$ and in X, we say that r' is a *subray of* r if there exists $t_0 \geq 0$ such that $r'(t) = r(t + t_0)$ for all $t \geq 0$.

Note that there are no geodesic rays in a metric space of finite diameter. However, if X is a complete geodesic space of infinite diameter, then every point in X is the origin of a geodesic ray.

1.2 Group Actions

In this section we briefly recall some terminology and general remarks concerning discrete group actions on topological spaces. Particular attention will be given to proper actions and actions by isometries on metric spaces. Our definitions and notations follow mainly [11]. For a more detailed introduction to actions of discrete groups the reader may consult [9].

Basic Definitions and Notations

We will begin with some formal definitions. Suppose Γ is a group and X is a space.

An *action* of Γ on X is a map $\Gamma \times X \to X,\ (\gamma, x) \mapsto \gamma.x$ such that, for all $x \in X$

(*i*) $(\gamma \cdot \delta).x = \gamma.(\delta.x)$ for all $\gamma, \delta \in \Gamma$, and

(*ii*) $1.x = x$, where 1 is the identity element of Γ.

The space X with an action by the group Γ is called a Γ-space.

Given an action $\Gamma \times X \to X$, for each $\gamma \in \Gamma$, we have a function $\Phi_\gamma : X \to X$, defined by

$$\Phi_\gamma(x) := \gamma.x, \text{ for all } x \in X.$$

Using the defining properties of the action we can easily see that each of the functions Φ_γ is a bijection with inverse $\Phi_{\gamma^{-1}}$, and that $\Phi_\gamma \circ \Phi_\delta = \Phi_{\gamma\delta}$. Thus the map $\gamma \mapsto \Phi_\gamma$ defines a group homomorphism from Γ to $\mathrm{Sym}(X)$, the symmetric group of X (i.e. the group of bijections from X to itself). Conversely, any group homomorphism $\Phi : \Gamma \to \mathrm{Sym}(X)$ yields an action of Γ on X defined by $\gamma.x = \Phi(\gamma)(x)$.

Thus, an action of a group Γ on a space X can be thought of as a homomorphism $\Phi : \Gamma \to \mathrm{Sym}(X)$. If the space X has additional structure, say X is a smooth or a Riemannian manifold, then we distinguish several types of actions of a group Γ by letting Φ have its image in various subgroups of symmetries that preserve the extra structure. For instance, we say that Γ acts on X by *homeomorphisms* (or *diffeomorphisms*, or *isometries*) if the image of the homomorphism Φ is contained in $\mathrm{Homeo}(X)$ (resp. $\mathrm{Diffeo}(X)$ or $\mathrm{Isom}(X)$). That is, for each $\gamma \in \Gamma$ the map $x \mapsto \gamma.x$ is a homeomorphism (reps. diffeomorphism or isometry) of X.

Given two Γ-spaces X and Y, a function $f : X \to Y$ is said to be Γ-*equivariant* if it "commutes with the action", i.e. if for all $x \in X$ and all $\gamma \in \Gamma$, we have $f(\gamma.x) = \gamma.f(x)$.

There are some other standard notions associated with the action of a group Γ on a space X.

The *orbit* of a point $x \in X$ is the set $\Gamma.x = \{\gamma.x \mid \gamma \in \Gamma\} \subseteq X$. The action of Γ induces an equivalence relation on X defined by $x \sim y$ if and only if x and y belong to the same orbit. The space of equivalence classes is called the *space of orbits* and will be denoted X/Γ (we agree to denote it like this even if the action here is considered from the left).

The elements of Γ which leave an element $x \in X$ fixed form a subgroup $\Gamma_x = \{\gamma \in \Gamma \mid \gamma.x = x\}$ called the *isotropy group* (or *stabilizer group*) at x. It is easy to see that if x and y are on the same orbit, say $y = \gamma.x$, then their isotropy groups are conjugate: $\Gamma_y = \gamma.\Gamma_x.\gamma^{-1}$; and in fact any conjugate subgroup to Γ_x occurs as an isotropy group Γ_y of some element y in the orbit of x. If $\Gamma_x = \Gamma$, then x is said to be a *fixed point* of the action. The set of all fixed points of the action is often denoted X^Γ. A subset $Y \subset X$ is called Γ-*invariant* if it is left invariant by the action of Γ, i.e. $\gamma.Y = Y$ for every $\gamma \in \Gamma$.

The action of Γ on X is said to be *effective* if no element of the group, besides the identity element, fixes all the elements of the space, i.e. if $\gamma.x = x$ for all $x \in X$ implies $\gamma = 1$. Equivalently, the group Γ acts effectively if the representation $\Phi : \Gamma \to \mathrm{Sym}(X)$ is an injective homomorphism. In this case, by identifying Γ with its image $\Phi(\Gamma) \subset \mathrm{Sym}(X)$ we can regard Γ as a subgroup of symmetries of X.

The action of Γ on X is called *free* if no point of M is fixed by an element of Γ other than the identity, i.e. $\Gamma_x = \{1\}$, for all $x \in X$.

Assume Γ acts effectively on X. A point $x \in X$ is called a *singular point* of the Γ-action if the isotropy group Γ_x at x is non-trivial. The collection of all singular points in X is denoted Σ_Γ and is called the *singular set* of the Γ action on X. Thus

$$\Sigma_\Gamma = \{x \in X \mid \gamma.x = x, \text{ for some } \gamma \in \Gamma, \gamma \neq 1\} = \bigcup_{\gamma \in \Gamma, \gamma \neq 1} \Sigma_\gamma,$$

where $\Sigma_\gamma = X^\gamma$ denotes the set of points in X fixed by γ. Note that if the action is free, then $\Sigma_\Gamma = \varnothing$.

Proper Actions

Let X denote a topological space and Γ a group acting by homeomorphisms on X. Let $\Phi : \Gamma \to \mathrm{Homeo}(X)$ denote this action. We say that the action of Γ on X is *discrete* if $\Phi(\Gamma)$ is a discrete subgroup of $\mathrm{Homeo}(X)$ endowed with the compact-open topology. The action is said to have *discrete orbits* if every $x \in X$ has a neighbourhood U that contains only finitely many translates of x, i.e. the set $\{\gamma \in \Gamma \mid \gamma.x \in U\}$ is finite.

The action of Γ on X is called *discontinuous* at a point $x \in X$ if there exists a neighbourhood U of x such that the set $\{\gamma \in \Gamma \mid \gamma.U \cap U \neq \varnothing\}$ is finite. The action of Γ is said to be *properly discontinuous* at a point x in X, if there exists a neighbourhood U of x which is *nice*, i.e. all the nontrivial element $\gamma \in \Gamma$ move U outside itself: $\gamma.U \cap U = \varnothing$.

The action of Γ on X is said to be *discontinuous* (resp. *properly discontinuous*) if it is discontinuous (resp. properly discontinuous) at every point $x \in X$. Note that properly discontinuous actions are necessarily free.

If the action of Γ on X is discontinuous, then the orbits of the action have no accumulation points: for any sequence of elements $\gamma_n \in \Gamma$ and any point $x \in X$, the sequence $\gamma_n.x$ has no limit points in X. If X is locally compact, then the converse holds as well.

The action of Γ on X is said to be *proper* (alternatively, "Γ acts *properly* on X") if the map $\Gamma \times X \to X \times X,\ (\gamma, x) \mapsto (\gamma.x, x)$ is proper. Recall that a map is proper if the preimages of compact sets are compact; and recall also that a proper map between locally compact Hausdorff spaces is closed. Thus, if Γ is endowed with the discrete topology, the action of Γ on X is proper if and only if for any compact sets K_1, K_2 in X, the set $\{\gamma \in \Gamma \mid \gamma.K_1 \cap K_2 \neq \varnothing\}$ is finite.

Note that on a locally compact space any proper action is discontinuous and any discontinuous action has discrete orbits, but the converses are not true in general.

The following characterization for proper actions will be useful (see also [11, Excercise (2), page 132]). Here we do not require that X is a Hausdorff space.

Proposition 1.1. *The action of a discrete group Γ on a locally compact topological space X is proper if and only if all of the following hold:*

(*i*) *the space of orbits X/Γ is Hausdorff with the quotient topology;*

(*ii*) *each $x \in X$ has finite isotropy group;*

(*iii*) *each $x \in X$ has a Γ_x-invariant neighbourhood U such that*

$$\{\gamma \in \Gamma \mid \gamma.U \cap U \neq \varnothing\} = \Gamma_x.$$

Proof. ($\Rightarrow$) (*i*) The orbit equivalence relation is the image of the map $\Gamma \times X \to X \times X$, hence it is closed. Since the projection $X \to X/\Gamma$ is open, it follows that the quotient space X/Γ is Hausdorff (see [40, Theorem 11, page 98]).

(ii) follows directly from the definition by letting $K_1 = K_2 = \{x\}$.

(iii) Let $K_1 = \{x\}$ and K_2 be a compact neighbourhood of x. Since the action is proper, the set $\{\gamma \in \Gamma \mid \gamma.x \in K_2\}$ is finite and contains Γ_x. Thus we can find a compact neighbourhood of x, say K_2', such that $\{\gamma \in \Gamma \mid \gamma.x \in K_2'\} = \Gamma_x$. Consider now the set $K = \bigcup_{\gamma\in\Gamma_x} \gamma.K_2'$. Then K is a compact Γ_x-invariant neighbourhood of x. Applying the definition for $K_1 = K_2 = K$, the set $\{\gamma \in \Gamma \mid \gamma.K \cap K \neq \varnothing\}$ is finite and contains Γ_x. Then we can find a neighbourhood of x in K, say U', such that $\{\gamma \in \Gamma \mid \gamma.U' \cap U' \neq \varnothing\} = \Gamma_x$ and by taking $U = \bigcup_{\gamma\in\Gamma_x} \gamma.U'$ we obtain the Γ_x-invariant neighbourhood satisfying (iii).

($\Leftarrow$) By (i), if $x, x' \in X$ do not belong to the same orbit, then we can find neighbourhoods U for x and U' for x' and x', such that $\gamma.U \cap U' = \varnothing$ for all $\gamma \in \Gamma$. Assume now that x and x' belong to the same orbit, say $x = \delta.x'$ for some $\delta \in \Gamma \smallsetminus \Gamma_x$. Let U be a neighbourhood of x as given by (iii) and let $U' = \delta.U$. Then $\{\gamma \in \Gamma \mid \gamma.U \cap U' \neq \varnothing\} = \{\gamma \in \Gamma \mid \gamma.U \cap \delta.U \neq \varnothing\} = \{\gamma \in \Gamma \mid U \cap (\gamma^{-1} \cdot \delta).U \neq \varnothing\} = \delta.\Gamma_x$ which by (ii) is finite. Thus, for any $x, x' \in X$ we can find neighbourhoods U and U' such that $\{\gamma \mid \Gamma.U \cap U' \neq \varnothing\}$ is at most finite. Let now K be a compact set in X. Then $K \times K$ is compact in $X \times X$ and thus it has a finite cover with sets of the form $U \times U'$ where $\{\gamma \mid \Gamma.U \cap U' \neq \varnothing\}$ is finite. Hence the set $\{\gamma \in \Gamma \mid \Gamma.K \cap K \neq \varnothing\}$ is finite, i.e. the action is proper. $\square$

Remark 1.2. If in the above proposition the action is free, then the quotient map $X \to X/\Gamma$ is a covering map and the group of automorphisms of the covering is Γ itself: $\mathrm{Aut}(X \to X/\Gamma) = \Gamma$. If in addition X is assumed to be connected and simply connected, then $\pi_1(X/\Gamma) \cong \Gamma$.

Proposition 1.3 (Armstrong, [7])**.** *Let Γ act properly by homeomorphisms on a connected, simply connected, locally compact metric space X, and let Γ' be the normal subgroup of Γ generated by the elements which have fixed points in X. Then the fundamental group of the orbit space X/Γ is isomorphic to the factor group Γ/Γ'.*

The idea of the proof is a s follows. Note first that the action of Γ on X factors into an action by Γ' on X followed by a free action of Γ/Γ' on X/Γ'. Using that Γ' is generated by the elements having fixed points, the author shows that the quotient X/Γ' is simply connected. Furthermore, since Γ/Γ' acts freely on X/Γ, it follows that $\pi_1(X/\Gamma) \cong \pi_1((X/\Gamma')/(\Gamma/\Gamma')) \cong \Gamma/\Gamma'$.

Proper Actions by Isometries

In the next two propositions we collect some facts about proper actions by isometries on metric spaces (see [11, Proposition I.8.5 and Proposition II.6.10]).

Proposition 1.4. *Suppose that the group Γ acts properly by isometries on a metric space (X, d). Then*

(i) for each $x \in X$ there exists $\varepsilon > 0$ such that if $\gamma.B(x,\varepsilon) \cap B(x,\varepsilon) \neq \varnothing$ then $\gamma \in \Gamma_x$;

(*ii*) *the orbit space* X/Γ *is naturally a metric space;*

(*iii*) *if a subspace* X' *of* X *is invariant under the action of a subgroup* Γ' *of* Γ*, then the action of* Γ' *on* X' *is proper.*

Proof. (*i*) Let $x \in X$. Since Γ acts by isometries, any ball $B(x, r)$ centred at x is Γ_x-invariant. Then, the claim follows from Proposition 1.1 (*iii*).
(*ii*) The function $d'(\Gamma.x, \Gamma.y) = \inf\{d(x', y') \mid (x', y') \in \Gamma.x \times \Gamma.y\}$ on $X/\Gamma \times X/\Gamma$ defines a pseudometric on the orbit space. For $d'(\cdot, \cdot)$ to be a metric on X/Γ it suffices to show that it is positive definite. Assume $d'(\Gamma.x, \Gamma.y) = 0$. Then for any $n > 0$ there exists $\gamma_n \in \Gamma$ such that $d(x, \gamma_n y) < 1/n$. Since the action is proper, there exists $\gamma \in \Gamma$ such that $\gamma_n = \gamma$ for all n larger then some n_0. In particular $\gamma.y = x$, i.e. $\Gamma.x = \Gamma.y$.
(*iii*) follows from the definition. □

Recall that the action of a group Γ on X is called *cocompact* if there exists a compact set $K \subseteq X$ such that $X = \Gamma.K$.

Proposition 1.5. *Suppose* Γ *acts properly and cocompactly by isometries on the metric space* X*. Then*

(*i*) *there are finitely many conjugacy classes of isotropy subgroups in* Γ*;*

(*ii*) *every element of* Γ *is a semi-simple isometry of* X*;*

(*iii*) *the set of translation distances* $\{|\gamma| \mid \gamma \in \Gamma\}$ *is a discrete subset of* $\mathbb{R}$*.*

Proof. (*i*) Let $K \subseteq X$ be a compact set such that $X = \Gamma.K$. Since the action is proper, the sets $S = \{\gamma \in \Gamma \mid \gamma.K \cap K \neq \varnothing\}$ is finite. In particular, $\Gamma_x \subseteq S$ for all $x \in K$. Since for each $y \in X$ there exists $\gamma \in \Gamma$ such that $\gamma.y \in K$, we have that $\gamma\Gamma_y\gamma^{-1} = \Gamma_{\gamma.y} \subseteq S$.

(*ii*) Fix $\gamma \in \Gamma$. We will show that $\mathrm{Min}(\gamma) \neq \varnothing$ (see 1.1 for definitions). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X such that $d_\gamma(x_n) \to |\gamma|$ as $n \to \infty$. Since $\Gamma.K = X$, for each x_n there exists $\gamma_n \in \Gamma$ such that $y_n = \gamma_n.x_n \in K$. Note that $d_\gamma(x_n) = d(\gamma\gamma_n^{-1}.y_n, \gamma_n^{-1}.y_n) = d_{\gamma_n\gamma\gamma_n^{-1}}(y_n) \to |\gamma|$ as $n \to \infty$. Since $y_n \in K$ and K is compact, any point $x \in K$ is within finite distance from y_n. It follows that $d(\gamma_n\gamma\gamma_n^{-1}.x, x)$ is finite for all $x \in K$ and all n. In particular, the sequence $\gamma_n\gamma\gamma_n^{-1}.x$ is bounded and therefore it has a convergent subsequence which we will continue indexing by n. Using the properness of the action, there exists $\delta \in \Gamma$ such that $\delta = \gamma_n\gamma\gamma_n^{-1}$ for all n. On the other hand, since the sequence $(y_n) \subset K$, and K is compact, we can assume (eventually passing to a subsequence) that $y_n \to y \in K$, as $n \to \infty$. But $d_\gamma(\gamma_n^{-1}.y) = d(\gamma_n^{-1}.y, \gamma\gamma_n^{-1}.y) = d_\delta(y) = \lim_{n\to\infty} d_\delta(y_n) = \lim_{n\to\infty} d_\gamma(x_n) = |\gamma|$, which implies that $\gamma_n^{-1}.y \in \mathrm{Min}(\gamma)$ for all n, i.e. $\mathrm{Min}(\gamma) \neq \varnothing$.

(*iii*) Assume that the set of translations of Γ has an accumulation point. That is, there exists $a \geq 0$ and a sequence γ_n of elements of Γ such that $|\gamma_n| \neq |\gamma_m|$ for all $n \neq m$ and $|\gamma_n| \to a$ as $n \to \infty$. By (*ii*), since all the elements of Γ are semi-simple, $\mathrm{Min}(\gamma_n) \neq \varnothing$ for all n. Thus we can choose $x_n \in \mathrm{Min}(\gamma_n)$. Replacing γ_n by a suitable conjugate we can assume that all the x_n are contained in K. Since K is compact, it is bounded, say contained in a ball $B(x, r)$. It follows that $\gamma_n.B(x, r+a+1) \cap B(x, r) \neq \varnothing$ for arbitrarily large n and this contradicts the properness of the action. $\square$

Recall that a presentation of a group Γ is a pair $\langle \mathcal{A} \mid \mathcal{R} \rangle$ that consists of a set of generators $\mathcal{A}$ and a subset $\mathcal{R} \subset F(\mathcal{A})$ of the free group on $\mathcal{A}$, such that Γ is isomorphic to $F(\mathcal{A})/\langle\!\langle \mathcal{R} \rangle\!\rangle$, the factor group of $F(\mathcal{A})$ by the normal closure of $\mathcal{R}$ in $F(\mathcal{A})$. A group Γ is said to be *finitely generated* if it admits a presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ such that the set of generators $\mathcal{A}$ is finite; and Γ is said to be *finitely presented* if both the sets $\mathcal{A}$ and $\mathcal{R}$ can be taken to be finite.

The following result gives necessary and sufficient conditions for a group acting on a metric space to be finitely presented:

Proposition 1.6 ([11, Corollary I.8.11])**.** *A group is finitely presented if and only if it acts properly and cocompactly by isometries on a simply connected geodesic space.*

A presentation for the group Γ can be given as follows (for details see the proof of [11, Corollary I.8.11]). Using the fact that Γ acts cocompactly on X, we can find a compact set $K \subset X$ such that $\Gamma.K = X$. Let $U = B(x, r)$ denote an open ball in X containing K and let

$$S = \{\gamma \in \Gamma \mid U \cap \gamma.U \neq \varnothing\} \subset \Gamma.$$

Since Γ acts properly on X the set S is finite, and since X is connected, S generates Γ. Let now $\mathcal{A}$ be a set of symbols a_γ indexed by $\gamma \in S$, and let $\mathcal{R}$

be the finite subset of the free group on $\mathcal{A}$ given by

$$\mathcal{R} = \{a_{\gamma_1} a_{\gamma_2} a_{\gamma_3}^{-1} \mid \gamma_1, \gamma_2, \gamma_3 \in S; U \cap \gamma_1.U \cap \gamma_3.U \neq \varnothing; \gamma_1\gamma_2 = \gamma_3 \text{ in } \Gamma\}.$$

Since both U and X are connected and X is simply connected, $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is a presentation for Γ, and this presentation is finite.

1.3 Riemannian Manifolds

In the first part of this section we present some basic notions from Riemannian geometry with an emphasis on the metric viewpoint. A classic reference on the background material for Riemannian geometry is [20]. The second part of this section contains some results concerning finite actions by diffeomorphisms on smooth manifolds. As we will see in the next chapter, these results play an important role in understanding the local properties of orbifolds.

Riemannian Manifolds as Metric Spaces

Let M denote a connected smooth manifold. Recall that a *Riemannian metric* on M is a map ρ that assigns to each $x \in M$ an inner product ρ_x on the tangent space T_xM which varies smoothly from point to point: for any open subset U of M and for any pair of tangent smooth vector fields $\mathcal{X}$ and $\mathcal{Y}$ on U, the map $\rho : U \to \mathbb{R}$ given by $x \mapsto \rho_x(\mathcal{X}(x), \mathcal{Y}(x))$ is smooth. The pair (M, ρ) will denote the manifold M with a given Riemannian metric ρ.

For a differentiable path $t \mapsto c(t)$ on M, we write $\dot{c}(t) \in T_{c(t)}M$ for the velocity vector at t, and $|\dot{c}(t)| = \rho_{c(t)}(\dot{c}(t), \dot{c}(t))^{1/2}$ for its norm with respect to the given inner product on $T_{c(t)}M$. The *Riemannian length* of a piecewise smooth path $c : [a, b] \to M$, is defined by $\ell_a^b(c) = \int_a^b |\dot{c}(t)|dt$.

We introduce a distance function d on the Riemannian manifold (M, ρ) in the following way: for any two points $x, y \in M$ we define $d(x, y)$ (the distance from x to y) to be the infimum of the Riemannian length of all piecewise smooth paths $c : [a, b] \to M$ with $c(a) = x$ and $c(b) = y$.

Using this definition of distance, a *Riemannian geodesic* is a differentiable path $c : I \subseteq \mathbb{R} \to M$ which is locally distance-minimizing in the following sense: there exists a constant $v \geq 0$ such that for any interior point t of I there exists a neighbourhood $J \subseteq I$ of t such that $\ell_s^{s'}(c) = d(c(s), c(s')) = v|s - s'|$ for all $s, s' \in J$. We say that the geodesic $c : I \to M$ is *normalized* (or has *unit speed*) if $v = 1$, and that it is *minimal* if the above holds for all $s, s' \in I$. In particular, the length of a minimal geodesic segment is equal to the distance between its endpoints.

In general, a minimal geodesic between two points if it exists, needs not be unique. For example, given two antipodal points on the round 2-sphere, the unit speed parametrization of an arc of any great circle connecting the two points is a minimal geodesic.

It is important to notice that Riemannian geodesics need not be geodesics in the metric sense; in general they are only local geodesics in the sense defined in the previous section and, a Riemannian geodesic is a metric geodesic if and only if it is minimal. Note also that given two points on a connected manifold M, it may be possible that there is no minimal geodesic connecting them or even no geodesic at all containing them. For example, if we consider the Euclidean space with the origin removed then there are no geodesics containing antipodal points. Another situation is that of two points x and y on the round unit sphere which are not antipodal. Then there are infinitely many unit speed Riemannian geodesics connecting them: any unit speed path starting at x and ending at y and whose image is contained in the great circle through x and y (possibly going more than once around the sphere). Among these geodesics there is only one minimal geodesic (which is the unique path with length smaller than π). If we remove a point z of the great circle through x and y such that $d(x,z) + d(z,y) < \pi$, then on $\mathbb{S}^2 \smallsetminus \{z\}$ there is a unique geodesic connecting x to y and this geodesic is not minimal.

Any point x of a Riemannian manifold M has a neighbourhood (called *normal neighbourhood*) U such that for any $y \in U$ there exists a unique minimal geodesic from x to y. Even more is true. At any point $x \in M$ we can find a neighbourhood W (called *totally normal neighbourhood*) such that any two distinct points in W can be connected through a unique minimizing geodesic. A consequence of the existence of normal neighbourhoods at each point of M is that the topology induced by the distance d coincides with the original topology on M.

Recall that a manifold is said to be *geodesically complete* if any geodesic segment can be extended indefinitely. By the Hopf-Rinow theorem [20, Theorem 7.2.8], a manifold M is geodesically complete if and only if it is complete (as a metric space). An important consequence of this is that every complete connected Riemannian manifold M is a geodesic metric space, i.e. any two points M can be connected through a minimal geodesic.

A *Riemannian isometry* of a Riemannian manifold M is, by definition, a diffeomorphism of M whose differential preserves the inner product on each tangent space. From the definition of the distance d it is clear that any Riemannian isometry is an isometry of the associated metric structure (M, d). If the Riemannian metric is smooth (or at least C^2) then the converse holds, and in that case we denote by $\mathrm{Isom}(M)$ the group of isometries of M.

Let now (M, ρ) be a Riemannian manifold and $N \subseteq M$ a smoothly embedded submanifold. The restriction to T_xN of the given inner product ρ_x on T_xM, defines an inner product on the tangent space T_xN at each point $x \in N$. Thus N inherits a Riemannian structure from M which we denote by $\rho|_N$. Then the distance d_N on N associated to $\rho|_N$ coincides with the induced distance on N by the distance on M associated to ρ.

A submanifold N is said to be *totally geodesic* if any geodesic in N is also a geodesic in M. Note that a totally geodesic submanifold need not be (metrically) convex. For example, on the round unit sphere any arc of a great circle which has length at least π is totally geodesic but not convex. However, if the manifold M has unique geodesic property, then a submanifold is totally geodesic if and only if it is (metrically) convex.

Finite Group Actions on Smooth Manifolds

Let M be a smooth manifold and Γ a group acting effectively by diffeomorphisms on M. It is convenient to regard Γ as a subgroup of $\mathrm{Diffeo}(M)$. The action of Γ on M induces an action on the tangent bundle TM given by

$$\gamma.v := (d\gamma)_x(v) \in T_{\gamma.x}M,$$

for any $\gamma \in \Gamma$, $x \in M$ and $v \in T_xM$. In particular, for any $x \in M$ there is an induced action of the isotropy group Γ_x of x on the tangent space T_xM at x. By definition, the action of Γ_x on T_xM is linear and let $d_x : \Gamma_x \to \mathrm{Aut}(T_xM)$ denote the homomorphism associated to this action.

Proposition 1.7. *Suppose M is a connected, paracompact smooth manifold and Γ is a finite subgroup of Diffeo(M). Then the homomorphism $d_x : \Gamma_x \to$ Aut(T_xM) is injective for every $x \in M$.*

The proof of the above proposition relies on the following simple lemma:

Lemma 1.8. *If M and Γ are as in Proposition 1.7, then M admits a Γ-invariant Riemannian metric.*

Proof. Since the manifold M is connected and paracompact, it admits a Riemannian metric, say g. Since Γ is finite we can obtain a Γ-invariant Riemannian metric ρ by averaging g over Γ. That is, for every $x \in M$ and every $v, w \in T_xM$ define

$$\rho_x(v, w) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} g_{\gamma.x}(\gamma.v, \gamma.w),$$

where $|\Gamma|$ denotes the order of Γ. Clearly ρ is Γ-invariant. □

Proof. (Prop. 1.7) Let ρ denote a Γ-invariant Riemannian metric on M given by lemma 1.8, and consider the exponential map associated with this metric. Fix $x \in M$ and let $\varepsilon > 0$ and $U \subseteq M$ an open neighbourhood of x, such that $\exp_x : B(0, \varepsilon) \subset T_xM \to U$ is a diffeomorphism. Since the metric is Γ-invariant, the induced action of Γ_x on T_xM is by orthogonal transformations. In particular, U is Γ_x-invariant and the exponential map is Γ_x-equivariant: $\exp_x \circ (d\gamma)_x = \gamma \circ \exp_x$ for all $\gamma \in \Gamma_x$. If we assume that $\gamma \in \mathrm{Ker}(d_x)$, then $(d\gamma)_x = I \in \mathrm{Aut}(T_xM)$ and the equivariance of $\exp_x$ implies that the restriction $\gamma|_U$ is the identity on U. Since M is connected it follows that γ is the identity on M. To see this, let $A := \{y \in M |\ \gamma.y = y \text{ and } \gamma \in \mathrm{Ker}(d_y)\}$. Then, $A \neq \varnothing$ (since $x \in A$) and it is obviously closed. Let now $y \in A$. The condition $\gamma.y = y$ implies that $\gamma \in \Gamma_y$ and the condition that $\gamma \in \mathrm{Ker}(d_y)$ implies that the restriction of γ to an open neighbourhood V of y is the identity. Thus $V \subseteq A$ and A is open. Since M is connected, $A = M$ and since the action is effective γ is the identity on M. □

Corollary 1.9. *The only diffeomorphism of finite order on a connected, paracompact smooth manifold which fixes an open set is the identity.*

Another consequence of Proposition 1.7 is the following result concerning the singular set.

Proposition 1.10. *Suppose M is a connected, paracompact, smooth manifold and Γ a finite subgroup of Diffeo(M). Then Σ_Γ is closed and nowhere dense in M.*

Proof. The set Σ_Γ is obviously closed as it is the finite union $\cup_{\gamma\neq 1}\Sigma_\gamma$ and each of Σ_γ is closed. To see that Σ_Γ has empty interior let $x \in \Sigma_\Gamma$ and assume that there exists an open neighbourhood U of x which is contained in Σ_Γ. Since Γ is finite and $\Sigma_\Gamma = \cup_{\gamma\neq 1}\Sigma_\gamma$, the neighbourhood U has to be in one of the Σ_γ's. Thus there exists $\gamma \neq 1$ such that the restriction $\gamma|_U = 1$. Since M is connected, by Corollary 1.9, γ has to be the identity on the whole M, a contradiction. Thus Σ_Γ has empty interior and the proof is complete. $\square$

As we will see in the next chapter, the following proposition plays an important role in the definition of the orbifold structure.

Proposition 1.11. *Suppose M is a connected, paracompact, smooth manifold and Γ is a finite subgroup of Diffeo(M). Let $V \neq \varnothing$ be a connected open subset of M and $f : V \to M$ be a diffeomorphism onto its image such that $\varphi \circ f = \varphi|_V$, where $\varphi : M \to M/\Gamma$ is the natural projection. Then there exists a unique $\gamma \in \Gamma$ such that $f = \gamma|_V$.*

Proof. Consider on M a Γ-invariant Riemannian metric ρ as given by Lemma 1.8. Then Γ acts by isometries on (M, ρ). By Proposition 1.10, the regular set $M_0 = M \setminus \Sigma_\Gamma$ is open and dense in M.

Let $x \in V \cap M_0$. Since Γ_x is trivial, by Proposition 1.1 there exists a connected neighbourhood U of x contained in $V \cap M_0$ such that $\gamma.U \cap U = \varnothing$ for all $\gamma \neq 1$. Then the condition $\varphi \circ f = \varphi|_V$ implies that there is a unique $\gamma \in \Gamma$ such that $f(x) = \gamma.x$ and on U. In particular we have $(df)_x = (d\gamma)_x$ and $f|_U$ is a Riemannian isometry. By Corollary 1.9 $f = \gamma$ on the whole connected component of $V \cap M_0$ containing x.

If the codimension of Σ_Γ in M is at least two, then M_0 is connected and so is $V \cap M_0$. Moreover, since $V \cap M_0$ is dense in V, by continuity $f = \gamma|_V$.

Assume now that Σ_Γ has components of codimension one, and let Σ' be such a component that intersects V. Let V_1 and V_2 be two neighbouring connected components of $V \smallsetminus \Sigma'$. Then, as before, for V_1 and V_2 there exists two unique elements γ_1 and γ_2 in Γ such that $f = \gamma_1$ on V_1 and $f = \gamma_2$ on V_2. We want to show that $\gamma_1 = \gamma_2$.

First note that by continuity $\gamma_1 = \gamma_2$ on $\Sigma' \cap V$. Let now $z \in \Sigma' \cap V$. Since Σ' has codimension one and the action of Γ is by isometries, the isotropy group of z is generated by an element of order two (the reflection in the hyperplane $T_z\Sigma' \subset T_zM$).

Denote by δ the generator of Γ_z and let $U \subseteq V$ be a δ-invariant totally normal neighbourhood of z.

Let $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$. Then U_1 and U_2 are nonempty , open and connected and $\delta.U_1 = U_2$ (and of course $\delta.U_2 = U_1$). Let now $x \in U_1$ and let $c : [0, 1] \to M$ be the unique geodesic connecting $x \in U_1$ to $\delta.x \in U_2$. Then $c(1/2) \in \Sigma' \cap V$ is the unique point where the geodesic c intersects Σ'

and notice that c is perpendicular to Σ' at this point.

Since γ_1 and γ_2 are isometries, the translates $\gamma_1.c$ and $\gamma_2.c$ of c are geodesics and moreover they are both perpendicular to $\gamma_1.\Sigma' = \gamma_2.\Sigma'$ at the same point $\gamma_1.c(1/2) = \gamma_2.c(1/2)$. Therefore the two geodesics $\gamma_1.c$ and $\gamma_2.c$ must have the same image. There are only two possibilities for the end points to match: either $\gamma_1.x = \gamma_2.x$ or $\gamma_1.x = \gamma_2\delta.x$. Using our assumption that $f = \gamma_1$ on $U_1 \ni x$ and $f = \gamma_2$ on $U_2 \ni \delta.x$, the later equality rewrites: $f(x) = f(\delta.x)$. Since $x \neq \delta.x \in U \subseteq V$ it contradicts the fact that f is a diffeomorphism on V. Thus $\gamma_1.x = \gamma_2.x$ and since x was an arbitrary point in U_1, it follows that $\gamma_1|_{U_1} = \gamma_2|_{U_1}$. Furthermore, since U_1 is open and connected, by Corollary 1.9 we have that $\gamma_1 = \gamma_2$ on M and the proof is complete. $\square$

As the next result shows, the fixed point set of a family of isometries of a Riemannian manifold has a nice geometric structure.

Proposition 1.12 ([42, Theorem 5.1])**.** *Suppose M is a Riemannian manifold and $\mathcal{G}$ is a set of isometries of M. Let F be the set of points of M which are fixed by all elements of $\mathcal{G}$. Then each connected component of F is a closed totally geodesic submanifold of M.*

Chapter 2

Orbifolds

This chapter gives a detailed introduction to the foundational material on orbifolds, including orbifold structures, isotropy groups, $\mathcal{H}$-paths, covering spaces of orbifolds, orbifold fundamental groups, tangent bundles and Riemannian structures for orbifolds, and geodesics on orbifolds. The last section presents several illustrative examples of low-dimensional orbifolds where these concepts can be worked out in detail.

The material in this chapter is well-known and there are many excellent sources. Besides the original work of Satake [57] and Thurston [66], other good introductions to the classical theory of orbifolds include [1, Chapter 1], [11, Appendix $\mathcal{G}$.1], [38, Chapter 6] and [50, Section 2.4]. A great deal of information on the differential geometry of orbifolds can be found in the appendix of [17], and a detailed presentation of the geometric structure of 2-dimensional orbifolds is provided in [60].

2.1 The Orbifold Structure

We begin with the formal definition of a smooth ($\mathcal{C}^\infty$-differentiable) orbifold structure as given in [1] and [51] and which is equivalent to the ones given in [17] and [57].

The Orbifold Atlas

Let Q denote a paracompact Hausdorff topological space and let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open cover of Q which is closed under finite intersections. Fix a non-negative integer n.

Definition 2.1. (orbifold chart) *An n-dimensional* smooth orbifold chart *(also called* smooth uniformizing system*) associated to an open set $U_i \in \mathcal{U}$ is given by a triple $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ where*

- *$\widetilde{U}_i$ is a connected open subset of $\mathbb{R}^n$,*
- *Γ_i is a finite group acting smoothly by diffeomorphisms on $\widetilde{U}_i$, and*
- *$\varphi_i : \widetilde{U}_i \to U_i$ is a continuous surjective map that induces a homeomorphism from $\widetilde{U}_i/\Gamma_i$ onto the open set U_i.*

If Γ_i acts effectively on $\widetilde{U}_i$, then the orbifold chart is said to be reduced.

Let now U_i and U_j be two open subsets such that $U_i \subseteq U_j$ and let $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(\widetilde{U}_j, \Gamma_j, \varphi_j)$ be orbifold charts over U_i and U_j, respectively.

Definition 2.2. (embeddings of charts) *An* embedding *between such charts is a pair*

$$(\widetilde{\varphi}_{ij}, \lambda_{ij}) : (\widetilde{U}_i, \Gamma_i, \varphi_i) \hookrightarrow (\widetilde{U}_j, \Gamma_j, \varphi_j)$$

consisting of a smooth embedding $\widetilde{\varphi}_{ij} : \widetilde{U}_i \hookrightarrow \widetilde{U}_j$ and a monomorphism $\lambda_{ij} : \Gamma_i \hookrightarrow \Gamma_j$ such that $\widetilde{\varphi}_{ij}$ is λ_{ij}-equivariant. Furthermore, if the charts are not reduced, we require that λ_{ij} induces an isomorphism from

$$K_i := ker(\Gamma_i \to \mathrm{Diff}(\widetilde{U}_i)) \text{ to } K_j := \ker(\Gamma_j \to \mathrm{Diff}(\widetilde{U}_j)).$$

Remark 2.3. It is important to note that the maps $\widetilde{\varphi}_{ij}$ (resp. the morphisms λ_{ij}) are defined up to composition (resp. conjugation) with an element of Γ_j, in the following sense. Given two embeddings

$$(\widetilde{\varphi}_{ij}, \lambda_{ij}), (\widetilde{\psi}_{ij}, \mu_{ij}) : (\widetilde{U}_i, \Gamma_i, \varphi_i) \hookrightarrow (\widetilde{U}_j, \Gamma_j, \varphi_j),$$

between reduced orbifolds charts, there exists a unique $\gamma_j \in \Gamma_j$ such that

$$\widetilde{\psi}_{ij} = \gamma_j \cdot \widetilde{\varphi}_{ij} \text{ and } \mu_{ij} = \gamma_j \lambda_{ij} \gamma_j^{-1}.$$

In particular if $i = j$ then $\widetilde{\varphi}_{ii}$ is an element $\gamma_i \in \Gamma_i$ and λ_{ii} is conjugation by γ_i (see [1], [51]).

For reduced charts, the existence and uniqueness of the element $\gamma_j \in \Gamma_j$ above follows directly from Proposition 1.11. To see this, let $\widetilde{U}_i' = \widetilde{\varphi}_{ij}(\widetilde{U}_i) \subseteq$

$\widetilde{U}_j$ and $\widetilde{U}_i'' = \widetilde{\psi}_{ij}(\widetilde{U}_i) \subseteq \widetilde{U}_j$. Since $\widetilde{\varphi}_{ij}$ and $\widetilde{\psi}_{ij}$ are diffeomorphisms onto their images, the composition $f := \widetilde{\psi}_{ij} \circ \widetilde{\varphi}_{ij}^{-1} : \widetilde{U}_i' \to \widetilde{U}_i''$ is a diffeomorphism. Moreover, since the embeddings $\widetilde{\varphi}_{ij}$ and $\widetilde{\psi}_{ij}$ are λ_{ij} and μ_{ij}-equivariant, it follows that $f \circ \varphi_j = \varphi_j|_{\widetilde{U}_i'}$. Then by Proposition 1.11, there is a unique element $\gamma_j \in \Gamma_j$ such that $\gamma_j|_{\widetilde{U}_i'} = f = \widetilde{\psi}_{ij} \circ \widetilde{\varphi}_{ij}^{-1}$, i.e. $\widetilde{\psi}_{ij} = \gamma_j.\widetilde{\varphi}_{ij}$.

If the orbifold charts are not reduced, then the uniqueness in the above property holds up to composition (resp. conjugation) with elements of K_j (which, by definition, is isomorphic to K_i).

Remark 2.4. Another important technical point regarding the embedding between reduced charts is the following: given an embedding $(\widetilde{\varphi}_{ij}, \lambda_{ij})$ as in Definition 2.2, if $\gamma_j \in \Gamma_j$ is such that $\widetilde{\varphi}_{ij}(\widetilde{U}_i) \cap \gamma_j.\widetilde{\varphi}_{ij}(\widetilde{U}_i) \neq \varnothing$, then there exists a unique $\gamma_i \in \Gamma_i$ such that $\gamma_j = \lambda_{ij}(\gamma_i)$. As before, this follows from Proposition 1.11 and Remark 1.9.

It is easy to see that the composition of any two embeddings of charts is again an embedding. However, it is worth noticing that it is not true in general that if $U_i \subseteq U_j \subseteq U_k$ then $\widetilde{\varphi}_{ik} = \widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij}$, but there exists an element $\gamma \in \Gamma_k$ such that $\gamma \circ \widetilde{\varphi}_{ik} = \widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij}$ and $\gamma \circ \lambda_{ik} \circ \gamma^{-1} = \lambda_{jk} \circ \lambda_{ij}$.

Definition 2.5. (orbifold atlas) *An* orbifold atlas $\mathcal{A}$ *on* Q *associated to* $\mathcal{U}$ *is a collection of orbifold charts* $\{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}$ *which are locally compatible in the following way: given any two charts* $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ *and* $(\widetilde{U}_j, \Gamma_j, \varphi_j)$*, there exists an open set* $U_k \subseteq U_i \cap U_j$ *and an associated orbifold chart* $(\widetilde{U}_k, \Gamma_k, \varphi_k)$ *that embeds in both* $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ *and* $(\widetilde{U}_j, \Gamma_j, \varphi_j)$.

If $\mathcal{U}'$ is another open cover for Q that refines $\mathcal{U}$, we say that the associated orbifold atlas $\mathcal{A}'$ *refines* $\mathcal{A}$ if every orbifold chart in $\mathcal{A}'$ can be embedded in some orbifold chart in $\mathcal{A}$. Two orbifold atlases are *equivalent* if they have a common refinement.

Definition 2.6. (orbifold) *A smooth n-dimensional* orbifold $\mathcal{Q}$ *is a space Q as above together with an equivalence class of orbifold atlases $\mathcal{A}$ on it.*

We will denote the orbifold structure on Q by the calligraphic $\mathcal{Q}$. The topological space Q is called the underlying topological space of the orbifold $\mathcal{Q}$. Similar to the manifold case, where it is possible to have non-diffeomorphic smooth structures on a fixed topological manifold, there may be more than one (non-equivalent) orbifold structures on the same topological space Q (see Examples 2.34 – 2.37). It is also important to notice that except for dimension 2, the underlying topological space Q of a smooth orbifold $\mathcal{Q}$ need not have the structure of a topological manifold (see Example 2.38).

Note that if all the groups Γ_i are trivial, or if they act freely on the $\widetilde{U}_i$'s, then $\mathcal{Q}$ has the structure of a smooth manifold.

Remark 2.7. (gluing charts) Similar to the manifold case, given an orbifold atlas $\mathcal{A}$ it is sometimes useful to describe the gluing maps (or transition functions) between charts.

Given two orbifold charts $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(\widetilde{U}_j, \Gamma_j, \varphi_j)$ and a point $x \in U_i \cap U_j$, by definition there exists a third orbifold chart $(\widetilde{U}_k, \Gamma_k, \varphi_k)$ uniformizing an open set $U_k \subseteq U_i \cap U_j$ containing x; together with embeddings of charts

$$(\widetilde{U}_i, \Gamma_i, \varphi_i) \xleftarrow{(\widetilde{\varphi}_{ki}, \lambda_{ki})} (\widetilde{U}_k, \Gamma_k, \varphi_k) \xrightarrow{(\widetilde{\varphi}_{kj}, \lambda_{kj})} (\widetilde{U}_j, \Gamma_j, \varphi_j).$$

The maps $\widetilde{\varphi}_{ki}$ and $\widetilde{\varphi}_{kj}$ are diffeomorphisms onto their images and are equivariant with respect to the injective homomoprhisms $\lambda_{ki} : \Gamma_k \to \Gamma_i$ and $\lambda_{kj} : \Gamma_k \to \Gamma_j$. The groups $\lambda_{ki}(\Gamma_k)$ and $\lambda_{kj}(\Gamma_k)$ are isomorphic to Γ_k, and the maps

$$\widetilde{U}_i \supseteq \widetilde{\varphi}_{ki}(\widetilde{U}_k) \xleftarrow{\widetilde{\varphi}_{ki}} \widetilde{U}_k \xrightarrow{\widetilde{\varphi}_{kj}} \widetilde{\varphi}_{kj}(\widetilde{U}_k) \subseteq \widetilde{U}_j$$

can be regarded as equivariant diffeomorphisms of Γ_k-manifolds. Moreover, the composition

$$\left(\widetilde{\varphi}_{ki}(\widetilde{U}_k), \lambda_{ki}(\Gamma_k), \varphi_{i_{|\widetilde{\varphi}_{ki}(\widetilde{U}_k)}}\right) \xrightarrow{(\widetilde{\varphi}_{kj}\widetilde{\varphi}_{ki}^{-1}, \lambda_{kj}\lambda_{ki}^{-1})} \left(\widetilde{\varphi}_{kj}(\widetilde{U}_k), \lambda_{kj}(\Gamma_k), \varphi_{j_{|\widetilde{\varphi}_{kj}(\widetilde{U}_k)}}\right)$$

gives an isomorphism between the induced uniformizing systems for U_k.

Thus the transition functions $\widetilde{\varphi}_{kj}\widetilde{\varphi}_{ki}^{-1}$ are Γ_k-equivariant differentiable maps and we can use them to recover the underlying topological space from the orbifold atlas. We glue the spaces $\widetilde{U}_i/\Gamma_i$ and $\widetilde{U}_j/\Gamma_j$ by identifying the points $\varphi_i(\tilde{y}) \sim \varphi_j(\tilde{z})$ whenever $\tilde{y} \in \widetilde{\varphi}_{ki}(\widetilde{U}_k) \subseteq \widetilde{U}_i$ and $\tilde{z} \in \widetilde{\varphi}_{kj}(\widetilde{U}_k) \subseteq \widetilde{U}_j$ satisfy $\widetilde{\varphi}_{kj}\widetilde{\varphi}_{ki}^{-1}(\tilde{y}) = \tilde{z}$. Denote by

$$Y = \bigsqcup_{\widetilde{U}_i \in \mathcal{U}} (\widetilde{U}_i/\Gamma_i)/\sim$$

the space obtained by performing these gluing. The maps $\varphi_i : \widetilde{U}_i \to Q$ induce a homeomorphism $\phi : Y \to Q$.

Isotropy Groups

To each point x of an orbifold $\mathcal{Q}$ we can associate a finite group Γ_x well-defined up to isomorphism in the following way. Let $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ be an orbifold chart at x. For a fixed $\tilde{x} \in \widetilde{U}_i$ such that $\varphi_i(\tilde{x}) = x$, we denote by $\Gamma^i_{\tilde{x}}$ the isotropy group of $\tilde{x}$ in Γ_i. The isomorphism class of $\Gamma^i_{\tilde{x}}$ is independent of the lift $\tilde{x}$ of x in $\widetilde{U}_i$. Moreover, if $(\widetilde{U}_j, \Gamma_j, \varphi_j)$ is another orbifold chart at x with embedding $(\widetilde{\varphi}_{ji}, \lambda_{ji}) : (\widetilde{U}_j, \Gamma_j, \varphi_j) \to (\widetilde{U}_i, \Gamma_i, \varphi_i)$, then λ_{ji} maps $\Gamma^i_{\tilde{x}}$ isomorphically into $\Gamma^j_{\widetilde{\varphi}_{ji}(\tilde{x})}$. Thus, for a given $x \in \mathcal{Q}$ the isomorphism class of the isotropy groups $\Gamma^i_{\tilde{x}}$ is independent on both the orbifold chart at x and its lift within the orbifold chart. We will denote this isomorphism class by Γ_x and we will refer to it as the *isotropy group of* x.

An orbifold $\mathcal{Q}$ is said to be *effective (or "reduced")* if each of the orbifold charts $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ is reduced, i.e. each action Γ_i on $\widetilde{U}_i$ is effective. Given a noneffective orbifold $\mathcal{Q}$ we can always associate to it an effective one denoted $\mathcal{Q}_{\text{eff}}$, by redefining the groups in each orbifold chart $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ to be Γ_i/K_i.

A *singular point* x of an orbifold $\mathcal{Q}$ is a point whose isotropy group Γ_x is nontrivial with respect to the orbifold structure of $\mathcal{Q}_{\text{eff}}$. A nonsingular point is also called a *regular point*. For an orbifold $\mathcal{Q}$, we denote by Σ the collection of its singular points, and by Q_{reg} the set of the regular ones. If $\Sigma = \varnothing$ then the orbifold $\mathcal{Q}$ is in fact a manifold.

Proposition 2.8. *The singular set of an effective orbifold is closed and nowhere dense.*

Proof. Let $(\widetilde{U}, \Gamma, \phi)$ be any orbifold chart over an open set U which has nonempty intersection with the singular set. Then

$$\begin{aligned}
\Sigma \cap U &= \{x \in U \mid \Gamma_x \neq 1\} \\
&= \{x \in U \mid \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\tilde{x} = \tilde{x} \text{ for some } \tilde{x} \in \varphi^{-1}(x) \subset \widetilde{U}\} \\
&= \bigcup_{\tilde{x} \in \widetilde{U}} \{\varphi(\widetilde{x}) \mid \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\widetilde{x} = \widetilde{x}\} \\
&= \varphi(\bigcup_{\tilde{x} \in \widetilde{U}} \{\tilde{x} \mid \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\tilde{x} = \tilde{x}\}) \\
&= \varphi(\Sigma_\Gamma).
\end{aligned}$$

By Proposition 1.10 Σ_Γ is closed and has empty interior. Since φ is a homeomorphism , $\Sigma_\Gamma \cap U$ is closed and has empty interior. Hence Σ is closed and since Q is locally compact and Hausdorff, $\Sigma = \bigcup_i \Sigma \cap U_i$ has empty interior. $\square$

For an effective orbifold, it is easy to see that the regular set, $Q_{reg} = Q \setminus \Sigma$ is a dense open set in Q which has the structure of a smooth manifold. From the definition, we can see that any orbifold is locally compact. If we further assume that the codimension of the singular locus is at least two, then the regular set is also locally path connected. In this case the orbifold is connected if and only if its regular set is path connected.

Unlike the regular set, in general, the singular locus of an orbifold is not a manifold and it may have several components of different dimensions. We will return to describing in more detail the structure of the singular locus in section 3.1.

Remark 2.9. Each point $x \in \mathcal{Q}$ has an open neighbourhood U_x (called *fundamental neighbourhood at* x) such that the group of the associated orbifold chart can be chosen to be the isotropy group Γ_x of x. We will denote such chart by $(\widetilde{U}_x, \Gamma_x, \varphi_x)$ and will refer to it as the *fundamental chart at* x.

From the compatibility condition of charts it follows that the isotropy group of any point $y \in U_x$ is isomorphic to a subgroup of the isotropy group Γ_x of x. Moreover, a fundamental chart $(\widetilde{U}_y, \Gamma_y, \varphi_y)$ at y can be chosen such that $\widetilde{U}_y \subset \widetilde{U}_x$, $\Gamma_y \leq \Gamma_x$ and $\varphi_y = \varphi_x|_{\widetilde{U}_y}$. In particular, note that any point contained in a fundamental neighbourhood of a regular point is again regular.

2.2 Developable Orbifolds

Proposition 2.10 ([66, Proposition 13.2.1])**.** *Let Γ be a discrete group acting properly on a manifold M. Then the quotient space M/Γ has a natural orbifold structure.*

Proof. Let Q denote the quotient space M/Γ. Since the action by Γ is proper, by Proposition 1.1 (i), Q is Hausdorff. We will now construct an orbifold atlas on Q.

Let $\pi : M \to Q$ denote the quotient map and let $x \in Q$. Choose $\tilde{x} \in M$ such that $\pi(\tilde{x}) = x$ and let $\Gamma_{\tilde{x}} = \{\gamma \in \Gamma | \; \gamma.\tilde{x} = \tilde{x}\}$ denote the isotropy group of $\tilde{x}$. By Proposition 1.1 (iii), there exists an open connected neighbourhood $\widetilde{U}_{\tilde{x}}$ of $\tilde{x}$, which is invariant to $\Gamma_{\tilde{x}}$ and disjoint from all its translates by elements of Γ not in $\Gamma_{\tilde{x}}$. Let $U_x := \pi(\widetilde{U}_{\tilde{x}}) \subseteq Q$. Then the restriction $\pi|_{\widetilde{U}_{\tilde{x}}} : \widetilde{U}_{\tilde{x}} \to Q$ induces a homeomorphism between $\widetilde{U}_{\tilde{x}}/\Gamma_{\tilde{x}}$ and U_x.

Consider now an open cover $\widetilde{\mathcal{U}}$ of M associated to a maximal manifold atlas for M. Without loss of generality we may assume that all the elements of $\widetilde{\mathcal{U}}$ are simply connected and by eventually shrinking $\widetilde{U}_{\tilde{x}}$, we can assume that $\widetilde{U}_{\tilde{x}} \in \widetilde{\mathcal{U}}$. Then $\{U_x \mid x \in Q\}$ is an open cover for Q and each U_x has associated an orbifold chart $(\widetilde{U}_{\tilde{x}}, \Gamma_{\tilde{x}}, \pi|_{\widetilde{U}_{\tilde{x}}})$ as in Remark 2.9. In order to get a suitable cover of Q we augment the cover $\{U_x | \; x \in Q\}$ by adjoining finite intersections. Let now $x_1, \ldots, x_k \in Q$ such that the corresponding sets U_{x_i} as above satisfy $U_{x_1} \cap \ldots \cap U_{x_k} \neq \varnothing$. Since Γ acts by permutations on the set of connected components of $\pi^{-1}(U_{x_1} \ldots \cap U_{x_k})$, there exist $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $\gamma_1.\widetilde{U}_{\tilde{x}_1} \cap \ldots \cap \gamma_k.\widetilde{U}_{\tilde{x}_k} \neq \varnothing$, where $\widetilde{U}_{\tilde{x}_i}$ denote $\Gamma_{\tilde{x}_i}$-neighbourhoods of $\tilde{x}_i \in \pi^{-1}(x_i)$. This intersection may be taken to be

$$U_{x_1} \widetilde{\cap \ldots} \cap U_{x_k}$$

which is clearly invariant to the action by the finite subgroup

$$\gamma_1 \cdot \Gamma_{\tilde{x}_1} \cdot \gamma_1^{-1} \cap \ldots \cap \gamma_k \cdot \Gamma_{\tilde{x}_k} \cdot \gamma_k^{-1}.$$

In this way we obtain a cover $\mathcal{U}$ of Q which is closed under finite intersections elements and satisfies the conditions in Definition 2.1.

We will next show the local compatibility of these charts. Let U and U' be two open sets in $\mathcal{U}$ satisfying $U' \subseteq U$. Let $x \in U'$ and fix $\tilde{x} \in \pi^{-1}(x) \subset M$. For x and U, consider $\widetilde{U}_{\tilde{x}}$ and $\Gamma_{\tilde{x}}$ as above and choose $\widetilde{U}'_{\tilde{x}}$ such that it contains $\tilde{x}$. In order to prove that there is an embedding between the two charts, it suffices to prove that $\widetilde{U}'_{\tilde{x}}$ is contained in $\widetilde{U}_{\tilde{x}}$. To see this, assume there exists a point $\tilde{y} \in \widetilde{U}'_{\tilde{x}} \setminus \widetilde{U}_{\tilde{x}}$. Then there should exist $\gamma \in \Gamma_{\tilde{x}}$ such that $\gamma.\tilde{y} \in \widetilde{U}' \cap \widetilde{U}$, since $\pi(\tilde{y}) = y \in U' \subset U$. But both $\widetilde{U}'_{\tilde{x}}$ and $\widetilde{U}_{\tilde{x}}$ are $\Gamma_{\tilde{x}}$-invariant and hence so is $\widetilde{U}'_{\tilde{x}} \cap \widetilde{U}_{\tilde{x}}$. This means that $\widetilde{y} \in \widetilde{U}'_{\tilde{x}} \cap \widetilde{U}_{\tilde{x}}$ which contradicts the fact that $\tilde{y} \in \widetilde{U}'_{\tilde{x}} \setminus \widetilde{U}_{\tilde{x}}$ and proves that $\widetilde{U}'_{\tilde{x}} \subset \widetilde{U}_{\tilde{x}}$.

Finally, notice that the orbifold structure on $Q = M/\Gamma$ is *natural* in the sense that it depends only on the action of the group Γ and not on the choice of the atlas $\widetilde{\mathcal{U}}$ on M. $\square$

The orbifold $\mathcal{Q} = M/\Gamma$ is called *the orbifold quotient of M by the proper action of* Γ.

Definition 2.11. (developable orbifolds) *An orbifold is called* developable *if it arises as the global quotient by a discrete group acting properly on a manifold. Orbifolds that are developable are sometimes called* good, *and those that are not are called* bad.

Remark 2.12. In Definition 2.1 we can require that the uniformizing charts $\widetilde{U}_i$ have the structure of simply connected smooth n-manifolds instead of being open subsets of $\mathbb{R}^n$. By Proposition 2.10 we obtain equivalent definitions. We can also replace the condition on the Γ_i's to be finite by requiring instead that they act properly on the $\widetilde{U}_i$'s such that the maps φ_i induce homeomorphisms between $\widetilde{U}_i/\Gamma_i$ and U_i. In this case a connected orbifold is developable if and only if can be defined by an atlas consisting of a single orbifold chart (see also [11, Definition III.$\mathcal{G}$.1.1, Remark III.$\mathcal{G}$.1.6(1)]).

2.3 The Pseudogroup of Change of Charts

Recall that a *pseudogroup of local diffeomorphisms* of a differentiable manifold X is a collection $\mathcal{H}$ of diffeomorphisms $h : V \to W$ of open sets of X such that:

(*i*) $\mathcal{H}$ contains the identity map $1_X : X \to X$;

(*ii*) the restriction of an element of $\mathcal{H}$ to any open subset of X belongs to $\mathcal{H}$;

(*iii*) $\mathcal{H}$ is closed under taking inverses, compositions (whenever possible) and unions of its elements.

Given a family H of local diffeomorphisms of a manifold X containing the identity of X, we can form the *pseudogroup generated by H* which is obtained by taking the restrictions of the elements in H to open subset of X, together with their inverses, compositions and unions.

Two points $x, y \in X$ are said to belong to the same *orbit* of $\mathcal{H}$ if there exists an element $h \in \mathcal{H}$ such that $h(x) = y$. This defines an equivalence relation on X whose classes are called the *orbits* of $\mathcal{H}$. The quotient of X by this equivalence relation, with the quotient topology is denoted by $\mathcal{H}\backslash X$.

Two pseudogroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of local diffeomorphisms of differentiable manifolds X_1 and X_2, respectively, are said to be *equivalent* if there exists a pseudogroup $\mathcal{H}$ of local diffeomorphisms of the disjoint union $X = X_1 \sqcup X_2$ whose restriction to X_j is equal to $\mathcal{H}_j$ and such that the inclusion of $X_j \hookrightarrow X$ induces a homeomorphism $\mathcal{H}_j\backslash X_j \to \mathcal{H}\backslash X, j = 1, 2$.

Let now $\mathcal{Q}$ be an orbifold and let $\mathcal{A} = \{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}_{i\in\mathcal{I}}$ an orbifold atlas on $\mathcal{Q}$. We define the pseudogroup of change of charts of the orbifold atlas $\mathcal{A}$. Following [33] (see section 2.1.2), we denote by X the disjoint union of the $\widetilde{U}_i$ and define $\psi : X \to Q$ to be the union of the maps $\varphi_i : \widetilde{U}_i \to Q$. A diffeomorphism h from an open set V of X into an open subset of X which satisfies $\psi \circ h = \psi|_V$ is called a change of chart (see also Remark 2.7). The collection of change of charts of the atlas $\mathcal{A}$ generates a pseudogroup $\mathcal{H}$ of local diffeomorphisms of X.

The pseudogroup $\mathcal{H}$ obtained in this way is called the *pseudogroup of change of charts of the orbifold atlas* $\mathcal{A}$. It contains in particular all the elements of the groups Γ_i: if V is a connected open set of a component $\widetilde{U}_i$ of X and h is a change of chart defined on V such that $h(V) \subset \widetilde{U}_i$, then h is the restriction to V of an element of Γ_i (cf. Remark 2.3). The map ψ induces a homeomorphism from the orbit space $\mathcal{H}\backslash X$ to Q.

As before, the pseudogroups of change of charts of two equivalent orbifold atlases on Q are equivalent. Thus the orbifold structure on Q gives an equivalence class of pseudogroups of local diffeomorphisms of a manifold.

More generally, if a pseudogroup $\mathcal{H}$ of local diffeomorphisms of a differentiable manifold X is such that

(i) each point x of X has an open neighbourhood V such that the restriction of $\mathcal{H}$ to V is generated by a finite group Γ_V of diffeomorphisms of V, and

(ii) the quotient space $\mathcal{H}\backslash X$ is Hausdorff,

then the quotient space $\mathcal{H}\backslash X$ has a natural smooth orbifold structure whose pseudogroup of change of charts is equivalent to $\mathcal{H}$.

In particular, an orbifold $\mathcal{Q}$ is developable if and only if there exists a manifold M and a subgroup Γ of the group of diffeomorphisms of M such that the pseudogroup $\mathcal{H}$ of change of charts of an orbifold atlas $\mathcal{A}$ of $\mathcal{Q}$ is generated by Γ.

2.4 Orbifold Paths

We begin by defining the notion of smooth maps between orbifolds.

Definition 2.13. (orbifold map) *A* smooth map *between two orbifolds $\mathcal{P}$ and $\mathcal{Q}$ is a continuous map $f : P \to Q$ between their underlying topological* *spaces, such that for each point $x \in P$ and $y = f(x) \in Q$ there are coordinate charts $(\widetilde{V}_x, \Gamma^*_x, \varphi^*_x)$ and $(\widetilde{U}_y, \Gamma_y, \varphi_y)$ with the property that f maps $V_x = \varphi^*_x(\widetilde{V}_x)$ into $U_y = \varphi_y(\widetilde{U}_y)$ and can be lifted to a smooth map $\widetilde{f}_{xy} : \widetilde{V}_x \to \widetilde{U}_y$ such that $\varphi^* \circ \tilde{f} = f \circ \varphi$.*

We will denote by $\tilde{f} : \mathcal{P} \to \mathcal{Q}$ an orbifold map whose underlying continuous map is $f : P \to Q$. As noted in [17] (Example 4.1.6b), it is possible for non-isomorphic orbifold maps to have the same underlying continuous map.

The real line $\mathbb{R}$ as a smooth manifold has naturally a trivial orbifold structure. The smooth orbifold maps $\tilde{f} : \mathcal{P} \to \mathbb{R}$ are called *smooth functions* on the orbifold $\mathcal{P}$. Note that an orbifold function $\tilde{f}$ with underlying continuous map $f : P \to \mathbb{R}$ is smooth if and only if the map $f \circ \varphi^*$ is smooth for any orbifold chart $(\widetilde{V}, \Gamma^*, \varphi^*)$ in an orbifold atlas of $\mathcal{P}$. For a developable orbifold $\mathcal{Q} = M/\Gamma$ the smooth orbifold functions on $\mathcal{Q}$ are precisely the Γ-invariant smooth functions on M.

Definition 2.14. *A* path *on an orbifold $\mathcal{Q}$ is an orbifold map from $\mathbb{R}$ (or an interval I) with the trivial orbifold structure into the orbifold $\mathcal{Q}$.*

A more concrete way of describing paths on orbifolds can be given using the pseudogroup of change of charts of the orbifold.

Let $\mathcal{Q}$ be a connected orbifold and let $\mathcal{H}$ be the pseudogroup of change of charts of an orbifold atlas $\mathcal{A} = \{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}_{i \in \mathcal{I}}$ of $\mathcal{Q}$ as defined in the section 2.3. Let $X = \bigsqcup_{\mathcal{I}} \widetilde{U}_i$ and let x and y be two points in X.

Definition 2.15. *A* continuous $\mathcal{H}$-path *from x to y over a subdivision* $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ *of* $[0, 1]$ *is a sequence* $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \ldots \tilde{c}_k, h_k)$ *where:*

- *each* $\tilde{c}_i : [t_{i-1}, t_i] \to \widetilde{U}_j$ *is a continuous map into some* $\widetilde{U}_j$*,*
- $h_1, \ldots, h_k$ *are elements of* $\mathcal{H}$ *defined on neighbourhoods* V_i *of* $\tilde{c}_i(t_i)$ *such that* $h_i(\tilde{c}_i(t_i)) = \tilde{c}_{i+1}(t_i)$ *for each* $i = 1, 2, \ldots, k-1$ *and* $h_k(\tilde{c}_k(t_k)) = y$*; and* h_0 *is defined on a neighbourhood* V *of* x *such that* $h_0(x) = \tilde{c}_1(0)$*.*

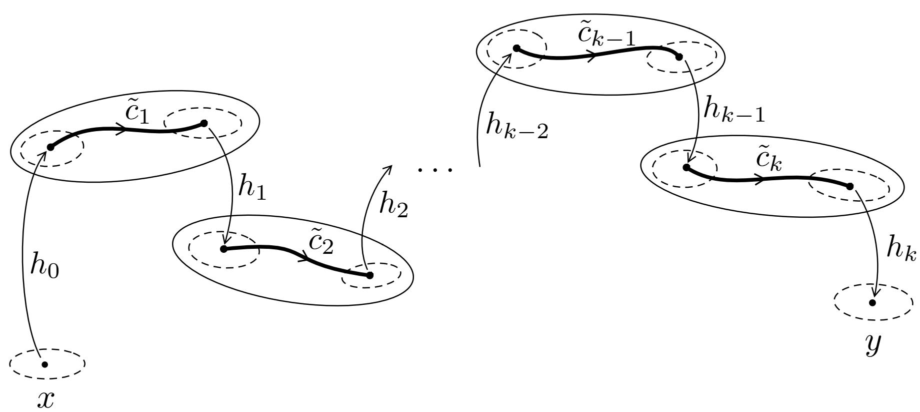


Figure 2.1: An $\mathcal{H}$-path joining x to y.

Among the $\mathcal{H}$-paths from x to y parametrized on $[0, 1]$ we can define an equivalence relation given by the following operations:

(i) Given a $\mathcal{H}$-path $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \ldots, \tilde{c}_k, h_k)$ over the subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$, we can add a new point $t' \in (t_{i-1}, t_i)$ together with the identity map $h' = 1_{\widetilde{U}_i}$ to get a new sequence, replacing $\tilde{c}_i$ in

$\tilde{c}$ by $\tilde{c}_i', h', \tilde{c}_i''$, where $\tilde{c}_i'$ and $\tilde{c}_i''$ are the restriction of $\tilde{c}_i$ to the intervals $[t_{i-1}, t']$ and $[t', t_i]$ respectively. See figure 2.2 below.

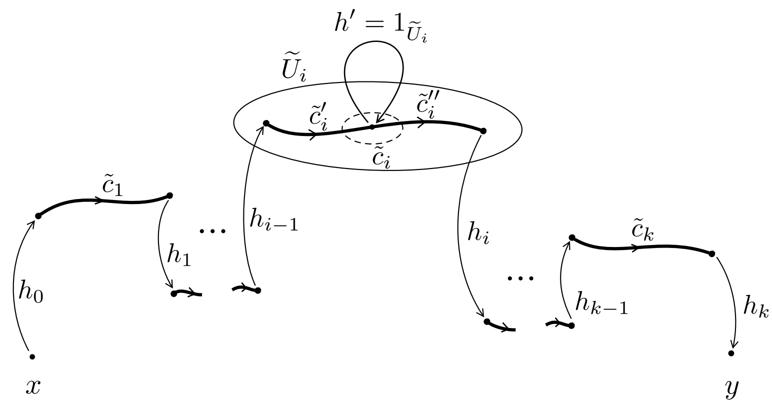


Figure 2.2: Equivalent $\mathcal{H}$-paths joining x to y obtained by adding a point to the subdivision.

(*ii*) Replace the $\mathcal{H}$-path $\tilde{c}_{x,y}$ by a new one $\tilde{c}'_{x,y} = (h_0', \tilde{c}_1', h_1', \dots, \tilde{c}_k', h_k')$ over the same subdivision as follows (figure 2.3 below): for each $i = 1, \dots, k$ choose $g_i \in \mathcal{H}$ defined in a neighbourhood of the paths $\tilde{c}_i$ such that

- $g_i \circ \tilde{c}_i = \tilde{c}_i'$ for $i = 1, \dots, k$,
- $h_i' \circ gi$ and $g_{i+1} \circ h_i$ have the same germ at $\tilde{c}_i(t_i)$ for $i = 1, \dots, k-1$,
- h_0' and $g_1 \circ h_0$ have the same germ at x,
- $h_k' \circ g_k$ and h_k have the same germ at $\tilde{c}_k(1)$.

Remark 2.16. (*a*) If two $\mathcal{H}$-paths on different subdivisions are equivalent, then we can pass from one to another first by considering their equiv-

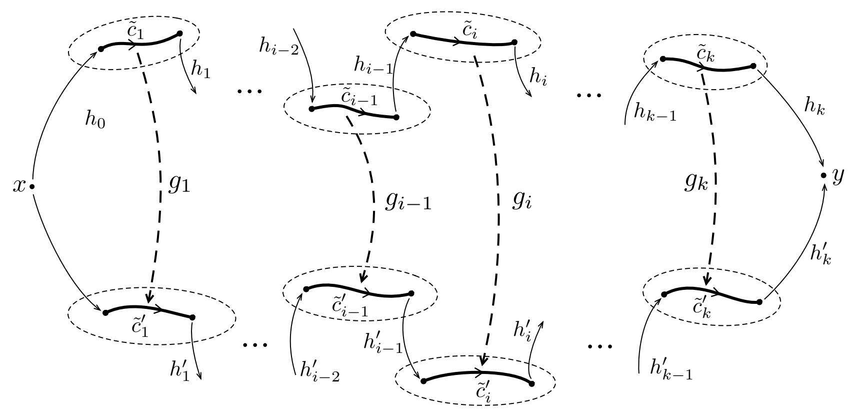


Figure 2.3: Equivalent $\mathcal{H}$-paths from x to y defined over the same subdivision.

alent paths by (i) on a suitable common subdivision, and then by an operation similar to (ii).

(b) Note that two equivalent $\mathcal{H}$-paths have the same initial and terminal point.

(c) For any $\mathcal{H}$-path $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \ldots, \tilde{c}_k, h_k)$ from x to y, we can find equivalent paths $\tilde{c}'_{x,y} = (h'_0, \tilde{c}'_1, h'_1, \ldots, \tilde{c}'_l, h'_l)$ such that h'_0 or h'_l are identity maps.

(d) The germs of the maps g_i in (ii) above are uniquely defined by $\tilde{c}_{x,y}$ and $\tilde{c}'_{x,y}$.

(e) If a $\mathcal{H}$-path $\tilde{c}_{x,y}$ is such that all the $\tilde{c}_i$'s are constant, then the equivalence class of $\tilde{c}_{x,y}$ is completely characterized by an element $h \in \mathcal{H}$ such that $h(x) = y$.

The equivalence class of a $\mathcal{H}$-path $\tilde{c}_{x,y}$ between two points x and y in X will be denoted by $[\tilde{c}]_{x,y}$. Clearly any $\mathcal{H}$-path $\tilde{c}_{x,y}$ from x to y projects via $\psi : X \to Q$ to a continuous map $c : [0,1] \to Q$ from $p = \psi(x)$ to $q = \psi(y)$. The underlying map c depends only on the equivalence class $[\tilde{c}]_{x,y}$ and not the particular choice of $\tilde{c}_{x,y} \in [\tilde{c}]_{x,y}$.

We note $\Omega_{x,y}(\mathcal{H})$ the set of equivalence classes of $\mathcal{H}$-paths joining x to y, and write $\Omega(\mathcal{H})$ for the union of all $\Omega_{x,y}(\mathcal{H})$, with $(x,y) \in X \times X$.

Remark 2.17. If $\mathcal{H}$ is the pseudogroup of change of charts of an developable orbifold $\mathcal{Q} = M/\Gamma$, then any $\mathcal{H}$-path $(h_0, \tilde{c}_1, h_1 \ldots, \tilde{c}_k, h_k)$ between two points x and y in M is equivalent to a unique $\mathcal{H}$-path $(\tilde{c}_x, \gamma)$, where $\tilde{c}_x : [0,1] \to M$ is a continuous path with $\tilde{c}_x(0) = x$, and $\gamma \in \Gamma$ satisfies $\gamma.\tilde{c}_x(1) = y$. Indeed, since $\mathcal{H}$ is generated by Γ, from Proposition 1.11 it follows that for each h_i there exists an unique element $\gamma_i \in \Gamma$ such that h_i is the restriction of γ_i to the domain of h_i. Define $\tilde{c}_x(t) = \gamma_0 \ldots \gamma_{i-1}.\tilde{c}_i(t)$ for $t \in [t_{i-1}, t_i]$ and $\gamma = \gamma_0 \ldots \gamma_k$. It is easy to see that $\tilde{c}_x$ and γ satisfy the required properties and that the given $\mathcal{H}$-path is equivalent to the $\mathcal{H}$-path $(\tilde{c}_x, \gamma)$.

Before giving the definition of paths on orbifolds (that agrees with Definition 2.14) we will introduce the following relation on $\Omega(\mathcal{H})$.

Let $x, y \in X$ and $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \ldots, \tilde{c}_k, h_k)$ be a $\mathcal{H}$-path joining x to y. Suppose x' and y' are two points in the $\mathcal{H}$-orbits through x and y, respectively, and let g_0 and g_1 be elements of $\mathcal{H}$ such that $g_0(x') = x$ and $g_1(y) = y'$. Then $\tilde{c}_{x',y'} = (g_0 \circ h_0, \tilde{c}_1, h_1, \ldots, \tilde{c}_k, g_1 \circ h_k)$ is a $\mathcal{H}$-path joining x' to y'.

If $\tilde{c}'_{x',y'}$ is another $\mathcal{H}$-path from x' to y', we say that $\tilde{c}_{x,y}$ and $\tilde{c}'_{x',y'}$ are equivalent as $\mathcal{H}$-paths in $\Omega(\mathcal{H})$ if and only if $\tilde{c}_{x',y'}$ and $\tilde{c}'_{x'y'}$ belong to the same equivalence class as $\mathcal{H}$-paths joining x' to y', i.e. if and only if $\tilde{c}_{x',y'} \in [\tilde{c}']_{x',y'}$.

Given a $\mathcal{H}$-path $\tilde{c}_{x,y}$ we denote by $\tilde{c}$ its equivalence class in $\Omega(\mathcal{H})$. As before, the projection $\psi : X \to Q$ gives rise to a continuous path $c : [0,1] \to Q$, which we will refer to as the underlying path of $\tilde{c}$.

Finally, if $\mathcal{H}$ and $\mathcal{H}'$ are two equivalent pseudogroups of local diffeomorphisms of differentiable manifolds X and X', then there is a one-to-one correspondence between the set of equivalence classes of $\Omega(\mathcal{H})$ and $\Omega(\mathcal{H}')$. In particular this is true when $\mathcal{H}$ and $\mathcal{H}'$ are the pseudogroups of change of charts of two atlases defining the same orbifold structure $\mathcal{Q}$ on Q. Thus we can give the following definition.

Definition 2.18. *Let $\mathcal{Q}$ be a connected orbifold and let p and q be two points in Q. Suppose $\mathcal{H}$ is the pseudogroup of change of charts of an orbifold atlas $\mathcal{A}$ for $\mathcal{Q}$. An orbifold path joining p to q is an equivalence class of $\mathcal{H}$-paths $\tilde{c}$ on $[0,1]$ such that the underlying map $c : [0,1] \to Q$ satisfies $c(0) = p$ and $c(1) = q$.*

For a developable orbifold $\mathcal{Q} = M/\Gamma$, the orbifold paths on $\mathcal{Q}$ are in one-to-one correspondence with equivalence classes of pairs $(\tilde{c}, \gamma)$, where $\tilde{c}$ is a path in M and $\gamma \in \Gamma$. Two pairs $(\tilde{c}, \gamma)$ and $(\tilde{c}', \gamma')$ are equivalent if and only if there exists an element $\delta \in \Gamma$ such that $\tilde{c'} = \delta.\tilde{c}$ and $\gamma' = \delta^{-1}\gamma\delta$.

Similarly, a loop on the developable orbifold $\mathcal{Q} = M/\Gamma$ is the equivalence class of a pair $(\tilde{c}, \gamma)$ as before with $\tilde{c} : [0,1] \to M$ and $\gamma \in \Gamma$ such that $\gamma.\tilde{c}(1) = \tilde{c}(0)$. An interesting situation is that of orbifold loops that project to a point in Q. Given $p \in \mathcal{Q}$, a constant orbifold loop at p is represented by a pair (x, γ), where x is a point in the fiber above p and γ an element of Γ_x the isotropy group at x. The set of constant orbifold loops at p is in one-to-one correspondence with the conjugacy classes of Γ_p.

2.5 Orbifold Fundamental Group

The Fundamental Group $\pi_1(\mathcal{H}, x)$

Inverse $\mathcal{H}$-paths. Let $x, y \in X$ and $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \ldots, \tilde{c}_k, h_k)$ be a $\mathcal{H}$-path from x to y, defined over the subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$. We can define the *inverse of* $\tilde{c}$ to be the $\mathcal{H}$-path from y to x given by

$$\tilde{c}_{y,x} = \tilde{c}_{x,y}^{-1} = (h'_0, \tilde{c}'_1, h'_1, \ldots, \tilde{c}'_k, h'_k)$$

defined over the subdivision $0 = t'_0 < t'_1 < \cdots < t'_k = 1$, where for each $i = 0, \ldots, k$ we have $t'_i = 1 - t_{k-i}$, $h'_i = h_{k-i}^{-1}$ and $\tilde{c}'_i(t) = \tilde{c}_{k-i}(1-t)$ for $t \in [t'_{i-1}, t'_i]$ and $i = 1, \ldots, k$.

It is easy to see that the inverses of equivalent $\mathcal{H}$-paths are equivalent.

Composition of $\mathcal{H}$-paths. Given two $\mathcal{H}$-paths $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \ldots, \tilde{c}_k, h_k)$ over a subdivision $0 = t_0 < t_1 \leq \cdots \leq t_k = 1$ and $\tilde{c}'_{y,z} = (h'_0, \tilde{c}'_1, h'_1, \ldots, \tilde{c}'_{k'}, h'_{k'})$ over $0 = t'_0 \leq t'_1 \leq \cdots \leq t'_{k'} = 1$, we can define their *composition* (or concatenation) to be the $\mathcal{H}$-path

$$\tilde{c}_{x,y} * \tilde{c}'_{y,z} = \tilde{c}''_{x,z} = (h''_0, \tilde{c}''_1, h''_1, \ldots, \tilde{c}''_k, h''_k)$$

over a subdivision $0 = t''_0 \leq t''_1 \leq \cdots \leq t''_{k+k'} = 1$, where

$$t''_i = \begin{cases} \frac{t_i}{2} & i = 0, \ldots, k \\ \frac{1+t'_{i-k}}{2} & i = k+1, \ldots, k+k' \end{cases}$$

$$\tilde{c}''_i(t) = \begin{cases} \tilde{c}_i(t/2) & i = 1, \ldots, k \\ \tilde{c}'_{i-k}(2t-1) & i = k+1, \ldots, k+k' \end{cases}$$

$$h''_i = \begin{cases} h_i & i = 0, \ldots, k-1 \\ h'_0 h_k & i = k \\ h'_{i-k} & i = k+1, \ldots, k+k'. \end{cases}$$

Again, if $\tilde{c}_{x,y}$ is equivalent to $\tilde{c}'_{x,y}$ and $\tilde{c}_{y,z}$ is equivalent to $\tilde{c}'_{y,z}$, then the composition $\tilde{c}_{x,y} * \tilde{c}_{y,z}$ is equivalent to $\tilde{c}'_{x,y} * \tilde{c}'_{yz}$.

Homotopies of $\mathcal{H}$-paths. An *elementary homotopy* between two $\mathcal{H}$-paths $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \ldots, \tilde{c}_k, h_k)$ over $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ and $\tilde{c}'_{x,y} = (h'_0, \tilde{c}'_1, h'_1, \ldots, \tilde{c}'_k, h'_k)$ over $0 = t'_0 \leq t'_1 \leq \cdots \leq t'_k = 1$, is a family of $\mathcal{H}$-paths joining x to y parametrized by $s \in [s_0, s_1]$,

$$\tilde{c}^s_{x,y} = (h^s_0, \tilde{c}^s_1, h^s_1, \ldots, \tilde{c}^s_k, h^s_k)$$

over $0 = t_0^s \leq t_1^s \leq \cdots \leq t_k^s = 1$, where t_i^s, $\tilde{c}_i^s$ and h_i^s depend continuously on the parameter s, h_0^s and h_k^s are independent of s and $\tilde{c}_{x,y}^{s_0} = \tilde{c}_{x,y}$, $\tilde{c}_{x,y}^{s_1} = \tilde{c}'_{x,y}$.

We say that two $\mathcal{H}$-paths are *homotopic (relative to their end points)* if one can be obtained from the other by a finite sequence of equivalence of $\mathcal{H}$-paths and elementary homotopies.

The homotopy class of a $\mathcal{H}$-path $\tilde{c}_{x,y}$ will be denoted by $[\tilde{c}_{x,y}]$. Given two composable $\mathcal{H}$-paths $\tilde{c}_{x,y}$ and $\tilde{c}'_{y,z}$, the homotopy class $[\tilde{c}_{x,y} * \tilde{c}'_{y,z}]$ depends only on the homotopy classes $[\tilde{c}_{x,y}]$ and $[\tilde{c}'_{y,z}]$ and will be denoted $[\tilde{c}_{x,y}] * [\tilde{c}'_{y,z}]$. If $\tilde{c}_{x,y}$, $\tilde{c}'_{y,z}$ and $\tilde{c}''_{z,w}$ are composable $\mathcal{H}$-paths, then

$$[\tilde{c}_{x,y} * \tilde{c}'_{y,z}] * [\tilde{c}''_{z,w}] = [\tilde{c}_{x,y}] * [\tilde{c}'_{y,z} * \tilde{c}''_{z,w}] = [\tilde{c}_{x,y}] * [\tilde{c}'_{y,z}] * [\tilde{c}''_{z,w}].$$

Definition 2.19. *The set of homotopy classes of $\mathcal{H}$-loops based at a point $x \in X$ together with the operation induced by the composition of $\mathcal{H}$-paths, forms a group $\pi_1(\mathcal{H}, x)$ called* the fundamental group of the pseudogroup $\mathcal{H}$ *based at x.*

Similar to topological spaces, if y is different point in X and $\tilde{a}_{x,y}$ is a $\mathcal{H}$-path joining x to y, then the map that associates to each $\mathcal{H}$-loop $\tilde{c}_x$ based at x the $\mathcal{H}$-loop $\tilde{a}_{x,y}^{-1} * \tilde{c}_x * \tilde{a}_{x,y}$ based at y, induces an isomorphism from $\pi_1(\mathcal{H}, x)$ to $\pi_1(\mathcal{H}, y)$. Thus, if the orbit space $\mathcal{H} \backslash X$ is connected, then up to isomorphism the fundamental group $\pi_1(\mathcal{H}, x)$ is independent of the choice of base point. In this case we will write $\pi_1(\mathcal{H})$ for the isomorphism class of the fundamental group of $\mathcal{H}$.

Definition 2.20. *If $\mathcal{Q}$ is a connected orbifold and $\mathcal{H}$ the pseudogroup of change of charts of an orbifold atlas for $\mathcal{Q}$, we define $\pi_1^{orb}(\mathcal{Q})$, the* orbifold fundamental group of *$\mathcal{Q}$ to be the fundamental group $\pi_1(\mathcal{H})$ of the pseudogroup $\mathcal{H}$.*

An orbifold is called *simply connected* if it is connected and has trivial orbifold fundamental group.

It is important to note that in general, the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ is not the same as $\pi_1(Q)$ the fundamental group of its underlying topological space (see Proposition 1.3 and section 2.7 Examples).

As for topological spaces, a useful result for computing the orbifold fundamental group is the Seifert van Kampen theorem

Theorem 2.21 ([34])**.** *Let $\mathcal{Q}$ be a connected orbifold and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be open connected suborbifolds such that*

- $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$
- $\mathcal{Q}_1 \cap \mathcal{Q}_2$ *is connected*
- *the closures of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are suborbifolds with boundary in $\mathcal{Q}$ such that the boundary of $\mathcal{Q}_i$ equals the frontier of Q_i in Q.*

Then

$$\pi_1^{orb}(\mathcal{Q}) \cong \pi_1^{orb}(\mathcal{Q}_1) *_{\pi_1^{orb}(\mathcal{Q}_1 \cap \mathcal{Q}_2)} \pi_1^{orb}(\mathcal{Q}_2)$$

is the amalgamated free product of the orbifold fundamental groups of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ over the orbifold fundamental group of the intersection $\mathcal{Q}_1 \cap \mathcal{Q}_2$.

Orbifold Covers

Orbifold covering spaces are defined similarly to the ones for topological spaces. Given two orbifolds $\mathcal{Q}$ and $\mathcal{Q}'$, an orbifold projection $p : \mathcal{Q}' \to \mathcal{Q}$ is called a *covering map* if it satisfies the condition that, for each point $x \in \mathcal{Q}$, there exists a neighbourhood U uniformized by $(\widetilde{U}, \Gamma, \varphi)$ such that for each connected component U_i of $p^{-1}(U)$ in Q', the uniformizing systems of U_i is $(\widetilde{U}, \Gamma_i, \varphi_i')$ for some subgroup $\Gamma_i \leq \Gamma$. Note that the underlying space Q' is not generally a covering space of Q.

The *universal covering* $p : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ of a connected orbifold $\mathcal{Q}$ is the initial object in the category of orbifold coverings, i.e. it is a covering such that for any other covering $p' : \mathcal{Q}' \to \mathcal{Q}$ there exists a covering $\overline{p} : \widetilde{\mathcal{Q}} \to \mathcal{Q}'$ such that $p = p' \circ \overline{p}$. If $p : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ is is the universal covering then $\widetilde{\mathcal{Q}}$ is called the *universal covering space* of $\mathcal{Q}$.

Thurston proved that each orbifold $\mathcal{Q}$ has a universal cover (see Proposition 13.2.4 in [66]) and also defined the *orbifold fundamental group* $\pi_1^{orb}(\mathcal{Q})$ as the group of deck transformations of its universal covering.

In the case of a developable orbifold, the quotient $M \to M/\Gamma$ can be regarded as an orbifold covering with Γ as the group of deck transformations. Similarly, any subgroup Γ' induces an intermediate orbifold covering $M/\Gamma' \to M/\Gamma$. On the other hand, any manifold covering $\widetilde{M} \to M$ gives an orbifold covering by composing with the quotient map $M \to M/\Gamma$. In particular, the universal covering of M gives rise to a universal orbifold covering of $\mathcal{Q}$, and the orbifold fundamental group belongs in a short exact sequence

$$1 \to \pi_1(M) \to \pi_1^{orb}(\mathcal{Q}) \to \Gamma \to 1.$$

Note that an orbifold is developable if and only if its universal covering space is a manifold.

As mentioned before, the orbifold fundamental group of an orbifold $\mathcal{Q}$ is different from the fundamental group of the underlying topological space Q. For a developable orbifold $\mathcal{Q} = M/\Gamma$, with M simply connected, the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q}) \cong \Gamma$ while, by Proposition 1.3, the fundamental group of the topological quotient $\pi_1(Q) \cong \Gamma/\Gamma_0$, where Γ_0 is the normal subgroup generated by all the elements of Γ which act non freely on M.

2.6 Riemannian Orbifolds

The Tangent Bundle

We start with a differentiable developable n-orbifold $\mathcal{Q} = M/\Gamma$. Since the action of Γ on M is smooth, it can be extended to an action on the tangent bundle TM of M by setting $\gamma.(\widetilde{x}, v) := \big(\gamma.\widetilde{x}, d(\gamma)_{\tilde{x}}(v)\big)$, for all $\gamma \in \Gamma$ and $(\widetilde{x}, v) \in TM$, and where $d(\gamma)_{\tilde{x}} : T_{\tilde{x}}M \to T_{\gamma\tilde{x}}M$ denotes the differential of γ at $\tilde{x}$. It is easy to see that this action is proper and therefore, by Proposition 2.10, the quotient of TM by this action inherits a natural orbifold structure. In this case, we define the tangent bundle $\mathcal{TQ}$ of the orbifold $\mathcal{Q}$ to be the

$2n$-orbifold obtained as the quotient TM/Γ. The underlying space of $\mathcal{TQ}$ will be denoted by TQ. As before, the space TQ is generally not a manifold.

Unlike for the tangent bundle of manifolds, the fibers of the orbifold tangent bundle $\mathcal{TQ}$ need not have the structure of a vector space. Let x be a point in Q and let $\widetilde{x} \in M$ denote one of its lifts. By taking the differentials $(d\gamma)_{\tilde{x}} : T_{\tilde{x}}M \to T_{\tilde{x}}M$ of the elements γ in the isotropy group $\Gamma_{\tilde{x}}$ of $\tilde{x}$, we obtain a new group which acts linearly on the fiber $T_{\widetilde{x}}M$. By Proposition 1.11 this new group is naturally isomorphic to $\Gamma_{\tilde{x}}$, and since up to isomorphism this group is independent of the choice of the lift we will denote it by Γ_x. The fiber in $\mathcal{TQ}$ above $x \in Q$ is then is isomorphic to $T_{\widetilde{x}}M/\Gamma_x$. We denote this fiber by $T_x\mathcal{Q}$ and refer to it as the *tangent cone to $\mathcal{Q}$ at x*. It has the structure of a vector space if and only if x is a regular point in Q.

Since any orbifold is locally good, the construction above gives a local way to work with tangent cones to orbifolds.

For any differentiable n-orbifold $\mathcal{Q}$ with an orbifold atlas $\mathcal{A} = \{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}$, given an open set $U_i \subseteq Q$ uniformized by an orbifold chart $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ we can form the tangent bundle $T\widetilde{U}_i/\Gamma_i$ over U_i. By patching together these bundles we obtain a $2n$-dimensional orbifold $\mathcal{TQ}$ with an atlas given by $\{(T\widetilde{U}_i, \Gamma_i, \psi_i)\}$. As in Remark 2.7, we can use the transition functions of this orbifold atlas to obtain a space TQ the underlying topological space of $\mathcal{TQ}$. The natural projection $p : TQ \to Q$ defines a smooth orbifold map, with fibers $p^{-1}(x) \simeq T_{\tilde{x}}\widetilde{U}/\Gamma_x$ (see [1, Proposition 1.21]).

Riemannian Metrics on Orbifolds

Let $\mathcal{Q}$ be a smooth orbifold and let $\mathcal{U} = \{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}_{i\in\mathcal{I}}$ be a maximal orbifold atlas on $\mathcal{Q}$.

Definition 2.22. *A* Riemannian metric *on the orbifold $\mathcal{Q}$ is a collection $\rho = (\rho_i)$, where each ρ_i is a Γ_i-invariant Riemannian metric on $\widetilde{U}_i$ and such that any embedding $\widetilde{\phi}_{ij}$ coming from an injection between orbifold charts $(\widetilde{U}_i, \Gamma_i, \varphi_i) \hookrightarrow (\widetilde{U}_j, \Gamma_j, \varphi_j)$ is an isometry as a map from $(\widetilde{U}_i, \rho_i)$ to $(\widetilde{U}_j, \rho_j)$. An orbifold with such a Riemannian metric is called a* Riemannian orbifold.

Remark 2.23. Note that the Riemannian metrics ρ_i on $\widetilde{U}_i$ are Γ_i-invariant, so locally, Riemannian orbifolds look like the quotient of a Riemannian manifold by a (finite) group of isometries (see also Remark 2.12) . By a suitable choice of coordinate charts it can be assumed that the local group actions are by finite subgroups of $O(n)$ for a general n-dimensional Riemannian orbifold, and by finite subgroups of $SO(n)$ for orientable Riemannian n-orbifolds.

As in the manifold case, the following proposition holds.

Proposition 2.24. *Any smooth orbifold admits a Riemannian metric.*

Proof. Let $\mathcal{Q}$ be an orbifold and let $\{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}_{i\in\mathcal{I}}$ denote an orbifold atlas on it. Since the underlying topological space Q is paracompact we may assume that the cover $\{U_i\}_{i\in\mathcal{I}}$ is locally finite. We can define a 'smooth' partition of unity $\{f_i : U_i \to \mathbb{R}\}$, subordinate to the cover $\{U_i\}$ in the following way: on each $\widetilde{U}_i$ choose a Γ_i-invariant, non-negative smooth function

$\tilde{h}_i : \widetilde{U}_i \to \mathbb{R}$ such that the functions $h_i = \tilde{h}_i \circ \varphi_i$ can be extended over Q by zero and such that $\{supp(h_i) \subset U_i : i \in \mathcal{I}\}$ still covers Q. The function $h(x) = \sum_{i \in \mathcal{I}} h_i(x)$ is non-zero on Q and we define $f_i := h_i/h$. Then $\{f_i : i \in \mathcal{I}\}$ is a smooth partition of unity subordinate to the cover U_i.

Consider now an arbitrary Riemannian metric g_i on each $\widetilde{U}_i$. By Lemma 1.8 there exists a Γ_i-invariant Riemannian metric α_i on each $\widetilde{U}_i$ obtained from g_i by averaging over Γ_i. For any $i \in \mathcal{I}$, define a new Riemannian metric ρ_i on $\widetilde{U}_i$ as follows:

$$(\rho_i)_{\tilde{x}}\big(v, w\big) := \sum_{j \in \mathcal{I}} f_j(\varphi_i(\tilde{x}))(\alpha_j)_{\widetilde{\varphi}_{ij}(\tilde{x})}\big(d(\widetilde{\varphi}_{ij})_{\tilde{x}}(v), d(\widetilde{\varphi}_{ij})_{\tilde{x}}(w)\big)$$

for any $\tilde{x} \in \widetilde{U}_i$ and any $v, w \in T_{\tilde{x}}\widetilde{U}_i$ and where $\widetilde{\phi}_{ij}$ is an embedding coming from an injection between $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ and any $(\widetilde{U}_j, \Gamma_j, \varphi_j)$, $j \in \mathcal{I}$. Then, Lemma 1.8 together with the second part of the Remark 2.3 guarantee that the Riemannian metric defined in this way is well defined, i.e. it is independent of the choice of the embedding between the uniformizing charts. It is also easy to check that each embedding is an isometry, hence the collection $\rho = (\rho_i)$ is defines a Riemannian metric in the sense of the definition above. $\square$

Remark 2.25. It follows from the definition, that the change of charts of an orbifold atlas of a Riemannian orbifold $\mathcal{Q}$ are isometries. As in Remark 2.3, the collection of change of charts of an orbifold atlas $\{\widetilde{U}_i, \Gamma_i, \varphi_i)\}$ on $\mathcal{Q}$

generates a pseudogroup $\mathcal{H}$ of local isometries of the Riemannian manifold $X = \bigsqcup_i \widetilde{U}_i$. The map $\psi : X \to Q$ given by $\psi = \sqcup_i \varphi_i$ induces a homeomorphism between the quotient space $\mathcal{H}\backslash X$ and the underlying topological space Q of $\mathcal{Q}$.

Alternatively, a Riemannian orbifold can be defined as an equivalence class of pseudogroups $\mathcal{H}$ of local isometries of a Riemannian manifold X, satisfying: (i) the quotient space $\mathcal{H}\backslash X$ is Hausdorff; (ii) each point $x \in X$ has an open neighbourhood V such that the restriction $\mathcal{H}|_V$ is generated by a finite group Γ_V of isometries of V.

If $\mathcal{Q} = \mathcal{H}\backslash X$ is a Riemannian orbifold and $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \dots, \tilde{c}_k, h_k)$ is a piecewise differentiable $\mathcal{H}$-path (i.e. each of the paths $\tilde{c}_i$ is piecewise differentiable) then we define its length $\ell(\tilde{c}_{x,y})$ to be the sum of the lengths of the paths $\tilde{c}_i$. It depends only on the equivalence class $\tilde{c}$ of $\tilde{c}_{x,y}$.

Given two points $p, q \in Q$ we define the distance $d(p, q)$ to be the infimum of the lengths of all piecewise differentiable $\mathcal{H}$-paths $\tilde{c}$ whose underlying continuous path c joins p to q. This distance defines a metric on the underlying topological space Q.

Definition 2.26. *We say that the Riemannian orbifold $\mathcal{Q}$ is* complete *if (Q, d) is complete as a metric space.*

Geodesics on Orbifolds

Let $\mathcal{Q}$ be a connected Riemannian orbifold and let $\mathcal{H}$ be the pseudogroup of change of charts of an atlas $\{(\widetilde{U}_i, \Gamma_i \varphi_i)\}$ as described in Remark 2.25. Since Q is connected, the Riemannian manifold $X = \bigsqcup_i \widetilde{U}_i$ is *$\mathcal{H}$-path connected.* That is, for any two points $x, y \in X$ there exists a $\mathcal{H}$-path $\tilde{c}_{x,y}$ joining them.

Definition 2.27. *Given two points $x, y \in X$, a* geodesic $\mathcal{H}$-path *joining x to y is an $\mathcal{H}$-path $\tilde{c}_{x,y} = (h_0, \tilde{c}_1, h_1, \dots, \tilde{c}_k, h_k)$ over some subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ of $[0,1]$ such that:*

(i) each $\tilde{c}_i : [t_{i-1}, t_i] \to \widetilde{U}_j$ is a constant speed geodesic segment in some $\widetilde{U}_j$;

(ii) the differential $d(h_i)_{\tilde{c}_i(t_i)} : T_{\tilde{c}_i(t_i)}\widetilde{U}_i \to T_{\tilde{c}_{i+1}(t_i)}\widetilde{U}_{i+1}$ maps the velocity vector $\dot{\tilde{c}}_i(t_i)$ to the vector $\dot{\tilde{c}}_{i+1}(t_i)$.

Since any $\mathcal{H}$-path which is equivalent to a geodesic $\mathcal{H}$-path is also a geodesic, in the vein of Definition 2.18 we define an *orbifold geodesic in $\mathcal{Q}$* as the equivalence class $\tilde{c}$ of a geodesic $\mathcal{H}$-path as above.

The projection $\psi : X \to Q$ associates to each equivalence class of geodesic $\mathcal{H}$-paths $\tilde{c}$ parametrized over $[0,1]$ a continuous path $c : [0,1] \to Q$. When there is no place for confusion, we will also refer to the image $c([0,1])$ as of a geodesic on the orbifold.

Note that since on smooth manifolds geodesics are uniquely determined by the tangent vector at one of its points, each of the geodesic segments $\tilde{c}_{i+1}$ depends only on the tangent vector $\dot{\tilde{c}}_{i+1}(t_i)$, which in turn, by Definition 2.27

(ii), depends only on the vector $\dot{\tilde{c}}_i(t_i)$ and the germ of the isometry h_i at $\tilde{c}_i(t_i)$. It follows that the vector $\dot{\tilde{c}}_1(0)$ is an invariant of the equivalence class $\tilde{c}$ of the geodesic $\mathcal{H}$-path $\tilde{c}_{x,y}$.

The image of $\dot{\tilde{c}}_i(t)$ under the projection $T_{\tilde{c}_i(t)}\widetilde{U}_i \to T_{\tilde{c}_i(t)}\widetilde{U}_i/\Gamma_{c(t)}$ is independent of the choice of $\mathcal{H}$-path in the equivalence class $\tilde{c}$. We denote this image by $\dot{c}(t)$. It belongs to the fiber of the tangent bundle over the point $c(t)$. It is easy to see that the norm $|\dot{c}(t)|$ is constant in t. The vector $\dot{c}(0)$ is called the *initial vector of the geodesic $\mathcal{H}$-path $\tilde{c}$.*

Let now x be a point in X and T_xX be the tangent space at x to the manifold X. Suppose h_0 is an element of $\mathcal{H}$ defined on a neighbourhood of x. For any vector $v \in T_x\widetilde{U}_i$ there exists $\varepsilon_0 > 0$ and a geodesic $\mathcal{H}$-path $\tilde{c}_x = (h_0, \tilde{c}_1)$ on $[0, \varepsilon_0)$ issuing from x and which satisfies $\dot{\tilde{c}}_i(0) = d(h_0)_x(v)$. If $y = \tilde{c}_1(t_1)$ for $0 < t_1 < \varepsilon_0)$ and $h_1 \in \mathcal{H}$ is a local isometry defined in a neighbourhood of y, then as above, there exists $\varepsilon_1 > 0$ and a geodesic $\mathcal{H}$-path $\tilde{c}_y = (h_1, \tilde{c}_2)$ on $[0, \varepsilon_1)$ issuing at y and such that the differential of h_1 at y maps the tangent vector $\dot{\tilde{c}}_1(t_1)$ to the vector $\dot{\tilde{c}}_2(0)$. It is easy to see that the concatenation of the two geodesic $\mathcal{H}$-paths gives a geodesic $\mathcal{H}$-path issuing at x and with initial vector v. We can continue this process until we obtain a geodesic $\mathcal{H}$-path defined on a maximal interval $[0, \varepsilon)$ issuing at x and with initial velocity v. Passing to equivalence classes of $\mathcal{H}$-paths we have the following:

Proposition 2.28. *If $\mathcal{Q}$ is a Riemannian n-orbifold, then for any point*

$p \in Q$ and any vector $v \in T_p\mathcal{Q} \simeq \mathbb{R}^n/\Gamma_p$ in the tangent cone at p to $\mathcal{Q}$ there exists a maximal parametrized orbifold geodesic $\tilde{c} : (-\varepsilon', \varepsilon) \to \mathcal{Q}$ such that $c(0) = x$ and $\dot{c}(0) = v$.

Definition 2.29. *We say that a Riemannian orbifold $\mathcal{Q}$ is* geodesically complete *if $\varepsilon = \varepsilon' = \infty$ for all $p \in Q$.*

Just as for manifolds, one can prove that a connected Riemannian orbifold is complete if and only if it is geodesically complete (see [21]). For manifolds, this result is known as the Hopf-Rinow Theorem ([20]). Also, given any two points p and q in a connected complete Riemannian orbifold $\mathcal{Q}$ there is an orbifold geodesic $\tilde{c}$ connecting them. Moreover, the geodesic $\tilde{c}$ can be chosen such that it is of minimal length among all the piecewise differentiable orbifold paths joining the two points. We call such geodesic a *minimal geodesic*. Note that the underlying path of a minimal geodesic realizes the distance between the end points, and thus it is a minimal geodesic in (Q, d).

We will now define the closed geodesics on an orbifold.

Definition 2.30. *An $\mathcal{H}$-loop $\tilde{c}_x = (h_0, \tilde{c}_1, h_1, \dots, \tilde{c}_k, h_k)$ based at x represents a closed orbifold geodesic $\tilde{c}$ on $\mathcal{Q}$ if in addition to (i) and (ii) in Definition 2.27 it satisfies that the differential of $h_0 \circ h_k$ at $\tilde{c}_k(1)$ maps the velocity vector $\dot{\tilde{c}}_k(1)$ to the vector $\dot{\tilde{c}}_1(0)$.*

For a developable orbifold $\mathcal{Q} = M/\Gamma$ obtained as the quotient of a connected Riemannian manifold M by the proper action of discrete subgroup

$\Gamma \subset \mathrm{Isom}(M)$, a closed geodesic on $\mathcal{Q}$ is represented by a pair $(\tilde{c}, \gamma)$, where $\tilde{c} : [0,1] \to M$ is a geodesic in M and γ is an element of Γ such that the differential of γ at $\tilde{c}(1)$ maps the velocity vector $\dot{\tilde{c}}(1)$ to $\dot{\tilde{c}}(0)$. Another such pair $(\tilde{c}', \gamma')$ represents the same closed geodesic if and only if there is an element $\delta \in \Gamma$ such that $\tilde{c}' = \delta.\tilde{c}$ and $\gamma' = \delta^{-1}\gamma\delta$.

The underlying map $c : [0,1] \to Q$ of a minimal orbifold geodesic $\tilde{c}$ is a geodesic in the metric sense on Q with respect to $d(\cdot,\cdot)$ the induced length metric by the Riemannian structure on $\mathcal{Q}$.

2.7 Examples

We will now exhibit some of the notions introduced in this chapter through some simple and well known examples of orbifolds.

Example 2.31. We begin by noticing that any manifold M is naturally an orbifold, where all points have trivial isotropy.

When M is an n-manifold with boundary ∂M, then it can be also given the structure of an n-orbifold without boundary, which we will denote by $\mathcal{M}$. If $x \in \mathrm{Int}(M)$ is an interior point, then a manifold chart at x becomes an orbifold chart with trivial local group. If $y \in \partial M$ is a boundary point, then an orbifold chart at y is given by $(\mathbb{R}^n, \mathbb{Z}_2, \varphi)$, where the local group $\mathbb{Z}_2$ is generated by the reflection in the hyperplane corresponding to the tangent space to boundary $T_y\partial M$. Thus the singular locus of $\mathcal{M}$ is the boundary ∂M

and every singular point has isotropy $\mathbb{Z}_2$. The manifold $DM := M \bigcup_{\partial M} M$, obtained as the double of M along its boundary is a cover for the orbifold $\mathcal{M}$. Thus $\mathcal{M}$ is a developable orbifold, with universal cover $\widetilde{DM}$.

If $(M, \partial M)$ is a compact Riemannian manifold with boundary, then $\mathcal{M}$ can be given the structure of a compact Riemannian orbifold. It follows directly from the definition that any smooth closed geodesic contained in $\mathrm{Int}(M)$ or ∂M is also a closed geodesic in $\mathcal{M}$. Recall that an *geodesic chord* in $(M, \partial M)$ is a geodesic segment $c : [a, b] \to M$ such that $c(]a, b[) \subset \mathrm{Int}(M)$ and $c(a), c(b) \in \partial M$. A geodesic cord is said to be *orthogonal* if $\dot{c}(a^+) \in (T_{c(a)}\partial M)^\perp$ and $\dot{c}(b^-) \in (T_{c(b)}\partial M)^\perp$, where $\dot{c}(\cdot^\pm)$ denote the lateral derivatives and $(T_y\partial M)^\perp$ is the orthogonal complement of $T_y\partial M$ in T_yM. Any orthogonal geodesic chord $c : [a, b] \to M$ gives rise to a closed geodesic of positive length $\tilde{c}$ in $\mathcal{M}$ in the sense of Definition 2.30 by letting $\tilde{c}_{c(a)} = (1_{c(a)}, c, \delta, c^-, \gamma)$, where c^- denotes the geodesic obtained from c by reversing the orientation, and δ and γ are the generators of the isotropy groups at $c(b)$ and $c(a)$, respectively. Note that the double Dc of c in DM is the image of a smooth closed geodesic in DM that projects to $\tilde{c}$ via the orbifold covering map $DM \to \mathcal{M}$.

Example 2.32. The only 1-dimensional compact connected effective orbifolds are the circle $\mathbb{S}^1$ and the 'mirrored interval', i.e. the orbifold $\mathcal{I}$ associated to a compact interval I as in the previous example, or equivalently, as the quotient of the circle by an orientation-reversing involution.

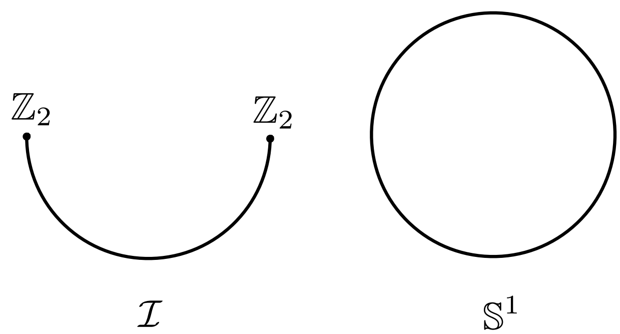


Figure 2.4: 1-dimensional orbifolds.

Example 2.33. [*The cone*] Let $\Gamma = \mathbb{Z}_n$ be the cyclic group of order n acting on the Euclidean plane $\mathbb{R}^2$ by rotations of angle $2\pi/n$ about the origin. The quotient space $\mathbb{R}^2/\mathbb{Z}_n$ is topologically $\mathbb{R}^2$, but metrically it is the flat cone with angle $2\pi/n$ at the vertex. $\mathbb{R}^2/\mathbb{Z}_n$ has a natural orbifold structure with one singular point (the cone point) with isotropy $\mathbb{Z}_n$.

Clearly the orbifold $\mathbb{R}^2/\mathbb{Z}_n$ is developable: its universal cover is $\mathbb{R}^2$ and its orbifold fundamental group is isomorphic to $\mathbb{Z}_n$. Note that the orbifold $\mathbb{R}^2/\mathbb{Z}_n$ admits no closed geodesics of positive length (see Remark 4.4).

Example 2.34. [*The $\mathbb{Z}_n$-football*] Let $\mathbb{S}^2 = \{x^2+y^2+z^2 = 1 \mid (x,y,z) \in \mathbb{R}^3\}$ be the unit sphere in $\mathbb{R}^3$, and let $\Gamma = \langle\gamma\rangle \cong \mathbb{Z}_n$ be the cyclic group of order n generated by

$$\gamma = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the rotation of angle $2\pi/n$ around the z-axis. The quotient space $\mathbb{S}^2/\Gamma$ has

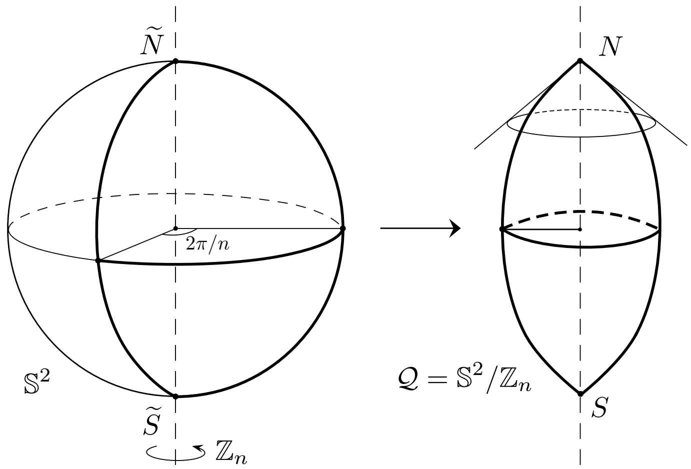


Figure 2.5: The $\mathbb{Z}_n$-football orbifold

a natural orbifold structure $\mathcal{Q}$. The singular locus $\Sigma_{\mathcal{Q}} = \{N, S\}$ consists of two isolated singular points with isotropy $\mathbb{Z}_n$ (see Figure 2.5). The fiber of the tangent bundle $\mathcal{TQ}$ over each of the singular points is isomorphic to the cone $\mathbb{R}^2/\mathbb{Z}_n$ of angle $2\pi/n$ (see the previous example), and it is isomorphic to $\mathbb{R}^2$ over the regular points.

The sphere $\mathbb{S}^2$ is the universal orbifold cover of $\mathcal{Q}$ and the orbifold fundamental group is the cyclic group of order n: $\pi_1^{orb}(\mathbb{S}^2/\Gamma) = \Gamma \cong \mathbb{Z}_n$. The underlying topological space Q of $\mathcal{Q}$ is homeomorphic to the 2-sphere, thus $\pi_1(\mathbb{S}^2/\Gamma) = 1$.

Since the round metric on the sphere $\mathbb{S}^2$ is Γ-invariant, the quotient $\mathbb{S}^2/\Gamma$ has the structure of a Riemannian orbifold as in Definition 2.22. Any closed geodesic on the sphere $\mathbb{S}^2$ projects to a closed geodesic on the orbifold $\mathbb{S}^2/\Gamma$.

Such a geodesic homotopically trivial, and if it is not the constant loop, then its length is an integral multiple of 2π. We can find closed geodesics of positive length in every homotopy class of free loops on $\mathbb{S}^2/\Gamma$. For instance, given an integer k which is not a multiple on n, a closed geodesic in the homotopy class of $\gamma^k \in \Gamma$ is represented by a pair (c, γ^k), where $c : [0,1] \to \mathbb{S}^2$ is either the constant map to N or S, or is a geodesic arc on the equator $\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ of length $2(m - \frac{k}{n})\pi$, with $m \in \mathbb{Z}$, $m \geq 1$.

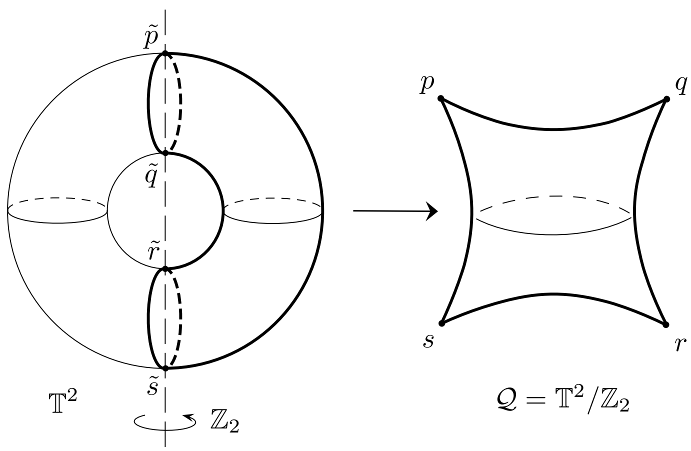


Figure 2.6: The pillowcase orbifold

Example 2.35. [*The pillow case*] Consider the 2-torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$ together with the action by the group $\Gamma = \langle\gamma\rangle \cong \mathbb{Z}_2$, where γ is the rotation of angle π around one of the axis of $\mathbb{T}^2$ as in Figure 2.6. The quotient space $\mathbb{T}^2/\Gamma$ is an orbifold $\mathcal{Q}$ whose underlying space is homeomorphic to the 2-sphere and whose singular locus Σ consists of four singular points $\{p, q, r, s\}$, each with isotropy $\mathbb{Z}_2$.

The universal cover $\mathbb{R}^2 \to \mathbb{T}^2$ is also the orbifold universal cover for $\mathcal{Q}$. Thus the orbifold fundamental group is a semi-direct product: $\pi_1^{orb}(\mathbb{T}^2/\mathbb{Z}_2) \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_2$.

Example 2.36. [*The $\mathbb{Z}_n$-teardrop*] Let n be an integer, $n > 1$. The $\mathbb{Z}_n$-teardrop orbifold has the 2-sphere as the underlying space and one singular point O with isotropy $\mathbb{Z}_n$ (see Figure 2.7). Let $\mathcal{U} = \{U_1, U_2\}$ be an open

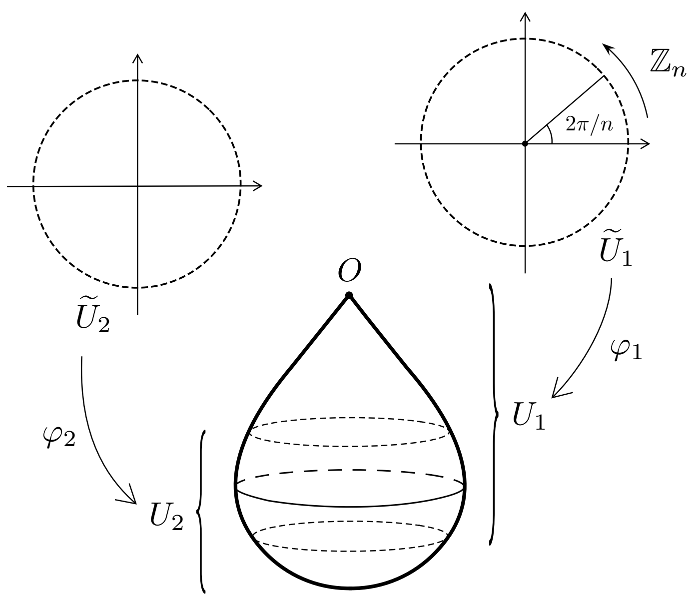


Figure 2.7: The $\mathbb{Z}_n$-teardrop orbifold.

cover of the sphere. The orbifold atlas for the teardrop orbifold $\mathcal{Q}$ associated to $\mathcal{U}$ consists of two orbifold charts $(\mathbb{R}^2, 1, \varphi_2)$ over U_2, and $(\mathbb{R}^2, \mathbb{Z}_n, \varphi_1)$ over U_1, where the group $\mathbb{Z}_n$ acts on $\mathbb{R}^2$ by rotations of angle $2\pi/n$ around the origin, and $\varphi_1(0) = O$.

To compute the orbifold fundamental group we use Theorem 2.21. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ denote the suborbifolds obtained by restricting the orbifold structure on $\mathcal{Q}$ to the open sets U_1 and U_2, respectively. Then

$$\pi_1^{orb}(\mathcal{Q}_1) \cong \pi_1^{orb}(\mathbb{R}^2/\mathbb{Z}_n) = \mathbb{Z}_n,$$

$$\pi_1^{orb}(\mathcal{Q}_2) \cong \pi_1^{orb}(\mathbb{R}^2) = \pi_1(\mathbb{R}^2) = 1, \text{ and}$$

$$\pi_1^{orb}(\mathcal{Q}_1 \cap \mathcal{Q}_2) \cong \pi_1^{orb}(\mathbb{S}^1 \times (0,1)) = \pi_1(\mathbb{S}^1 \times (0,1)) = \mathbb{Z}.$$

By Theorem 2.21

$$\pi_1^{orb}(\mathcal{Q}) \cong \mathbb{Z}_n *_{\mathbb{Z}} 1 = 1,$$

which shows that the $\mathbb{Z}_n$-teardrop orbifold is simply connected and therefore it is its own orbifold universal cover. In particular, the $\mathbb{Z}_n$-teardrop orbifold is not developable.

Example 2.37. [*The $\mathbb{Z}_m$-$\mathbb{Z}_n$-football*] Let $n, m \in \mathbb{Z}$, $n, m > 1$. The $\mathbb{Z}_m$-$\mathbb{Z}_n$-football orbifold $\mathcal{Q}$ is the 2-sphere with two cone points (N and S in Figure 2.8) with isotropy $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. Note that for $n = m$ we obtain the global quotient orbifold in example 2.34 above. However, if $n \neq m$, then the orbifold $\mathcal{Q}$ is not developable. To see this we can proceed as in the previous example and compute the orbifold fundamental group of the $\mathbb{Z}_m$-$\mathbb{Z}_n$-football:

$$\pi_1^{orb}(\mathcal{Q}) \cong \mathbb{Z}_n *_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d,$$

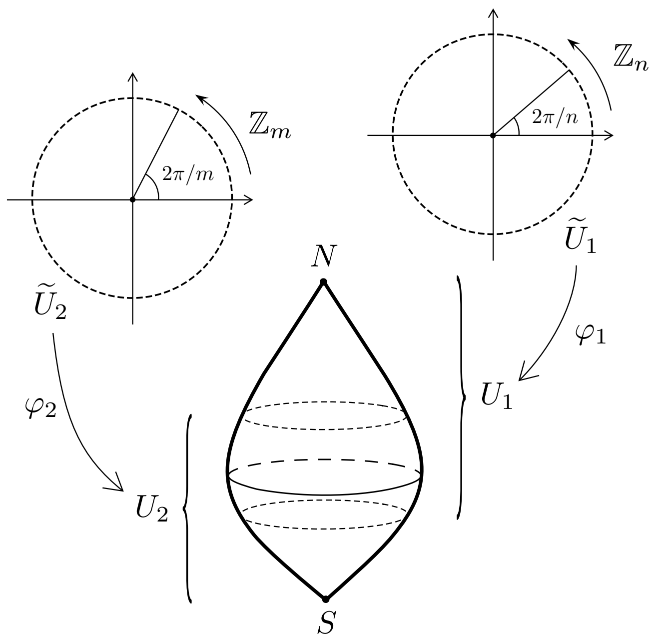


Figure 2.8: The $\mathbb{Z}_m$-$\mathbb{Z}_n$-football orbifold.

where $d = \gcd(m, n)$. Thus the universal orbifold covering of $\mathcal{Q}$ is the $\mathbb{Z}_p$-$\mathbb{Z}_q$-football orbifold with $p = m/d$ and $q = n/d$, which is a manifold if and only if $p = q = 1$ (or equivalently $m = n$).

Example 2.38. Let $M = \mathbb{R}^3$ and $\Gamma = \mathbb{Z}_2$ acting by the antipodal map $x \mapsto -x$. The quotient $\mathbb{R}^3/\mathbb{Z}_2$ is a 3-dimensional orbifold whose underlying topological space is a cone over $\mathbb{RP}^2$ and therefore fails to be a manifold at the singular point.

Chapter 3

Closed Geodesics

As we have seen in the previous chapter, the orbifold structure induces a natural stratification of the underlying topological space: the stratification by orbit type, where strata correspond to the conjugacy classes of isotropy groups. An important geometric feature of this stratification is that the strata are totally geodesic (Proposition 3.4). In this chapter we study the existence of closed geodesics on compact orbifolds by considering the structure of the singular locus. For example, we will show that every compact orbifold whose singular locus contains a connected component of dimension one admits at least one closed geodesic of positive length.

We begin by giving a description of this stratification from a slightly different perspective, which was also used by Seaton in [61].

3.1 Stratification by Singular Dimension

Let $\mathcal{Q}$ be a n-dimensional effective smooth orbifold and let $\mathcal{T}\mathcal{Q}$ denote its tangent bundle. As noted in section 2.6, the fiber $T_x\mathcal{Q}$ above a point $x \in \mathcal{Q}$ is not in general a vector space.

If $(\widetilde{U}, \Gamma_x, \varphi)$ is a fundamental chart at x and $\tilde{x} = \varphi^{-1}(x)$ (see Remark 2.9), then $T_x\mathcal{Q}$ is the quotient of the n-dimensional vector space $T_{\tilde{x}}\widetilde{U}$ by the linear action of the isotropy group Γ_x. Denote by $T_{\tilde{x}}\widetilde{U}^{\Gamma_x}$ the vector subspace of $T_{\tilde{x}}\widetilde{U}$ fixed by Γ_x. It is easy to see that up to isomorphism, the subspace $T_{\tilde{x}}\widetilde{U}^{\Gamma_x}$ is independent of the choice of the chart at x. We denote by $T_x\mathcal{Q}^{\Gamma_x}$ the isomorphism class of this vector subspace and, as in [61], we will refer to it as the *space of tangent vectors* at x. Thus $T_x\mathcal{Q}^{\Gamma_x}$ depends only on the point x and the action of its isotropy group Γ_x.

Definition 3.1. *The* singular dimension *of a point x of an effective orbifold $\mathcal{Q}$ is the dimension of $T_x\mathcal{Q}^{\Gamma_x}$, the space of tangent vectors at x.*

For each $k = 0, 1, \ldots, n$, denote by Σ_k the set of points in Q with singular dimension k. Thus the underlying space is the disjoint union $Q = \bigsqcup_{k=0}^{n} \Sigma_k$.

Example 3.2. Let $\Gamma = \langle \gamma, \delta \rangle$ act on $\mathbb{R}^3$, where

$$\gamma = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) & 0 \\ \sin(2\pi/3) & \cos(2\pi/3) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

denotes the rotation of angle $2\pi/3$ about the z-axis in $\mathbb{R}^3$ and

$$\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

is the reflection in the xy-plane. Clearly $\gamma^3 = \delta^2 = 1$ and $\gamma\delta = \delta\gamma$, thus $\Gamma \cong \mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$. The 3-dimensional orbifold $\mathcal{Q}$ obtained as the quotient $\mathbb{R}^3/\Gamma$ is geometrically a cone over the closed 2-disk (see figure 3.1 below).

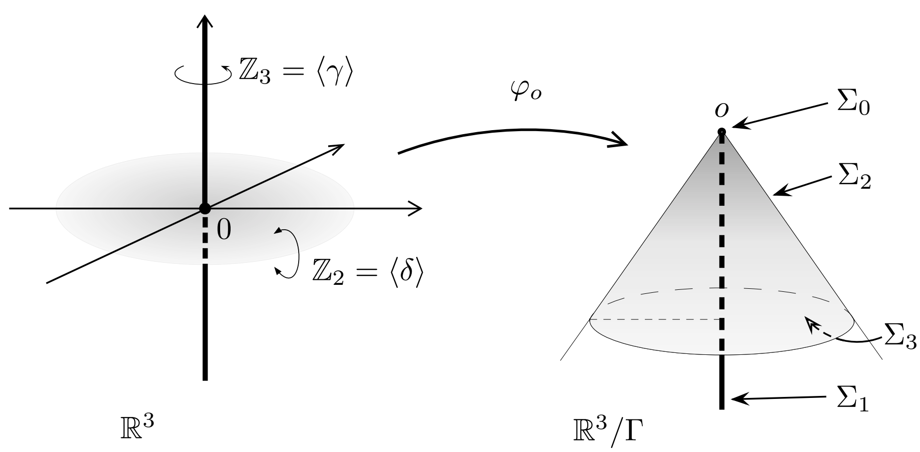


Figure 3.1: Stratification by singular dimension.

In this case $Q = \bigsqcup_{k=0}^{3} \Sigma_k$, where $\Sigma_0 = \{o\}$ is the vertex of the cone with isotropy $\mathbb{Z}_6$; the set $\Sigma_1 \simeq \mathbb{R}$ is a line whose points have isotropy $\mathbb{Z}_3$, the set $\Sigma_2 \simeq \mathbb{R} \times \mathbb{S}^1$ has points with isotropy $\mathbb{Z}_2$; and $Q_{reg} = \Sigma_3 \simeq \mathbb{R} \times (0,1) \times \mathbb{S}^1$ consists of points with trivial isotropy.

Example 3.3. Let Γ be the subgroup of $O(6)$ generated by two elements δ and γ given by:

$$\delta = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $\delta^2 = \gamma^2 = (\delta\gamma)^3 = 1$ and $\gamma\delta \neq \delta\gamma$. Thus $\Gamma \cong S_3$ is the symmetric group of degree 3. Γ has one normal subgroup of order 3 generated by $\delta\gamma$, and three subgroups of order 2 with generators δ, γ and $\delta\gamma\delta$, respectively. Note that the three subgroups of order two are conjugate in Γ.

Let Γ act on the unit sphere $\mathbb{S}^5 = \{\mathbf{x} \in \mathbb{R}^6 \mid x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2 = 1\}$ and let $\mathcal{Q}$ denote the orbifold quotient $\mathbb{S}^5/\Gamma$. Let $\pi : \mathbb{S}^5 \to Q$ be the quotient map. Corresponding to each of the subgroups of Γ the action has the following fixed point sets:

$$\widetilde{\Sigma}_0 := (\mathbb{S}^5)^{\Gamma} = \mathbb{S}^5 \cap \big\{\mathbf{x} \in \mathbb{R}^6 \mid x_1 = x_2 = x_3 = x_4 = x_5 = x_6\big\} \simeq \mathbb{S}^0$$

$$\widetilde{\Sigma}_1 := (\mathbb{S}^5)^{\langle\delta\gamma\rangle} = \mathbb{S}^5 \cap \big\{\mathbf{x} \in \mathbb{R}^6 \mid x_1 = x_2 = x_3;\ x_4 = x_5 = x_6\big\} \simeq \mathbb{S}^1$$

$$\widetilde{\Sigma}_2 := (\mathbb{S}^5)^{\langle\delta\rangle} = \mathbb{S}^5 \cap \big\{\mathbf{x} \in \mathbb{R}^6 \mid x_1 = x_4;\ x_2 = x_6;\ x_3 = x_5\big\} \simeq \mathbb{S}^2$$

$$\widetilde{\Sigma}_2' := (\mathbb{S}^5)^{\langle\gamma\rangle} = \mathbb{S}^5 \cap \big\{\mathbf{x} \in \mathbb{R}^6 \mid x_1 = x_6;\ x_2 = x_5;\ x_3 = x_4\big\} \simeq \mathbb{S}^2$$

$$\widetilde{\Sigma}_2'' := (\mathbb{S}^5)^{\langle\delta\gamma\delta\rangle} = \mathbb{S}^5 \cap \big\{\mathbf{x} \in \mathbb{R}^6 \mid x_1 = x_5;\ x_2 = x_4;\ x_3 = x_6\big\} \simeq \mathbb{S}^2.$$

Thus the fixed point set of the Γ-action consists of three embedded two-spheres and one circle, all meeting at the two antipodal points in $\widetilde{\Sigma}_0$.

In this case $Q = Q_{reg} \bigsqcup \Sigma$, where the singular locus $\Sigma = \bigsqcup_{k=0}^{2} \Sigma_k$ has components of singular dimension $0, 1$ and 2. Thus $\Sigma_3 = \Sigma_4 = \varnothing$.

The stratum Σ_2 consists of one connected component homeomorphic to a 2-sphere with two points deleted, and it corresponds to the conjugacy class of the subgroups of order two in Γ. The isotropy groups at points in Σ_2 are cyclic of order two. The preimage $\pi^{-1}(\Sigma_2) \subset \mathbb{S}^5$ is the disjoint union of three two-spheres each having two points deleted:

$$\pi^{-1}(\Sigma_2) = (\widetilde{\Sigma}_2 \smallsetminus \widetilde{\Sigma}_0) \sqcup (\widetilde{\Sigma}_2' \smallsetminus \widetilde{\Sigma}_0) \sqcup (\widetilde{\Sigma}_2'' \smallsetminus \widetilde{\Sigma}_0).$$

The stratum Σ_1 has two connected components, both homeomorphic to the interval $(0, 1)$. The isotropy groups of points in Σ_1 are cyclic of order three, and both the connected components of Σ_1 correspond to the fixed point set of the normal subgroup generated by $\delta\gamma$. The preimage $\pi^{-1}(\Sigma_1)$ in $\mathbb{S}^5$ is a circle with two pints deleted:

$$\pi^{-1}(\Sigma_1) = \widetilde{\Sigma}_1 \smallsetminus \widetilde{\Sigma}_0.$$

Finally, the zero-dimensional stratum Σ_0, consists of two points, corresponding to the two antipodal points in $\widetilde{\Sigma}_0$, both having isotropy $\Gamma \cong S_3$.

The following proposition shows that the sets Σ_k have a particularly nice structure (see also [39] and [61]).

Proposition 3.4. *Suppose $\mathcal{Q}$ is an effective n-orbifold without boundary. Then for each $k = 0, \ldots, n$, the set Σ_k has naturally the structure of a k-dimensional manifold without boundary. The tangent space $T_x\Sigma_k$ at a point $x \in \Sigma_k$ is canonically identified with $T_x\mathcal{Q}^{\Gamma_x}$, the space of tangent vectors at x. Furthermore, if $\mathcal{Q}$ is a Riemannian orbifold, then each connected component of Σ_k is totally geodesic in $\mathcal{Q}$.*

Proof. Let x be a point in Q and $(\widetilde{U}_x, \Gamma_x, \varphi_x)$ a fundamental chart at x. Since the orbifold $\mathcal{Q}$ is assumed to be effective, the preimage of x by φ_x consists of a single point $\tilde{x} = \varphi_x^{-1}(x) \in \widetilde{U}_x$.

If $x \in Q_{reg}$ is a regular point, then the isotropy group Γ_x at x is trivial and the map $\varphi_x : \widetilde{U}_x \to Q$ gives a homeomorphism from $\widetilde{U}_x$ onto its image $\varphi_x(\widetilde{U}_x) \subseteq Q_{reg}$. The space of tangent vectors at a regular point x is isomorphic to $T_{\tilde{x}}\widetilde{U}_x$ and has dimension n. Thus $Q_{reg} \subseteq \Sigma_n$.

If $x \in \Sigma$ is a singular point, then the isotropy group Γ_x is non-trivial and acts effectively on $\widetilde{U}_x$. By Proposition 1.7, the induced Γ_x-action on $T_{\tilde{x}}\widetilde{U}_x$ is effective and the space $T_{\tilde{x}}\widetilde{U}_x^{\Gamma_x}$ of the tangent vectors at x has dimension $k < n$. This implies that the singular locus Σ is contained in $\bigsqcup_{k=0}^{n-1} \Sigma_k$. In particular, Σ_n contains no singular points and thus

$$Q_{reg} = \Sigma_n \text{ and } \Sigma = \bigsqcup_{k=0}^{n-1} \Sigma_k.$$

To show that Σ_n is a smooth manifold, note first that any atlas $\mathcal{A}$ defining the orbifold structure on $\mathcal{Q}$, can be refined to an equivalent orbifold atlas $\mathcal{A}'$

that contains only fundamental charts. It is easy to see that the restriction of such an atlas $\mathcal{A}'$ to the regular part gives $Q_{reg} = \Sigma_n$ the structure of a smooth n-manifold without boundary.

If $\mathcal{Q}$ is a Riemannian orbifold, then Q_{reg} inherits the Riemannian structure from $\mathcal{Q}$, and the maps $\varphi_x : \widetilde{U}_x \to \varphi_x(\widetilde{U}_x)$ are isometries. If $c : [0,1] \to Q_{reg}$ is a parametrized geodesic in Q_{reg}, then for each $t \in [0,1]$ the local lift $\tilde{c}_t = \varphi_{c(t)}^{-1} \circ c$ of c in a chart $(\widetilde{U}_{c(t)}, 1, \varphi_{c(t)})$ at $c(t)$ is a geodesic in $\widetilde{U}_{c(t)}$. This shows that any geodesic c in Q_{reg} defines an orbifold geodesic in $\mathcal{Q}$, and hence Q_{reg} is totally geodesic in $\mathcal{Q}$.

Fix now k such that $0 \le k \le n-1$. Let $x \in \Sigma_k$ be a point of singular dimension k. Denote by $\widetilde{\Sigma}_{\Gamma_x}$ the set of points in $\widetilde{U}_x$ fixed by Γ_x. Clearly $\tilde{x} = \varphi_x^{-1}(x) \in \widetilde{\Sigma}_{\Gamma_x}$ and since $(\widetilde{U}_x, \Gamma_x, \varphi_x)$ is a fundamental chart at x, the set $\widetilde{\Sigma}_{\Gamma_x}$ is connected. By Theorem 1.12, $\widetilde{\Sigma}_{\Gamma_x}$ has the structure of a closed totally geodesic submanifold of $\widetilde{U}_x$. Also, if V is a neighbourhood of the origin in tangent space $T_{\tilde{x}}\widetilde{U}_x$ such that the exponential map $\exp_{\tilde{x}}|_V : V \to \widetilde{U}_x$ is a diffeomorphism, then

$$\widetilde{\Sigma}_{\Gamma_x} \cap \exp_{\tilde{x}}(V) = \exp_{\tilde{x}}\left(V \cap T_{\tilde{x}}\widetilde{U}_x^{\Gamma_x}\right).$$

This shows that the tangent space $T_{\tilde{x}}\widetilde{\Sigma}_{\Gamma_x} = T_{\tilde{x}}\widetilde{U}^{\Gamma_x}$, and hence $\dim(\widetilde{\Sigma}_{\Gamma_x}) = k$.

If $k = 0$ then $\widetilde{\Sigma}_{\Gamma_x} = \{\widetilde{x}\}$ and the open set $\varphi_x(\widetilde{U}_x) \subseteq Q$ contains x as the only point with singular dimension zero. Thus Σ_0 is a discrete set in Q and clearly has the structure of a 0-manifold.

Assume $k \geq 1$. Since the map $\varphi_x : \widetilde{U}_x \to Q$ is injective on $\widetilde{\Sigma}_{\Gamma_x}$, the restriction $\varphi_x|_{\widetilde{\Sigma}_{\Gamma_x}}$ of φ_x to $\widetilde{\Sigma}_{\Gamma_x}$ induces a homeomorphism between $\widetilde{\Sigma}_{\Gamma_x}$ and the open neighbourhood $\Sigma_k \cap \varphi_x(\widetilde{U}_x)$ of x in Σ_k. It is clear from the construction that such neighbourhoods exists for any point $x \in \Sigma_k$. Given two points $x, y \in \Sigma_k$ and fundamental charts $(\widetilde{U}_x, \Gamma_x, \varphi_x)$ and $(\widetilde{U}_y, \Gamma_y, \varphi_y)$ such that

$$\Sigma_k \cap \varphi_x(\widetilde{U}_x) \cap \varphi_y(\widetilde{U}_y) \neq \varnothing,$$

the local groups Γ_x and Γ_y are isomorphic and the transition map between the two orbifold charts (as defined in Remark 2.7) is an equivariant differentiable map that induces a diffeomorphism between $\widetilde{\Sigma}_{\Gamma_x} \subseteq \widetilde{U}_x$ and $\widetilde{\Sigma}_{\Gamma_y} \subseteq \widetilde{U}_y$. The collection of all neighbourhoods $\{\Sigma_k \cap \varphi_x(\widetilde{U}_x)\}_{x \in \Sigma_k}$ together with the induced transition functions as above give the desired manifold atlas on Σ_k.

Thus Σ_k has the structure of a differentiable k-manifold, and for each $x \in \Sigma_k$ there is a natural diffeomorphism $\Sigma_k \cap \varphi_x(\widetilde{U}_x) \simeq \widetilde{\Sigma}_{\Gamma_x}$. In particular, by Proposition 1.12, the connected components of Σ_k have no boundary.

It is easy to see that if $\mathcal{Q}$ is a Riemannian orbifold, then the manifold Σ_k inherits a Riemannian structure such that for all $x \in \Sigma_k$ the maps $\Sigma_k \cap \varphi_x(\widetilde{U}_x) \simeq \widetilde{\Sigma}_{\Gamma_x}$ are isometries. To show that Σ_k is totally geodesic in $\mathcal{Q}$, let $c : [0,1] \to \Sigma_k$ be a parametrized geodesic contained in one of the connected components of Σ_k. Since, for each $t \in [0,1]$, the restriction $\varphi|_{\widetilde{\Sigma}_{\Gamma_{c(t)}}} : \widetilde{\Sigma}_{\Gamma_{c(t)}} \to \Sigma_k$ defined as above is an isometry, the local lifts $\tilde{c}_t := \varphi_{c(t)}^{-1} \circ c|_{I_t}$ is a geodesic in $\widetilde{\Sigma}_{\Gamma_{c(t)}}$. Here I_t denotes an open connected subinterval of $[0,1]$ such that

$c(I_t) \subseteq \Sigma_k \cap \varphi_{c(t)}(\widetilde{U}_{c(t)})$. Using the fact that $\widetilde{\Sigma}_{\Gamma_{c(t)}}$ is a totally geodesic submanifold of $\widetilde{U}_{c(t)}$, the lift $\tilde{c}_t$ is also a geodesic in $\widetilde{U}_{c(t)}$. As this holds for all $t \in [0,1]$, the path $c : [0,1] \to \Sigma_k$ is a geodesic in $\mathcal{Q}$ and thus Σ_k is totally geodesic in $\mathcal{Q}$. □

Remark 3.5. (a) Note that in general, the components of Σ_k need not be compact (see Examples 3.2 and 3.3).
(b) Letting Γ_x denote the isotropy group at $x \in Q$, then the isomorphism class of Γ_x is constant along the connected component of Σ_k containing x. For a developable orbifold $\mathcal{Q} = M/\Gamma$, this isomorphism class is just the conjugacy class of Γ_x in Γ.
(c) Suppose $\mathcal{Q}$ is an effective n-orbifold and $x \in \Sigma_\ell$. Let U be a fundamental neighbourhood of x with uniformizing chart $(\widetilde{U}, \Gamma_x, \varphi_x)$, where $\widetilde{U}$ is the n-disk D^n and Γ_x acts linearly and orthogonally on D^n (cf. Remark 2.23). To each connected component in $\Sigma_k \cap U$ with $k \geq \ell$, one can associate a conjugacy class of subgroups of Γ_x (see also Remark 2.9), and if Γ'_x is such a subgroup, then its fixed point set $(\widetilde{U})^{\Gamma'_x}$ will have dimension k. In particular, the smooth points in U correspond to a full orbit, and the singular points are those with nontrivial isotropy. Moreover, for each subgroup Γ'_x of Γ_x with fixed point set $(\widetilde{U})^{\Gamma'_x}$ having dimension k and different from that of Γ_x, the corresponding component of $\Sigma_k \cap U$ will have dimension $k \geq \ell$ and will have x in its closure (see also Example 3.3).

The following is a direct consequence of the Remark 3.5(b):

Corollary 3.6. *If $c : [0,1] \to \Sigma_k$ is a smooth parametrized geodesic contained in a connected component of Σ_k, then the parametrized orbifold geodesics $\tilde{c} : [0,1] \to \mathcal{Q}$ covering c are in one-to-one correspondence with the conjugacy classes of elements in $\Gamma_{c(0)}$. In particular, for each parametrized geodesic $c : [0,1] \to Q_{reg}$ there exists a unique orbifold geodesic $\tilde{c} : [0,1] \to \mathcal{Q}$ with underlying path c.*

It is clear that the manifolds Σ_k need not be connected. The following proposition shows that in a compact orbifold the number of connected components of each of the manifolds Σ_k is finite.

Proposition 3.7. *If $\mathcal{Q}$ is an effective compact connected n-orbifold, then for each $k = 0, \dots, n$, the manifolds Σ_k have finitely many connected components.*

Proof. Note first that it follows from the proof of Proposition 3.4 that Σ_0 is a discrete subset of Q. Since Q is compact, Σ_0 consists of a finite collection of points.

Let now $\mathcal{U} = \{U_x \mid x \in Q\}$ be an open cover of Q by fundamental neighbourhoods U_x at points $x \in Q$ (see Remark 2.9). Using the compactness of Q there exists a finite collection of points $x_1, \dots, x_j \in Q$ such that $Q \subseteq U_{x_1} \cup \cdots \cup U_{x_j}$ and each U_{x_i} is uniformized by a fundamental chart $(\widetilde{U}_{x_i}, \Gamma_{x_i}, \varphi_{x_i})$ at x_i. Note in particular that $\Sigma_0 \subseteq \{x_1, \dots, x_j\}$.

In each U_{x_i} the number of connected components of $\Sigma_k \cap U_{x_i}$ is bounded from above by the number of conjugacy classes of subgroups of Γ_{x_i} for $k \geq 2$, and by twice this number when $k = 1$. Since each of the isotropy groups Γ_{x_i} is finite, this number is finite and the conclusion of the lemma follows. $\square$

3.2 Closed Geodesics in the Singular Locus

In general, the connected components of the singular locus Σ need not be manifolds. If S is a connected component of Σ_k for some $k > 0$, we define the *frontier of* S to be the set $\mathrm{fr}(S)$ consisting of points in $Q \setminus S$ which are the limit points of sequences in S; and the *closure of* S to be the union $\mathrm{cl}(S) = S \cup \mathrm{fr}(S)$.

Remark 3.8. It is important to note that the points in the frontier of an open connected component of Σ_k belong to singular strata of singular dimension strictly less than k, and also that the intersection of the closures of two connected components Σ_k and $\Sigma_{k'}$ belongs to the disjoint union $\bigsqcup_{j<\min(k,k')} \Sigma_j$.

Together with Proposition 3.4 this implies the following.

Proposition 3.9. *Let $\mathcal{Q}$ be a compact connected Riemannian n-orbifold. Suppose that there exists a connected component $S \subset \Sigma_k$ for some $0 < k \leq n$ such that $S = \mathrm{cl}(S)$ is closed. Then there exists a closed geodesic of positive length in $\mathcal{Q}$.*

Indeed, since $\mathcal{Q}$ is compact, any closed component S as in the above proposition has the structure of a compact manifold and therefore by Fet's result in [44] it admits closed geodesics of positive length. Since S is totally geodesic in $\mathcal{Q}$, any of these closed geodesics in S gives rise to closed orbifold geodesics of positive length in $\mathcal{Q}$.

A particular situation when the conditions of Proposition 3.9 are met is when there are no points of singular dimension zero. In this case if $k > 0$ is the minimal singular dimension such that Σ_k is nonempty, then every connected component S of Σ_k is necessarily closed. Thus we have:

Corollary 3.10. *Suppose $\mathcal{Q}$ is a compact connected Riemannian effective n-orbifold such that $\Sigma_0 = \varnothing$. Then there exists at least one closed geodesic of positive length in $\mathcal{Q}$.*

Another direct consequence of Proposition 3.4 is that for an orbifold $\mathcal{Q}$ with $\Sigma_1 \neq \varnothing$, constant speed parametrizations of the connected components of Σ_1 are orbifold geodesic paths in $\mathcal{Q}$. As the following proposition shows, if the orbifold $\mathcal{Q}$ is compact, then the closure of Σ_1 contains a totally geodesic compact 1-orbifold. In particular, $\mathcal{Q}$ contains a closed geodesic of positive length.

Proposition 3.11. *Suppose $\mathcal{Q}$ is a compact effective Riemannian orbifold with $\Sigma_1 \neq \varnothing$. Then there exists at least one closed geodesic of positive length in $\mathcal{Q}$ contained in the closure* $\mathrm{cl}(\Sigma_1)$ *of Σ_1.*

Proof. Let $S \neq \varnothing$ be a connected component of Σ_1. By Proposition 3.4, S has the structure of a 1-dimensional manifold without boundary which is totally geodesic in $\mathcal{Q}$. If S is compact, then it is diffeomorphic to a circle and as in Proposition 3.9, any non-zero constant speed parametrized orbifold path whose image is contained in S gives a closed geodesic of positive length in $\mathcal{Q}$.

If S is not compact, then it is diffeomorphic to the open interval $(0,1)$. Choose an orientation for S and let $c:(0,1)\to S$ be a smooth orientation preserving parametrization of S. Without loss of generalization we can assume that c has constant speed.

The frontier $\mathrm{fr}(S)$ of S consists of two points defined by $c(0)=\lim_{t\to 0} c(t)$ and $c(1)=\lim_{t\to 1} c(t)$. Clearly the frontier of S is contained in Σ_0 and the points $c(0)$ and $c(1)$ could map either to two distinct points or to same point in Q. Assume the former, and let $x,y\in Q$ such that $c(0)\mapsto x$ and $c(1)\mapsto y$.

Let $(\widetilde{U}_x,\Gamma_x,\varphi_x)$ be a fundamental chart at x and choose $\tilde{c}$ to be a local lift of c to $\widetilde{U}_x$. That is, $\tilde{c}:[0,\varepsilon_0)\to\widetilde{U}_x$ be such that $\tilde{c}(0)=\tilde{x}=\varphi_x^{-1}(x)$ and $\varphi_x\circ\tilde{c}=c|_{[0,\varepsilon_0)}$, for some $0<\varepsilon_0<1$. Since the image of $c|_{(0,1)}$ is contained in Σ_1, there exists a proper subgroup $\Gamma_0\leq\Gamma_x$ which fixes $\tilde{c}$ pointwise. Note that $\tilde{c}$ is a geodesic path starting at x, and therefore it is uniquely determined by its initial vector $v=\dot{\tilde{c}}(0)\in T_{\tilde{x}}\widetilde{U}_x$. Since the the induced action of Γ_0 on $T_{\tilde{x}}\widetilde{U}_x$ is linear and fixes v, Γ_0 fixes the 1-dimensional vector subspace V spanned by v. Thus $V=(T_{\tilde{x}}\widetilde{U}_x)^{\Gamma_0}$ and the geodesic $\tilde{c}:[0,\varepsilon_0)\to\widetilde{U}_x$ can be extended through $\tilde{x}$ to a Γ_0-invariant geodesic $\tilde{c}:(-\varepsilon_0',\varepsilon_0)\to\widetilde{U}_x$ for some $\varepsilon_0>0$.

We distinguish two possible situations. The first one is when there exists a subgroup $\Gamma_0'\leq\Gamma_x$ that leaves invariant the subspace spanned by v. This happens precisely when there exists an element $\gamma\in\Gamma_x\smallsetminus\Gamma_0$ of order two and such that $\gamma.v = -v$. Then Γ_0' is an extension of order two of Γ_0, and thus $\Gamma_0 \lhd \Gamma_0' \leq \Gamma_x$. Note that in this case

$$\varphi_x(\tilde{c}(t)) = \varphi_x(\tilde{c}(-t)) = c(t) \text{ for all } 0 \leq t \leq \min(\varepsilon_0, \varepsilon_0'),$$

i.e. in the underlying topological space the image of $c(0,1) \subset \Sigma_1$ terminates at the left at $x = c(0)$. In this case we will say that x is an *end of* c.

The second possible situation is of course when there are no elements $\gamma \in \Gamma_x \setminus \Gamma_0$ of order two which satisfy $\gamma.v = -v$; or equivalently when Γ_0 is the only subgroup of Γ_x that leaves invariant the subspace spanned by v in $T_{\tilde{x}}\widetilde{U}_x$. In this case for each $0 < t < \min(\varepsilon_0, \varepsilon_0')$ we have

$$\varphi_x(\tilde{c}(t)) \neq \varphi_x(\tilde{c}(-t)),$$

which implies that there exists a connected component $S' \subset \Sigma_1$ containing $(\varphi_x \circ \tilde{c})(-\varepsilon_0', 0)$, which is different that S and which 'extends' $c(0,1)$ to the left beyond $x = c(0) \in \Sigma_0$. Note that the isomorphism class of the isotropy groups of the components S and S' are the same.

Similarly, at $y = c(1)$, we have one of the two possible situations: either y is an end of c, or $c(0,1)$ can be 'extended' to the right beyond $y = c(1) \in \Sigma_0$ to a component S'' of Σ_1 which is different than S (but possibly the same as S' above).

Consider now the case when $c(0) = c(1) = x$, and let $(\widetilde{U}_x, \Gamma_x, \varphi_x)$ be a fundamental chart at x. Let $\tilde{c} : [0, \varepsilon_0) \to \widetilde{U}_x$ and $\tilde{c}' : (1-\varepsilon_1, 1] \to \widetilde{U}_x$ be lifts to $\widetilde{U}_x$ of $c|_{[0,\varepsilon_0)}$ and $c|_{(1-\varepsilon_1,1]}$, respectively. Here, for $\widetilde{U}_x$ small enough we can assume that $0 < \varepsilon_0 < 1 - \varepsilon_1 < 1$. Let Γ_0 and Γ_1 be the subgroups of Γ_x that fix the vector subspaces in $T_{\tilde{x}}\widetilde{U}_x$ spanned by $\dot{\tilde{c}}(0)$ and $\dot{\tilde{c}}'(1)$, respectively. Although the groups Γ_0 and Γ_1 are isomorphic (see Remark 3.5(b)), in general they need not be conjugate in Γ_x. Note that $\Gamma_0 = \gamma\Gamma_1\gamma^{-1}$ for some $\gamma \in \Gamma_x$ if and only if γ satisfies $\gamma.\dot{\tilde{c}}'(1) = \dot{\tilde{c}}(0)$.

The possible situations described above apply to this situation and consequently, x can be either be an end point for c at both sides or just one of the sides of c.

Finally, note that by Proposition 3.7, the process of ‘extending’ the component S within $\mathrm{cl}(\Sigma_1)$ terminates after finitely many steps. Denote by $\overline{S}$ the maximal closed extension of S in $\mathrm{cl}(\Sigma_1) \subseteq \Sigma_1 \cup \Sigma_0$. It is clear that $\overline{S}$ can only have either two ends or no end, and as we will now show, in either case, $\overline{S}$ contains a closed geodesic of positive length in $\mathcal{Q}$.

Let $a, b \in \mathbb{R}$ with $a \leq 0$ and $1 \leq b$ and such that $c : [a, b] \to Q$ is the parametrization of $\overline{S}$ that extends the parametrization $c : (0, 1) \to Q$ of S. Then $c([a, b]) \subseteq \Sigma_1 \cup \Sigma_0$, and let $a = s_0 < s_1 < \cdots < s_m = b$ be such that $c(s_i) \in \Sigma_0$ for $i = 0, \ldots, m$ are the points of singular dimension zero along the image $c([a, b])$. Note that if $\overline{S}$ has no ends, then $c(a) = c(b)$.

The image of the restrictions $c^i = c|_{(s_i,s_{i+1})}$ for $i = 0, \ldots, m-1$ are constant speed parametrizations of connected components S_i of Σ_1 and by Proposition 3.4, there exist orbifold geodesic paths $\tilde{c}^i : (s_i, s_{i+1}) \to \mathcal{Q}$ whose underlying paths are the maps $c^i : (s_i, s_{i+1}) \to Q$. For each $i = 0, \ldots, m-1$, let $s_i = t_0^i \leq t_1^i \leq \cdots \leq t_{k_i}^i = s_{i+1}$ be a subdivision of the interval $[s_i, s_{i+1}]$ and let $\tilde{c}^i = (h_0^i, \tilde{c}_1^i, h_1^i, \ldots, \tilde{c}_{k_i}^i, h_{k_i}^i)$ be a $\mathcal{H}$-path representing the orbifold geodesic $\tilde{c}^i$. For simplicity, we can assume that h_0^i and $h_{k_i}^i$ are identity maps (see Remark 2.16 (c)).

Clearly, for each $i = 0, \ldots, m-1$, the points $\tilde{c}_{k_i}^i(s_{i+1})$ and $\tilde{c}_0^{i+1}(s_{i+1})$ belong to the $\mathcal{H}$-orbit above $c(s_{i+1})$ and since each S_{i+1} is an extension of S_i, there exist elements $g_i \in \mathcal{H}$ such that $g_i(\tilde{c}_{k_i}^i(s_{i+1})) = \tilde{c}_0^{i+1}(s_{i+1})$ and the differential of g_i at $\tilde{c}_{k+i}^i(s_{i+1})$ maps the tangent vector $\dot{\tilde{c}}_{k_i}^i(s_{i+1})$ to the vector $\dot{\tilde{c}}_0^{i+1}(s_{i+1})$. Thus the $\mathcal{H}$-path

$$(h_0^0, \tilde{c}_1^0, \ldots, h_{k_0-1}^0, \tilde{c}_{k_0}^0, g_1, \tilde{c}_1^1, \ldots, h_{k_1-1}^1, \tilde{c}_{k_1}^1, g_2, \tilde{c}_1^2, \ldots, h_{k_m-1}^m, \tilde{c}_{k_m}^m, h_{k_m}^m)$$

defined over the subdivision of $[a, b]$ obtained as the union of the subdivisions of the intervals $[s_i, s_{i+1}]$, is a geodesic $\mathcal{H}$-path. Denote by $\tilde{c} : [a, b] \to \mathcal{Q}$ the orbifold geodesic given by the equivalence class of this geodesic $\mathcal{H}$-path. Note that $\tilde{c}$ has the path $c : [a, b] \to Q$ as its underlying continuous map.

If $\overline{S}$ has no ends, then the component S_0 is an extension of the component S_m and hence there exists an element $g_m \in \mathcal{H}$ such that $g_m(\tilde{c}_{k_m}^m(b)) = \tilde{c}_1^0(a)$ and the differential of g_m at $\tilde{c}_{k_m}^m(b)$ maps the vector $\dot{\tilde{c}}_{k_m}^m(b)$ to $\dot{\tilde{c}}_1^0(a)$. Thus the orbifold path $\tilde{c}' : [a, b] \to \mathcal{Q}$ represented by the $\mathcal{H}$-path

$$(h_0^0, \tilde{c}_1^0, h_1^0, \ldots, \tilde{c}_{k_0}^0, g_1, \tilde{c}_1^1, \ldots, h_{k_m-1}^m, \tilde{c}_{k_m}^m, g_m)$$

is a closed geodesic of positive length in $\mathcal{Q}$. Note that $\tilde{c}'$ is obtained as the composition of the orbifold paths $\tilde{c} * [g_m]$, where $[g_m]$ denotes the equivalence class of the $\mathcal{H}$-path of length zero $(1_{\tilde{c}^m_{k_m}(b)}, \tilde{c}^m_{k_m}(b), g_m)$.

If $\overline{S}$ has two ends, then there exist elements of order two g_0 and g_m in $\mathcal{H}$ fixing $\tilde{c}^0_1(a)$ and $\tilde{c}^m_{k_m}(b)$ respectively, and such that the differential of g_0 maps $\dot{\tilde{c}}^0_1(a)$ to $-\dot{\tilde{c}}^0_1(a)$ and the differential of g_m maps $\dot{\tilde{c}}^m_{k_m}(b)$ to $-\dot{\tilde{c}}^m_{k_m}(b)$. Let $\tilde{c}''$ be the orbifold path obtained as the composition

$$\tilde{c} * [g_m] * \tilde{c}^{-1} * [g_0],$$

where $[g_0]$ and $[g_m]$ denote the equivalence classes of the $\mathcal{H}$-paths $(1_{\tilde{c}^0_1(a)}, \tilde{c}^0_1(a), g_0)$ and $(1_{\tilde{c}^m_{k_m}(b)}, \tilde{c}^m_{k_m}(b), g_m)$, respectively; and $\tilde{c}^{-1}$ denotes the inverse path of $\tilde{c}$. Then $\tilde{c}''$ is a closed orbifold geodesic whose length is twice the length of $\tilde{c}$. $\square$

Example 3.12. Suppose the singular locus of a 3-orbifold $\mathcal{Q}$ is as in Figure 3.2, where the set Σ_1 of points of singular dimension 1 consists of three connected components S, S' and S'', each of which is homeomorphic to the open interval $(0, 1)$; and the set Σ_0 of points of singular dimension zero consists of two points x and y.

Assume that a fundamental neighbourhood U_x at x is uniformized by an orbifold chart $(\widetilde{U}, \Gamma_x, \varphi_x)$, where $\widetilde{U}$ is a ball in $\mathbb{R}^3$ centred at the origin, and Γ_x is the subgroup of $SO(3)$ generated by a rotation γ of angle $2\pi/3$ around the axis $0x_3$ and a rotation δ of angle π around the axis $0x_1$. We can easily see that the isotropy group Γ_x has the presentation $\langle \gamma, \delta | \gamma^3 = \delta^2 = \gamma\delta\gamma^{-1}\delta = 1 \rangle$

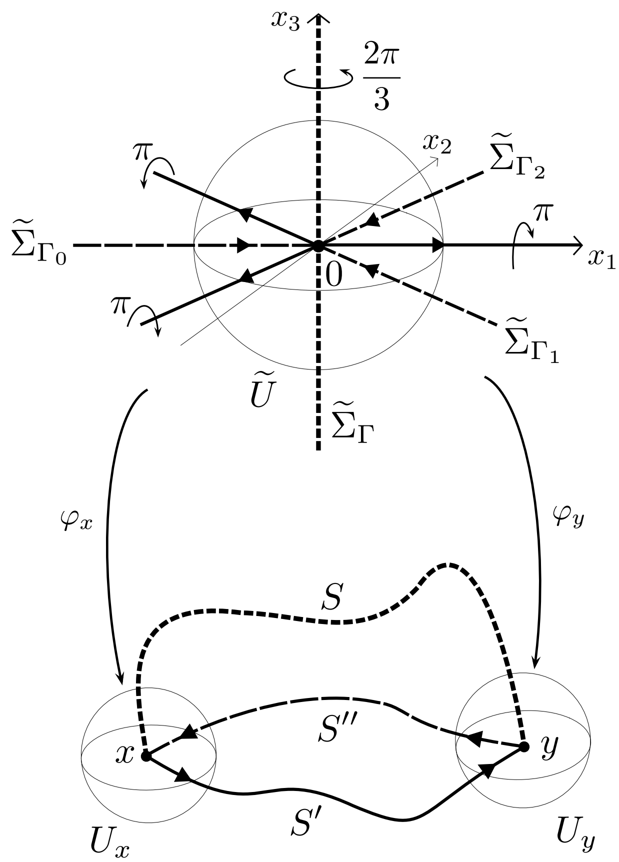


Figure 3.2: Closed geodesics contained in $\mathrm{cl}(\Sigma_1)$.

and thus it is isomorphic to D_3, the dihedral group with six elements. The group Γ_x has four proper subgroups: one subgroup $\Gamma = \langle\gamma\rangle$ of order 3, and three subgroups of order two $\Gamma_0 = \langle\delta\rangle$, $\Gamma_1 = \langle\gamma\delta\rangle$ and $\Gamma_2 = \langle\gamma^{-1}\delta\rangle$, which are conjugate to each other: $\Gamma_0 = \gamma\Gamma_1\gamma^{-1} = \gamma^{-1}\Gamma_2\gamma$.

If $\widetilde{\Sigma}_\Gamma$ denotes the one dimensional subspace fixed by Γ, then the action by each of the elements of order two in Γ_x leaves the subspace $\widetilde{\Sigma}_\Gamma$ invariant and maps any vector $v \in \widetilde{\Sigma}_\Gamma$ into $-v$. Thus $\widetilde{\Sigma}_\Gamma \cap \widetilde{U}_x$ projects via φ_x to

$\{x\} \cup (S \cap U_x)$ and we can see that x is an end of the component S. Let now $\widetilde{\Sigma}_{\Gamma_i}$ be the one dimensional vector subspace fixed by Γ_i, for $i = 0, 1, 2$. Note that $\widetilde{\Sigma}_{\Gamma_1} = \gamma \widetilde{\Sigma}_{\Gamma_0}$ and $\widetilde{\Sigma}_{\Gamma_2} = \gamma^{-1} \widetilde{\Sigma}_{\Gamma_0}$. Since $\varphi_x(\widetilde{\Sigma}_{\Gamma_0} \cap \widetilde{U}_x) = \varphi_x(\widetilde{\Sigma}_{\Gamma_1} \cap \widetilde{U}_x) = \varphi_x(\widetilde{\Sigma}_{\Gamma_2} \cap \widetilde{U}_x) = ((S' \cup S'') \cap U_x) \cup \{x\}$, the point $x \in \Sigma_0$ is not an end of either S' or S'', i.e. the component S' extends S"$'$ through x and vice-versa.

Assume that a fundamental neighbourhood U_y at y is also modeled on an orbifold chart as above. Then just as before y is an end of the component S of Σ_1 and the components S' and S'' are each-others extensions through y.

The closure $\mathrm{cl}(S) = S \cup \{x, y\}$ has the structure of a compact one dimensional orbifold with orbifold points x and y with isotropy $\mathbb{Z}_2$. To obtain a closed geodesic contained in $\mathrm{cl}(S)$ consider a constant speed parametrization of $\mathrm{cl}(S)$ that starts at x goes along S toward y, gets reflected at y by the element of order two in its isometry group, travels then along S in the reversed orientation towards x where again gets reflected by the element of order two in the isotropy group at x.

We can distinguish another closed geodesic in the closure $\mathrm{cl}(\Sigma_1)$. The union $S' \cup S'' \cup \{x, y\}$ is a totally geodesic embedded circle. Any constant speed parametrization of a path going around $S' \cup S'' \cup \{x, y\}$ gives a closed geodesic of positive length in $\mathcal{Q}$.

Example 3.13. In Figure 3.3 assume that x is a singular point in a 3-orbifold as in the example 3.12, and that the isotropy group at y is the Klein group of four elements, $\mathbb{Z}_2 \times \mathbb{Z}_2$, which acts on $\mathbb{R}^3$ by rotations of angle π around

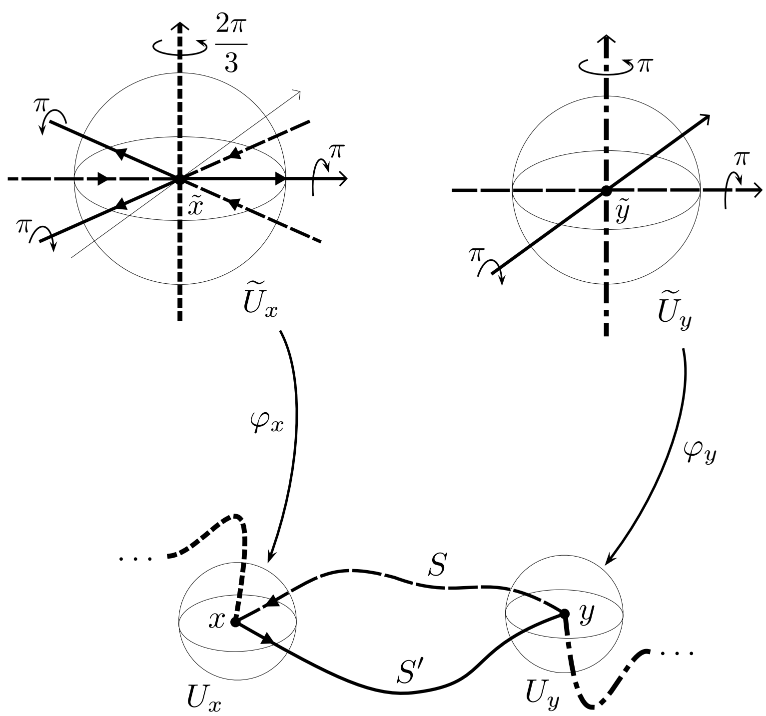


Figure 3.3: Examples of closed geodesics contained in $\mathrm{cl}(\Sigma_1)$.

the three orthogonal axes. As in the previous example, $x \in \mathrm{fr}(S) \cap \mathrm{fr}(S')$ is not an end for either S or S' and the components S and S' extend each other through x. However, this time the point y is an end of both S and S'. The union $S \cup S' \cup \{x, y\}$ has the structure of a 1-orbifold with underlying space homeomorphic to $[0, 1]$, and gives rise to a closed geodesic of positive length.

It is clear by now that in general, the singular locus of an orbifold need not have the structure of an orbifold. However, as we have already seen in Proposition 3.11, it is possible to put an orbifold structure on the union of some of the components in the singular locus. In the next proposition we see that this is always the case when we consider the components of minimal singular dimension.

Proposition 3.14. *Let $\mathcal{Q}$ be a compact effective n-orbifold and suppose $k \geq 2$ is the minimal positive dimension such that $\Sigma_k \neq \varnothing$. Then the closure* $\mathrm{cl}(S)$ *of each of the connected component $S \subseteq \Sigma_k$ has a natural structure of a compact orbifold $\mathcal{S}$ such that the associated effectve orbifold $\mathcal{S}_{\mathrm{eff}}$ has only zero-dimensional singular locus or is a smooth manifold. If $\mathcal{Q}$ is a Riemannian orbifold, then $\mathcal{S}$ is totally geodesic in $\mathcal{Q}$.*

Proof. Note that if $k = n$ is the minimal positive dimension such that $\Sigma_k \neq \varnothing$, then the conclusion follows trivially with $S = Q_{reg}$ and $\mathcal{S} = \mathcal{Q}$.

Assume that $2 \leq k \leq n-1$ and let $S \subseteq \Sigma_k$ be a connected component of singular dimension k. If $S = \mathrm{cl}(S)$ is closed, then by Proposition 3.4, S has the structure of a k-dimensional manifold, which is compact since $\mathcal{Q}$ is compact. If $\mathcal{Q}$ is a Riemannian orbifold, then S is totally geodesic in $\mathcal{Q}$.

Assume now that $S \neq \mathrm{cl}(S)$ is not closed. Then any point $x \in \mathrm{fr}(S)$ in the frontier of S belongs to a singular stratum of dimension strictly less than k. In our case, since k is the smallest positive singular dimension, any such point x would belong to Σ_0. Since $\mathcal{Q}$ is compact, the set Σ_0 of singular points of zero singular dimension is finite. Thus $\mathrm{fr}(S) = \mathrm{cl}(S) \smallsetminus S \subseteq \Sigma_0$ is finite.

Fix $x \in \mathrm{fr}(S)$ and let $(\widetilde{U}_x, \Gamma_x, \varphi_x)$ be a fundamental orbifold chart at x. Let $\tilde{x} = \varphi_x^{-1}(x) \in \widetilde{U}_x$ be the preimage of x by φ_x and U_x be the image $\varphi(\widetilde{U}_x)$. Denote by $S_x^\circ = S \cap U_x$ and let $S_x = S_x^\circ \cup \{x\}$. Since $k \geq 2$, S_x° is connected

(S_x° is homeomorphic to a punctured k-disk). For any $y \in S_x^\circ$ the isotropy group Γ_y is isomorphic to a proper subgroup of Γ_x (see Remark 2.9).

Choose $\tilde{y} \in \widetilde{U}_x$ such that $\varphi_x(\tilde{y}) = y$. Denote by $\widetilde{S}_{\tilde{y}}^\circ \subset \widetilde{U}_x$ the connected component of $\varphi_x^{-1}(S_x^\circ)$ containing $\tilde{y}$ and let $\Gamma_{\tilde{y}} \leq \Gamma_x$ be the isotropy group of $\tilde{y}$. Of course $\Gamma_{\tilde{y}} \cong \Gamma_y$. Let $\widetilde{S}_{\tilde{y}} = \widetilde{S}_{\tilde{y}}^\circ \cup \{\tilde{x}\}$. Clearly $\widetilde{S}_{\tilde{y}} = \widetilde{U}_x \cap \widetilde{\Sigma}_{\Gamma_{\tilde{y}}}$ has the structure of a k-dimensional totally geodesic submanifold of $\widetilde{U}_x$. Denote by $\Gamma'_{\tilde{y}}$ the maximal subgroup of Γ_x that leaves $\widetilde{S}_{\tilde{y}}$ invariant. The restriction of the action of $\Gamma'_{\tilde{y}}$ to $\widetilde{S}_{\tilde{y}}$ is $\Gamma_{\tilde{y}}$ and hence $\Gamma_{\tilde{y}} \trianglelefteq \Gamma'_{\tilde{y}} \leq \Gamma_x$. Consequently, the restriction of φ_x to $\widetilde{S}_{\tilde{y}}$ gives a continuous surjective map $\varphi'_x : \widetilde{S}_{\tilde{y}} \to S_x$ which in turn induces a homeomorphism between $\widetilde{S}_{\tilde{y}}/\Gamma'_{\tilde{y}}$ and the open set S_x.

Thus, the point $x \in \mathrm{cl}(S)$ is an orbifold point with isotropy group $\Gamma'_x \cong \Gamma'_{\tilde{y}}$ and fundamental uniformizing chart $(\widetilde{S}_{\tilde{y}}, \Gamma'_x, \varphi'_x)$ over the open set $S_x \subset \mathrm{cl}(S)$.

Note that the construction of the orbifold chart at x over S_x is independent of the choice of the lift of y in $\widetilde{U}$. If $\tilde{y}' \in \varphi_x^{-1}(y)$ is a different such choice, then $\tilde{y}' = \gamma\tilde{y}$ for some $\gamma \in \Gamma_x \smallsetminus \Gamma_{\tilde{y}}$. In the same way as before we obtain an orbifold chart $(\widetilde{S}_{\tilde{y}'}, \Gamma''_x, \varphi''_x)$ over S_x, where $\Gamma''_x \cong \Gamma'_{\tilde{y}'}$ and $\varphi''_x = (\varphi_x)|_{\widetilde{S}_{\tilde{y}'}} : \widetilde{S}_{\tilde{y}'} \to S_x$. Since $\Gamma'_{\tilde{y}'} = \gamma\Gamma'_{\tilde{y}}\gamma^{-1}$ and $\widetilde{S}_{\tilde{y}'} = \gamma\widetilde{S}_{\tilde{y}}$, we can see that γ induces an isomorphism between the charts $(\widetilde{S}_{\tilde{y}}, \Gamma'_x, \varphi'_x)$ and $(\widetilde{S}_{\tilde{y}'}, \Gamma''_x, \varphi''_x)$ at $x \in \mathrm{cl}(S)$.

We can proceed as in the proof of Proposition 3.4 to obtain an orbifold structure $\mathcal{S}$ on $\mathrm{cl}(S)$, by augmenting the atlas defining the manifold structure on S with the orbifold charts at points in the frontier $\mathrm{fr}(S)$. Moreover, from the construction of the charts it follows that if $\mathcal{Q}$ is a Riemannian orbifold, then $\mathcal{S}$ is totally geodesic in $\mathcal{Q}$.

It is clear that if $k < n$ then the orbifold $\mathcal{S}$ is not effective. However, since S is connected, by Remark 3.5(b), there is a finite group Γ_S such that the isotropy group Γ_y of each point $y \in S$ is isomorphic to Γ_S. If $x \in \mathrm{fr}(S)$, then Γ_S is isomorphic with a normal subgroup of the isotropy group Γ'_x of x in $\mathcal{S}$ as above. Therefore, in the associated effective orbifold $\mathcal{S}_{\text{eff}}$ of $\mathcal{S}$, we have that $S \subseteq (\mathcal{S}_{\text{eff}})_{reg}$ and the singular locus of $\mathcal{S}_{\text{eff}}$, $\Sigma_{\mathcal{S}} \subseteq \mathrm{fr}(S) \subseteq \Sigma_0$. Note that if $\Gamma'_x \cong \Gamma_S$ then x is actually a regular point in the effective orbifold $\mathcal{S}_{\text{eff}}$. $\square$

Proposition 3.14 holds for complete orbifolds. In that case, the orbifold $\mathcal{S}$ given in the proposition is not necessarily compact, but a complete orbifold.

Remark 3.15. An important consequence of Proposition 3.14 is that we can reduce the problem of existence of closed geodesics of positive length on compact orbifolds to the case of orbifolds with only zero dimensional singular locus. Note also that throughout this section the orbifold $\mathcal{Q}$ has not been assumed to be developable.

Chapter 4

Geodesics on Developable Orbifolds

This chapter presents geometric proofs of the existence of closed geodesics of positive length for a large class of compact developable orbifolds. The first proves existence whenever the orbifold fundamental group is finite or contains a hyperbolic isometry. The second, applying the techniques developed in the previous section for general compact orbifolds (see Remark 3.15), reduces the problem to even-dimensional developable compact orbifolds with only finitely many orbifold points and with orbifold fundamental group infinite torsion of odd exponent. Further existence results are obtained under the assumption that the orbifold admits a metric satisfying various curvature conditions, and these are established by showing that an infinite torsion group cannot act properly and cocompactly by elliptic isometries on a complete simply connected Riemannian manifold whose sectional curvature is everywhere nonpositive (or nonnegative).

4.1 The Setup

Let $\mathcal{Q}$ be an effective n-dimensional compact connected developable Riemannian orbifold (with a fixed Riemannian structure of class at least C^2).

We write $\mathcal{Q}$ as the orbifold quotient M/Γ, where M denotes the universal covering of $\mathcal{Q}$ and $\Gamma = \pi_1^{orb}(\mathcal{Q})$ is the orbifold fundamental group. Thus M is a connected, simply connected complete Riemannian manifold (with the natural Riemannian structure pulled back from $\mathcal{Q}$) and Γ is a discrete subgroup of the group $\mathrm{Isom}(M)$ acting properly and cocompactly by isometries on M. In short, we will say that Γ acts *geometrically* on M. We denote by Q the underlying topological space of $\mathcal{Q}$ and let $\pi : M \to Q$ be the natural projection map.

Our requirement on the differentiability of the Riemannian metric on M allows us to identify the Riemannian isometries of M with the metric isometries of (M, d), where d denotes the induced length metric on M. Since Γ acts geometrically on M, by Proposition 1.5 (ii), every element of Γ is a semi-simple isometry of M (see also section 1.1). We distinguish two classes of elements in Γ: the *elliptic* elements, which are the isometries with nonempty fix point set in M, and the *hyperbolic* elements which are the semi-simple isometries that act on M without fixed points.

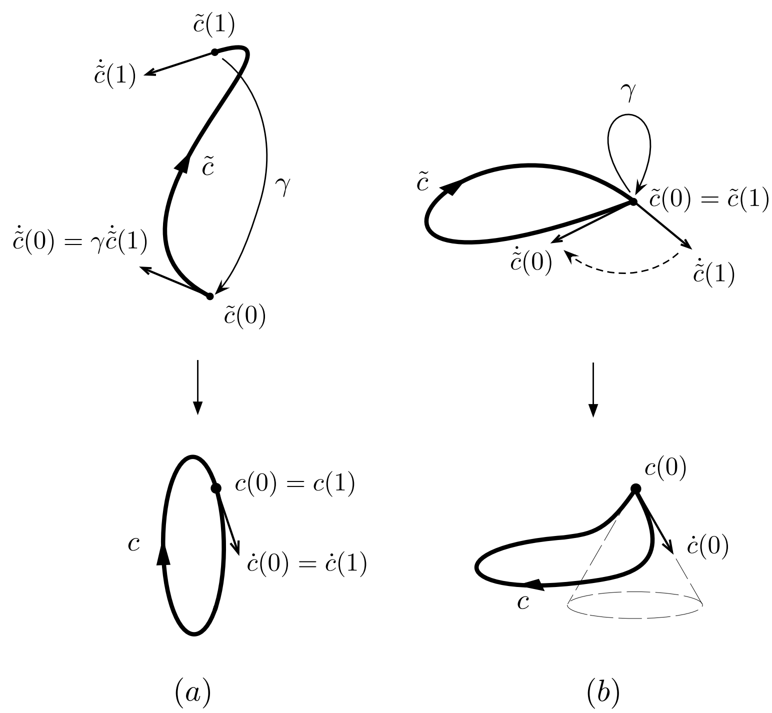


Figure 4.1: Examples of closed geodesics in $\mathcal{Q} = M/\Gamma$.

As in Definition 2.30, the closed geodesics of positive length on the developable orbifold $\mathcal{Q}$ are in one-to-one correspondence with the equivalence classes of pairs $(\tilde{c}, \gamma)$, where $\tilde{c} : [0, 1] \to M$ is a non-constant geodesic segment in M and $\gamma \in \Gamma$ is an isometry of M such that:

$$\gamma\tilde{c}(1) = \tilde{c}(0) \text{ and } \gamma\dot{\tilde{c}}(1) = \dot{\tilde{c}}(0)$$

(see Figure 4.1). Two pairs $(\tilde{c}, \gamma)$ and $(\tilde{c}', \gamma')$ are equivalent if and only if there is an element $\delta \in \Gamma$ such that $\tilde{c}' = \delta.\tilde{c}$ and $\gamma' = \delta\gamma\delta^{-1}$.

4.2 Existence Results I

In this section we present an elementary proof of an existence result for closed geodesics on developable orbifolds that implies part (b) of the following theorem of Guruprasad and Haefliger in [33].

Theorem 4.1 ([33, Theorem 5.1.1]). *Let $\mathcal{Q}$ be a compact connected Riemannian orbifold. There exists at least one closed geodesic on $\mathcal{Q}$ of positive length in the following cases:*

(a) *$\mathcal{Q}$ is not developable,*

(b) *$\mathcal{Q}$ is developable and the fundamental group of $\mathcal{Q}$ has an element of infinite order or is finite.*

We begin by noticing that it follows easily from the definition of closed geodesics on developable orbifolds that any nontrivial closed geodesic $\tilde{c}$ in the universal covering M gives rise to a closed geodesic of positive length on $\mathcal{Q}$ represented by the pair $(\tilde{c}, 1)$, where 1 denotes the identity of Γ. Fet's result [44] on the existence of closed geodesics on compact manifolds can be used to show the existence of a closed geodesic of positive length in the case when the universal cover M of $\mathcal{Q}$ is compact. This is precisely the case when the orbifold fundamental group Γ is finite.

For compact developable orbifolds $\mathcal{Q}$ with infinite fundamental group, the result of Fet can still be successfully employed whenever the orbifold $\mathcal{Q}$ has a compact intermediate manifold cover M'. Using the one-to-one correspondence between the covers of $\mathcal{Q}$ and the subgroups of the fundamental group (see section 2.5, page 55), the existence of an intermediate compact manifold cover for $\mathcal{Q}$ is equivalent to the existence of a finite index subgroup $\Gamma' \leq \Gamma$ which acts freely on the universal covering M. Indeed, in this case since Γ' is of finite index in Γ and acts freely on M, the quotient $M' = M/\Gamma'$ is a compact manifold. Since $\pi_1(M') \cong \Gamma'$, there exists a nontrivial closed geodesic on M' in the free homotopy class of each nontrivial element of $\pi_1(M')$. Thus if $\gamma \in \Gamma'$ is not the identity and $\tilde{c}' : [0,1] \to M'$ is a closed geodesic in $[\gamma]$, then for any $x \in M$ with $\pi(x) = \pi'(\tilde{c}'(0))$, the unique lift $\tilde{c} : [0,1] \to M$ at x is a geodesic and the pair $(\tilde{c}, \gamma)$ represents a closed geodesic in $\mathcal{Q}$.

Remark 4.2. This happens, for instance, whenever Γ is a virtually torsion free group (i.e. has a torsion free finite index subgroup), for any semi-simple isometry which has infinite order is hyperbolic. According to a result by Malćev [46] and Selberg [62] and known as " the Selberg lemma", any finitely generated matrix group with entries in a field is virtually torsion free, and thus it is either finite or contains an element of infinite order.

As noted in [60] the orbifold fundamental group of any compact developable 2-dimensional orbifold can be realized as a finitely presented subgroup of $PSL(2, \mathbb{R})$ and therefore is virtually torsion free. Thus any compact connected developable 2-orbifold is finitely covered by a manifold [60, Theorem 2.5]. This argument shows that *all compact developable* 2*-dimensional orb-*

ifolds admit at least one closed geodesic of positive length.

In general, for Γ infinite, one cannot expect the existence of closed geodesics on M (as M is not compact) or that of a compact intermediate manifold cover (which at the level of the orbifold fundamental group is equivalent to the presence of a virtually torsion free subgroup). However, as we will next show, the existence of closed geodesics of positive length in $\mathcal{Q}$ follows whenever the orbifold $\mathcal{Q}$ admits a nontrivial intermediate manifold cover M' which is not necessarily compact. This is precisely the case when the orbifold fundamental group Γ of $\mathcal{Q}$ contains an element γ which acts without fixed points on M. In this case the intermediate manifold cover M' is the quotient M/Γ' of the universal cover M by the free action of the subgroup $\Gamma' \leq \Gamma$ generated by γ.

The idea of the proof for this case follows that of [8, Lemma 6.5]. Let $\gamma \in \Gamma$ be a hyperbolic isometry of M. By definition (see section 1.1) the minimal set $\mathrm{Min}(\gamma)$ is non-empty. Let $x \in \mathrm{Min}(\gamma)$. Since $\mathrm{Min}(\gamma)$ is γ-invariant (Proposition 1.5), the translate γx of x belongs to $\mathrm{Min}(\gamma)$. Let now $\tilde{c} : [0, 1] \to M$ be a minimizing geodesic in M connecting x to γx (such geodesic exists since M is complete) and denote by $y \in M$ the midpoint of $\tilde{c}$ (i.e. $y = \tilde{c}(1/2)$). Then the translate $\gamma\tilde{c}$ is a minimizing geodesic connecting γx to $\gamma^2 x$, and γy is the midpoint of $\gamma\tilde{c}$ (see Figure 4.2 below).

From the triangle inequality we have that

$$0 < d(y, \gamma y) \leq d(y, \gamma x) + d(\gamma x, \gamma y) = d(x, \gamma x) = |\gamma|,$$

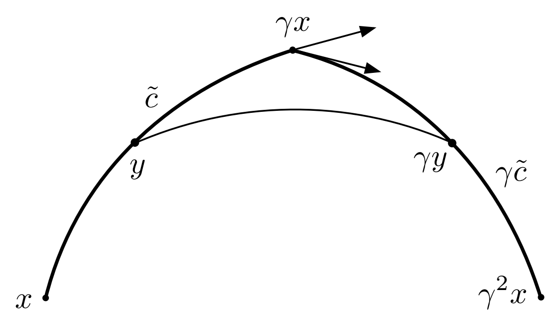


Figure 4.2: Midpoint argument.

which implies that $y \in \mathrm{Min}(\gamma)$ or equivalently that $d(y, \gamma y) = |\gamma|$. Since the distance between y and γy measured along $\tilde{c}$ and then $\gamma\tilde{c}$ is also $|\gamma|$, it follows that the concatenation of the two geodesics is a smooth geodesic, i.e. $\gamma.\dot{\tilde{c}}(0) = \dot{\tilde{c}}(1)$. Thus the pair $(\tilde{c}, \gamma)$ represents a closed geodesic of positive length in $\mathcal{Q}$.

We have thus proved the following existence result:

Theorem 4.3. *A developable compact connected Riemannian orbifold $\mathcal{Q}$ has a closed geodesic of positive length if the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ is finite or if it contains a hyperbolic element.*

We would like to mention that this result is more general than the one in [33] in the following sense. As we have seen earlier, if a semi-simple isometry has infinite order then it is hyperbolic. In general, the converse is not necessarily true, as it is possible for hyperbolic isometries to have finite order. Therefore, by our theorem the existence of closed geodesics of positive length follows whenever Γ has an element that acts without fixed point, which can be of finite order. Examples of complete Riemannian manifolds with hyperbolic isometries of finite order can be found in section 4.4.

Remark 4.4. It is easy to see that the translate $\gamma\tilde{c}$ of a geodesic $\tilde{c} : [0,1] \to M$ by an isometry $\gamma \in \Gamma$, is again a geodesic. Given a pair $(\tilde{c}, \gamma)$ representing

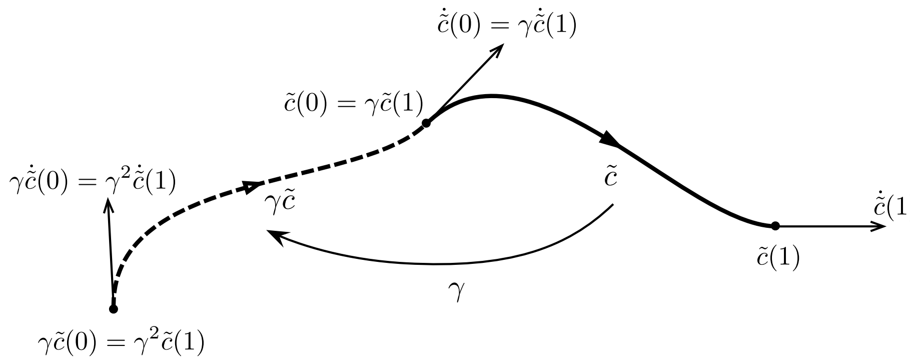
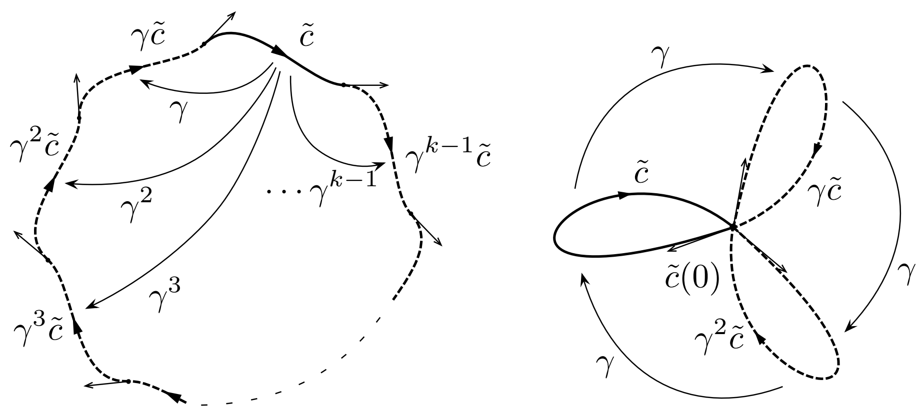


Figure 4.3: Collinear geodesics.

a closed geodesic in $\mathcal{Q}$, the condition that $\gamma\tilde{c}(1) = \tilde{c}(0)$ implies that in M the two geodesic segments $\tilde{c}$ and $\gamma\tilde{c}$ have the point $\tilde{c}(0)$ in common; and the condition that $\gamma\dot{\tilde{c}}(1) = \dot{\tilde{c}}(0)$ implies that the union of the two geodesic segments is smooth at this point.

If $(\tilde{c}, \gamma)$ represents a closed geodesic with γ of finite order, say $\gamma^k = 1$, then the path $\tilde{c}^{(k-1)} := \tilde{c} * \gamma\tilde{c} * \ldots * \gamma^{k-1}\tilde{c}$, obtained by successively concatenating the translates of $\tilde{c}$ by γ, is a smooth closed geodesic in M (see Figure 4.4).

The only situation not covered by Theorem 4.3 is when Γ is infinite and each of its elements is an elliptic isometry. Since elliptic isometries have finite order, Γ is an *infinite torsion group*. Moreover, since the action is



(a) (b)

Figure 4.4: Closed geodesic in the universal cover.

cocompact, by Proposition 1.5, Γ is finitely presented and also has finitely many conjugacy classes of isotropy groups.

Remark 4.5. An important consequence of the fact that each of the elements of Γ is elliptic, is that Γ has to have finite exponent. This follows from the fact that each of the elements of Γ is in one of the isotropy groups, and these groups are finite. Then the least common multiple of the orders of the isotropy groups (which clearly are preserved under conjugation) gives an upper bound for the exponent of Γ. Note that one cannot deduce that Γ has finite exponent from the results of [33].

While examples of infinite torsion groups that are finitely generated and even of finite exponent are known to exist, there are no examples known to be finitely presentable (as also noted in [33, Remark 5.1.2]). The existence problem for closed geodesics of positive length on compact orbifolds is therefore intimately related (but not equivalent) to the following question:

Question 4.6. *Can an infinite torsion group* Γ *act properly and cocompactly by elliptic isometries on a complete simply connected Riemannian manifold* M*?*

Clearly, a negative answer to this question would imply the existence of closed geodesics on all compact orbifolds. On the other hand, if such actions were to exist, then by Remark 4.4, the existence of a closed geodesic on M/Γ would be equivalent to the existence of a closed smooth geodesic in M. However, there are many examples of complete simply connected non-compact manifolds that are uniquely geodesic and thus have no closed geodesics (e.g. Euclidean space, simply connected manifolds of nonpositive curvature, or more generally any simply connected Riemannian manifold without conjugate points). An interesting problem is then, whether any of these spaces can admit geometric actions as in Question 4.6, for an affirmative answer would give rise to a compact orbifold with no closed geodesics of positive length. We will return to this problem in section 4.4.

We would like to mention, however, that if such an action were to exist, then the quotient space would be a compact orbifold $\mathcal{Q} = M/\Gamma$ with $\pi_1^{orb}(M/\Gamma) \cong \Gamma$ nontrivial, but whose underlying topological space Q has trivial $\pi_1(Q)$ (see Proposition 1.3). In particular, if M were contractible, and every simply connected manifold with the unique geodesic property is contractible, then $\mathcal{Q}$ would be an aspherical orbifold, namely a developable orbifold with contractible universal cover (cf. [19, Lecture 5]).

As a special case of Question 4.6 one could ask whether there exist compact aspherical orbifolds with infinite torsion fundamental group. As noted before, the orbifold universal cover of an orbifold $\mathcal{Q}$ need not be a covering for the underlying topological space Q. In particular, the underlying topological space of an aspherical orbifold need not be aspherical (or contractible). For example the pillowcase orbifold in Example 2.35 is aspherical but has underlying topological space homeomorphic to the two sphere. An interesting related problem is to construct an aspherical orbifold $\mathcal{Q}$ whose underlying topological space Q is contractible, or to show that such examples do not exist.

4.3 Existence Results II

Throughout this section $\mathcal{Q}$ will denote an effective compact connected developable Riemannian n-orbifold with universal cover M and orbifold fundamental group $\pi_1^{orb}(\mathcal{Q}) = \Gamma$. As noted in section 3.1, the orbifold structure on $\mathcal{Q}$ induces a natural stratification $Q = \bigsqcup_{k=0}^{n} \Sigma_k$, where each Σ_k inherits naturally the structure of a k-dimensional manifold without boundary. Since $\mathcal{Q}$ is developable, each of the connected components of Σ_k is covered by k-dimensional submanifolds of M whose connected components are totally geodesic in M (see also Proposition 1.12).

In the following theorem we compile a series of conditions on the structure of the singular locus in a developable orbifold $\mathcal{Q}$ that give the existence of at least one closed geodesic of positive length on $\mathcal{Q}$. Many of these conditions have already been used in Chapter 3 for general compact orbifolds.

Theorem 4.7. *Suppose $\mathcal{Q}$ is a compact connected effective Riemannian developable n-orbifold, and let $Q = \bigsqcup_{k=0}^{n} \Sigma_k$ be the natural stratification by singular dimension induced by the orbifold structure. There exists at least one closed geodesic of positive length on $\mathcal{Q}$ in any of the following situations:*

(i) *there exists $0 < k \leq n$ such that Σ_k has a compact connected component;*

(ii) $\Sigma_0 = \varnothing$;

(iii) *n is odd and $\Sigma_k = \varnothing$ for all $0 < k < n$;*

(iv) *$\Sigma_{n-1} \neq \varnothing$ and $\Sigma_k = \varnothing$ for all $0 < k < n-1$;*

(v) *the smallest positive singular dimension ℓ such that $\Sigma_\ell \neq \varnothing$ equals 2 or is an odd number;*

(vi) *$\mathcal{Q}$ has only zero-dimensional singular locus and $\pi_1^{orb}(\mathcal{Q})$ contains an element of order two.*

Remark 4.8. Note that if $\Sigma_0 = \varnothing$, then the set Σ_ℓ in part (v) of the above theorem has the structure of an ℓ-manifold (possibly disconnected). Otherwise, it is an ℓ-orbifold with only zero-dimensional singular locus.

Proof. (i) This is Proposition 3.9. Note that $k = n$ if and only if $\mathcal{Q}$ is a manifold and the existence of closed geodesics follows from the classical result of Lyusternik and Fet [44]. Assume $k < n$ and let S be a compact connected component contained in Σ_k. We will next exhibit an equivalence class of pairs $(\tilde{c}, \gamma)$ representing a closed geodesic in $\mathcal{Q}$.

Assume first that S is not simply connected and let $c : [0, 1] \to S$ be a closed geodesic in the free homotopy class of a nontrivial element $\delta \in \pi_1(S)$. Let $\gamma \in \Gamma$ be the image of δ by $i_* : \pi_1(S) \to \Gamma$. If γ is trivial then any lift $\tilde{c}$ of c in M is closed, and since it is contained in $\pi^{-1}(S)$ whose components are totally geodesic, it is also a geodesic. Thus $\tilde{c}$ is a closed geodesic in M and therefore $(\tilde{c}, 1)$ is a closed geodesic in $\mathcal{Q}$. If γ is nontrivial then $\gamma\tilde{c}(0) = \tilde{c}(1)$ for some lift $\tilde{c} : [0, 1] \to M$ of c. As before, $\tilde{c}$ is a geodesic and since it projects to a smooth closed curve in $\mathcal{Q}$, we have $\gamma\dot{\tilde{c}}(0) = \dot{\tilde{c}}(1)$. Thus the pair $(\tilde{c}, \gamma)$ represents a closed geodesic in $\mathcal{Q}$.

If S is simply connected, then each of the connected components of $\pi^{-1}(S) \subset M$ is diffeomorphic to S and has the structure of a closed (compact without boundary) totally geodesic submanifold of M. Any closed geodesic c in $\pi^{-1}(S)$ (or equivalently in S) is again a closed geodesic in M which projects to a closed geodesic in $\mathcal{Q}$ in the equivalence class of $(c, 1)$.

(ii) This is Corollary 3.10 and also a direct consequence of (i) above, since in this case, the components of smallest singular dimension in $\mathcal{Q}$ are closed and thus have the structure of a compact manifold.

(iii) If $\Sigma_0 = \varnothing$, then $\mathcal{Q}$ is a compact manifold and therefore contains closed geodesics of positive length. Otherwise $\mathcal{Q}$ is an odd-dimensional compact orbifold with only zero-dimensional singular locus. Using the fact that at each singular point $x \in \Sigma_0$, the isotropy group Γ_x acts orthogonally and freely on the unit sphere in the tangent space $T_{\tilde{x}}M$, together with the assumption that $\mathcal{Q}$ is odd-dimensional (i.e. the unit tangent sphere is even-dimensional), it follows that all isotropy groups Γ_x are cyclic of order two. If we further assume that the orbifold fundamental group Γ contains only elliptic elements (by Theorem 4.3), then it has exponent two (cf. Remark 4.5). In particular Γ is abelian and since it is finitely generated it must be finite. The existence of closed geodesics follows then from Theorem 4.3.

(iv) If $\Sigma_0 = \varnothing$ then each connected component of Σ_{n-1} has the structure of a compact manifold and therefore admits a closed geodesic of positive length. Assume then that $\Sigma_0 \neq \varnothing$. We will show that the connected components of Σ_{n-1} are closed.

Assume to the contrary that there exists a component $S \subseteq \Sigma_{n-1}$ which is not closed, and let $x \in \mathrm{fr}(S) \subseteq \Sigma_0$. Let $\tilde{x} \in \pi^{-1}(x)$ be a lift of x in M and denote by $\widetilde{S} \subset M$ the closure in M of a lift of S such that $\tilde{x} \in \widetilde{S}$. Let Γ_x denote the isotropy group at x and let Γ'_x be the maximal subgroup of Γ_x that fixes $\widetilde{S}$. Then Γ'_x is a cyclic group generated by an element γ of order two which is given by the reflection in the hyperplane $T_{\tilde{x}}\widetilde{S} \subset T_{\tilde{x}}M$ (see Proposition 3.4 and Remark 3.5). More precisely, γ satisfies $\gamma w = w$ for all

$w \in T_{\tilde{x}}\widetilde{S}$ and $\gamma v = -v$ for $v \in (T_{\tilde{x}}\widetilde{S})^{\perp}$. Note that the subgroup Γ'_x is normal in Γ_x. Otherwise if $\delta \in \Gamma_x \smallsetminus \Gamma'_x$ is such that $\delta^{-1}\Gamma'_x\delta \neq \Gamma'_x$, then the fixed point set $\widetilde{S}'$ of $\delta^{-1}\Gamma'_x\delta$ has codimension one and the intersection $\widetilde{S}' \cap \widetilde{S}$ gives rise to a component in Σ_{n-2} and contradicts the assumption that $\Sigma_k = \varnothing$ for all $0 < k < n-1$.

Let $\delta \in \Gamma_x \smallsetminus \Gamma'_x$. Since δ leaves invariant the subspace $T_{\tilde{x}}\widetilde{S} = (T_{\tilde{x}}M)^{\gamma}$, the fixed point set of the restriction of δ to $T_{\tilde{x}}\widetilde{S}$ is a local model for the fixed point set in M of the subgroup generated by δ and γ. The latter set necessarily sits in Σ_k with $k < n-1$, and by hypothesis, it follows that the restriction of δ to $T_{\tilde{x}}\widetilde{S}$ has the zero vector as fixed point set. Note now that if $v \in (T_{\tilde{x}}\widetilde{S})^{\perp}$, then δ must map v to $-v$, since otherwise δ would fix $(T_{\tilde{x}}\widetilde{S})^{\perp}$ and then the fixed point set in M of the subgroup of Γ_x generated by δ would have dimension one and contradict the assumption that $\Sigma_1 = \varnothing$. This also shows that δ has order two and since it has zero-dimensional fixed point set, δ is the inversion in the origin in $T_{\tilde{x}}M$. But then $\delta\gamma v = v$ if $v \in (T_{\tilde{x}}\widetilde{S})^{\perp}$ and $\delta\gamma w = -w$ for $w \in T_{\tilde{x}}\widetilde{S}$, which shows that the subgroup generated by $\delta\gamma$ has one-dimensional fixed point set, contradicting again the assumption that $\Sigma_1 = \varnothing$.

This shows that $\Gamma_x = \Gamma'_x$ and thus $\mathrm{fr}(S) = \varnothing$. The conclusion then follows from the part (i) of the theorem.

(v) The case $\ell = 2$ follows from Remark 4.2 together with Proposition 3.14. For $\ell > 1$ odd, the conclusion follows from (iii) and Remark 4.8 above. The case when $\ell = 1$ is Proposition 3.11. We include the proof for this latter case.

Let $q \in \Sigma_1$ and fix $\tilde{q} \in \pi^{-1}(q)$. Denote by Γ' the isotropy group at $\tilde{q}$ (that is, the image of Γ_q into Γ by the natural homomorphism). The fixed point set of Γ' is a closed totally geodesic embedded 1-dimensional submanifold N of M containing $\tilde{q}$. Then N is either homeomorphic to the circle or to the real line, and any path with constant speed parametrization whose image is N, is a geodesic in M. If N is homeomorphic to the circle, then it is a closed geodesic; and if it is homeomorphic to the real line, it is a geodesic line in M. Assume the latter and let $r : \mathbb{R} \to M$ be a constant speed parametrization it. If the projection of r onto $\mathcal{Q}$ is entirely contained in Σ_1, then its image is a compact component of Σ_1 and as above it is a closed geodesic in $\mathcal{Q}$.

Assume now that the projection of r contains points of zero singular dimension. There are two possible situations that can happen at a point p in the closure of a component S of Σ_1: either S 'terminates' at p or it can be extended beyond p to another component of Σ_1. In the former case we will refer to p as of the end of S.

To see this, let S and p as above and let $(\widetilde{U}, \Gamma_p, \varphi_p)$ be a fundamental orbifold chart at p. We can shrink U if necessarily so that the restriction of the exponential map at $\tilde{p} \in \pi^{-1}(p) \cap \widetilde{U}$ to a ball centred at $0 \in T_{\tilde{p}}\widetilde{U}$ is a diffeomorphism onto its image. Moreover we can assume that $\tilde{q} \in \widetilde{U}$. Then Γ' is isomorphic to a proper subgroup of $\Gamma_{\tilde{p}}$. For simplicity assume $\Gamma' \leq \Gamma_{\tilde{p}}$. Let $v \in T_{\tilde{p}}\widetilde{U}$ be a nonzero vector such that $\exp_{\tilde{p}}(v) = \tilde{q}$. Clearly v spans the one-dimensional subspace fixed by Γ'. The two possible situations are as follows. If there exists $\gamma \in \Gamma_{\tilde{p}} \setminus \Gamma'$ such that $\gamma v = -v$ (in particular this implies that γ has order two), then the projection of $\exp_{\tilde{p}}(\epsilon v)$ for $\epsilon \in [-1, 1]$ is contained in S, i.e. S 'terminates' at p. If $\gamma v \neq -v$ for all $\gamma \in \Gamma_{\tilde{p}} \setminus \Gamma'$, then the point $\tilde{q}' = \exp_{\tilde{p}}(-v)$ is fixed by Γ' and projects to a point $q' \in Q \setminus S$. Then there exists a unique extension of S beyond p to a component of Σ_1 that contains q'; namely the projection of $\exp_{\tilde{p}}(\epsilon v)$ for $\epsilon \in [-1, 0)$.

Using the fact that each Σ_k has only finitely many components the projection of N onto Q has either 0 or 2 ends.

Assume first that $\pi(N)$ has no ends, and let $q \in \pi(N) \cap \Sigma_1$. Choose two points $\tilde{q}_1 = r(t_1)$ and $\tilde{q}_2 = r(t_2)$ in $\pi^{-1}(q)$ such that $t_1 < t_2$. Since $\tilde{q}_1, \tilde{q}_2 \in N$ they are both fixed by Γ', and since $\tilde{q}_1$ and $\tilde{q}_2$ are in the same orbit, there exists $\delta \in \Gamma$ such that $\delta\tilde{q}_1 = \tilde{q}_2$. Moreover, $\delta\Gamma'\delta^{-1} = \Gamma'$ and thus $\delta r \subset N$. Since $\delta : T_{\tilde{q}_1}N \to T_{\tilde{q}_2}N$, there are two possibilities: either $\delta\dot{r}(t_1) = \dot{r}(t_2)$ or $\delta\dot{r}(t_1) = -\dot{r}(t_2)$. In the first case $(r|_{[t_1,t_2]}, \delta)$ is a closed geodesic in Q. In the second case, since $\delta r \subset N$, δ fixes the midpoint of $\widetilde{m} = r([t_1, t_2])$, has order two and the projection $m = \pi(\widetilde{m}) \in Q$ is an end point. This contradicts the assumption that the projection of N has no ends.

If $\pi(N)$ has two ends, say p_1 and p_2 in Σ_0 (not necessarily distinct), then we can use two reflecting elements in the isotropy groups of two lifts of p_1 and p_2 to construct a closed geodesic in $\mathcal{Q}$.

(vi) Again by Theorem 4.3, we could further assume that the orbifold fundamental group Γ is infinite and contains only elliptic isometries. Since Γ has an element of order two, Γ is torsion and of finite even exponent. Let $\gamma \in \Gamma$ be an element of order two. Then there exists $\delta \in \Gamma$ such that $\delta x \neq x$ (otherwise Γ would be finite) and δ has odd order (otherwise Γ would have exponent two and therefore would be abelian, thus finite). Assume that $x \in M$ is fixed by γ. As before consider two cases: one when γ fixes δx (i.e. $[\gamma, \delta]x = x$, or even stronger condition when γ and δ commute) and the case when γ does not fix δx (i.e. $[\gamma, \delta]x \neq x$, or more generally when γ and δ do not commute).

In the first case, consider any geodesic segment $c : [0, 1] \to M$ connecting x to δx and its translate $\gamma.c$ by γ. The concatenation of the two geodesic segments, one of which is to be considered with the reversed orientation is a smooth closed geodesic in M, that projects to a closed geodesic of positive length in $\mathcal{Q}$.

In the second case let $c : [0, 1] \to M$ be a geodesic segment connecting x to δx and let $c' : [0, 1] \to M$ be a reparametrization of the concatenation $\gamma c^- * c$. Then c' is a smooth geodesic segment connecting $\gamma\delta x$ to δx and passing through x; and which has as initial vector $\dot{c}'(0) = -\gamma\dot{c}(1)$ and $\dot{c}'(1) = \dot{c}(1)$. Note that the commutator $[\delta^{-1}, \gamma]$ takes $\gamma\delta x$ to $[\delta^{-1}, \gamma]\gamma\delta x = \delta\gamma\delta^{-1}\gamma\gamma\delta x = \delta x$ and $\dot{c}'(0)$ to

$$[\delta^{-1}, \gamma].\dot{c}'(0) = [\delta^{-1}, \gamma].(-\gamma\dot{c}(1)) = -\delta\gamma\delta^{-1}\gamma\gamma\dot{c}(1) = -\delta\gamma\delta^{-1}\dot{c}(1) = \dot{c}(1)$$

since $\delta\gamma\delta^{-1} \in \Gamma_{\delta x}$ and has order two. Thus, the pair $(c', [\delta^{-1}, \gamma])$ represents a closed geodesic of positive length. $\square$

Remark 4.9. (a) In the proof of (v) and (vi) above we used that an orthogonal involution that does not fix a nontrivial vector subspace is an inversion in the origin, i.e. it maps $v \mapsto -v$ for any v.
(b) The construction in the proof of (vi) above gives a closed geodesic in any compact developable orbifold $\mathcal{Q}$ having an isolated singular point whose isotropy group has even order. In particular, any odd-dimensional compact developable orbifold with at least one isolated singular point admits a closed geodesic of positive length.

As a direct application of the Theorem 4.7 we obtain the existence of closed geodesics on compact developable orbifolds in dimension 3, 5 and 7. Together with the result of [33] for non-developable orbifolds, this implies:

Corollary 4.10. *If $\mathcal{Q}$ is a compact orbifold with* $\dim(\mathcal{Q})$ *equal to 3, 5 or 7, then $\mathcal{Q}$ admits a closed geodesic of positive length.*

Proof. By Lyusternik and Fet [44], we can assume that the singular locus is nonempty, so $\Sigma \neq \varnothing$, and by part (a) of Theorem 4.1 of Guruprasad and Haefliger [33, Theorem 5.1.1], we can assume that $\mathcal{Q}$ is developable. Let $Q = \bigsqcup_{k=0}^{n} \Sigma_k$ be the natural stratification by singular dimension. By part (ii) of Theorem 4.7, we can assume that $\Sigma_0 \neq \varnothing$.

Suppose firstly that $\dim(\mathcal{Q}) = 3$. By part (v) of Theorem 4.7, we can assume that $\Sigma_\ell = \varnothing$ for $\ell = 1, 2$, and now the conclusion follows from part (iii) of Theorem 4.7.

Now suppose $\dim(\mathcal{Q}) = 5$. Applying part (v) of Theorem 4.7, we can assume that $\Sigma_\ell = \varnothing$ for $\ell = 1, 2, 3$ and the conclusion follows from part (iv) of the Theorem 4.7 if $\Sigma_4 \neq \varnothing$ and from part (iii) otherwise.

Lastly, suppose $\dim(\mathcal{Q}) = 7$. Let $\ell > 0$ denote the smallest positive singular dimension such that $\Sigma_\ell \neq \varnothing$. We first explain how to deduce the conclusion from Theorem 4.7 when $\ell \neq 4$. For example, if $\ell = 7$ then $\Sigma = \Sigma_0$ and we apply part (iii). If $\ell = 6$, we apply part (iv). Otherwise, if $\ell = 1, 2, 3$ or 5, we apply part (v).

The only remaining case is therefore when $\ell = 4$, and the proof is similar to that of part (iv) in Theorem 4.7. We will show that if $\Sigma_k = \varnothing$ for $0 < k < 4$ and $\Sigma_4 \neq \varnothing$, then each connected component of Σ_4 is closed. Then, the conclusion follows from part (i) of Theorem 4.7.

Note first that the condition $\Sigma_k = \varnothing$ for $k = 1, 2, 3$ together with the Remark 3.8 implies that the points in the frontier of the components in Σ_4 have singular dimension zero. In particular, if Σ_0 is empty, then each connected component of Σ_4 is closed. Assume then that $\Sigma_0 \neq \varnothing$ and let S be a connected component in Σ_4. Assume further that S is not closed and let $x \in \mathrm{fr}(S) \subseteq \Sigma_0$. Let Γ_x denote the isotropy group at x and let Γ'_x be the maximal subgroup of Γ_x that fixes $T_{\tilde{x}}\widetilde{S}$, where $\widetilde{S} \subset M$ denotes the closure in M of a lift of S (see Remark 3.5). The subgroup Γ'_x is normal in Γ_x, for otherwise, the fixed point set of the conjugate $\delta^{-1}\Gamma'_x\delta$, with $\delta \in \Gamma_x \smallsetminus \Gamma'_x$, is again a 4-dimensional submanifold $\widetilde{S}' \subset M$ and the intersection $\widetilde{S} \cap \widetilde{S}'$ gives rise to a component of singular dimension at least one, which contradicts the assumption that $\Sigma_k = \varnothing$ for all $0 < k < 4$. Note that the orthogonal complement $(T_{\tilde{x}}\widetilde{S})^{\perp}$ in $T_{\tilde{x}}M$ is invariant under Γ'_x, and in fact Γ'_x restricted to $(T_{\tilde{x}}\widetilde{S})^{\perp}$ has isolated fixed point at zero. In particular, Γ'_x induces a free orthogonal action on the unit sphere in $(T_{\tilde{x}}\widetilde{S})^{\perp}$, and thus Γ'_x is the cyclic group of order two, whose generator γ satisfies $\gamma w = w$ if $w \in T_{\tilde{x}}\widetilde{S}$ and $\gamma v = -v$ if $v \in (T_{\tilde{x}}\widetilde{S})^{\perp}$.

Note now that while all the elements in $\Gamma_x \setminus \Gamma'_x$ leave $T_{\tilde{x}}\widetilde{S}$ invariant, there has to be at least one element that does not fix $T_{\tilde{x}}\widetilde{S}$. Otherwise the fixed point set of Γ_x would have dimension 4 and would therefore contradict our assumption that $x \in \mathrm{fr}(S) \subset \Sigma_0$. Let $\delta \in \Gamma_x \setminus \Gamma'_x$ be an element that does not fix $T_{\tilde{x}}\widetilde{S}$ and let $k < 4$ be the dimension of the fixed point set of δ in $T_{\tilde{x}}\widetilde{S}$. This implies that the fixed point set in M of subgroup of Γ_x generated by δ and γ has dimension equal to k, and since there are no components in $\mathcal{Q}$ of singular dimension 1, 2 or 3, we conclude that k has to equal zero. This shows that the restriction of δ to $T_{\tilde{x}}\widetilde{S}$ has only the zero vector as fixed point, and we will see that this is actually true for the restriction of δ to the orthogonal complement $(T_{\tilde{x}}\widetilde{S})^{\perp}$ in $T_{\tilde{x}}M$ as well. Indeed, if δ had nontrivial fixed point set in $(T_{\tilde{x}}\widetilde{S})^{\perp}$, then the fixed point set of the the group generated by δ would have dimension k for some $0 < k \leq 3 = \dim(T_{\tilde{x}}\widetilde{S})^{\perp}$, and this would contradict the assumption that $\Sigma_k = \varnothing$ for such k. Since $\dim(M) = 7$, we see that δ must be an inversion in the origin, i.e. that $\delta v = -v$ for $v \in T_{\tilde{x}}M$, and it follows that $\delta\gamma$ satisfies $\delta\gamma w = -w$ for $w \in T_{\tilde{x}}\widetilde{S}$ and $\delta\gamma v = v$ for $v \in (T_{\tilde{x}}\widetilde{S})^{\perp}$. But this would imply that the fixed point set of the subgroup generated by $\delta\gamma$

has dimension 3, in contradiction to our assumption that $\Sigma_3 = \varnothing$.

This shows that if $S \subset \Sigma_4$, then $\mathrm{fr}(S) = \varnothing$. Hence S has the structure of a compact manifold and the conclusion then follows from part (i) of 4.7. $\square$

Remark 4.11. The simplest example not covered by the above theorem is that of a connected compact developable 4-dimensional orbifold Q with zero-dimensional singular locus, and $\Gamma \cong \pi_1^{orb}(\mathcal{Q})$ is an infinite torsion group of odd exponent whose action on the universal cover M satisfies the property that each element $\gamma \in \Gamma$ has nonempty fixed point set. Note that in this case, each isotropy group admits a free action on the three sphere and thus it must be cyclic (see also Remark 5.16).

4.4 Geometric Conditions

In this section we continue to denote by $\mathcal{Q}$ a compact connected Riemannian developable orbifold, obtained as the quotient M/Γ of a simply connected manifold M by the geometric action of a discrete group $\Gamma \subset \mathrm{Isom}(M)$.

As noted at the beginning of section 4.2, any closed geodesic of positive length in M projects to a closed geodesic of positive length in the quotient $\mathcal{Q}$. Thus the existence of closed geodesics on compact developable orbifolds can be reduced to the case when M has no closed geodesics. On the other hand, as we have seen in Remark 4.4, any closed geodesic of positive length $(\tilde{c}, \gamma)$ in $\mathcal{Q} = M/\Gamma$ for which $\gamma \in \Gamma$ has finite order gives rise to a closed geodesic in M.

A particularly interesting situation is when the orbifold fundamental group Γ is infinite torsion and the universal covering M is a manifold without closed geodesics. For such manifolds M, the existence of closed geodesics on any compact orbifold quotient of M would follow if one could show that infinite torsion subgroups of $\mathrm{Isom}(M)$ cannot act properly and cocompactly on M (see Question 4.6). Clearly, if a discrete infinite torsion group Γ acts on such a manifold geometrically, then the orbifold quotient $\mathcal{Q} = M/\Gamma$ does not contain a closed geodesic of positive length.

Note that in dimension two it follows directly from the Selberg lemma that infinite torsion groups cannot act properly and cocompactly by isometries on simply connected manifolds without closed geodesics (see Remark 4.2). In dimensions 3, 5 and 7, the same conclusion follows from Corollary 4.10. Thus the following partial result holds:

Proposition 4.12. *Suppose M is a simply connected complete Riemannian manifold without closed geodesics and let $\Gamma \subset \mathrm{Isom}(M)$ be a discrete group of isometries of M acting properly and cocompactly on M. If the dimension of M is 2, 3, 5 or 7, then Γ contains at least one element of infinite order.*

If one believes that compact orbifolds are similar to manifolds, then the existence of closed geodesics would suggest that infinite torsion groups do not act properly and cocompactly by isometries on simply connected manifolds without closed geodesics in all the dimensions. The purpose of this section is to study certain similar situations and to show that such actions cannot exist under the assumption of certain curvature conditions.

There are many examples of complete, connected, simply connected Riemannian non-compact manifolds that do not admit closed geodesics. Examples include manifolds with positive curvature and manifolds with the unique geodesic property (e.g. Hadamard manifolds, the Euclidean space, manifolds without conjugate points). In general, there are no topological restrictions (like on homotopy or homology groups) that are independent of the dimension of the manifold and that can be forced upon a complete manifold to obtain the existence of closed geodesics with respect to all Riemannian metrics.

For instance, given any (non-compact) complete Riemannian manifold N, the product $M = \mathbb{R} \times N$ with the (complete) metric

$$\langle X, Y \rangle = xy + e^r \langle X^*, Y^* \rangle^*,$$

where $X = (x, X^*)$ and $Y = (y, Y^*)$ are in $T_{(r,p)}(\mathbb{R} \times N)$, and $\langle \cdot, \cdot \rangle^*$ denotes the metric on N, has no closed geodesics of positive length. This holds regardless of whether the factor N has closed geodesics or not.

However, not all simply connected manifolds can be realized as the universal cover of a compact orbifold. In other words not all simply connected complete Riemannian manifolds admit geometric actions by discrete (infinite) groups of isometries. In particular, product manifolds as above cannot be the universal cover of a compact orbifold.

To see this, assume $\Gamma \subset \mathrm{Isom}(M)$ acts geometrically on $M = N \times \mathbb{R}$

and let $K \subset M$ be a fundamental domain for the action. Since Γ acts cocompactly, the set K is compact and of course $M = \Gamma.K$. Then any point $x \in M$ has a neighbourhood $U_x \in M$ which is isometric (via an element $\gamma \in \Gamma$) with a neighbourhood of the point $\gamma^{-1}x \in K$. If M has the above metric, we can see that two points (r_1, p) and (r_2, q) cannot be in the same orbit of an isometry, provided $r_1 \neq r_2$.

This shows in particular that the universal cover of a compact connected orbifold has in some sense bounded and uniform geometry.

Constant Sign Curvature

Of some interest are manifolds with sectional curvature κ of constant sign. Given a Riemannian orbifold $\mathcal{Q}$ and a point $x \in Q$, we define the sectional curvature κ_x at x to be the sectional curvature $\kappa_{\tilde{x}}$ at one of its lifts $\tilde{x}$ in a orbifold chart at x. Furthermore, we say that the orbifold $\mathcal{Q}$ is of positive (resp. nonnegative, zero, negative, nonpositive) sectional curvature if the sectional curvature κ_x at any point $x \in Q$ has the appropriate sign.

Proposition 4.13. *(a) If $\mathcal{Q}$ is a compact connected Riemannian developable orbifold of positive sectional curvature, then the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ cannot be an infinite torsion group.*

(b) If $\mathcal{Q}$ is a compact connected Riemannian orbifold of negative sectional curvature, then the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ cannot be an infinite torsion group.

Corollary 4.14. *If $\mathcal{Q}$ is a compact connected developable orbifold with Riemannian metric of positive or negative sectional curvature, then $\mathcal{Q}$ admits a closed geodesic of positive length.*

Positive curvature. In the case when $\mathcal{Q}$ has positive sectional curvature, its universal cover M is a complete connected and simply connected manifold of positive sectional curvature. The classical Bonnet–Meyer theorem states that if the curvature of M is bounded from below by a positive constant $\varepsilon > 0$, then M is compact (see [52] for the stronger form involving the Ricci curvature). In this case the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ is necessarily finite. In particular M and therefore $\mathcal{Q}$ admit closed geodesics of positive length.

If the curvature is not bounded away from zero, M needs no longer be compact. However, in this case, by a well known theorem of Gromoll and Meyer, the full group $\mathrm{Isom}(M)$ of isometries of M is compact (see [30, Theorem 3]). Any discrete group that acts by isometries on M is finite and the action is necessarily non-free. The latter claim follows from the fact any complete open manifold of positive curvature is contractible (see [30, Theorem 2]). Thus any quotient by a group of isometries of a complete non-compact manifold of positive curvature is a non-compact orbifold (which is not a manifold, i.e. has nonempty singular locus).

In conclusion, if a compact orbifold has positive sectional curvature, then either it is not developable or if developable then its universal covering is a compact manifold of positive curvature. In either case such an orbifold admits closed geodesics of positive length. This follows in the developable case by using the result of Lyusternik and Fet [44] for the compact universal cover and in the non-developable one by using part (a) of Theorem 4.1. □

Negative curvature. Orbifolds of negative sectional curvature are developable and their universal cover is diffeomorphic to $\mathbb{R}^n$, where n is the dimension of the orbifold. Although the fundamental group on such orbifolds is always infinite, they cannot be torsion (see for instance [22, Theorem 3.4.1 and Example 2.5.12]). □

Proposition 4.15. (a) *If $\mathcal{Q}$ is a compact connected Riemannian orbifold of nonpositive sectional curvature, then the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ cannot be an infinite torsion group.*

(b) *If $\mathcal{Q}$ is a compact connected developable Riemannian orbifold of nonnegative sectional curvature, then the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ cannot be an infinite torsion group.*

Corollary 4.16. *If $\mathcal{Q}$ is a compact connected developable orbifold with Riemannian metric of nonpositive or nonnegative sectional curvature, then $\mathcal{Q}$ admits a closed geodesic of positive length.*

Nonpositive curvature. It is well known that all orbifolds of nonpositive curvature are developable [11]. If $\mathcal{Q}$ is such an orbifold, then its universal cover M is a Hadamard manifold. That is, a simply connected complete manifold of nonpositive curvature.

If M is a Hadamard manifold, then M with the length metric induced by the Riemannian structure is a Hadamard space. In fact, a Riemannian manifold is a Hadamard manifold if and only if the associated metric space is a Hadamard space. It was shown by Swenson in [64] that infinite torsion groups cannot act geometrically on Hadamard spaces. $\square$

Remark 4.17. If $\mathcal{Q}$ has zero curvature, i.e. M is the Euclidean space $\mathbb{R}^n$, then the fact that Γ cannot be infinite torsion follows directly from a celebrated theorem of Bieberbach [10] (see also [14]). According to this theorem any discrete group of isometries of the Euclidean space $\mathbb{R}^n$ with compact fundamental domain is virtually abelian. Thus, if Γ is the orbifold fundamental group of a compact flat orbifold then Γ is virtually abelian. Since Γ is also finitely generated (finitely presented), it cannot be torsion.

Nonnegative curvature. Assume now that $\mathcal{Q}$ is a compact developable orbifold of nonnegative sectional curvature. As before, its universal cover M is a complete simply connected manifold of nonnegative curvature. Unlike the positive curvature case, if M is non-compact, then its isometry group $\mathrm{Isom}(M)$ needs not be compact and there are compact manifolds of nonnegative curvature whose universal cover is not compact. Thus, if $\mathcal{Q}$ is a compact developable orbifold of nonnegative curvature, then it is possible for its universal cover to be non-compact.

A key result concerning the manifolds of nonnegative curvature is the Toponogov Splitting Theorem [67], that states that any complete manifold M

of nonnegative sectional curvature may be written uniquely as the isometric product $\overline{M} \times \mathbb{R}^k$, where $\mathbb{R}^k$ has the standard flat metric and $\overline{M}$ has nonnegative sectional curvature and contains no line (that is, a normal geodesic $\tilde{c} : (-\infty, \infty) \to M$, any segment of which is a minimal geodesic). Furthermore, Cheeger and Gromoll showed in [16] that if the isometry group of a manifold M of nonnegative sectional curvature is not compact, then M contains at least a line. In consequence, any complete manifold M of nonnegative curvature admits a unique isometric splitting $M = \overline{M} \times \mathbb{R}^k$ such that the isometry group of $\overline{M}$ is compact, and $\mathrm{Isom}(M) = \mathrm{Isom}(\overline{M}) \times \mathrm{Isom}(\mathbb{R}^k)$. Note that cf. [15] the same results hold in the more general case of manifolds with nonnegative Ricci curvature.

If Γ is a discrete group acting geometrically on M, then $\Gamma = \Gamma' \times \Gamma''$, where Γ' and Γ'' are discrete subgroups of $\mathrm{Isom}(\overline{M})$ and $\mathrm{Isom}(\mathbb{R}^k)$, respectively. Since $\mathrm{Isom}(\overline{M})$ is compact, the factor Γ' is necessarily finite. Note that if $\overline{M}$ is not compact then Γ cannot act cocompactly on M. On the other hand, the group Γ'' acts geometrically on the Euclidean factor $\mathbb{R}^n$ and therefore Γ'' has elements of infinite order (see Remark 4.17). This implies that Γ contains elements of infinite order, and so it cannot be torsion.

In conclusion, if $\mathcal{Q}$ is a compact developable orbifold of nonnegative sectional curvature, then its orbifold fundamental group is either finite or contains elements of infinite order. By Theorem 4.3, $\mathcal{Q}$ admits closed geodesics of positive length. $\square$

Chapter 5

Infinite Torsion Groups

As we have seen in the previous chapter, the problem of existence of closed geodesics of positive length on compact developable orbifolds reduces to a particular class of orbifolds that satisfy a number of restrictions regarding both their geometry and topology. Perhaps the biggest challenge in completely solving this problem is the lack of examples of orbifolds that satisfy these conditions and, as formulated in Question 4.6, especially those conditions regarding the orbifold fundamental group.

In this section, we deal exclusively with compact developable orbifolds $\mathcal{Q}$ for which the existence of closed geodesics is not covered by the results in the previous chapter, specifically Theorems 4.3 and 4.7, and we investigate the various group-theoretic properties that one can assume about the orbifold fundamental group $\Gamma = \pi_1^{orb}(\mathcal{Q})$ of such orbifolds. Along the way, we present an idiosyncratic summary of the Burnside problem, including variants of the original problem, progress made, and open questions on infinite torsion groups related to the problem of existence of closed geodesics for compact orbifolds.

Restrictions on the Orbifold Fundamental Group

We first recall a fundamental property concerning the orbifold fundamental group of a general compact connected developable orbifold.

Property 5.1. *The orbifold fundamental group of a compact connected developable orbifold is finitely presented.*

Proof. Let $\mathcal{Q}$ be a compact connected developable orbifold, and let M be its universal cover. Then M is a simply connected manifold on which the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ acts properly and cocompactly. Let ρ be a Riemannian metric on the orbifold $\mathcal{Q}$ obtained as in Proposition 2.24. The pullback of ρ by the covering map gives a natural Riemannian metric $\tilde{\rho}$ on the universal cover M, and $\pi_1^{orb}(\mathcal{Q})$ acts as a group of isometries in M. Let d denote the distance on M induced by the Riemannian metric $\tilde{\rho}$. Since $\mathcal{Q}$ is compact, (M, d) is a complete metric space and by the Hopf-Rinow theorem any two points in M can be connected through a minimizing geodesic. Thus (M, d) is a geodesic space. Finally note that any Riemannian isometry of M is also a metric isometry of the space (M, d), and therefore, by Proposition 1.6, the group $\pi_1^{orb}(\mathcal{Q})$ is finitely presented. $\square$

Hereforth, justified by Theorem 4.3, we make the following assumptions on the orbifold $\mathcal{Q}$ and the orbifold fundamental group $\Gamma = \pi_1^{orb}(\mathcal{Q})$:

Assumption A. *$\mathcal{Q}$ is a compact Riemannian orbifold obtained as the quotient $\mathcal{Q} = M/\Gamma$, where M is a complete connected and simply connected Riemannian manifold and $\Gamma \subset \mathrm{Isom}(M)$ is an infinite discrete subgroup consisting entirely of elliptic isometries.*

The following group-theoretic properties of Γ are a direct consequence of Assumption A. To begin, we have:

Property 5.2. *Γ is an infinite torsion group.*

Combined with the fact that each element of Γ is an elliptic isometry of M, we deduce that:

Property 5.3. *Γ has only finitely many conjugacy classes of elements.*

To see this, recall first that a group that acts properly and cocompactly by isometries on a simply connected geodesic space has only finitely many conjugacy classes of isotropy groups (see Proposition 1.5). Denote by $\Gamma_1, \ldots, \Gamma_k$ the representatives up to conjugacy of the isotropy groups of $\mathcal{Q}$ in Γ (i.e. the isotropy types of $\mathcal{Q}$). Since each of the isotropy groups Γ_i is finite, the union $\Gamma_1 \cup \ldots \cup \Gamma_k$ is a finite set. Furthermore, since each element $\gamma \in \Gamma$ is elliptic, it is conjugate to an element in $\Gamma_1 \cup \ldots \cup \Gamma_k$. Thus Γ has only finitely many conjugacy classes of elements.

Recall that a torsion group G is said to have finite (or bounded) exponent, if there exists a positive integer n such that $g^n = 1$ for all $g \in G$, and in this case, the exponent of the group G is defined to be the smallest positive such n. Otherwise the group G is said to have unbounded exponent. Clearly, if a torsion group G has exponent n, then n is the least common multiple of the orders of all elements in G. In particular, if G is a finite group, then its exponent is a divisor of the order of G.

Using now Property 5.3 we can further conclude that:

Property 5.4. Γ *has finite exponent.*

Indeed, since each element of Γ has finite order and the order of an element is preserved under conjugation, the exponent of Γ is finite and equal to the least common multiple of the orders of all elements in $\Gamma_1 \cup \ldots \cup \Gamma_k$.

As it turns out, from the perspective of our work, Property 5.1 is the most stringent condition that Γ has to satisfy, in the sense that for all the latter properties (5.2, 5.3, 5.4) in this section, examples of finitely generated groups can be found, but none of them is known to be finitely presented.

The Burnside Problem and its Variants

The Burnside problem is an old and influential question in group theory and concerns finitely generated groups. The original problem dates back to the beginning of the twentieth century, when William Burnside [12] posed two questions, the first of which is known today as the *general Burnside problem* and asks:

Question 5.5. *Is every finitely generated torsion group finite?*

It is an easy exercise to see that any finitely generated *abelian* torsion group is finite. The quotient group $\mathbb{Q}/\mathbb{Z}$ shows the necessity of the condition that G be finitely generated, since it is an example of an infinite torsion group which is abelian but not finitely generated.

The general Burnside problem was answered in the negative in 1964 by Golod and Shafarevich [25], and Golod provided in [24] the first example of a finitely generated infinite torsion group. Golod showed that for each prime p there exists a residually finite infinite p-group that can be generated by three elements.

Later more examples of finitely generated infinite torsion groups with unbounded exponent were found by Aleshin [5], Grigorchuk [27], and Gupta and Sidki [32]. These groups, known as the *Grigorchuk type groups*, are examples of residually finite infinite p-groups generated by two elements of prime odd order p. They are realizable as subgroups of the automorphism group of a regular tree of degree p. Grigorchuk's group in [27] is a residually finite 2-group with three generators that can be presented as a group of automorphisms of an infinite binary rooted tree. Using these examples Rozhkov constructed in [55] a two generator torsion group that contains elements of order n for each natural number n.

Grigorchuk type groups play an important role in geometric group theory.

Recall that for a finitely generated group G, the *growth function* of G is defined to be the function $k \mapsto \beta_G(k)$, where $\beta_G(k)$ is the number of different elements of G that can be represented as a product of at most k of generators (we assume here that the generating set is symmetric, i.e. if g belongs to the generating set, then so does its inverse g^{-1}). The group G has *polynomial growth of degree* d if there exists a constant C such that $\beta_G(k) \leq Ck^d$ for all $k \geq 1$; and G has *exponential growth* if there exists $C > 0$ such that $\beta_G(k) \geq e^{Ck}$, for large $k > 1$. Finally, the group G has *intermediate growth* if the function $\beta_G(k)$ grows faster than any polynomial in k and slower than any exponential in k.

Note that every finitely generated group has at most exponential growth, and that every free group of finite rank ≥ 2 has exponential growth. Moreover, Wolf [70] showed that every finitely generated nilpotent group has polynomial growth. Milnor posed the following question in [49]:

Question 5.6. *Is there a finitely generated group of intermediate growth?*

An important contribution to this problem is the celebrated result of Gromov [31], which states that a finitely generated group has polynomial growth if and only if it is virtually nilpotent, that is it has a nilpotent finite index subgroup. But virtually nilpotent finitely generated torsion groups are finite. One easy way to see this is by using the fact that nilpotent groups are solvable, and finitely generated solvable torsion groups are indeed finite: the commutator subgroup $[G, G]$ of a finitely generated torsion group G is of finite index, as the quotient $G/[G,G]$ is finitely generated, abelian and torsion. Using then that the derived series of G is finite, it follows that G itself is finite. Thus, in light of Gromov's result and the fact Golod's group has exponential growth, it is natural to ask whether there exists a finitely generated infinite torsion group of intermediate growth. Grigorchuk showed in [28] and [29] that the group constructed in [27] has sub-exponential growth, providing thus the first example of a finitely generated group of intermediate growth.

None of these infinite torsion groups is finitely presented and so far there is no general idea how to prove or disprove that such groups exist. However, it is conjectured that there are no finitely presented groups of intermediate growth [26].

The second problem posed by William Burnside in [12] and known today as the *bounded Burnside problem* is the following:

Question 5.7. *Is every finitely generated group of finite exponent finite?*

As before, the condition for the group to be finitely generated is essential. The additive group $\mathbb{Z}_2[x]$ of all polynomials with $\mathbb{Z}_2$ coefficients is an example of an infinite group of exponent 2 which is not finitely generated (as it is abelian). In fact it is easy to see that any group of exponent 2 is abelian, thus all finitely generated groups of exponent 2 are finite.

Recall that the free Burnside group $B(m,n)$ of rank m and exponent n is the quotient F_m/F_m^n of the free group F_m on m generators by the subgroup F_m^n

generated by the n-th powers of all the elements in F_m. Thus, $B(m,n)$ can be regarded as the universal group G with m generators in which $g^n = 1$ holds for all $g \in G$, and consequently, any other group generated by m elements and of exponent n can be obtained as a factor of the group $B(m,n)$. With this notation, the bounded Burnside problem asks: *are the groups $B(m,n)$ finite for all pairs (m,n)?*

The initial results pointed toward a positive answer to the bounded Burnside problem. In his paper [12], Burnside showed that $B(1,n)$ is cyclic for all n and that the groups $B(m,2)$ and $B(m,3)$ are finite for all $m \geq 1$. Burnside also showed in [13] that finitely generated subgroups of $GL(k,\mathbb{C})$ of finite exponent are finite, a result which was soon generalized by Schur [59] by removing the finite exponent condition and showing that all finitely generated torsion subgroups of $GL(k,\mathbb{C})$ are finite. The groups $B(m,4)$ and $B(m,6)$ with $m \geq 1$ were also known to be finite by [56] and [35], respectively.

By far the most significant contribution to the bounded Burnside problem came in 1968 with the result of S.I. Adian and P.S. Novikov in [53], which states that the groups $B(m,n)$ are infinite for all $m > 1$ and all n odd, $n \geq 4381$; and thus providing a negative answer to the bounded Burnside problem. Later the bound on the exponent n was improved by Adian [3] to n odd, $n \geq 665$. Although the work of Adian and Novikov suggested that the groups $B(m,n)$ are infinite for sufficiently large exponent n regardless of its parity, it was not until 1992 that S. Ivanov [37] and I. Lysenok [43] (independently) announced that $B(m,n)$ are infinite for some large even exponents.

Adian provided a series of remarkable results concerning the (infinite) free Burnside groups $B(m, n)$ with $m > 1$ and n odd, $n \geq 665$. For instance all the finite subgroups and all the abelian subgroups of $B(m, n)$ are cyclic and their orders divide the exponent n. In particular, the groups $B(m, n)$ have trivial center. He also showed [2] that the infinite free Burnside group $B(m, n)$ contains infinite decreasing and infinite increasing chains of embedded subgroups, so it does not satisfy the minimal and maximal conditions.

The groups $B(m, n)$ contain infinite decreasing and infinite increasing chains of embedded normal subgroups if either $m > 1$ and $n = rs$, where $r \geq 665$ is odd and $s > 1$, or $m \geq 66$ and $n \geq 665$ is prime (see [4]). In particular, these free Burnside groups admit infinitely many non-isomorphic factor groups.

The infinite free Burnside groups $B(m, n)$ with $m > 1$ and odd $n \geq 665$ have exponential growth, and do not admit a finite presentation.

The question of existence of an infinite finitely presented group of bounded exponent is attributed to P.S. Novikov. To date there is no known finitely presented infinite torsion group (of bounded exponent or not). A possible way to obtain a finitely presented group is by using the Higman embedding theorem [36]. That is, a finitely generated group G embeds in a finitely presented group if and only if it is recursively presented. Since the free Burnside groups $B(m, n)$ with $m > 1$ and odd $n \geq 665$ are recursively presented, they embed in a finitely presented group. However, all the known constructions of such embeddings involve using HNN extensions and amalgamated products, and these methods do not provide torsion groups, as they require the introduction of elements of infinite order. It is also known that finitely generated infinite torsion groups satisfy Serre's (FA) property and therefore are both indecomposable (i.e. not a free product of two groups) and strongly indecomposable (i.e. not an amalgam) [63].

In 1950 a solution to the bounded Burnside problem was not yet known and Magnus [45] proposed another problem closely related to the Burnside problem, which he called the *restricted Burnside problem*:

Question 5.8. *Is there a maximal finite group with m generators and a given exponent n?*

This hypothetical group, denoted $R(m,n)$, is then a maximal finite quotient of the free Burnside group $B(m,n)$. That is, $R(m,n)$ is the quotient of $B(m,n)$ by a minimal normal subgroup N_0 of finite index in $B(m,n)$. Note that N_0 can be obtained as the intersection of all normal subgroups of finite index in $B(m,n)$. Note also that such minimal normal subgroup need not exist. For instance for a residually finite group, the intersection of all the normal subgroups of finite index is trivial. A reformulation of the restricted Burnside problem is whether the free Burnside group $B(m,n)$, for a fixed (m,n), has a minimal normal subgroup of finite index.

In [72] and [73], Zel′manov provides a positive solution to the restricted Burnside problem.

Further Restrictions on the Orbifold Fundamental Group

Using the results in Theorem 4.7 we see that the problem of existence of closed geodesics on compact developable orbifolds can be reduced to those orbifolds that in addition to the conditions stated in the Assumption A satisfy the following assumption:

Assumption B. *$\mathcal{Q}$ is a compact developable orbifold such that*

(i) the singular locus Σ has dimension zero;

(ii) the dimension $\dim(\mathcal{Q}) \geq 4$ and is even;

(iii) $\Gamma = \pi_1^{orb}(\mathcal{Q})$ contains no element of order two.

The condition (*iii*) in Assumption B above, together with Properties 5.2 and 5.4, imply that the group Γ is an infinite torsion group of odd exponent. Thus Γ is a factor of the free Burnside group $B(m, n)$ for some $m > 1$ and large odd $n \gg 1$. Note that since these infinite free Burnside groups are not finitely presentable, Γ is necessarily a proper quotient of $B(m, n)$. Using the positive solution to the restricted Burnside problem for odd exponent [72], we can further assume that:

Property 5.9. *Γ has no non-trivial finite-index subgroups.*

Proof. Suppose that Γ has m generators and exponent n, with n odd and sufficiently large so that Γ is infinite. Then Γ is isomorphic to the quotient $B(m, n)/\langle\langle R \rangle\rangle$ of the free Burnside group of rank m and exponent n by the subgroup $\langle\langle R\rangle\rangle$ normally generated by a set of relations R. Since the normal subgroups of Γ are in one-to-one correspondence with the normal subgroups of $B(m,n)$ containing $\langle\langle R\rangle\rangle$, there exits a minimal normal subgroup of finite index Γ_0 of Γ (which is unique up to isomorphism), corresponding to the minimal finite index normal subgroup N_0 of $B(m,n)$. Note that N_0 has to contain the subgroup $\langle\langle R\rangle\rangle$ in order for Γ to be infinite. Then Γ_0 is a finitely generated infinite torsion group of odd exponent which has no normal subgroups of finite index. (Indeed, if H would be a normal subgroup of Γ_0 of finite index, then H would also be a subgroup of finite index in Γ, although not necessarily normal. However, we can find a normal subgroup K of Γ of finite index which is contained in H, and use then the minimality of Γ_0.) Finally, note that any proper subgroup H of finite index in Γ always contains a normal subgroup N (the normal core of H) which is also normal and of finite index in Γ. Thus no normal subgroups of finite index implies no subgroups of finite index.

We will now see that this assumption on Γ is still relevant to our problem (that of describing the properties of the orbifold fundamental group of compact orbifolds for which the existence of closed geodesics is not known), in the sense that we can replace Γ with the subgroup Γ_0. First, using the one-to-one correspondence between the normal subgroups of $\Gamma = \pi_1^{orb}(\mathcal{Q})$ and the regular orbifold covers of $\mathcal{Q}$, we can associate a finite orbifold cover $\mathcal{Q}_0$ of $\mathcal{Q}$ to the subgroup Γ_0 of Γ. Thus $\mathcal{Q}_0$ is a compact developable orbifold and the existence of a closed geodesic in $\mathcal{Q}_0$ implies the existence of a closed geodesic in $\mathcal{Q}$ as well. Note now that a compact developable orbifold with fundamental group consisting only of elliptic elements has a closed geodesic of positive length if and only if the universal cover has a closed geodesic of positive length (see Remark 4.4 and the discussion following Question 4.6). Note also that this closed geodesic in the universal cover will push down as a closed geodesic to any quotient, compact or not. Thus we can reduce the existence of closed geodesics problem to those compact orbifolds whose orbifold fundamental group satisfies (in addition to all previously mentioned properties) the condition that it has no proper subgroup of finite index. $\square$

Using now Property 5.9 we can easily deduce that:

Property 5.10. Γ *is perfect and not residually finite.*

To see that Γ is perfect, let $\Gamma' = [\Gamma, \Gamma]$ denote the commutator subgroup of Γ. Since the factor group Γ/Γ' is finitely generated, abelian and torsion, it is finite. Hence Γ' is a normal subgroup of finite index in Γ. But Property 5.9 implies that $\Gamma' = \Gamma$, i.e. Γ is perfect.

Recall that a group is residually finite if and only if the intersection of all its subgroups of finite index is trivial. By Property 5.9, since Γ has no proper subgroup of finite index, this intersection is the whole group. Thus Γ is not residually finite. This also implies that Γ is not solvable.

Remark 5.11. It follows directly from the positive solution to the restricted Burnside problem that the free Burnside groups are not residually finite. The fact that finitely generated solvable groups of finite exponent are finite was first proved by P. Hall.

Define the FC-center of a group G to be the set

$$FC(G) = \{g \in G \mid [G : C(g)] < \infty\},$$

where $C(g)$ denotes the centralizer of g in G. Since for any element $g \in G$, the map $h \mapsto hgh^{-1}$ gives a bijective correspondence between the set of right cosets of $C(g)$ and the set of all conjugates of g, the set $FC(G)$ consists of all elements in G with finitely many conjugates. Note that $FC(G)$ is a normal subgroup of G that contains $Z(G)$, the center of G. A group with trivial FC-center is called an *icc*-group (infinite conjugacy classes).

Property 5.9 shows that Γ does not contain any proper subgroup of finite index, and this implies that $FC(\Gamma) = 1$. Thus Γ is an *icc*-group, and in particular Γ is centerless.

A *threading set* for a group G is defined to be a set $\{g_1, g_2, \ldots\} \subset G$ such that for any nontrivial element $g \in G$ there exists n such that g^n is conjugate to one of the elements g_i. We say that the group G has a *treading tuple* if G has a finite threading set $\{g_1, \ldots, g_k\}$. Furthermore, we say that a group G has a *strong threading tuple* if there exists a set $\{g_1, \ldots, g_k\} \subset G$ such that any nontrivial $g \in G$ is conjugate to one of the elements g_i. Note that, in general G having a threading tuple does not imply that G has a strong threading tuple. Note also that any group with finitely many conjugacy classes of elements has a strong threading tuple. In this case, a minimal strong threading tuple can be obtained by choosing one representative for each nontrivial conjugacy class.

It follows from Property 5.3 that the group Γ has a strong threading tuple. Together with the fact that Γ has trivial FC-center, this implies the following:

Property 5.12. Γ *has no finite normal subgroups and no infinite normal series.*

Proof. Let $\{\gamma_1, \dots, \gamma_k\}$ be a strong threading tuple for Γ. Without loss of generality we can assume that this set is minimal. For any normal subgroup $H \leq \Gamma$, we have $H \cap \{\gamma_1, \dots, \gamma_k\} \neq \varnothing$ and if H were finite, then any threading element contained in H would have finite conjugacy class. This contradicts the fact that Γ has trivial FC-center.

It is easy to see that Γ has only finitely many normal subgroups (up to isomorphism) since the normal subgroups of Γ can be indexed by their intersection with the threading set, which is finite. Note that a group cannot be the union of two normal subgroups whose intersection is the trivial group. Thus, if H is a proper normal subgroup of Γ such that $H \cap \{\gamma_1, \dots, \gamma_m\} = \{\gamma_1, \dots, \gamma_j\}$, then there is no normal subgroup whose intersection with the threading tuple is $\{\gamma_{j+1}, \dots, \gamma_m\}$. $\square$

Examples of finitely generated infinite torsion groups with finitely many conjugacy classes were constructed by Ol′shanskii in [54]. These so-called

Tarski monsters are infinite simple groups which are finitely generated (generated by any two non-commuting elements) and satisfy the property that every proper subgroup is a cyclic group of large prime order p. Ol′shanskii also showed that for each prime $p > 10^{75}$ there are continuum-many non-isomorphic Tarski p-groups. To date it is not known whether or not finitely presented Tarski monsters exist.

We will now see that the conditions in Assumption B allows us to determine a presentation for the isotropy groups of $\mathcal{Q}$. Let $N = \dim(\mathcal{Q})$.

Note first that since the orbifold $\mathcal{Q}$ is compact, the condition (iii) of Assumption B implies that the singular locus Σ consists of a finite collection of points, say $\Sigma = \{x_1, \ldots, x_l\}$. For each $i \in \{1, \ldots, l\}$, let Γ_{x_i} denote the isotropy group at $x_i \in \Sigma$. If $\tilde{x}_i \in M$ is a lift of x_i in M, then the isotropy group $\Gamma_{\tilde{x}_i} \leq \Gamma$ at $\tilde{x}_i$ is naturally isomorphic to Γ_{x_i}. The local action of $\Gamma_{\tilde{x}_i}$ at $\tilde{x}_i \in M$ induces a free orthogonal action on the unit sphere in the tangent space $T_{\tilde{x}_i}M$ at $\tilde{x}_i$. Since $N = \dim(M)$ is even, the isotropy group $\Gamma_{\tilde{x}_i}$ acts freely and orthogonally on the odd-dimensional sphere $\mathbb{S}^{N-1}$. Note now that that condition (i) implies that the isotropy group $\Gamma_{\tilde{x}_i}$ has odd order. Since by Feit-Thompson Theorem [23], any finite group of odd order is solvable, the group $\Gamma_{\tilde{x}_i}$ is solvable.

Thus each of the isotropy groups Γ_{x_i} of $\mathcal{Q}$ is solvable, has odd order and admits a free orthogonal action on $\mathbb{S}^{N-1}$.

The problem of classifying the finite groups that admit orthogonally free actions on spheres, known as the *Clifford-Klein spherical space problem*, has been settled mainly in the works of H. Zassenhaus [71] (1935), G. Vincent [68] (1947), and J.A. Wolf [69] (1972).

It is known (cf. [69, Theorem 5.3.1]) that every finite group G which acts orthogonally on a sphere without fixed points satisfies the so-called *pq-condition*. That is, for p and q primes, not necessarily distinct, any subgroup of G of order pq is cyclic. Zassenhaus showed that the converse of the above statement holds as well when restricting to the class of solvable groups: any finite solvable group that satisfies the pq-conditions acts orthogonally freely on some sphere.

Thus, if $\mathcal{Q}$ is an orbifold with only zero dimensional singular locus, then each of its isotropy groups satisfies the pq-condition.

Zassenhaus, Vincent and Wolf also gave a complete classification of the finite solvable groups that satisfy the pq-condition (see [69, Theorem 6.1.11]). Accordingly a finite group G of odd order satisfies the pq-condition if and only if G is of *type I* and has the following presentation:

$$\langle \alpha, \beta \mid \alpha^m = \beta^k = 1, \beta\alpha\beta^{-1} = \alpha^r \rangle \tag{5.1}$$

where,

(*a*) $k, m \geq 1$ are odd;

(*b*) $\gcd(k(r-1), m) = 1$ and $r^k \equiv 1 (\mathrm{mod}\ m)$;

(c) if d is the order of r in the multiplicative group of residues modulo m, of integers prime to m, then d divides k and k/d is divisible by every prime divisor of d.

The group G given by the presentation (5.1) above, is a semidirect product $\mathbb{Z}_k \ltimes \mathbb{Z}_m$, and thus it has order km. The condition (a) guarantees that G has odd order. Note then that d must also be odd. The condition (b) alone implies, that any abelian subgroup of G is cyclic. It is a necessary condition for a finite group to act topologically freely on some sphere. The condition (c) is required for the action to be orthogonally free. Any group G satisfying (b) and (c) admits an orthogonally free action on a sphere $\mathbb{S}^{2sd-1}$, with $s \geq 1$.

Note that in the presentation (5.1) if $k = 1$, then $G = 1$ is trivial; whereas if $m = 1$, then $G \cong \mathbb{Z}_k$ is the cyclic group of order k. Notice that if $d = 1$, then $r \equiv 1 (\mathrm{mod}\ m)$ and by condition (b), $m = 1$.

Property 5.13. *Suppose $\mathcal{Q}$ satisfies the conditions of the Assumption B and let $N = \dim(\mathcal{Q})$. Then each of the isotropy groups of $\mathcal{Q}$ has the presentation (5.1) for some (k, m, r) with km odd and d a divisor of $N/2$.*

We note the following simple fact:

Proposition 5.14. *[69, 7.4.14] Suppose a finite group G acts freely and orthogonally on $\mathbb{S}^{N-1}$. If the order of G is relatively prime to N, then G is cyclic.*

Proof. If N is odd, then $N - 1$ is even and G is either trivial or it is cyclic of order two. If N is even, then the order of G is odd and therefore G admits a presentation (5.1) for some k, m, r. Such groups act freely and orthogonally on spheres $\mathbb{S}^{2sd-1}$ of dimension $2sd-1$, so $N = 2sd$ and we can see that d divides both N and k. The coprime condition implies that $d = 1$, and it follows that G must be cyclic of order k. □

Corollary 5.15. *Suppose $\mathcal{Q}$ is an orbifold satisfying both the Assumptions A and B above, and let $N = \dim(\mathcal{Q})$. If the exponent of Γ is relatively prime to N, then all the isotropy groups of $\mathcal{Q}$ are cyclic.*

Indeed, in this case N is relatively prime to the order of any finite subgroup of Γ, and in particular to the order of the isotropy subgroups.

Remark 5.16. The case $N = 4$ in Proposition 5.14 was first proved by Hopf (see for instance [48, Theorem 2]), and the case $N = 2^\ell$, $\ell \geq 2$ by Vincent in [68]. Note that in all these cases, the isotropy groups of Γ are cyclic.

We conclude this chapter with a list of group theoretic questions which are related to or motivated by the problem of the existence of closed geodesics on compact orbifolds. While some of these questions are well known open questions, some even with a long history, other questions just arise naturally (or so we believe) with each of the properties that $\Gamma = \pi_1^{orb}(\mathcal{Q})$ has to satisfy, as we have seen in this chapter.

1. Does there exist an infinite group with finite exponent that is finitely presented? Note that such a group would be a factor of a free Burnside group $B(m, n)$ for some $m > 1$ and $n \gg 1$.

2. Given two positive integers m and n, are there only finitely many finitely presented groups with m generators and exponent n? An equivalent formulation is: For a fixed pair (m, n), do the free Burnside groups $B(m, n)$ admit finitely many finitely presented factors? Note that by Zel′manov's solution to the restricted Burnside problem, the answer to this question is yes if the previous question has a negative answer.

3. Assume that both questions 1 and 2 have a positive answer. Is there an order on the set of all finitely presented factors of $B(m, n)$, for a given pair (m, n)? Is there a maximal finitely presented quotient of $B(m, n)$?

4. Is it true that any finite subgroup of Γ satisfies the pq-condition? Note that for $m > 1$ and $n \geq 665$, every finite subgroup of the free Burnside group $B(m, n)$ is cyclic. The answer is yes if any finite subgroup of Γ is isomorphic to a (subgroup of) an isotropy group.

5. Suppose Γ_1 and Γ_2 are isotropy groups in $\Gamma = \pi_1^{orb}(\mathcal{Q})$ corresponding to two distinct points in $\mathcal{Q}$. In particular this implies that Γ_1 and Γ_2 are not conjugate in Γ. Let $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ be non-trivial elements and assume that γ_1 is not conjugate to γ_2. Can the subgroup $\langle \gamma_1, \gamma_2 \rangle$ of Γ generated by γ_1 and γ_2 be finite?

   We can assume that γ_1 and γ_2 are not conjugate in Question 5 since otherwise, by [33, Remark 5.1.3], one can show that $\mathcal{Q}$ admits a closed geodesic of positive length.

References

[1] Alejandro Adem, Johann Leida, and Yongbin Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007. MR 2359514 (2009a:57044)

[2] S. I. Adian, *The subgroups of free periodic groups of odd exponent*, Trudy Mat. Inst. Steklov. **112** (1971), 64–72, 386, Collection of articles dedicated to Academician Ivan Matveevič Vinogradov on his eightieth birthday, I. MR 0323913 (48 #2266)

[3] ______, *The Burnside problem and identities in groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 95, Springer-Verlag, Berlin, 1979, Translated from the Russian by John Lennox and James Wiegold. MR 537580 (80d:20035)

[4] ______, *Normal subgroups of free periodic groups*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 5, 931–947, 1198. MR 637610 (84g:20050)

[5] S. V. Alešin, *Finite automata and the Burnside problem for periodic groups*, Mat. Zametki **11** (1972), 319–328. MR 0301107 (46 #265)

[6] M. M. Alexandrino and M. A. Javayoles, *On closed geodesics in the leaf spaces of singular Riemannian foliations*, Glasgow Mathematical Journal **53** (2011), no. 03, 555–568.

[7] M. A. Armstrong, *The fundamental group of the orbit space of a discontinuous group*, Proc. Cambridge Philos. Soc. **64** (1968), 299–301. MR 0221488 (36 #4540)

[8] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston Inc., Boston, MA, 1985. MR 823981 (87h:53050)

[9] Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1995, Corrected reprint of the 1983 original. MR 1393195 (97d:22011)

[10] Ludwig Bieberbach, *Über einige Extremalprobleme im Gebiete der konformen Abbildung*, Math. Ann. **77** (1916), no. 2, 153–172. MR 1511853

[11] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038)

[12] W. Burnside, *On an unsettled question in the theory of discontinuous groups*, Quart. J. Math. **33** (1902), 230–238.

[13] ———, *On criteria for the finiteness of the order of a group of linear substitutions*, Proc. London Math. Soc. **3** (1905), no. 2, 435–440.

[14] Leonard S. Charlap, *Bieberbach groups and flat manifolds*, Universitext, Springer-Verlag, New York, 1986. MR 862114 (88j:57042)

[15] Jeff Cheeger and Detlef Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128. MR 0303460 (46 #2597)

[16] ———, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443. MR 0309010 (46 #8121)

[17] Weimin Chen and Yongbin Ruan, *Orbifold Gromov-Witten theory*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 25–85. MR 1950941 (2004k:53145)

[18] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR 2360474 (2008k:20091)

[19] ———, *Lectures on orbifolds and reflection groups*, Transformation Groups and Moduli Spaces of Curves (S-T Yau L. Ji, ed.), International Press, 2010, pp. 63–93.

[20] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty. MR 1138207 (92i:53001)

[21] G. Dragomir, *Orbifolds of nonpostive curvature and their loop space*, Master's thesis, McMaster University, 2005.

[22] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. MR 1161694 (93i:20036)

[23] Walter Feit and John G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029. MR 0166261 (29 #3538)

[24] E. S. Golod, *On nil-algebras and finitely approximable p-groups*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 273–276. MR 0161878 (28 #5082)

[25] E. S. Golod and I. R. Šafarevič, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 261–272. MR 0161852 (28 #5056)

[26] Rostislav Grigorchuk and Igor Pak, *Groups of intermediate growth: an introduction*, Enseign. Math. (2) **54** (2008), no. 3-4, 251–272. MR 2478087 (2009k:20101)

[27] R. I. Grigorčuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 53–54. MR 565099 (81m:20045)

[28] ———, *On the Milnor problem of group growth*, Dokl. Akad. Nauk SSSR **271** (1983), no. 1, 30–33. MR 712546 (85g:20042)

[29] ———, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985. MR 764305 (86h:20041)

[30] Detlef Gromoll and Wolfgang Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. (2) **90** (1969), 75–90. MR 0247590 (40 #854)

[31] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73. MR 623534 (83b:53041)

[32] Narain Gupta and Saïd Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), no. 3, 385–388. MR 696534 (84g:20075)

[33] K. Guruprasad and A. Haefliger, *Closed geodesics on orbifolds*, Topology **45** (2006), no. 3, 611–641. MR 2218759 (2006m:53056)

[34] André Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal structure of foliations (Toulouse, 1982). MR 755163 (86c:57026a)

[35] Marshall Hall, Jr., *Solution of the Burnside problem for exponent six*, Illinois J. Math. **2** (1958), 764–786. MR 0102554 (21 #1345)

[36] G. Higman, *Subgroups of finitely presented groups*, Proc. Roy. Soc. Ser. A **262** (1961), 455–475. MR 0130286 (24 #A152)

[37] Sergei V. Ivanov, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra Comput. **4** (1994), no. 1-2, ii+308. MR 1283947 (95h:20051)

[38] Michael Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, vol. 183, Birkhäuser Boston Inc., Boston, MA, 2001. MR 1792613 (2002m:57018)

[39] Tetsuro Kawasaki, *The signature theorem for V-manifolds*, Topology **17** (1978), no. 1, 75–83. MR 0474432 (57 #14072)

[40] John L. Kelley, *General topology*, Springer-Verlag, New York, 1975, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27. MR 0370454 (51 #6681)

[41] Wilhelm Klingenberg, *Lectures on closed geodesics*, Springer-Verlag, Berlin, 1978, Grundlehren der Mathematischen Wissenschaften, Vol. 230. MR 0478069 (57 #17563)

[42] Shoshichi Kobayashi, *Transformation groups in differential geometry*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70. MR 0355886 (50 #8360)

[43] I. G. Lysënok, *Infinite Burnside groups of even period*, Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 3, 3–224. MR 1405529 (97j:20037)

[44] L. A. Lyusternik and A. I. Fet, *Variational problems on closed manifolds*, Doklady Akad. Nauk SSSR (N.S.) **81** (1951), 17–18. MR 0044760 (13,474c)

[45] Wilhelm Magnus, *A connection between the Baker-Hausdorff formula and a problem of Burnside*, Ann. of Math. (2) **52** (1950), 111–126. MR 0038964 (12,476c)

[46] A. Malcev, *On isomorphic matrix representations of infinite groups*, Rec. Math. [Mat. Sbornik] N.S. **8 (50)** (1940), 405–422. MR 0003420 (2,216d)

[47] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. MR 0163331 (29 #634)

[48] John Milnor, *Groups which act on S^n without fixed points*, Amer. J. Math. **79** (1957), 623–630. MR 0090056 (19,761d)

[49] ———, *Problems and Solutions: Advanced Problems: 5603*, Amer. Math. Monthly **75** (1968), no. 6, 685–686. MR 1534960

[50] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003. MR 2012261 (2005c:58039)

[51] I. Moerdijk and D. A. Pronk, *Orbifolds, sheaves and groupoids*, K-Theory **12** (1997), no. 1, 3–21. MR 1466622 (98i:22004)

[52] S. B. Myers, *Riemannian manifolds with positive mean curvature*, Duke Math. J. **8** (1941), 401–404. MR 0004518 (3,18f)

[53] P. S. Novikov and S. I. Adian, *Infinite periodic groups. I,II,III*, Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 212–244; 251–524; 709–731. MR 0240178–80 (39 #1532a,b,c)

[54] A. Yu. Ol′shanskiĭ, *Groups of bounded period with subgroups of prime order*, Algebra i Logika **21** (1982), no. 5, 553–618. MR 721048 (85g:20052)

[55] A. V. Rozhkov, *On the theory of groups of Aleshin type*, Mat. Zametki **40** (1986), no. 5, 572–589, 697. MR 886178 (88f:20053)

[56] I. N. Sanov, *Solution of Burnside's problem for exponent 4*, Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10** (1940), 166–170. MR 0003397 (2,212c)

[57] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363. MR 0079769 (18,144a)

[58] ______, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492. MR 0095520 (20 #2022)

[59] I. Schur, *Über gruppen periodischer substitutionen*, Sitzungsber. Preuss. Akad. Wiss. (1911), 619–627.

[60] Peter Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487. MR 705527 (84m:57009)

[61] Christopher Seaton, *Two Gauss-Bonnet and Poincaré-Hopf theorems for orbifolds with boundary*, Differential Geom. Appl. **26** (2008), no. 1, 42–51. MR 2393971 (2009b:53135)

[62] Atle Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164. MR 0130324 (24 #A188)

[63] Jean-Pierre Serre, *Trees*, Springer-Verlag, Berlin, 1980, Translated from the French by John Stillwell. MR 607504 (82c:20083)

[64] Eric L. Swenson, *A cut point theorem for* CAT(0) *groups*, J. Differential Geom. **53** (1999), no. 2, 327–358. MR 1802725 (2001i:20083)

[65] Gudlaugur Thorbergsson, *Closed geodesics on non-compact Riemannian manifolds*, Math. Z. **159** (1978), no. 3, 249–258. MR 0493872 (58 #12833)

[66] W. P. Thurston, *The geometry and topology of 3-manifolds*, ch. 13, Princeton University, 1978-1979.

[67] V. A. Toponogov, *The metric structure of Riemannian spaces with non-negative curvature which contain straight lines*, Amer. Math. Soc. Translations Ser. 2, Vol. 70: 31 Invited Addresses (8 in Abstract) at the Internat. Congr. Math. (Moscow, 1966), Amer. Math. Soc., Providence, R.I., 1968, pp. 225–239. MR 0231319 (37 #6874)

[68] Georges Vincent, *Les groupes linéaires finis sans points fixes*, Comment. Math. Helv. **20** (1947), 117–171. MR 0021936 (9,131d)

[69] Joseph A. Wolf, *Spaces of constant curvature*, McGraw-Hill Book Co., New York, 1967. MR 0217740 (36 #829)

[70] ______, *Growth of finitely generated solvable groups and curvature of Riemanniann manifolds*, J. Differential Geometry **2** (1968), 421–446. MR 0248688 (40 #1939)

[71] Hans J. Zassenhaus, *Über endliche fastkörper*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität **11** (1935), 187–220.

[72] E. I. Zel′manov, *Solution of the restricted Burnside problem for groups of odd exponent*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 42–59, 221. MR 1044047 (91i:20037)

[73] ______, *Solution of the restricted Burnside problem for 2-groups*, Mat. Sb. **182** (1991), no. 4, 568–592. MR 1119009 (93a:20063)