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Quantum Groups

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This is a report on recent works on Hopf algebras (or quantum groups, which is more or less the same) motivated by the quantum inverse scattering method (QISM), a method for constructing and studying integrable quantum systems, which was developed mostly by L. D. Faddeev and his collaborators. Most of the definitions, constructions, examples, and theorems in this paper are inspired by the QISM. Nevertheless I will begin with these definitions, constructions, etc. and then explain their relation to the QISM. Thus I reverse the history of the subject, hoping to make its logic clearer.

1. What is a quantum group? Recall that both in classical and in quantum mechanics there are two basic concepts: state and observable. In classical mechanics states are points of a manifold M and observables are functions on M. In the quantum case states are 1-dimensional subspaces of a Hilbert space H and observables are operators in H (we forget the self-adjointness condition). The relation between classical and quantum mechanics is easier to understand in terms of observables. Both in classical and in quantum mechanics observables form an associative algebra which is commutative in the classical case and noncommutative in the quantum case. So quantization is something like replacing commutative algebras by noncommutative ones.

Now let us consider elements of a group G as states and functions on G as observables. The notion of group is usually defined in terms of states. To quantize it one has to translate it first into the language of observables. This translation is well known, but let us recall it nevertheless. Consider the algebra $A = \mathrm{Fun}(G)$ consisting of functions on G which are supposed to be smooth if G is a Lie group, regular if G is an algebraic group, and so on. A is a commutative associative unital algebra. Clearly $\mathrm{Fun}(G \times G) = A \otimes A$ if one understands the sign "$\otimes$" in the appropriate sense (e.g. if G is a Lie group then $\otimes$ should be understood as the topological tensor product). So the group operation considered as a mapping $f\colon G \times G \to G$ induces an algebra homomorphism $\Delta\colon A \to A \otimes A$ called "comultiplication." To formulate the associativity property of the group operation in terms of Δ one expresses it as the commutativity of the diagram

$$\begin{tikzcd} & G\times G \arrow[dr, "f"] & \\ G\times G\times G \arrow[ur, "f\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times f"'] & & G \\ & G\times G \arrow[ur, "f"'] & \end{tikzcd}$$

and then the functor $X \mapsto \mathrm{Fun}(X)$ is applied. The result is the coassociativity property of Δ, i.e., the commutativity of the diagram

$$\begin{tikzcd} & A\otimes A \arrow[dr, "\mathrm{id}\otimes\Delta"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A\otimes A\otimes A \\ & A\otimes A \arrow[ur, "\Delta\otimes\mathrm{id}"'] & \end{tikzcd}$$

The translation of the notions of the group unit e and the inversion mapping $g \mapsto g^{-1}$ is quite similar. We consider the mappings $\varepsilon\colon A \to k$, $S\colon A \to A$ given by $\varphi \mapsto \varphi(e)$ and $\varphi(g) \mapsto \varphi(g^{-1})$ respectively (here k is the ground field, i.e. $k = \mathbf{R}$ if G is a real Lie group, $k = \mathbf{C}$ if G is a complex Lie group, etc). Then we translate the identities $g\cdot e = e\cdot g = g$, $g\cdot g^{-1} = g^{-1}\cdot g = e$ into the language of commutative diagrams and apply the functor $X \mapsto \mathrm{Fun}(X)$. As a result we obtain the commutative diagrams

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & A \arrow[d, equal] \\ A\otimes A \arrow[r, "\mathrm{id}\otimes\varepsilon"] & A\otimes k \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & A \arrow[d, equal] \\ A\otimes A \arrow[r, "\varepsilon\otimes\mathrm{id}"] & k\otimes A \end{tikzcd} \tag{1}$$

$$\left.\begin{array}{c} \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[drr, "\varepsilon"] & A\otimes A \arrow[r, "\mathrm{id}\otimes S"] & A\otimes A \arrow[r, "m"] & A \\ & & k \arrow[ur, "i"'] & \end{tikzcd} \\ \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[drr, "\varepsilon"] & A\otimes A \arrow[r, "S\otimes\mathrm{id}"] & A\otimes A \arrow[r, "m"] & A \\ & & k \arrow[ur, "i"'] & \end{tikzcd} \end{array}\right\} \tag{2}$$

Here m is the multiplication (i.e. $m(a\otimes b) = ab$) and $i(c) = c\cdot 1_A$. The commutativity of (1) (resp. (2)) is expressed by the words "ε is the counit" (resp. "S is the antipode"). The properties of $(A, m, \Delta, i, \varepsilon, S)$ listed above mean that A is a commutative Hopf algebra.

Now there is a general principle: the functor $X \mapsto \mathrm{Fun}(X)$ from the category of "spaces" to the category of commutative associative unital algebras (perhaps, with some additional structures or properties) is an antiequivalence. This principle becomes a theorem if "space" is understood as "affine scheme," or if "space" is understood as "compact topological space" and "algebra" is understood as "C^*-algebra." From the above principle it follows that the category of groups is antiequivalent to the category of commutative Hopf algebras.

Now let us define the category of quantum spaces to be dual to the category of (not necessarily commutative) associative unital algebras. Denote by Spec A (spectrum of A) the quantum space corresponding to an algebra A. Finally define a quantum group to be the spectrum of a (not necessarily commutative) Hopf algebra. So the notions of Hopf algebra and quantum group are in fact equivalent, but the second one has some geometric flavor.

Let me make some general remarks on the definition of Hopf algebra. First of all, it is known that for given A, m, Δ the counit $\varepsilon\colon A \to k$ is unique and is a homomorphism. The antipode $S\colon A \to A$ is also unique and it is an antihomomorphism (with respect to both multiplication and comultiplication). Secondly, in the noncommutative case one also requires the existence of the "skew antipode" $S'\colon A \to A$ which is in fact the antipode for the opposite multiplication and the same comultiplication. S' is also the antipode for the opposite comultiplication and the same multiplication. It is known that $SS' = S'S = \mathrm{id}$. In the commutative or cocommutative case $S' = S$, but in general $S' \neq S$ and $S^2 \neq \mathrm{id}$. The proofs of these statements can be found in standard texts on Hopf algebras [**1**–**3**]. An informal discussion of some aspects of the notion of Hopf algebra can be found in [**4**]. The reader should keep in mind that our usage of the term "Hopf algebra" is not generally accepted: some people do not require the existence of the unit, the counit and the antipode (so their Hopf algebras correspond to quantum semigroups).

It is important that a quantum group is not a group, nor even a group object in the category of quantum spaces. This is because for noncommutative algebras the tensor product is not a coproduct in the sense of category theory.

Now the question arises whether there exist natural examples of noncommutative Hopf algebras. An easy way of constructing such an algebra is to consider A^*, A being a commutative but not cocommutative algebra (recall that the dual space of a Hopf algebra A has a Hopf algebra structure, the multiplication mapping $A^* \otimes A^* \to A^*$ being induced by the comultiplication of A and the comultiplication of A^* being induced by the multiplication of A). In this way more or less all cocommutative noncommutative Hopf algebras are obtained. The words "more or less" are due to the fact that $V^{**} \neq V$ for a general vector space V. But at the heuristic level $V^{**} = V$ and therefore any cocommutative Hopf algebra is of the form $(\mathrm{Fun}(G))^*$ for some group G. Clearly $(\mathrm{Fun}(G))^*$ is commutative iff G is commutative. Note that $(\mathrm{Fun}(G))^*$ is nothing but the group algebra of G. An important class of cocommutative Hopf algebras is formed by universal enveloping algebras (the comultiplication $U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ is defined by the formula $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$). If $\mathfrak{g}$ is the Lie algebra of a Lie group G then $U\mathfrak{g}$ may be considered as the subalgebra of $(C^\infty(G))^*$ consisting of distributions $\varphi \in C_0^{-\infty}(G)$ such that $\mathrm{Supp}\,\varphi \subset \{e\}$. One can identify $(U\mathfrak{g})^*$ with the completion of the local ring of $e \in G$ or with $\mathrm{Fun}(\hat{G})$ where $\hat{G}$ is the formal group corresponding to $\mathfrak{g}$ (the second realization of $(U\mathfrak{g})^*$ makes sense even if G does not exist, which may well happen if $\dim \mathfrak{g} = \infty$).

The most interesting and mysterious Hopf algebras are those which are neither commutative nor cocommutative. Though Hopf algebras were intensively studied both by "pure" algebraists [**2**–**12**] and by specialists in von Neumann algebras [**13**–**26**], I believe that most of the examples of noncommutative noncocommutative Hopf algebras invented independently of integrable quantum system theory are counterexamples rather than "natural" examples (however there are remarkable exceptions, e.g. [**12**], [**16**], and [**2**, pp. 89-90]). We are going to discuss a general method for constructing noncommutative noncocommutative Hopf algebras, which was proposed in [**27**, **28**] under the influence of the QISM. This method is based on the concept of quantization. It can be considered as a realization of the ideas of G. I. Kac and V. G. Palyutkin (see the end of [**16**]).

2. Quantization. Roughly speaking, a quantization of a commutative associative algebra A_0 over k is a (not necessarily commutative) deformation of A_0 depending on a parameter h (Planck's constant), i.e. an associative algebra A over $k[[h]]$ such that $A/hA = A_0$ and A is a topologically free $k[[h]]$-module. Given A, we can define a new operation on A_0 (the Poisson bracket) by the formula

$$\{a \bmod h, b \bmod h\} = \frac{[a,b]}{h} \bmod h. \tag{3}$$

Thus A_0 becomes a Poisson algebra (i.e. a Lie algebra with respect to $\{\ ,\ \}$ and a commutative associative algebra with respect to multiplication, these two structures being compatible in the following sense:

$$\{a, bc\} = \{a,b\}c + b\{a,c\}).$$

Now we shall slightly change our point of view on quantization.

DEFINITION. A *quantization* of a Poisson algebra A_0 is an associative algebra deformation A of A_0 over $k[[h]]$ such that the Poisson bracket on A_0 defined by (3) is equal to the bracket given a priori.

Of course, this approach to quantization is as old as quantum mechanics. It was explained to mathematicians by F. A. Berezin, J. Vey, A. Lichnerowicz, M. Flato, D. Sternheimer, and others.

We shall need a Hopf algebra version of the above definition. In this case A_0 is a Poisson-Hopf algebra (i.e. a Hopf algebra structure and a Poisson algebra structure on A_0 are given such that the multiplication is the same for both structures and the comultiplication $A_0 \to A_0 \otimes A_0$ is a Poisson algebra homomorphism, the Poisson bracket on $A_0 \otimes A_0$ being defined by $\{a\otimes b, c\otimes d\} = ac\otimes\{b,d\} + \{a,c\}\otimes bd$) and A is a Hopf algebra deformation of A_0. We shall also use the dual notion of quantization of co-Poisson-Hopf algebras (a co-Poisson-Hopf algebra is a cocommutative Hopf algebra B with a Poisson cobracket $B \to B \otimes B$ compatible with the Hopf algebra structure).

We discuss the structure of Poisson-Hopf algebras and co-Poisson-Hopf algebras in §§3 and 4. Then we consider the quantization problem.

3. Poisson groups and Lie bialgebras. A *Poisson group* is a group G with a Poisson bracket on $\mathrm{Fun}(G)$ which makes $\mathrm{Fun}(G)$ a Poisson-Hopf algebra. In other words the Poisson bracket must be compatible with the group operation, which means that the mapping $\mu\colon G \times G \to G$, $\mu(g_1, g_2) = g_1 g_2$, must be a Poisson mapping in the sense of [**33**], i.e. $\mu^*\colon \mathrm{Fun}(G) \to \mathrm{Fun}(G \times G)$ must be a Lie algebra homomorphism. Specifying the meaning of the word "group" and the symbol $\mathrm{Fun}(G)$, we obtain the notions of Poisson-Lie group, Poisson formal group, Poisson algebraic group, etc. According to our general principles the notions of Poisson group and Poisson-Hopf algebra are equivalent.

There exists a very simple description of Poisson-Lie groups in terms of *Lie bialgebras.*

DEFINITION. A *Lie bialgebra* is a vector space $\mathfrak{g}$ with a Lie algebra structure and a Lie coalgebra structure, these structures being compatible in the following sense: the cocommutator mapping $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ must be a 1-cocycle ($\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by means of the adjoint representation).

If G is a Poisson-Lie group then $\mathfrak{g} = \mathrm{Lie}\,(G)$ has a Lie bialgebra structure. To define it write the Poisson bracket on $C^\infty(G)$ as

$$\{\varphi, \psi\} = \eta^{\mu\nu} \partial_\mu \varphi \cdot \partial_\nu \psi, \qquad \varphi, \psi \in C^\infty(G), \tag{4}$$

where $\{\partial_\mu\}$ is a basis of right-invariant vector fields on G. The compatibility of the bracket with the group operation means that the function $\eta\colon G \to \mathfrak{g} \otimes \mathfrak{g}$ corresponding to $\eta^{\mu\nu}$ is a 1-cocycle. The 1-cocycle $f\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ corresponding to η defines a Lie bialgebra structure on $\mathfrak{g}$ (the Jacobi identity for $f^*\colon \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ holds because f^* is the infinitesimal part of the bracket (4)).

THEOREM 1. *The category of connected and simply-connected Poisson-Lie groups is equivalent to the category of finite dimensional Lie bialgebras.*

The analogue of Theorem 1 for Poisson formal groups over a field of characteristic 0 can be proved in the following way. The algebra of functions on the formal group corresponding to $\mathfrak{g}$ is nothing but $(U\mathfrak{g})^*$. A Poisson-Hopf structure on $(U\mathfrak{g})^*$ is equivalent to a co-Poisson-Hopf structure on $U\mathfrak{g}$. So it suffices to prove the following easy theorem.

THEOREM 2. *Let $\delta\colon U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ be a Poisson cobracket which makes $U\mathfrak{g}$ a co-Poisson-Hopf algebra (the Hopf structure on $U\mathfrak{g}$ is usual). Then $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$ and $(\mathfrak{g}, \delta|\mathfrak{g})$ is a Lie bialgebra. Thus we obtain a one-to-one correspondence between co-Poisson-Hopf structures on $U\mathfrak{g}$ inducing the usual Hopf structure and Lie bialgebra structures on $\mathfrak{g}$ inducing the given Lie algebra structure.*

Now let us discuss the notion of Lie bialgebra. First of all there is a one-to-one correspondence between Lie bialgebras and *Manin triples.* A Manin triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ consists of a Lie algebra $\mathfrak{p}$ with a nondegenerate invariant scalar product on it and isotropic Lie subalgebras $\mathfrak{p}_1, \mathfrak{p}_2$ such that $\mathfrak{p}$ is the direct sum of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ as a vector space. The correspondence mentioned above is constructed in the following way: if $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple then we put $\mathfrak{g} = \mathfrak{p}_1$ and define the cocommutator $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ to be dual to the commutator mapping $\mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p}_2$ (note that $\mathfrak{p}_2$ is naturally isomorphic to $\mathfrak{g}^*$). Conversely, if a Lie bialgebra $\mathfrak{g}$ is given we put $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\mathfrak{p}_1 = \mathfrak{g}$, $\mathfrak{p}_2 = \mathfrak{g}^*$ and define the commutator $[x, l]$ for $x \in \mathfrak{g}$, $l \in \mathfrak{g}^*$ so that the natural scalar product on $\mathfrak{p}$ should be invariant. Note that if $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple then so is $(\mathfrak{p}, \mathfrak{p}_2, \mathfrak{p}_1)$. Therefore the notion of Lie bialgebra is self-dual.

Here are some examples of Lie bialgebras. Examples 3.2–3.4 are important for the inverse scattering method.

EXAMPLE 3.1. If $\dim \mathfrak{g} = 2$ then any linear mappings $\bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ define a Lie bialgebra structure on $\mathfrak{g}$. A 2-dimensional Lie bialgebra is called nondegenerate if the composition $\bigwedge^2 \mathfrak{g} \to \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ is nonzero. In this case there exists a basis $\{x_1, x_2\}$ of $\mathfrak{g}$ such that $[x_1, x_2] = \alpha x_2$ and the cocommutator is given by $x_1 \mapsto 0, x_2 \mapsto \beta x_2 \wedge x_1$. Here $\alpha\beta \neq 0$ and $\alpha\beta$ does not depend on the choice of x_1, x_2.

EXAMPLE 3.2. Let $\mathfrak{g}$ be a Kac-Moody algebra (in the sense of [**55**]) with a fixed invariant scalar product $\langle\ ,\ \rangle$, $\mathfrak{h}$ the Cartan subalgebra, $\mathfrak{b}_\pm \supset \mathfrak{h}$ the Borel subalgebras. Put $\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}$, $\mathfrak{p}_1 = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} | x = y\} \approx \mathfrak{g}$, $\mathfrak{p}_2 = \{(x, y) \in \mathfrak{b}_- \times \mathfrak{b}_+ | x_\mathfrak{h} + y_\mathfrak{h} = 0\}$. Define the scalar product of $(x_1, y_1) \in \mathfrak{p}$ and $(x_2, y_2) \in \mathfrak{p}$ to equal $\langle x_1, x_2\rangle - \langle y_1, y_2\rangle$. Since $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple, $\mathfrak{g}$ has a Lie bialgebra structure. The cocommutator $\varphi\colon \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ can be described explicitly in terms of the canonical generators X_i^+, X_i^-, H_i (here $X_i^\pm \in [\mathfrak{b}_\pm, \mathfrak{b}_\pm]$ and H_i is the image of the simple root $\alpha_i \in \mathfrak{h}^*$ under the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$): $\varphi(H_i) = 0$, $\varphi(X_i^\pm) = \frac{1}{2} X_i^\pm \wedge H_i$. Note that $\mathfrak{b}_+$ and $\mathfrak{b}_-$ are subbialgebras of $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{sl}(2)$ then $\mathfrak{b}_+$ and $\mathfrak{b}_-$ are of the type described in Example 3.1. The Manin triple corresponding to $\mathfrak{b}_+$ is $(\mathfrak{g} \times \mathfrak{h}, \mathrm{Im}\psi_+, \mathrm{Im}\psi_-)$ where $\psi_\pm\colon \mathfrak{b}_\pm \hookrightarrow \mathfrak{g} \times \mathfrak{h}$ is defined by $\psi_\pm(a) = (a, \pm a_\mathfrak{h})$ and the scalar product on $\mathfrak{g} \times \mathfrak{h}$ equals $\langle\ ,\ \rangle_\mathfrak{g} - \langle\ ,\ \rangle_\mathfrak{h}$.

EXAMPLE 3.3. Fix a simple Lie algebra $\mathfrak{a}$ ($\dim \mathfrak{a} < \infty$) and an invariant scalar product on it. Set $\mathfrak{p} = \mathfrak{a}((u^{-1}))$, $\mathfrak{p}_1 = \mathfrak{a}[u]$, $\mathfrak{p}_2 = u^{-1}\mathfrak{a}[[u^{-1}]]$ and define the scalar product on $\mathfrak{p}$ by $(f, g) = \mathrm{res}_{u=\infty}(f(u), g(u))\, du$. The Manin triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ defines a Lie bialgebra structure on $\mathfrak{g} = \mathfrak{a}[u]$. The cocommutator in $\mathfrak{g}$ is given by the formula

$$a(u) \mapsto [a(u) \otimes 1 + 1 \otimes a(v), r(u, v)]. \tag{5}$$

Here $a \in \mathfrak{a}[u]$, $\mathfrak{g} \otimes \mathfrak{g}$ is identified with $(\mathfrak{a} \otimes \mathfrak{a})[u, v]$ and $r(u, v) = t/(u - v)$ where t is the element of $\mathfrak{a} \otimes \mathfrak{a}$ corresponding to our scalar product. The right-hand side of (5) has no pole at $u = v$ because t is invariant.

EXAMPLE 3.4 (cf. [**29–31**]). Fix a nonsingular irreducible projective algebraic curve X over $\mathbf{C}$. Denote by E (resp. A, O_x) the field of rational functions on X (resp. the adèle ring, the completion of the local ring of X at x). Fix an absolutely simple Lie algebra G over E ($\dim G < \infty$) and a rational differential ω on X, $\omega \neq 0$. Define an invariant $\mathbf{C}$-valued scalar product on $G \otimes_E A$ by

$$(u, v) = \sum_{x \in X} \mathrm{res}_x\{\omega \cdot \mathrm{Tr}\rho(u_x)\rho(v_x)\}$$

where $\rho\colon G \to \mathfrak{gl}(n, E)$ is an exact representation, $u = (u_x)_{x\in X}$, $v = (v_x)_{x\in X}$. Then G is a maximal isotropic subspace of $G \otimes_E A$. If there exists an open isotropic $\mathbf{C}$-subalgebra $\Lambda \subset G\otimes_E A$ such that $G\otimes_E A = G \oplus \Lambda$ then we obtain a Lie bialgebra structure on G. To every subset $S \subset X$, $S \neq \varnothing$, there corresponds a subbialgebra $G_S = \{a \in G |$ the image of a in $G \otimes A^S$ belongs to the image of Λ in $G\otimes A^S\}$, where A^S is the ring of adèles without x-components, $x \in S$. The bialgebra of Example 3.3 is, in fact, G_S, where

$$X = \mathbf{P}^1, \quad \omega = d\lambda, \quad S = \{\infty\}, \quad \mathfrak{g} = \mathfrak{a}\otimes E, \quad \Lambda = \mathfrak{a}\otimes\left(m_\infty \times \prod_{x\in X\setminus S} O_x\right),$$

and m_∞ is the maximal ideal of O_∞. If $\mathfrak{g}$ is an affine Kac-Moody algebra with the bialgebra structure of Example 3.2 then $\mathfrak{g}' \overset{\text{def}}{=} [\mathfrak{g}, \mathfrak{g}]/(\text{the center of } \mathfrak{g})$ is, in fact, G_S for $X = \mathbf{P}^1$, $\omega = \lambda^{-1}d\lambda$, $S = \{0, \infty\}$, $G = \mathfrak{g}'\otimes_{\mathbf{C}[\lambda,\lambda^{-1}]} \mathbf{C}(\lambda)$ ($\mathfrak{g}'$ has a natural structure of a $\mathbf{C}[\lambda, \lambda^{-1}]$-algebra),

$$\Lambda = \mathfrak{c} \times \prod_{x\in X\setminus S} (\mathfrak{g}'\otimes_{\mathbf{C}[\lambda,\lambda^{-1}]} O_x),$$

where $\mathfrak{c} \subset (\mathfrak{g}'\otimes_{\mathbf{C}[\lambda,\lambda^{-1}]} \mathbf{C}((\lambda^{-1}))) \times (\mathfrak{g}'\otimes_{\mathbf{C}[\lambda,\lambda^{-1}]} \mathbf{C}((\lambda)))$ is the closure of

$$\mathfrak{p}_2' = \{(x, y)\} \in \mathfrak{g}' \times \mathfrak{g}' | x \in \mathfrak{b}'_-,\ y \in \mathfrak{b}'_+,\ x_{\mathfrak{h}} + y_{\mathfrak{h}} = 0\}.$$

4. Classical Yang-Baxter equation. Let $\mathfrak{g}$ be a Lie algebra and suppose that $\varphi\colon \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ is the coboundary of $r \in \bigwedge^2 \mathfrak{g}$. It can be shown that $(\mathfrak{g}, \varphi)$ is a Lie bialgebra iff

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \text{ is } \mathfrak{g}\text{-invariant.} \tag{6}$$

Here, for instance,

$$[r^{12}, r^{13}] \overset{\text{def}}{=} \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j$$

where $r = \sum_i a_i \otimes b_i$ (one may imagine that $r^{12} = \sum_i a_i \otimes b_i \otimes 1 \in (U\mathfrak{g})^{\otimes 3}$, $r^{13} = \sum_i a_i \otimes 1 \otimes b_i \in (U\mathfrak{g})^{\otimes 3}$, $r^{23} = \sum_i 1 \otimes a_i \otimes b_i \in (U\mathfrak{g})^{\otimes 3}$). In particular, if

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \tag{7}$$

then $(\mathfrak{g}, \varphi)$ is a Lie bialgebra. Equation (7) is just the classical Yang-Baxter equation (CYBE), or the classical triangle equation. There is an important case which is intermediate between (6) and (7). Suppose that $r \in \mathfrak{g}\otimes\mathfrak{g}$ satisfies (7) but the skew-symmetry condition $r^{12} + r^{21} = 0$ is replaced by "$r^{12} + r^{21}$ is $\mathfrak{g}$-invariant" (then the coboundary $\varphi = \partial r$ is skew-symmetric). In this situation $(\mathfrak{g}, \varphi)$ is also a Lie bialgebra. If $r^{12} + r^{21} = P$, $\rho = r - \frac{1}{2}P$ then $\rho \in \bigwedge^2 \mathfrak{g}$, $\varphi = \partial\rho$ and $[\rho^{12}, \rho^{13}] + [\rho^{12}, \rho^{23}] + [\rho^{13}, \rho^{23}] = \frac{1}{4}[P^{12}, P^{23}]$.

DEFINITION. A *coboundary Lie bialgebra* is a pair $(\mathfrak{g}, r)$, where $\mathfrak{g}$ is a Lie bialgebra, $r \in \bigwedge^2 \mathfrak{g}$, $\partial r =$ the cocommutator of $\mathfrak{g}$. A coboundary Lie bialgebra $(\mathfrak{g}, r)$ is said to be *triangular* if r satisfies (7). A *quasitriangular Lie bialgebra* is a pair $(\mathfrak{g}, r)$, where $\mathfrak{g}$ is a Lie bialgebra, $r \in \mathfrak{g}\otimes\mathfrak{g}$, $\partial r =$ the cocommutator of $\mathfrak{g}$, and r satisfies (7).

It was noticed in [**32**, **27**] that the expression $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$, $r \in \bigwedge^2 \mathfrak{g}$, is equal to $\frac{1}{2}\{r, r\}$ where $\{\ ,\ \}$ is the bilinear operation on $\bigwedge^* \mathfrak{g}$ such that $\{a, b\} = [a, b]$ for $a, b \in \mathfrak{g}$ and

$$\{a,b\} = -(-1)^{(k+1)(l+1)}\{b,a\},$$
$$\{a, b\wedge c\} = \{a,b\}\wedge c + (-1)^{(k+1)l} b\wedge\{a,c\}$$

for $a \in \bigwedge^k \mathfrak{g}$, $b \in \bigwedge^l \mathfrak{g}$, $c \in \bigwedge^* \mathfrak{g}$ ($\bigwedge^* \mathfrak{g}$ is a Poisson superalgebra with an odd Poisson bracket; if elements of $\bigwedge^* \mathfrak{g}$ are considered as left-invariant polyvector fields on the Lie group corresponding to $\mathfrak{g}$ then the operation $\{\ ,\ \}$ in $\bigwedge^* \mathfrak{g}$ is the Schouten bracket [**34**]).

The Poisson bracket on the Poisson-Lie group G corresponding to a coboundary bialgebra $(\mathfrak{g}, r)$ is of the form

$$\{\varphi, \psi\} = r^{\mu\nu}(\partial'_\mu\varphi\cdot\partial'_\nu\psi - \partial_\mu\varphi\cdot\partial_\nu\psi), \tag{8}$$

where ∂_μ (respectively ∂'_μ) are the right-invariant (respectively left-invariant) vector fields on G corresponding to a basis of $\mathfrak{g}$ and $r^{\mu\nu}$ are the coordinates of r. The following property of the bracket (8) is crucial for the Hamiltonian version of the classical inverse scattering method (a systematic exposition of this method is given in [**35**]; see also [**36–38**]):

$$\varphi, \psi \in C^\infty(G), \quad \varphi(gxg^{-1}) = \varphi(x), \quad \psi(gyg^{-1}) = \psi(y) \Rightarrow \{\varphi,\psi\} = 0. \tag{9}$$

Let us reconsider Examples 3.1–3.4. The cocommutator of a nondegenerate 2-dimensional bialgebra is not a coboundary. Now let us consider the cocommutator (5) on $\mathfrak{g} = \mathfrak{a}[u]$. First suppose that $r(u, v)$ is an $\mathfrak{a}\otimes\mathfrak{a}$-valued polynomial. Then we can consider r as an element of $\mathfrak{g}\otimes\mathfrak{g}$ and the cocommutator (5) as the coboundary of r. The CYBE and the condition $r \in \bigwedge^2 \mathfrak{g}$ take the form

$$\begin{aligned}[]
[r^{12}(u_1,u_2), r^{13}(u_1,u_3)] + [r^{12}(u_1,u_2), r^{23}(u_2,u_3)]&\\
+[r^{13}(u_1,u_3), r^{23}(u_2,u_3)] = 0&,
\end{aligned} \tag{10}$$

$$r^{12}(u_1,u_2) + r^{21}(u_2,u_1) = 0. \tag{11}$$

So every polynomial solution of (10), (11) defines a triangular bialgebra structure on $\mathfrak{g}$. In Example 3.3, $r(u, v)$ is a nonpolynomial solution of (10), (11). Therefore the corresponding bialgebra is "pseudotriangular" (nevertheless (9) holds).

The bialgebra $\mathfrak{g}$ of Example 3.2 has a quasitriangular structure if $\dim\mathfrak{g} < \infty$: consider the element $t \in \mathfrak{g}\otimes\mathfrak{g}$ corresponding to the scalar product on $\mathfrak{g}$, represent t as $t_{+-}+t_0+t_{-+}$, $t_{+-} \in [\mathfrak{b}_+, \mathfrak{b}_+]\otimes[\mathfrak{b}_-, \mathfrak{b}_-]$, $t_{-+} \in [\mathfrak{b}_-, \mathfrak{b}_-]\otimes[\mathfrak{b}_+, \mathfrak{b}_+]$, $t_0 \in \mathfrak{h}\otimes\mathfrak{h}$. and set $r = t_{+-}+\frac{1}{2}t_0$. If $\dim\mathfrak{g} = \infty$ then r does not belong to the algebraic tensor product $\mathfrak{g}\otimes\mathfrak{g}$. Therefore $(\mathfrak{g}, r)$ is "pseudoquasitriangular." Let us examine more attentively the non-twisted affine case (the twisted affine case is similar). In this case $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]/(\text{the center of }\mathfrak{g}) = \mathfrak{a}[\lambda, \lambda^{-1}]$ for some simple algebra $\mathfrak{a}$, $\dim\mathfrak{a} < \infty$. Denote by $\bar{r}$ the analogue of r for $\mathfrak{g}'$. Consider $\bar{r}$ as a power series in $\lambda = \lambda\otimes 1$ and

$\mu = 1 \otimes \lambda$ with coefficients in $\mathfrak{a} \otimes \mathfrak{a}$. Then $\bar{r}(\lambda, \mu)$ is the Laurent decomposition of a rational function of λ/μ having a single pole at $\lambda/\mu = 1$ with residue $t_\mathfrak{a}$ ($t_\mathfrak{a} \in \mathfrak{a} \otimes \mathfrak{a}$ corresponds to the scalar product on $\mathfrak{a}$). This rational function satisfies (10) and (11). Moral [**39**, §3]: if a solution $r(u, v)$ of (10), (11) has a pole at $u = v$ then the corresponding Lie bialgebra is "pseudoquasitriangular" rather than "pseudotriangular"; in other words $r^{12}(u_1, u_2) + r^{21}(u_2, u_1)$ is $t_\mathfrak{a} \cdot \delta(u_1 - u_2)$ rather than 0.

I. V. Cherednik proved [**31**] that the bialgebras of Example 4 also correspond to solutions $r(u, v)$ of (10), (11) having a pole at $u = v$. Moreover, it follows from [**40–42**] that there is a one-to-one correspondence between "nondegenerate" solutions of (10), (11) up to an equivalence relation (such solutions always have a pole at $u = v$) and quadruples (X, ω, G, Λ) of the type described in Example 3.4. Besides, a classification of nondegenerate solutions of (10), (11) is given in [**40, 41**]. In terms of (X, ω, G, Λ) the results of [**40, 41**] can be stated in the following way.

(1) For Λ to exist ω should have no zeros and therefore there are three possibilities: (a) X is an elliptic curve and ω is regular, (b) $X = \mathbf{P}^1$, $\omega = \lambda^{-1} d\lambda$, (c) $X = \mathbf{P}^1$, $\omega = d\lambda$. The corresponding solutions $r(u, v)$ are elliptic, trigonometric or rational functions of $u - v$.

(2) In case (a) for Λ to exist G has to be isomorphic to $\mathfrak{sl}(n)$, and for every n there are finitely many choices of Λ all of which are listed. In case (b) G is of the type described at the end of Example 3.4, and all the choices of Λ are listed (they are similar to the Λ described at the end of Example 3.4). In case (c) little is known.

Some important ideas and results of M. A. Semenov-Tian-Shansky concerning Poisson-Lie groups and the CYBE are in [**39, 43**].

5. Some historical remarks. The Poisson bracket (8) and the CYBE were introduced by E. K. Sklyanin [**44, 37**] as the classical limits of the corresponding quantum objects. The compatibility of the bracket (8) with the group operation was formulated by him almost explicitly. The abstract notion of Poisson-Lie group appeared later [**27**]. The classification of the solutions of the CYBE was based on vast experimental material due to many authors (see the Appendix of [**45**]). In particular, the solution $r(u, v) = t/(u - v)$ was found in [**45, 46**].

6. Quantization of Lie bialgebras (examples). By definition, a quantization of a Lie bialgebra $\mathfrak{g}$ is a quantization of $U\mathfrak{g}$ in the sense of §2, where $U\mathfrak{g}$ is considered as a co-Poisson-Hopf algebra according to Theorem 2. If A is a quantization of $\mathfrak{g}$ we call $\mathfrak{g}$ the *classical limit* of A.

EXAMPLE 6.1 (cf. [**12**], [**2**, pp. 89-90]). Consider the $\mathbf{C}[[h]]$-algebra A generated (in the h-adic sense, i.e. as an algebra complete in the h-adic topology) by x_1, x_2 with the defining relation $[x_1, x_2] = \alpha x_2$, $\alpha \in \mathbf{C}^*$. Define $\Delta \colon A \to A \otimes A$ by $\Delta(x_1) = x_1 \otimes 1 + 1 \otimes x_1$, $\Delta(x_2) = x_2 \otimes \exp(\frac{1}{2} h\beta x_1) + \exp(-\frac{1}{2} h\beta x_1) \otimes x_2$, $\beta \in \mathbf{C}^*$. Then A is a quantization of the bialgebra of Example 3.1. From the results of §9 it follows that the quantization is unique up to substitutions $h \mapsto h + \sum_{i=2}^{\infty} c_i h^i$, $c_i \in \mathbf{C}$.

EXAMPLE 6.2. Let $\mathfrak{g}, \mathfrak{h}, \alpha_i, \langle\ ,\ \rangle, H_i$ denote the same objects as in Example 3.2. Consider the $\mathbf{C}[[h]]$-algebra $U_h\mathfrak{g}$ generated (in the h-adic sense) by $\mathfrak{h}$ and X_i^+, X_i^- with the defining relations

$$[a_1, a_2] = 0 \quad \text{for } a_1, a_2 \in \mathfrak{h}, \qquad [a, X_i^\pm] = \pm\alpha_i(a)X_i^\pm \quad \text{for } a \in \mathfrak{h},$$

and also

$$[X_i^+, X_j^-] = 2\delta_{ij}h^{-1}\mathrm{sh}(hH_i/2),$$

$$i \neq j,\ n = 1 - A_{ij}, q = \exp\frac{h}{2}\langle H_i, H_i\rangle \Rightarrow$$
$$\Rightarrow \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q q^{-k(n-k)/2}(X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k} = 0.$$

Here (A_{ij}) is the Cartan matrix and $\binom{n}{k}_q$ is the Gauss polynomial [**47**, §3.3], i.e., $\binom{n}{k}_q = (q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)/(q^k - 1)/(q^{k-1} - 1)\cdots(q - 1)$. It can be shown that $U_h\mathfrak{g}$ is a topologically free $\mathbf{C}[[h]]$-module and that there exists a homomorphism $\Delta\colon U_h\mathfrak{g} \to U_h\mathfrak{g} \otimes U_h\mathfrak{g}$ such that

$$\Delta(X_i^\pm) = X_i^\pm \otimes \exp(hH_i/4) + \exp(-hH_i/4) \otimes X_i^\pm,$$

$\Delta(a) = a \otimes 1 + 1 \otimes a$ for $a \in \mathfrak{h}$. $(U_h\mathfrak{g}, \Delta)$ is a quantization of $\mathfrak{g}$. It can be shown using the results of §9 that $U_h\mathfrak{g}$ is the unique (up to substitutions $h \mapsto h + \sum_{i=2}^{\infty} c_i h^i$, $c_i \in \mathbf{C}$) quantization A of $\mathfrak{g}$ for which there exist a cocommutative Hopf subalgebra $C \subset A$ and a mapping $\theta\colon A \to A$ such that the mapping $C/hC \to U\mathfrak{g}$ is injective and its image is equal to $U\mathfrak{h}$, $\theta^2 = \mathrm{id}$, $\theta(C) = C$, θ is an algebra automorphism and a coalgebra antiautomorphism, $\theta \bmod h$ is the Cartan involution (if $A = U_h\mathfrak{g}$ put $C = U\mathfrak{h}[[h]]$, $\theta(X_i^\pm) = -X_i^\mp$, $\theta|_\mathfrak{h} = -\mathrm{id}$). So it is natural to call $U_h\mathfrak{g}$ *the* quantized Kac-Moody algebra corresponding to $\mathfrak{g}$. $U_h\mathfrak{sl}(2)$ was introduced by Kulish-Reshetikhin [**48**] and E. K. Sklyanin [**49**]. For general $\mathfrak{g}$ the algebra $U_h\mathfrak{g}$ was introduced by M. Jimbo [**50, 51**] and the author [**28**]. All these works were motivated by the QISM. For $\mathfrak{g}$ affine or finite dimensional $U_h\mathfrak{g}$ is closely related to trigonometric solutions of the quantum Yang-Baxter equation (QYBE); see [**50**–**52**] and §13.

EXAMPLE 6.3 [**28, 54**]. Let $\mathfrak{a}$ and $\mathfrak{g}$ denote the same objects as in Example 3.3. Since $\mathfrak{g}$ is graded ($\deg au^k = k$ for $a \in \mathfrak{a}$) and the cocommutator has degree -1, it is natural to look for a quantization Q which is a graded Hopf algebra over $\mathbf{C}[[h]]$, where $\mathbf{C}[[h]]$ is graded so that $\deg h = 1$. It can be shown using the results of §9 that Q exists and is unique. As an h-adic algebra, Q is generated by the Lie algebra $\mathfrak{a}$ ($\deg a = 0$ for $a \in \mathfrak{a}$) and elements $J(a)$ of degree 1, $a \in \mathfrak{a}$, with the defining relations

$$J(\lambda a + \mu b) = \lambda J(a) + \mu J(b), \quad J([a, b]) = [a, J(b)] \quad \text{for } a, b \in \mathfrak{a},\ \lambda, \mu \in \mathbf{C}, \tag{12}$$

$$\sum_i [a_i, b_i] = 0 \Rightarrow \sum_i [J(a_i), J(b_i)] \\ = \frac{h^2}{12} \sum_i ([[a_i, I_\alpha], [b_i, I_\beta]], I_\gamma) \cdot \{I_\alpha, I_\beta, I_\gamma\}, \quad a_i, b_i \in \mathfrak{a}, \tag{13}$$

$$\sum_i [[a_i, b_i], c_i] = 0 \Rightarrow \sum_i [[J(a_i), J(b_i)], J(c_i)] \\ = \frac{h^2}{4} \sum_i \sum_{\alpha,\beta,\gamma} f(a_i, b_i, c_i, I_\alpha, I_\beta, I_\gamma)\{I_\alpha, I_\beta, J(I_\gamma)\}, \quad a_i, b_i, c_i \in \mathfrak{a}, \tag{14}$$

where $\{I_\alpha\}$ is an orthonormal basis of $\mathfrak{a}$,

$$f(a, b, c, x, y, z) = \underset{a,b}{\mathrm{Alt}}\, \underset{a,c}{\mathrm{Sym}}\, ([a, [b, x]], [[c, y], z]),$$

$$\{x_1, x_2, x_3\} = \frac{1}{6} \sum_{i \neq j \neq k} x_i x_j x_k.$$

$\Delta\colon Q \to Q \otimes Q$ is defined by the formulas

$$\begin{aligned} &\Delta(a) = a \otimes 1 + 1 \otimes a, \\ &\Delta(J(a)) = J(a) \otimes 1 + 1 \otimes J(a) + \tfrac{h}{2}[a \otimes 1, t] \qquad a \in \mathfrak{a}, \end{aligned} \tag{15}$$

where $t = \sum_\alpha I_\alpha \otimes I_\alpha$. Note that the classical limit of $J(a)$ is $au \in \mathfrak{g}$. So the formula (12) and the left-hand sides of (13), (14) are quite natural. (15) is also natural because $[a \otimes 1, t]$ is the image of au under the cocommutator mapping. The terrific right-hand sides of (13), (14) are derived from (12), (15). Note also that if $\mathfrak{a} = \mathfrak{sl}(2)$ then (13) is superfluous and if $\mathfrak{a} \neq \mathfrak{sl}(2)$ then (14) is superfluous. The Hopf algebra over $\mathbf{C}$ obtained from Q by setting $h = 1$ is denoted by $Y(\mathfrak{a})$ and called the Yangian of $\mathfrak{a}$. $Y(\mathfrak{a})$ is related to rational solutions of the QYBE (see §12), the simplest of which was found by C. N. Yang [**53**].

Fortunately Q has another system of generators $\xi^+_{ik}, \xi^-_{ik}, \varphi_{ik}$, $i \in \Gamma, k = 0, 1, 2, \ldots$, $\deg \xi^\pm_{ik} = \deg \varphi_{ik} = k$, where Γ is the set of simple roots of $\mathfrak{a}$. The classical limits of $\xi^\pm_{ik}$ and φ_{ik} are equal to $X^\pm_i u^k$ and $H_i u^k$, where $X^\pm_i$ and H_i are the generators of $\mathfrak{a}$ used in Example 3.2. Here is the complete set of relations between $\xi^\pm_{ik}$ and φ_{ik}:

$$\begin{gathered} [\varphi_{ik}, \varphi_{jl}] = 0, \qquad [\varphi_{i0}, \xi^\pm_{jl}] = \pm 2B_{ij}\xi^\pm_{jl}, \qquad [\xi^+_{ik}, \xi^-_{jl}] = \delta_{ij}\varphi_{i,k+l}, \\ [\varphi_{i,k+1}, \xi^\pm_{jl}] - [\varphi_{ik}, \xi^\pm_{j,l+1}] = \pm h B_{ij}(\varphi_{ik}\xi^\pm_{jl} + \xi^\pm_{jl}\varphi_{ik}) \\ [\xi^\pm_{i,k+1}, \xi^\pm_{jl}] - [\xi^\pm_{ik}, \xi^\pm_{j,l+1}] = \pm h B_{ij}(\xi^\pm_{ik}\xi^\pm_{jl} + \xi^\pm_{jl}\xi^\pm_{ik}), \\ i \neq j, n = 1 - A_{ij} \Rightarrow \mathrm{Sym}_{k_1,\ldots,k_n}[\xi^\pm_{ik_1}, [\xi^\pm_{ik_2}, \ldots [\xi^\pm_{ik_n}, \xi^\pm_{jl}]], \ldots] = 0, \end{gathered}$$

where (A_{ij}) is the Cartan matrix of $\mathfrak{a}$, $B_{ij} = \frac{1}{2}(H_i, H_j)$. The connection between the two presentations of Q is described in [**54**].

A similar realization exists for quantized affine Kac-Moody algebras [**54**]. For instance, if $\mathfrak{g}$ is untwisted, i.e. $[\mathfrak{g}, \mathfrak{g}]/(\text{the center of } \mathfrak{g}) = \mathfrak{a}[\lambda, \lambda^{-1}]$, then $U_h\mathfrak{g}$ has the following presentation. Generators: $c, D, \xi_{ik}^{\pm}, \kappa_{ik}$, where $i \in \Gamma$, $k \in \mathbf{Z}$. Defining relations:

$$[c, \kappa_{ik}] = [c, \xi_{ik}^{\pm}] = [c, D] = 0, \qquad [D, \kappa_{ik}] = k\kappa_{ik}, \qquad [D, \xi_{ik}^{\pm}] = k\xi_{ik}^{\pm},$$
$$[\kappa_{ik}, \kappa_{jl}] = 4\delta_{k,-l} k^{-1} h^{-2} \mathrm{sh}\,(khB_{ij}) \mathrm{sh}(khc/2),$$
$$[\kappa_{ik}, \xi_{jl}^{\pm}] = \pm 2(kh)^{-1} \mathrm{sh}(khB_{ij}) \exp(\mp|k| \cdot hc/4) \xi_{j,k+l}^{\pm},$$
$$\xi_{i,k+1}^{\pm}\xi_{jl}^{\pm} - e^{\pm hB_{ij}} \xi_{jl}^{\pm}\xi_{i,k+1}^{\pm} = e^{\pm hB_{ij}} \xi_{ik}^{\pm}\xi_{j,l+1}^{\pm} - \xi_{j,l+1}^{\pm}\xi_{ik}^{\pm},$$
$$[\xi_{ik}^{+}, \xi_{jl}^{-}] = \delta_{ij} h^{-1}\{\psi_{i,k+l} e^{hc(k-l)/4} - \varphi_{i,k+l} e^{hc(l-k)/4}\},$$
$$i \neq j,\ n = 1 - A_{ij},\ q = \exp \tfrac{h}{2}(H_i, H_i) \Rightarrow$$
$$\Rightarrow \mathrm{Sym}_{k_1,\ldots,k_n} \sum_{r=0}^{n} (-1)^r \binom{n}{r}_q q^{-r(n-r)/2} \xi_{ik_1}^{\pm} \cdots \xi_{ik_r}^{\pm} \xi_{jl}^{\pm} \xi_{ik_{r+1}}^{\pm} \cdots \xi_{ik_n}^{\pm} = 0.$$

Here φ_{ip}, ψ_{ip} are defined from the relations

$$\sum_p \varphi_{ip} u^p = \exp h \left(\frac{1}{2}\kappa_{i0} + \sum_{p<0} \kappa_{ip} u^p \right),$$
$$\sum_p \psi_{ip} u^p = \exp h \left(\frac{1}{2}\kappa_{i0} + \sum_{p>0} \kappa_{ip} u^p \right).$$

The connection between the two presentations of $U_h\mathfrak{g}$ is described in [**54**]. Note that $U_h\mathfrak{g}'$ (i.e. $U_h\mathfrak{g}$ "without D" and with the auxiliary relation $c = 0$) degenerates into $Y(\mathfrak{a})$: if A is the subalgebra of $U_h\mathfrak{g}' \otimes_{\mathbf{C}[[h]]} \mathbf{C}((h))$ generated by $U_h\mathfrak{g}'$ and $h^{-1} \cdot \mathrm{Ker}\, f$, where f is the composition $U_h\mathfrak{g}' \to U\mathfrak{g}' = U(\mathfrak{a}[\lambda, \lambda^{-1}]) \xrightarrow{\lambda=1} U\mathfrak{a}$, then $A/hA = Y(\mathfrak{a})$.

The problem of describing Δ in terms of $\xi_{ik}^{\pm}$ is not solved for Yangians nor for quantized affine algebras.

7. Quantized universal enveloping algebras and h-formal groups. A *quantized universal enveloping algebra* (QUE-algebra) is a Hopf algebra A over $\mathbf{C}[[h]]$ such that A is a topologically free $\mathbf{C}[[h]]$-module and A/hA is a universal enveloping algebra. In other words, a QUE-algebra is a quantization of some Lie bialgebra. It is easy to show that a cocommutative QUE-algebra A is the h-adic completion of $U\mathfrak{g}$ for some Lie algebra $\mathfrak{g}$ over $\mathbf{C}[[h]]$ which is a topologically free $\mathbf{C}[[h]]$-module (in fact, $\mathfrak{g} = \{a \in A |\ \Delta(a) = a \otimes 1 + 1 \otimes a\}$).

Another important class of Hopf algebras over $\mathbf{C}[[h]]$ consists of *quantized formal series Hopf algebras* (QFSH-algebras). A QFSH-algebra is a topological Hopf algebra B over $\mathbf{C}[[h]]$ such that (1) as a topological $\mathbf{C}[[h]]$-module B is isomorphic to $\mathbf{C}[[h]]^I$ for some set I, (2) B/hB is isomorphic to $\mathbf{C}[[u_1, u_2, \ldots]]$ as a topological algebra.

An *h-formal group* is the spectrum of a QFSH-algebra. To understand h-formal groups in down-to-earth terms let us fix a "coordinate system," i.e. elements $x_i \in B$ such that (1) B/hB is the ring of formal power seris in $\bar{x}_i = x_i \bmod h$, and (2) $\varepsilon(x_i) = 0$, where $\varepsilon\colon B \to \mathbf{C}[[h]]$ is the counit. $\Delta(x_i)$ can be expressed as $F_i(x_1, x_2, \ldots; y_1, y_2, \ldots; h)$, where $y_i = 1 \otimes x_i \in B \otimes B$ and x_i is identified with $x_i \otimes 1 \in B \otimes B$. So an h-formal group is defined by a "quantum group law" $F(x, y, h)$ and "commutation relations," i.e., formulas expressing $h^{-1}[x_i, x_j]$ as a formal series in x_i and h.

Now let us discuss the relation between QUE-algebras and QFSH-algebras. First of all, the dual of a QUE-algebra is a QFSH-algebra and vice versa (in both cases the dual should be understood in an appropriate sense). For instance, the dual of the QUE algebra of Example 6.1 is generated as an algebra by ξ_1 and ξ_2 where

$$\xi_1(f(x_1, x_2)) = \frac{\partial}{\partial x_1} f(x_1, 0)|_{x_1=0}, \qquad \xi_2\left(\sum_i \varphi_i(x_1)x_2^i\right) = \varphi_1\left(\frac{\alpha}{2}\right).$$

The commutation relation between ξ_1 and ξ_2 is of the form $[\xi_1, \xi_2] = -h\beta\xi_2$ and the comultiplication is given by $\Delta(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1$, $\Delta(\xi_2) = \xi_2 \otimes \exp(-\frac{\alpha}{2}\xi_1) + \exp(\frac{\alpha}{2}\xi_1) \otimes \xi_2$. The dual of $U_h\mathfrak{sl}(n)$ also has an explicit description. Consider the representation $\rho\colon U_h\mathfrak{sl}(n) \to \mathrm{Mat}\,(n, \mathbf{C}[[h]])$ such that $\rho(H_i) = e_{ii} - e_{i+1,i+1}$, $\rho(X_i^+) = e_{i,i+1}$, $\rho(X_i^-) = 2h^{-1}\mathrm{sh}(h/2)e_{i+1,i}$ where e_{ij} are the matrix units (the scalar product on $\mathfrak{sl}(n)$ is supposed to equal $\mathrm{Tr}(XY)$). The matrix elements ρ_{ij}, $1 \le i, j \le n$, belong to $(U_h\mathfrak{sl}(n))^*$. It is easy to show that

$$\rho_{ij}\rho_{il} = e^{-h/2}\rho_{il}\rho_{ij} \quad \text{if } j < l, \tag{16}$$

$$\rho_{ij}\rho_{kj} = e^{-h/2}\rho_{kj}\rho_{ij} \quad \text{if } i < k, \tag{17}$$

$$\rho_{il}\rho_{kj} = \rho_{kj}\rho_{il} \quad \text{if } i < k,\ l > j, \tag{18}$$

$$[\rho_{il}, \rho_{kj}] = (e^{h/2} - e^{-h/2})\rho_{ij}\rho_{kl} \quad \text{if } i > k, l > j, \tag{19}$$

$$\sum_{i_1,\dots,i_n} \rho_{1i_1}\rho_{2i_2}\cdots\rho_{ni_n} \cdot (-e^{-h/2})^{l(i_1,\dots i_n)} = 1,$$

where $l(i_1, \dots, i_n)$ is the number of inversions in the permutation $(i_1, \dots, i_n)$. In fact $(U_h\mathfrak{sl}(n))^*$ is the quotient of the ring of noncommutative power series in $\tilde{\rho}_{ij} = \rho_{ij} - \delta_{ij}$ by the ideal generated by the above relations. The comultiplication is usual: $\Delta(\rho_{ij}) = \sum_k \rho_{ik} \otimes \rho_{kj}$. It is natural to call $\mathrm{Spec}(U_h\mathfrak{sl}(n))^*$ the quantized formal $\mathrm{SL}(n)$. One can also consider the subalgebra of $(U_h\mathfrak{sl}(n))^*$ consisting of the matrix elements of finite-dimensional representations of $U_h\mathfrak{sl}(n)$, i.e. of the polynomials in ρ_{ij}. The spectrum of this subalgebra can be considered as a quantization of the algebraic group $\mathrm{SL}(n)$.

Somewhat unexpectedly the category of QFSH-algebras finitely generated as topological algebras is *equivalent* (not only antiequivalent) to the category of QUE-algebras whose classical limit is a finite dimensional Lie bialgebra (the finiteness conditions are not very essential here). The following example can help to understand this equivalence. Consider the QUE-algebra of Example 6.1, i.e. $[x_1, x_2] = \alpha x_2$, $\Delta(x_1) = x_1 \otimes 1 + 1 \otimes x_1$, $\Delta(x_2) = x_2 \otimes \exp(\frac{1}{2}h\beta x_1) + \exp(-\frac{1}{2}h\beta x_1) \otimes x_2$. Put $y_i = hx_i$. Then $[y_1, y_2] = \alpha h y_2$, $\Delta(y_1) = y_1 \otimes 1 + 1 \otimes y_1$, $\Delta(y_2) = y_2 \otimes \exp(\frac{1}{2}\beta y_1) + \exp(-\frac{1}{2}\beta y_1) \otimes y_2$, i.e. we have obtained a QFSH-algebra. Here is the general construction. If A is a QUE-algebra we put $\delta_1(a) = a - \varepsilon(a) \cdot 1$, $\delta_2(a) = \Delta(a) - a \otimes 1 - 1 \otimes a + \varepsilon(a) \cdot (1 \otimes 1)$, etc. ($\varepsilon$ is the counit).

Then $\{a \in A \mid \delta_n(a) \in h^n A^{\otimes n} \text{ for any } n\}$ is a QFSH-algebra. Conversely, if B is a QFSH-algebra and m is its maximal ideal then the h-adic completion of $\sum_n h^{-n} m^n$ is a QUE-algebra.

DEFINITION. The QUE-dual of a QUE-algebra A is the QUE-algebra corresponding to A^* in the above sense (or, which is the same, the dual of the QFSH-algebra corresponding to A).

It can be shown that the classical limit of the QUE-dual of A is a Lie bialgebra dual to the classical limit of A.

8. The square of the antipode. Let A be a quantization of a Lie bialgebra $\mathfrak{g}$. Denote by S the antipode of A. In general $S^2 \neq \mathrm{id}$, but $S^2 \equiv \mathrm{id} \bmod h$. Here is a description of $h^{-1}(S^2 - \mathrm{id}) \bmod h$ in terms of $\mathfrak{g}$. Denote by D the composition of the cocommutator $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ and the commutator $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. It is easy to show that $D\colon \mathfrak{g} \to \mathfrak{g}$ is a bialgebra derivation (this means that e^{tD} is a bialgebra automorphism) and $h^{-1}(S^2 - \mathrm{id}) \bmod h$ is the unique derivation of $U\mathfrak{g}$ whose restriction to $\mathfrak{g}$ is equal to $-D/2$.

If $\mathfrak{g} = \mathfrak{a}[u]$ is the bialgebra of Example 3.3 and the scalar product on $\mathfrak{a}$ is the Killing form then $D = d/du$. For bialgebras corresponding to nondegenerate solutions of (10), (11), D becomes equal to d/du after a suitable normalization.

9. Obstruction theory. Let $\mathfrak{g}$ be a Lie bialgebra. Denote by $\mathrm{Der}^i \mathfrak{g}$ the $(i+2)$th cohomology group of $\mathrm{Ker}(C^*(\mathfrak{g} \oplus \mathfrak{g}^*) \to C^*(\mathfrak{g}) \oplus C^*(\mathfrak{g}^*))$ where $\mathfrak{g} \oplus \mathfrak{g}^*$ is equipped with a Lie algebra structure defined in §3 and $C^*(\mathfrak{a})$ is the standard complex $0 \to \mathfrak{a}^* \to \bigwedge^2 \mathfrak{a}^* \to \bigwedge^3 \mathfrak{a}^* \to \cdots$. Obviously $\mathrm{Der}^i \mathfrak{g} = 0$ for $i < 0$. $\mathrm{Der}^0 \mathfrak{g}$ is the set of bialgebra derivations of $\mathfrak{g}$. It can be shown that (1) if two quantizations of $\mathfrak{g}$ coincide modulo h^n then their difference $\bmod h^{n+1}$ "belongs" to $\mathrm{Der}^1 \mathfrak{g}$, and (2) if $\mathrm{Der}^2 \mathfrak{g} = 0$ then $\mathfrak{g}$ has a quantization. I do not know whether every Lie bialgebra can be quantized.

10. Coboundary, triangular, and quasitriangular Hopf algebras. A pair (A, R) consisting of a Hopf algebra A and an invertible element $R \in A \otimes A$ will be called a *coboundary Hopf algebra* if

$$\Delta'(a) = R\Delta(a)R^{-1}, \qquad a \in A, \tag{20}$$

and $R^{12}R^{21} = 1, R^{12} \cdot (\Delta \otimes \mathrm{id})(R) = R^{23} \cdot (\mathrm{id} \otimes \Delta)(R)$, $(\varepsilon \otimes \varepsilon)(R) = 1$. Here $\Delta' = \sigma \circ \Delta$, $\sigma\colon A \otimes A \to A \otimes A$, $\sigma(x \otimes y) = y \otimes x$ and the symbols R^{12}, R^{21}, R^{23} have the following meaning: if $R = \sum_i a_i \otimes b_i$ then

$$R^{21} = \sum_i b_i \otimes a_i, \qquad R^{23} = \sum_i 1 \otimes a_i \otimes b_i, \qquad R^{12} = \sum_i a_i \otimes b_i \otimes 1$$

(and sometimes $R^{12} = R$). (A, R) is called a *quasitriangular Hopf algebra* if R satisfies (20) and the equations

$$(\Delta \otimes \mathrm{id})(R) = R^{13}R^{23}, \qquad (\mathrm{id} \otimes \Delta)(R) = R^{13}R^{12}. \tag{21}$$

A quasitriangular Hopf algebra is said to be *triangular* if $R^{12}R^{21} = 1$. By definition, a coboundary (resp. triangular, quasitriangular) QUE-algebra is a coboundary (triangular, quasitriangular) Hopf algebra (A, R) such that A is a QUE-algebra and $R \equiv 1 \bmod h$.

It can be shown that if (A, R) is a coboundary (triangular, quasitriangular) QUE-algebra, $r = h^{-1}(R-1) \bmod h$ and $\mathfrak{g}$ is the classical limit of A then $r \in \mathfrak{g} \otimes \mathfrak{g}$ (a priori $r \in U\mathfrak{g} \otimes U\mathfrak{g}$) and $(\mathfrak{g}, r)$ is a coboundary (triangular, quasitriangular) Lie bialgebra. There are also other arguments in favor of the above definitions.

It is easy to show that (1) a triangular Hopf algebra is a coboundary Hopf algebra, (2) if (A, R) is a quasitriangular QUE-algebra and $\overline{R} = (R^{12}R^{21})^{-1/2}R$ then $(A, \overline{R})$ is a coboundary QUE-algebra, (3) if (A, R) is quasitriangular then R satisfies the QYBE, i.e.

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

(4) if (A, R) is a coboundary QUE-algebra and R satisfies the QYBE then (A, R) is triangular, (5) equations (21) mean that the mapping $A^* \to A$ given by $l \mapsto (l \otimes \mathrm{id})(R)$ is an algebra homomorphism and a coalgebra antihomomorphism.

To understand the various types of Hopf algebras one may use the Tannaka-Krein approach [**56**], i.e., instead of considering a Hopf algebra A one may consider the category Rep_A of its representations together with the forgetful functor $F\colon \mathrm{Rep}_A \to \{\text{vector spaces}\}$. Recall that a representation of a Hopf algebra A is a representation of A as an algebra, and the tensor product of representations $\rho_1\colon A \to \operatorname{End} V_1$ and $\rho_2\colon A \to \operatorname{End} V_2$ is equal to the composition $A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho_1 \otimes \rho_2} \operatorname{End}(V_1 \otimes V_2)$. For any A the functor $\otimes\colon \mathrm{Rep}_A \times \mathrm{Rep}_A \to \mathrm{Rep}_A$ is associative. More precisely, for this functor there is an associativity constraint [**56**] compatible with F. In general the A-modules $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are not isomorphic, but if (20) holds then R defines an isomorphism between them. Moreover, if $R^{12}R^{21} = 1$ we obtain a commutativity constraint [**56**] for $\otimes\colon \mathrm{Rep}_A \times \mathrm{Rep}_A \to \mathrm{Rep}_A$. It is not compatible with F unless $R = 1$. The condition (21) is the compatibility of the commutativity and the associativity constraints [**56**]. So if (A, R) is triangular then Rep_A is a tensor category [**56**], but F is not a tensor functor unless $R = 1$. V. V. Lyubashenko's result [**57**] means that a tensor category C with a functor $F\colon C \to \{\text{vector spaces}\}$ having the above properties is more or less Rep_A for a unique triangular Hopf algebra A.

If (A, Δ_0) is a cocommutative Hopf algebra and ϕ is an invertible element of $A \otimes A$ such that $\phi^{12} \cdot (\Delta_0 \otimes \mathrm{id})(\phi) = \phi^{23} \cdot (\mathrm{id} \otimes \Delta_0)(\phi)$ then we can obtain a triangular Hopf algebra (A, Δ, R) by setting $\Delta(a) = \phi \Delta_0(a) \phi^{-1}$, $R = \phi^{21}(\phi^{12})^{-1}$. It can be shown that all triangular QUE-algebras can be obtained in this way (this is natural from the Tannaka-Krein point of view). In particular, a triangular QUE-algebra is isomorphic (as an algebra) to a universal enveloping algebra.

Finally I am going to mention a version of formula (20) which plays a crucial role in the QISM (e.g., see [**71**]) and which was in fact the origin of (20). Given a representation $\rho\colon A \to \mathrm{Mat}(n, \mathbf{C})$, its matrix elements $t_{ij} \in A^*$ satisfy the commutation relations $R_\rho \tilde{T} \overset{\approx}{T} = \overset{\approx}{T} \tilde{T} R_\rho$, where $R_\rho = (\rho \otimes \rho)(R)$,

$T = (t_{ij})$, $\tilde{T} = T \otimes E_n$, $\tilde{\tilde{T}} = E_n \otimes T$ (E_n is the unit matrix). Moreover, for a family of representations $\rho_\lambda \colon A \to \mathrm{Mat}(n, \mathbf{C})$, $\lambda \in \mathbf{C}$, we have

$$R(\lambda,\mu)\tilde{T}(\lambda)\tilde{\tilde{T}}(\mu) = \tilde{\tilde{T}}(\mu)\tilde{T}(\lambda)R(\lambda,\mu), \tag{22}$$

where $R(\lambda,\mu) = (\rho_\lambda \otimes \rho_\mu)(R)$, $T(\lambda) = (t_{ij}(\lambda))$, $t_{ij}(\lambda)$ are the matrix elements of ρ_λ. The more traditional view on (22) is to start from a function $R(\lambda,\mu) \in \mathrm{End}(\mathbf{C}^n \otimes \mathbf{C}^n)$ and consider the algebra B with generators $t_{ij}(\lambda), \lambda \in \mathbf{C}$, $1 \leq i, j \leq n$ and defining relations (22). Note that the formula

$$\Delta(t_{ij}(\lambda)) = \sum_k t_{ik}(\lambda) \otimes t_{kj}(\lambda)$$

defines a Hopf algebra structure on B (in the weak sense, i.e. without antipode), and if $R(\lambda,\mu) = (\rho_\lambda \otimes \rho_\mu)(R)$ for some $A, \{\rho_\lambda\}$ and $R \in A \otimes A$ such that (20) holds, then there is a natural homomorphism $B \to A$. So Hopf algebras have already appeared implicitly in [**71**]. Precise statements concerning the connection between triangular Hopf algebras and the QYBE can be found in [**57, 79**].

11. QISM from the Hopf algebra point of view. The QISM is a method for constructing and studying integrable quantum systems, which was created in 1978–1979 by L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan [**69–71, 37**]. It is closely related to the classical inverse scattering method, Baxter's works on exactly solvable models of classical statistical mechanics (e.g. see [**58**]), and one-dimensional relativistic factorized scattering theory (C. N. Yang, A. B. Zamolodchikov, ...). Here is an exposition of the basic construction of the QISM, which slightly differs from the standard expositions [**70–72, 36, 37**], where Hopf algebras occur implicitly and (22) is used instead of (20).

Let A be a Hopf algebra, $C = \{l \in A^* \mid l(ab) = l(ba) \text{ for any } a, b \in A\}$. Then C is a subalgebra of A^*. Now suppose that (20) holds. Then it is easy to show that C is commutative (this is the quantum analogue of (9)). Fix a representation $\pi \colon A^* \to \mathrm{End}\, V$ ("the quantum L-operator") and a number N ("the number of sites of a 1-dimensional lattice"). Elements of $V^{\otimes N}$ are considered as quantum states. The image of C under the representation $\pi^{\otimes N} \colon A^* \to \mathrm{End}\, V^{\otimes N}$ is a commutative algebra of observables. Physicists usually fix an element $H \in A^*$ and consider $\pi^{\otimes N}(H)$ as a Hamiltonian.

If (A, R) is quasitriangular then R induces a homomorphism $A^* \to A$ and therefore a representation of A induces that of A^*. Usually only those representations of A^* are considered which come from representations of A. This means that instead of $\pi^{\otimes N}$ and C one works with $\rho^{\otimes N}$ and C', where ρ is a representation of A and C' is the image of C in A, i.e. $C' = \{(l \otimes \mathrm{id})(R) \mid l \in C\}$. To use this scheme one has to construct a quasitriangular (A, R) and as many representations of A as possible (these representations can be taken as ρ, and their characters belong to C). The Hopf algebras A used in the QISM are quantizations of Lie bialgebras G_S of Example 3.4 (i.e. of Lie bialgebras corresponding to nondegenerate solutions of the CYBE). Yangians and quantized affine algebras are typical examples. On these algebras there exists not a quasitriangular but a "pseudotriangular" structure (see §§12,13), but this suffices to use the above scheme.

Deeper aspects of the QISM, such as the description of the spectrum of $\rho^{\otimes N}(C')$ and the limit $N \to \infty$ (see [**65**, **36**, **71**]), are not yet understood from the Hopf algebra point of view.

12. The pseudotriangular structure on Yangians and rational solutions of the QYBE. Recall that $Y(\mathfrak{a})$ is obtained by quantizing the bialgebra $\mathfrak{g} = \mathfrak{a}[u]$ of Example 3.3 and setting $h = 1$. Since $\mathfrak{g}$ is pseudotriangular it is natural to expect $Y(\mathfrak{a})$ to be pseudotriangular too. To be more precise let us consider the operators $T_\lambda \colon \mathfrak{g} \to \mathfrak{g}$ given by $T_\lambda(f(u)) = f(u+\lambda)$. Note that though $r = t/(u-v)$ does not belong to $\mathfrak{g}\otimes\mathfrak{g}$, the expression

$$(T_\lambda \otimes \mathrm{id})r = t/(u+\lambda-v) = \sum_{k=0}^{\infty} t(v-u)^k \lambda^{-k-1}$$

is a power series in λ^{-1} whose coefficients belong to $\mathfrak{g}\otimes\mathfrak{g}$. Now define the automorphism $T_\lambda \colon Y(\mathfrak{a}) \to Y(\mathfrak{a})$ by the formulas $T_\lambda \mid_{\mathfrak{a}} = \mathrm{id}, T_\lambda(J(a)) = J(a)+\lambda a$ for $a \in \mathfrak{a}$. It is natural to expect that there exists a formal series

$$\mathcal{R}(\lambda) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k \lambda^{-1}, \qquad \mathcal{R}_k \in Y(\mathfrak{a}) \otimes Y(\mathfrak{a}),$$

which behaves as if $\mathcal{R}(\lambda)$ were equal to $(T_\lambda \otimes \mathrm{id})R$ for some R satisfying (20), (21) and the equation $R^{12}R^{21} = 1$. This is really so [**28**]. More precisely, there is a unique $\mathcal{R}(\lambda)$ such that $(\Delta\otimes\mathrm{id})\mathcal{R}(\lambda) = \mathcal{R}^{13}(\lambda)\mathcal{R}^{23}(\lambda)$ and $(T_\lambda\otimes\mathrm{id})\Delta'(a) = \mathcal{R}(\lambda)((T_\lambda\otimes\mathrm{id})\Delta(a))\mathcal{R}(\lambda)^{-1}$ for $a \in Y(\mathfrak{a})$. Besides, $\mathcal{R}(\lambda_1-\lambda_2)$ satisfies the QYBE, i.e.

$$\begin{aligned}\mathcal{R}^{12}(\lambda_1-\lambda_2)&\mathcal{R}^{13}(\lambda_1-\lambda_3)\mathcal{R}^{23}(\lambda_2-\lambda_3)\\ &= \mathcal{R}^{23}(\lambda_2-\lambda_3)\mathcal{R}^{13}(\lambda_1-\lambda_3)\mathcal{R}^{12}(\lambda_1-\lambda_2).\end{aligned}$$

Moreover, $\mathcal{R}^{12}(\lambda)\mathcal{R}^{21}(-\lambda) = 1$, $(T_\mu\otimes T_\nu)\mathcal{R}(\lambda) = \mathcal{R}(\lambda+\mu-\nu)$, and the coefficient $\mathcal{R}_1$ is equal to t.

By the way, we are lucky that $Y(\mathfrak{a})$ is pseudotriangular and not triangular: otherwise $Y(\mathfrak{a})$ would be isomorphic (as an algebra) to a universal enveloping algebra and life would be dull.

Given a representation $\rho\colon Y(\mathfrak{a}) \to \mathrm{Mat}(n, \mathbf{C})$ we obtain a matrix solution $R(\lambda_1-\lambda_2) = (\rho\otimes\rho)(\mathcal{R}(\lambda_1-\lambda_2))$ of the QYBE. It is of the form

$$R(\lambda_1-\lambda_2) = 1 + (\rho\otimes\rho)(t)\cdot(\lambda_1-\lambda_2)^{-1} + \sum_{k=2}^{\infty} A_k(\lambda_1-\lambda_2)^{-k}. \tag{23}$$

If ρ is irreducible then $R(\lambda)$ is a rational function up to a scalar factor [**28**].

Now let us temporarily forget about $Y(\mathfrak{a})$ and adopt the point of view of a person who wants to construct matrix solutions of the QYBE. According to [**45**] it is natural to look for solutions of the form

$$R(\lambda_1 - \lambda_2, h) = 1 + h(\rho \otimes \rho)(r(\lambda_1 - \lambda_2)) + \sum_{k=2}^{\infty} h^k a_k(\lambda_1 - \lambda_2),$$

where ρ is a representation of $\mathfrak{a}$ and r is an $\mathfrak{a} \otimes \mathfrak{a}$-valued solution of the CYBE. The simplest r is given by $r(\lambda) = t/\lambda$. Since this r is homogeneous it is natural to require that $R(\alpha\lambda, \alpha h) = R(\lambda, h)$. In this case $R(\lambda, h)$ is uniquely determined by $R(\lambda, 1)$, which is of the form (23). So the natural problem is to describe for a given $\rho\colon \mathfrak{a} \to \mathrm{Mat}(n, \mathbf{C})$ all the solutions of the QYBE of the form (23). Theorem [**28**]: *if ρ is irreducible then all such solutions are of the form $(\tilde{\rho} \otimes \tilde{\rho})(\mathcal{R}(\lambda_1 - \lambda_2))$ for some representation $\tilde{\rho}\colon Y(\mathfrak{a}) \to \mathrm{Mat}(n, \mathbf{C})$ such that $\tilde{\rho}|_{\mathfrak{a}} = \rho$.*

So it is clear that the problem of describing all the irreducible finite-dimensional representations of $Y(\mathfrak{a})$ is very important. It was solved for $\mathfrak{a} = \mathfrak{sl}(2)$ by V. E. Korepin and V. O Tarasov [**59**–**61**]. For classical $\mathfrak{a}$ specialists in the QISM have constructed by more or less elementary means a lot of R-matrices of the form (23) (see, for instance, [**62**, **63**] and the Appendix of [**45**]) and therefore a lot of representations of $Y(\mathfrak{a})$. Some results concerning this problem were obtained in [**28**, **64**]. In [**54**] a parametrization of the set of irreducible finite-dimensional representations of $Y(\mathfrak{a})$ (in the spirit of Cartan's highest weight theorem) is obtained using the second presentation of $Y(\mathfrak{a})$ (see §6) and the results for $Y(\mathfrak{sl}(2))$.

$Y(\mathfrak{a})$ has two "large" commutative subalgebras: the "Cartan" subalgebra generated by φ_{ik} (see §6) and the subalgebra $\mathcal{A}$ generated by the coefficients of $(l \otimes \mathrm{id})\mathcal{R}(\lambda)$, $l \in C$, where $C = \{l \in (Y(\mathfrak{a}))^* |\ l(ab) = l(ba)$ for any $a, b \in Y(\mathfrak{a})\}$. It is desirable to describe the spectra of these subalgebras in irreducible representations of $Y(\mathfrak{a})$. For $\mathcal{A}$ this problem is especially important (see §11). For classical $\mathfrak{a}$ and at least some representations of $Y(\mathfrak{a})$ the spectrum of $\mathcal{A}$ is known [**65**–**68**]. The analogy between the structure of the Bethe equations and the corresponding Cartan matrices (N. Yu. Reshetikhin [**68**]) suggests that the spectrum of $\mathcal{A}$ can be described for all $\mathfrak{a}$.

Finally, I am going to mention a realization of $Y(\mathfrak{a})$ which is often useful and which appeared much earlier than the general definition of $Y(\mathfrak{a})$. Fix a nontrivial irreducible representation $\rho\colon Y(\mathfrak{a}) \to \mathrm{Mat}(n, \mathbf{C})$. Set $R(\lambda) = (\rho \otimes \rho)(\mathcal{R}(\lambda))$. Then $Y(\mathfrak{a})$ is almost isomorphic to the Hopf algebra A_ρ with generators $t_{ij}^{(k)}$, $1 \le i, j \le n$, $k = 1, 2, \ldots$, defining relations

$$\tilde{T}(\lambda)\tilde{\tilde{T}}(\mu)R(\mu - \lambda) = R(\mu - \lambda)\tilde{\tilde{T}}(\mu)\tilde{T}(\lambda)$$

and comultiplication

$$\Delta(t_{ij}(\lambda)) = \sum_k t_{ik}(\lambda) \otimes t_{kj}(\lambda),$$

where $t_{ij}(\lambda) = \delta_{ij} + \sum_{k=1}^{\infty} t_{ij}^{(k)} \lambda^{-k}$, $T(\lambda) = (t_{ij}(\lambda))$, $\tilde{T}(\lambda) = T(\lambda) \otimes E_n$, $\tilde{\tilde{T}}(\lambda) = E_n \otimes T(\lambda)$. More precisely, there is an epimorphism $A_\rho \to Y(\mathfrak{a})$ given by $T(\lambda) \mapsto (\mathrm{id} \otimes \rho)(\mathcal{R}(\lambda))$, and to obtain $Y(\mathfrak{a})$ from A_ρ one has to add an auxiliary relation of the form $c(\lambda) = 1$, where $c(\lambda) \in A_\rho[[\lambda^{-1}]]$, $\Delta(c(\lambda)) = c(\lambda) \otimes c(\lambda)$, and $[a, c(\lambda)] = 0$ for any $a \in A_\rho$. For instance, if $\mathfrak{a} = \mathfrak{sl}(n)$ and $\rho\colon Y(\mathfrak{a}) \to \mathrm{Mat}(n, \mathbf{C})$ is the "vector" representation then $c(\lambda)$ is the "quantum determinant" of $T(\lambda)$ in the sense of [**73**] and $R(\lambda)$ is proportional to the Yang R-matrix $1 + \lambda^{-1}\sigma$, where $\sigma\colon \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n$, $\sigma(x \otimes y) = y \otimes x$.

13. The double. Trigonometric solutions of the QYBE. If $\mathfrak{g}$ is a Lie bialgebra denote by $D(\mathfrak{g})$ the space $\mathfrak{g} \oplus \mathfrak{g}^*$ with the Lie algebra structure defined in §3. It can be shown that (1) the image r of the canonical element of $\mathfrak{g} \otimes \mathfrak{g}^*$ under the embedding $\mathfrak{g} \otimes \mathfrak{g}^* \hookrightarrow D(\mathfrak{g}) \otimes D(\mathfrak{g})$ satisfies (7) and therefore $(D(\mathfrak{g}), r)$ is a quasitriangular Lie bialgebra, and (2) the embedding $\mathfrak{g} \hookrightarrow D(\mathfrak{g})$ is a Lie bialgebra homomorphism and the embedding $\mathfrak{g}^* \hookrightarrow D(\mathfrak{g})$ is an algebra homomorphism and coalgebra antihomomorphism. $D(\mathfrak{g})$ is called the *double* of $\mathfrak{g}$. This construction has important applications [**43**].

Now let us define the *quantum double.* Let A be a Hopf algebra. Denote by A° the algebra A^* with the opposite comultiplication. It can be shown that there is a unique quasitriangular Hopf algebra $(D(A), R)$ such that (1) $D(A)$ contains A and A° as Hopf subalgebras, (2) R is the image of the canonical element of $A \otimes A^\circ$ under the embedding $A \otimes A^\circ \hookrightarrow D(A) \otimes D(A)$, and (3) the linear mapping $A \otimes A^\circ \to D(A)$ given by $a \otimes b \mapsto ab$ is bijective. As a vector space, $D(A)$ can be identified with $A \otimes A^\circ$, and the Hopf algebra structure on $A \otimes A^\circ$ can be found from the commutation relations

$$e_s e^t = \mu_s^{kjn} m_{plk}^t \sigma_n^p e^l e_j \quad \text{and} \quad e^t e_s = \mu_s^{njk} m_{klp}^t \sigma_n^p e_j e^l,$$

which follow from (1)–(3). Here $\{e_s\}$ and $\{e^t\}$ are dual bases of A and A°, while σ, m, and μ are the matrices of the skew antipode $A \to A$, the multiplication map $A \otimes A \otimes A \to A$, and the comultiplication map $A \to A \otimes A \otimes A$. If A is a QUE-algebra we will understand A° in the QUE-sense. Then $D(A)$ is a QUE-algebra, too. If $\mathfrak{g}$ is the classical limit of A then $(D(\mathfrak{g}), r)$ is the classical limit of $(D(A), R)$

Let $\mathfrak{g}, \mathfrak{b}_+, \mathfrak{b}_-, \mathfrak{h}$ denote the same objects as in Example 3.2. Then $D(\mathfrak{b}_+) = \mathfrak{g} \times \mathfrak{h}$ with the trivial Lie bialgebra structure on $\mathfrak{h}$. The quasitriangular (or pseudoquasitriangular) structure on $\mathfrak{g}$ (see §4) is defined by the image of $r \in D(\mathfrak{b}_+) \otimes D(\mathfrak{b}_+)$ under the natural mapping $D(\mathfrak{b}_+) \otimes D(\mathfrak{b}_+) \to \mathfrak{g} \otimes \mathfrak{g}$. It can be shown that in the quantum case the situation is quite similar. Denote by $U_h\mathfrak{b}_+$ the subalgebra of $U_h\mathfrak{g}$ generated by $\mathfrak{h}$ and X_i^+. Then $U_h\mathfrak{b}_- = (U_h\mathfrak{b}_+)^\circ$ and $D(U_h\mathfrak{b}_+) = U_h\mathfrak{g} \otimes U\mathfrak{h}$. Denote by $\overline{R}$ the image of $R \in D(U_h\mathfrak{b}_+) \otimes D(U_h\mathfrak{b}_+)$ under the natural mapping $D(U_h\mathfrak{b}_+) \otimes D(U_h\mathfrak{b}_+) \to U_h\mathfrak{g} \otimes U_h\mathfrak{g}$. Then $\overline{R}$ defines a quasitriangular structure on $U_h\mathfrak{g}$ if $\dim \mathfrak{g} < \infty$ and a "pseudoquasitriangular" structure if $\dim \mathfrak{g} = \infty$.

If $\mathfrak{g} = \mathfrak{sl}(2)$ with the scalar product $\mathrm{Tr}(XY)$ then

$$\overline{R} = \sum_{k=0}^{\infty} h^k Q_k(h) \{\exp \frac{h}{4}[H \otimes H + k(H \otimes 1 - 1 \otimes H)]\} (X^+)^k \otimes (X^-)^k,$$

where $Q_k(h) = e^{-kh/2}\prod_{r=1}^{k}(e^h-1)/(e^{rh}-1)$. In general,

$$\overline{R} = \sum_{\beta\in\mathbf{Z}_+^r}\left\{\exp h\left[\frac{1}{2}t_0 + \frac{1}{4}(H_\beta\otimes 1 - 1\otimes H_\beta)\right]\right\}P_\beta,$$

where t_0 is the element of $\mathfrak{h}\otimes\mathfrak{h}$ corresponding to the canonical scalar product on $\mathfrak{h}$, $\mathbf{Z}_+ = \{n\in\mathbf{Z} \mid n\geq 0\}$, r is the rank of $\mathfrak{g}$, for every $\beta = (\beta_1, \beta_2, \dots)\in\mathbf{Z}^r$ the symbol H_β denotes $\sum_i \beta_i H_i$, and $P_\beta \in U_h\mathfrak{b}_+\otimes U_h\mathfrak{b}_-$ is a polynomial in $u_i = X_i^+\otimes 1$ and $v_i = 1\otimes X_i^-$ homogeneous with respect to each variable and such that $\deg_{u_i} P_\beta = \det_{v_i} P_\beta = \beta_i$. In addition: (1) $P_0 = 1$, (2) $P_\beta \equiv 0 \bmod h^{k_\beta}$, $P_\beta \not\equiv 0 \bmod h^{k_\beta - 1}$, where $k_\beta = \inf\{n \mid H_\beta$ is representable as a sum of n positive roots of $\mathfrak{g}\}$, and (3) the coefficients of $h^{-|\beta|}P_\beta$ are rational functions of e^{ch} for $c\in\mathbf{C}$, where $|\beta| = \sum_i\beta_i$.

If $\dim\mathfrak{g}<\infty$ then $R_\rho \stackrel{\text{def}}{=} (\rho\otimes\rho)(\overline{R})$ makes sense and satisfies the QYBE for any representation $\rho\colon U_h\mathfrak{g}\to\mathrm{Mat}(n, \mathbf{C}[[h]])$. For instance, if $\mathfrak{g} = \mathfrak{sl}(n)$ and ρ is the representation mentioned in §7 then

$$R_\rho = e^{-h/2n}\left\{\sum_{i\neq j} e_{ii}\otimes e_{jj} + e^{h/2}\sum_i e_{ii}\otimes e_{ii} + (e^{h/2}-e^{-h/2})\sum_{i<j} e_{ij}\otimes e_{ji}\right\},$$

where e_{ij} are the matrix units (this formula is implicit in [**52**]). Note that the relations (16)–(19) simply mean that $R_\rho\tilde{T}\tilde{\tilde{T}} = \tilde{\tilde{T}}\tilde{T}R_\rho$, where $T = (\rho_{ij})$.

Now let $\mathfrak{g}$ be affine and set $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]/(\text{the center of } \mathfrak{g})$. Denote by R' the analogue of $\overline{R}$ for $U_h\mathfrak{g}'$. Suppose that a representation $\rho\colon U_h\mathfrak{g}'\to\mathrm{Mat}(n, \mathbf{C}[[h]])$ is given. Strictly speaking, the expression $(\rho\otimes\rho)(R')$ is meaningless. But if we define $T_\lambda\in\mathrm{Aut}\, U_h\mathfrak{g}$ setting $T_\lambda(X_i^\pm) = \lambda^{\pm 1}X_i^\pm$ and put $R'(\lambda) = (T_\lambda\otimes\mathrm{id})R'$ then $R_\rho(\lambda) \stackrel{\text{def}}{=} (\rho\otimes\rho)(R'(\lambda))$ is a well-defined formal series in λ such that

$$R_\rho^{12}(\lambda_1/\lambda_2)R_\rho^{13}(\lambda_1/\lambda_3)R_\rho^{23}(\lambda_2/\lambda_3) = R_\rho^{23}(\lambda_2/\lambda_3)R_\rho^{13}(\lambda_1/\lambda_3)R_\rho^{12}(\lambda_1/\lambda_2).$$

Using an idea of M. Jimbo [**50**] it can be proved that if ρ is irreducible then $R_\rho(\lambda)$ is a rational function of λ up to a scalar factor. Moreover, for a given ρ this rational function can be found by solving certain linear equations [**50**]. Substituting $\lambda = e^u$ we obtain trigonometric solutions of the QYBE. Explicit formulas for the solutions corresponding to the classical $\mathfrak{g}$ and some ρ were found by elementary methods [**48**, **74**–**77**].

14. Final remarks. Because of the lack of space I cannot discuss here the relation between $Y(\mathfrak{sl}(n)), U_h\mathfrak{sl}(n)$ and Hecke algebras (see [**52**, **64**]), between Yangians and Hermitian symmetric spaces (see [**62**] and [**28**, Theorem 7]), between $Y(\mathfrak{sl}(n))$ and representations of infinite-dimensional classical groups (see [**80**]). By the same reason I cannot discuss compact quantum groups [**81**–**83**] as well as the attempts to generalize the notion of superalgebra, which are based on the QYBE [**78**] or triangular Hopf algebras [**79**].

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