

Drinfeld Centers

*n*Lab materials

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Abstract

A reference account of Drinfeld centers of monoidal categories in general and of group-graded vector spaces with cocycle-twisted associator in particular, with special attention to the realization of the latter by pull-push operations on local systems over free loop spaces.

associated *n*Lab entry: <https://ncatlab.org/nlab/show/Drinfeld+center>

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1. Idea

The *Drinfeld center*, or *monoidal center*, of a monoidal category $\mathcal{C}$ has as its objects pairs (V, γ) , consisting of an object V of $\mathcal{C}$ together with a half-braiding γ : a coherent family of isomorphisms that commute V past every object of $\mathcal{C}$. For a monoid, centrality is the condition $xz = zx$ for all x . For a monoidal category, equality is replaced by a specified natural isomorphism

$$\gamma_X : X \otimes V \xrightarrow{\cong} V \otimes X,$$

called a *half-braiding*. Compatibility with tensor products says that commuting past $X \otimes Y$ agrees with commuting past Y and then X , with the associators inserted. Centrality is therefore additional structure on V , and morphisms must preserve that structure. The center $\mathcal{Z}(\mathcal{C})$ is canonically braided even when $\mathcal{C}$ has no chosen braiding [JS91; Maj91, Example 3.4].

For a finite group G , an important example is $\mathcal{C} = \text{Vec}_G$, the category of finite-dimensional complex G -graded vector spaces with the convolution tensor product. Its center can be described by graded vector spaces with a compatible conjugation action of G . Replacing the usual associator by a scalar group 3-cocycle ω gives Vec_G^ω and hence the *twisted Drinfeld center* $\mathcal{Z}(\text{Vec}_G^\omega)$. The degree of the cocycle matters: the monoidal category is twisted by a 3-cocycle, while the associated projective representations of centralizers involve transgressed 2-cocycles [Eti+15, Example 8.5.4; Wil08, § 3].

Historically, the construction is connected with Drinfeld’s quantum double of a Hopf algebra, described in his 1986 ICM lecture, published in 1987 [Dri87]. The categorical center is attributed to Drinfeld’s unpublished work, with published constructions by Joyal–Street and Majid in 1991 [JS91; Maj91, Example 3.4]. This attribution is also recorded in the historical notes of [Eti+15, pp. 190, 261]. The relationship is precise: for a finite-dimensional Hopf algebra H , its quantum double has a representation category braided equivalent to the center of $\text{Rep}(H)$ [Eti+15, Example 8.5.6].

For finite groups, Dijkgraaf–Pasquier–Roche introduced the quasi-Hopf algebras now called twisted quantum doubles, built from a group and a scalar 3-cocycle [DPR91]. Willerton identified their underlying algebras with twisted algebras of inertia groupoids and displayed the associated transgression formula [Wil08]. A detailed derivation of that formula from homotopy-theoretic looping, using simplicial evaluation and the Eilenberg–Zilber map, is given by Sati–Schreiber [SS24, arXiv v1, § 3, Lem. 3.3 and Thm. 3.4]. Ganter–Usher related this description to inertia groupoids of categorical groups [GU16, arXiv version, Prop. 3.5 and Cor. 4.16]. Complementary treatments of the tensor and coherence data appear in Galindo–Gutiérrez–Uribe’s cochain description [GGU15, §§ 3.5, 4] and in Morton’s linearization of cocycle-decorated spans [Mor15, arXiv version, §§ 5.4–5.5].

Sati–Schreiber explicitly relate the two-punctured-disk mapping-space correspondence to untwisted Drinfeld-center fusion when $\pi_2(\mathcal{A})$ is trivial [SS26b, § 3.2 and Lem. A.6]. This is a direct precedent for the untwisted surface realization below. For finite 2-groups with arbitrary action, Huang–Xu–Zhang identify loop-groupoid functors with centers of 2-group algebras [HXZ24, arXiv v3, Cor. 3.20]; Huang–Zhang compute the associated character-stabilizer categories [HZ23, arXiv v4, § 3.3, Prop. 3.19].

This reference account develops the definitions and their historical background, and then proves the homotopical realization in § 3.1. That statement constructs the braided fusion category for each orbit of characters of the second homotopy group, and identifies it with a

twisted center on the character stabilizer. The trivial-action and untwisted cases are recorded in Cor. 3.2. The comparison with earlier descriptions is made explicit in the proof and in Rem. 3.4.

2. Definition

2.1. The Drinfeld center of a monoidal category

Fix a locally small monoidal category

$$(\mathcal{C}, \otimes, \mathbf{1}, a, l, r).$$

We use the directions

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z), \quad l_X : \mathbf{1} \otimes X \xrightarrow{\cong} X, \quad r_X : X \otimes \mathbf{1} \xrightarrow{\cong} X.$$

The associator and unitors satisfy the pentagon and triangle axioms. Composition is written right to left. All categories may be taken in a fixed universe, enlarged for the center if necessary. No braiding, linearity, semisimplicity, or rigidity of $\mathcal{C}$ is assumed. Our convention is the one in [Eti+15, Def. 7.13.1]; original constructions are in [JS91; Maj91, Example 3.4].

Definition 2.1 (Half-braiding). A *half-braiding* on $V \in \mathcal{C}$ is a natural isomorphism of functors

$$\gamma : (-) \otimes V \Longrightarrow V \otimes (-).$$

Thus, for every morphism $u : X \rightarrow Y$,

$$(\text{id}_V \otimes u) \circ \gamma_X = \gamma_Y \circ (u \otimes \text{id}_V). \quad (1)$$

We require compatibility with the tensor product:

$$\begin{aligned} \gamma_{X \otimes Y} &= a_{V,X,Y} \circ (\gamma_X \otimes \text{id}_Y) \circ a_{X,V,Y}^{-1} \\ &\quad \circ (\text{id}_X \otimes \gamma_Y) \circ a_{X,Y,V}, \end{aligned} \quad (2)$$

as maps $(X \otimes Y) \otimes V \rightarrow V \otimes (X \otimes Y)$, and compatibility with the unit:

$$\gamma_{\mathbf{1}} = r_V^{-1} \circ l_V. \quad (3)$$

For clarity, the right-hand side of (2) is the following composite:

$$\begin{aligned} (X \otimes Y) \otimes V &\xrightarrow{a_{X,Y,V}} X \otimes (Y \otimes V) \xrightarrow{\text{id}_X \otimes \gamma_Y} X \otimes (V \otimes Y) \\ &\xrightarrow{a_{X,V,Y}^{-1}} (X \otimes V) \otimes Y \xrightarrow{\gamma_X \otimes \text{id}_Y} (V \otimes X) \otimes Y \xrightarrow{a_{V,X,Y}} V \otimes (X \otimes Y). \end{aligned}$$

In a strict monoidal category this reduces to $\gamma_{X \otimes Y} = (\gamma_X \otimes \text{id}_Y) \circ (\text{id}_X \otimes \gamma_Y)$.

Definition 2.2 (Drinfeld center). The *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ has objects (V, γ) with γ a half-braiding on V . A morphism $f : (V, \gamma) \rightarrow (W, \eta)$ is a morphism $f : V \rightarrow W$ in $\mathcal{C}$ such that, for every $X \in \mathcal{C}$,

$$(f \otimes \text{id}_X) \circ \gamma_X = \eta_X \circ (\text{id}_X \otimes f). \quad (4)$$

Identities and composition are those of $\mathcal{C}$.

Definition 2.3 (Tensor product and braiding of the center). For $(V, \gamma), (W, \eta) \in \mathcal{Z}(\mathcal{C})$, set

$$(V, \gamma) \otimes (W, \eta) = (V \otimes W, \gamma \star \eta),$$

where

$$(\gamma \star \eta)_X = a_{V,W,X}^{-1} \circ (\text{id}_V \otimes \eta_X) \circ a_{V,X,W} \circ (\gamma_X \otimes \text{id}_W) \circ a_{X,V,W}^{-1}. \quad (5)$$

This is a map $X \otimes (V \otimes W) \rightarrow (V \otimes W) \otimes X$. The tensor product of morphisms is the one in $\mathcal{C}$. The unit object is $(\mathbf{1}, \epsilon)$, with

$$\epsilon_X = l_X^{-1} \circ r_X : X \otimes \mathbf{1} \longrightarrow \mathbf{1} \otimes X.$$

The associator and unitors are inherited from $\mathcal{C}$. The braiding is

$$c_{(V,\gamma),(W,\eta)} = \eta_V : V \otimes W \longrightarrow W \otimes V. \quad (6)$$

These formulas equip $\mathcal{Z}(\mathcal{C})$ with a braided monoidal structure [JS91; Eti+15, Prop. 8.5.1]. They also specify the faithful strong monoidal forgetful functor $(V, \gamma) \mapsto V$. In particular, both half-braidings enter the tensor product, while the braiding in (6) uses the half-braiding of its second object evaluated on its first.

Remark 2.4 (Direction conventions). Some authors use half-braidings $V \otimes X \rightarrow X \otimes V$. Inverting the maps converts between the two directions, but the formula for the chosen braiding must be converted as well. We fix (2) and (6) throughout; these conventions will also govern later cocycle comparisons.

2.2. The untwisted group-graded case

Definition 2.5 (Group-graded vector spaces). Let G be a discrete group with identity e . Write Vec_G for the category of finite-dimensional complex vector spaces

$$V = \bigoplus_{g \in G} V_g$$

and degree-preserving linear maps. Finite-dimensionality implies that only finitely many V_g are nonzero. Its tensor product has grading

$$(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes_{\mathbb{C}} W_b. \quad (7)$$

The unit is δ_e , where δ_g denotes a copy of $\mathbb{C}$ concentrated in degree g . The associator and unitors are the usual ones for vector spaces. Its *untwisted Drinfeld center* is $\mathcal{Z}(\text{Vec}_G)$ with the structure of Def. 2.3.

For finite G , Vec_G is a fusion category (Def. A.2): it is a rigid, semisimple $\mathbb{C}$ -linear monoidal category with simple unit, finite-dimensional morphism spaces, and finitely many simple isomorphism classes. Its simple objects are the δ_g , and the dual of δ_g is $\delta_{g^{-1}}$. The definitions also make sense for infinite G , but the finiteness condition for a fusion category then fails.

Example 2.6 (Equivariant description). For Vec_G , a half-braiding is determined by its values on the objects δ_x . Writing

$$\gamma_{\delta_x}(1_x \otimes v) = \rho_x(v) \otimes 1_x, \quad v \in V_g,$$

preservation of degree gives

$$\rho_x : V_g \xrightarrow{\cong} V_{xgx^{-1}}.$$

The half-braiding axioms become $\rho_{xy} = \rho_x \rho_y$ and $\rho_e = \text{id}$. Conversely, these identities extend the displayed formula by linearity to a half-braiding on every graded vector space.

Thus the center consists of G -graded vector spaces with a G -action carrying V_g to $V_{xgx^{-1}}$, and equivariant degree-preserving maps. Under this description the tensor product uses (7) with diagonal G -action, and

$$c_{V,W}(v_g \otimes w_h) = \rho_g^W(w_h) \otimes v_g.$$

This is the standard description by equivariant bundles on G under conjugation, with finite-dimensional fibers and finite support; compare [Eti+15, Example 8.5.4].

2.3. The cocycle-twisted group-graded case

Definition 2.7 (Normalized group 3-cocycle). A normalized 3-cocycle with trivial coefficient action is a function $\omega : G^3 \rightarrow \mathbb{C}^\times$ satisfying

$$\omega(h, k, \ell) \omega(g, hk, \ell) \omega(g, h, k) = \omega(gh, k, \ell) \omega(g, h, k\ell) \quad (8)$$

for all $g, h, k, \ell \in G$, and

$$\omega(e, g, h) = \omega(g, e, h) = \omega(g, h, e) = 1. \quad (9)$$

The set of such cocycles is denoted $Z^3(G; \mathbb{C}^\times)$, with normalization understood.

Definition 2.8 (Twisted graded category and its center). For such a cocycle, Vec_G^ω has the same objects, morphisms, tensor bifunctor, and unit object as Vec_G , but associator

$$a_{U,V,W}^\omega((u_g \otimes v_h) \otimes w_k) = \omega(g, h, k) u_g \otimes (v_h \otimes w_k) \quad (10)$$

on homogeneous vectors. The unitors remain the usual ones. The ω -twisted Drinfeld center of G -graded vector spaces means

$$\mathcal{Z}_\omega(G) := \mathcal{Z}(\text{Vec}_G^\omega). \quad (11)$$

Explicitly, its half-braidings obey (1) to (3) with $a = a^\omega$. Its morphisms, tensor product, and braiding are given by (4) to (6) with that same substitution. In particular its associator is the map (10) on underlying graded vector spaces.

The two routes around the pentagon on degrees (g, h, k, ℓ) multiply by the two sides of (8); thus the cocycle identity is precisely the pentagon condition. Normalization makes the triangle condition hold with the ordinary unitors. This construction is recalled in [Eti+15, Example 2.3.8], and the corresponding twisted-double algebra originates in [DPR91]. Setting $\omega = 1$ recovers Def. 2.5 and its center.

Remark 2.9 (Meaning of the twist). Here the twist changes the associator of the input monoidal category, after which one takes its ordinary Drinfeld center. It is not an extra 2-cocycle in the definition of the center. The centralizer 2-cocycles arise when this category is subsequently described in representation-theoretic terms [Wil08]. Nor does the term refer here to a center twisted by a chosen monoidal endofunctor, which is a different use of the terminology.

3. Properties

3.1. Homotopical Pull-Push Realization

We give a purely homotopy-theoretic realization of twisted Drinfeld centers (Thm. 3.1), equivalently as categories of local systems (flat vector bundles) on free loop spaces and

equipped with the fusion product given by the natural pull-tensor-push integral transformation through the pair-of-pants cobordism. Aspects of this statement may be recognized scattered in the literature (cf. Rem. 3.4) but the full and explicit statement we give in Thm. 3.1 may be new as such. It is the evident (if more tedious to prove) twisted generalization of the presentation given in [SS26b].

3.1.1. Preliminaries

The homotopy data. We consider

$$\mathcal{A} \in \text{Top}^* \quad (12)$$

a pointed homotopy type with

- (i) $\pi_0(\mathcal{A}) \simeq *$ ($\mathcal{A}$ is connected),
- (ii) $\pi_1(\mathcal{A})$ is finite.

The abelian group $\pi_2(\mathcal{A})$ may be infinite, and its canonical $\pi_1(\mathcal{A})$ -action may be nontrivial.

The underlying 2-type and its k-invariant. The following constructions and Thm. 3.1 depend only on the Postnikov 2-truncation $[\mathcal{A}]_2$ of $\mathcal{A}$ (12), as proved in Lem. B.3. No vanishing of $\pi_3(\mathcal{A})$ or of higher homotopy groups is assumed: the construction is performed on $L\mathcal{A}$ itself and agrees with that for $L[\mathcal{A}]_2$. By connectedness (item (i)) this is equivalently the delooping of a 2-group

$$[\mathcal{A}]_2 \simeq B\mathcal{G}. \quad (13)$$

A skeletal model of the 2-group $\mathcal{G}$ has:

- (i) objects $g \in \pi_1(\mathcal{A})$
- (ii) automorphism groups $\pi_2(\mathcal{A})$ at every object,
with no morphisms between distinct objects,
- (iii) tensor product

$$g \otimes h = gh, \quad (a : g \rightarrow g) \otimes (b : h \rightarrow h) = (a + g \cdot b : gh \rightarrow gh).$$

We fix the *Postnikov sign convention* of recording the inverse associator by a normalized cocycle

$$k \in Z^3(\pi_1(\mathcal{A}); \pi_2(\mathcal{A})). \quad (14)$$

The coefficient action in this group cohomology is the given action. Thus

$$g \cdot k(h, j, \ell) - k(gh, j, \ell) + k(g, hj, \ell) - k(g, h, j\ell) + k(g, h, j) = 0. \quad (15)$$

The arrow $g \otimes (h \otimes j) \rightarrow (g \otimes h) \otimes j$ has label $k(g, h, j)$; the forward associator has label $-k(g, h, j)$. This convention fixes the signs of the scalar twists below.

Fix a character

$$\pi_2(\mathcal{A}) \xrightarrow{\phi} \mathbb{C}^\times, \quad \phi(a + b) = \phi(a)\phi(b). \quad (16)$$

The character group and its action are

$$\widehat{\pi_2(\mathcal{A})} := \text{Hom}(\pi_2(\mathcal{A}), \mathbb{C}^\times), \quad (x \cdot \phi)(a) := \phi(x^{-1} \cdot a).$$

Write $[\phi]$ for the orbit of ϕ , and define its stabilizer by

$$\text{Stab}(\phi) := \{x \in \pi_1(\mathcal{A}) : x \cdot \phi = \phi\}. \quad (17)$$

Both the orbit and its stabilizer are finite because $\pi_1(\mathcal{A})$ is finite. On this stabilizer the function

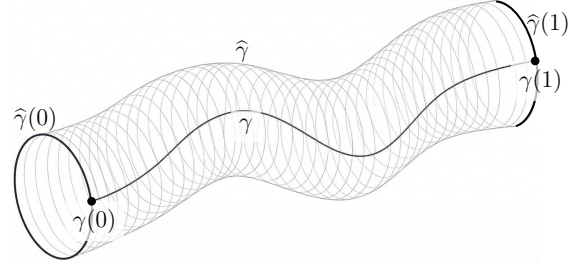
$$\omega_\phi := (\phi \circ k)|_{\text{Stab}(\phi)^3} \in Z^3(\text{Stab}(\phi); \mathbb{C}^\times) \quad (18)$$

is an ordinary scalar cocycle: applying ϕ to (15) removes the coefficient action. Unless ϕ is invariant under the whole fundamental group, $\phi \circ k$ need not be a scalar cocycle on that whole group.

The categories on the loop space. Write $L\mathcal{A} := \text{Map}(S^1, \mathcal{A})$ for the free loop space.

Its connected components are indexed by conjugacy classes in $\pi_1(\mathcal{A})$, cf. Fig. 1.

Figure 1: A path $\hat{\gamma}$ in a free loop space $L\mathcal{A}$ and the path γ traced out by the basepoint of the loops in $\mathcal{A}$. The loop $\hat{\gamma}(1)$ is based-homotopic to the conjugate concatenation of $\hat{\gamma}(0)$ by γ .



For a loop γ_g representing g , evaluation at the basepoint gives an extension

$$1 \longrightarrow \frac{\pi_2(\mathcal{A})}{(g-1)\pi_2(\mathcal{A})} \longrightarrow \pi_1(L\mathcal{A}, \gamma_g) \longrightarrow Z_{\pi_1(\mathcal{A})}(g) \longrightarrow 1. \quad (19)$$

The centralizer acts on its kernel through the given action on $\pi_2(\mathcal{A})$; the extension need not be central. Let ι_g denote the composite from $\pi_2(\mathcal{A})$ to this kernel and then into the middle group. See Rem. B.5.

For a local system $\mathcal{F}$, define its character subspace at a representative loop by

$$\mathcal{F}_{g,\psi} := \{v \in \mathcal{F}_{\gamma_g} : \text{mon}_{\mathcal{F}}(\iota_g(a))v = \psi(a)v \text{ for all } a \in \pi_2(\mathcal{A})\}. \quad (20)$$

It can be nonzero only if $g \cdot \psi = \psi$. Transport whose marked-point path represents x carries $\mathcal{F}_{g,\psi}$ to $\mathcal{F}_{xgx^{-1}, x \cdot \psi}$.

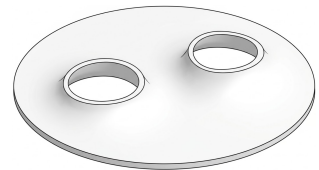
Define $\text{Loc}_{[\phi]}^{\text{fd}}(L\mathcal{A})$ to be the full subcategory of finite-dimensional local systems satisfying

$$\mathcal{F}_{\gamma_g} = \bigoplus_{\psi \in [\phi]} \mathcal{F}_{g,\psi} \quad (g \in \pi_1(\mathcal{A})). \quad (21)$$

This is an intrinsic condition because transport permutes the indicated character spaces. For infinite $\pi_2(\mathcal{A})$ the equality is an actual semisimplicity requirement, not a condition merely on generalized eigencharacters. If $[\phi] = \{\phi\}$, we also write $\text{Loc}_{\phi}^{\text{fd}}(L\mathcal{A})$.

The surface correspondence. Let P be an oriented disk with the interiors of two disjoint disks removed (cf. Fig. 2), with labeled and parametrized boundary circles. Choose connecting arcs and incoming loop conventions so that the outgoing based word is $g_1 g_2$.

Figure 2: Homotopical realization of the fusion product is given by pull-push through the cobordism known as the *trinion* or *pair-of-pants*, which may be understood as the process of a pair of circles approaching each other, touching, then merging to a single circle.



Boundary restriction gives

$$L\mathcal{A} \times L\mathcal{A} \xleftarrow{p} \text{Map}(P, \mathcal{A}) \xrightarrow{q} L\mathcal{A}. \quad (22)$$

For local systems define

$$\mathcal{F} \star \mathcal{G} := q_! p^*(\mathcal{F} \boxtimes \mathcal{G}). \quad (23)$$

Here $\boxtimes$ is external tensor product and $q_!$ is ordinary left Kan extension on fundamental groupoids, initially with values in all complex vector spaces. The relation to the general mapping-space pull-push for ∞ -local systems is explained in Rem. B.2.

Let $d : \text{Map}(D^2, \mathcal{A}) \rightarrow L\mathcal{A}$ be boundary restriction. Let $Q_{[\phi]}$ be the maximal quotient satisfying the character decomposition for $[\phi]$, defined in Def. B.6, and put

$$\mathbf{1}_{[\phi]} := Q_{[\phi]}(d_! \mathbb{C}). \quad (24)$$

It is supported on the constant-loop component and has one character line for every $\psi \in [\phi]$, permuted by $\pi_1(\mathcal{A})$. In particular it has finite-dimensional fibers even when $\pi_2(\mathcal{A})$ is infinite. For a singleton orbit write $\mathbf{1}_\phi$. The geometric associator, unitors, and braiding use gluing and the chosen exchange of incoming holes; see Rem. B.9.

3.1.2. Realization Theorem

With this language in hand, we have the following transparent homotopical realization of twisted Drinfeld centers:

Theorem 3.1 (Homotopical pull-push realization of Drinfeld centers). *For $\mathcal{A}$ (12) with k (14), ϕ (16) and ω_ϕ (18) as above:*

- (i) *The pull-tensor-push functor (23) preserves the orbit sector (21). With the geometric constraints and unit (24) it defines a braided fusion category*

$$(\text{Loc}_{[\phi]}^{\text{fd}}(L\mathcal{A}), \star, \mathbf{1}_{[\phi]}) \in \text{BrFusCat}. \quad (25)$$

- (ii) *Taking the ϕ -character spaces gives a braided monoidal equivalence*

$$\boxed{(\text{Loc}_{[\phi]}^{\text{fd}}(L\mathcal{A}), \star, \mathbf{1}_{[\phi]}) \simeq \mathcal{Z}(\text{Vec}_{\text{Stab}(\phi)}^{\omega_\phi})}. \quad (26)$$

It identifies the surface product, disk unit, associator, and exchange of holes with the corresponding structures of the center.

- (iii) *If $\pi_2(\mathcal{A})$ is finite, the entire category is braided multifusion, with unit $d_! \mathbb{C}$, and*

$$(\text{Loc}^{\text{fd}}(L\mathcal{A}), \star) \simeq \bigoplus_{[\phi] \in \widehat{\pi_2(\mathcal{A})}/\pi_1(\mathcal{A})} \mathcal{Z}(\text{Vec}_{\text{Stab}(\phi)}^{\omega_\phi}). \quad (27)$$

The sum uses one representative per character orbit; tensor products between distinct summands vanish.

The proof is given in § 3.1.3 below. Here we first highlight the main special cases (Cor. 3.2) and where aspects of Thm. 3.1 may be recognized in the literature (Rem. 3.4).

Corollary 3.2 (Homotopical realization for trivial action and in the untwisted case).

- (i) *If $\pi_1(\mathcal{A})$ acts trivially on $\pi_2(\mathcal{A})$, every character orbit is a singleton and $\text{Stab}(\phi) = \pi_1(\mathcal{A})$. The equivalence (26) restricts to*

$$(\text{Loc}_\phi^{\text{fd}}(L\mathcal{A}), \star, \mathbf{1}_\phi) \simeq \mathcal{Z}(\text{Vec}_{\pi_1(\mathcal{A})}^{\phi \circ k}). \quad (28)$$

The loop extension (19) is then central with kernel $\pi_2(\mathcal{A})$. No finiteness assumption on $\pi_2(\mathcal{A})$ is needed.

- (ii) If moreover $\pi_2(\mathcal{A}) = 0$, the unique character and Postnikov cocycle are trivial, the sector is the whole category, and

$$(\mathrm{Loc}^{\mathrm{fd}}(L\mathcal{A}), \star, d! \mathbb{C}) \simeq \mathcal{Z}(\mathrm{Vec}_{\pi_1(\mathcal{A})}). \quad (29)$$

Here $\pi_1(L\mathcal{A}, \gamma_g) \cong Z_{\pi_1(\mathcal{A})}(g)$, and the surface pull-push is the usual restriction–induction description of untwisted fusion.

For an example in which a nontrivial action yields a character orbit with a proper stabilizer, whose sector is the Drinfeld center associated to that smaller group, see Example 3.5 below.

Proof. In the first case every stabilizer is the whole fundamental group, so apply parts (i) and (ii) of the theorem. In the second case the character condition is vacuous and the scalar associator is 1. $\square$

Remark 3.3.

- (i) In fact (28) already holds for any *invariant character* ϕ , even if the action on $\pi_2(\mathcal{A})$ is nontrivial: its orbit is still a singleton. Every normalized scalar 3-cocycle on a finite group is covered by the first case, by taking $\pi_2(\mathcal{A}) = \mathbb{C}^\times$ as a discrete abelian group, with trivial action and $\phi = \mathrm{id}$.
- (ii) Fusion and multifusion categories are recalled in Appendix A, especially Defs. A.2 and A.3. In part (iii) of Thm. 3.1, each orbit sector has a simple unit, but their sum need not. The resulting direct sum is a braided multifusion category, as explained in Prop. A.4.
- (iii) For infinite $\pi_2(\mathcal{A})$, arbitrary finite-dimensional local systems need not be semisimple under its action, and the unrestricted disk pushforward is infinite-dimensional. Parts (i) and (ii) avoid both issues by imposing the stated orbit condition; part (iii) is not asserted in that generality. Choices of orbit representative, cocycle, and skeletal presentation affect the displayed equivalences, not the resulting braided equivalence classes.

Remark 3.4 (Comparison to the literature). The statement separates three levels: the general character-orbit realization in Thm. 3.1, its trivial-action specialization in Cor. 3.2, item (i), and the untwisted specialization in item (ii). Several sources address these at different levels of generality. The value of the present formulation is to give their mapping-space realization in one explicit statement and proof, retaining the geometric product and all its constraints, and allowing an infinite second homotopy group within a specified character orbit.

- (i) *The general mapping-space pull-push formalism.* Sati–Schreiber, in *Complete Topological Quantization of Higher Gauge Fields*, § 2.3.3, (155)–(159), construct pull-push for ∞ -local systems along boundary-restriction correspondences [SS26a]. Our pants span is its two-input instance. The ordinary coefficient version and its relation to degree-zero homology are spelled out in Rem. B.2; no identification of ordinary and entire derived pushforwards is implicit.

- (ii) *The center for finite 2-groups, including nontrivial actions.* Huang–Xu–Zhang identify functors on the loop 2-groupoid with the center of the 2-group algebra, and describe convolution and braiding [HXZ24, arXiv v3, § 3.4 and Cor. 3.20]. Here the 2-group algebra is the additive, idempotent-complete linearization of the categorical group modeling $\Omega[\mathcal{A}]_2$. This is a direct categorical precedent for the finite- $\pi_2(\mathcal{A})$ part of Thm. 3.1, with arbitrary action. Our proof identifies those operations through the explicit surface correspondence and then passes to the stabilizer description. Ordinary vector-space functors on the loop 2-groupoid factor through its fundamental groupoid, so their coefficients match ours.
- (iii) *Character orbits and stabilizer categories.* Huang–Zhang compute the character summands of the 2-group algebra and identify their diagonal corners as pointed categories on character stabilizers, with associator obtained by evaluating the 3-cocycle [HZ23, arXiv v4, § 3.3, Prop. 3.19]. Taking centers and using Morita invariance gives the finite decomposition in (27). Thus the stabilizer description has a concrete algebraic antecedent. The proof here derives it directly from the coinvariant relations of surface pushforward, and does so for each finite character orbit even when $\pi_2(\mathcal{A})$ is infinite. In translating conventions, the covariant Fourier comparison in [HXZ24, arXiv v3, § 3.4] uses inversion of morphisms: a character of the linearized categorical group becomes its inverse as a transport character. Together with our inverse-associator convention, this gives the scalar twist ω_ϕ used here.
- (iv) *The direct untwisted surface construction.* Sati–Schreiber’s *Drinfeld Center as Quantum State Monodromy over Bloch Hamiltonians around Defects* explicitly makes the geometric observation in (29) [SS26b, Props. 3.2 and 3.4, § 3.2, and Lem. A.6]. With $\pi_2(\mathcal{A}) = 0$, the loop and pants monodromy groups are centralizers and joint centralizers. Lemma A.6 gives the categorical restriction–tensor–induction formula; Proposition A.8 gives fusion coefficients. The general theorem extends this surface description to the character-orbit sectors and their Postnikov twists.
- (v) *The homotopy 2-type.* The classification by $\pi_1(\mathcal{A})$, its action on $\pi_2(\mathcal{A})$, and a Postnikov class is the homotopy-theoretic form of the classification of categorical groups attributed to Sinh; see [BL04, § 8.3]. This supplies the model of $[\mathcal{A}]_2$ used in the proof; Lem. B.3 explains why it suffices for an untruncated target. Nontrivial coefficient actions are part of this classification, not an additional structure introduced here.
- (vi) *Transgression on cochains and its homotopy-theoretic identification.* Willerton’s Parmesan construction compares the classifying space of an inertia groupoid with the free loop space; Theorems 2 and 4 establish the homotopy equivalence. The further cochain-level transgression identification in § 1.3.3 is presented through a prism picture and a signed formula [Wil08, arXiv version, §§ 1.3–1.4]. Its chain calculation is evaluation composed with Eilenberg–Zilber.

Sati–Schreiber give the detailed simplicial argument in *Cyclification of Orbifolds*: Lemma 3.3 computes evaluation, and Theorem 3.4 proves the looping-to-transgression identification using Eilenberg–Zilber [SS24, arXiv v1, § 3]. This supplies the detailed justification of the identification sketched in Willerton’s account. It applies directly to the trivial-coefficient situation of (28) and to the scalar cocycle on each stabilizer. The general Postnikov cocycle has local coefficients; our square-pasting argument gives the

corresponding quotient-valued extension formula without treating it as an untwisted coefficient map.

- (vii) *The central inertia extension in the trivial-action case.* Ganter–Usher identify the fundamental groupoid of the inertia 2-groupoid with the central extension determined by transgression under the trivial-action hypothesis [GU16, arXiv version, Prop. 3.5]. This supplies the extension part of Cor. 3.2, item (i). The general theorem instead uses the quotient kernel in (19) and its potentially nontrivial centralizer action.
- (viii) *Twisted doubles and their projective centralizer descriptions.* Dijkgraaf–Pasquier–Roche construct twisted quantum doubles from scalar 3-cocycles [DPR91]. Willerton’s inertia-algebra description and Ganter–Usher’s representation-category comparison explain the projective centralizer representations [Wil08; GU16, Cor. 4.16]. For trivial action this comparison uses the representations in which the coefficient subgroup acts through the specified character ϕ ; unrestricted representations of the central extension also contain other character sectors. The relevant centralizers lie in $\pi_1(\mathcal{A})$; in the general orbit sector they lie in $\text{Stab}(\phi)$, as in (50). Our projective multiplier is the inverse of Willerton’s transgression of ω_ϕ , with the sign conversion explained in Rem. B.8.
- (ix) *Tensor product and coherence.* Galindo–Gutiérrez–Uribe describe separately the cocycles controlling equivariance, multiplication, and associativity [GGU15, arXiv version, §§ 3.5, 4]. Morton develops pull-push linearization of groupoid spans and its cocycle-decorated version [Mor11; Mor15, arXiv version, §§ 5.4–5.5]. Here the correspondence carries no extra scalar decoration: the Postnikov data are already in the mapping spaces, and the calculation extracts their transport and geometric coherence maps.
- (x) *The representation-ring comparison.* Kaufmann–Pham identify an orbifold K -theory push-pull product with the representation ring of the Drinfeld double and develop twisted counterparts [KP09, arXiv version, Thm. 3.13 and §§ 4.7–4.8]. This is a comparison at the Grothendieck-ring level; the present theorem retains the category, associator, and braiding.
- (xi) *Centers and loop-space convolution in another coefficient theory.* Ben-Zvi–Francis–Nadler identify centers of quasi-coherent sheaf categories on perfect derived stacks with sheaves on their loop stacks and describe operations by mapping-space correspondences [BFN10, Thm. 1.7 and § 6]. This is a broader geometric precedent with different coefficients. Our statement concerns ordinary finite-dimensional local systems on free loop spaces of arbitrary connected homotopy types; its character sectors and braided structure depend only on their Postnikov 2-truncations (Lem. B.3).

Together these comparisons place the theorem as an explicit geometric synthesis and extension of the stated constructions. In the finite case its categorical center and stabilizer descriptions are already recognizable in the 2-group literature; the proof here exhibits their common surface pull-push, including the disk quotient and geometric coherence. The orbit formulation also preserves the earlier allowance of arbitrary abelian $\pi_2(\mathcal{A})$, under the precise semisimple character condition (21).

3.1.3. Proof of Realization

Proof of Thm. 3.1. First apply Lem. B.3: postcomposition with $\mathcal{A} \rightarrow [\mathcal{A}]_2$ identifies the fundamental groupoids of all the circle and surface mapping spaces used here, together with boundary restriction and the geometric constraints. It also preserves the character sectors and their units. We may therefore carry out the computation for $[\mathcal{A}]_2$ and transfer it back to $L\mathcal{A}$.

We compute the fundamental groupoids and the Kan extension before extracting one character space. This makes explicit why nontrivial actions require character orbits, and why the resulting fusion category belongs to the stabilizer. We then check the geometric constraints against the defining formulas for the center.

1. Paths between loops and their character spaces. Use the skeletal categorical group specified above, with forward associator $-k$. Its one-object delooping models $[\mathcal{A}]_2$. By the preceding reduction, its fundamental mapping groupoids and their pull-push compute the required operations for $\mathcal{A}$.

An object of a model for $\Pi_1 L\mathcal{A}$ is $g \in \pi_1(\mathcal{A})$. An arrow to $g' = xgx^{-1}$ is represented by

$$(a, x) : g \longrightarrow g', \quad a \in \pi_2(\mathcal{A}).$$

Here x is the path of the marked point and a labels a square $x \otimes g \rightarrow g' \otimes x$. A modification with label u changes a by $(g' - 1)u$. Thus the actual arrow has a defined modulo $(g' - 1)\pi_2(\mathcal{A})$. This follows by cutting a cylinder along its marked-point path; squares give paths of loops and modifications give homotopies of those paths.

For two composable arrows, pasting squares gives

$$(a, x) \circ (b, y) = (a + x \cdot b + t(x, y; g), xy), \quad (30)$$

with t as in (47), interpreted in the quotient at the target. The three rebracketings contribute

$$-k(x, y, g), \quad k(x, ygy^{-1}, y), \quad -k((xy)g(xy)^{-1}, x, y).$$

The square labeled b is tensored on the left by x , hence contributes $x \cdot b$. More explicitly, substitution into (15) gives the associativity defect

$$\begin{aligned} & t(x, y; zgz^{-1}) + t(xy, z; g) \\ & - x \cdot t(y, z; g) - t(x, yz; g) \\ & = ((xyz)g(xyz)^{-1} - 1) \cdot k(x, y, z). \end{aligned} \quad (31)$$

The right-hand side is a modification at the final target, so composition is associative in the quotient. Normalization gives identity arrows, and reversing squares gives inverses. This proves the extension (19) as well as its stated action on the kernel.

Write $V_g = \mathcal{F}_{\gamma_g}$, let $D_g(a)$ denote transport along (a, e) , and let R_x denote transport along $(0, x)$. The defining relations for a local system are

$$\begin{aligned} D_g(a + b) &= D_g(a)D_g(b), & D_g((g - 1)a) &= \text{id}, \\ R_x D_g(a) &= D_{xgx^{-1}}(x \cdot a)R_x, & R_e &= \text{id}, \\ (R_x R_y)|_{V_g} &= D_{(xy)g(xy)^{-1}}(t(x, y; g))R_{xy}|_{V_g}. \end{aligned} \quad (32)$$

Transport along (a, x) is $D_{xgx^{-1}}(a)R_x$. Conversely these relations give a functor on the

groupoid. In an orbit sector they imply

$$V_g = \bigoplus_{\psi \in [\phi]} V_{g,\psi}, \quad V_{g,\psi} = 0 \text{ unless } g \cdot \psi = \psi, \quad R_x V_{g,\psi} = V_{xgx^{-1}, x \cdot \psi}.$$

2. Restricting to one character and identifying the center. Consider pairs (g, ψ) with $\psi \in [\phi]$ and $g \in \text{Stab}(\psi)$. Conjugation by x sends this pair to $(xgx^{-1}, x \cdot \psi)$. The preceding transport relations describe projective vector bundles on this action groupoid, with multiplier obtained by evaluating t in the target character. The quotient relations on the labels do not change that multiplier because the target degree fixes the target character.

The full subgroupoid on pairs (g, ϕ) is an equivalent groupoid: every character in the orbit can be moved to ϕ , and arrows between such pairs are precisely $x \in \text{Stab}(\phi)$. This remains an equivalence with the multiplier included, since the multiplier is simply restricted along the full inclusion. Explicitly, choose for each $\psi \in [\phi]$ an element carrying ϕ to ψ , transport the vector spaces along these choices, and use (32) to define all remaining arrows. The cocycle identity ensures independence under composition; different choices give naturally isomorphic extensions. Restriction is fully faithful because these same transports determine every component of an intertwiner. This proves essential surjectivity as well as full faithfulness without any finiteness assumption on $\pi_2(\mathcal{A})$.

On the restricted subgroupoid the data are $V_{g,\phi}$ for $g \in \text{Stab}(\phi)$ and maps R_x for $x \in \text{Stab}(\phi)$ satisfying

$$(R_x R_y)|_{V_{g,\phi}} = \phi(t(x, y; g)) R_{xy}|_{V_{g,\phi}}.$$

Form the stabilizer-graded vector space $\bigoplus_{g \in \text{Stab}(\phi)} V_{g,\phi}$. Its half-braiding is defined on homogeneous vectors by

$$\gamma_X(u_x \otimes v_g) = R_x(v_g) \otimes u_x. \quad (33)$$

The degrees agree and the map is natural in X . Substitution into (2) gives

$$R_{xy}|_{V_{g,\phi}} = \frac{\omega_\phi(x, y, g) \omega_\phi((xy)g(xy)^{-1}, x, y)}{\omega_\phi(x, ygy^{-1}, y)} (R_x R_y)|_{V_{g,\phi}},$$

which is precisely the preceding transport relation. Normalization gives the unit condition. Values on the one-dimensional objects δ_x recover every R_x , and morphisms agree. This proves the linear-category equivalence in (26).

3. The two boundary maps on the pants groupoid. A spine of P is a wedge of two circles. Its mapping groupoid has objects (g, h) and arrows

$$(a, b, x) : (g, h) \longrightarrow (g', h'), \quad g' = xgx^{-1}, \quad h' = xhx^{-1}.$$

Simultaneous modification identifies

$$(a, b, x) \sim (a + (g' - 1)u, b + (h' - 1)u, x). \quad (34)$$

Composition acts by x on the two labels of the arrow being composed on the right and adds the respective corrections $t(x, y; g)$ and $t(x, y; h)$. For three successive arrows, (31) gives the same modification parameter $u = k(x, y, z)$ for both labels. Thus associativity holds modulo the *simultaneous* relation (34), not only after passing separately to the two loop quotients. Incoming restriction sends an arrow to $((a, x), (b, x))$ in the two loop groupoids.

Outgoing restriction is

$$q(g, h) = gh, \quad q(a, b, x) = (a + g' \cdot b + s(x; g, h), x), \quad (35)$$

where

$$s(x; g, h) = k(x, g, h) - k(g', x, h) + k(g', h', x). \quad (36)$$

Indeed, paste the incoming squares along

$$\begin{aligned} x \otimes (g \otimes h) &\longrightarrow (x \otimes g) \otimes h \longrightarrow (g' \otimes x) \otimes h \\ &\longrightarrow g' \otimes (x \otimes h) \longrightarrow g' \otimes (h' \otimes x) \longrightarrow (g' \otimes h') \otimes x. \end{aligned}$$

The three rebracketings contribute s ; the two squares contribute a and $g' \cdot b$. The change in the outgoing label under (34) is $(g'h' - 1)u$, so the formula descends to the outgoing loop groupoid. All these corrections are part of boundary restriction itself.

4. The Kan extension selects equal characters. Allow initially two orbit sectors, and decompose the incoming fibers into $V_{g,\psi} \otimes W_{h,\chi}$. Fix an outgoing degree z . Every comma object consists of (g, h) and an arrow $(c, x) : gh \rightarrow z$. It is isomorphic to one with outgoing arrow the identity: the arrow lifts through q by taking $b = 0$ and $a = c - s(x; g, h)$. The representatives with identity outgoing arrow are the factorizations $gh = z$.

A morphism between such representatives must have $x = e$, so cannot change the factorization. Its labels satisfy $a + g \cdot b \in (z - 1)\pi_2(\mathcal{A})$. By a modification (34) they can be chosen to satisfy $a + g \cdot b = 0$. Thus the stabilizer relations in the comma colimit are generated by

$$(a, -g^{-1} \cdot a, e), \quad a \in \pi_2(\mathcal{A}).$$

Its action on the indicated tensor factor is the scalar

$$\psi(a) \chi(-g^{-1} \cdot a) = \psi(a)(g \cdot \chi)(a)^{-1}.$$

The coinvariants are zero unless $\psi = g \cdot \chi$. Since $V_{g,\psi} \neq 0$ already implies $g \cdot \psi = \psi$, this condition is equivalent to $\chi = \psi$. When the characters agree all these relations act trivially. Simultaneous modifications also act trivially on the incoming tensor factors, by their loop quotient relations, so impose no additional relations. Therefore

$$(\mathcal{F} \star \mathcal{G})_{z,\psi} \cong \bigoplus_{gh=z} V_{g,\psi} \otimes_{\mathbb{C}} W_{h,\psi}. \quad (37)$$

The nonzero terms have $g, h \in \text{Stab}(\psi)$. This proves preservation of an orbit sector, finite-dimensionality, and vanishing of products between distinct orbits.

For transport along (c, x) , use the lift $(c - s(x; g, h), 0, x)$. On a summand of character ψ , with outgoing target character $x \cdot \psi$, the transport is

$$(x \cdot \psi)(c - s(x; g, h))(R_x^V \otimes R_x^W). \quad (38)$$

Another lift differs by a relation already imposed. On the ϕ -character spaces, for $g, h, x \in \text{Stab}(\phi)$ and $c = 0$, this becomes

$$R_x^{V \star W}|_{V_{g,\phi} \otimes W_{h,\phi}} = \frac{\omega_\phi(g', x, h)}{\omega_\phi(x, g, h) \omega_\phi(g', h', x)} (R_x^V \otimes R_x^W). \quad (39)$$

The three scalar factors in (5) are exactly those in this formula. Together with (37), this identifies the binary Kan-extension functor with the center's tensor product. The natural comparison is given by the comma representatives with outgoing arrow the identity.

5. Gluing, base change, and the associator. Mapping spaces take surface gluing to homotopy pullbacks. Ordinary local-system pull-push satisfies base change for these homotopy pullbacks, as proved in Lem. B.1. This does not require Π_1 to preserve the pullbacks: the

possible extra fundamental-group kernel acts trivially on the coefficient system being pushed. This point is essential with a nontrivial action. For example, after reduction to the 2-type, the second homotopy group of a connected graph mapping space is the subgroup of $\pi_2(\mathcal{A})$ fixed by all edge monodromies, and boundary restriction need not be surjective on these subgroups. For the original untruncated target there can also be contributions from $\pi_3(\mathcal{A})$ (Rem. B.4); the base-change argument requires neither their absence nor a computation of them.

Base change and the preservation of colimits by tensoring identify the two iterated products with sums over triples g, h, j , all of the same character ψ . On the chosen ϕ -spaces the degrees lie in $\text{Stab}(\phi)$. The outgoing words of the two gluings are $(g \otimes h) \otimes j$ and $g \otimes (h \otimes j)$; their geometric comparison is the forward associator with label $-k(g, h, j)$.

For a natural isomorphism $\eta : q_L \Rightarrow q_R$, the comparison on left Kan extensions sends

$$[u : q_L(y) \rightarrow z, v] \longmapsto [u \circ \eta_y^{-1} : q_R(y) \rightarrow z, v].$$

The inverse is necessary because these are outgoing comma arrows. Hence the geometric associator on the ϕ -spaces is

$$(v_g \otimes w_h) \otimes u_j \longmapsto \omega_\phi(g, h, j) v_g \otimes (w_h \otimes u_j). \quad (40)$$

This is (10) with its stated direction. Since ϕ is invariant under the stabilizer, applying ϕ to the module cocycle identity gives the ordinary scalar cocycle identity. Its pentagon is exactly the pentagon of (40). Naturality follows either from the pasted square comparison or by substituting (39) into that scalar identity. By full faithfulness in step 2, these checks on the ϕ -spaces verify the coherence on the whole orbit sector.

6. The disk and the unit. The constant-loop extension is the semidirect product in (46). The disk pushforward has fiber $\mathbb{C}[\pi_2(\mathcal{A})]$ there: $\pi_2(\mathcal{A})$ acts by translation and $x \in \pi_1(\mathcal{A})$ sends the basis element $[a]$ to $[x \cdot a]$. Its orbit quotient is

$$\bigoplus_{\psi \in [\phi]} \mathbb{C}_\psi,$$

where x permutes the character lines by $\psi \mapsto x \cdot \psi$. Restriction to the ϕ -line gives a one-dimensional space on which $\text{Stab}(\phi)$ acts trivially. Thus it corresponds to the center's unit under step 2.

One can see the disk constraints directly in the comma colimit. On a character- ψ summand, the relation for multiplying the disk by a local system identifies

$$[a + b] \otimes v = [a] \otimes \psi(b)v.$$

The map $[a] \otimes v \mapsto \psi(a)v$ is an isomorphism after imposing these relations. On the other side the analogous map is $v \otimes [a] \mapsto \psi(a)v$, since its degree fixes ψ . These maps factor through exactly the orbit quotient, and after selecting ϕ they are the usual scalar unitors. Normalization of k makes the associator the identity when a degree is e , proving the triangle identity. The unit is simple because its image in the center is simple, even though its constant-loop fiber may have dimension $||[\phi]||$.

7. Exchanging the holes and the braiding. Use the exchange taking (g, h) to (ghg^{-1}, g) . Its first new incoming circle is the old second circle transported along the first, represented by the square $(0, g)$; the other circle keeps its label. Boundary restriction to the outgoing

circle is fixed. Thus on the summands selected by the Kan extension the geometric exchange is

$$v_g \otimes w_h \longmapsto R_g^W(w_h) \otimes v_g. \quad (41)$$

Both input characters are ψ and $g \cdot \psi = \psi$, so the displayed target has the same character.

For completeness, its compatibility with outgoing restriction can be checked on the scalar ϕ -spaces. For $g, h, x \in \text{Stab}(\phi)$, the exchanged first square acquires correction

$$t(xgx^{-1}, x; h) - t(x, g; h),$$

and the second keeps its label. The needed cancellation is obtained by evaluating the following expression under ϕ :

$$\begin{aligned} & t(xgx^{-1}, x; h) - t(x, g; h) \\ & + s(x; ghg^{-1}, g) - s(x; g, h). \end{aligned}$$

Substitution of (36) and (47) cancels its terms. Coefficient actions on square labels disappear after evaluation under ϕ , because every group element here lies in its stabilizer. Hence there is no extra scalar from the outgoing comparison, and (41) is a map of local systems.

Under (33) this map is exactly η_V , the center braiding (6). Its hexagons hold with the already identified tensor product and associator. Full faithfulness then verifies the hexagons on the orbit sector. We have checked the geometric maps, rather than choosing constraints solely by transport across an abstract equivalence.

8. Finiteness, the full category, and choices. The stabilizer is finite and ω_ϕ is a scalar 3-cocycle. Its graded category and its center are fusion categories [Eti+15, Thm. 9.3.2]. Steps 2–7 give a braided strong monoidal equivalence, so establish rigidity, semisimplicity, and the remaining finiteness assertions for each orbit sector. The explicit character-space argument used no finiteness of $\pi_2(\mathcal{A})$.

If $\pi_2(\mathcal{A})$ is finite, every finite-dimensional representation of it is a direct sum of characters. The transport relation groups these characters into orbits. Thus every local system is a direct sum of its orbit subsystems, and there are only finitely many possible orbits. Step 4 proves that their products vanish across distinct orbits. The regular disk representation is the direct sum of their units. This proves (27) and braided multifusion.

Finally, if $k' = k + \delta b$ for a normalized 2-cochain, the coefficient differential is

$$(\delta b)(g, h, j) = g \cdot b(h, j) - b(gh, j) + b(g, hj) - b(g, h).$$

The skeletal models are monoidally equivalent by the identity on objects and arrows with tensor comparison labeled b . On the stabilizer, applying ϕ gives a scalar coboundary. The corresponding identity functor between the graded categories with associators ω_ϕ and ω'_ϕ has tensor comparison $\phi(b(g, h))^{-1}$. It induces a braided equivalence of centers compatible with the mapping-space calculation. Changing the character representative is handled by the conjugation transports in step 2. These comparisons involve the indicated choices; no preferred equivalence independent of all such choices is asserted. $\square$

3.1.4. Examples

Example 3.5 (An orbit with a proper stabilizer). Let $\pi_1(\mathcal{A}) = \mathbb{Z}/2$ act on $\pi_2(\mathcal{A}) = \mathbb{Z}/3$ by inversion, and take $k = 0$. The trivial character has stabilizer $\mathbb{Z}/2$. The two nontrivial

characters form a single orbit with trivial stabilizer. Hence

$$(\mathrm{Loc}^{\mathrm{fd}}(LA), \star) \simeq \mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}/2}) \oplus \mathrm{Vec}_{\mathbb{C}}.$$

The same categorical decomposition is exhibited in [HXZ24, arXiv v3, Ex. 4.24]. The non-trivial character orbit contributes a fusion sector with just one simple object; its geometric unit has a two-dimensional fiber at the constant loop.

(...)

A. Fusion Categories

A fusion category abstracts the finite semisimple tensor structure familiar from representations of a finite group; a multifusion category allows the unit to split into several simple objects. All categories here are over $\mathbb{C}$. We recall the definitions and the unit decomposition used in Thm. 3.1, following [Eti+15, §§ 2.10, 4.1–4.3]. Neither a braiding nor a pivotal or ribbon structure is part of the definition. A guide to introductory and research references appears in Rem. A.5.

A.1. Ordinary fusion

Definition A.1 (Finite semisimplicity and rigidity). Following [Eti+15, §§ 1.2–1.3, 4.1], a *finite semisimple complex linear category* is an essentially small $\mathbb{C}$ -linear abelian category with finite-dimensional morphism spaces, finitely many isomorphism classes of simple objects, and every object a finite direct sum of simple objects. Here a simple object is nonzero and has no proper nonzero subobject. Over $\mathbb{C}$, its endomorphisms are precisely the scalar multiples of its identity.

A monoidal category is *rigid* [Eti+15, § 2.10] if every object V has both a left dual V^* and a right dual *V . A left dual comes with maps

$$\mathrm{ev}_V : V^* \otimes V \longrightarrow \mathbf{1}, \quad \mathrm{coev}_V : \mathbf{1} \longrightarrow V \otimes V^*$$

satisfying the two triangle identities

$$\begin{aligned} (\mathrm{id}_V \otimes \mathrm{ev}_V)(\mathrm{coev}_V \otimes \mathrm{id}_V) &= \mathrm{id}_V, \\ (\mathrm{ev}_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes \mathrm{coev}_V) &= \mathrm{id}_{V^*}. \end{aligned}$$

Associators and unitors are understood in these equations. For a right dual the evaluation and coevaluation instead have directions

$$\widetilde{\mathrm{ev}}_V : V \otimes {}^*V \longrightarrow \mathbf{1}, \quad \widetilde{\mathrm{coev}}_V : \mathbf{1} \longrightarrow {}^*V \otimes V,$$

with the corresponding triangle identities. For finite-dimensional vector spaces these are the usual evaluation and coevaluation maps involving the linear dual. In particular, rigidity does not require evaluation to be an isomorphism.

Definition A.2 (Fusion category). A *fusion category* over $\mathbb{C}$ [Eti+15, § 4.1; ENO05, § 2] is a finite semisimple complex linear category equipped with a rigid monoidal structure whose tensor product is $\mathbb{C}$ -bilinear on morphisms and whose unit $\mathbf{1}$ is simple. The last condition is equivalent to

$$\mathrm{End}_{\mathbb{C}}(\mathbf{1}) \cong \mathbb{C}.$$

A *braided fusion category* is a fusion category together with a braiding: natural isomorphisms $c_{V,W} : V \otimes W \rightarrow W \otimes V$ satisfying the two hexagon identities. The notation BrFusCat in (25) refers to braided fusion categories.

Examples include finite-dimensional representations of a finite group, with their usual tensor product, and the categories Vec_G^ω of Def. 2.8 for finite G ; see [ENO05, § 2]. In the latter example every simple object is tensor-invertible, whereas a fusion category can also have noninvertible simple objects. Thus “fusion” refers to finite semisimple tensor decompositions, not to invertibility of all objects.

The tensor product preserves finite direct sums and is exact in each variable, since every short exact sequence in a semisimple category splits; compare [Eti+15, § 4.2]. If $\{V_i\}_{i \in I}$ is a set of representatives of the simple objects, its *fusion rules* are the finite decompositions

$$V_i \otimes V_j \cong \bigoplus_{\ell \in I} V_\ell^{\oplus N_{ij}^\ell}, \quad N_{ij}^\ell = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{C}}(V_\ell, V_i \otimes V_j).$$

These multiplicities are the structure constants of the *Grothendieck ring*: the free abelian group on the simple isomorphism classes, with multiplication induced by tensor product [Eti+15, § 3.1]. They describe the tensor product on isomorphism classes; the associator and, when present, the braiding are further data. In Thm. 3.1 the character-stabilizer categories are fusion because the stabilizers are finite, and their centers are braided fusion [Eti+15, Thm. 9.3.2].

A.2. Multifusion

Definition A.3 (Multifusion category). A *multifusion category* [Eti+15, § 4.1; ENO05, §§ 2, 2.4] is a nonzero finite semisimple complex linear category equipped with a rigid monoidal structure whose tensor product is $\mathbb{C}$ -bilinear on morphisms. Its unit is not required to be simple. A *braided multifusion category* is such a category equipped with a braiding. Thus a multifusion category is fusion precisely when its unit is simple.

The unit decomposes as

$$\mathbf{1} \cong \bigoplus_{i \in I} \mathbf{1}_i, \quad \mathbf{1}_i \otimes \mathbf{1}_j \cong \begin{cases} \mathbf{1}_i, & i = j, \\ 0, & i \neq j, \end{cases} \quad (42)$$

where the $\mathbf{1}_i$ are pairwise nonisomorphic simple objects. To see the absence of multiplicities, recall that composition and tensor product make $\text{End}(\mathbf{1})$ commutative by the interchange law. Semisimplicity expresses this algebra as a product of matrix algebras; commutativity forces every matrix size to be one. Its primitive idempotents then give the tensor identities in (42). This is the semisimple case of the unit decomposition in [Eti+15, § 4.3].

Let $\mathcal{C}_{ij}$ be the full subcategory of objects isomorphic to $\mathbf{1}_i \otimes V \otimes \mathbf{1}_j$. Every object is the direct sum of these components, and

$$\mathcal{C}_{ij} \otimes \mathcal{C}_{k\ell} = 0 \quad (j \neq k), \quad \mathcal{C}_{ij} \otimes \mathcal{C}_{j\ell} \subseteq \mathcal{C}_{i\ell}.$$

This matrix of components is described in [ENO05, § 2.4] and [Eti+15, § 4.3]. In general the off-diagonal components need not vanish, so a multifusion category need not be a direct sum of fusion categories. For example, matrices of finite-dimensional vector spaces, with

$$(V \otimes W)_{i\ell} = \bigoplus_j V_{ij} \otimes_{\mathbb{C}} W_{j\ell},$$

form a multifusion category whose unit is the diagonal matrix of copies of $\mathbb{C}$; for matrices of size greater than one it has nonzero off-diagonal components.

Proposition A.4 (The braided case). *Every braided multifusion category is a finite direct sum of braided fusion categories. The summands have units $\mathbf{1}_i$, and tensor products between distinct summands are zero.*

Proof. The braiding gives

$$\mathbf{1}_i \otimes V \otimes \mathbf{1}_j \cong V \otimes \mathbf{1}_i \otimes \mathbf{1}_j = 0 \quad (i \neq j).$$

Thus only the diagonal components $\mathcal{C}_{ii}$ remain. Each is closed under tensor product and has simple unit $\mathbf{1}_i$. Left and right duality send $\mathcal{C}_{ij}$ to $\mathcal{C}_{ji}$, so preserve each diagonal component. The restricted associator and braiding therefore make each $\mathcal{C}_{ii}$ a braided fusion category, and the displayed tensor identities give the asserted direct sum. $\square$

In (27) these summands are exactly the character-orbit sectors, with units $\mathbf{1}_{[\phi]}$. The simplicity of each sector unit is a statement about subobjects in the category of local systems; it does not require its fiber at a loop to be one-dimensional. Indeed, that fiber has dimension $|\phi|$ at the constant loop, as illustrated in Ex. 3.5.

Remark A.5 (Pointers to the literature). The following sources serve different purposes. For a first reading, begin with the survey in item (i) and use the monograph in item (ii) for definitions and proofs.

- (i) *An introductory survey.* Müger, *Tensor categories: A selective guided tour* [Müg10], explains the basic structures through examples and diagrams. Section 1 introduces monoidal categories, duality, and linearity; § 4 turns to braidings and centers. It is a useful orientation before undertaking a systematic study.
- (ii) *A systematic reference, including multifusion categories.* Etingof–Gelaki–Nikshych–Ostrik, *Tensor Categories* [Eti+15], is the main reference for the conventions here. Read § 2.10 for duality, §§ 4.1–4.3 for tensor and multitensor categories and their units, and § 3.1 for the rings recording fusion rules. For the subject of this note, continue with § 7.13 on the center, § 8.5 on its braiding, and § 9.3 on centers of fusion categories. Their term *multitensor category* also allows nonsemisimple categories; *multifusion* imposes finite semisimplicity.
- (iii) *Braided, ribbon, and modular structures.* Bakalov–Kirillov, *Lectures on Tensor Categories and Modular Functors* [BK01, Chs. 1–3], develops these successive structures in a textbook treatment. It is particularly useful for learning the graphical language and the additional structures that arise beyond the definition of a fusion category. A fusion category by itself is not required to have any of them; the definitions in this appendix stop before imposing ribbon or modular structure.
- (iv) *A foundational research account.* Etingof–Nikshych–Ostrik, *On fusion categories* [ENO05], develops structural and finiteness results. The beginning of § 2 gives the definitions and examples of fusion and multifusion categories; § 2.4 describes their matrix of components. Theorem 2.15 includes the fusion property of the Drinfeld center in characteristic zero. This paper is useful for tracing results to the research literature after consulting the preceding introductions.

For the categorical prerequisites, Mac Lane's *Categories for the Working Mathematician* [Mac98, Chs. VII, XI] treats monoidal and braided categories and their coherence. The same book's Chapter X supplies background for the Kan extensions in Appendix B.

B. Local Systems

This appendix explains the coefficients and operations used in § 3.1. We first recall ordinary local systems and their pull-push for general maps of spaces. We then specialize to free loop spaces, explaining the reduction to the 2-type, the choice of a character orbit, and the cocycles governing its monodromy. All vector spaces are over $\mathbb{C}$.

B.1. Basics and pull-push

Local systems as representations of a fundamental groupoid. A local system assigns a vector space to each point and an isomorphism to each path, depending only on its homotopy class relative to the endpoints. Formally, for a space Y , its fundamental groupoid $\Pi_1 Y$ has points as objects and endpoint-preserving homotopy classes of paths as arrows. Write $\text{Vect}_{\mathbb{C}}$ for all complex vector spaces and $\text{Vec}_{\mathbb{C}}$ for finite-dimensional ones. With covariant parallel transport, set

$$\text{Loc}(Y) := \text{Fun}(\Pi_1 Y, \text{Vect}_{\mathbb{C}}), \quad \text{Loc}^{\text{fd}}(Y) := \text{Fun}(\Pi_1 Y, \text{Vec}_{\mathbb{C}}).$$

Morphisms are natural transformations. A path $u : y \rightarrow y'$ acts by $\mathcal{F}(u) : \mathcal{F}_y \rightarrow \mathcal{F}_{y'}$, and transport along a composite is the composite of transports. Choosing a basepoint in every component gives

$$\text{Loc}^{\text{fd}}(Y) \simeq \prod_{[y] \in \pi_0 Y} \text{Rep}^{\text{fd}}(\pi_1(Y, y)).$$

Here Rep^{fd} denotes finite-dimensional complex representations and their intertwining maps. The equivalence uses paths to the chosen basepoints; it is not a preferred identification without these choices. For finitely many components the product is a finite direct sum. Higher homotopy groups of Y have no separate action on an ordinary local system, but they can affect the fundamental groupoids of mapping spaces formed from Y .

For example, a local system on S^1 is specified, up to isomorphism, by a vector space and an invertible linear map: transport once around the circle. A morphism intertwines these maps. This illustrates how a local system can have nontrivial monodromy even when all its fibers have the same dimension.

For local systems $\mathcal{F}$ on X and $\mathcal{G}$ on Y , their *external tensor product* is the local system on $X \times Y$ with fibers

$$(\mathcal{F} \boxtimes \mathcal{G})_{(x,y)} = \mathcal{F}_x \otimes_{\mathbb{C}} \mathcal{G}_y, \quad (x, y) \in X \times Y,$$

and tensor-product transport. The constant local system $\underline{\mathbb{C}}$ has fiber $\mathbb{C}$ and identity transport everywhere.

Restriction, Kan extension, and base change. A map $f : Y \rightarrow Z$ induces a pullback functor $f^* : \text{Loc}(Z) \rightarrow \text{Loc}(Y)$ by precomposition with $\Pi_1 f$: its fiber at y is the original fiber at $f(y)$. Its left adjoint $f_! : \text{Loc}(Y) \rightarrow \text{Loc}(Z)$ is *left Kan extension*. The adjunction

means that maps $f_! \mathcal{F} \rightarrow \mathcal{G}$ correspond naturally to maps $\mathcal{F} \rightarrow f^* \mathcal{G}$. Its pointwise formula is [Mac98, Ch. X]

$$(f_! \mathcal{F})_z = \operatorname{colim}_{(y, u: f(y) \rightarrow z)} \mathcal{F}_y. \quad (43)$$

A comma morphism $(y, u) \rightarrow (y', u')$ is a path class $v : y \rightarrow y'$ with $u' \circ f(v) = u$. The colimit is the direct sum of the fibers indexed by objects, modulo the relations identifying a vector with its transport along every arrow. A small skeleton may be used.

For a group homomorphism $H \rightarrow K$, the map $BH \rightarrow BK$ gives restriction and

$$f_! V = \mathbb{C}[K] \otimes_{\mathbb{C}[H]} V.$$

An inclusion gives the usual induced representation. A surjection gives the quotient by the span of $hv - v$ for h in its kernel, with the induced action of K ; this is the space of *coinvariants*. Thus pushforward includes the relations imposed by automorphisms and is not merely a sum over components. The finite-groupoid formalism is developed in [Mor11].

These Kan extensions always exist with values in all vector spaces but need not preserve finite-dimensionality. The notation $f_!$ here refers to this left adjoint, with no extra compact-support condition.

Consequently a correspondence $Y \xleftarrow{p} W \xrightarrow{q} Z$ gives a functor $q_! p^* : \operatorname{Loc}(Y) \rightarrow \operatorname{Loc}(Z)$. For a two-input correspondence, first take the external tensor product of the two local systems. This is the meaning of *pull-tensor-push* in (23). The following base-change statement is what makes composition of correspondences agree with composition of these functors.

Lemma B.1 (Ordinary base change for homotopy pullbacks). *Let $f : Y \rightarrow Z$ and $r : X \rightarrow Z$, and write p_Y, p_X for the projections from their homotopy pullback $Y \times_Z X$. For every $\mathcal{F} \in \operatorname{Loc}(Y)$ there is a natural isomorphism*

$$(p_X)_! p_Y^* \mathcal{F} \cong r^* f_! \mathcal{F}.$$

It is compatible with iterated homotopy pullbacks and composition of Kan extensions.

Proof. One may compute (43) over the actual homotopy fiber Y_z instead. The comparison from $\Pi_1 Y_z$ to the comma groupoid is bijective on components and surjective on the automorphism group of each component. Its kernel is the image of $\pi_2(Z, z)$ in $\pi_1(Y_z)$, as follows from the homotopy-fiber exact sequence. This kernel acts trivially on the pulled-back system, since its image in $\pi_1(Y)$ is trivial. Quotienting by it therefore does not change the colimit. The two sides of base change at $x \in X$ now use the same homotopy fiber $Y_{r(x)}$ and the same coefficients. Compatibility follows from this identification and the universal property of successive colimits. In particular this argument does not assert that Π_1 preserves arbitrary homotopy pullbacks. $\square$

Remark B.2 (Relation to pull-push of ∞ -local systems). The general mapping-space construction in [SS26a, arXiv v2, § 2.3.3, (155)–(159)] assigns pullback and homotopy-colimit pushforward to boundary restriction. The pants correspondence (22) is its two-input instance, specialized here to ordinary local systems.

More precisely, regard an ordinary local system $\mathcal{F}$ as a local system $\mathcal{F}[0]$ of complex chain complexes concentrated in degree zero. If $f_!^{\text{der}}$ denotes the derived left Kan extension, there is a natural identification

$$H_0(f_!^{\text{der}} \mathcal{F}[0]) \cong f_! \mathcal{F}.$$

Indeed, degree-zero homology of chains on a homotopy fiber with these coefficients is its ordinary coinvariant colimit. Derived pushforward has no homology in negative degrees (it is connective), but it may have homology in positive degrees. Pullback commutes with H_0 ; on connective complexes over $\mathbb{C}$, the tensor product satisfies $H_0(U \otimes V) = H_0(U) \otimes H_0(V)$. Thus ordinary pull-push is the degree-zero specialization of this formalism, with compatible composition and tensor products. It should not be identified with the entire derived output without an additional vanishing argument.

B.2. Over free loop spaces

For the rest of this appendix, let $\mathcal{A}$ be a connected pointed homotopy type with finite $\pi_1(\mathcal{A})$, as in Thm. 3.1. Its second homotopy group may be infinite and may carry a nontrivial fundamental-group action. Write $L\mathcal{A} = \text{Map}(S^1, \mathcal{A})$. No truncation assumption is needed, for the following reason.

Reduction to the second Postnikov truncation. The Postnikov 2-truncation $[\mathcal{A}]_2$ retains the components, fundamental group, and second homotopy group of $\mathcal{A}$, together with their action and Postnikov class, and has vanishing homotopy groups in degrees greater than two. The following lemma explains why this retains all the information needed by the ordinary local systems and surface operations considered here.

Lemma B.3 (The surface construction depends only on the 2-type). *Let $\mathcal{A}$ be a connected pointed homotopy type, with no truncation assumption, and let $r : \mathcal{A} \rightarrow [\mathcal{A}]_2$ be its Postnikov projection. For a space X of the homotopy type of a finite graph, postcomposition induces an equivalence*

$$\Pi_1 \text{Map}(X, \mathcal{A}) \xrightarrow{\simeq} \Pi_1 \text{Map}(X, [\mathcal{A}]_2). \quad (44)$$

These equivalences are natural under precomposition and homotopies between maps of source spaces. In the setting of Thm. 3.1, they induce

$$(Lr)^* : \text{Loc}_{[\phi]}^{\text{fd}}(L[\mathcal{A}]_2) \xrightarrow{\simeq} \text{Loc}_{[\phi]}^{\text{fd}}(L\mathcal{A}) \quad (45)$$

preserving the surface product, sector unit, associator, unitors, and braiding. Once established for the 2-type, these make (45) a braided monoidal equivalence. The full-category construction is likewise preserved whenever defined.

Proof. Represent r by a fibration. Its fibers are 2-connected, meaning connected with trivial π_1 and π_2 , by the homotopy exact sequence. We use the relative lifting method for Postnikov truncations recalled in [Hat02, § 4.3].

Replace X by a finite graph. The homotopy fiber of $\text{Map}(X, r)$ is a section space of the pulled-back fibration. It is nonempty and simply connected: constructing a section uses cells of dimension at most one; connecting sections uses $(X \times I, X \times \partial I)$, of relative dimension at most two; filling a loop uses $(X \times D^2, X \times S^1)$, of relative dimension at most three. The obstruction over an m -cell lies in π_{m-1} of the fiber, so vanishes in each case. Hence $\text{Map}(X, r)$ induces bijections on components and isomorphisms on fundamental groups, proving (44).

Postcomposition commutes with boundary restriction and its homotopies. The circle, disk, and disks with holes used in the construction have finite graph spines (allowing a point).

Their fundamental-groupoid diagrams are therefore naturally equivalent. Pullback identifies ordinary local systems, and the universal property of left Kan extension gives

$$(Lr)^*(\mathcal{F} \star \mathcal{G}) \cong (Lr)^*\mathcal{F} \star (Lr)^*\mathcal{G}.$$

The projection preserves $\pi_1(\mathcal{A})$, $\pi_2(\mathcal{A})$, their action, and the first Postnikov invariant, hence also character sectors and their quotient functors. The disk pushforward and sector unit are preserved. Naturality for gluing and exchange, including their homotopies, identifies the constraints and coherence identities. No equivalence of the full mapping spaces is required. $\square$

Remark B.4 (Higher homotopy is allowed). No vanishing of $\pi_3(\mathcal{A})$ is needed. The full mapping spaces can nevertheless change: evaluation and its constant-loop section give

$$\pi_2(L\mathcal{A}, \text{const}) \cong \pi_2(\mathcal{A}) \oplus \pi_3(\mathcal{A}).$$

An ordinary local system on the loop space does not detect this extra homotopy group of that space. The reduction does not identify the full derived pushforwards of Rem. B.2.

The loop extension and character-orbit quotients.

Remark B.5. Choose a based loop γ_g representing $g \in \pi_1(\mathcal{A})$. Evaluation $L\mathcal{A} \rightarrow \mathcal{A}$ at the marked point has based loop space $\Omega\mathcal{A}$ as fiber, based here at γ_g . The relevant homotopy exact sequence is

$$\pi_2(\mathcal{A}) \xrightarrow{g \cdot (-) - \text{id}} \pi_2(\mathcal{A}) \longrightarrow \pi_1(L\mathcal{A}, \gamma_g) \longrightarrow \pi_1(\mathcal{A}).$$

Changing the sign of the first arrow does not change its image. The image at the right is the centralizer of g , proving (19). Conjugation on the kernel is the induced centralizer action on the quotient. For trivial action the quotient is $\pi_2(\mathcal{A})$ itself and is central; in general neither simplification is valid.

The constant-loop section splits the extension at e , giving

$$\pi_1(L\mathcal{A}, \gamma_e) \cong \pi_2(\mathcal{A}) \rtimes \pi_1(\mathcal{A}), \quad (a, x)(b, y) = (a + x \cdot b, xy). \quad (46)$$

Write $d : \text{Map}(D^2, \mathcal{A}) \rightarrow L\mathcal{A}$ for boundary restriction. Under $\text{Map}(D^2, \mathcal{A}) \simeq \mathcal{A}$, its homomorphism on fundamental groups is $x \mapsto (0, x)$. Hence $d_! \mathbb{C}$ has fiber $\mathbb{C}[\pi_2(\mathcal{A})]$ on this component, with translation by $\pi_2(\mathcal{A})$ and permutation $[a] \mapsto [x \cdot a]$ by x . It is zero on the other components.

Definition B.6 (The character-orbit quotient). Fix a character $\phi : \pi_2(\mathcal{A}) \rightarrow \mathbb{C}^\times$. Its *character orbit* is the set

$$[\phi] := \{x \cdot \phi : x \in \pi_1(\mathcal{A})\}, \quad (x \cdot \phi)(a) = \phi(x^{-1} \cdot a).$$

This is a finite set of distinct characters, since $\pi_1(\mathcal{A})$ is finite. Let $\mathcal{H}$ be any local system on $L\mathcal{A}$, allowing arbitrary-dimensional fibers. Denote by ι_g the composite

$$\pi_2(\mathcal{A}) \longrightarrow \pi_2(\mathcal{A})/(g-1)\pi_2(\mathcal{A}) \hookrightarrow \pi_1(L\mathcal{A}, \gamma_g).$$

Define its *character-orbit quotient* by

$$(Q_{[\phi]}\mathcal{H})_{\gamma_g} := \bigoplus_{\psi \in [\phi]} \mathcal{H}_{\gamma_g} / \left\langle \mathcal{H}(\iota_g(a))v - \psi(a)v : a \in \pi_2(\mathcal{A}), v \in \mathcal{H}_{\gamma_g} \right\rangle.$$

The angle brackets denote the linear span of the displayed relations. The ψ -summand is the largest quotient of the fiber on which every $a \in \pi_2(\mathcal{A})$ acts as multiplication by $\psi(a)$. It is zero unless g fixes ψ , because ι_g kills $(g-1)\pi_2(\mathcal{A})$.

Transport along a loop-space path whose marked point traces x carries the quotient at (g, ψ) to the quotient at $(xgx^{-1}, x \cdot \psi)$. Thus the formula defines a local system. It depends only on the set $[\phi]$, not on its chosen representative. When the orbit is a singleton, write Q_ϕ .

There is a canonical quotient map $\mathcal{H} \rightarrow Q_{[\phi]}\mathcal{H}$, sending a vector to its class in every summand. Although each component is a quotient map, their joint surjectivity merits explanation: distinct characters define pairwise comaximal maximal ideals of $\mathbb{C}[\pi_2(\mathcal{A})]$, and the finite Chinese remainder theorem gives joint surjectivity for any module over this algebra. Every map from $\mathcal{H}$ to a system with the orbit decomposition factors uniquely through it. Thus $Q_{[\phi]}$ is the left adjoint to inclusion of the subcategory satisfying that decomposition, allowing arbitrary-dimensional character spaces.

For infinite $\pi_2(\mathcal{A})$ this need not be a projection onto eigenvectors already present in $\mathcal{H}$. For example, let a generator of $\pi_2(\mathcal{A}) = \mathbb{Z}$ act on a two-dimensional fiber by a nontrivial unipotent Jordan block. The quotient for the trivial character is the quotient by the image of this operator minus the identity, whereas the invariant vectors form its kernel. In this example the invariant line maps to zero in the quotient. Thus the quotient construction and the invariant-subspace construction must be distinguished; the quotient functor need not be exact.

For finite $\pi_2(\mathcal{A})$ the quotient is canonically the isotypic summand obtained by summing the idempotents

$$e_\psi = \frac{1}{|\pi_2(\mathcal{A})|} \sum_{a \in \pi_2(\mathcal{A})} \psi(a)^{-1} [a]$$

over the orbit. In the disk representation, its ψ -quotient is the line given by $[a] \mapsto \psi(a)$. Consequently the orbit quotient of the disk is finite-dimensional for every $\pi_2(\mathcal{A})$ and has the fiber and action stated before Thm. 3.1.

Extension cocycles, transgression, and projective multipliers. Recall that the normalized cocycle $k \in Z^3(\pi_1(\mathcal{A}); \pi_2(\mathcal{A}))$ records the inverse 2-group associator ((14)). Write $\pi_2(\mathcal{A})$ additively. For composable conjugation arrows $g \xrightarrow{y} h \xrightarrow{x} j$, where $h = ygy^{-1}$ and $j = xhx^{-1}$, put

$$t(x, y; g) := k(x, h, y) - k(x, y, g) - k(j, x, y). \quad (47)$$

In general this label is used modulo $(j-1)\pi_2(\mathcal{A})$, with the coefficient action in (30). For $x, y \in Z_{\pi_1(\mathcal{A})}(g)$, where this centralizer consists of the elements commuting with g , it yields

$$t_g(x, y) := k(x, g, y) - k(x, y, g) - k(g, x, y) \pmod{(g-1)\pi_2(\mathcal{A})}. \quad (48)$$

It is a 2-cocycle with coefficients in the centralizer module $\pi_2(\mathcal{A})/(g-1)\pi_2(\mathcal{A})$, classifying (19). The extension multiplication is

$$(\bar{a}, x)(\bar{b}, y) = (\bar{a} + x \cdot \bar{b} + t_g(x, y), xy).$$

For trivial action these formulas are ordinary $\pi_2(\mathcal{A})$ -valued groupoid and group cocycles. Their signs agree with [GU16, arXiv version, Prop. 3.5] after replacing that source's forward-associator cocycle by $-k$.

For the homotopy-theoretic identification of ordinary transgression, [SS24, arXiv v1, Lem. 3.3 and Thm. 3.4] computes simplicial evaluation followed by Eilenberg–Zilber. For trivial coefficients it identifies looping a cocycle map $B\pi_1(\mathcal{A}) \rightarrow B^3\pi_2(\mathcal{A})$ and projection to $B^2\pi_2(\mathcal{A})$ with the cochain operation. With nontrivial action the Postnikov class has

local coefficients and is not such an untwisted cocycle map. The square-pasting calculation in the proof gives the quotient-valued formula directly; after restriction to a character stabilizer and evaluation by that character it reduces to ordinary scalar transgression of the forward-associator cocycle, as specified below.

Definition B.7 (Projective multiplier convention). For a group H , a normalized scalar 2-cocycle is a function $\beta : H \times H \rightarrow \mathbb{C}^\times$ satisfying

$$\beta(x, y)\beta(xy, z) = \beta(y, z)\beta(x, yz), \quad \beta(e, x) = \beta(x, e) = 1.$$

A β -projective representation is a vector space with invertible operators R_x satisfying $R_e = \text{id}$ and $R_x R_y = \beta(x, y) R_{xy}$. The cocycle equation ensures associativity of these relations. Morphisms are linear maps intertwining every R_x ; write $\text{Rep}_\beta^{\text{fd}}(H)$ for the finite-dimensional representations. For $\beta = 1$ this is the ordinary representation category.

Return to the chosen character ϕ . Its stabilizer $\text{Stab}(\phi)$ consists of the elements fixing it under the action above. The scalar cocycle $\omega_\phi = (\phi \circ k)|_{\text{Stab}(\phi)^3}$ is defined in (18). For $g \in \text{Stab}(\phi)$ and $x, y \in Z_{\text{Stab}(\phi)}(g)$, evaluation of (48) gives

$$\beta_{g,\phi}(x, y) = \phi(t_g(x, y)) = \frac{\omega_\phi(x, g, y)}{\omega_\phi(x, y, g)\omega_\phi(g, x, y)}. \quad (49)$$

Indeed, g fixes ϕ , so ϕ is trivial on $(g - 1)\pi_2(\mathcal{A})$, and the stabilizer acts trivially after applying ϕ . Restriction of a local system to its ϕ -character space therefore gives these projective representations. The sector $\text{Loc}_{[\phi]}^{\text{fd}}(L\mathcal{A})$ consists of the finite-dimensional local systems whose fibers decompose as actual character spaces for characters in $[\phi]$ ((21)). Its underlying linear category is

$$\text{Loc}_{[\phi]}^{\text{fd}}(L\mathcal{A}) \simeq \bigoplus_{[g] \subset \text{Stab}(\phi)} \text{Rep}_{\beta_{g,\phi}}^{\text{fd}}(Z_{\text{Stab}(\phi)}(g)). \quad (50)$$

Here conjugacy in (50) is taken inside the stabilizer. In particular simple objects are indexed by such a conjugacy class and an irreducible projective representation of its centralizer. If the action is trivial, these are the familiar centralizers in the whole $\pi_1(\mathcal{A})$; if $\pi_2(\mathcal{A}) = 0$, the multiplier is 1.

Remark B.8 (Transgression sign conventions). To compare signs with the literature, let τ denote Willerton's multiplicative transgression [Wil08, arXiv version, Thm. 3]. On this centralizer it is

$$(\tau\omega_\phi)_g(x, y) = \frac{\omega_\phi(x, y, g)\omega_\phi(g, x, y)}{\omega_\phi(x, g, y)}.$$

Thus $\beta_{g,\phi} = (\tau\omega_\phi)_g^{-1}$, equivalently the transgression of ω_ϕ^{-1} . This is required by our conventions: k records the inverse 2-group associator, while the graded category has forward associator ω_ϕ . The extension label t transgresses the forward 2-group associator $-k$. Consequently one must convert these conventions when comparing twisted-double parameters; the scalar 3-cocycle and its projective multiplier should not be identified without this inverse.

The half-braiding identity gives precisely the same sign: on a degree- g vector, with $h = ygy^{-1}$ and $j = xhx^{-1}$, it reads

$$R_{xy} = \frac{\omega_\phi(x, y, g)\omega_\phi(j, x, y)}{\omega_\phi(x, h, y)} R_x R_y.$$

A different choice of lifts of centralizer elements into the loop extension changes the multiplier by a coboundary and rescales the operators R_x . This comparison determines the underlying

category; the surface calculation in the theorem’s proof supplies its tensor product and coherence.

Remark B.9 (Surface operations and their coherence data). A disk with n holes retracts onto a wedge of n circles. Boundary restriction gives the n -input version of (22); parametrizations and connecting arcs determine the multiplication order. Gluing gives homotopy pullbacks of mapping spaces and the base-change maps of Lem. B.1.

The comparison of the two gluings of two pairs of pants gives the associator; disk gluings give the unitors; their isotopies give the pentagon and triangle. Choose the exchange

$$(g, h) \mapsto (ghg^{-1}, g), \quad v_g \otimes w_h \mapsto R_g^W(w_h) \otimes v_g.$$

This agrees with (6); the opposite exchange gives the opposite braiding. Compatibility with gluing gives the hexagons. The proof verifies these geometric maps on the character spaces.

Here v_g and w_h are vectors in the degree- g and degree- h fibers, and R_g^W denotes transport in the second local system along the conjugation path labeled by g , as in the theorem’s proof. The distinction from pointwise tensor product is visible on the kernel labels. Outgoing restriction over (g, h) sends

$$(a, b) \mapsto a + g \cdot b,$$

with simultaneous-modification quotients understood. Its coinvariant relation selects $\psi = g \cdot \chi$ on $V_{g,\psi} \otimes W_{h,\chi}$; since g fixes ψ , this means $\psi = \chi$. The fusion product consequently preserves a character orbit and is zero across distinct orbits. Pointwise tensor product instead multiplies the characters, and is a different operation.

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