provided

$$[b, c, \gamma] \cdot \delta = 0 \tag{12.53}$$

where $\cdot$ denotes the inner product on $\mathbb{O}$ defined in (4.25); this condition is satisfied for compatible M since γ and δ lie in the same complex subspace of $\mathbb{O}$. Finally, by construction we have

$$f_M(w) = B_0 C_0^{-1} \tag{12.54}$$

and putting this all together results in

$$BC^{-1} = f_M(w) \tag{12.55}$$

or equivalently

$$f_M(w) = (\alpha w + \beta)(\gamma w + \delta)^{-1} = (\alpha b + \beta c)(\gamma b + \delta c)^{-1}. \tag{12.56}$$

This is the desired result, since b and c were arbitrary (satisfying (12.44)).

We have shown that the finite octonionic Lorentz transformations in ten dimensions as given by Manogue & Schray [5] can be used to define octonionic Möbius transformations, thus recovering (and correcting) the earlier results of Dündarer, Gürsey, & Tze [16, 17]. However, our approach differs significantly from theirs, as theirs corresponds to using (12.36), while ours uses (12.38). We have thus shown that octonionic Möbius transformations extend to the octonionic projective space $\mathbb{OP}^1$, defined by (12.2). This description could be a key ingredient when attempting to generalize 4-dimensional twistor theory to ten dimensions. Much recent research in superstrings, supergravity, and M-theory has emphasized the importance of lightlike objects in ten dimensions. An appropriate octonionic generalization of twistor theory to ten dimensions might allow powerful twistor techniques to be applied to these other theories.

A key role in our argument is the use of two fundamental properties of the octonionic Lorentz transformations in [5], namely *nesting* and *compatibility*. Our results here support our view that these are essential features of any computation involving octonions. Otherwise, repeated transformations of the form (12.36) are not equivalent to those of the form (12.38), due to the lack of associativity.

12.5 The Octonionic Projective Plane

In Section 12.2, we considered projective *lines*, which are equivalence classes of points in $\mathbb{K}^2$. Those equivalence classes, in turn, correspond to the (normalized) "squares" ($vv^\dagger$) of the "points" (v). Here, we extend the discussion to projective *planes*, which are described in $\mathbb{K}^3$.

So what is a projective plane? Starting with $\mathbb{R}$, we would like to define an equivalence relation between points, of the form

$$(b, c, r) \sim (b\chi, c\chi, r\chi) \tag{12.57}$$

where $0 \neq \chi \in \mathbb{R}$. Again, it is more natural to consider squares of normalized columns. If we normalize $v \in \mathbb{R}^3$ by

$$v^\dagger v = 1 \tag{12.58}$$

where we have again written $\dagger$ instead of T for transpose, then

$$(vv^\dagger)(vv^\dagger) = v(v^\dagger v)v^\dagger = vv^\dagger. \tag{12.59}$$

Thus, normalized column vectors v have matrix squares $vv^\dagger$ that are projection operators, that is, which square to themselves. Thus, we define

$$\mathbb{RP}^2 = \{\mathcal{X} \in \mathbf{H}_3(\mathbb{R}) : \mathcal{X}^2 = \mathcal{X}, \operatorname{tr}\mathcal{X} = 1\} \tag{12.60}$$

where again the trace condition guarantees that $\mathcal{X} = vv^\dagger$ for some v.

We expect this construction to go through for the other division algebras, and it does, but there's a catch. Over $\mathbb{C}$ and $\mathbb{H}$, everything goes through as expected, yielding projective planes $\mathbb{CP}^2$ and $\mathbb{HP}^2$. What happens over $\mathbb{O}$? The octonionic projective plane, discovered by Ruth Moufang and known as the *Cayley plane*, is indeed defined by

$$\mathbb{OP}^2 = \{\mathcal{X} \in \mathbf{H}_3(\mathbb{O}) : \mathcal{X}^2 = \mathcal{X}, \operatorname{tr}\mathcal{X} = 1\}. \tag{12.61}$$

However, we used associativity in (12.59)! We must therefore ask, which matrices in $\mathbf{H}_3(\mathbb{O})$ square to themselves? The answer is a bit surprising.

Writing

$$\mathcal{X} = \begin{pmatrix} p & \overline{a} & c \\ a & m & \overline{b} \\ \overline{c} & b & n \end{pmatrix} \tag{12.62}$$

and squaring, we see that, for instance, the $(1,3)$ component of $\mathcal{X}^2 = \mathcal{X}$ yields

$$(p+n)c + \overline{a}\,\overline{b} = c. \tag{12.63}$$

Since $p, n \in \mathbb{R}$, a, b, c must associate! Thus, the components of matrices in the Cayley plane must lie in a quaternionic subalgebra of $\mathbb{O}$! The components of different matrices can of course lie in different quaternionic subalgebras; nonetheless, this is a significant restriction on the possible matrices.

If we recall the definition of the Freudenthal product given in Section 11.2, we see that

$$\mathcal{X} * \mathcal{X} = \mathcal{X}^2 - \mathcal{X}\,\mathrm{tr}\,(\mathcal{X}) + \frac{1}{2}\big(\mathrm{tr}\,(\mathcal{X})^2 - \mathrm{tr}\,(\mathcal{X}^2)\big) \tag{12.64}$$

so if $\mathcal{X}^2 = \mathcal{X}$ and $\mathrm{tr}\,\mathcal{X} = 1$, we obtain

$$\mathcal{X} * \mathcal{X} = 0. \tag{12.65}$$

Conversely, if the Freudenthal square vanishes, then so does its trace

$$\mathrm{tr}\,(\mathcal{X} * \mathcal{X}) = \frac{1}{2}\big(\mathrm{tr}\,(\mathcal{X}^2) - \mathrm{tr}\,(\mathcal{X})^2\big) \tag{12.66}$$

so that (12.65) reduces to

$$\mathcal{X}^2 = \mathcal{X}\mathrm{tr}\,(\mathcal{X}). \tag{12.67}$$

Thus, points in the Cayley plane are precisely the normalized solutions of (12.65), that is,

$$\mathbb{OP}^2 = \{\mathcal{X} \in \mathbf{H}_3(\mathbb{O}) : \mathcal{X} * \mathcal{X} = 0, \mathrm{tr}\,\mathcal{X} = 1\}. \tag{12.68}$$

Furthermore, since the components of $\mathcal{X} \in \mathbf{H}_3(\mathbb{O})$ lie in some quaternionic subalgebra $\mathbb{H} \subset \mathbb{O}$ (which depends on $\mathcal{X}$), simple linear algebra shows that

$$\mathcal{X} = vv^\dagger \tag{12.69}$$

for some $v \in \mathbb{H}^3$. Thus, the elements of $\mathbb{OP}^2$ the *squares* of (quaternionic!) triples in $\mathbb{O}^3$.

Now that we know what the *points* in the projective plane are, we need to find the *lines*. We first briefly return to the real case. Let $v, w \in \mathbb{R}^3$ be (real) vectors in three Euclidean dimensions. In Section 11.2, we introduced the Jordan ($\circ$) and Freudenthal ($*$) products on the Albert algebra of octonionic Hermitian 3×3 matrices. We can construct such matrices from v and w, namely vv^T and ww^T. We than have the identities

$$\begin{aligned}(vv^T) \circ (ww^T) &= (v^T w)(vw^T) + (w^T v)(wv^T) \\ &= (v^T w)(vw^T + wv^T), \end{aligned}\tag{12.70}$$

$$\mathrm{tr}\,(vv^T \circ ww^T) = (v^T w)^2, \tag{12.71}$$

$$(vv^T) * (ww^T) = (v \times w)(v \times w)^T, \tag{12.72}$$

where $v \times w$ denotes the usual cross product in $\mathbb{R}^3$, and where of course $v^T w = w^T v \in \mathbb{R}$. Equations (12.70)–(12.72) justify regarding the trace of the Jordan product as a generalized dot product, and the Freudenthal product as a generalized cross product.

Over $\mathbb{C}$ and $\mathbb{H}$, (12.71) becomes

$$\mathrm{tr}\,(vv^\dagger \circ ww^\dagger) = |v^\dagger w|^2 \tag{12.73}$$

since now $v^\dagger w = \overline{w^\dagger v}$.

Continuing the analogy to vector analysis, a "vector" "cross" itself must vanish. In fact, if we think of $\mathbb{OP}^2 \otimes \mathbb{R}$ as the space of (non-normalized) "vectors", henceforth called "1-squares", so that

$$\mathbb{OP}^2 \otimes \mathbb{R} = \{\mathcal{X} \in \mathbf{H}_3(\mathbb{O}) : \mathcal{X} * \mathcal{X} = 0\} \tag{12.74}$$

then for $\mathcal{A}, \mathcal{B} \in \mathbb{OP}^2 \otimes \mathbb{R}$ we have

$$\mathcal{A} * \mathcal{B} = 0 \Longleftrightarrow \mathcal{B} = \lambda\mathcal{A} \tag{12.75}$$

for some $\lambda \in \mathbb{R}$, so that $\mathcal{A}$ and $\mathcal{B}$ are "parallel".

Similar analogies can be made with the dot product. In $\mathbb{HP}^2$ (and hence also in $\mathbb{RP}^2$ and $\mathbb{CP}^2$),

$$\mathrm{tr}\,(vv^\dagger \circ ww^\dagger) = 0 \Longrightarrow vv^\dagger \circ ww^\dagger = 0 \tag{12.76}$$

which follows from (12.73). Remarkably, (12.76) still holds in $\mathbb{OP}^2$ even though (12.73) does not, as can be checked by an explicit but lengthy computation. That is,

$$\mathrm{tr}\,(\mathcal{X} \circ \mathcal{Y}) = 0 \Longrightarrow \mathcal{X} \circ \mathcal{Y} = 0 \tag{12.77}$$

for $\mathcal{X}, \mathcal{Y} \in \mathbb{OP}^2$. An interesting consequence of (12.77) is that

$$\mathcal{X} * \mathcal{X} = 0 = \mathrm{tr}\,\mathcal{X} \Longleftrightarrow \mathcal{X} = 0. \tag{12.78}$$

Using our intuition about points and lines in the real projective plane, our knowledge of vector analysis in $\mathbb{R}^3$, and our notions of generalized dot and cross products, we are now ready to study the properties of the octonionic projective plane.

So what are the lines in the projective plane? Recall that the *points* of $\mathbb{OP}^2$ are given by (12.61). Consider by analogy the matrices

$$\Lambda = \{\mathcal{X} \in \mathbf{H}_3(\mathbb{O}) : \mathcal{X}^2 = \mathcal{X}, \mathrm{tr}\,\mathcal{X} = 2\} \tag{12.79}$$

and call the elements of Λ *lines*.

Why is this terminology reasonable? In algebraic terms, $\mathcal{P} \in \mathbb{OP}^2$ is a *primitive idempotent* of $\mathbf{H}_3(\mathbb{O})$. Restricting temporarily to $\mathbb{H}$ to avoid associativity issues, $\mathcal{P}$ is a projection operator on $\mathbb{H}^3$ into a 1-dimensional subspace, whereas $\mathcal{L} \in \Lambda$ is a projection operator into a 2-dimensional subspace (and $\mathcal{I}$ "projects" into a 3-dimensional subspace, namely all of $\mathbb{H}^3$). In the projective plane, (equivalence classes of) such 1-dimensional subspaces are "points", and (equivalence classes of) such 2-dimensional subspaces are lines. The definitions (12.61) and (12.79) therefore represent plausible generalizations of these concepts to the nonassociative case.

Using definition (12.79), if $\mathcal{Q} \in \mathbb{O}\mathbb{P}^2$ is a point, then

$$\mathcal{L} = \mathcal{I} - \mathcal{Q} \in \Lambda \tag{12.80}$$

is a line, where $\mathcal{I}$ is the 3×3 identity matrix, which is also the unique matrix that squares to itself and has trace 3. There is thus a natural duality relationship between the lines and points of $\mathbb{O}\mathbb{P}^2$. Using our intuition from $\mathbb{R}^3$, where there is a unique line (through the origin) perpendicular to each plane (through the origin), and vice versa, we can use the point $\mathcal{I}-\mathcal{L}$, which is "perpendicular" to $\mathcal{L}$, to define the points $\mathcal{P}$ that are *on* the line $\mathcal{L}$ as being those points which are orthogonal to $\mathcal{I}-\mathcal{L}$. What does orthogonality mean for points? Use the trace of the Jordan product, that is, define $\mathcal{P}$ to be on $\mathcal{L}$ if

$$\operatorname{tr}\left((\mathcal{I} - \mathcal{L}) \circ \mathcal{P}\right) = 0. \tag{12.81}$$

Since $\mathcal{P}, \mathcal{I} - \mathcal{L} \in \mathbb{O}\mathbb{P}^2$, we can use (12.77) to drop the trace; $\mathcal{P}$ is on $\mathcal{L}$ if

$$(\mathcal{I} - \mathcal{L}) \circ \mathcal{P} = 0 \tag{12.82}$$

or equivalently if

$$\mathcal{L} \circ \mathcal{P} = \mathcal{P}. \tag{12.83}$$

What is the line determined by the points $\mathcal{P}$ and $\mathcal{Q}$? Use the cross product! If $\mathcal{P}, \mathcal{Q} \in \mathbb{O}\mathbb{P}^2$, then it does indeed follow that

$$(\mathcal{P} * \mathcal{Q}) * (\mathcal{P} * \mathcal{Q}) = 0 \tag{12.84}$$

that is, $\mathcal{P} * \mathcal{Q}$ is a 1-square if $\mathcal{P}$ and $\mathcal{Q}$ are 1-squares (and is nonzero so long as $\mathcal{P}$ and $\mathcal{Q}$ are not "parallel"). However, $\mathcal{P} * \mathcal{Q}$ will not in general be normalized. The point "orthogonal" to $\mathcal{P}$ and $\mathcal{Q}$ is therefore $\mathcal{P}*\mathcal{Q}/\operatorname{tr}(\mathcal{P}*\mathcal{Q})$, so the line determined by $\mathcal{P}$ and $\mathcal{Q}$ must be[8]

$$\mathcal{L}_{\mathcal{P}\mathcal{Q}} = \mathcal{I} - \frac{\mathcal{P} * \mathcal{Q}}{\operatorname{tr}(\mathcal{P} * \mathcal{Q})}. \tag{12.85}$$

[8]Some authors, such as Baez [18], consider lines to be given by $\mathcal{P} * \mathcal{Q}$, rather than by expressions such as (12.85).

$\mathcal{P}$ and $\mathcal{Q}$ are on this line, since

$$(\mathcal{P} * \mathcal{Q}) \circ \mathcal{Q} = 0 \tag{12.86}$$

for any 1-squares $\mathcal{P}$ and $\mathcal{Q}$. More generally, three points $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{S}$ are collinear if

$$(\mathcal{P} * \mathcal{Q}) \circ \mathcal{S} = 0. \tag{12.87}$$

Recall from Sections 11.2–11.4 that E_6 acts on $\mathbf{H}_3(\mathbb{O})$. We briefly summarize some properties of the induced action of E_6 on $\mathbb{OP}^2$.

- The trace of the triple product is the polarization of the determinant, and hence preserved by E_6. Thus, E_6 is precisely the symmetry group which preserves the notion of collinear points in $\mathbb{OP}^2$.
- In fact, E_6 takes p-squares to p-squares, but the boosts in E_6 do not preserve the normalization condition on $\mathbb{OP}^2$. Thus, E_6 takes points to points, and lines to lines *up to normalization* (which can be corrected by slightly modifying the action).
- The action of E_6 on lines is *not* the usual action of E_6 on 2-squares, but rather the "squared" action induced by

  $$\mathcal{P} * \mathcal{Q} \longmapsto (\mathcal{M}\mathcal{P}\mathcal{M}^\dagger) * (\mathcal{M}\mathcal{Q}\mathcal{M}^\dagger),$$

  with subsequent renormalization as needed.
- Since F_4 is the automorphism group of the Jordan product, it is also the automorphism group of the Freudenthal product, so that

  $$(\mathcal{M}\mathcal{P}\mathcal{M}^\dagger) * (\mathcal{M}\mathcal{Q}\mathcal{M}^\dagger) = \mathcal{M}(\mathcal{P} * \mathcal{Q})\mathcal{M}^\dagger \tag{12.88}$$

  for $\mathcal{M} \in F_4$.
- The remaining elements ("boosts") in E_6 are generated by complex Hermitian matrices, and for such matrices direct computation shows that

  $$(\mathcal{M}\mathcal{P}\mathcal{M}^\dagger) * (\mathcal{M}\mathcal{Q}\mathcal{M}^\dagger) = (\mathcal{M} * \mathcal{M})(\mathcal{P} * \mathcal{Q})(\mathcal{M} * \mathcal{M})^\dagger. \tag{12.89}$$

  This operation yields a "dual" action of boosts on 1-squares, given by $\mathcal{M} * \mathcal{M}$ rather than $\mathcal{M}$—and note that $\mathcal{M} * \mathcal{M}$ is (a multiple of) $\mathcal{M}^{-1}$; these boosts go "the other way".
- Any projective line is a 2-square with repeated eigenvalue 1, and can therefore be written (in several ways) as $\mathcal{L} = \mathcal{P} + \mathcal{Q}$, where $\mathcal{P}$, $\mathcal{Q}$ are projective points satisfying $\mathcal{P} \circ \mathcal{Q} = 0$.
- But if $\mathcal{P} \circ \mathcal{Q} = 0$, the Freudenthal product simplifies to

  $$2\,\mathcal{P} * \mathcal{Q} = \mathcal{I} - \mathcal{P} - \mathcal{Q}. \tag{12.90}$$

- Thus, a projective line can be written as $\mathcal{L} = \mathcal{I} - 2\mathcal{P} * \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are ("orthogonal") points on the line.
- If $\mathcal{L} \in \Lambda$ is a line in $\mathbb{OP}^2$, then the action of E_6 is given by

$$\mathcal{L} \longmapsto \mathcal{L}' = \begin{cases} \mathcal{I} - \mathcal{M}(\mathcal{I} - \mathcal{L})\mathcal{M}^\dagger & \text{(rotations)} \\ \mathcal{I} - \frac{1}{N}(\mathcal{M} * \mathcal{M})(\mathcal{I} - \mathcal{L})(\mathcal{M} * \mathcal{M})^\dagger & \text{(boosts)} \end{cases} \tag{12.91}$$

where N is a normalization constant.
- Untangling these definitions, the condition that a point $\mathcal{A}$ is on $\mathcal{L}$ is preserved by boosts, since

$$\begin{aligned} (\mathcal{M} * \mathcal{M})(\mathcal{I} - \mathcal{L})(\mathcal{M} * \mathcal{M})^\dagger &= 2(\mathcal{M} * \mathcal{M})(\mathcal{P} * \mathcal{Q})(\mathcal{M} * \mathcal{M})^\dagger \\ &= 2(\mathcal{M}\mathcal{P}\mathcal{M}^\dagger) * (\mathcal{M}\mathcal{Q}\mathcal{M}^\dagger) \end{aligned} \tag{12.92}$$

and therefore

$$\begin{aligned} \mathcal{L}' \circ \mathcal{A}' = \mathcal{A}' \Longleftrightarrow 0 &= (\mathcal{M} * \mathcal{M})(\mathcal{I} - \mathcal{L})(\mathcal{M} * \mathcal{M})^\dagger \circ \mathcal{M}\mathcal{A}\mathcal{M}^\dagger \\ &= 2\big((\mathcal{M}\mathcal{P}\mathcal{M}^\dagger) * (\mathcal{M}\mathcal{Q}\mathcal{M}^\dagger)\big) \circ \mathcal{M}\mathcal{A}\mathcal{M}^\dagger \\ &= 2(\mathcal{P} * \mathcal{Q}) \circ \mathcal{A} \\ \Longleftrightarrow \mathcal{L} \circ \mathcal{A} &= \mathcal{A}. \end{aligned} \tag{12.93}$$

- E_6 is therefore the symmetry group which takes points to points and lines to lines in $\mathbb{OP}^2$, while preserving the incidence relation.

12.6 Quaternionic Integers

What are integers? Over $\mathbb{R}$, the answer is easy, namely the infinite set

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}. \tag{12.94}$$

It is straightforward to extend this definition to the complex numbers, resulting in the *Gaussian integers*

$$\mathbb{Z}[\ell] = \mathbb{Z} \oplus \mathbb{Z}\ell = \{m + n\ell : m, n \in \mathbb{Z}\}. \tag{12.95}$$

where we continue to use ℓ rather than i for the complex unit. The Gaussian integers form a *lattice* in two dimensions. The *units* of $\mathbb{Z}[\ell]$ are the elements with norm one, namely the set $\{\pm 1, \pm \ell\}$.

What happens over the other division algebras?

We can of course simply extend the construction of the Gaussian integers. Over $\mathbb{H}$, we obtain the *Lipschitz integers*

$$\mathbb{Z}[i, j, k] = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k \tag{12.96}$$