

A GENERALIZATION OF THE WHITEHEAD THEOREM

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1. INTRODUCTION

Let $f: X \rightarrow Y$ be a map of CW-complexes. If f induces an isomorphism $H_*f: H_*(X;Z) \rightarrow H_*(Y;Z)$ and X and Y are simply-connected, then $\pi_*f: \pi_*X \rightarrow \pi_*Y$ is also an isomorphism. This is the classical Whitehead theorem. It is well known that if either X or Y is not simply-connected then the Whitehead theorem need not be true.

Quillen [3] proved that in the category of simplicial pro-p-groups the Whitehead theorem holds. Stallings [4] showed that if π_1X and π_1Y are nilpotent groups, then a low dimensional integral homology isomorphism implies an isomorphism on π_1 .

In the present work the methods of Quillen and Stallings are developed and used to get a generalization of the Whitehead Theorem: We assume that $H_*(f,Z)$ is an isomorphism and ask what further conditions on π_*f will guarantee that π_*f is also an isomorphism.

We work in the category of spaces with a base point which have the homotopy type of a connected CW-complex. The notation H_* is used for the integral homology functor.

The paper is organized as follows: The functors needed for stating the main theorem are defined in Section 2, and the main theorem itself is formulated in Section 3. Section 4 is devoted to examples and corollaries and Sections 5 and 6 to the proof of the main theorem. The paper closes with an appendix on the nilpotent and solvable cases. I would like to thank Professor D. M. Kan and A. K. Bousfield for numerous conversations, ideas, and hints.

2. PRELIMINARIES

2.1 Definition

Let π be a group and G a π -group, i.e., a group together with a homomorphism $\pi \xrightarrow{\psi} \text{aut } G$.

We define the lower central series of π -groups

$$\dots \subseteq \Gamma_{n+1}G \subseteq \Gamma_n G \subseteq \dots \subseteq \Gamma_2G \subseteq \Gamma_1G = G$$

as follows:

- (i) Γ_2G is the normal π -subgroup of G generated by all elements of the form $((\psi x)g)g^{-1}$ with $x \in \pi$ and $g \in G$.
- (ii) $\Gamma_n G = \Gamma_2 \Gamma_{n-1} G$.

Thus, the group $\Gamma_n G$ is the kernel of the projection of $\Gamma_{n-1} G$ onto its largest quotient which is a trivial π -group. Clearly the lower central series is functorial in both π and G in the obvious sense.

2.2 Note

In this paper we are only interested in the lower central series filtration for two cases of π -groups. The non-abelian case is where G is not abelian: here we will only consider the case when $G = \pi$ and $(\psi x)g = xgx^{-1}$ for all $x, g \in G$. In this case $\Gamma_n G$ is the usual lower central series. The abelian case is where G is abelian and can be thought of as π -module or $Z(\pi)$ -module, where $Z(\pi)$ is the group ring of π . In this case the filtration $\Gamma_n G$ is the usual filtration of G induced by the powers I^n of the augmentation ideal $I = \ker(Z(\pi) \rightarrow Z)$.

Note that in both cases $\Gamma_n G$ are normal π -subgroups of G . Thus the following definitions make sense.

2.3 Definition

The π -completion \hat{G} is defined to be the inverse limit of the following tower of epimorphisms

$$\dots \rightarrow G/\Gamma_{n+1} G \rightarrow G/\Gamma_n G \rightarrow \dots \rightarrow G/\Gamma_2 G \rightarrow (e) \quad .$$

The π -completion \hat{G} , being the inverse limit of the tower, comes with a canonical map $i: G \rightarrow \hat{G}$.

2.4 Definition

The kernel of $i: G \rightarrow \hat{G}$ is denoted by $\Gamma_\infty G$. Alternatively, we have $\Gamma_\infty G = \bigcap_{k < \infty} \Gamma_k G$.

2.5 Definition

If G is an abelian π -group, then so is \hat{G} and we can define $\Gamma'_\infty G = \text{coker}(i: G \rightarrow \hat{G})$. If G is not abelian then $\Gamma'_\infty G$ is the pointed set whose elements are the left cosets of $i(G)$ in \hat{G} , and with the obvious distinguished element.

2.6 Definition

The maximal π -perfect subgroup of G .

In general we say that a π -group H is (π) -perfect if $\Gamma_2 H = H$. It is clear that the π -subgroup generated by a family of π -perfect subgroups of G is again π -perfect. So it follows that every π -group G contains a unique maximal π -perfect subgroup which we denote by ΓG . Clearly $\Gamma G \subseteq \Gamma_\infty G$. Note that in the non-abelian case when $\pi = G$ the subgroup ΓG need not be a perfect group.

2.7 Examples

Let $\pi = \mathbb{Z}_2$ (the integers mod 2) and let us look at three different \mathbb{Z}_2 -modules. (i) \mathbb{Z} , (ii) \mathbb{Z}_4 , (iii) \mathbb{Z}_3 . In all cases \mathbb{Z}_2 acts by sending 1 to -1.

(i) The lower central series is

$$\dots \subseteq 2^r \mathbb{Z} \subseteq \dots \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \quad .$$

The completion $\hat{\mathbb{Z}}$ is the group of the 2-adics, and $\Gamma\mathbb{Z} = \Gamma_\infty\mathbb{Z} = 0$.

(ii) The lower central series is

$$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_4 \quad .$$

One has $\hat{\mathbb{Z}}_4 = \mathbb{Z}_4$ and $\Gamma_\infty\mathbb{Z}_4 = \Gamma'_\infty\mathbb{Z}_4 = \Gamma\mathbb{Z}_4 = 0$.

(iii) The lower central series is constant:

$$\dots = \mathbb{Z}_3 = \mathbb{Z}_3 \quad .$$

Thus, we have $\Gamma'_\infty\mathbb{Z}_3 = 0$ and $\Gamma_\infty\mathbb{Z} = \Gamma\mathbb{Z}_3 = \mathbb{Z}_3$. Thus \mathbb{Z}_3 is an example of a perfect \mathbb{Z}_2 -module.

3. THE MAIN THEOREM

If X is a space then $\pi_n X$ is naturally a $\pi_1 X$ -group. Therefore, we can regard $\hat{\pi}_*, \Gamma_\infty\pi_*, \Gamma'_\infty\pi_*, \Gamma\pi_*$ as functors on spaces.

In terms of these functors we can proceed to formulate the following generalized Whitehead theorem.

3.1 Theorem

Let $f: X \rightarrow Y$ be a map of connected spaces such that H_*f is an isomorphism. Then π_*f is also an isomorphism if, in addition, f satisfies the following three conditions:

- (i) $\Gamma_\infty\pi_*f$ is an epimorphism,
- (ii) $\Gamma'_\infty\pi_*f$ is a monomorphism, and
- (iii) $\Gamma\pi_*f$ is a monomorphism.

The theorem follows by induction from the following proposition.

3.2 Proposition

Let $f: X \rightarrow Y$ be a map of connected spaces such that for some $n \geq 1$, $\pi_j f: \pi_j X \rightarrow \pi_j Y$ is an isomorphism for $0 \leq j \leq n-1$. If $H_j f$ is an isomorphism for $j = n$ and an epimorphism for $j = n+1$, then $\hat{\pi}_n f: \hat{\pi}_n X \rightarrow \hat{\pi}_n Y$ is an isomorphism. If in addition f satisfies

- (i) $\Gamma_\infty\pi_n f$ is an epimorphism,
- (ii) $\Gamma'_\infty\pi_n f$ is a monomorphism, and

(iii) $\Gamma_{\pi_n} f$ is a monomorphism,

then $\pi_n f$ is an isomorphism.

Before going into the proof of Proposition 3.2, which is given in Sections 5 and 6, let us look at some examples and corollaries.

4. EXAMPLES

Let $f: X \rightarrow Y$ be a map of connected spaces.

4.1 The Simply-Connected Case

For any simply-connected space Z , $\Gamma_{\infty} \pi_{\star} Z = \Gamma'_{\infty} \pi_{\star} Z = \Gamma_{\pi_{\star}} Z = 0$. Thus if both X and Y are simply-connected, we recover from Theorem 2.1 the classical Whitehead Theorem.

4.2 The Simple Case

If both X and Y are simple (i.e., $\Gamma_2 \pi_{\star} X = \Gamma_2 \pi_{\star} Y = 0$) then again conditions 3.1 (i)-(iii) are trivially satisfied and thus $\pi_{\star} f$ is an isomorphism if $H_{\star} f$ is. This example includes any map between H-spaces. Note that in this case obstruction theory arguments or a careful relative Hurewicz argument would do. For dimensions > 1 the same proof will work in the next two examples 4.3 and 4.4.

4.3 The Nilpotent Case

We call a connected space X nilpotent if for all $n \geq 0$ there exists r such that $\Gamma_r \pi_n X = 0$. If X is nilpotent, then $\Gamma_{\infty} \pi_{\star} X = \Gamma'_{\infty} \pi_{\star} X = \Gamma_{\pi_{\star}} X = 0$. Thus, it follows that if both X and Y are nilpotent space and $H_{\star} f$ is an isomorphism, then so is $\pi_{\star} f$. The nilpotent case is discussed in more detail in the Appendix.

4.4 The Complete Case

The most general case in which Γ_{∞} , Γ'_{∞} and Γ vanish, is when the map $\pi_{\star} X \rightarrow \hat{\pi}_{\star} X$ (see 2.3) is an isomorphism, in which case we say that X is (π -)complete. So for a map $f: X \rightarrow Y$ between two complete spaces, $\pi_{\star} f$ is isomorphic if $H_{\star} f$ is. As an example of a π -complete space one can take the geometric realization of the pro- p -completion [3] of a free semi-simplicial group which is finitely generated in each dimension (Bousfield).

4.5 Perfect Fundamental Group

If $\pi_1 X$ is a perfect group, i.e., $\Gamma_2 \pi_1 X = \pi_1 X$ (that is, $H_1 X = 0$) then $\Gamma_{\pi_{\star}} X = \Gamma_{\infty} \pi_{\star} X = \Gamma_2 \pi_{\star} X$ which means that $\Gamma_2 \pi_{\star} X$ is $\pi_1 X$ -perfect (see 2.6). To see this let us recall that for any abelian π -group A , $H_0(\pi, A) = A/\Gamma_2 A$. Hence it follows

from the exactness of the coefficient sequence

$$H_1(\pi; A/\Gamma_2 A) \rightarrow H_0(\pi; \Gamma_2 A) \rightarrow H_0(\pi; A) \xrightarrow{\cong} H_0(\pi; A/\Gamma_2 A) \quad ,$$

in which the leftmost group is trivial by the assumption on π , that $\Gamma_2 A/\Gamma_3 A \cong H_0(\pi; \Gamma_2 A) = 0$, which means that $\Gamma A = \Gamma_\infty A = \Gamma_2 A$. Further this implies $\Gamma'_\infty A = 0$ and so one gets:

4.6 Corollary

Assume that $H_1 X = H_1 Y = 0$ and $H_ f$ is an isomorphism. Then, if $\Gamma_{\pi_*} f$ is an isomorphism, so is $\pi_* f$.*

As a special case, one gets the following result.

4.7 Corollary

Let $f: X \rightarrow Y$ be a map of acyclic spaces (i.e., $\tilde{H}_ X = \tilde{H}_* Y = 0$). Then f is a homotopy equivalence if $\Gamma_{\pi_*} f$ is an isomorphism.*

Corollary 4.7 suggests that in the category of acyclic spaces the functor Γ_{π_*} plays a role similar to that played by the functor π_* in the category of all CW-complexes. This idea is basic in the analysis of acyclic spaces given in [1].

4.8 The Case Where f is A Retraction

If the map $f: X \rightarrow Y$ is a retraction, i.e., there exists a one-sided inverse $i: Y \rightarrow X$ such that $fi \sim id_Y$, then it is easy to see that conditions 3.1 (i) and (ii) are satisfied, if $H_* f$ is an isomorphism. That $\Gamma_\infty \pi_* f$ is an epimorphism is clear. To show that $\Gamma'_\infty \pi_* f$ is a monomorphism, one needs to use, inductively, Proposition 3.2. In fact one just applies the five-lemma and 3.2 to the exact sequence

$$\pi_n \rightarrow \hat{\pi}_n \rightarrow \Gamma_\infty \pi_n \rightarrow 0 \quad .$$

A map which is an isomorphism on $\hat{\pi}_n$ and epimorphic on π_n must be an isomorphism on $\Gamma_\infty \pi_n$. Thus, one gets the useful result

4.9 Corollary

Let f be a retraction and assume that $H_ f$ is bijective. Then $\pi_* f$ is bijective if $\Gamma_{\pi_*} f$ is injective.*

5. AN ALGEBRAIC LEMMA

Proposition 3.2 is based on corresponding algebraic propositions for the abelian and non-abelian cases (see 2.2).

Since the non-abelian case was proved by Stallings [4], complete proofs will be given only for the abelian case.

5.1 Proposition (Stallings)

Let $f: \pi \rightarrow \pi'$ be a map of groups and assume that $H_1 f$ is bijective and $H_2 f$ is surjective. Then the induced map $\pi/\Gamma_r \pi \rightarrow \pi'/\Gamma_r \pi'$ is an isomorphism for all r ; thus so is \hat{f} (2.2 and 2.3).

Proof. The proof is sketched after the proof of the following proposition.

5.2 Proposition

Let $f: A \rightarrow A'$ be a map of π -modules. If $H_0(\pi; f)$ is bijective, and $H_1(\pi; f)$ is surjective, then $f/\Gamma_r: A/\Gamma_r A \rightarrow A'/\Gamma_r A'$ is an isomorphism for all $r \geq 1$ and thus so is \hat{f} (see 2.3). The crucial step in the proof of Proposition 5.2 is the following lemma.

5.3 Lemma

Let $f: A \rightarrow A'$ be a map of π -modules which carries the submodule $B \subseteq \Gamma_2 A$ into the submodule $B' \subseteq \Gamma_2 A'$. Assume that $H_1(\pi; f): H_1(\pi; A) \rightarrow H_1(\pi; A')$ is surjective and that f induces an isomorphism $A/B \xrightarrow{\cong} A'/B'$. Then f induces an isomorphism $A/\Gamma_2 B \xrightarrow{\cong} A'/\Gamma_2 B'$.

Proof of 5.3. Consider the (twisted) coefficient exact sequence

$$H_1(\pi; A) \rightarrow H_1(\pi; A/B) \rightarrow H_0(\pi; B) \rightarrow H_0(\pi; A) \rightarrow H_0(\pi; A/B) \quad .$$

Since $H_0(\pi; B) = B/\Gamma_2 B$ and $A/\Gamma_2 A \xrightarrow{\cong} (A/B)/\Gamma_2(A/B)$, one gets the map of exact sequences

$$\begin{array}{ccccccc} H_1(\pi; A) & \rightarrow & H_1(\pi; A/B) & \rightarrow & B/\Gamma_2 B & \rightarrow & 0 \\ \downarrow & & \downarrow \cong & & \downarrow & & \\ H_1(\pi; A') & \rightarrow & H_1(\pi; A'/B') & \rightarrow & B'/\Gamma_2 B' & \rightarrow & 0 \end{array} \quad .$$

Hence, it follows from the assumptions and the five-lemma that $B/\Gamma_2 B \xrightarrow{\cong} B'/\Gamma_2 B'$.

Applying the five-lemma again to the sequence $0 \rightarrow B/\Gamma_2 B \rightarrow A/\Gamma_2 B \rightarrow A/B \rightarrow 0$, Lemma 5.3 follows from the assumptions.

Proof of Proposition 5.2. Since $\hat{A} = \varprojlim A/\Gamma_n A$, it is enough to prove that $A/\Gamma_n A \xrightarrow{\cong} A'/\Gamma_n A'$ for all $n \geq 1$. In fact since $H_0(\pi; f)$ is an isomorphism, so is $A/\Gamma_2 A \xrightarrow{\cong} A'/\Gamma_2 A'$. Assuming the isomorphism, by induction, for $n = r$, one simply applies Lemma 5.3 to get it for $n = r + 1$, since $\Gamma_{r+1} A = \Gamma_2 \Gamma_r A$. Q.E.D.

Exactly the same method is used to prove Proposition 5.1, except that instead of the coefficient sequence used in the proof of Lemma 5.3 one uses the low-dimensional homology exact sequence associated with the extension $0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0$:

$$H_2 G \rightarrow H_2(G/H) \rightarrow (G/H)/\Gamma_2(G/H) \rightarrow 0$$

for $H \subseteq \Gamma_2 G$ a normal subgroup of G .

This completes the algebraic preparation and we can now turn to the proof of the main theorem.

6. PROOF OF 3.2

For $n = 1$ the main homotopy-theoretical observation is that the canonical map $H_2X \rightarrow H_2\pi_1X$ is always surjective. This is due to Hopf [2]. Under the conditions of 3.2 this implies that the map $H_2\pi_1X \rightarrow H_2\pi_1Y$ is surjective, and enables us to apply 5.1. In the higher dimensions the situation is slightly more complicated since we do not have, in general, a surjection of $H_{n+1}X$ onto $H_1(\pi_1X, \pi_nX)$. Still we have the following situation.

6.1 Lemma

Let $f: X \rightarrow Y$ be a map of connected spaces such that $\pi_j f$ is an isomorphism for $0 \leq j \leq n-1$. Assume that $H_n f$ is bijective and that $H_{n+1} f$ is surjective. Then:

- (i) For $n = 1$, $H_i f$ is bijective for $i = 1$ and surjective for $i = 2$.
- (ii) For $n > 1$, $H_i(\pi_1X, \pi_n f)$ is bijective for $i = 0$ and surjective for $i = 1$.

Proof: We use the spectral sequence of the fibration $K(\pi_n, n) \rightarrow P_n X \rightarrow P_{n-1} X$ where $P_i X$ denotes the i -th stage of the Postnikov tower of X . We have

$$E_{p,q}^2 = H_p(P_{n-1}X, H_q K(\pi_n, n)) \Rightarrow H_{p+q} P_n X$$

where $H_q K(\pi_n, n)$ is regarded as a $\pi_1 P_{n-1} X$ -module. Since $H_{n+1} K(\pi, n) = 0$ for all π and $n > 1$, we have for $n > 1$ only two non-zero groups in the E^2 term with total degree $n+1$, namely $E_{1,n}^2$ and $E_{n+1,0}^2$. It follows easily from the usual arguments that the classical Serre exact sequence can be extended to get:

$$\begin{aligned} H_{n+2} P_n X &\rightarrow H_{n+2} P_{n-1} X \rightarrow H_1(\pi_1 X; \pi_n X) \rightarrow H_{n+1} P_n X \rightarrow H_{n+1} P_{n-1} X \rightarrow \\ &\rightarrow H_0(\pi_1 X; \pi_n X) \rightarrow H_n X \rightarrow H_n P_{n-1} X \rightarrow 0 \end{aligned}$$

It follows from the last two terms of this exact sequence for the case $H_{n+1} X \rightarrow H_{n+1} P_n X \rightarrow 0$ that if $H_{n+1} f$ is surjective $H_{n+1} P_n X \rightarrow H_{n+1} P_n Y$ is surjective, too. Since we assume that $P_{n-1} X \xrightarrow{\sim} P_{n-1} Y$ is an equivalence, we may apply the map f to the extended Serre exact sequence above, and Lemma 6.1 easily follows from the five-lemma. Note that (i) follows from the last two terms of the sequence for $n = 2$.

Proof of 3.2. The conditions imply that $\pi_n f$ satisfies, by Lemma 6.1, the assumptions of Proposition 5.1 and 5.2. Hence, by those propositions $\hat{\pi}_n f$ is an isomorphism. Further, since one has the exact sequence of functors

$$0 \rightarrow \Gamma_\infty \pi_n \rightarrow \pi_n \rightarrow \hat{\pi}_n \rightarrow \Gamma'_\infty \pi_n \rightarrow 0 \quad ,$$

the assumptions imply that $\pi_n f$ is an epimorphism. It remains to show that $\pi_n f$ is a monomorphism. For this we use the exact sequence of functors

$$0 \rightarrow \Gamma \pi_n \rightarrow \pi_n \rightarrow \pi_n / \Gamma \pi_n \rightarrow 0$$

Since by assumption $\Gamma \pi_n f$ is injective, by the five-lemma one need only check that $\pi_n X / \Gamma \pi_n X \rightarrow \pi_n Y / \Gamma \pi_n Y$ is a monomorphism. This follows from the fact that $\Gamma \pi_n X$ is in fact $\Gamma_\eta \pi_n X$ for some transfinite ordinal η . One defines the transfinite lower central series of a π -group G as the following series of π -subgroups:

- (a) If $\alpha = \beta + 1$, then $\Gamma_\alpha G = \Gamma_2 \Gamma_\beta G$.
- (b) If α is a limit ordinal then $\Gamma_\alpha G = \bigcap_{\beta < \alpha} \Gamma_\beta G$.

For cardinality reasons it is clear that for some ordinal η , $\Gamma_{\eta+1} G = \Gamma_\eta G$. In fact this is true for any η whose cardinality is greater than the cardinality of the set of π -subgroups of G . On the other hand, $\Gamma G \subseteq \Gamma_\alpha G$ for all α , thus $\Gamma G = \Gamma_\eta G$ for the above η . Now since $\pi_n f$ is an epimorphism one easily sees, using Lemma 5.3 for non-limit ordinals and the epimorphism $\pi_n f$ for the limit ones, that for any transfinite ordinal α the induced map $\pi_n X / \Gamma_\alpha \pi_n X \rightarrow \pi_n Y / \Gamma_\alpha \pi_n Y$ is an isomorphism. This makes the map $\pi_n X / \Gamma \pi_n X \rightarrow \pi_n Y / \Gamma \pi_n Y$ an isomorphism if 3.2 (i)-(ii) are satisfied. Assumption 3.2 (iii) now completes the proof of 3.2, and thus the proof of 3.1.

7. APPENDIX: NILPOTENT SPACES (see 4.3)

A π -group G is called (π -)nilpotent if $\Gamma_r G = 0$ for some $r \geq 1$. For example, if both π and G are finite p -groups for some prime p , then G must be a nilpotent π -group. It is well known that any finite nilpotent group is the product of p -groups. Likewise any nilpotent space with finite homotopy groups is homotopy-equivalent to a product of p -spaces X_p (i.e., $\pi_* X_p$ is a p -group). This can be shown using the following description of nilpotent spaces.

Geometrically speaking, a space is nilpotent iff its Postnikov tower can be refined to a tower of principal fibrations with connected abelian fibers.

One of the advantages of the category of nilpotent spaces over that of simply-connected spaces is that it is closed under certain constructions.

7.1 A Pull-back Lemma

Let $f: X \rightarrow B$, $g: Y \rightarrow B$ be fibrations, and let E be the fiber-product in the pull-back diagram:

$$\begin{array}{ccc} E & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & B \end{array}$$

Consider $\pi_* X$, $\pi_* Y$, and $\pi_* B$ as $\pi_1 E$ -groups via this diagram. If $\pi_* X$, $\pi_* Y$, and $\pi_* B$ are nilpotent as $\pi_1 E$ -groups and E is connected then E is nilpotent.

Proof. We first prove that $\pi_1 E$ is a nilpotent group. We look at the sequence

$$\Omega(X \times Y) \xrightarrow{d} \Omega B \rightarrow E \xrightarrow{fg} X \times Y \xrightarrow[\begin{smallmatrix} f \\ g \end{smallmatrix}]{\begin{smallmatrix} f \\ g \end{smallmatrix}} B \quad .$$

Thus, $\pi_1 E$ is given as an extension: $0 \rightarrow \text{coker } \pi_1 d \rightarrow \pi_1 E \rightarrow \sigma_1 \rightarrow 0$, where $\sigma_1 = \{\alpha \in \pi_1 X \times Y \mid \pi_1 f(\alpha) = \pi_1 g(\alpha)\}$. Since by assumption σ_1 is nilpotent, and acts nilpotently on $\pi_1 \Omega B = \pi_2 B$, it follows that $\pi_1 E$ is nilpotent. Likewise in higher dimensions $\pi_n E$ is given as an extension of $\pi_1 E$ -groups: $0 \rightarrow \text{coker } \pi_n d \rightarrow \pi_n E \rightarrow \ker \pi_{n+1} d \rightarrow 0$. We only need to see that $\pi_1 E$ acts nilpotently on $\text{coker } \pi_n d$. Since $\pi_1 \Omega B$ acts trivially on $\pi_n \Omega B$ the map $\pi_1 \rightarrow \text{aut}(\text{coker } \pi_n d)$ factors through $\sigma_1 \rightarrow \text{aut}(\text{coker } \pi_n d)$ which is, by assumption, a nilpotent action. This completes the proof.

One interesting corollary is:

7.2 Corollary (Bousfield)

Let $F \rightarrow E \xrightarrow{P} B$ be a fibration, with nilpotent E and connected fiber F . Then F is also nilpotent.

7.3 "Transfinitely Solvable" Fundamental Group

There is an interesting analogue to Stallings' Theorem 5.1 and Theorem 3.1, involving the derived series of the fundamental group and homology with module coefficients.

7.4 Proposition

Let $f: \pi \rightarrow \sigma$ be a map of groups and let $\Lambda = \mathbb{Z}(\sigma)$ be the group ring of σ . If the twisted coefficients map $H_0(f; \Lambda)$ is monic then f is epic. If, in addition, $H_1(f; \Lambda)$ is isomorphic, then for any ordinal $\alpha \geq 0$, the map $f/D_\alpha: \pi/D_\alpha \pi \xrightarrow{\cong} \sigma/D_\alpha \sigma$ is an isomorphism, where D_α denotes the α -term of the transfinite derived series functor. Thus, $\pi/P \cong \sigma/P\sigma$ is also an isomorphism, where P denotes "the maximal perfect subgroup" functor.

Note that $H_*(f; \Lambda)$ is an isomorphism iff for any σ -module M , $H_*(f; M)$ is an isomorphism.

Proof. The first part is clear since under the condition $\Lambda \otimes_{\mathbb{Z}(\pi)} \mathbb{Z} = \mathbb{Z}$; thus $\mathbb{Z}(\pi) \rightarrow \Lambda$ is onto. The proof of the second part parallels Stallings' transfinite induction argument in [4], using the equation

$$H_1(G, \mathbb{Z}(G/D_\alpha G)) \cong D_\alpha G/D_{\alpha+1} G \quad .$$

As an immediate corollary to the proposition one has

7.5 Corollary

Let $f: X \rightarrow Y$ be a map with $H_*(f; \Lambda)$ an isomorphism, where $\Lambda = \mathbb{Z}(\pi_1 Y)$.
 Then $\pi_* f$ is an isomorphism if and only if $P\pi_1 f$ is a monomorphism.

Thus if $\pi_1 X$ contains no perfect subgroups one has

$$H_*(f; \Lambda) \text{ bijective} \Leftrightarrow \pi_* f \text{ bijective} \quad .$$

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