Model Categories and More General Abstract Homotopy Theory: A Work in What We Like to Think of as Progress

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Part 1

Basic definitions and examples

In [**14**] Quillen introduced the notion of a *model category*, i.e. a category with three distinguished classes of maps (called *weak equivalences*, *cofibrations* and *fibrations*) satisfying a few simple axioms and observed that in such a model category one can "do homotopy theory". Our aim in part 1 of this monograph (i.e. the first three chapters) is to give an updated version of his definitions and to discuss some examples. Chapter I contains the definitions of *model categories* and *Quillen functors* between them, as well as a few immediate consequences of these definitions. In chapter II we discuss some of the original examples of model categories, *simplicial sets*, *topological spaces* and *simplicial algebras* and their *diagram categories* and we note that these model categories are all *cofibrantly generated*. And in chapter III we show that the *simplicial* and *cosimplicial* diagrams in a model category admit a so-called Reedy model category structure and that the same holds for any category of diagrams in a model category, indexed by a *Reedy category*.

CHAPTER I

Model categories and Quillen functors

1. Introduction

1.1. Summary. In this first chapter we

(i) introduce the notion of a *model category*, i.e. a category $\boldsymbol{M}$ with three distinguished classes of maps (which are actually subcategories), called *weak equivalences*, *fibrations* and *cofibrations*, satisfying a few simple axioms which enable one to "do homotopy theory" in $\boldsymbol{M}$, and

(ii) define an appropriate kind of "morphisms between model categories" called *Quillen functors* and corresponding "equivalences between model categories" called *Quillen equivalences*.

Various examples will be given in chapters II and III.

In more detail:

1.2. Model categories. In dealing with model categories, the first question is to decide what is the "right" generality in which to work. Quillen [**14**] already noticed that "closed" model categories (i.e. model categories in which any two of the three distinguished classes of maps (1.1) determine the third) can be characterized by five particularly nice axioms [**16**] and that moreover the requirement that a model category be closed is not a serious one. In fact he showed that a model category is closed iff all three of its distinguished classes of maps are closed under retracts and from this it readily follows that one can turn any model category in which (as always seems to be the case) the class of the weak equivalences is closed under retracts, into a closed model category just by closing the other two classes under retracts. However the first and the last of these five axioms are weaker than one would expect; the first axiom assumes the existence of finite limits and colimits, but not of *arbitrary small* ones and the last axiom assumes the existence of certain factorizations of maps, but does not insist on their *functoriality.* While this allows for the inclusion of such categories as the various categories of finitely generated chain complexes, it also considerably complicates much of the theory. **We will therefore throughout use the term model category for a closed model category which satisfies the above suggested stronger versions of Quillen's first and fifth axioms**.

1.3. Quillen functors and Quillen equivalences. It turns out that the useful notion of "morphism between model categories" is not, as one would expect, a functor which is compatible with the model category structures in the sense that it preserves weak equivalences, cofibrations and fibrations, but a functor which is one of a pair of adjoint functors (called *Quillen functors*), each of which is compatible with one *half* of the model category structures in the sense that the left adjoint (the *left Quillen functor*) preserves cofibrations and trivial cofibrations (i.e. cofibrations which are also weak equivalences) and the right adjoint (the *right Quillen functor*) preserves fibrations and trivial fibrations.

There is a corresponding notion of "equivalences between model categories" (called *Quillen equivalences*). These are Quillen functors which, as we will see in the last part of this monograph, induce "equivalences of homotopy theories".

We end with some comments on the

1.4. Organization of the chapter. In order to keep our account of model categories more or less self contained, we start (in §2) with a review of several categorical notions. In §3 we then define *model categories* (1.2) and discuss some of the immediate consequences of this definition, while the last section (§4) deals with *Quillen functors* and *Quillen equivalences*.

2. Categorical preliminaries

We recall here some categorical terminology and notation. But first some remarks about

2.1. Universes. Unless otherwise stated we will work in an arbitrary but fixed **Grothendieck universe** $\mathcal{U}$, i.e. [**12**, Ch I] a set of sets (called **small sets** or **$\mathcal{U}$-sets**) with a few simple properties which ensure that the standard operations of set theory, when applied to $\mathcal{U}$-sets, produce again $\mathcal{U}$-sets. The exceptions are the few occasions in which we make a construction that does not necessarily again produce $\mathcal{U}$-sets. If that happens we will work in some fixed universe $\mathcal{U}'$ which is **higher** than $\mathcal{U}$, i.e. for which $\mathcal{U}$ is a $\mathcal{U}'$-set.

Accordingly we define

2.2. Categories and small categories. In view of 2.1 we use the term **category** for **category in $\mathcal{U}$**, i.e. a category $\boldsymbol{C}$ such that

(i) for every pair of objects $X, Y \in \boldsymbol{C}$, the hom-set $\boldsymbol{C}(X, Y)$ of the maps $X \to Y \in \boldsymbol{C}$ is *small*, i.e. a $\mathcal{U}$-set, and
(ii) the set of objects of $\boldsymbol{C}$ is not necessarily small, but is still a subset of (the set of sets) $\mathcal{U}$,

we call such a category **small** if

(ii)′ the set of objects of $\boldsymbol{C}$ is actually *small*, i.e. a $\mathcal{U}$-set,

and refer to a functor between two small categories as a **small functor**. As usual we denote by **Cat** the category of the small categories and the (small) functors between them.

Note however that (2.1) *every category or functor in $\mathcal{U}$ is a small category or functor in our higher universe $\mathcal{U}'$.*

2.3. Diagrams. Given a category $\boldsymbol{C}$ and a small (2.2) category $\boldsymbol{D}$, a **$\boldsymbol{D}$-diagram** in $\boldsymbol{C}$ is just a functor $\boldsymbol{D} \to \boldsymbol{C}$. They give rise to a **diagram category** $\boldsymbol{C}^{\boldsymbol{D}}$ which has these functors as objects and which has as maps the natural transformations between them. A functor $d\colon \boldsymbol{D} \to \boldsymbol{D}'$ between two small categories then clearly induces a functor $d^*\colon \boldsymbol{C}^{\boldsymbol{D}'} \to \boldsymbol{C}^{\boldsymbol{D}}$. If the category $\boldsymbol{D}$ is not small, this construction of $\boldsymbol{C}^{\boldsymbol{D}}$ still makes sense, except that $\boldsymbol{C}^{\boldsymbol{D}}$ then is not a category in our chosen universe $\mathcal{U}$, but only in our higher universe $\mathcal{U}'$ (2.2).

A trivial example is the category $\boldsymbol{C}^{\mathbf{0}}$ where $\mathbf{0}$ denotes the category with one object and no non-identity maps. It clearly is *canonically isomorphic* to $\boldsymbol{C}$. For every small category $\boldsymbol{D}$, there is a unique functor $c\colon \boldsymbol{D} \to \boldsymbol{0}$ and the induced functor $c^*\colon \boldsymbol{C} \to \boldsymbol{C}^{\boldsymbol{D}}$ is the **constant diagram functor**, i.e. the functor which sends every object of $\boldsymbol{C}$ to the corresponding constant $\boldsymbol{D}$-diagram.

Another example is the **category of maps** of a category $\boldsymbol{C}$, i.e. the diagram category $\boldsymbol{C}^{\boldsymbol{1}}$ where $\boldsymbol{1}$ denotes the category with two objects, 0 and 1, and one map $0 \to 1$.

Next we discuss colimits and limits, which we describe in terms of

2.4. Under and over categories. Given two categories $\boldsymbol{A}$ and $\boldsymbol{B}$, a functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ and an object $B \in \boldsymbol{B}$, the **under category** $(B \downarrow u)$ (resp. the **over category** $(u \downarrow B)$) of u at B is the category which has as objects the pairs (A, b), where A is an object of $\boldsymbol{A}$ and b is a map $b\colon B \to uA \in \boldsymbol{B}$ (resp. $b\colon uA \to B \in \boldsymbol{B}$), and which has a map $(A, b) \to (A', b')$ for every map $a\colon A \to A' \in \boldsymbol{A}$ such that $(ua)b = b'$ (resp. $b'(ua) = b$). These under and over categories come with an obvious **forgetful functor** to the category $\boldsymbol{A}$.

If $\boldsymbol{A} = \boldsymbol{B}$ and $u = 1_{\boldsymbol{B}}$ (the identity functor of $\boldsymbol{B}$), then one usually writes $(B \downarrow \boldsymbol{B})$ and $(\boldsymbol{B} \downarrow B)$ instead of $(B \downarrow 1_{\boldsymbol{B}})$ and $(1_{\boldsymbol{B}} \downarrow B)$. Furthermore, given a map $f\colon B_1 \to B_2 \in \boldsymbol{B}$, one denotes by $f_*\colon (\boldsymbol{B} \downarrow B_1) \to (\boldsymbol{B} \downarrow B_2)$ and $f^*\colon (B_2 \downarrow \boldsymbol{B}) \to (B_1 \downarrow \boldsymbol{B})$ the functors obtained by "composing with f".

2.5. Colimits and limits [**12**, Ch III]**.** Given categories $\boldsymbol{C}$ and $\boldsymbol{D}$ and an object $X \in \boldsymbol{C}^{\boldsymbol{D}}$ (2.3), a $\boldsymbol{D}$**-colimit** or **colimit** of X consists of an object $\operatorname{colim}^{\boldsymbol{D}} X \in \boldsymbol{C}$ and a map $t\colon X \to c^* \operatorname{colim}^{\boldsymbol{D}} X \in \boldsymbol{C}^{\boldsymbol{D}}$ such that the pair $(\operatorname{colim}^{\boldsymbol{D}} X, t)$ is an *initial object* (if such exists) of the under category $(X \downarrow c^*)$ (2.4). Clearly such a colimit (if it exists) is unique up to a canonical isomorphism and, if X and $X' \in \boldsymbol{C}^{\boldsymbol{D}}$ are objects with colimits $(\operatorname{colim}^{\boldsymbol{D}} X, t)$ and $(\operatorname{colim}^{\boldsymbol{D}} X', t')$ respectively, then there is, for every map $f\colon X \to X' \in \boldsymbol{C}^{\boldsymbol{D}}$, a unique map $\operatorname{colim}^{\boldsymbol{D}} f\colon \operatorname{colim}^{\boldsymbol{D}} X \to \operatorname{colim}^{\boldsymbol{D}} X' \in \boldsymbol{C}$ such that $t'(c^* \operatorname{colim}^{\boldsymbol{D}} f) = ft$. If every object of $\boldsymbol{D}^{\boldsymbol{D}}$ has a colimit, then the resulting function $\operatorname{colim}^{\boldsymbol{D}}\colon \boldsymbol{C}^{\boldsymbol{D}} \to \boldsymbol{C}$ is a functor (called the **colimit functor**)which is a *left adjoint* of the constant diagram functor $c^*\colon \boldsymbol{C} \to \boldsymbol{C}^{\boldsymbol{D}}$ and conversely, if c^* has a left adjoint, then every object of $\boldsymbol{C}^{\boldsymbol{D}}$ has a colimit. One says that the category $\boldsymbol{C}$ is **cocomplete** if, for every small category $\boldsymbol{D}$ (2.2), such a left adjoint exists.

Dually a $\boldsymbol{D}$**-limit** or **limit** of an object $X \in \boldsymbol{C}^{\boldsymbol{D}}$ consists of an object $\lim^{\boldsymbol{D}} X \in \boldsymbol{C}$ and a map $t\colon c^* \lim^{\boldsymbol{D}} X \to X \in \boldsymbol{C}^{\boldsymbol{D}}$ such that the pair $(\lim^{\boldsymbol{D}} X, t)$ is a *terminal object* (if such exists) of the over category $(c^* \downarrow X)$. As above one then has that every object of $\boldsymbol{C}^{\boldsymbol{D}}$ has a limit iff the constant diagram functor has a *right adjoint* (denoted by $\lim^{\boldsymbol{D}}\colon \boldsymbol{C}^{\boldsymbol{D}} \to \boldsymbol{C}$ and called the **limit functor**) and one says that the category $\boldsymbol{C}$ is **complete** if, for every small category $\boldsymbol{D}$, such a right adjoint exists.

If $\boldsymbol{D}$ is empty, then the existence of $\operatorname{colim}^{\boldsymbol{D}}$ (resp. $\lim^{\boldsymbol{D}}$) is equivalent to the existence of an *initial* (resp. a *terminal*) object in $\boldsymbol{C}$.

If $\boldsymbol{D}$ consists of a set of objects and their identity maps only, then $\operatorname{colim}^{\boldsymbol{D}}$ is the *coproduct* which we denote by $\amalg$ and $\lim^{\boldsymbol{D}}$ is the *product* which we denote by Π or $\times$.

Other important examples of colimits and limits are

2.6. Pushouts and pullbacks. If $\boldsymbol{D}$ is the category $(0 \leftarrow 1 \to 2)$ (resp. $(0 \to 1 \leftarrow 2)$) with three objects, 0, 1 and 2, and the indicated non-identity maps, then $\operatorname{colim}^{\boldsymbol{D}}$ (resp. $\lim^{\boldsymbol{D}}$) is often called the **pushout** (resp. **pullback**) functor. Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{k} & D \end{array}$$

in $\boldsymbol{C}$, one

(i) often denotes, when no confusion can arise, the pushout of the diagram $B \xleftarrow{f} A \xrightarrow{h} C$ by $B \amalg_A C$ or even $B \amalg_\bullet C$, refers to the induced map $B \amalg_A C \to D$ as the **pushout corner map** of the square and calls g the **pushout** (or the cobase extension) **of** f **along** h if the above square is a **pushout square**, i.e. if its pushout corner map is an isomorphism, and dually

(ii) often denotes, when no confusion can arise, the pullback of the diagram $B \xrightarrow{k} D \xleftarrow{g} C$ by $B \prod_D C$ or $B \times_D C$ or even $B \prod_\bullet C$, refers to the induced map $A \to B \prod_D C$ as the **pullback corner map** of the square and calls f the **pullback** (or the base extension) **of** g **along** k if the above square is a **pullback square**, i.e. if its pullback corner map is an isomorphism.

Using colimits and limits one can define

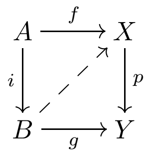
2.7. Direct and inverse (transfinite) compositions. Recall that an **ordinal** γ is an ordered isomorphism class of well ordered sets (in our universe $\mathcal{U}$ (2.1)) and can be identified with the well ordered set of all the preceding ordinals. We will, for every infinite ordinal γ, use the same symbol γ to denote the associated category which has these ordinals as objects and which has exactly one map $\alpha \to \beta$ whenever $\alpha \le \beta$ $(\alpha, \beta < \gamma)$. Clearly this category has an initial object, the **empty ordinal** 0.

Given a cocomplete (2.5) category $\boldsymbol{C}$ and an infinite ordinal γ, a functor $V\colon \gamma \to \boldsymbol{C}$ is called a **γ-sequence** if, for every limit ordinal β in γ (i.e. $\beta < \gamma$), the natural map $\operatorname{colim}^\beta (V|\beta) \to V\beta$ is an isomorphism and the resulting map $V0 \to \operatorname{colim}^\gamma V \in \boldsymbol{C}$ is called the **direct (transfinite) composition** of (the maps of) the γ-sequence V. A subcategory $\boldsymbol{C}_1 \subset \boldsymbol{C}$ then is said to be **closed under direct (transfinite) compositions** if, for every infinite ordinal γ and every γ-sequence $V\colon \gamma \to \boldsymbol{C}$ such that, for every ordinal α with $\alpha + 1 < \gamma$, the map $V\alpha \to V(\alpha+1)$ is in $\boldsymbol{C}_1$, the induced map $V0 \to \operatorname{colim}^\gamma V$ is in $\boldsymbol{C}_1$.

There are of course obvious dual notions of an **inverse (transfinite) composition** and of a subcategory being **closed under inverse (transfinite) compositions**.

We end with

2.8. Lifting properties. Given two maps $i\colon A \to B$ and $p\colon X \to Y$ in a category $\boldsymbol{C}$, one says that i has the **left lifting property** with respect to p and that p has the **right lifting property** with respect to i if, for every commutative solid arrow diagram



in $\boldsymbol{C}$, the dotted arrow "exists", i.e. there exists a map $k\colon B \to X \in \boldsymbol{C}$ such that $ki = f$ and $pk = g$.

These definitions readily imply the

2.9. Retract lemma. *Given a category $\boldsymbol{C}$, a map $h\colon P \to Q \in \boldsymbol{C}$ and a factorization $h = pi$,*

(i) *if h has the left lifting property with respect to p, then p has a right inverse p' such that $p'h = i$ and hence h is a retract of i (in $(P \downarrow \boldsymbol{C})$ (2.3) and therefore in the category of maps of $\boldsymbol{C}$) and dually*

(ii) *if h has the right lifting property with respect to i, then i has a left inverse i' such that $hi' = p$ and hence h is a retract of p (in $(\boldsymbol{C} \downarrow Q)$ and therefore in the category of maps of $\boldsymbol{C}$).*

2.10. Corollary. Given a category $\boldsymbol{C}$, maps $h\colon P \to Q \in \boldsymbol{C}$ and $k\colon Q \to R \in \boldsymbol{C}$ and factorizations $h = pi$ and $k = qj$,

(i) if h and k have the left lifting property with respect to p and q respectively, then kh is a retract (in $(P \downarrow \boldsymbol{C})$ and hence in the category of maps of $\boldsymbol{C}$) of the composition of i with the pushout of j along a right inverse p' of p such that $p'h = i$, and dually

(ii) if h and k have the right lifting property with respect to i and j respectively, then kh is a retract (in $(\boldsymbol{C} \downarrow R)$ and hence in the category of maps of $\boldsymbol{C}$) of the composition with q of the pullback of p along a left inverse j' of j such that $kj' = q$.

3. Model categories

In this section we define model categories and discuss a few immediate consequences of the definition. Various examples will be given in chapters II and III As mentioned in 1.2, we will use the term *model category* for a closed model category that satisfies a strengthened version of the limit axiom and of the factorization axiom. This strengthening of the axioms simplifies many statements and arguments and the closure implies that

(i) *any two of the three distinguished classes of maps (weak equivalences, cofibrations and fibrations) determine the third, and*

(ii) *the cofibrations and the trivial fibrations (i.e. fibrations which are also weak equivalences) determine each other and dually, so do the fibrations and the trivial cofibrations.*

3.1. Model categories. A **model category** is a category $\boldsymbol{M}$, together with three classes of maps (**weak equivalences**, **fibrations** and **cofibrations**) satisfying the following five axioms.

M1: Limit axiom. The category $\boldsymbol{M}$ is complete and cocomplete (2.5).

M2: Two out of three axiom. If f and g are maps in $\boldsymbol{M}$ such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.

M3: Retract axiom. If f and g are maps in $\boldsymbol{M}$ such that f is a *retract* of g (in the category of maps of $\boldsymbol{M}$ (2.3)) and g is a weak equivalence, a fibration or a cofibration, then so is f.

M4: Lifting axiom. Given a commutative diagram in $\boldsymbol{M}$,

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

i has the left lifting property (2.8) with respect to p and that p has the right lifting property with respect to i if

(i) i is a cofibration and p is a **trivial** fibration (i.e. a fibration which is also a weak equivalence), or

(ii) p is a fibration and i is a **trivial** cofibration (i.e. a cofibration which is also a weak equivalence).

M5: Factorization axiom. The maps $f \in \boldsymbol{M}$ admit *functorial* factorizations

(i) $f = qi$, where i is a cofibration and q is a trivial fibration, and

(ii) $f = pj$, where p is a fibration and j is a trivial cofibration.

An object $A \in \boldsymbol{M}$ is called **cofibrant** if the (unique) map to it from the *initial object* (which exists in view of M1 and 2.5) is a cofibration and dually an object $X \in \boldsymbol{M}$ is called **fibrant** if the (unique) map from it to the *terminal object* is a fibration.

Some immediate consequences of the axioms are:

(i) *The notion of a model category is self dual, i.e. if $\boldsymbol{M}$ is a model category, then so is its opposite $\boldsymbol{M}^{\mathrm{op}}$, with the "opposites" of the fibrations as cofibrations and the "opposites" of the cofibrations as fibrations.*

(ii) *For every object X in a model category $\boldsymbol{M}$, the under category $(X \downarrow \boldsymbol{M})$* (2.4) *and the over category $(\boldsymbol{M} \downarrow X)$ inherit from $\boldsymbol{M}$ a model category structure in which a map is a weak equivalence, a cofibration or a fibration whenever its image in $\boldsymbol{M}$ under the forget functor* (2.4) *is so.*

(iii) *Every small* (2.1) *product of model categories is again a model category, with as weak equivalences, cofibrations and fibrations the "coordinatewise" ones.*

(iv) *Every category admits a "trivial" model category structure in which the weak equivalences are the isomorphisms and all maps are cofibrations as well as fibrations.*

We also note the

3.2. Closure properties for (trivial) cofibrations and (trivial) fibrations.

(i) *A map $f \in \boldsymbol{M}$ is a cofibration (resp. a fibration) iff it has the left (resp. the right) lifting property with respect to all trivial fibrations (resp. all trivial cofibrations), and*

(ii) *a map $f \in \boldsymbol{M}$ is a trivial cofibration (resp. a trivial fibration) iff it has the left (resp. the right) lifting property with respect to all fibrations (resp. all cofibrations).*

Proof. Let $f \in \boldsymbol{M}$ have the left lifting property with respect to all trivial fibrations. As (M5 (i)) $f = gi$ where g is a trivial fibration and i is a cofibration, the first part of (i) then follows from M3 and 2.9. The proofs of the other parts of the proposition are similar.

3.3. Corollary. *The cofibrations and the trivial cofibrations in a model category $\boldsymbol{M}$ form subcategories of $\boldsymbol{M}$ which are closed under*

(i) *retracts*
(ii) *pushouts* (2.6)
(iii) *small coproducts* (2.5), *and*
(iv) *direct compositions* (2.7),

and dually the fibrations and the trivial fibrations form subcategories of $\boldsymbol{M}$ which are closed under

(i)′ *retracts*
(ii)′ *pullbacks*
(iii)′ *small products, and*
(iv)′ *inverse compositions.*

We end with a brief discussion of

3.4. Some other useful subcategories. Clearly (M2) the weak equivalences form a subcategory of $\boldsymbol{M}$.

Other important subcategories are the full subcategories of $\boldsymbol{M}$ spanned by the cofibrant, the fibrant and the cofibrant fibrant objects (3.1). They will be denoted by $\boldsymbol{M}_c$, $\boldsymbol{M}_f$ and $\boldsymbol{M}_{cf}$ respectively. The weak equivalences in $\boldsymbol{M}_c$ and $\boldsymbol{M}_f$ have the following useful property.

3.5. Proposition.

(i) *Every functor between model categories which preserves cofibrations and trivial cofibrations preserves weak equivalences between cofibrant objects, and dually*
(ii) *every functor between model categories which preserves fibrations and trivial fibrations preserves weak equivalences between fibrant objects.*

This is an easy consequence of the fact that, in a model category $\boldsymbol{M}$, the maps of the subcategories $\boldsymbol{M}_c$ and $\boldsymbol{M}_f$ admit especially nice functorial factorizations (M5), namely

3.6. Ken Brown's lemma [3].

(i) *The maps $f \in \boldsymbol{M}_c$ admit a functorial factorization $f = qi$, where i is a cofibration and q is a trivial fibration which has an (also functorial) trivial cofibration as a right inverse, and dually*
(ii) *the maps $f \in \boldsymbol{M}_f$ admit a functorial factorization $f = pj$, where p is a fibration and j is a trivial cofibration which has an (also functorial) trivial fibration as a left inverse.*

Proof. It clearly suffices to prove (i).

Given a map $f\colon A \to B \in \boldsymbol{M}_c$, let $k\colon A \amalg B \to B \in \boldsymbol{M}_c$ be the map such that $ki_0 = f$ and $ki_1 = 1_B$, where $i_0\colon A \to A \amalg B$ and $i_1\colon B \to A \amalg B$ are the injections (2.5). As A and B are cofibrant, the maps i_0 and i_1 are cofibrations (3.3), and the desired result now follows readily from the fact that (M5 (i)) $k = gi$ where i is a cofibration and g is a trivial fibration and that $f = ki_0 = gii_0$ and $gii_1 = ki_1 = 1_B$.

4. Quillen functors

In this last section we investigate what might be a good notion of "morphism between model categories".

Given two model categories $\boldsymbol{M}$ and $\boldsymbol{N}$, the obvious notion of a morphism between $\boldsymbol{M}$ and $\boldsymbol{N}$ would seem to be a functor $\boldsymbol{M} \to \boldsymbol{N}$ which is compatible with the model category structures of $\boldsymbol{M}$ and $\boldsymbol{N}$, i.e. a functor which preserves cofibrations, fibrations and weak equivalences or equivalently a functor which preserves cofibrations, trivial cofibrations, fibrations and trivial fibrations. However most of the functors between model categories that one usually runs into do *not* have this property. But many of these are *one of a pair* of *adjoint functors* of which the *left adjoint* is compatible with one half of the model category structures of $\boldsymbol{M}$ and $\boldsymbol{N}$ in the sense that it *preserves cofibrations and trivial cofibrations* while the *right adjoint* is compatible with the other halves and *preserves fibrations and trivial fibrations*. We therefore define

4.1. Quillen functors. Given two model categories $\boldsymbol{M}$ and $\boldsymbol{N}$, a functor $f\colon \boldsymbol{M} \to \boldsymbol{N}$ will be called a **left Quillen functor** if

(i) f has a right adjoint, and

(ii) f preserves cofibrations and trivial cofibrations and hence (3.6) weak equivalences between cofibrant objects

and dually a functor $g\colon \boldsymbol{N} \to \boldsymbol{M}$ will be called a **right Quillen functor** if

(i)′ g has a left adjoint, and

(ii)′ g preserves fibrations and trivial fibrations and hence weak equivalences between fibrant objects.

By an **adjoint pair of Quillen functors** we mean a pair of adjoint functors between model categories for which the left adjoint is a left Quillen functor and (hence) the right adjoint is a right Quillen functor.

There is a special kind of Quillen functors which, as we will see in the last part of this monograph, induce "equivalences of homotopy theories" and which we therefore call

4.2. Quillen equivalences. An adjoint pair of Quillen functors $f : \boldsymbol{M} \leftrightarrow \boldsymbol{N} : g$ will be called an **adjoint pair of Quillen equivalences** and we call the left adjoint a **left Quillen equivalence** and the right adjoint a **right Quillen equivalence** if

(i) for every pair of objects $A \in \boldsymbol{M}_c$ and $X \in \boldsymbol{N}_f$ (3.4), a map $A \to gX \in \boldsymbol{M}$ is a weak equivalence iff its adjoint $fA \to X \in \boldsymbol{N}$ is so,

or equivalently if

(ii) for every pair of objects $B \in \boldsymbol{M}_{cf}$ and $Y \in \boldsymbol{N}_{cf}$ (3.4), a map $B \to gY \in \boldsymbol{M}$ is a weak equivalence iff its adjoint $fB \to Y \in \boldsymbol{N}$ is so.

If f and g both preserve weak equivalences (and not merely the trivial cofibrations or trivial fibrations), then this is also equivalent to requiring that

(iii) for every pair of objects $B \in \boldsymbol{M}_{cf}$ and $Y \in \boldsymbol{N}_{cf}$, the adjunction maps

$$B \to gfB \in \boldsymbol{M} \quad \text{and} \quad fgY \to Y \in \boldsymbol{N}$$

are weak equivalences.

Clearly the above definitions imply

4.3. Elementary properties.

(i) *The composition of two left Quillen functors is a left Quillen functor and the composition of two right Quillen functors is a right Quillen functor.*
(ii) *The identity functor of a model category is both a left Quillen functor and a right Quillen functor.*
(iii) *The right adjoint of a left Quillen functor is a right Quillen functor and the left adjoint of a right Quillen functor is a left Quillen functor.*
(iv) *The opposite of a left Quillen functor is a right Quillen functor and the opposite of a right Quillen functor is a left Quillen functor.*
(v) *The above four statements remain valid if one replaces everywhere "Quillen functor" by "Quillen equivalence".*

Also not difficult to verify is the

4.4. Two out of three property for Quillen equivalences. *If f and g are left (resp. right) Quillen functors such that gf is defined and two of f, g and gf are left (resp. right) Quillen equivalences, then so is the third.*

4.5. Remark. Probably the most useful of the above properties of Quillen functors is 4.3 (iii) as, for a pair of adjoint functors $f\colon \boldsymbol{M} \leftrightarrow \boldsymbol{N} : g$ between model categories, often one of the two equivalent statements

(i) *f preserves cofibrations and trivial cofibrations* (and hence weak equivalences between cofibrant objects) and
(ii) *g preserves fibrations and trivial fibrations* (and hence weak equivalences between fibrant objects)

is much easier to verify than the other.

We end with discussing an interesting

4.6. Example. For every map $f\colon X \to Y$ in a model category $\boldsymbol{M}$,

(i) the pair of adjoint functors (2.4)

$$\underset{\rightarrow}{f}\colon (X \downarrow \boldsymbol{M}) \leftrightarrow (Y \downarrow \boldsymbol{M}) : f^*$$

where $\underset{\rightarrow}{f}$ denotes "pushing out along f" (2.6), is clearly a Quillen pair which moreover consists of *Quillen equivalences* whenever f is a *trivial cofibration*, and dually
(ii) the pair of adjoint functors

$$f_*\colon (\boldsymbol{M} \downarrow X) \leftrightarrow (\boldsymbol{M} \downarrow Y) : \underset{\leftarrow}{f}$$

where $\underset{\leftarrow}{f}$ denotes "pulling back along f", is a Quillen pair which moreover consists of *Quillen equivalences* whenever f is a *trivial fibration.*

A better understanding of this example requires the notion of

4.7. Properness. Given a model category $\boldsymbol{M}$, a weak equivalence in $\boldsymbol{M}$ will be called a **left proper equivalence** if all its pushouts along cofibrations (2.6) are again weak equivalences and it will be called a **right proper equivalence** if all its pullbacks along fibrations are so. Furthermore $\boldsymbol{M}$ will be called **left proper** if all weak equivalences are left proper equivalences, **right proper** if all weak equivalences are right proper equivalences and **proper** if $\boldsymbol{M}$ is both left proper and right proper.

Clearly (3.3) *all trivial cofibrations in a model category are left proper equivalences and all trivial fibrations are right proper equivalences.*

A straightforward argument then yields [**18**]

4.8. Rezk's lemma. *Given a map $f\colon X \to Y$ in a model category $\boldsymbol{M}$,*

(i) *the induced Quillen functors of* 4.6 *(i) are Quillen equivalences iff f is a left proper equivalence, and dually*

(ii) *the induced Quillen functors of* 4.6 *(ii) are Quillen equivalences iff f is a right proper equivalence.*

This implies, in view of 3.6 and 4.4

4.9. Corollary. *Every weak equivalence between cofibrant objects in a model category is a left proper equivalence and every weak equivalence between fibrant objects is a right proper equivalence.*

CHAPTER II

Cofibrantly generated model categories

5. Introduction

5.1. Summary. In this chapter we

(i) describe two of the original examples of model categories, *simplicial sets* and *topological spaces* and obtain a related model category structure for their *diagram categories*,

(ii) note that these model category structures are all *cofibrantly generated* in the sense that in each there are *small* sets of cofibrations and trivial cofibrations which, in a certain precise manner, determine the whole model category structure, and

(iii) formulate a *recognition lemma* and a *lifting lemma* which we then use to obtain other cofibrantly generated model categories, e.g. various categories of *simplicial algebras*.

There is of course a dual notion of fibrantly generated model categories, but these seem to be much less prevalent.

In more detail:

5.2. The usual model category structures for simplicial sets and topological spaces and their diagrams. The usual model category structures on the categories $\boldsymbol{S}$ of *simplicialsets* and $\boldsymbol{T}$ of *compactly generated topological spaces* are connected by an adjoint pair of *Quillen equivalences* which consists of the *geometric realization* $|-|\colon \boldsymbol{S} \to \boldsymbol{T}$ and the *singular functor* $\operatorname{Sin}\colon \boldsymbol{T} \to \boldsymbol{S}$. The weak equivalences in $\boldsymbol{S}$ are the maps $f \in \boldsymbol{S}$ such that $|f| \in \boldsymbol{T}$ is a homotopy equivalence and the weak equivalences in $\boldsymbol{T}$ are the maps $g \in \boldsymbol{T}$ such that $\operatorname{Sin} g \in \boldsymbol{S}$ is a weak equivalence. The fibrations in $\boldsymbol{S}$ and $\boldsymbol{T}$ are the Kan fibrations and the Serre fibrations respectively and the cofibrations in $\boldsymbol{S}$ are the monomorphisms, while those in $\boldsymbol{T}$ are the retracts of the "relative cell complexes". The main problem in verifying the model category axioms then is in constructing the required functorial factorizations and this is overcome by a so-called *small object argument.* One can also use this argument to show that, for every small category $\boldsymbol{D}$, the above model category structures induce model category structures on the *diagram categories* $\boldsymbol{S}^{\boldsymbol{D}}$ and $\boldsymbol{T}^{\boldsymbol{D}}$ in which the weak equivalences and the fibrations are the objectwise ones.

5.3. Cofibrantly generated model categories. The model categories mentioned in 5.2 are examples of *cofibrantly generated* model categories, i.e. model categories in which there exists a small set I of cofibrations (called *generating cofibrations*) and a small set J of trivial cofibrations (called *generating trivial cofibrations*) such that

(i) a map is a trivial fibration (resp. a fibration) iff it has the right lifting property with respect to the generating cofibrations (resp. the generating trivial cofibrations), and
(ii) the required functorial factorizations can be obtained from the sets I and J by means of a possibly transfinite version of the *small object argument* that was mentioned in 5.2.

A convenient characterization of cofibrantly generated model categories is contained in the following

5.4. Recognition lemma. This is a lemma which gives necessary and sufficient conditions on a category $\boldsymbol{C}$, a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ and two small sets I and J of maps in $\boldsymbol{C}$, in order that $\boldsymbol{C}$ admits a cofibrantly generated model category structure with $\boldsymbol{W}$ as its category of weak equivalences and I and J as its sets of generating cofibrations and generating trivial cofibrations.

Using this lemma it is not difficult to show that, for every small category $\boldsymbol{D}$, the diagram category $\boldsymbol{S}^{\boldsymbol{D}}$ also admits an "unusual" model category structure in which the weak equivalences and the *cofibrations* (and not, as in 5.2, the fibrations) are the objectwise ones.

Another application of the recognition lemma is a

5.5. Lifting lemma. This is a lemma which, for a pair of adjoint functors $F\colon \boldsymbol{B} \leftrightarrow \boldsymbol{C} : U$ and a cofibrantly generated model category structure on $\boldsymbol{B}$, gives necessary and sufficient conditions in order that $\boldsymbol{C}$ admits a cofibrantly generated model category structure with as generating cofibrations and trivial cofibrations the images under F of the generating cofibrations and trivial cofibrations in $\boldsymbol{B}$ and in which a map is a weak equivalence of a fibration wherever its image under U is so.

One can use this lemma to obtain a topological version of the "unusual" model category structure which we just mentioned (5.4). Other applications are

(i) For every cocomplete and complete category $\boldsymbol{A}$ of *universal algebras*, i.e. algebras with a specified small set of finitary operations and identities (e.g. groups, monoids, rings, modules, Lie algebras, etc.), the category $\boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of the *simplicial objects* in $\boldsymbol{A}$ admits a cofibrantly generated model category structure in which a map is a weak equivalence or a fibration whenever the induced map between the underlying simplicial sets is so.
(ii) A similar result holds, more generally, for the simplicial objects in a cocomplete and complete category of so-called *indexed* universal algebras, i.e. graded versions of universal algebras, where the grading is indexed by an arbitrary small set and not necessarily a set of integers. Examples of such algebras are not only the graded versions of the algebras mentioned in (i), but also, for every small set O, the category O-**Cat** of the small categories which have O as their sets of objects (and are thus $(O\times O)$-indexed universal algebras).
(iii) For every cofibrantly generated model category $\boldsymbol{N}$ and every small category $\boldsymbol{D}$, the *diagram category* $\boldsymbol{N}^{\boldsymbol{D}}$ admits a cofibrantly generated model category structure in which the weak equivalences and the fibrations are the objectwise ones.

5.6. Organization of the chapter. After devoting the rest of this section to some simplicial preliminaries, we obtain (in §6) the usual model category structures for *simplicial sets* and *topological spaces* and their *diagrams* (5.2). The notion of a *cofibrantly generated* (5.3) model category then is introduced in §7, while the last section (§8) deals with the *recognition* and *lifting lemmas* (5.4 and 5.5) and their applications.

We thus end this section with a brief discussion of

5.7. Simplicial and cosimplicial objects. Let $\mathbf{\Delta}$ be the category which has as objects the *finite ordered sets of integers* $[n] = (0, \ldots, n)$ $(n \geq 0)$ and as maps the *order preserving functions* between them and for every pair of integers (i, n) with $0 \leq i \leq n$, let $d^i\colon [n-1] \to [n]$ $(n > 0)$ and $s^i\colon [n+1] \to [n]$ denote the weakly monotone functions given by

$$\begin{array}{llll} d^i j = j & \text{for } j < i & s^i j = j & \text{for } j \leq i \\ d^i j = j+1 & \text{for } j \geq i & s^i j = j-1 & \text{for } j > i. \end{array}$$

Given a category $\boldsymbol{C}$ one refers to $\boldsymbol{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ as the category of **simplicial objects** in $\boldsymbol{C}$ and we will therefore call $\mathbf{\Delta}^{\mathrm{op}}$ the **simplicial indexing category**. For an object $K \in \boldsymbol{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, one usually writes K_n instead of $K[n]$ $(n \geq 0)$ and $d_i\colon K_n \to K_{n-1}$ and $s_i\colon K_n \to K_{n+1}$ instead of $Kd^i\colon K[n] \to K[n-1]$ and $Ks^i\colon K[n] \to K[n+1]$ and one calls d_i and s_i the i**-face** and the i**-degeneracy** map. Furthermore one refers to $\boldsymbol{C}^{\mathbf{\Delta}}$ as the category of **cosimplicial objects** in $\boldsymbol{C}$ and we will therefore call $\mathbf{\Delta}$ the **cosimplicial indexing category**.

5.8. Simplicial sets. As usual we denote by **Set** the category of small sets (2.1) and by $\boldsymbol{S}$ the category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ of **simplicial sets**. Given an object $K \in \boldsymbol{S}$, an element $k \in K_n$ $(n \geq 0)$ is called an n**-simplex**, which is **degenerate** if it is of the form $k = s_i k'$ for some $k' \in K_{n-1}$ and $0 \leq i < n$ and **non-degenerate** otherwise.

There is an obvious functor $\Delta[-] = \mathbf{\Delta}(-,-)\colon \mathbf{\Delta} \to \boldsymbol{S} = \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$, the **diagram of standard simplices**. The resulting object $\Delta[n] = \mathbf{\Delta}(-,[n])$ $(n \geq 0)$, the **standard** n**-simplex**, has exactly one non-degenerate n-simplex, the identity map $1_{[n]}\colon [n] \to [n]$. Furthermore it has the *universal property* that, for every object $K \in \boldsymbol{S}$ and n-simplex $k \in K$, there is a *unique* map $i_k\colon \Delta[n] \to K \in \boldsymbol{S}$ such that $i_k 1_{[n]} = k$. Closely related is the notion of

5.9. The category of simplices of an object $K \in \boldsymbol{S}$. This is the over category $(\Delta[-] \downarrow K)$ (2.4) which has as objects the maps $\Delta[n] \to K \in \boldsymbol{S}$ $(n \geq 0)$ and as maps the obvious commutative triangles. We will denote this category by $\mathbf{\Delta}\, K$ and its opposite by $\mathbf{\Delta}^{\mathrm{op}}\, K$ (so that $\mathbf{\Delta}\, \Delta[0] \approx \mathbf{\Delta}$ and $\mathbf{\Delta}^{\mathrm{op}}\, \Delta[0] \approx \mathbf{\Delta}^{\mathrm{op}}$). Of course we denote, for a map $f\colon K \to L \in \boldsymbol{S}$, by $\mathbf{\Delta}\, f\colon \mathbf{\Delta}\, K \to \mathbf{\Delta}\, L$ and $\mathbf{\Delta}^{\mathrm{op}}\, f\colon \mathbf{\Delta}^{\mathrm{op}}\, K \to \mathbf{\Delta}^{\mathrm{op}}\, L$ the functors induced by f. The category of simplices $\mathbf{\Delta}\, K$ comes with a forgetful functor $\Delta[K]\colon \mathbf{\Delta}\, K \to \boldsymbol{S}$, **the diagram of simplices** of K, which has the property that *the induced map* $\operatorname{colim}^{\mathbf{\Delta}\, K} \Delta[K] \to K \in \boldsymbol{S}$ *is an isomorphism.* This readily implies

5.10. Proposition. *Let $\boldsymbol{C}$ be a category which is complete and cocomplete* (2.5)*. Then*

(i) *for every object $A^* \in \boldsymbol{C}^{\boldsymbol{\Delta}}$ (5.7), the functor $\boldsymbol{C}(A^*, -)\colon \boldsymbol{C} \to \boldsymbol{S}$ has a left adjoint, which is the unique functor $\boldsymbol{S} \to \boldsymbol{C}$ which preserves all small colimits and sends $\Delta[-]$ to A^*, and dually*

(ii) *for every object $X_* \in \boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$, the functor $\boldsymbol{C}(-, X)\colon \boldsymbol{C} \to \boldsymbol{S}^{\mathrm{op}}$ has a right adjoint, which is the unique functor $\boldsymbol{S}^{\mathrm{op}} \to \boldsymbol{C}$ which preserves all small limits and sends $\Delta[-]$ to X_*.*

6. The usual model category structures for simplicial sets and topological spaces and their diagrams

We now

(i) obtain the usual model category structures on the categories $\boldsymbol{S}$ of *simplicial sets* and $\boldsymbol{T}$ of *compactly generated topological spaces* and note that these are connected by an adjoint pair of *Quillen equivalences* consisting of the geometric realization and the singular functor, and

(ii) note that, for every small category $\boldsymbol{D}$, there is a corresponding result for the diagram categories $\boldsymbol{S}^{\boldsymbol{D}}$ and $\boldsymbol{T}^{\boldsymbol{D}}$.

We will freely use some basic results on simplicial sets and topological spaces which, for instance, can be found in [**6**], [**13**] and/or [**9**].

We first recall the definitions of

6.1. The geometric realization and the singular functor. The category $\boldsymbol{S}$ of simplicial sets is related to the category $\boldsymbol{T}$ of **compactly generated topological spaces** [**13**, Ch VII] by a pair of adjoint functors $|-|\colon \boldsymbol{S} \leftrightarrow \boldsymbol{T}$:Sin with very useful properties. The left adjoint $|-|$, the **geometric realization**, is determined by the requirement (5.10) that

(i) for every integer $n \geq 0$, it sends the object $\Delta[n] \in \boldsymbol{S}$ to the *topological n-simplex* $|\Delta[n]|$ which is the subspace of $(n+1)$-dimensional Euclidean space consisting of the points $(t_0, \ldots, t_n)$ such that $\Sigma t_i = 1$ and $0 \leq t_i \leq 1$ for all i, and

(ii) for every map $a\colon [n] \to [n'] \in \boldsymbol{\Delta}$, it sends the map $\Delta a\colon \Delta[n] \to \Delta[n'] \in \boldsymbol{S}$ to the *linear* map $|\Delta a|\colon |\Delta[n]| \to |\Delta[n']| \in \boldsymbol{T}$ which, for every integer i with $0 \leq i \leq n$, maps the point $(t_0, \ldots, t_n) \in |\Delta[n]|$ for which $t_i = 1$ (and hence $t_j = 0$ for $j \neq i$) to the point $(t'_0, \ldots, t'_{n'}) \in |\Delta[n']|$ for which $t_{ai} = 1$.

The right adjoint, the **singular functor** Sin, is the functor $\mathrm{Sin} = \boldsymbol{T}(|\Delta-|, -)\colon \boldsymbol{T} \to \boldsymbol{S}$.

A convenient property of this pair of adjoint functors is

6.2. Proposition. *The geometric realization, the singular functor and both adjunction transformations $1_{\boldsymbol{S}} \to \mathrm{Sin}|-|$ and $|\mathrm{Sin}\,-| \to 1_{\boldsymbol{T}}$ commute with finite products.*

To formulate another useful property we first have to introduce

6.3. Weak homotopy equivalences. A map $f \in \boldsymbol{S}$ is called a **weak homotopy equivalence** if its geometric realization $|f| \in \boldsymbol{T}$ (6.1) is a homotopy equivalence and a map $g \in \boldsymbol{T}$ is called a **weak homotopy equivalence** if its singular map $\mathrm{Sin}\, g \in \boldsymbol{S}$ (6.1) is a weak homotopy equivalence.

Now we can state

6.4. Proposition. *Both the geometric realization and the singular functor preserve weak homotopy equivalences and the adjunction transformations are natural weak homotopy equivalences.*

Next we define

6.5. Kan fibrations and Serre fibrations. For every integer $n \geq 0$, let $\partial\Delta[n] \subset \Delta[n]$ be the largest subobject not containing the n-simplex $i_{[n]}$ (5.8) and for every pair of integers (k, n) with $n > 0$ and $0 \leq k \leq n$, let $\Delta^k[n] \subset \Delta[n]$ be the largest subobject not containing the $(n-1)$-simplex $d_k i_{[n]}$. Then

(i) a map $f \in \boldsymbol{S}$ is called a **Kan fibration** if it has the right lifting property (3.1) with respect to the inclusions $\Delta^k[n] \to \Delta[n]$ $(n > 0,\ 0 \leq k \leq n)$, and
(ii) a map $g \in \boldsymbol{T}$ is called a **Serre fibration** if it has the right lifting property with respect to the induced inclusions $\left|\Delta^k[n]\right| \to \left|\Delta[n]\right|$, or equivalently (6.1) if the map $\operatorname{Sin} g \in \boldsymbol{S}$ is a Kan fibration.

There is a similar characterization for those fibrations which are also weak homotopy equivalences (6.3), namely

6.6. Proposition. *A map $f \in \boldsymbol{S}$ is a Kan fibration as well as a weak homotopy equivalence iff it has the right lifting property with respect to the inclusions $\partial\Delta[n] \to \Delta[n]$ $(n \geq 0)$ (and hence with respect to all monomorphisms).*

6.7. Proposition. *A map $g \in \boldsymbol{T}$ is a Serre fibration as well as a weak homotopy equivalence iff it has the right lifting property with respect to the induced inclusions $\left|\partial\Delta[n]\right| \to \left|\Delta[n]\right|$ $(n \geq 0)$.*

Now we can describe

6.8. Model category structure for simplicial sets. *The category $\boldsymbol{S}$ of simplicial sets admits a model category structure in which*

(i) *a map is a weak equivalence if it is a weak homotopy equivalence* (6.3)*,*
(ii) *a map is a fibration if it is a Kan fibration* (6.5)*, and*
(iii) *a map is a cofibration if it is* $1-1$ *(but not necessarily onto).*

Moreover in this model category structure

(iv) *a map is a trivial fibration iff* (6.6) *it has the right lifting property with respect to the inclusions $\partial\Delta[n] \to \Delta[n]$ $(n \geq 0)$, and*
(v) *a map is a trivial cofibration iff it is a retract of a countable composition $X_0 \to X_1 \to \cdots \to X_i \to \cdots$ in which each map $X_i \to X_{i+1}$ is a pushout of a disjoint union of copies of the inclusions $\Delta^k[n] \to \Delta[n]$ $(n > 0,\ 0 \leq k \leq n)$.*

6.9. Model category structure for compactly generated topological spaces. *The category $\boldsymbol{T}$ of compactly generated topological spaces admits a model category structure* (3.1) *in which*

(i) *a map is a weak equivalence if it is a weak homotopy equivalence* (6.3)*,*
(ii) *a map is a fibration if it is a Serre fibration, i.e.* (6.5) *if it has the right lifting property with respect to the inclusions $\left|\Delta^k[n]\right| \to \left|\Delta[n]\right|$ $(n > 0,\ 0 \leq k \leq n)$, and*

(iii) *a map is a cofibration if it is a retract of a countable composition* $X_0 \to X_1 \to \cdots \to X_i \to \cdots$ *in which each map* $X_i \to X_{i+1}$ *is a pushout* (2.5) *of a disjoint union of copies of the inclusion maps* $|\partial\Delta[n]| \to |\Delta[n]| \in \boldsymbol{T}$ *(*$n \geq 0$*) (i.e. each* X_{i+1} *is obtained from* X_i *by attaching cells).*

Moreover in this model category structure

(iv) *a map is a trivial fibration iff* (6.7) *it has the right lifting property with respect to the inclusions* $|\partial\Delta[n]| \to |\Delta[n]|$ *(*$n \geq 0$*), and*

(v) *a map is a trivial cofibration iff it is a retract of a countable composition* $X_0 \to X_1 \to \cdots \to X_i \to \cdots$ *in which each map* $X_i \to X_{i+1}$ *is a pushout of a disjoint union of copies of the inclusions* $|\Delta^k[n]| \to |\Delta[n]|$ *(*$n > 0$*,* $0 \leq k \leq n$*).*

And more generally

6.10. Model category structures for diagrams in $\boldsymbol{T}$ and $\boldsymbol{S}$. *Let* $\boldsymbol{D}$ *be a small category* (2.2)*. Then the diagram categories* $\boldsymbol{T}^{\boldsymbol{D}}$ *and* $\boldsymbol{S}^{\boldsymbol{D}}$ (2.3) *admit a model category structure in which*

(i) *the weak equivalences are the objectwise ones,*

(ii) *the fibrations are the objectwise ones, and*

(iii) *a map is a cofibration if it is a retract of a countable composition* $X_0 \to X_1 \to \cdots \to X_i \to \cdots$ *in which each map* $X_i \to X_{i+1}$ *(*$i \geq 0$*) is a pushout of a disjoint union of copies of the inclusion maps*

$$|\partial\Delta[n]| \times \boldsymbol{D}(D,-) \to |\Delta[n]| \times \boldsymbol{D}(D,-) \in \boldsymbol{T}^{\boldsymbol{D}}$$

or

$$\partial\Delta[n] \times \boldsymbol{D}(D,-) \to \Delta[n] \times \boldsymbol{D}(D,-) \in \boldsymbol{S}^{\boldsymbol{D}}$$

($n \geq 0$*,* $D \in \boldsymbol{D}$*) (which are the maps in* $\boldsymbol{T}^{\boldsymbol{D}}$ *or* $\boldsymbol{S}^{\boldsymbol{D}}$*, freely generated by a copy of the inclusion map* $|\partial\Delta[n]| \to |\Delta[n]|$ *or* $\partial\Delta[n] \to \Delta[n]$ *"at the object* D*").*

Moreover in this model category structure

(iv) *the trivial fibrations are the objectwise ones, and*

(v) *a map is a trivial cofibration iff it is a retract of a countable composition* $X_0 \to X_1 \to \cdots \to X_i \to \cdots$ *as in* (iii)*, but with* $|\Delta^k[n]|$ *or* $\Delta^k[n]$ *(*$n > 0$*,* $0 \leq k \leq n$*) instead of* $|\partial\Delta[n]|$ *or* $\partial\Delta[n]$*.*

In view of 6.4 and 6.5 one then has

6.11. Proposition. *For every small category* $\boldsymbol{D}$ *(e.g.,* $\boldsymbol{D} = \boldsymbol{0}$*), the geometric realization and the singular functor* (6.1) *induce an adjoint pair of Quillen equivalences* (4.2)

$$|-|^{\boldsymbol{D}} \colon \boldsymbol{S}^{\boldsymbol{D}} \leftrightarrow \boldsymbol{T}^{\boldsymbol{D}} : \mathrm{Sin}^{\boldsymbol{D}} .$$

Furthermore

6.12. Proposition. *The singular functor* $\mathrm{Sin}\colon \boldsymbol{T} \to \boldsymbol{S}$ *preserves cofibrations and the geometric realization* $|-|\colon \boldsymbol{S} \to \boldsymbol{T}$ *preserves fibrations.*

Proof. The first part is easy and for the last part we refer to reader to [**15**] or [**9**].

We also note

6.13. Proposition. *The model category structures of* 6.8*,* 6.9 *and* 6.10 *above are all proper* (4.7)*.*

Proof. This is easy to verify for $\boldsymbol{S}$ and its diagram categories, The topological proof is somewhat more delicate and for this we refer the reader to [**11**].

It thus remains to prove 6.8, 6.9 and 6.10, but we will only prove 6.8 as the proofs of 6.9 and 6.10 are essentially the same. A key step in the proof of 6.8 is

6.14. A (countable) small object argument. Let $\{A_i \to B_i\}_{i\in I}$ be s small (2.2) set of *monomorphisms* in $\boldsymbol{S}$ such that each A_i $(i \in I)$ is *finite* (i.e. has only a finite number of non-degenerate simplices). Then one can, for every map $f \in \boldsymbol{S}$, construct a functorial commutative diagram

$$\begin{array}{ccccc} X = X_0 \longrightarrow \cdots \longrightarrow & X_k & \longrightarrow \cdots \longrightarrow & \operatorname{colim} X_k = Z \\ {\scriptstyle f=f_0}\searrow & \downarrow{\scriptstyle f_k} & \swarrow & \\ & Y & & \end{array}$$

where each map $X_k \to X_{k+1}$ $(k \geq 0)$ is the pushout of a coproduct of copies of the maps $A_i \to B_i$ $(i \in I)$, which contains one copy of the map $A_i \to B_i$ $(i \in I)$ for every commutative square in $\boldsymbol{S}$ of the form

$$\begin{array}{ccc} A_i & \longrightarrow & X_k \\ \downarrow & & \downarrow{\scriptstyle f_k} \\ B_i & \longrightarrow & Y \end{array}$$

and which is taken along the obvious map into X_k from a similar coproduct of copies of the A_i $(i \in I)$, and where the map f_{k+1} is the obvious one. The finiteness of the A_i $(i \in I)$ then readily implies that *in the resulting factorization* $X \xrightarrow{i} Z \xrightarrow{q} Y$ *of* f

(i) *the map* $q\colon Z \to Y$ *has the right lifting property with respect to the maps* $A_i \to B_i$ *(*$i \in I$*), so that* (2.9)

(ii) *if* f *has the left lifting property with respect to* q*, then* f *is a retract of* i *(in* $(X \downarrow \boldsymbol{C})$*).*

Using this we now give a

Proof of 6.8. Clearly axioms M1, M2 and M3 are satisfied, while 6.8 (iv) is just 6.6 and implies M4 (i) and the small object argument (6.14 (i)) using the maps $\partial\Delta[n] \to \Delta[n]$ $(n \geq 0)$ therefore implies M5 (i).

To prove M4 (ii), M5 (ii) and 6.8 (v), one first notes that the "if" part of 6.8 (v) is easy. The small object argument (6.14 (i)) using the maps $\Delta^k[n] \to \Delta[n]$ $(0 \leq k \leq n,\ n > 0)$ then yields, for every map $f \in \boldsymbol{S}$, a factorization $f = qi$ in which i is a trivial cofibration and q is a fibration, proving M5 (ii). If f is a weak equivalence, then so is q and if f is also a cofibration, then (6.14 (ii)) f is a retract of i, which implies the "only if" part of 6.8 (v) as well as M4 (ii).

7. Cofibrantly generated model categories

In this section we introduce the useful class of *cofibrantly generated model categories*, i.e. model categories for which, as in the examples of §5.2, there exist a small set of *generating cofibrations* and a small set of *generating trivial cofibrations* which completely determine the model category structure and which give rise to the required functorial factorizations by means of a possibly transfinite version of the *small object argument* (6.14) that was used in the proofs of 6.8, 6.9 and 6.10. To precisely formulate this we need a few definitions.

7.1. Relatively small objects. Recall that an ordinal κ (2.7) is called a **cardinal** if the cardinality of any preceding ordinal is less than the cardinality of κ, and such a cardinal κ is called **regular** if, for every set of sets $\{X_j\}_{j\in J}$ indexed by a set J of cardinality less than the cardinality of κ, in which the cardinality of each X_j ($j \in J$) is less than the cardinality of κ, the coproduct $\amalg_{j\in J} X_j$ has also cardinality less than the cardinality of κ. One then readily verifies that, *for every infinite cardinal, its successor cardinal is regular.*

Given a cocomplete category $\boldsymbol{C}$ (2.5) and a subcategory $\boldsymbol{C}_1 \subset \boldsymbol{C}$ which is closed under direct compositions (2.7), an object $C \in \boldsymbol{C}$ is said to be **small rel. $\boldsymbol{C}_1$** if there exists a regular cardinal κ such that C is **κ-small rel. $\boldsymbol{C}_1$**, i.e. for every regular cardinal $\lambda \geq \kappa$ and every functor $V\colon \lambda \to \boldsymbol{C}_1$ which is a λ-sequence in $\boldsymbol{C}$, the obvious map $\operatorname{colim}^\lambda \boldsymbol{C}(C,V) \to \boldsymbol{C}(C, \operatorname{colim}^\lambda V)$ is an isomorphism. A convenient property of *objects which are small rel.* $\boldsymbol{C}_1$ is that [**1**, 4.3] they *are closed under small colimits.*

7.2. I-injectives and (regular) I-cofibrations. Given a cocomplete category $\boldsymbol{C}$ and a set I of maps in $\boldsymbol{C}$, we

(i) denote by I-**inj** the subcategory of $\boldsymbol{C}$ consisting of the maps which have the right lifting property (3.1) with respect to the maps in I and which we call **I-injectives**,
(ii) denote by I-**cof** the subcategory of $\boldsymbol{C}$ consisting of the maps which have the left lifting property with respect to the I-injectives and which we call **I-cofibrations** (and we call similarly an object of $\boldsymbol{C}$ **I-cofibrant** if the map from the initial object of $\boldsymbol{C}$ into it is an I-cofibration),
(iii) denote by I-$\mathbf{cof_{reg}} \subset I$-**cof** the subcategory consisting of the maps which can be written as direct compositions (2.7) of pushouts (2.6) of coproducts of maps in I and which we call **regular I-cofibrations**, and
(iv) note that *the subcategories I-$\mathbf{cof_{reg}}$ and I-**cof** of $\boldsymbol{C}$ are closed under direct compositions* (2.7) and that, if the category $\boldsymbol{C}$ is complete (2.5), then *the subcategory I-**inj** is closed under inverse compositions* (2.7).

Now we can discuss

7.3. The small object argument. Given a cocomplete category $\boldsymbol{C}$ and a set I of maps in $\boldsymbol{C}$, one says that I *permits the small object argument* if

- the set I is small (2.1), and
- the domains of the maps in I are small rel. I-$\mathbf{cof_{reg}}$ (7.1 and 7.2).

If κ is a regular cardinal such that the domains of the maps in I are κ-small rel. I-$\mathbf{cof_{reg}}$, then a κ-version of the countable small object argument of 6.14 yields

(i) *for every map* $f\colon X \to Y \in \boldsymbol{C}$*, a functorial factorization* $f = qi$ *in which* $q \in I$-**inj** (7.2) *and* i *is the colimit of a* κ*-sequence* $\kappa \to \boldsymbol{C}$ *which lies in* I-$\mathbf{cof_{reg}}$ *and therefore an* I*-cofibration,*

which implies (2.9) that

(ii) *if* f *is an* I*-cofibration, then* f *is a retract of* i *in* $(X \downarrow \boldsymbol{C})$*.*

Combining this last result with a κ-version of 2.10 one then gets that

(iii) *all objects of* $\boldsymbol{C}$ *which are small rel.* I-$\mathbf{cof_{reg}}$ *(e.g. the domains of the maps in* I*) are also small rel.* I-**cof**.

The last sentence of 7.1 implies that, *if the codomains of the maps in* I *are also small rel.* I-$\mathbf{cof_{reg}}$, then all I-cofibrant objects of $\boldsymbol{C}$ are small rel. I-$\mathbf{cof_{reg}}$ and hence rel. I-**cof**. In most of the cases we consider one has the even stronger result (8.5) that *all the objects are small rel.* I-**cof**.

Finally we are ready to define

7.4. Cofibrantly generated model categories. A **cofibrantly generated model category** is a model category $\boldsymbol{N}$ in which there exist

(i) a set I of cofibrations which permits *the small object argument* (7.3) and for which the I*-cofibrations* are exactly the *cofibrations* in $\boldsymbol{N}$ (so that (7.3 (ii)) every cofibration in $\boldsymbol{N}$ is a retract of a regular I-cofibration), and

(ii) a set J of trivial cofibrations which permits *the small object argument* and for which the J*-cofibrations* are exactly the *trivial cofibrations* in $\boldsymbol{N}$ (so that every trivial cofibrations in $\boldsymbol{N}$ is a retract of a regular J-cofibration).

In view of 3.2 this implies that

(iii) the *trivial fibrations* in $\boldsymbol{N}$ are exactly the I*-injectives* and the *fibrations* in $\boldsymbol{N}$ are exactly the J*-injectives* so that

(iv) the required *functorial factorizations* (3.1) can be obtained by applying the small object arguments to I and J (7.3 (i)), and

(v) if $\boldsymbol{W} \subset \boldsymbol{N}$ denotes the subcategory of the weak equivalences, then

$$J\text{-}\mathbf{cof} = I\text{-}\mathbf{cof} \cap \boldsymbol{W} \qquad \text{and} \qquad I\text{-}\mathbf{inj} = J\text{-}\mathbf{inj} \cap \boldsymbol{W}.$$

The sets I and J will be called **sets of generating cofibrations** and **generating trivial cofibrations** respectively and their elements will be referred to as **generating cofibrations** and **generating trivial cofibrations**.

7.5. Examples. As mentioned above, the categories $\boldsymbol{S}$ of *simplicial sets* and $\boldsymbol{T}$ of *compactly generated topological spaces* (6.8 and 6.9) are examples of cofibrantly generated model categories and so are their *diagram categories* (6.10).

Other examples will come up on the next section, as well as in later chapters.

We end with a few remarks on the dual notion of

7.6. Fibrantly generated model categories. Dualizing the above one clearly can

(i) define, for a complete category $\boldsymbol{C}$ and (2.7) a subcategory $\boldsymbol{C}_1 \subset \boldsymbol{C}$ which is closed under inverse compositions, the notion of an object which is **cosmall rel.** $\boldsymbol{C}_1$,

(ii) construct, for a set K of maps in $\boldsymbol{C}$, subcategories

$$K\text{-}\mathbf{proj} = (K^{\mathrm{op}}\text{-}\mathbf{inj})^{\mathrm{op}}, K\text{-}\mathbf{fib} = (K^{\mathrm{op}}\text{-}\mathbf{cof})^{\mathrm{op}}, K\text{-}\mathbf{fib}_{\mathbf{reg}} = (K^{\mathrm{op}}\text{-}\mathbf{cof}_{\mathbf{reg}})^{\mathrm{op}}$$

of $\boldsymbol{C}$, the maps of which will be called **K-projectives**, **K-fibrations** and **regular K-fibrations** respectively,

(iii) formulate the **cosmall object argument** and the **cosmallness criterion** (which involves an effective underlying set functor $\boldsymbol{C}^{\mathrm{op}} \to \mathbf{Set}$) and finally

(iv) define a **fibrantly generated model category** as a model category $\boldsymbol{N}'$ in which there exist sets of maps K and L which permit the cosmall object argument and for which the K-fibrations are exactly the fibrations in $\boldsymbol{N}'$ and the L-fibrations are the trivial fibrations.

8. A recognition lemma

In this section we discuss a *recognition lemma* which lists necessary and sufficient conditions on a category $\boldsymbol{C}$, a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ and two small sets I and J of maps in $\boldsymbol{C}$, in order that $\boldsymbol{C}$ admits a cofibrantly generated model category structure with $\boldsymbol{W}$ as its category of weak equivalence and I and J as its sets of generating cofibrations and trivial cofibrations,

We start with

8.1. Recognition lemma. *Let $\boldsymbol{C}$ be a category which is complete and cocomplete* (2.5)*, let $\boldsymbol{W} \subset \boldsymbol{C}$ be a subcategory which* (3.1) *is closed under retracts and satisfies the "two out of three" axiom and let I and J be small sets of maps in $\boldsymbol{C}$ such that*

(i) *the sets I and J permit a small object argument* (7.3)

(ii) $J\text{-}\mathbf{cof} \subset I\text{-}\mathbf{cof} \cap \boldsymbol{W}$ *and* $I\text{-}\mathbf{inj} \subset J\text{-}\mathbf{inj} \cap \boldsymbol{W}$*, and*

(iii) $J\text{-}\mathbf{cof} \supset I\text{-}\mathbf{cof} \cap \boldsymbol{W}$ *or* $I\text{-}\mathbf{inj} \supset J\text{-}\mathbf{inj} \cap \boldsymbol{W}$.

Then $\boldsymbol{C}$ admits a cofibrantly generated model category structure with I and J as its sets of generating cofibrations and trivial cofibrations respectively and $\boldsymbol{W}$ as the subcategory of the weak equivalences.

Proof. Define weak equivalences, fibrations and cofibrations in $\boldsymbol{C}$ as the maps in $\boldsymbol{W}$, the J-injective maps and the I-cofibrations respectively. Then axioms M1, M2 and M3 clearly hold, while axiom M5 follows from (i), (ii) and the small object argument 7.3. Furthermore (iii) implies M4 (ii) (or M4 (i)) and one then obtains M4 (i) (or M4 (ii)) using M5 and 2.9.

The verifications of the axioms for the model category structures of 6.8, 6.9 and 6.10 are essentially applications of this recognition lemma. Another easy application is

8.2. The Heller model category structure for $\boldsymbol{S}^{\boldsymbol{D}}$ [10]. *For every small category $\boldsymbol{D}$, the category $\boldsymbol{S}^{\boldsymbol{D}}$ of the $\boldsymbol{D}$-diagrams of simplicial sets admits a cofibrantly generated model category structure in which*

(i) *the weak equivalences and the cofibrations are the objectwise ones.*

Moreover, if κ is a regular cardinal (7.1) *$\geq$ the cardinality of the set of maps of $\boldsymbol{D}$, then*

(iii) *a map in* $\boldsymbol{S}^{\boldsymbol{D}}$ *is a fibration (resp. a trivial fibration) iff it has the right lifting property with respect to the objectwise trivial cofibrations (resp. the objectwise cofibrations) involving simplicial sets of cardinality* $\leq \kappa$.

Proof. This follows readily from the recognition lemma, once one realizes the "well known" fact that, *for every simplicial set* Y*, simplex* $u \in Y$ *and simplicial subset* $X \subset Y$ *for which the inclusion* $X \to Y$ *is a weak equivalence, there exists a countable simplicial subset* $K \subset Y$ *containing* u *such that the inclusion* $X \cap K \to K$*, and hence the inclusions* $X \to X \cup K$ *and* $X \cup K \to Y$*, are weak equivalences.*

8.3. Corollary. *The identity functor of* $\boldsymbol{S}^{\boldsymbol{D}}$ *is a left Quillen equivalence* (4.2) *from the model category structure on* 6.8 *to the one of* 8.2 *and a right Quillen equivalence in the other direction.*

8.4. Remark. In most applications of the recognition lemma (8.1), condition (iii) is considerably more difficult to verify than condition (ii), while condition (i) is either obvious or a consequence of the following

8.5. Smallness criterion. Given a set I of maps in a cocomplete and complete category $\boldsymbol{C}$, we will call a functor $U\colon \boldsymbol{C} \to \mathbf{Set}$ an I**-effective underlying set functor** if

(i) U is faithful,

(ii) U preserves sufficiently long sequential colimits of regular I-cofibrations (7.3), i.e. there exists a regular cardinal κ (7.1) such that, for every regular cardinal $\lambda \geq \kappa$ and every λ-sequence $V\colon \lambda \to I\text{-}\mathbf{cof_{reg}}$ (7.1 and 7.2), the obvious map $\operatorname{colim}^{\lambda} UV \to U \operatorname{colim}^{\lambda} V$ is an isomorphism, and

(iii) every regular I-cofibration $f\colon X \to Y \in \boldsymbol{C}$ is an *effective monomorphism*, i.e. f is the equalizer of the obvious pair of maps $Y \rightrightarrows Y \amalg_X Y \in \boldsymbol{C}$, and U preserves these equalizers, i.e. Uf is the equalizer of the induced pair of maps $UY \rightrightarrows U(Y \amalg_X Y) \in \mathbf{Set}$.

One then readily verifies that *the existence of such an* I*-effective underlying set functor implies that every object of* $\boldsymbol{C}$ *is small rel.* I-$\mathbf{cof_{reg}}$ *and hence* (7.3) *rel.* I-$\mathbf{cof}$.

We end with a remark on

8.6. The fibrantly generated case. For a *fibrantly generated* model category 7.6 there clearly is a dual version of the *recognition lemma* involving (in the notation of 7.6) the inclusions

$$K\text{-}\mathbf{proj} \subset L\text{-}\mathbf{proj} \cap \boldsymbol{W} \quad \text{and} \quad L\text{-}\mathbf{fib} \subset K\text{-}\mathbf{fib} \cap \boldsymbol{W}, \text{ and}$$
$$K\text{-}\mathbf{proj} \supset L\text{-}\mathbf{proj} \cap \boldsymbol{W} \quad \text{or} \quad L\text{-}\mathbf{fib} \supset K\text{-}\mathbf{fib} \cap \boldsymbol{W}.$$

9. A lifting lemma

In this last section we use the recognition lemma (8.1) to prove a *lifting lemma* which, for a pair of adjoint functors $F\colon \boldsymbol{B} \leftrightarrow \boldsymbol{C} : U$ and a cofibrantly generated model category structure on $\boldsymbol{B}$, gives necessary and sufficient conditions in order that $\boldsymbol{C}$ admits a cofibrantly generated model category structure with as generating cofibrations and trivial cofibrations the images under F of the generating cofibrations and trivial cofibrations of $\boldsymbol{B}$ and in which a map is a weak equivalence or a fibration whenever its image under U is so, and use this lemma to obtain the cofibrantly generated model categories mentioned in 5.5.

9.1. Lifting lemma (c.f. [5]). *Let $\boldsymbol{B}$ be a cofibrantly generated model category with I and J as its sets of generating cofibrations and trivial cofibrations, let $\boldsymbol{C}$ be a complete and cocomplete* (2.5) *category and let $F\colon \boldsymbol{B} \leftrightarrow \boldsymbol{C} :U$ be a pair of adjoint functors. Then $\boldsymbol{C}$ admits a cofibrantly generated model category structure with FI and FJ as sets of generating cofibrations and trivial cofibrations, in which a map is a weak equivalence or a fibration whenever its image under U is so, iff*

(i) *the sets FI and FJ permit the small object argument and*
(ii) *one (and hence both) of the following conditions are satisfied:*
  (ii)′ *Uf is a weak equivalence for every $f \in FJ$-**cof*** (7.2)*, or*
  (ii)″ *every map $f \in \boldsymbol{C}$ admits a factorization $f = pj$ such that Uj is a weak equivalence and Up is a fibration.*

If these conditions ((i) and (ii)) are satisfied, then F and U form an adjoint pair of Quillen functors.

Proof. One readily verifies that (ii)′ follows from (ii)″, (i) and 2.9. The rest of the proof then is a rather straightforward application of the above recognition lemma (8.1).

In view of the left properness of the usual model category structure on the category $\boldsymbol{T}$ of compactly generated topological spaces (6.9 and 6.13), the lifting lemma (9.1) and 8.5 readily imply the existence of the following topological version of the Heller model category structure for diagrams of simplicial sets (8.2).

9.2. The Heller model category structure for $\boldsymbol{T}^{\boldsymbol{D}}$. *Let $\boldsymbol{D}$ be a small category and let κ be a regular cardinal* (7.1) *$\geq$ the cardinality of the set of the maps of $\boldsymbol{D}$. Then the category $\boldsymbol{T}^{\boldsymbol{D}}$ of the $\boldsymbol{D}$-diagrams of compactly generated topological spaces admits a cofibrantly generated model category structure in which*

(i) *the weak equivalences are the objectwise ones, and*
(ii) *the set of generating (trivial) cofibrations consists of the geometric realizations of the objectwise (trivial) cofibrations in the diagram category $\boldsymbol{S}^{\boldsymbol{D}}$ which involve only simplicial sets of cardinality $\leq \kappa$.*

9.3. Remark. Unlike in 8.2, we do *not* claim that the (trivial) cofibrations are the objectwise ones. However we will show in 11.13 that this *is* the case if one assumes that $\boldsymbol{D}$ is a so-called *inverse category* (11.8).

Of course, one has, as in 6.11 and 8.3

9.4. Corollary. *The identity functor of $\boldsymbol{T}^{\boldsymbol{D}}$ is a left Quillen equivalence from the model category structure of* 6.9 *to the one of* 9.2 *and a right Quillen equivalence in the opposite direction, and the geometric realization and the singular functor induce an adjoint pair of Quillen equivalences*

$$\left|-\right|^{\boldsymbol{D}}\colon \boldsymbol{S}^{\boldsymbol{D}} \leftrightarrow \boldsymbol{T}^{\boldsymbol{D}} :\mathrm{Sin}^{\boldsymbol{D}}$$

between the model category structures of 8.2 *and* 9.2.

To describe the other applications of the lifting lemma we need

9.5. A tensor product. Given a cocomplete (2.5) category $\boldsymbol{C}$ we denote by

$$\otimes\colon \boldsymbol{C} \times \mathbf{Set} \to \boldsymbol{C}$$

the **tensor product**, i.e. the functor which sends a pair of objects $C \in \boldsymbol{C}$ and $X \in \mathbf{Set}$ to the coproduct of as many copies of C as there are elements in X.

Now we can generalize 6.10 to

9.6. A model category structure for diagram categories. *Let $\boldsymbol{N}$ be a cofibrantly generated model category with maps u_i $(i \in I)$ and v_j $(j \in J)$ as its generating cofibrations and trivial cofibrations and let $\boldsymbol{D}$ be a small category. Then*

(i) *the diagram category $\boldsymbol{N}^{\boldsymbol{D}}$* (2.3) *admits a cofibrantly generated model category structure with the maps*

$$u_i \otimes \boldsymbol{D}(D,-) \quad \textit{and} \quad v_j \otimes \boldsymbol{D}(D,-) \qquad (i \in I, j \in J, D \in \boldsymbol{D})$$

9.5 *as its generating cofibrations and trivial cofibrations, in which*

(ii) *the weak equivalences and the fibrations are the objectwise ones.*

Proof. The proof is easy if $\boldsymbol{D}$ is discrete, i.e. if $\boldsymbol{D}$ has no non-identity maps.

To prove the general case, let $\boldsymbol{O} \subset \boldsymbol{D}$ be its maximal discrete subcategory, let $U\colon \boldsymbol{N}^{\boldsymbol{D}} \to \boldsymbol{N}^{\boldsymbol{O}}$ denote the functor induced by the inclusion $\boldsymbol{O} \to \boldsymbol{D}$ and let $F\colon \boldsymbol{N}^{\boldsymbol{O}} \to \boldsymbol{N}^{\boldsymbol{D}}$ be its left adjoint. Then the desired result is a ready consequence of lemma 9.1 with $\boldsymbol{B} = \boldsymbol{N}^{\boldsymbol{O}}$ and $\boldsymbol{C} = \boldsymbol{N}^{\boldsymbol{D}}$.

For our last application of the lifting lemma we first recall what is meant by

9.7. Universal algebras. Recall [**12**, p. 120] that a *universal algebra* A consists of an *underlying set* UA together with an action thereon by a specified small set of *finitary operations* satisfying a small set of *identities*. Thus A is *a certain type of diagram of maps between the finite powers* $(UA)^n$ $(n \geq 0)$ of UA.

Similarly, given a small (indexing) set I, an *I-indexed universal algebra* A will consists of an *underlying I-indexed set* $UA = \coprod_{i \in I} U_i A$ together with an action on the $U_i A$ $(i \in I)$ by a specified small set of finitary operations satisfying a small set of identities. In other words A is a certain type of diagram of maps between the finite products $U_{i_1} A \times \cdots \times U_{i_n} A$ $(i_1, \ldots, i_n \in I,\ n \geq 0)$ of the $U_i A$.

Obvious examples of universal algebras are *sets*, *groups*, *monoids*, *rings*, *modules*, *Lie algebras*, etc. and their *graded* versions are examples of indexed universal algebras. Other examples of the latter are various diagrams of (possibly indexed) universal algebras (e.g. *differential graded modules*, *rings*, *Lie algebras* etc.) and, for every small set O, the *O-graphs* and *O-categories*, i.e. the graphs and categories with O as set of objects (which are $(O \times O)$-indexed).

Now we can describe a

9.8. Model category structure for simplicial algebras. *Let I be a small (indexing) set, let $\boldsymbol{A}$ be a complete and cocomplete* (2.5) *category of I-indexed universal algebras* (9.7) *and, for every element $i \in I$, let G_i denote the free algebra on a single generator of index i. then*

(i) *the category $\boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$* (5.7) *admits a cofibrantly generated model category structure with the inclusions* (9.5)

$$G_i \otimes \partial\Delta[n] \to G_i \otimes \Delta[n] \qquad (i \in I,\ n \geq 0)$$
$$G_i \otimes \Delta^k[n] \to G_i \otimes \Delta[n] \qquad (i \in I,\ n > 0,\ 0 \leq k \leq n)$$

as the generating cofibrations and trivial cofibrations, in which

(ii) *a map is a weak equivalence or a fibration whenever the underlying map of (indexed) simplicial sets is so.*

Proof. This follows readily from the lifting lemma, 8.5, 9.7 and

9.9. Functorial factorizations which commute with finite products. *Every map f in $\boldsymbol{T}$ or $\boldsymbol{S}$ admits a functorial factorization $f = pj$ which commutes with finite products and in which j is a weak equivalence and p is a fibration.*

Proof. If $f\colon X \to Y \in \boldsymbol{T}$, then the desired factorization $X \to Z \to Y$ is the "usual" one in which Z is the space of pairs (x, g) consisting of a point $x \in X$ and a path g in Y which starts at fx.

If $f\colon X \to Y \in \boldsymbol{S}$, then one obtains the factorization $X \to Z \xrightarrow{p} Y$ in $\boldsymbol{S}$ from the above factorization $|X| \to Z' \xrightarrow{p'} |Y|$ of $|f|$ in $\boldsymbol{T}$, by taking for p the pullback (2.5) of $\operatorname{Sin} p'$ along the adjunction map $Y \to \operatorname{Sin}|Y|$ (6.1). That this factorization in $\boldsymbol{S}$ commutes with finite products follows readily from the fact that this adjunction map does (6.2).

CHAPTER III

Reedy model category structures

10. Introduction

10.1. Summary. It seems that, for a model category $\boldsymbol{M}$ and a small category $\boldsymbol{D}$, the diagram category $\boldsymbol{M}^{\boldsymbol{D}}$ need *not* admit a model category structure in which the weak equivalences are the objectwise ones, unless one imposes some restriction on either $\boldsymbol{M}$ or $\boldsymbol{D}$ (or maybe both). We saw in the previous chapter that (6.10, 8.2 and 9.2) for simplicial sets and compactly generated topological spaces, the diagram categories admit (at least) two such model category structures.

Our aim in this chapter is to describe a sufficient restriction on $\boldsymbol{D}$, namely that $\boldsymbol{D}$ be a so-called *Reedy category.* Important examples of such Reedy categories are

(i) the cosimplicial and simplicial indexing categories $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^{\mathrm{op}}$ (5.7),
(ii) more generally, for every simplicial set K, the category of simplices $\boldsymbol{\Delta} K$ and its opposite $\boldsymbol{\Delta}^{\mathrm{op}} K$ and
(iii) the coprismatic and prismatic indexing categories $\boldsymbol{\Delta\!\!\!\!\Delta}$ and $\boldsymbol{\Delta\!\!\!\!\Delta}^{\mathrm{op}}$ (13.1) which are kind of "reduced product" categories on $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^{\mathrm{op}}$ respectively.

They give rise to Reedy model category structures which (in parts II and III of this monograph) will be used to construct *function complexes* and *homotopy colimits* and *limits.*

In more detail:

10.2. Direct and inverse categories. We start with a discussion of two special kinds of Reedy categories, *direct categories* and their opposites *inverse categories*, where a direct category is a small category $\boldsymbol{B}$ for which there exists a degree function which assigns to every object of $\boldsymbol{B}$ an ordinal in such a manner that all non-identity maps of $\boldsymbol{B}$ raise this degree. We then show that

(i) *for every model category $\boldsymbol{M}$ and every direct category $\boldsymbol{B}$, the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ admits a model category structure in which the weak equivalences and the fibrations are the objectwise ones,*

which readily implies that

(ii) *the pair of adjoint functors* $\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M} :c^*$ *is a Quillen pair* (4.1) *and hence the functor* $\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ *preserves weak equivalences between cofibrant objects.*

Of course, the dual result holds if $\boldsymbol{B}$ is an inverse category.

10.3. Reedy categories. A *Reedy category* will now consist of a small category $\boldsymbol{B}$, together with a direct (10.2) subcategory $\overrightarrow{\boldsymbol{B}}$ and an inverse (10.2) subcategory $\overleftarrow{\boldsymbol{B}}$ such that

(i) $\overrightarrow{\boldsymbol{B}}$ and $\overleftarrow{\boldsymbol{B}}$ have a common degree function, i.e. there exists a degree function which assigns to every object of $\boldsymbol{B}$ and ordinal in such a manner that all the non-identity maps of $\overrightarrow{\boldsymbol{B}}$ and $\overleftarrow{\boldsymbol{B}}$ respectively raise or lower this degree, and
(ii) every map $b \in \boldsymbol{B}$ has a unique factorization $b = \overrightarrow{b}\overleftarrow{b}$ with $\overrightarrow{b} \in \overrightarrow{\boldsymbol{B}}$ and $\overleftarrow{b} \in \overleftarrow{\boldsymbol{B}}$.

Generalizing 10.2 (i) and its dual, we then show that *for every model category $\boldsymbol{M}$ and Reedy category $\boldsymbol{B}$, the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ admits a model category structure in which a map is a weak equivalence, a cofibration or a fibration whenever its restrictions to both $\boldsymbol{M}^{\overrightarrow{\boldsymbol{B}}}$ and $\boldsymbol{M}^{\overleftarrow{\boldsymbol{B}}}$ are so in the model category structures of* 10.2.

10.4. Direct and inverse Reedy categories. The useful property 10.2 (ii) not only holds for all direct categories but also for certain other Reedy categories (e.g. $\boldsymbol{\Delta}$, $\boldsymbol{\Delta} K$ and $\boldsymbol{\Delta}$ (10.1)) which we therefore call *direct Reedy categories.* They turn out to be exactly those Reedy categories $\boldsymbol{B}$ for which the inverse subcategory $\overleftarrow{\boldsymbol{B}}$ (10.3) is a coproduct f categories with a terminal object. There is of course also a dual notion of *inverse Reedy categories.*

10.5. Organization of the chapter. After fixing some notation and terminology (in 10.6 and 10.7), we discuss (in §11) *direct* and *inverse diagrams*, i.e. diagrams indexed by direct or inverse categories, and then use these (in §12) to deal with *Reedy diagrams.* The last section (§13) deals with the *coprismatic* and *prismatic indexing categories.*

We end with a brief review of some notions that will be needed in this chapter. First the very simple but useful notion of

10.6. The nerve of a category. For every integer $n \geq 0$, let $\boldsymbol{n}$ denote the category which has as objects the integers $0, \ldots, n$ and which has exactly one map $i \to j$ whenever $i \leq j$. Given a small category $\boldsymbol{B}$ (2.2), its **nerve** $\mathrm{N}\boldsymbol{B}$ then is [**2**] the simplicial set which has as n-simplices ($n \geq 0$) the functors $\boldsymbol{n} \to \boldsymbol{B}$ (with the obvious faces and degeneracies, i.e. the i-face is the functor $\boldsymbol{n-1} \to \boldsymbol{B}$ obtained by composition with the functor $\boldsymbol{n-1} \to \boldsymbol{n}$ which "skips" the integer i) and one says that $\boldsymbol{B}$ is **discrete**, **connected** or **contractible** if (the geometric realization (6.1) of) $\mathrm{N}\boldsymbol{B}$ is so. Thus a small category $\boldsymbol{B}$ is connected if every two of its objects can be connected by a finite zigzag of maps and contractible if $\mathrm{N}\boldsymbol{B}$ is weakly equivalent (6.3) to $\Delta[0]$ (e.g. if $\boldsymbol{B}$ has an initial or a terminal object). Clearly $\mathrm{N}\boldsymbol{n} = \Delta[n]$ (5.8).

We also recall that initial and terminal objects can be considered special cases of

10.7. Initial and terminal functors and subcategories. Given two small categories $\boldsymbol{A}$ and $\boldsymbol{B}$, a functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$, a complete category $\boldsymbol{C}$ and an object $X \in \boldsymbol{C}^{\boldsymbol{B}}$, there is an obvious map $\lim^{\boldsymbol{B}} X \to \lim^{\boldsymbol{A}} u^* X \in \boldsymbol{C}$ (2.3 and 2.5) and [**12**] this map is an isomorphism for every complete category $\boldsymbol{C}$ and object $X \in \boldsymbol{C}^{\boldsymbol{B}}$ iff, for every object $B \in \boldsymbol{B}$, the over category $(u \downarrow B)$ (2.4) is *connected* (10.6). In [**12**] such a functor u was called initial but we prefer 0**-initial**, because we want to use the adjective **initial** for such a functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ which is left cofinal, i.e. u has the stronger property that, for every object $B \in \boldsymbol{B}$, the over category $(u \downarrow B)$ is *contractible* (10.6). Similarly we call a subcategory $\boldsymbol{B}_0 \subset \boldsymbol{B}$ a 0**-initial**

subcategory or an **initial subcategory** if the inclusion functor $\boldsymbol{B}_0 \to \boldsymbol{B}$ is 0-initial or initial.

Dually we call a functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ between small categories **0-terminal** or **terminal** (instead of right cofinal) if, for every object $B \in \boldsymbol{B}$, the under category $(B \downarrow u)$ is connected or contractible and a subcategory $\boldsymbol{B}_0 \subset \boldsymbol{B}$ will be called **0-terminal** or **terminal** if the inclusion functor $\boldsymbol{B}_0 \to \boldsymbol{B}$ is so.

Clearly a functor $f\colon \mathbf{0} \to \boldsymbol{B}$ (10.6) is initial iff the object $f0 \in \boldsymbol{B}$ is an initial object and terminal iff $f0 \in \boldsymbol{B}$ is a terminal object.

11. Direct and inverse diagrams

In preparation for our discussion of Reedy diagrams in a model category (in §12) we consider here first the two special cases of *direct* and *inverse diagrams* and show that

(i) given a model category $\boldsymbol{M}$ and a *direct* (11.1) category $\boldsymbol{B}$, the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ inherits from $\boldsymbol{M}$ a model category structure in which the weak equivalences and the fibrations are the objectwise ones, so that the pair of adjoint functors $\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M} :c^*$ is a Quillen pair (4.1), and dually

(ii) given a model category $\boldsymbol{M}$ and an *inverse* (11.8) category $\boldsymbol{B}$, the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ inherits from $\boldsymbol{M}$ a model category structure in which the weak equivalences and the cofibrations are the objectwise ones, so that the pair of adjoint functors $c^*\colon \boldsymbol{M} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}} :\lim^{\boldsymbol{B}}$ is a Quillen pair.

We first deal with the direct case.

11.1. Direct categories. A **direct category** consists of a small (2.2) category $\boldsymbol{B}$ for which there exists a **degree function** which assigns to every object $B \in \boldsymbol{B}$ an ordinal (2.7) $\deg B$, such that all non-identity maps of $\boldsymbol{B}$ raise the degree.

Obvious examples are

(i) all subcategories of a direct category
(ii) all over and under categories (2.3) of a direct category,
(iii) all finite products of direct categories, and
(iv) for every infinite ordinal γ, the associated (2.7) category (which is also denoted by γ).

Other examples are

11.2. Finite dimensional categories. A **finite dimensional** category is a small category $\boldsymbol{F}$ for which there exists an integer n such that (the nerve (10.6) of) $\boldsymbol{F}$ is n-dimensional, i.e. $\boldsymbol{F}$ contains composable sequences $F_0 \to \cdots \to F_j$ of j non-identity maps for $j = n$, but not for $j > n$. If $\boldsymbol{F}$ is n-dimensional, then there clearly exists a degree function (11.1) with $\deg F \le n$ for every object $F \in \boldsymbol{F}$, which makes $\boldsymbol{F}$ a direct category, and conversely, if $\boldsymbol{F}$ is a direct category and $\deg F \le n$ for some fixed integer n and every object $F \in \boldsymbol{F}$, then $\boldsymbol{F}$ is finite dimensional and its dimension is $\le n$. Some obvious examples of finite dimensional categories are

(i) for every integer $n \ge 0$, the category $\boldsymbol{n}$ (10.6),
(ii) every subcategory of a finite dimensional category, and
(iii) for every direct category $\boldsymbol{B}$ and every object $B \in \boldsymbol{B}$, the over category $(\boldsymbol{B} \downarrow B)$ (2.4).

Now we turn to

11.3. Direct diagrams and their latching objects and maps. Given a cocomplete (2.5) category $\boldsymbol{C}$ and a **direct diagram** in $\boldsymbol{C}$, i.e. a diagram $X\colon \boldsymbol{B} \to \boldsymbol{C}$ where $\boldsymbol{B}$ is a direct category (11.1), one can, for every object $B \in \boldsymbol{B}$, form the maximal subcategory $\partial(\boldsymbol{B}\downarrow B) \subset (\boldsymbol{B}\downarrow B)$ which does *not* contain the identity map of B, and define

(i) the **latching object** $\mathrm{L}XB$ of X at B as the object

$$\mathrm{L}XB = \operatorname{colim}^{\partial(\boldsymbol{B}\downarrow B)} j^* X \in \boldsymbol{C}$$

where $j\colon \partial(\boldsymbol{B}\downarrow B) \to \boldsymbol{B}$ denotes the forgetful functor, and

(ii) the **latching map** as the induced map $\mathrm{L}XB \to XB \in \boldsymbol{C}$.

As the definition of the latching object $\mathrm{L}XB$ only involves objects of $\boldsymbol{C}$ of the form XB' with $\deg B' < \deg B$, this readily implies that *the diagram X is completely determined by the function which, inductively* (on the degree), *assigns to each object $B \in \boldsymbol{B}$, the object $XB \in \boldsymbol{C}$ and the latching map $\mathrm{L}XB \to XB \in \boldsymbol{C}$.* This explains the appearance of the latching objects and maps in

11.4. The direct model category structure. *Given a model category $\boldsymbol{M}$* (3.1) *and a direct category $\boldsymbol{B}$* (11.1)*, the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ admits a model category structure in which*

(i) *the weak equivalences and the fibrations are the objectwise ones, and*

(ii) *a map $X \to Y \in \boldsymbol{M}^{\boldsymbol{B}}$ is a cofibration if, for every object $B \in \boldsymbol{B}$ the induced map*

$$XB \amalg_{\mathrm{L}XB} \mathrm{L}YB \to YB \in \boldsymbol{M}$$

is so.

Moreover

(iii) *a map $X \to Y \in \boldsymbol{M}^{\boldsymbol{B}}$ is a trivial cofibration iff, for every object $B \in \boldsymbol{B}$, the induced map*

$$XB \amalg_{\mathrm{L}XB} \mathrm{L}YB \to YB \in \boldsymbol{M}$$

is so, and

(iv) *this model category structure does not depend on the choice of the degree function* (11.1)*.*

11.5. Corollary. *The pair of adjoint functors* $\operatorname{colim}\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M} : c^*$ (2.5) *is a Quillen pair* (4.1) *and hence* (3.6) *the left adjoint* $\operatorname{colim}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ *preserves weak equivalences between cofibrant objects.*

11.6. Corollary. *Every cofibration in $\boldsymbol{M}^{\boldsymbol{B}}$ is an objectwise cofibration.*

Proof of 11.4. The proof of 11.4 is rather straightforward, by induction on the degree, using the fact that 11.4, and hence 11.5, holds for Reedy categories with degree functions which involve only lesser degrees.

We end with observing that 11.4 does not yield anything new in

11.7. The cofibrantly generated case. If $\boldsymbol{M}$ is cofibrantly generated, then clearly the above model category structure coincides with the (cofibrantly generated) one of 9.6. In particular, if $\boldsymbol{M} = \boldsymbol{S}$ or $\boldsymbol{T}$ (§6), then the model category structure of 11.4 coincides with the one of 6.10.

Dualizing this one has

11.8. Inverse categories. An **inverse category** is a category $\boldsymbol{B}$ such that $\boldsymbol{B}^{\mathrm{op}}$ is direct (11.1). Thus every *subcategory* of an inverse category is inverse and so are all its *over* and *under categories*. Also (11.2) every *finite dimensional* small category can be turned into an inverse category.

Furthermore one has dualizing 11.3

11.9. Inverse diagrams and their matching objects and maps. Given a complete (2.5) category $\boldsymbol{C}$ and an **inverse diagram** in $\boldsymbol{C}$, i.e. a diagram $X\colon \boldsymbol{B} \to \boldsymbol{C}$ where $\boldsymbol{B}$ is an inverse category, one can, for every object $B \in \boldsymbol{B}$, form the maximal subcategory $\partial(B \downarrow \boldsymbol{B}) \subset (B \downarrow \boldsymbol{B})$ which does *not* contain the identity map of B and define

(i) the **matching object** of X at B as the object

$$\mathrm{M}XB = \lim{}^{\partial(B\downarrow\boldsymbol{B})}\, j^*X \in \boldsymbol{C}$$

where $j\colon \partial(B \downarrow \boldsymbol{B}) \to \boldsymbol{B}$ denotes the forgetful functor, and

(ii) the *matching map* as the induced map $XB \to \mathrm{M}XB \in \boldsymbol{C}$.

Again it readily follows that *the diagram X is completely determined by the function which inductively* (on the degree) *assigns to each object $B \in \boldsymbol{B}$, the object $XB \in \boldsymbol{C}$ and the matching map $XB \to \mathrm{M}XB \in \boldsymbol{C}$.*

Finally one gets, dualizing 11.4

11.10. The inverse model category structure. *Given a model category $\boldsymbol{M}$ and an inverse category $\boldsymbol{B}$* (11.8)*, the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ admits a model category structure in which*

(i) *the weak equivalences and the cofibrations are the objectwise ones, and*

(ii) *a map $X \to Y \in \boldsymbol{M}^{\boldsymbol{B}}$ is a fibration if, for every object $B \in \boldsymbol{B}$, the induced map*

$$XB \to YB\, \Pi_{\mathrm{M}YB}\, \mathrm{M}XB \in \boldsymbol{M}$$

is so.

Moreover

(iii) *a map $X \to Y \in \boldsymbol{M}^{\boldsymbol{B}}$ is a trivial fibration iff, for every object $B \in \boldsymbol{B}$, the induced map*

$$XB \to YB\, \Pi_{\mathrm{M}YB}\, \mathrm{M}XB \in \boldsymbol{M}$$

is so, and

(iv) *this model category structure does not depend on the choice of the degree function (*11.1 *and* 11.8*).*

11.11. Corollary. *The pair of adjoint functors $c^*: \boldsymbol{M} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}}$* : lim (2.5) *is a Quillen pair* (4.1) *and hence* 3.6 *the right adjoint* lim: $\boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ *preserves weak equivalences between fibrant objects.*

11.12. Corollary. *Every fibration in $\boldsymbol{M}^{\boldsymbol{B}}$ is an objectwise fibration.*

We end again with a discussion of the cofibrantly generated case. First we note

11.13. Proposition. *If* $\boldsymbol{M} = \boldsymbol{S}$ *or* $\boldsymbol{T}$ *(§6), then the model category structures of* 11.10 *coincide with the Heller model category structures of* 8.2 *and* 9.2.

Proof. This is obvious for $\boldsymbol{M} = \boldsymbol{S}$ and the case $\boldsymbol{M} = \boldsymbol{T}$ follows readily from 9.2 and the following result for

11.14. The cofibrantly generated case. *Let* $\boldsymbol{M}$ *be a cofibrantly generated model category* (7.4) *with maps* $P_i \to Q_i$ $(i \in I)$ *and* $R_j \to S_j$ $(j \in J)$ *as generating cofibrations and trivial cofibrations and let* $\boldsymbol{B}$ *be an inverse category. Then the inverse model category structure* (11.10) *on the diagram category* $\boldsymbol{M}^{\boldsymbol{B}}$ *is also cofibrantly generated and has the maps* (9.5)

$$u_{i,B}\colon P_i \otimes \boldsymbol{B}(B,-) \amalg_{P_i \otimes \partial\boldsymbol{B}(B,-)} Q_i \otimes \partial\boldsymbol{B}(B,-) \to Q_i \otimes \boldsymbol{B}(B,-) \qquad i \in I, B \in \boldsymbol{B}$$

and

$$v_{j,B}\colon R_j \otimes \boldsymbol{B}(B,-) \amalg_{R_j \otimes \partial\boldsymbol{B}(B,-)} S_j \otimes \partial\boldsymbol{B}(B,-) \to S_j \otimes \boldsymbol{B}(B,-) \qquad j \in J, B \in \boldsymbol{B}$$

as generating cofibrations and trivial cofibrations, where $\partial\boldsymbol{B}(B,-)$ *denotes the sub* $\boldsymbol{B}$*-diagram of sets of* $\boldsymbol{B}(B,-)$ *consisting of the maps* $B \to B' \in \boldsymbol{B}$ *such that* $\deg B' < \deg B$.

Moreover the identity functor of $\boldsymbol{M}^{\boldsymbol{B}}$ *is a left Quillen equivalence* (4.2) *from the model category structure of* 9.6 *to the one of* 11.10 *and a right Quillen equivalence in the opposite direction.*

Proof. Choose a degree function and an ordinal γ such that $\deg B < \gamma$ for every object $B \in \boldsymbol{B}$. Every map $f\colon X \to Y \in \boldsymbol{M}^{\boldsymbol{B}}$ then can be written as a direct composition (2.7) of maps $Z_\alpha \to Z_{\alpha+1}$ $(\alpha < \gamma)$, where each Z_α agrees with Y in degrees $< \alpha$ and with X in degrees $\geq \alpha$. If f is a cofibration, then so are these maps $Z_\alpha \to Z_{\alpha+1}$ and this readily implies that each map $Z_\alpha \to Z_{\alpha+1}$ is an I'_α-cofibration (7.2) where I'_α consists of the maps $u_{i,B}$ with $i \in I$ and $\deg B = \alpha$. A similar argument applies if f is a trivial cofibration.

12. Reedy diagrams

We now extend the results of the previous section to *Reedy diagrams*, which are a common generalization of direct and inverse diagrams. They are indexed by

12.1. Reedy categories. A **Reedy category** is a small category $\boldsymbol{B}$, together with two subcategories $\overrightarrow{\boldsymbol{B}}$ and $\overleftarrow{\boldsymbol{B}}$ which each contain all the objects, for which there exists a degree function which assigns to every object $B \in \boldsymbol{B}$ an ordinal (2.7), such that

(i) all non-identity maps of $\overrightarrow{\boldsymbol{B}}$ raise the degree (and hence $\overrightarrow{\boldsymbol{B}}$ is a direct category),
(ii) all non-identity maps of $\overleftarrow{\boldsymbol{B}}$ lower the degree (and hence $\overleftarrow{\boldsymbol{B}}$ is an inverse category), and
(iii) every map $b \in \boldsymbol{B}$ has a unique factorization $b = \overrightarrow{b}\overleftarrow{b}$ with $\overrightarrow{b} \in \overrightarrow{\boldsymbol{B}}$ and $\overleftarrow{b} \in \overleftarrow{\boldsymbol{B}}$.

This definition clearly implies that *direct and inverse categories* are Reedy categories, that the *opposite* as well as the *under and over categories* of a Reedy category are again Reedy categories, that a finite product of Reedy categories is a Reedy category and that, for every Reedy category $\boldsymbol{B}$, degree function on $\boldsymbol{B}$ and ordinal α, the resulting α-skeleton $\boldsymbol{B}^\alpha$ (i.e. the full subcategory spanned by the objects of degree $\leq \alpha$) is a Reedy category. Other examples of Reedy categories are provided by the *category of simplices* $\boldsymbol{\Delta}\, K$ of a simplicial set K (5.9) and its *opposite* $\boldsymbol{\Delta}^{\mathrm{op}}\, K$ and in particular the categories $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^{\mathrm{op}}$ (5.7), as well as the *coprismatic* and *prismatic* indexing categories $\Delta\!\!\!\Delta$ and $\Delta\!\!\!\Delta^{\mathrm{op}}$ which will be discussed in §13.

Reedy categories give rise to (11.3) and (11.9)

12.2. Reedy diagrams. Given a complete and cocomplete category $\boldsymbol{C}$ and a **Reedy diagram** in $\boldsymbol{C}$, i.e. a diagram $X\colon \boldsymbol{B} \to \boldsymbol{C}$ where $\boldsymbol{B}$ is a Reedy category, define, for every object $B \in \boldsymbol{B}$,

(i) the **latching object** $\mathrm{L}XB$ of X at B as the latching object (11.3) of the restriction $X|\overrightarrow{\boldsymbol{B}}$ at B, i.e.

$$\mathrm{L}XB = \operatorname{colim}^{\partial(\overrightarrow{\boldsymbol{B}}\downarrow B)} j^* X \in \boldsymbol{C}$$

where $j\colon \partial(\overrightarrow{\boldsymbol{B}} \downarrow B) \to \boldsymbol{B}$ denotes the forgetful functor, and the *latching map* as the induced map $\mathrm{L}XB \to XB \in \boldsymbol{C}$,

(ii) the **matching object** $\mathrm{M}XB$ of X at B as the matching object (11.9) of the restriction $X|\overleftarrow{\boldsymbol{B}}$ at B, i.e.

$$\mathrm{M}XB = \lim^{\partial(B\downarrow\overleftarrow{\boldsymbol{B}})} j^* X \in \boldsymbol{C}$$

where $j\colon \partial(B \downarrow \overleftarrow{\boldsymbol{B}}) \to \boldsymbol{B}$ denotes the forgetful functor, and the *matching map* as the induced map $XB \to \mathrm{M}XB \in \boldsymbol{C}$, and

(iii) the **connecting map** of X at B as the composition

$$\mathrm{L}XB \to XB \to \mathrm{M}XB \in \boldsymbol{C}$$

of the latching map and the matching map.

As, given a degree function, this definition of $\mathrm{L}XB$, $\mathrm{M}XB$ and their connecting map clearly only involves the restrictions to the α-skeleta (12.1) of $\boldsymbol{B}$ with $\alpha < \deg B$, *the diagram X is completely determined by the function which inductively* (on the degree) *assigns to every object $B \in \boldsymbol{B}$ an object $XB \in \boldsymbol{C}$ together with a factorization $\mathrm{L}XB \to XB \to \mathrm{M}XB$ of the connecting map at B.*

Using 12.1 (iii) one then readily shows

12.3. Proposition. *For every degree function on $\boldsymbol{B}$ and object $B \in \boldsymbol{B}$ of degree α, let $\partial(\boldsymbol{B}^\alpha \downarrow B)$ and $\partial(B \downarrow \boldsymbol{B}^\alpha)$ denote the maximal subcategories of $(\boldsymbol{B}^\alpha \downarrow B)$ and $(B \downarrow \boldsymbol{B}^\alpha)$* (12.1) *not containing the identity map of B. Then $\partial(\overrightarrow{\boldsymbol{B}} \downarrow B)$ is a terminal subcategory* (10.7) *of $\partial(\boldsymbol{B}^\alpha \downarrow B)$ and $\partial(B \downarrow \overleftarrow{\boldsymbol{B}})$ is an initial subcategory of $\partial(B \downarrow \boldsymbol{B}^\alpha)$.*

Using this 11.4 and 11.10 now generalize to (c.f. [**17**])

12.4. The Reedy model category structure. *Given a model category $\boldsymbol{M}$ and a Reedy category $\boldsymbol{B}$* (12.1)*, the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ admits a model category structure in which a map is a weak equivalence, a (trivial) fibration or a (trivial) cofibration whenever its restrictions to $\overrightarrow{\boldsymbol{B}}$ and $\overleftarrow{\boldsymbol{B}}$ both are so.*

This immediately implies

12.5. Proposition. *Given a model category $\boldsymbol{M}$ and two Reedy categories $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, the Reedy model category structure on $\boldsymbol{M}^{\boldsymbol{B}_1\times\boldsymbol{B}_2}$ obtained directly from the model category structure on $\boldsymbol{M}$ coincides with the ones obtained from the Reedy model category structures on $\boldsymbol{M}^{\boldsymbol{B}_1}$ and $\boldsymbol{M}^{\boldsymbol{B}_2}$.*

12.6. Proposition. *If $f\colon \boldsymbol{M} \to \boldsymbol{N}$ is a left or a right Quillen functor, then so is, for every Reedy category $\boldsymbol{B}$, the induced functor $\boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{N}^{\boldsymbol{B}}$.*

Furthermore the lifting lemma (9.1) and 11.14 readily imply the following result for

12.7. The cofibrantly generated case. *Let $\boldsymbol{M}$ be a cofibrantly generated model category* (7.4) *with maps $P_i \to Q_i$ $(i \in I)$ and $R_j \to S_j$ $(j \in J)$ as generating cofibrations and trivial cofibrations and let $\boldsymbol{B}$ be a Reedy category. Then the Reedy model category structure* (12.4) *on the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ is also cofibrantly generated and has the maps*

$$u_{i,B}\colon P_i \otimes \boldsymbol{B}(B,-) \amalg_{P_i\otimes\partial\boldsymbol{B}(B,-)} Q_i \otimes \partial\boldsymbol{B}(B,-) \to Q_i \otimes \boldsymbol{B}(B,-) \qquad i \in I, B \in \boldsymbol{B}$$

and

$$v_{j,B}\colon R_j \otimes \boldsymbol{B}(B,-) \amalg_{R_j\otimes\partial\boldsymbol{B}(B,-)} S_j \otimes \partial\boldsymbol{B}(B,-) \to S_j \otimes \boldsymbol{B}(B,-) \qquad j \in J, B \in \boldsymbol{B}$$

as generating cofibrations and trivial cofibrations, where $\partial\boldsymbol{B}(B,-)$ denotes the sub $\boldsymbol{B}$-diagram of sets of $\boldsymbol{B}(B,-)$ consisting of the maps $B \to B' \in \boldsymbol{B}$ such that $\deg B' < \deg B$.

Moreover the identity functor of $\boldsymbol{M}^{\boldsymbol{B}}$ is a left Quillen equivalence (4.2) *from the model category structure of* 9.6 *to the one of* 12.4 *and a right Quillen equivalence in the opposite direction.*

To obtain analogs of 11.5 and 11.11 we have to restrict ourselves to diagrams indexed by

12.8. Direct and inverse Reedy categories. A **direct Reedy category** is a Reedy category $\boldsymbol{B}$ for which the inverse subcategory $\overleftarrow{\boldsymbol{B}}$ (12.1) has a discrete terminal subcategory (10.6 and 10.7). With other words $\boldsymbol{B}$ *is a coproduct of categories with a terminal object.* Examples are *direct categories* and the *categories of simplices of simplicial sets* (5.9) and in particular *the category* $\boldsymbol{\Delta}$ (5.7), as well as *the coprismatic indexing category* $\boldsymbol{\Delta\!\!\!\Delta}$ (13.2). And of course any *skeleton* (12.1) *of a direct Reedy category* is again a direct Reedy category.

Similarly an **inverse Reedy category** is a Reedy category $\boldsymbol{B}$ whose opposite is a direct Reedy category, i.e. the direct subcategory $\overrightarrow{\boldsymbol{B}} \subset \boldsymbol{B}$ has a discrete initial subcategory (10.6 and 10.7).

One now readily verifies using the results of §11

12.9. Proposition. *Let $\boldsymbol{B}$ be a Reedy category. Then*

(i) *$\boldsymbol{B}$ is a direct Reedy category iff, for every model category $\boldsymbol{M}$, the pair of adjoint functors* $\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M} :c^*$ (2.5) *is a Quillen pair* (4.1) *(so that the functor* $\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ *preserves weak equivalences between cofibrant objects), and dually*

(ii) *$\boldsymbol{B}$ is an inverse Reedy category iff, for every model category $\boldsymbol{M}$, the pair of adjoint functors $c^*\colon \boldsymbol{M} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}} :\lim^{\boldsymbol{B}}$ is a Quillen pair (so that the functor $\lim^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ preserves weak equivalences between fibrant objects).*

As a *direct* Reedy category with an *initial* object is also an *inverse* Reedy category and dually an *inverse* Reedy category with a *terminal* object is also a *direct* Reedy category, proposition 12.9 readily implies

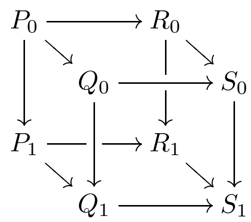
12.10. Proposition. *Given a model category* $\boldsymbol{M}$*,*

(i) *if* $\boldsymbol{B}$ *is a direct Reedy category with an initial object* B_0 *and* $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *is a cofibrant diagram of weak equivalences* (50.6)*, then the induced map* $XB_0 \to \operatorname{colim} X \in \boldsymbol{M}$ *is also a weak equivalence, and dually*

(ii) *if* $\boldsymbol{B}$ *is an inverse Reedy category with a terminal object* B_0 *and* $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *is a fibrant diagram of weak equivalences, then the induced map* $\lim X \to XB_0 \in \boldsymbol{M}$ *is also a weak equivalence.*

We end with another useful result [**17**].

12.11. Application. Let $\boldsymbol{B}$ be the category with three objects B_0, B_1 and B_2 and two non-identity maps $\overleftarrow{b} : B_1 \to B_0$ and $\overrightarrow{b} : B_2 \to B_0$. Then $\boldsymbol{B}$ (resp. $\boldsymbol{B}^{\mathrm{op}}$) is clearly a direct (resp. an inverse) Reedy category and 12.9 therefore implies that *for a commutative cube*



in a model category

(i) *the map* $S_0 \to S_1$ *is a weak equivalence whenever the other three vertical maps are weak equivalences, the horizontal squares are pushout squares* (2.5) *between cofibrant objects and the maps* $P_0 \to Q_0$ *and* $P_1 \to Q_1$ *are cofibrations*

and dually

(i)′ *the map* $P_0 \to P_1$ *is a weak equivalence whenever the other three vertical maps are weak equivalences, the horizontal squares are pullback squares between fibrant objects and the maps* $Q_0 \to S_0$ *and* $Q_1 \to S_1$ *are fibrations.*

13. The coprismatic and prismatic indexing categories

In this last section we introduce two Reedy categories, the *coprismatic indexing category* $\boldsymbol{\Delta}\!\!\!\boldsymbol{\Delta}$ and its opposite, the *prismatic indexing category* $\boldsymbol{\Delta}\!\!\!\boldsymbol{\Delta}^{\mathrm{op}}$, which we will use in the next chapter in the construction of *composable* function complexes in an arbitrary model category. They are kind of "reduced product" categories on the cosimplicial indexing category $\boldsymbol{\Delta}$ and the simplicial indexing category $\boldsymbol{\Delta}^{\mathrm{op}}$ respectively.

We also briefly discus the associated *prismatic sets* which are closely related to simplicial sets.

We start with

13.1. The definitions. The coprismatic indexing category $\triangle\!\!\!\triangle$ and the prismatic indexing category $\triangle\!\!\!\triangle^{\mathrm{op}}$ will be *monoids* [**12**, p. 75] *in the category* **Cat** *of small categories* (2.2). Let $F\mathbf{\Delta}$ be the free monoid in **Cat** on the cosimplicial indexing category $\mathbf{\Delta}$ (5.7), i.e. $F\mathbf{\Delta} = \coprod_{i\geq 0} \mathbf{\Delta}^i$, where $\mathbf{\Delta}^i$ $(i \geq 0)$ is the i-th cartesian power of $\mathbf{\Delta}$ (so that $\mathbf{\Delta}^1 = \mathbf{\Delta}$ and $\mathbf{\Delta}^0$ is the terminal object of **Cat**). The **coprismatic indexing category** $\triangle\!\!\!\triangle$ then will be the quotient monoid of $F\mathbf{\Delta}$ obtained by identifying the only object of $\mathbf{\Delta}^0$ with the object $[0] \in \mathbf{\Delta} = \mathbf{\Delta}^1$ (5.7) and the **prismatic indexing category** will be its opposite $\triangle\!\!\!\triangle^{\mathrm{op}}$.

In order to be able to use these categories one needs of course

13.2. A more explicit description. The coprismatic indexing category $\triangle\!\!\!\triangle$ can be described as the category which has as *objects* the partially ordered sets (5.7)

$$[n_1, \dots, n_k] = [n_1] \times \cdots \times [n_k] \qquad\qquad n_1, \dots, n_k > 0, \quad k \geq 0$$

(which for $k = 0$ should be interpreted as the empty product, i.e. the terminal partially ordered set $[0]$) and which has as *maps*

$$f \colon [n_1, \dots, n_k] \to [m_1, \dots, m_j]$$

between two such objects, the order preserving functions for which there exists a commutative diagram

$$\begin{array}{ccc}
[n_1, \dots, n_k] & \approx & [n_1'] \times \cdots \times [n_t'] \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_1 \times \cdots \times f_t} \\
[m_1, \dots, m_j] & \approx & [m_1'] \times \cdots \times [m_t']
\end{array}$$

in which the sequences $n_1', \dots, n_t'$ and $m_1', \dots, m_t'$ are obtained from the sequences $n_1, \dots, n_k$ and $m_1, \dots, m_j$ by inserting 0's, the horizontal isomorphisms are the obvious ones and the maps $f_1, \dots, f_t$ are in $\mathbf{\Delta}$. One then readily verifies that

(i) the composition of two such maps is again such a map, i.e. $\triangle\!\!\!\triangle$ is indeed a category,
(ii) the product of two such maps is again such a map, i.e. $\triangle\!\!\!\triangle$ is a monoid, and
(iii) the vertical map on the right determines the one on the left and the resulting function $q \colon F\mathbf{\Delta} \to \triangle\!\!\!\triangle$ is a functor which is moreover a map of monoids with the desired (13.1) universal property.

Clearly one can turn $\triangle\!\!\!\triangle$ into a Reedy category and in fact a *direct Reedy category* (11.8) by choosing for $\overrightarrow{\triangle\!\!\!\triangle}$ its subcategory of monomorphisms and for $\overleftarrow{\triangle\!\!\!\triangle}$ its subcategory of epimorphisms.

The categories $\triangle\!\!\!\triangle$ and $\triangle\!\!\!\triangle^{\mathrm{op}}$ also admit

13.3. A description in terms of ordinary monoids. The monoid structure on $\triangle\!\!\!\triangle$ turns the set of the maps of $\triangle\!\!\!\triangle$ into an ordinary monoid (i.e. a monoid in **Set**) with $1_{[0]}$ as its identity element. To give a description of this monoid in terms of generators and relations we note that, *in the above diagram* (13.2)*, the vertical map on the left determines a unique map on the right subject to the restriction that, for every integer i with* $1 \leq i \leq t$, $n_i' = 0$ *implies* $m_i' = 0$ *and* $m_{i+1}' > 0$. From this it then readily follows that

(i) the monoid of the maps of $\Delta\!\!\!\Delta$ is the monoid generated by the maps of $\boldsymbol{\Delta}$, with $1_{[0]}$ as identity element and with, for every map $a \in \boldsymbol{\Delta}$ with $[0]$ as its domain and every map $b \in \boldsymbol{\Delta}$ with $[0]$ as its codomain, the relation $ab = ba$, and

(ii) the submonoids of the identity maps, the monomorphisms and the epimorphisms of $\Delta\!\!\!\Delta$ are respectively the (free) monoids generated by the identity maps, the monomorphisms and the epimorphisms of $\boldsymbol{\Delta}$ with again $1_{[0]}$ as identity element.

We end with a brief discussion of

13.4. Prismatic sets. Let $\boldsymbol{P}$ denote the category $\mathbf{Set}^{\Delta\!\!\!\Delta^{\mathrm{op}}}$ of **prismatic sets**. Then, as in 5.7-5.10,

(i) for every object $[n_1, \dots, n_k] \in \Delta\!\!\!\Delta$, there is a **standard** $(n_1, \dots, n_k)$**-prism**

$$\Delta\!\!\!\Delta[n_1, \dots, n_k] = \boldsymbol{\Delta\!\!\!\Delta}\big(-, [n_1, \dots, n_k]\big) \in \boldsymbol{P}$$

which, together with the induced maps between them form a **diagram of standard prisms** $\Delta\!\!\!\Delta[-]\colon \boldsymbol{\Delta\!\!\!\Delta} \to \boldsymbol{P}$,

(ii) for every pair of objects $X \in \boldsymbol{P}$ and $[n_1, \dots, n_k] \in \boldsymbol{\Delta\!\!\!\Delta}$, the $(n_1, \dots, n_k)$**-prisms** of X, i.e. the elements of $X[n_1, \dots, n_k]$, are in a natural 1-1 correspondence with the maps $\Delta\!\!\!\Delta[n_1, \dots, n_k] \to X \in \boldsymbol{P}$,

(iii) for every object $X \in \boldsymbol{P}$, its **category of prisms** $\boldsymbol{\Delta\!\!\!\Delta} X$, i.e. the over category $(\Delta\!\!\!\Delta[-] \downarrow X)$ (2.4) comes with a forgetful functor $\Delta\!\!\!\Delta[X]\colon \boldsymbol{\Delta\!\!\!\Delta} X \to \boldsymbol{P}$, the **diagram of prisms** of X, which has the property that the induced map $\operatorname{colim}^{\boldsymbol{\Delta\!\!\!\Delta} X} \Delta\!\!\!\Delta[X] \to X \in \boldsymbol{P}$ is an isomorphism, which implies that

(iv) there exists a pair of adjoint functors

$$\big|-\big|_{\boldsymbol{S}}\colon \boldsymbol{P} \leftrightarrow \boldsymbol{S} :\!\operatorname{Sin}_{\boldsymbol{P}}$$

in which the left adjoint, the **simplicial realization**, sends each standard prism $\Delta\!\!\!\Delta[n_1, \dots, n_k] \in \boldsymbol{P}$ to the **simplicial standard prism**

$$\Delta[n_1, \dots, n_k] = \Delta[n_1] \times \cdots \times \Delta[n_k] \in \boldsymbol{S}$$

and the right adjoint, the **prismatic singular functor**, sends each simplicial set K to the prismatic set $\operatorname{Sin}_{\boldsymbol{P}} K$ which has as $(n_1, \dots, n_k)$-prisms the maps $\Delta[n_1, \dots, n_k] \to K \in \boldsymbol{S}$.

We also note that the category $\boldsymbol{P}$ comes with a **tensor product** $\otimes\colon \boldsymbol{P} \times \boldsymbol{P} \to \boldsymbol{P}$, which sends any two objects $X, Y \in \boldsymbol{P}$ to the object $X \otimes Y$ given by

$$(X \otimes Y)[n_1, \dots, n_k] = \coprod_{0 \le i \le k} X[n_1, \dots, n_i] \times Y[n_{i+1}, \dots, n_k]$$

and which is well behaved with respect to the simplicial realization in the sense that there exists an obvious natural isomorphism

$$\big|X \otimes Y\big|_{\boldsymbol{S}} \approx \big|X\big|_{\boldsymbol{S}} \times \big|Y\big|_{\boldsymbol{S}} \qquad X, Y \in \boldsymbol{P}.$$

We end with a comment on

13.5. A possible model category structure. It seems likely, although probably not easy to prove, that

(i) the category $\boldsymbol{P}$ admits a (cofibrantly generated) model category structure in which the cofibrations are the monomorphisms and in which a map is a weak equivalence whenever its simplicial realization is so, and

(ii) this model category structure turns the above pair of adjoint functors into Quillen equivalences (4.2).

Leftovers, awaiting recycling

March 28, 1997

CHAPTER XI

Homotopy categories

41. Introduction

41.1. Summary. We now come to the main aim of Part I of this monograph. Given a model category $\boldsymbol{M}$, we

(i) construct, mimicking what one usually does for topological spaces and simplicial sets, a *classical homotopy category of* $\boldsymbol{M}$ which consists of *homotopy classes* of maps between the objects of $\boldsymbol{M}$ which are both cofibrant and fibrant (3.1), and
(ii) prove that this *classical homotopy category of* $\boldsymbol{M}$ *is equivalent* (42.1) *to a Quillen homotopy category of* $\boldsymbol{M}$ which is obtained by *localizing* $\boldsymbol{M}$ with respect to the weak equivalences, i.e. by "formally inverting" the weak equivalences.

This strongly suggests that *the weak equivalences in a model category are somehow more essential than the cofibrations and the fibrations.*

We also show that an adjoint pair of *Quillen functors* (4.1) between two model categories gives rise to an *adjoint pair of functors between their homotopy categories* which, if the Quillen functors are *Quillen equivalences* (4.2) is actually an *inverse pair of equivalences* (42.1).

In more detail:

41.2. Homotopy relations and the classical homotopy category. Given a model category $\boldsymbol{M}$, there is a *left homotopy relation* and dually a *right homotopy relation* on the set of maps between any two objects and two maps are called *homotopic* if they are both left homotopic and right homotopic. These relations are not always equivalence relations or compatible with composition. However, if the *domain is cofibrant*, then the left homotopy relation is an equivalence relation and implies the right homotopy relation and dually, if *the codomain is fibrant*, then the right homotopy relation is an equivalence relation and implies the left homotopy relation. Hence on maps from a cofibrant object to a fibrant one all three of these relations are equivalence relations and coincide. Moreover on the full subcategory $\boldsymbol{M}_{cf} \subset \boldsymbol{M}$ spanned by the objects which are both cofibrant and fibrant, this equivalence relation is compatible with the composition and the quotient $(\boldsymbol{M}_{cf}/\sim)$ of $\boldsymbol{M}_{cf}$ by this composable equivalence relation $\sim$ is what we call the *classical homotopy category.*

41.3. Localizations and homotopy categories. It turns out that the *classical homotopy category* $(\boldsymbol{M}_{cf}/\sim)$ of a model category $\boldsymbol{M}$ (41.2) is the *localization of* $\boldsymbol{M}_{cf}$ *with respect to its weak equivalences*, i.e. the category obtained from $\boldsymbol{M}_{cf}$ by "formally inverting" the weak equivalences. To prove the equivalence of this category to the *localization of all of* $\boldsymbol{M}$ *with respect to the weak equivalences* we

(i) say that a subcategory $\boldsymbol{C}_0$ of a category $\boldsymbol{C}$ is a *left* (or *right*) *deformation retract of* $\boldsymbol{C}$ with respect to a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ if there exists a pair (r, s), consisting of a functor $r\colon \boldsymbol{C} \to \boldsymbol{C}$ and a natural transformation $s\colon r \to 1_{\boldsymbol{C}}$ (or $s\colon 1_{\boldsymbol{C}} \to r$), with some rather obvious properties which readily imply that *the inclusion* $\boldsymbol{C}_0 \to \boldsymbol{C}$ *induces an equivalence between the localization of* $\boldsymbol{C}_0$ *with respect to* $\boldsymbol{W} \cap \boldsymbol{C}_0$ *and the localization of* $\boldsymbol{C}$ *with respect to* $\boldsymbol{W}$, and then

(ii) note that the desired result follows from the fact that, if $\boldsymbol{M}_c$ and $\boldsymbol{M}_f$ denote the full subcategories of $\boldsymbol{M}$ spanned by the cofibrant and the fibrant objects respectively, the existence of *functorial factorizations* (3.1) readily implies that $\boldsymbol{M}_c$ and $\boldsymbol{M}_{cf}$ *are left deformation retracts of* $\boldsymbol{M}$ *and* $\boldsymbol{M}_f$ with respect to the weak equivalences and that similarly $\boldsymbol{M}_f$ *and* $\boldsymbol{M}_{cf}$ *are right deformation retracts of* $\boldsymbol{M}$ *and* $\boldsymbol{M}_c$.

The results on Quillen functors (41.1) are obtained by similar methods.

We end with some comments on the

41.4. Organization of the chapter. We start with three sections (§42, §43 and §44) on *localizations* of categories and so-called *total derived functors* between such localizations and then use the material of the first two of these to prove (in §45 and §46) the above mentioned results on *homotopy categories*. The results of §44 are included in this chapter for completeness' sake, but they will only be needed in Chapter VI.

42. Localizations of categories

42.1. Isomorphisms and equivalences of categories. An *invertible* map in a category is called an *isomorphism* and, given two categories $\boldsymbol{A}$ and $\boldsymbol{B}$, one therefore calls a natural transformation between two functors $\boldsymbol{A} \to \boldsymbol{B}$ which assigns to every object of $\boldsymbol{A}$ an invertible map of $\boldsymbol{B}$, a *natural isomorphism*. Similarly one calls a functor $\boldsymbol{A} \to \boldsymbol{B}$ an *isomorphism* (of categories) if it is an invertible map in the category (in $\mathcal{U}'$ (2.2)) of categories (in $\mathcal{U}$ (2.2)), i.e. if it is 1-1 and onto on objects and maps. There is however also the slightly more general and very useful notion of an *equivalence* (of categories) between $\boldsymbol{A}$ and $\boldsymbol{B}$, i.e. a functor $a\colon \boldsymbol{A} \to \boldsymbol{B}$ for which there exists a functor $b\colon \boldsymbol{B} \to \boldsymbol{A}$ such that the compositions ba and ab are not necessarily equal but only naturally isomorphic to the identity functors of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. These functors a and b then are often referred to as *inverse equivalences* of each other.

If $\boldsymbol{A}$ is a subcategory of $\boldsymbol{B}$ and the inclusion functor $\boldsymbol{A} \to \boldsymbol{B}$ is an equivalence of categories, then there exists a functor $\boldsymbol{B} \to \boldsymbol{A}$ which is naturally isomorphic to $1_{\boldsymbol{B}}$ and we therefore call $\boldsymbol{A}$ a *deformation retract* of $\boldsymbol{B}$.

42.2. Localizations of categories [8]. The *localization* of a category $\boldsymbol{C}$ with respect to a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ is, roughly speaking, the category obtained from $\boldsymbol{C}$ by "formally inverting" the maps of $\boldsymbol{W}$. More precisely, it is a category $\boldsymbol{C}[\boldsymbol{W}^{-1}]$ together with a functor $\gamma\colon \boldsymbol{C} \to \boldsymbol{C}[\boldsymbol{W}^{-1}]$ such that

(i) $\gamma w \in \boldsymbol{C}[\boldsymbol{W}^{-1}]$ is an isomorphism for every map $w \in \boldsymbol{W}$, and
(ii) if $\beta\colon \boldsymbol{C} \to \boldsymbol{B}$ is a functor which sends all maps of $\boldsymbol{W}$ to isomorphisms in $\boldsymbol{B}$, then there is a unique functor $b\colon \boldsymbol{C}[\boldsymbol{W}^{-1}] \to \boldsymbol{B}$ such that $b\gamma = \beta$.

When it can cause no misunderstanding, we will often suppress the $\boldsymbol{W}$ and write $\operatorname{Ho}\boldsymbol{C}$ instead of $\boldsymbol{C}[\boldsymbol{W}^{-1}]$.

As the localization is defined by means of a universal property it clearly is *unique*, at least if it exists. But before dealing with existence we discuss its

42.3. Naturality. The localization is *natural* in the sense that, given categories $\boldsymbol{C}$ and $\boldsymbol{C}'$, subcategories $\boldsymbol{W} \subset \boldsymbol{C}$ and $\boldsymbol{W}' \subset \boldsymbol{C}'$ with localization functors $\gamma\colon \boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}$ and $\gamma'\colon \boldsymbol{C}' \to \operatorname{Ho}\boldsymbol{C}'$ and a functor $f\colon \boldsymbol{C} \to \boldsymbol{C}'$ which sends $\boldsymbol{W}$ into $\boldsymbol{W}'$, there is a unique induced functor $\operatorname{Ho} f\colon \operatorname{Ho}\boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}'$ such that $(\operatorname{Ho} f)\gamma = \gamma' f$. But it is also *natural* in the sense that, given two such functors $f_0, f_1\colon \boldsymbol{C} \to \boldsymbol{C}'$ and a natural transformation $d\colon f_0 \to f_1$, there is an obvious induced natural transformation $\operatorname{Ho} d\colon \operatorname{Ho} f_0 \to \operatorname{Ho} f_1$, and the latter is a natural isomorphism (42.1) if for every object $C \in \boldsymbol{C}$, the map $dC \in \boldsymbol{C}'$ is in $\boldsymbol{W}'$, in which case we call d a *natural weak equivalence*.

Note also that, given a category $\boldsymbol{C}$ and a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ with localization functor $\gamma\colon \boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}$, one can form the *closure* of $\boldsymbol{W}$ in $\boldsymbol{C}$, i.e. the subcategory $\overline{\boldsymbol{W}} \subset \boldsymbol{C}$ consisting of the maps $v \in \boldsymbol{C}$ such that $\gamma v \in \operatorname{Ho}\boldsymbol{C}$ is an isomorphism, which has the useful property that *the identity functor of $\boldsymbol{C}$ induces an isomorphism of categories $\boldsymbol{C}[\boldsymbol{W}^{-1}] \approx \boldsymbol{C}[\overline{\boldsymbol{W}}^{-1}]$*. We therefore call a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ *closed in $\boldsymbol{C}$* if it coincides with its closure in $\boldsymbol{C}$.

Now we investigate the question of

42.4. Existence. If $\boldsymbol{C}$ is a small category (2.2), then the localization exists as the category $\operatorname{Ho}\boldsymbol{C}$ and the functor $\gamma\colon \boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}$ can be obtained from the pushout square (2.6) in the category of small categories

$$\begin{array}{ccc} \boldsymbol{J}_W & \longrightarrow & \boldsymbol{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ \boldsymbol{I}_W & \longrightarrow & \operatorname{Ho}\boldsymbol{C} \end{array}$$

in which $\boldsymbol{I}_W$ (resp. $\boldsymbol{J}_W$) is the disjoint union of copies of the category which has two objects, 0 and 1, and one isomorphism (resp. one map $0 \to 1$) between them, indexed by the maps of $\boldsymbol{W}$, and in which the maps are the obvious ones.

If (2.2) $\boldsymbol{C}$ is not small in our chosen universe $\mathcal{U}$, it still is small in our higher universe $\mathcal{U}'$ and the above construction thus yields a category $\operatorname{Ho}\boldsymbol{C}$ in $\mathcal{U}'$, which may or may not be a category in $\mathcal{U}$. If it is actually a category in $\mathcal{U}$, then one says that $\operatorname{Ho}\boldsymbol{C}$ *exists.* A convenient tool for deciding whether this is the case is lemma 42.6 below, which involves a relative version of

42.5. Deformation retracts. Let $\boldsymbol{C}$ be a category and let $\boldsymbol{C}_0, \boldsymbol{W} \subset \boldsymbol{C}$ be subcategories. Then $\boldsymbol{C}_0$ is called a *left* (resp. a *right*) *deformation retract* of $\boldsymbol{C}$ (with respect to $\boldsymbol{W}$) if there exist a functor $r\colon \boldsymbol{C} \to \boldsymbol{C}_0$ and a natural transformation $s\colon r \to 1_{\boldsymbol{C}}$ (resp. $s\colon 1_{\boldsymbol{C}} \to r$) such that

(i) r sends $\boldsymbol{W}$ into $\boldsymbol{W} \cap \boldsymbol{C}_0$,
(ii) for every object $C \in \boldsymbol{C}$, the map sC is in $\boldsymbol{W}$, and
(iii) for every object $C_0 \in \boldsymbol{C}_0$, the map sC_0 is in $\boldsymbol{W} \cap \boldsymbol{C}_0$,

and such a pair (r, s) will be called a *left* (resp. *right*) *deformation retraction from* $\boldsymbol{C}$ *to* $\boldsymbol{C}_0$.

If the subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ satisfies the "two out of three" axiom M2 (3.1), then (i) holds automatically and if $\boldsymbol{C}_0 \subset \boldsymbol{C}$ is a full subcategory, then so does (iii). *If* $\boldsymbol{W}$ *is the category of the isomorphisms of* $\boldsymbol{C}$*, then left and right deformation retracts are the same as the deformation retracts of* 42.1.

An immediate consequence of this definition is

42.6. Lemma. *Let* $\boldsymbol{C}$ *be a category and let* $\boldsymbol{C}_0, \boldsymbol{W} \subset \boldsymbol{C}$ *be subcategories such that* $\boldsymbol{C}_0$ *is a left (or right) deformation retract of* $\boldsymbol{C}$ *(with respect to* $\boldsymbol{W}$*). Then, in the notation of* 42.3 *and* 42.5, $\operatorname{Ho} s$ *and* $\operatorname{Ho}(s|\boldsymbol{C}_0)$ *are natural isomorphisms between* $\operatorname{Ho} r$ *and* $\operatorname{Ho} 1_{\boldsymbol{C}}$ *and* $\operatorname{Ho}(r|\boldsymbol{C}_0)$ *and* $\operatorname{Ho} 1_{\boldsymbol{C}_0}$ *respectively, so that*

(i) $\operatorname{Ho} \boldsymbol{C}_0$ *is a deformation retract* (42.1) *of* $\operatorname{Ho} \boldsymbol{C}$ *and hence*
(ii) $\operatorname{Ho} \boldsymbol{C}$ *exists iff* $\operatorname{Ho} \boldsymbol{C}_0$ *does.*

We end with an

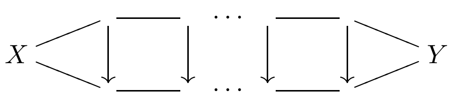
42.7. Explicit description of the localization. Given a category $\boldsymbol{C}$ and two objects $X, Y \in \boldsymbol{C}$, every finite sequence of maps (going in either direction) between X and Y

$$X \longrightarrow \cdots \longrightarrow \cdots \longrightarrow Y$$

can be *reduced*, by successively

(i) omitting identity maps, and
(ii) replacing adjacent maps which go in the same direction by their composition,

to a *unique* such sequence which is *reduced* in the sense that it contains no identity maps and that adjacent maps go in opposite directions. Similarly every *hammock* between X and Y, i.e. finite commutative diagram of the form



in which maps in the same column, i.e. which are above each other, go in the same direction, can be *reduced* to a *unique* such hammock which is *reduced* in the same sense, namely that no column contains only identity maps and that the maps in adjacent columns go in opposite directions.

In view of this it is not difficult to see that the localization $\operatorname{Ho} \boldsymbol{C}$ of a category $\boldsymbol{C}$ with respect to a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$, can be considered as the category which has as *objects* the objects of $\boldsymbol{C}$ and which, for every two objects $X, Y \in \boldsymbol{C}$, has as *hom-set* $\operatorname{Ho} \boldsymbol{C}(X, Y)$ the *set of components* of the (not necessarily small) 1-dimensional simplicial set which has as vertices and 1-simplices the reduced sequences of maps and the reduced hammocks between X and Y subject to the restriction that *all maps that go to the left are in* $\boldsymbol{W}$, and in which the vertices of each hammock are obtained by reducing its top and bottom sequences.

43. Total derived functors between localizations

Next we discuss

43.1. Induced functors. Given categories $\boldsymbol{C}$ and $\boldsymbol{C}'$, subcategories $\boldsymbol{W} \subset \boldsymbol{C}$ and $\boldsymbol{W}' \subset \boldsymbol{C}'$ with localization functors $\gamma\colon \boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}$ and $\gamma'\colon \boldsymbol{C}' \to \operatorname{Ho}\boldsymbol{C}'$ and a functor $f\colon \boldsymbol{C} \to \boldsymbol{C}'$, *there* (42.3) clearly *exists a (unique) functor* $\operatorname{Ho} f\colon \operatorname{Ho}\boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}'$ *such that* $(\operatorname{Ho} f)\gamma = \gamma' f$ *iff* f *sends* $\boldsymbol{W}$ *into* $\overline{\boldsymbol{W}}'$. If however this is not the case, then one can still hope for the existence of *Kan extensions* (43.2) of $\gamma' f$ along γ which are called *total derived functors*. To explain what we mean by this, we first recall the definition of

43.2. Kan extensions [12, Ch X]. These are a relative version of limits and colimits.

Given a category $\boldsymbol{C}$, a functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ and an object $X \in \boldsymbol{C}^{\boldsymbol{A}}$, a *u-limit* of X consists of an object $\lim^u X \in \boldsymbol{C}^{\boldsymbol{B}}$ and a map $t\colon u^* \lim^u X \to X \in \boldsymbol{C}^{\boldsymbol{A}}$ (2.3) such that the pair $(\lim^u X, t)$ is a *terminal object* (if such exists) of the over category $(u^* \downarrow X)$. Such a u-limit (if it exists) is unique up to a canonical isomorphism. It is often called the *right Kan extension* of X along u but, as $\lim^u X$ is such that $u^* \lim^u X$ is "closest to X from the left", it is also sometimes referred to as the *left derived functor* of X along u. Of course (2.5) every object $X \in \boldsymbol{C}^{\boldsymbol{A}}$ has a u-limit iff the functor $u^*\colon \boldsymbol{C}^{\boldsymbol{B}} \to \boldsymbol{C}^{\boldsymbol{A}}$ has a *right adjoint* (denoted by $\lim^u\colon \boldsymbol{C}^{\boldsymbol{A}} \to \boldsymbol{C}^{\boldsymbol{B}}$ and called the *u-limit functor*). If the category $\boldsymbol{C}$ is complete (2.5), then this right adjoint exists for every functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ between small categories as, for every object $X \in \boldsymbol{C}^{\boldsymbol{A}}$, the functor $\lim^u X\colon \boldsymbol{B} \to \boldsymbol{C}$ can be described by the formula $\lim^u X = \lim^{(-\downarrow u)} j^* X$, where j denotes the forgetful functor. Clearly $\lim^{\boldsymbol{B}} \lim^u = \lim^{\boldsymbol{A}}$ or more generally, given a functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between small categories, one has $\lim^v \lim^u = \lim^{vu}$.

Dually a *u-colimit* of X consists of an object $\operatorname{colim}^u X \in \boldsymbol{C}^{\boldsymbol{B}}$ and a map $t\colon X \to u^* \operatorname{colim}^u X \in \boldsymbol{C}^{\boldsymbol{A}}$ such that the pair $(\operatorname{colim}^u X, t)$ is an *initial object* (if such exists) of the under category $(X \downarrow u^*)$. It is also called the *left Kan extension* (or *right derived functor*) of X along u. As above every object $X \in \boldsymbol{C}^{\boldsymbol{A}}$ has a u-colimit iff the functor $u^*\colon \boldsymbol{C}^{\boldsymbol{B}} \to \boldsymbol{C}^{\boldsymbol{A}}$ has a *left adjoint* (denoted by $\operatorname{colim}^u\colon \boldsymbol{C}^{\boldsymbol{A}} \to \boldsymbol{C}^{\boldsymbol{B}}$ and called the *u-colimit functor*) and if $\boldsymbol{C}$ is cocomplete (2.5), then this left adjoint exists for every functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ between small categories as, for every object $X \in \boldsymbol{C}^{\boldsymbol{A}}$, the functor $\operatorname{colim}^u X\colon \boldsymbol{B} \to \boldsymbol{C}$ can be described by the formula $\operatorname{colim}^u X = \operatorname{colim}^{(u\downarrow -)} j^* X$, where j denotes the forgetful functor. Moreover $\operatorname{colim}^{\boldsymbol{B}} \operatorname{colim}^u = \operatorname{colim}^{\boldsymbol{A}}$ or more generally, given a functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between small categories, one has $\operatorname{colim}^v \operatorname{colim}^u = \operatorname{colim}^{vu}$.

Now we can define

43.3. Total derived functors. In the notation of 43.1 above, the *total left derived functor* of f (if it exists) consists of a functor $\mathbf{L} f\colon \operatorname{Ho}\boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}'$ and a natural transformation $\varepsilon\colon (\mathbf{L} f)\gamma \to \gamma' f$ such that the pair $(\mathbf{L} f, \varepsilon)$ is the *right* Kan extension (43.2) of $\gamma' f$ along γ and similarly the *total right derived functor* of f consists of a functor $\mathbf{R} f\colon \operatorname{Ho}\boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}'$ and a natural transformation $\eta\colon \gamma' f \to (\mathbf{R} f)\gamma$ such that the pair $(\mathbf{R} f, \eta)$ is the *left* Kan extension of $\gamma' f$ along γ (although the term total left (or right) derived functor is also sometimes used for just the functor $\mathbf{L} f$ (or $\mathbf{R} f$)). *If the functor* f *is as in* 43.1, *then clearly both total derived functors exist and "coincide", i.e.* $\mathbf{L} f = \mathbf{R} f = \operatorname{Ho} f$, *while* ε *and* η *are both the identity.*

Clearly the notion total derived functors is *natural*, in the sense that, given two functors $f_0, f_1\colon \boldsymbol{C} \to \boldsymbol{C}'$ whose total left (or right) derived functors exist, a natural transformation $d\colon f_0 \to f_1$ induces a *total derived natural transformation* $\mathbf{L}\, d\colon \mathbf{L}\, f_0 \to \mathbf{L}\, f_1$ (or $\mathbf{R}\, d\colon \mathbf{R}\, f_0 \to \mathbf{R}\, f_1$).

Next we observe that lemma 42.6 readily implies the following sufficient conditions for the *existence of total derived functors.*

43.4. Lemma. *Let $\boldsymbol{C}$ and $\boldsymbol{C}'$ be categories, let $\boldsymbol{W} \subset \boldsymbol{C}$ and $\boldsymbol{W}' \subset \boldsymbol{C}'$ be subcategories with localization functors $\gamma\colon \boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}$ and $\gamma'\colon \boldsymbol{C}' \to \operatorname{Ho}\boldsymbol{C}'$ and let (r,s) be a left (resp. a right) deformation retraction from $\boldsymbol{C}$ to $\boldsymbol{C}_0$* (42.5)*. If $f\colon \boldsymbol{C} \to \boldsymbol{C}'$ is a functor such that $f(\boldsymbol{W} \cap \boldsymbol{C}_0) \subset \boldsymbol{W}'$, then the total left (resp. right) derived functor of f exists and consists of the composition*

$$\operatorname{Ho}\boldsymbol{C} \xrightarrow{\operatorname{Ho} r} \operatorname{Ho}\boldsymbol{C}_0 \xrightarrow{\operatorname{Ho}(f|\boldsymbol{C}_0)} \operatorname{Ho}\boldsymbol{C}'$$

and the natural transformation $\gamma' f s$. Moreover, if $f_0, f_1\colon \boldsymbol{C} \to \boldsymbol{C}'$ are two such functors and $d\colon f_0 \to f_1$ is a natural transformation such that the restriction $d|\boldsymbol{C}_0$ is a natural weak equivalence (42.3)*, then the total left (or right) derived natural transformation of d* (43.3) *is a natural isomorphism* (42.1)*.*

We end with proving that similar conditions imply the *existence of adjoint pairs of total derived functors.*

43.5. Lemma. *Let $\boldsymbol{C}$ and $\boldsymbol{C}'$ be categories with subcategories $\boldsymbol{W} \subset \boldsymbol{C}$ and $\boldsymbol{W}' \subset \boldsymbol{C}'$, let $\boldsymbol{C}_{\mathrm{L}} \subset \boldsymbol{C}$ be a left deformation retract of $\boldsymbol{C}$, let $\boldsymbol{C}'_{\mathrm{R}} \subset \boldsymbol{C}'$ be a right deformation retract of $\boldsymbol{C}'$ and let*

$$f\colon \boldsymbol{C} \leftrightarrow \boldsymbol{C}' :g$$

be a pair of adjoint functors such that

(i) *$f(\boldsymbol{C}_{\mathrm{L}}) \subset \boldsymbol{W}'$ and $g(\boldsymbol{C}'_{\mathrm{R}}) \subset \boldsymbol{W}$.*

then the total derived functors (43.4) *$\mathbf{L}\, f$ and $\mathbf{R}\, g$ form a pair of adjoint functors*

$$\mathbf{L}\, f\colon \operatorname{Ho}\boldsymbol{C} \leftrightarrow \operatorname{Ho}\boldsymbol{C}' :\mathbf{R}\, g.$$

Moreover if in addition

(ii) *$\boldsymbol{W} \subset \boldsymbol{C}$ and $\boldsymbol{W}' \subset \boldsymbol{C}'$ are closed subcategories* (42.3)*, and*

(iii) *for every pair of objects $A \in \boldsymbol{C}_{\mathrm{L}}$ and $X \in \boldsymbol{C}'_{\mathrm{R}}$, a map $fA \to X \in \boldsymbol{C}'$ is in $\boldsymbol{W}'$ iff its adjoint $A \to gX \in \boldsymbol{C}$ is in $\boldsymbol{W}$, then $\mathbf{L}\, f$ and $\mathbf{R}\, g$ are actually inverse equivalences of categories* (42.1)*.*

Proof. We prove the first part by constructing an adjunction isomorphism

$$k\colon\ \operatorname{Ho}\boldsymbol{C}'\big((\mathbf{L}\, f)B, Y\big) \approx \operatorname{Ho}\boldsymbol{C}\big(B, (\mathbf{R}\, g)Y\big)$$

which is natural in $B \in \boldsymbol{C}$ and $Y \in \boldsymbol{C}'$, as follows, using 42.7 and 43.4. Choose a left deformation retraction $(r_{\mathrm{L}}, s_{\mathrm{L}})$ from $\boldsymbol{C}$ to $\boldsymbol{C}_{\mathrm{L}}$ and a right deformation retraction $(r'_{\mathrm{R}}, s'_{\mathrm{R}})$ from $\boldsymbol{C}'$ to $\boldsymbol{C}'_{\mathrm{R}}$. Given a map $u\colon (\mathbf{L}\, f)B \to Y \in \operatorname{Ho}\boldsymbol{C}'$ we then choose a finite sequence of maps

$$frB \;\text{------}\; \cdots \;\text{------}\; Y$$

which reduces to u and define $ku\colon B \to (\mathbf{R}\, g)Y \in \operatorname{Ho}\boldsymbol{C}$ as the reduction of the sequence

$$B \xleftarrow{\ s\ } rB \xrightarrow{h1_{frB}} gfrB \xrightarrow{gs'frB} gr'frB \xrightarrow{\ gr'-\ } \cdots \xrightarrow{\ gr'-\ } gr'Y$$

in which $h\colon \boldsymbol{C}'(f-,-) \approx \boldsymbol{C}(-,g-)$ is the adjunction isomorphism. A lengthy but essentially straightforward calculation then yields that

(i) ku is well defined, i.e. does not depend on the choice of the representative of u, and
(ii) k is indeed a natural isomorphism.

The second part of the lemma now follows readily using the fact that $\boldsymbol{W} \subset \boldsymbol{C}$ and $\boldsymbol{W}' \subset \boldsymbol{C}'$ are closed subcategories.

44. Total derived functors and compositions

Even though this will only really be needed in Chapter VI, when we investigate the total derived functors of the colimit and limit functors, we devote this third and last section on localizations to a brief discussion of the relationship between

44.1. Total derived functors of compositions and compositions of total derived functors. Let $\boldsymbol{C}$, $\boldsymbol{C}'$ and $\boldsymbol{C}''$ be categories with subcategories $\boldsymbol{W} \subset \boldsymbol{C}$, $\boldsymbol{W}' \subset \boldsymbol{C}'$ and $\boldsymbol{W}'' \subset \boldsymbol{C}''$ and localization functors $\gamma\colon \boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}$, $\gamma'\colon \boldsymbol{C}' \to \operatorname{Ho}\boldsymbol{C}'$ and $\gamma''\colon \boldsymbol{C}'' \to \operatorname{Ho}\boldsymbol{C}''$ and let $f\colon \boldsymbol{C} \to \boldsymbol{C}'$ and $f'\colon \boldsymbol{C}' \to \boldsymbol{C}''$ be functors whose total left (or right) derived functors exist. Then the total left (or right) derived functor of the composition $f'f$ need not exist and, if it exists, is not necessarily naturally isomorphic to the composition of the total left (or right) derived functors of f and f'. However one readily verifies

44.2. Proposition.

(i) *If f, f' and $f'f$ have total left derived functors $(\mathbf{L}\,f,\epsilon)$, $(\mathbf{L}\,f',\epsilon')$ and $(\mathbf{L}(f'f),\epsilon'')$, then there is a unique natural transformation*

$$\bar\epsilon\colon (\mathbf{L}\,f')(\mathbf{L}\,f) \to \mathbf{L}(f'f)$$

such that $\epsilon'' \circ \bar\epsilon\gamma = \epsilon' f \circ (\mathbf{L}\,f')\epsilon$, and dually

(ii) *if f, f' and $f'f$ have total right derived functors $(\mathbf{R}\,f,\eta)$, $(\mathbf{R}\,f',\eta')$ and $(\mathbf{R}(f'f),\eta'')$, then there is a unique natural transformation*

$$\bar\eta\colon \mathbf{R}(f'f) \to (\mathbf{R}\,f')(\mathbf{R}\,f)$$

such that $\bar\eta\gamma \circ \eta'' = (\mathbf{R}\,f')\eta \circ \eta' f$.

And in view of 43.4 this implies

44.3. Lemma. *Let $\boldsymbol{C}$, $\boldsymbol{C}'$ and $\boldsymbol{C}''$ be categories with subcategories $\boldsymbol{W} \subset \boldsymbol{C}$, $\boldsymbol{W}' \subset \boldsymbol{C}'$ and $\boldsymbol{W}'' \subset \boldsymbol{C}''$ and localization functors $\gamma\colon \boldsymbol{C} \to \operatorname{Ho}\boldsymbol{C}$, $\gamma'\colon \boldsymbol{C}' \to \operatorname{Ho}\boldsymbol{C}'$ and $\gamma''\colon \boldsymbol{C}'' \to \operatorname{Ho}\boldsymbol{C}''$, let (r,s) and (r',s') be left (resp. right) deformation retractions from $\boldsymbol{C}$ to a subcategory $\boldsymbol{C}_0 \subset \boldsymbol{C}$ and from $\boldsymbol{C}'$ to a subcategory $\boldsymbol{C}'_0 \subset \boldsymbol{C}'$, let $f\colon \boldsymbol{C} \to \boldsymbol{C}'$ and $f'\colon \boldsymbol{C}' \to \boldsymbol{C}''$ be functors such that*

$$f(\boldsymbol{W} \cap \boldsymbol{C}_0) \subset \boldsymbol{W}', \qquad f'(\boldsymbol{W}' \cap \boldsymbol{C}'_0) \subset \boldsymbol{W}'' \qquad \text{and} \qquad f'f(\boldsymbol{W} \cap \boldsymbol{C}_0) \subset \boldsymbol{W}'',$$

let the total derived functors $\mathbf{L}\,f$, $\mathbf{L}\,f'$ and $\mathbf{L}(f'f)$ (resp. $\mathbf{R}\,f$, $\mathbf{R}\,f'$ and $\mathbf{R}(f'f)$) be as in 43.4 and let $\bar\epsilon\colon (\mathbf{L}\,f')(\mathbf{L}\,f) \to \mathbf{L}(f'f)$ (resp. $\bar\eta\colon \mathbf{R}(f'f) \to (\mathbf{R}\,f')(\mathbf{R}\,f)$) be the resulting natural transformations (44.2). Then

$$\bar\epsilon\gamma \text{ (resp. } \bar\eta\gamma) = \gamma'' f' s' f r$$

If, in addition

$$f(\boldsymbol{W} \cap \boldsymbol{C}_0) \subset \boldsymbol{W}' \cap \boldsymbol{C}'_0$$

then $\bar\epsilon$ (resp. $\bar\eta$) is actually a natural isomorphism.

We end with the observation that an essentially straightforward calculation using 42.7 yields the following similar result for adjoint pairs of functors.

44.4. Lemma. *Let $\boldsymbol{C}$, $\boldsymbol{C}'$ and $\boldsymbol{C}''$ be categories with subcategories $\boldsymbol{W} \subset \boldsymbol{C}$, $\boldsymbol{W}' \subset \boldsymbol{C}'$ and $\boldsymbol{W}'' \subset \boldsymbol{C}''$, let $\boldsymbol{C}_{\mathrm{L}} \subset \boldsymbol{C}$ and $\boldsymbol{C}'_{\mathrm{L}} \subset \boldsymbol{C}'$ be left deformation retracts of $\boldsymbol{C}$ and $\boldsymbol{C}'$, let $\boldsymbol{C}'_{\mathrm{R}} \subset \boldsymbol{C}'$ and $\boldsymbol{C}''_{\mathrm{R}} \subset \boldsymbol{C}''$ be right deformation retracts of $\boldsymbol{C}'$ and $\boldsymbol{C}''$ and let*

$$f\colon \boldsymbol{C} \leftrightarrow \boldsymbol{C}' : g \qquad \text{and} \qquad f'\colon \boldsymbol{C}' \leftrightarrow \boldsymbol{C}'' : g'$$

be pairs of adjoint functors such that

$$\begin{array}{lllll} f(\boldsymbol{W} \cap \boldsymbol{C}_{\mathrm{L}}) \subset \boldsymbol{W}', & f'(\boldsymbol{W}' \cap \boldsymbol{C}'_{\mathrm{L}}) \subset \boldsymbol{W}'' & \text{and} & f'f(\boldsymbol{W} \cap \boldsymbol{C}_{\mathrm{L}}) \subset \boldsymbol{W}'', \\ g'(\boldsymbol{W}'' \cap \boldsymbol{C}''_{\mathrm{R}}) \subset \boldsymbol{W}', & g(\boldsymbol{W}' \cap \boldsymbol{C}'_{\mathrm{R}}) \subset \boldsymbol{W} & \text{and} & g'g(\boldsymbol{W}'' \cap \boldsymbol{C}''_{\mathrm{R}}) \subset \boldsymbol{W}. \end{array}$$

Then the resulting (44.2) *natural transformations*

$$\bar\epsilon\colon (\mathbf{L} f')(\mathbf{L} f) \to \mathbf{L}(f'f) \qquad \text{and} \qquad \bar\eta\colon\ \mathbf{R}(gg') \to (\mathbf{R} g)(\mathbf{R} g')$$

are conjugate, i.e. the diagram

$$\begin{array}{ccccc} & \operatorname{Ho}\boldsymbol{C}''\big(\mathbf{L}(f'f)-,-\big) & \approx & \operatorname{Ho}\boldsymbol{C}\big(-,\mathbf{R}(gg')-\big) & \\ {\scriptstyle \operatorname{Ho}\boldsymbol{C}''(\bar\epsilon-,-)}\swarrow & & & & \searrow{\scriptstyle \operatorname{Ho}\boldsymbol{C}(-,\bar\eta-)} \\ \operatorname{Ho}\boldsymbol{C}''\big((\mathbf{L} f')(\mathbf{L} f)-,-\big) & \approx & \operatorname{Ho}\boldsymbol{C}'\big((\mathbf{L} f)-,(\mathbf{R} g')-\big) & \approx & \operatorname{Ho}\boldsymbol{C}\big(-,(\mathbf{R} g)(\mathbf{R} g')-\big) \end{array}$$

in which the horizontal maps are the adjunction isomorphisms, commutes.

If, in addition,

$$f(\boldsymbol{W} \cap \boldsymbol{C}_{\mathrm{L}}) \subset \boldsymbol{W}' \cap \boldsymbol{C}'_{\mathrm{L}} \qquad \text{and/or} \qquad g(\boldsymbol{W}'' \cap \boldsymbol{C}''_{\mathrm{R}}) \subset \boldsymbol{W}' \cap \boldsymbol{C}'_{\mathrm{R}}$$

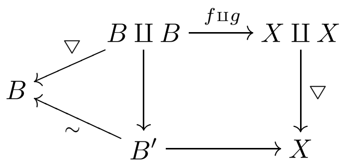
then $\bar\epsilon$ and $\bar\eta$ are actually natural isomorphisms.

45. The classical homotopy category

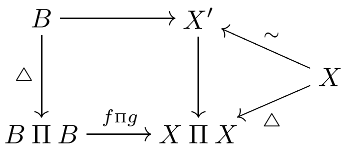
We now construct the *classical homotopy category* of a model category $\boldsymbol{M}$ as the quotient $(\boldsymbol{M}_{cf}/{\sim})$ of the full subcategory $\boldsymbol{M}_{cf} \subset \boldsymbol{M}$ spanned by the objects which are both cofibrant and fibrant, by an appropriate *homotopy relation* $\sim$ and prove that this category is canonically isomorphic to the *localization* (42.2) of $\boldsymbol{M}_{cf}$ with respect to its subcategory of weak equivalences.

We start with defining

45.1. The homotopy relation. Given a model category $\boldsymbol{M}$, objects $B, X \in \boldsymbol{M}$ and maps $f, g\colon B \to X \in \boldsymbol{M}$, one says that f is *left homotopic* to g and writes $f \overset{\mathrm{l}}{\sim} g$, if there exists a *left homotopy* between them, i.e. a commutative diagram in $\boldsymbol{M}$ of the form (2.5)



in which $\bigtriangledown$ denotes the folding map and the map $B' \to B$ is a weak equivalence, and dually one says that f is *right homotopic* to g and writes $f \overset{\mathrm{r}}{\sim} g$, if there exists a *right homotopy* between them, i.e.



in which $\bigtriangleup$ denotes the diagonal map and the map $X \to X'$ is a weak equivalence.

If $f \overset{\mathrm{l}}{\sim} g$ and $f \overset{\mathrm{r}}{\sim} g$, one says that f is *homotopic* to g and writes $f \sim g$, and one calls a map $f\colon X \to Y$ a *homotopy equivalence* if there exists a map $f'\colon Y \to X$ (called a *homotopy inverse* of f) such that $f'f \sim 1_X$ and $ff' \sim 1_Y$.

To obtain the basic properties of this homotopy relation, we first investigate

45.2. Properties of the left and right homotopy relations. Given a model category $\boldsymbol{M}$ and maps $e\colon A \to B$, $f, g\colon B \to X$ and $h\colon X \to Y \in \boldsymbol{M}$, it is not difficult to verify ([**14**, Ch. I, §1], [**7**]) that

(i) *if* $f \overset{\mathrm{l}}{\sim} g$, *then* $hf \overset{\mathrm{l}}{\sim} hg$,
(ii) *if* $f \overset{\mathrm{l}}{\sim} g$, *then there exists a left homotopy* 45.1 *in which the map* $B \amalg B \to B'$ *is a cofibration,*
(iii) *if* $X \in \boldsymbol{M}_f$, *then* $f \overset{\mathrm{l}}{\sim} g$ *implies* $fe \overset{\mathrm{l}}{\sim} ge$,
(iv) *if* $B \in \boldsymbol{M}_c$, *then* $\overset{\mathrm{l}}{\sim}$ *is an equivalence relation,*
(v) *if* $B \in \boldsymbol{M}_c$ *and* h *is a trivial fibration (or* (3.6) *a weak equivalence in* $\boldsymbol{M}_f$*), then* h *induces an isomorphism of quotient sets* $(\boldsymbol{M}(B, X)/\overset{\mathrm{l}}{\sim}) \approx (\boldsymbol{M}(B, Y)/\overset{\mathrm{l}}{\sim})$, *and*
(vi) *if* $B \in \boldsymbol{M}_c$, *then* $f \overset{\mathrm{l}}{\sim} g$ *implies* $f \overset{\mathrm{r}}{\sim} g$

and dually

(i)′ *if* $f \overset{\mathrm{r}}{\sim} g$ *then* $fe \overset{\mathrm{r}}{\sim} ge$,
(ii)′ *if* $f \overset{\mathrm{r}}{\sim} g$, *then there exists a right homotopy* 45.1 *in which the map* $X' \to X \Pi X$ *is a fibration,*
(iii)′ *if* $B \in \boldsymbol{M}_c$, *then* $f \overset{\mathrm{r}}{\sim} g$ *implies* $hf \overset{\mathrm{r}}{\sim} hg$,
(iv)′ *if* $X \in \boldsymbol{M}_f$, *then* $\overset{\mathrm{r}}{\sim}$ *is an equivalence relation,*

(v)′ *if $X \in \boldsymbol{M}_f$ and e is a trivial cofibration (or* (3.6) *a weak equivalence in $\boldsymbol{M}_c$), then e induces an isomorphism of quotient sets $(\boldsymbol{M}(B,X)/\overset{\mathrm{r}}{\sim}) \approx (\boldsymbol{M}(A,X)/\overset{\mathrm{r}}{\sim})$, and*

(vi)′ *if $X \in \boldsymbol{M}_f$, then $f \overset{\mathrm{r}}{\sim} g$ implies $f \overset{\mathrm{l}}{\sim} g$.*

This immediately implies

45.3. Proposition. *For every cofibrant object $A \in \boldsymbol{M}$ and every fibrant object $Y \in \boldsymbol{M}$, the homotopy relation is an equivalence relation on the maps $A \to Y \in \boldsymbol{M}$.*

45.4. Proposition. *The homotopy relation is an equivalence relation on the maps of $\boldsymbol{M}_{cf}$* (3.4) *which, moreover, is compatible with the composition.*

Somewhat harder to prove is

45.5. Proposition. *A map $f\colon X \to Y \in \boldsymbol{M}_{cf}$ is a weak equivalence iff it is a homotopy equivalence.*

Proof. The "only if" part follows readily from 45.2. to prove that a homotopy equivalence $f \in \boldsymbol{M}_{cf}$ is a weak equivalence, choose a factorization $f = qi$, where i is a cofibration and q a trivial fibration. By the "only if" part, q is a homotopy equivalence and so is therefore i. A straightforward argument then shows that i has the left lifting property with respect to all fibrations and thus is a weak equivalence.

In view of 45.4 we can now define

45.6. The classical homotopy category. The *classical homotopy category* of a model category $\boldsymbol{M}$ will be the quotient category $(\boldsymbol{M}_{cf}/\sim)$. This category is closely related with the *localization* $\operatorname{Ho}\boldsymbol{M}_{cf}$ of $\boldsymbol{M}_{cf}$ with respect to its subcategory of weak equivalences (42.2). In fact the "only if" part of 45.5 implies that there is a unique functor $\operatorname{Ho}\boldsymbol{M}_{cf} \to (\boldsymbol{M}_{cf}/\sim)$ such that the diagram

$$\begin{array}{ccc} & \boldsymbol{M}_{cf} & \\ {}^{\gamma}\swarrow & & \searrow^{\text{proj.}} \\ \operatorname{Ho}\boldsymbol{M}_{cf} & \longrightarrow & (\boldsymbol{M}_{cf}/\sim) \end{array}$$

commutes and, as every functor on $\boldsymbol{M}_{cf}$ which sends all weak equivalences to isomorphisms clearly identifies homotopic maps, one has

45.7. Proposition. *The canonical functor*

$$\operatorname{Ho}\boldsymbol{M}_{cf} \to (\boldsymbol{M}_{cf}/\sim) \qquad (45.6)$$

is an isomorphism of categories (42.1).

Combining this with the "if" part of 45.5 one also gets

45.8. Proposition. *The subcategory of the weak equivalences in $\boldsymbol{M}_{cf}$ is a closed* (42.3) *subcategory of $\boldsymbol{M}_{cf}$.*

We end with two obvious

45.9. Examples. If $\boldsymbol{M} = \boldsymbol{S}$ or $\boldsymbol{T}$ (§5.2), then the above homotopy relation reduces to the usual homotopy relation in these categories and the classical homotopy category becomes the usual homotopy categories of the fibrant simplicial sets and the cofibrant topological spaces.

46. The Quillen homotopy category

In this last section we show that the *Quillen homotopy category* $\operatorname{Ho}\boldsymbol{M}$ of a model category $\boldsymbol{M}$ (i.e. its localization with respect to the weak equivalences) is equivalent to the localization of the full subcategory $\boldsymbol{M}_{cf} \subset \boldsymbol{M}$ spanned by the objects which are both cofibrant and fibrant. Together with proposition 45.7 this implies the key result that the Quillen homotopy category $\operatorname{Ho}\boldsymbol{M}$ is equivalent to the classical homotopy category $(\boldsymbol{M}_{cf}/\!\sim)$ which consists of the homotopy classes of maps between the objects of $\boldsymbol{M}_{cf}$.

We also prove that an adjoint pair of *Quillen functors* (4.1) between two model categories gives rise to an *adjoint pair of functors between their homotopy categories* which, if the Quillen functors are *Quillen equivalences* (45.2) is actually an *inverse pair of equivalences.*

We start with fixing some

46.1. Notation and terminology. The *Quillen homotopy category* of a model category $\boldsymbol{M}$ will be its localization (42.2) $\operatorname{Ho}\boldsymbol{M}$ with respect to its subcategory of weak equivalences and we denote similarly by $\operatorname{Ho}\boldsymbol{M}_c$, $\operatorname{Ho}\boldsymbol{M}_f$ and $\operatorname{Ho}\boldsymbol{M}_{cf}$ the localizations of $\boldsymbol{M}_c$, $\boldsymbol{M}_f$ and $\boldsymbol{M}_{cf}$ (3.4) with respect to their subcategories of weak equivalences. These localizations are closely related. In fact

46.2. Proposition. *The inclusions of $\boldsymbol{M}_{cf}$, $\boldsymbol{M}_c$, $\boldsymbol{M}_f$ and $\boldsymbol{M}$ in each other induce* (42.3) *a commutative diagram of inclusions*

$$\begin{array}{ccccc} & & \operatorname{Ho}\boldsymbol{M}_c & & \\ & \nearrow & & \searrow & \\ \operatorname{Ho}\boldsymbol{M}_{cf} & & & & \operatorname{Ho}\boldsymbol{M} \\ & \searrow & & \nearrow & \\ & & \operatorname{Ho}\boldsymbol{M}_f & & \end{array}$$

in which all maps are equivalences of categories (42.1).

Proof. This follows from (42.6) and the following proposition which is an immediate consequence of the factorization axiom M5 3.1.

46.3. Proposition. *For every model category $\boldsymbol{M}$*

(i) *$\boldsymbol{M}_{cf}$ and $\boldsymbol{M}_c$ are left deformation retracts* (42.5) *of $\boldsymbol{M}_f$ and $\boldsymbol{M}$ respectively with respect to the weak equivalences, and similarly*
(ii) *$\boldsymbol{M}_{cf}$ and $\boldsymbol{M}_f$ are right deformation retracts of $\boldsymbol{M}_c$ and $\boldsymbol{M}$.*

Combining 46.2 with 45.7 we now finally reach the main aim of Part I of this monograph:

46.4. Theorem. *For every model category $\boldsymbol{M}$ the composition*

$$(\boldsymbol{M}_{cf}/\sim) \to \operatorname{Ho} \boldsymbol{M}_{cf} \to \operatorname{Ho} \boldsymbol{M}$$

in which the first map is the inverse of the canonical map of 45.7 *and the second map is as in* 46.2*, is an equivalence between the classical homotopy category of $\boldsymbol{M}$ and the Quillen homotopy category.*

We also note that 45.2(v) and (v)′ and 45.5 now readily imply the following two propositions.

46.5. Proposition. *If $B \in \boldsymbol{M}_c$ and $X \in \boldsymbol{M}_f$, then the obvious map*

$$(\boldsymbol{M}(B, X)/\sim) \to (\operatorname{Ho} \boldsymbol{M})(B, X)$$

is an isomorphism.

46.6. Closure property. *The category of the weak equivalences in a model category $\boldsymbol{M}$ is a closed* (42.3) *subcategory of $\boldsymbol{M}$.*

We end with a brief investigation of the extent to which a functor between model categories induces a functor between their homotopy categories.

We first note that 3.6, 42.6 and 46.3 imply

46.7. Proposition. *Let $f\colon \boldsymbol{M} \to \boldsymbol{N}$ be a functor between model categories.*

(i) *If f sends trivial cofibrations in $\boldsymbol{M}_c$ to weak equivalences in $\boldsymbol{N}$, then the total left derived functor $\mathbf{L}\, f\colon \operatorname{Ho} \boldsymbol{M} \to \operatorname{Ho} \boldsymbol{N}$ exists, and dually*

(ii) *If f sends trivial fibrations in $\boldsymbol{M}_f$ to weak equivalences in $\boldsymbol{N}$, then the total right derived functor $\mathbf{R}\, f\colon \operatorname{Ho} \boldsymbol{M} \to \operatorname{Ho} \boldsymbol{N}$ exists.*

Furthermore 4.1, 4.2 and 43.5 imply

46.8. Proposition. *Let $f\colon \boldsymbol{M} \leftrightarrow \boldsymbol{M}' : g$ be an adjoint pair of Quillen functors* (4.1) *between model categories. Then the total derived functors $\mathbf{L}\, f$ and $\mathbf{R}\, g$ both exist and form an adjoint pair*

$$\mathbf{L}\, f\colon \operatorname{Ho} \boldsymbol{M} \leftrightarrow \operatorname{Ho} \boldsymbol{M}' : \mathbf{R}\, g.$$

Furthermore the total derived functors $\mathbf{L}\, f$ and $\mathbf{R}\, g$ are actually (inverse) equivalences of categories if these Quillen functors are Quillen equivalences (4.2)*.*

And finally a simple calculation (or 44.3 yields

46.9. Proposition. *Let $f\colon \boldsymbol{M} \leftrightarrow \boldsymbol{M}' : g$ and $f'\colon \boldsymbol{M}' \leftrightarrow \boldsymbol{M} : g'$ be Quillen pairs* (4.1) *of adjoint functors between model categories. Then the natural transformations* (44.2)

$$\bar{\epsilon}\colon (\mathbf{L}\, f')(\mathbf{L}\, f) \to \mathbf{L}(f'f) \qquad \textit{and} \qquad \bar{\eta}\colon \mathbf{R}(gg') \to (\mathbf{R}\, g)(\mathbf{R}\, g')$$

are both natural isomorphisms.

CHAPTER XII

Leftovers

47. $\boldsymbol{S}$-categories

47.1. $\boldsymbol{S}$-categories. An $\boldsymbol{S}$-*category* is a "category with small simplicial hom-sets", i.e. (2.2 and 5.7) a simplicial object in the category (in $\mathcal{U}'$) of categories (in $\mathcal{U}$) for which the "simplicial set of objects" is discrete. Clearly the *opposite* of such an $\boldsymbol{S}$-category is again an $\boldsymbol{S}$-category. There are also obvious notions of a *functor* between two $\boldsymbol{S}$-categories and a *natural transformation* between two such functors. Given an $\boldsymbol{S}$-category $\boldsymbol{G}$ and a small $\boldsymbol{S}$-category $\boldsymbol{E}$ one can thus form the diagram category $\boldsymbol{G}^{\boldsymbol{E}}$ of the *simplicial* $\boldsymbol{E}$*-diagrams* in $\boldsymbol{G}$, i.e. the category which has as objects the functors $\boldsymbol{E} \to \boldsymbol{G}$ and as maps the natural transformations between them.

The *category of components* of an $\boldsymbol{S}$-category $\boldsymbol{G}$ is the category $\pi_0\boldsymbol{G}$ obtained from $\boldsymbol{G}$ be replacing each simplicial hom-set by its set of components. A 0-dimensional map $f\colon X \to Y \in \boldsymbol{G}$ then is called a *homotopy equivalence* if its image in $\pi_0\boldsymbol{G}$ is an isomorphism, i.e. if there exists a 0-dimensional map $g\colon Y \to X \in \boldsymbol{G}$ (called a *homotopy inverse* of f) such that the compositions $gf \in \boldsymbol{G}(X,X)$ and $fg \in \boldsymbol{G}(Y,Y)$ are in the same components as the identity maps of X and Y.

47.2. Example. Given a cocomplete (2.5) category $\boldsymbol{C}$, the category $\boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of the *simplicial objects in* $\boldsymbol{C}$ can be turned into an $\boldsymbol{S}$-category $\big(\boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}\big)_*$ as follows. Let

$$\otimes\colon \boldsymbol{C} \times \mathbf{Set} \to \boldsymbol{C}$$

be the *tensor product*, i.e. the functor which sends a pair of objects $C \in \boldsymbol{C}$ and $Y \in \mathbf{Set}$ to the coproduct of as many copies of C as there are elements in Y and let the same symbol denote the functor

$$\otimes\colon \boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}} \times \mathbf{Set} \to \boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$$

obtained by dimension-wise application of this tensor product. Then $\big(\boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}\big)_*$ will be the $\boldsymbol{S}$-category which has as objects the objects of $\boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ and which has, for every two objects $X, X' \in \boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$, as simplicial hom-set $\big(\boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}\big)_*(X,X')$ the simplicial set which has as n-simplices ($n \ge 0$) the maps

$$X \otimes \Delta[n] \to X' \in \boldsymbol{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}.$$

47.3. Fibrantly and mixed generated model categories. The above can be dualized as follows.

One defines, for a complete category $\boldsymbol{C}$, the notion of a subcategory which is *closed under inverse transfinite composition* and, for such a subcategory $\boldsymbol{C}_1 \subset \boldsymbol{C}$, the notion of an object of $\boldsymbol{C}$ which is *cosmall rel.* $\boldsymbol{C}_1$. For a small set K of maps in $\boldsymbol{C}$ one can then formulate a *cosmall object argument* involving the subcategories

Next, given a model category $\boldsymbol{N}$, one calls a set K of maps in $\boldsymbol{N}$ a *set of generating (trivial) fibrations* if the domains and codomains of these maps permit a cosmall object argument and K-**fib** coincides with the category of the (trivial) fibrations in $\boldsymbol{N}$. A *model category generated by fibrations and trivial fibrations* (for short *fibrantly generated model category*) then is defined as a model category for which there exist a set K of generating fibrations and a set L of generating trivial fibrations.

We end with noting that there are also *mixed* generated model categories, i.e. *model categories generated by cofibrations and fibrations* (for which there exist a set I of generating cofibrations and a set K of generating fibrations) and *model categories generated by trivial cofibrations and trivial fibrations* (for which there exist a set J of generating trivial cofibrations and a set L of generating trivial fibrations). For these model categories the small object argument yields one of the desired functorial factorizations and the cosmall object argument the other.

47.4. Model category structure for simplicial diagrams of simplicial algebras. *Let $\boldsymbol{E}$ be a small $\boldsymbol{S}$-category* (47.1) *and let I, A and G_i $(i \in I)$ be as in* 9.8. *Then*

(i) *the simplicial diagram category $\left(\boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}\right)^{\boldsymbol{E}}_{*}$ (*47.1 *and* 47.2*) admits a cofibrantly generated model category structure with the inclusions*

$$G_i \otimes \dot{\Delta}[n] \times \boldsymbol{E}(E,-) \to G_i \otimes \Delta[n] \times \boldsymbol{E}(E,-) \qquad (E \in \boldsymbol{E},\ i \in I,\ n \geq 0)$$

$$G_i \otimes \Delta^k[n] \times \boldsymbol{E}(E,-) \to G_i \otimes \Delta[n] \times \boldsymbol{E}(E,-) \qquad (E \in \boldsymbol{E},\ i \in I,\ n > 0,\ 0 \leq k \leq n)$$

as the generating cofibrations and trivial cofibrations, in which

(ii) *the weak equivalences and the fibrations are the objectwise ones.*

48. A model category structure for $\boldsymbol{S}$-categories

In this last section we describe a cofibrantly generated model category structure on the category $\boldsymbol{S}$-**Cat** of *small $\boldsymbol{S}$-categories*. Our main tools are the *recognition lemma* of the previous section and the already (9.8) obtained cofibrantly generated

48.1. Model category structure for simplicial O-categories. Let O be a small set (2.1), let O-**Cat** denote the category which has as objects the small categories with O as their sets of objects and which has as maps the functors between them which are the identity on O. Furthermore, for every two elements $O_1, O_2 \in O$, denote by

$$(O_1 \to O_2) \in O\text{-}\mathbf{Cat} \subset (O\text{-}\mathbf{Cat})^{\boldsymbol{\Delta}^{\mathrm{op}}}$$

the O-category which has a single map $O_1 \to O_2$ as its only non-identity map. then (9.8) the category $(O\text{-}\mathbf{Cat})^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of the (small) simplicial O-categories admits a cofibrantly generated model category structure in which

(i) the set I of the *generating cofibrations* consists of the inclusions

$$(O_1 \to O_2) \otimes \dot{\Delta}[n] \to (O_1 \to O_2) \otimes \Delta[n] \qquad (n \geq 0,\ O_1, O_2 \in O),$$

(ii) the set J of the *generating trivial cofibrations* consists of the inclusions

$$(O_1 \to O_2) \otimes \Delta^k[n] \to (O_1 \to O_2) \otimes \Delta[n] \qquad (n > 0,\ 0 \leq k \leq n,\ O_1, O_2 \in O),$$

(iii) a map $\boldsymbol{K} \to \boldsymbol{L} \in (O\text{-}\mathbf{Cat})^{\boldsymbol{\Delta}^{\mathrm{op}}}$ is a *weak equivalence* or a *fibration* whenever, for every two elements $O_1, O_2 \in O$, the restriction $\boldsymbol{K}(O_1, O_2) \to \boldsymbol{L}(O_1, O_2) \in \boldsymbol{S}$ is so, and

(iv) a map $f\colon \boldsymbol{K} \to \boldsymbol{L} \in (O\text{-}\mathbf{Cat})^{\boldsymbol{\Delta}^{\mathrm{op}}}$ is a *regular I-cofibration* (7.2) iff it is a *free* map, i.e. f is onto and there exists a set Γ of non-identity maps in $\boldsymbol{L}$, which is closed under the degeneracy operators, such that every non-identity map in $\boldsymbol{L}$ can *uniquely* be written as a finite composition of maps in Γ and non-identity maps in the image of f, in which no two of the latter are adjacent to each other.

To obtain the desired model category structure in the category $\boldsymbol{S}$-**Cat** of the small $\boldsymbol{S}$-categories we now start with describing what will be the *weak equivalences* and the *generating cofibrations* and *trivial cofibrations* in $\boldsymbol{S}$-**Cat**.

48.2. The weak equivalences in $\boldsymbol{S}$-Cat. Let $\boldsymbol{S}$-**Cat** denote the category of the small $\boldsymbol{S}$-categories and the functors between them (47.1). A map $f\colon \boldsymbol{G} \to \boldsymbol{H} \in \boldsymbol{S}$-**Cat** then will be called a *weak equivalence* if

(i) for every two objects $X, Y \in \boldsymbol{G}$, the restriction $\boldsymbol{G}(X, Y) \to \boldsymbol{H}(fX, fY) \in \boldsymbol{S}$ is a weak equivalence, and

(ii) the induced functor $\pi_0 f\colon \pi_0\boldsymbol{G} \to \pi_0\boldsymbol{H}$ between the categories of components (47.1) is an equivalence of categories (42.1) (which is a consequence of (i) if f is "onto on objects").

48.3. The generating cofibrations. Let $\boldsymbol{U}\colon \boldsymbol{S} \to \boldsymbol{S}$-**Cat** denote the functor which sends an object $K \in \boldsymbol{S}$ to the $\boldsymbol{S}$-category with 0 and 1 as its objects, K as its simplicial hom-set of maps from 0 to 1 and no other non-identity maps. The set I of the *generating cofibrations* then will consist of

(i) the inclusions $\boldsymbol{U}\dot{\Delta}[n] \to \boldsymbol{U}\Delta[n] \in \boldsymbol{S}$-**Cat** $(n \geq 0)$, and

(ii) the map $\emptyset \to * \in \boldsymbol{S}$-**Cat** from the initial object to the terminal object.

This definition readily implies that

(iii) *a map $f\colon \boldsymbol{G} \to \boldsymbol{H} \in \boldsymbol{S}$-**Cat** is an I-injective iff f is onto and, for every two objects $X, Y \in \boldsymbol{G}$, the restriction $\boldsymbol{G}(X, Y) \to \boldsymbol{H}(fX, fY) \in \boldsymbol{S}$ is a trivial fibration, and hence* (48.2)

(iv) *every I-injective is a weak equivalence.*

Furthermore we observe that the generating cofibrations are

48.4. Free map in $\boldsymbol{S}$-Cat. A map $f\colon \boldsymbol{G} \to \boldsymbol{H} \in \boldsymbol{S}$-**Cat** will be called *free* if

(i) f is 1-1 on objects and maps, and

(ii) there exists a set Γ of non-identity maps in $\boldsymbol{H}$ (called *generators*), which is closed under the degeneracy operators and which has the property that every non-identity map in $\boldsymbol{H}$ can *uniquely* be written as a finite composition of maps in Γ and non-identity maps in the image of f, in which no two of the latter appear next to each other,

and similarly an $\boldsymbol{S}$-category $\boldsymbol{G}$ will be called *free* if (48.3) the unique map $\emptyset \to \boldsymbol{G} \in \boldsymbol{S}$-**Cat** is free. Clearly (7.3)

(iii) *a map in $\boldsymbol{S}$-**Cat** is a regular I-cofibration iff it is the retract of a free map.*

Next we describe

48.5. The generating trivial cofibrations. For every integer $n \geq 1$, let $\boldsymbol{V}^n$ denote the free (48.4) $\boldsymbol{S}$-category with objects 0 and 1 and non-degenerate generators (5.7)

$$\begin{array}{lll} a_1, \dots, a_{2n-1} \in \boldsymbol{V}_0^n(0,0) & b_1, \dots, b_{2n} \in \boldsymbol{V}_1^n(0,0) & c \in \boldsymbol{V}_0^n(0,1) \\ a'_1, \dots, a'_{2n-1} \in \boldsymbol{V}_0^n(1,1) & b'_1, \dots, b'_{2n} \in \boldsymbol{V}_1^n(1,1) & c' \in \boldsymbol{V}_0^n(1,0) \end{array}$$

with faces given by the formulas

$$\begin{array}{lll} d_0 b_1 = 1_0 & d_0 b'_1 = 1_1 & \\ d_0 b_{2i} = d_0 b_{2i+1} = a_{2i} & d_0 b'_{2i} = d_0 b'_{2i+1} = a'_{2i} & 1 \leq i \leq n-1 \\ d_1 b_{2i+1} = d_1 b_{2i+2} = a_{2i+1} & d_1 b'_{2i+1} = d_1 b'_{2i+2} = a'_{2i+1} & 0 \leq i \leq n-1 \\ d_0 b_{2n} = c'c & d_0 b'_{2n} = cc' & \end{array}$$

One then readily verifies that

(i) *for every object* $\boldsymbol{G} \in \boldsymbol{S}$-**Cat**, *integer* $n \geq 1$ *and map* $f\colon \boldsymbol{V}^n \to \boldsymbol{G} \in \boldsymbol{S}$-**Cat**, *the map* $fc \in \boldsymbol{G}$ *is a homotopy equivalence* (47.1) *which has the map* $fc' \in \boldsymbol{G}$ *as a homotopy inverse, and*

(ii) *for every object* $\boldsymbol{G} \in \boldsymbol{S}$-**Cat** *and homotopy equivalence* $e \in \boldsymbol{G}$, *there is an integer* $k \geq 1$ *such that, for every integer* $n \geq k$, *there exists a map* $f\colon \boldsymbol{V}^n \to \boldsymbol{G} \in \boldsymbol{S}$-**Cat** *such that* $fc = e$.

The set J of the *generating trivial cofibrations* now will consist of (48.3)

(iii) the inclusions $\boldsymbol{U}\Delta^k[n] \to \boldsymbol{U}\Delta[n] \in \boldsymbol{S}$-**Cat** ($n > 0$, $0 \leq k \leq n$), and

(iv) the maps $* \to \boldsymbol{V}^n \in \boldsymbol{S}$-**Cat** ($n \geq 1$) which send the only object of the terminal object $*$ to the object $0 \in \boldsymbol{V}^n$

and a straightforward calculation yields that

(iv) *the generating trivial cofibrations are all free maps* (48.4) *as well as weak equivalences* (48.2).

Furthermore (i) and (ii) above readily imply that following

48.6. Characterization of the J-injectives. *A map* $f\colon \boldsymbol{G} \to \boldsymbol{H} \in \boldsymbol{S}$-**Cat** *is a J-injective* (7.2) *iff*

(i) *for every two objects* $X_1, X_2 \in \boldsymbol{G}$, *the restriction* $\boldsymbol{G}(X_1, X_2) \to \boldsymbol{H}(fX_1, fX_2) \in \boldsymbol{S}$ *is a fibration, and*

(ii) *for every two objects* $X_1 \in \boldsymbol{G}$ *and* $Y_2 \in \boldsymbol{H}$ *and homotopy equivalence* $e\colon fX_1 \to Y_2 \in \boldsymbol{H}$, *there exists an object* $X_2 \in \boldsymbol{G}$ *and a homotopy equivalence* $d\colon X_1 \to X_2 \in \boldsymbol{G}$ *such that* $fd = e$.

Finally we are ready to prove the existence of the desired cofibrantly generated

48.7. Model category structure on $\boldsymbol{S}$-Cat. *The category* $\boldsymbol{S}$-**Cat** *of small* $\boldsymbol{S}$*-categories admits a cofibrantly generated model category structure with* I (48.3) *and* J (48.5) *as sets of generating cofibrations and trivial cofibrations, in which*

(i) *the weak equivalences are as in* 48.2,

(ii) *the fibrations are the maps which satisfy* 48.6*(i) and (ii), and*

(iii) *the cofibrations are the retracts of the free maps* (48.4).

Proof. If $\boldsymbol{W} \subset \boldsymbol{S}\text{-}\mathbf{Cat}$ denotes the category of the weak equivalences, then (8.1) it suffices to show that

$$J\text{-}\mathbf{cof} \subset I\text{-}\mathbf{cof} \cap \boldsymbol{W} \qquad \text{and} \qquad I\text{-}\mathbf{inj} = J\text{-}\mathbf{inj} \cap \boldsymbol{W}.$$

The equality is an easy consequence of 48.2, 48.3 and 48.6 and as every J-cofibration is clearly an I-cofibration, it remains to show that $J\text{-}\mathbf{cof} \subset \boldsymbol{W}$ or equivalently, that *every pushout of a generating trivial cofibration is a weak equivalence.* It follows readily from 48.1 that this is the case for the generating trivial cofibrations of 48.5(iii). To deal with the others one notes that the map $* \to \boldsymbol{V}^n$ $(n \geq 1)$ admits a factorization $* \to \boldsymbol{V}^{n,0} \to \boldsymbol{V}^n$ in $\boldsymbol{W}$, where $\boldsymbol{V}^{n,0} \subset \boldsymbol{V}^n$ denotes the full sub-$\boldsymbol{S}$-category spanned by the object 0. The desired result now follows from the fact that for every object $G \in \boldsymbol{S}\text{-}\mathbf{Cat}$ and map $* \to \boldsymbol{G} \in \boldsymbol{S}\text{-}\mathbf{Cat}$, this factorization yields a double pushout diagram

$$\begin{array}{ccccc} * & \longrightarrow & \boldsymbol{V}^{n,0} & \longrightarrow & \boldsymbol{V}^n \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{G} & \longrightarrow & \boldsymbol{G}' & \longrightarrow & \boldsymbol{G}'' \end{array}$$

in which the map $\boldsymbol{G} \to \boldsymbol{G}'$ is a weak equivalence (in view of 48.1), while 48.5(i) readily implies that the map $\boldsymbol{G}' \to \boldsymbol{G}''$ is so too.

49. Quillen functors in two variables

49.1. Proposition. *Let $f\colon \boldsymbol{M} \to \boldsymbol{N}$ be a left or a right Quillen functor between cofibrantly generated model categories, then so is, for every small category $\boldsymbol{D}$, the induced functor $\boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{N}^{\boldsymbol{D}}$.*

49.2. Examples. Examples of Quillen pairs are

(i) the pair $|\ \ |\colon \boldsymbol{S} \leftrightarrow \boldsymbol{T} :\mathrm{Sin}$ consisting of the geometric realization and the singular functor (6.1),
(ii) the pair of adjoint functors $F\colon \boldsymbol{B} \leftrightarrow \boldsymbol{C} :U$ in the lifting lemma (9.1), and
(iii) for every cofibrantly generated model category $\boldsymbol{N}$ (5.3) and small category $\boldsymbol{D}$, the pair of adjoint functors $\mathrm{colim}^{\boldsymbol{D}}\colon \boldsymbol{N}^{\boldsymbol{D}} \leftrightarrow \boldsymbol{N} :c^*$ (2.5).

Next we define

49.3. Quillen functors in two variables. Given three model categories $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{N}$, a functor $f\colon \boldsymbol{M}_1 \times \boldsymbol{M}_2 \to \boldsymbol{N}$ will be called a *left Quillen functor* if

(i) f has both partial right adjoints $\boldsymbol{N} \times \boldsymbol{M}_2^{\mathrm{op}} \to \boldsymbol{M}_1$ and $\boldsymbol{M}_1^{\mathrm{op}} \times \boldsymbol{N} \to \boldsymbol{M}_2$, and
(ii) for every pair of cofibrations $i_\epsilon\colon A_\epsilon \to B_\epsilon \in \boldsymbol{M}_\epsilon$ $(\epsilon = 1, 2)$, the pushout corner map (2.5)

$$f(A_1, B_2) \amalg_\bullet f(B_1, A_2) \to f(B_1, B_2) \in \boldsymbol{N}$$

is a cofibration, which is trivial whenever at least one of i_1 and i_2 is trivial or equivalently (ii)$_*$ for every pair of cofibrations $i_\epsilon \colon A_\epsilon \to B_\epsilon \in \boldsymbol{M}_\epsilon$ ($\epsilon = 1, 2$) and every fibration $p \colon X \to Y \in \boldsymbol{N}$, at least *one* of which is *trivial*, the iterated pullback corner map in **Set**

$$\boldsymbol{N}\big(f(B_1, B_2), X\big) \to \boldsymbol{N}\big(f(A_1, B_2), X\big) \amalg_\bullet \boldsymbol{N}\big(f(B_1, A_2), X\big) \amalg_\bullet \boldsymbol{N}\big(f(B_1, B_2), Y\big)$$

is onto.

Right Quillen functors $\boldsymbol{M}_1 \times \boldsymbol{M}_2 \to \boldsymbol{N}$ are of course defined dually.

One readily verifies the following

49.4. Elementary properties.

(i) *If* $f \colon \boldsymbol{M}_1 \times \boldsymbol{M}_2 \to \boldsymbol{N}$ *is a left Quillen functor, then its partial right adjoints* $\boldsymbol{N} \times \boldsymbol{M}_2^{\mathrm{op}} \to \boldsymbol{M}_1$ *and* $\boldsymbol{M}_1^{\mathrm{op}} \times \boldsymbol{N} \to \boldsymbol{M}_2$ *and its opposite* $f^{\mathrm{op}} \colon \boldsymbol{M}_2^{\mathrm{op}} \times \boldsymbol{M}_1^{\mathrm{op}} \to \boldsymbol{N}^{\mathrm{op}}$ *are right Quillen functors and dually, if* f *is a right Quillen functor, then its partial left adjoints and its opposite are left Quillen functors.*

(ii) *If* $f \colon \boldsymbol{M}_1 \times \boldsymbol{M}_2 \to \boldsymbol{N}$ *is a left (resp. right) Quillen functor, then so is, for every cofibrant (resp. fibrant) object* $K_1 \in \boldsymbol{M}_1$*, the restriction* $f(K_1, -) \colon \boldsymbol{M}_2 \to \boldsymbol{N}$ *and, for every cofibrant (resp. fibrant) object* $K_2 \in \boldsymbol{M}_2$*, the restriction* $f(-, K_2) \colon \boldsymbol{M}_1 \to \boldsymbol{N}$.

One also readily verifies, generalizing 49.2

49.5. Proposition. *Let* $\boldsymbol{M}_1$ *and* $\boldsymbol{N}$ *be cofibrantly generated model categories. If* $f \colon \boldsymbol{M}_1 \times \boldsymbol{M}_2 \to \boldsymbol{N}$ *is a left or a right Quillen functor, then so is, for every small category* $\boldsymbol{D}$*, the induced functor* $\boldsymbol{M}_1^{\boldsymbol{D}} \times \boldsymbol{M}_2 \to \boldsymbol{N}^{\boldsymbol{D}}$.

49.6. Examples. Examples of *left* Quillen functors in two variables are

(i) the functor $\otimes \colon \boldsymbol{S} \times \boldsymbol{S} \to \boldsymbol{S}$ (5.8) given by

$$L \otimes K = L \times K \qquad K, L \in \boldsymbol{S}$$

(ii) the functor $\otimes \colon \boldsymbol{T} \times \boldsymbol{S} \to \boldsymbol{T}$ (6.1) given by

$$A \otimes K = A \times \big|K\big| \qquad A \in \boldsymbol{T}, K \in \boldsymbol{S}$$

(iii) for every category $\boldsymbol{A}$ of universal algebras the functor $\otimes \colon \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}} \times \boldsymbol{S} \to \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ given by (47.2)

$$(A \otimes K)_n = A_n \otimes K_n \qquad A \in \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}, K \in \boldsymbol{S}, n \geq 0$$

and examples of *right* Quillen functors in two variables are their partial right adjoints

(i)$'$ the functor $\hom \colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{S} \to \boldsymbol{S}$ given by

$$\hom(K, L)_n = \boldsymbol{S}(K \times \Delta[n], L) \qquad K, L \in \boldsymbol{S}, n \geq 0$$

(ii)$'$ the functor $\hom \colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{T} \to \boldsymbol{T}$ given by

$$\hom(K, Y) = Y^{\left|K\right|} \qquad Y \in \boldsymbol{T}, K \in \boldsymbol{S}$$

(iii)$'$ the functor $\hom \colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}} \to \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ which sends a pair of objects $K \in \boldsymbol{S}, Y \in \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ to the object $\hom(K, Y) \in \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ obtained by applying the functor $\hom(K, -) \colon \boldsymbol{S} \to \boldsymbol{S}$ of (i)$'$ to Y, considered as a diagram involving finite powers of the underlying simplicial set of Y (which is well defined as the functor $\hom(K, -) \colon \boldsymbol{S} \to \boldsymbol{S}$ preserves products).

We end with a few comments on

49.7. Quillen functors in several variables. The above definitions and elementary properties of Quillen functors readily extend to functors in more than two variables and it is not difficult to see that the resulting Quillen functors have the property that *for every* n *(*$n \geq 1$*) left (resp. right) Quillen functors in several variables*

$$f_i \colon \boldsymbol{M}_{i,1} \times \cdots \times \boldsymbol{M}_{i,k_i} \to \boldsymbol{N}_i \qquad (i = 1, \ldots, n)$$

and left (resp. right) Quillen functor

$$f \colon \boldsymbol{N}_1 \times \cdots \times \boldsymbol{N}_n \to \boldsymbol{P}$$

the "composition"

$$f(f_1 \times \cdots \times f_n) \colon \boldsymbol{M}_{1,1} \times \cdots \times \boldsymbol{M}_{n,k_n} \to \boldsymbol{P}$$

is again a left (resp. right) Quillen functor.

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CHAPTER XIII

Function complexes

50. Introduction

50.1. Summary. For topological spaces and simplicial sets the *homotopy classes of maps* between a cofibrant and a fibrant object are just the *components of the function complex* between them. Our main aim in this chapter is to show that a similar result holds in an arbitrary model category $\boldsymbol{M}$, i.e. we construct, for every two objects $A, Y \in \boldsymbol{M}$, *left function complexes* from A to Y and dually *right function complexes* from A to Y (which are simplicial sets with as vertices all the maps $A \to Y \in \boldsymbol{M}$ and as 1-simplices some of the left or right homotopies (45.1) between them) and show that, *for cofibrant* A *and fibrant* Y

(i) *these left and right function complexes from* A *to* Y *all have the same homotopy type* (i.e. they are in the same weak equivalence class),
(ii) *this homotopy type depends only on the homotopy types of* A *and* Y, and
(iii) *its set of components is canonically isomorphic to* $\operatorname{Ho}\boldsymbol{M}(A, Y)$ (46.1).

The usual function complexes for topological spaces and simplicial sets are left as well as right function complexes; they are also functorial in both variables and composable. In an arbitrary model category $\boldsymbol{M}$ one can always in a functorial manner select, for every pair of objects, a left function complex and a right function complex between them. However to ensure that one can do this in such a manner that these left and right function complexes *coincide* and are *composable* one has to impose some restrictions on $\boldsymbol{M}$, and we show that a sufficient such restriction is that $\boldsymbol{M}$ be a so-called *simplicial model category.* Examples of simplicial model categories are *simplicial sets*, *topological spaces*, *simplicial universal algebras* and their *diagram categories.*

In more detail:

50.2. Reedy model category structures. The construction of left and right function complexes in a model category $\boldsymbol{M}$ involves the *Reedy model category structures* on the categories $\boldsymbol{M}^{\boldsymbol{\Delta}}$ and $\boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of the cosimplicial and the simplicial diagrams in $\boldsymbol{M}$. These can be described as follows:

If $\boldsymbol{M}$ is a model category and $\boldsymbol{B}$ is a *direct* (resp. an *inverse*) *category* (i.e. the objects of $\boldsymbol{B}$ have a non-negative integral degree and all non-identity maps of $\boldsymbol{B}$ raise (resp. lower) this degree), then it is not difficult to see that the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ admits a model category structure in which the weak equivalences and the fibrations (resp. the cofibrations) are the objectwise ones, and more generally, if $\boldsymbol{B}$ is a *Reedy category* (i.e. $\boldsymbol{B}$ has a non-negative integral degree function on the objects, a direct subcategory $\overrightarrow{\boldsymbol{B}} \subset \boldsymbol{B}$ and an inverse subcategory $\overleftarrow{\boldsymbol{B}} \subset \boldsymbol{B}$ such that every map $b \in \boldsymbol{B}$ has a unique factorization $b = \overrightarrow{b}\overleftarrow{b}$ with $\overrightarrow{b} \in \overrightarrow{\boldsymbol{B}}$ and $\overleftarrow{b} \in \overleftarrow{\boldsymbol{B}}$), then the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ admits a so-called *Reedy model category structure* in which a map is a weak equivalence, a fibration or a cofibration whenever its restrictions to both diagram categories $\boldsymbol{M}^{\overrightarrow{\boldsymbol{B}}}$ and $\boldsymbol{M}^{\overleftarrow{\boldsymbol{B}}}$ are so.

Useful examples of Reedy categories are the *category of simplices* $\boldsymbol{\Delta}K$ of a simplicial set K (5.9) and its opposite $\boldsymbol{\Delta}^{\mathrm{op}}K$, and in particular the *cosimplicial* and *simplicial indexing categories* $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^{\mathrm{op}}$, which are used to define

50.3. Left and right function complexes. Given a model category $\boldsymbol{M}$ and objects $A, Y \in \boldsymbol{M}$, a *left function complex* from A to Y will be a simplicial set of the form $\boldsymbol{M}(A^*, Y)$, where A^* is a so-called *cosimplicial frame* on A, i.e. a cosimplicial object $A^* \in \boldsymbol{M}^{\boldsymbol{\Delta}}$ together with an isomorphism $A^*[0] \approx A \in \boldsymbol{M}$, such that the obvious map $A^* \to \mathrm{c}^*A$ (where c^*A denoted the constant diagram) is a *weak equivalence* in the (see 50.2) Reedy model category structure on the category $\boldsymbol{M}^{\boldsymbol{\Delta}}$ of simplicial objects in $\boldsymbol{M}$, and such that A^* is *cofibrant* in this model category structure whenever the object $A \in \boldsymbol{M}$ is cofibrant. *Right function complexes* are of course defined dually.

To obtain *functorial* (although not necessarily composable) left and right function complexes one has to choose the cosimplicial and simplicial frames on the objects of $\boldsymbol{M}$ in a functorial manner. This can indeed be done, thanks to the functoriality in the factorization axiom M5 (3.1). Such functorial choices lead to the useful notion of

50.4. Framed model categories. A *framed model category* will be a model category $\boldsymbol{M}$ with a *framing*, i.e. with a functorial choice of cosimplicial and simplicial frames on its objects (52.1) or equivalently (5.10) a (not necessarily adjoint) pair of functors $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ and $\hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{M}$ such that, for every pair of objects $A, Y \in \boldsymbol{M}$

(i) the diagrams (5.8) $A \otimes \Delta[-]$ and $\hom(\Delta[-], Y)$ are cosimplicial and simplicial frames on A and Y respectively, and
(ii) the restrictions $A \otimes -\colon \boldsymbol{S} \to \boldsymbol{M}$ and $\hom(-, Y)\colon \boldsymbol{S}^{\mathrm{op}} \to \boldsymbol{M}$ preserve colimits and limits respectively,

If the framing is *self adjoint*, i.e. if for every object $K \in \boldsymbol{S}$,

(iii) the restriction $-\otimes K\colon \boldsymbol{M} \to \boldsymbol{M}$ is left adjoint to the restriction $\hom(K, -)\colon \boldsymbol{M} \to \boldsymbol{M}$,

then the resulting left and right function complexes are *canonically isomorphic*, but *not* necessarily composable.

As we mentioned above (50.3) every model category admits a framing. But not every model category admits a self adjoint framing, let alone one for which the function complexes are composable. This leads us to the very useful notion of

50.5. Simplicial model categories. A *simplicial model category* will be a model category $\boldsymbol{M}$, together with a functor $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ which is

(i) a *cartesian $\boldsymbol{S}$-action*, i.e. compatible with the (cartesian) product in $\boldsymbol{S}$, and
(ii) a *left Quillen functor*, i.e. (4.1) it has both partial right adjoints and is, in a certain precise manner, comparable with the model category structures of $\boldsymbol{M}$ and $\boldsymbol{S}$.

We show that these two simple conditions imply that *the functor* $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ *and its partial right adjoint* $\hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{M}$ *form a self adjoint framing of* $\boldsymbol{M}$ *for which the function complexes are composable.*

Examples are *simplicial sets*, *compactly generated topological spaces* and *simplicial universal algebras*. Also for a simplicial model category $\boldsymbol{M}$ and a *Reedy* category $\boldsymbol{B}$ (50.2), the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ inherits from $\boldsymbol{M}$ a simplicial model category structure. The same holds for all diagram categories and even all *simplicial* diagram categories of *cofibrantly generated* (5.3) simplicial model categories (such as the above mentioned three examples).

Finally we mention the rather obvious notion of

50.6. Diagrams of weak equivalences. Given a model category $\boldsymbol{M}$ and a small category $\boldsymbol{B}$, a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$ will be called a *diagram of weak equivalences* if, for every map $b \in \boldsymbol{B}$, the map $Xb \in \boldsymbol{M}$ is a weak equivalence.

51. Reedy functors

Our aim here is to find some useful sufficient conditions of a Reedy functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ (12.1) in order that, for every model category $\boldsymbol{M}$, the induced functor $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$ preserves not only weak equivalences, but also cofibrations and/or fibrations, so that (43.2 and 4.1) the pairs of adjoint functors

$$v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}} :\lim^v \quad \text{and/or} \quad \operatorname{colim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M}^{\boldsymbol{D}} :v^*$$

are Quillen pairs.

We first observe

51.1. Proposition. *Let $\boldsymbol{M}$ be a model category. Then*

- (i) *for every functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between direct categories, the induced functor $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$ preserves fibrations,*
- (ii) *for every functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between inverse categories, the induced functor $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$ preserves cofibrations, and hence*
- (iii) *for every functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between small discrete* (10.6) *categories, the induced functor $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$ preserves both cofibrations and fibrations.*

This readily implies

51.2. Proposition. *Let $\boldsymbol{M}$ be a model category and let $v\colon \boldsymbol{B} \to \boldsymbol{D}$ be a Reedy functor* (12.1). *Then the induced functor $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$*

- (i) *preserves cofibrations if the restriction $(v|\overrightarrow{\boldsymbol{B}})^*\colon \boldsymbol{M}^{\overrightarrow{\boldsymbol{D}}} \to \boldsymbol{M}^{\overrightarrow{\boldsymbol{B}}}$ does so, and dually*
- (ii) *preserves fibrations if the restriction $(v|\overleftarrow{\boldsymbol{B}})^*\colon \boldsymbol{M}^{\overleftarrow{\boldsymbol{D}}} \to \boldsymbol{M}^{\overleftarrow{\boldsymbol{B}}}$ does so.*

The following two propositions, which are also not difficult to prove, then provide sufficient conditions in order that $(v|\overrightarrow{\boldsymbol{B}})^*$ preserve cofibrations or $(v|\overleftarrow{\boldsymbol{B}})^*$ preserve fibrations.

51.3. Proposition. *Let $\boldsymbol{M}$ be a model category. Then*

- (i) *a functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between direct categories induces a functor $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$ which preserves cofibrations if, for every object $B \in \boldsymbol{B}$, the induced functor $(\boldsymbol{B} \downarrow B) \to (\boldsymbol{D} \downarrow vB)$ does so, and dually*
- (ii) *a functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between inverse categories induces a functor $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$ which preserves fibrations if, for every object $B \in \boldsymbol{B}$, the induced functor $(B \downarrow \boldsymbol{B}) \to (vB \downarrow \boldsymbol{D})$ does so.*

51.4. Proposition. *Let* $\boldsymbol{M}$ *be a model category and let* $u\colon \boldsymbol{D} \leftrightarrow \boldsymbol{B} :v$ *be a pair of adjoint functors between small categories. Then*

(i) *the induced functors* u^* *and* v^* *form a pair of adjoint functors* $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}} :u^*$ *(so that* $u^* = \lim^v$ *and* $v^* = \operatorname{colim}^u$*) and hence (*4.1 *and* 51.1*)*

(ii) *if* $\boldsymbol{B}$ *and* $\boldsymbol{D}$ *are direct categories, then the induced functor* $v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \to \boldsymbol{M}^{\boldsymbol{B}}$ *preserves both cofibrations and fibrations, and dually*

(iii) *if* $\boldsymbol{B}$ *and* $\boldsymbol{D}$ *are inverse categories, then the induced functor* $u^*\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ *preserves both cofibrations and fibrations.*

A straightforward calculation using these two propositions now yields

51.5. Application. *Given a model category* $\boldsymbol{M}$ *and a map* $f\colon K \to L \in \boldsymbol{S}$*, the pairs of adjoint functors (*43.2 *and* 5.9*)*

$$(\boldsymbol{\Delta} f)^*\colon \boldsymbol{M}^{\boldsymbol{\Delta} L} \leftrightarrow \boldsymbol{M}^{\boldsymbol{\Delta} K} :\lim^{\boldsymbol{\Delta} f} \quad \text{and} \quad \operatorname{colim}^{\boldsymbol{\Delta}^{\mathrm{op}} f}\colon \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}} K} \leftrightarrow \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}} L} :(\boldsymbol{\Delta}^{\mathrm{op}} f)^*$$

are Quillen pairs (4.1) *and so are the pairs of adjoint functors*

$$\operatorname{colim}^{\boldsymbol{\Delta} f}\colon \boldsymbol{M}^{\boldsymbol{\Delta} K} \leftrightarrow \boldsymbol{M}^{\boldsymbol{\Delta} L} :(\boldsymbol{\Delta} f)^* \quad \text{and} \quad (\boldsymbol{\Delta}^{\mathrm{op}} f)^*\colon \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}} L} \leftrightarrow \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}} K} :\lim^{\boldsymbol{\Delta}^{\mathrm{op}} f}.$$

52. Function complexes

Given a model category $\boldsymbol{M}$ we now use the *Reedy model category structure* (12.4) on the categories $\boldsymbol{M}^{\boldsymbol{\Delta}}$ and $\boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of the cosimplicial and the simplicial objects in $\boldsymbol{M}$ to define *left* and *right function complexes* in $\boldsymbol{M}$ and to prove their main properties (50.1) as well as a *simplicial detection lemma for weak equivalences.* For this we need the notions of

52.1. Cosimplicial and simplicial frames. Given a model category $\boldsymbol{M}$ and an object $A \in \boldsymbol{M}$, a *cosimplicial frame* on A will consist of an object $A^* \in \boldsymbol{M}^{\boldsymbol{\Delta}}$ (5.7) together with an *isomorphism* $A^*[0] \approx A \in \boldsymbol{M}$ such that

(i) the induced map $A^* \to \mathrm{c}^*A \in \boldsymbol{M}^{\boldsymbol{\Delta}}$ (where c^*A denotes the constant diagram (2.3)) is a weak equivalence (12.4), and

(ii) if the object $A \in \boldsymbol{M}$ is cofibrant, then so is the object $A^* \in \boldsymbol{M}^{\boldsymbol{\Delta}}$ (12.4),

and a *map* between two such frames A^* and $\bar{A}^*$ of A will be a map (and in fact a weak equivalence) $A^* \to \bar{A}^* \in \boldsymbol{M}^{\boldsymbol{\Delta}}$ which is compatible with the isomorphisms $A^*[0] \approx A$ and $\bar{A}^*[0] \approx A$. That such frames indeed exist is not difficult to verify using the factorization axiom M5 (3.1).

Of course, for every object $Y \in \boldsymbol{M}$, there are dual notions of *simplicial frames* on Y and *maps* between them.

Now we can define

52.2. Left and right function complexes. Given a model category $\boldsymbol{M}$ and objects $A, Y \in \boldsymbol{M}$, we define a *left function complex* from A to Y as a simplicial set of the form $\boldsymbol{M}(A^*, Y)$, where A^* is a cosimplicial frame on A (52.1) and a *right function complex* from A to Y as a simplicial set of the form $\boldsymbol{M}(A, Y_*)$, where Y_* is a simplicial frame on Y.

If A *is cofibrant and* Y *is fibrant*, then the following propositions readily imply that *all left and right function complexes from* A *to* Y *have the same homotopy* *type and this homotopy type depends only on the homotopy types of A and Y* (i.e. the components of the nerve (10.6) of the category of the weak equivalences of $\boldsymbol{M}$, containing A and Y).

52.3. Proposition.

(i) *Let $A \to B \in \boldsymbol{M}$ be a cofibration, a trivial cofibration or a weak equivalence between cofibrant objects and let $Y_* \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ be a simplicial frame on a fibrant object $Y \in \boldsymbol{M}$. Then the induced map $\boldsymbol{M}(B, Y_*) \to \boldsymbol{M}(A, Y_*) \in \boldsymbol{S}$ is a fibration, a trivial fibration or a weak equivalence between fibrant objects.*

(ii) *Let $X \to Y \in \boldsymbol{M}$ be a fibration, a trivial fibration or a weak equivalence between fibrant objects and let $A^* \in \boldsymbol{M}^{\boldsymbol{\Delta}}$ be a cosimplicial frame on a cofibrant object $A \in \boldsymbol{M}$. Then the induced map $\boldsymbol{M}(A^*, X) \to \boldsymbol{M}(A^*, Y) \in \boldsymbol{S}$ is also a fibration, a trivial fibration or a weak equivalence between fibrant objects.*

Proof. A straightforward calculation using 3.6 and 12.4.

52.4. Proposition. *Let $A^* \to \bar{A}^* \in \boldsymbol{M}^{\boldsymbol{\Delta}}$ be a map between cosimplicial frames on a cofibrant object $A \in \boldsymbol{M}$ and let $Y_* \to \bar{Y}_* \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ be a map between simplicial frames on a fibrant object $Y \in \boldsymbol{M}$. Then the induced maps*

$$\begin{array}{lll} \boldsymbol{M}(\bar{A}^*, Y) \to \boldsymbol{M}(A^*, Y) & \textit{and} & \boldsymbol{M}(A, Y_*) \to \boldsymbol{M}(A, \bar{Y}_*) \\ \boldsymbol{M}(A^*, Y) \to \operatorname{diag} \boldsymbol{M}(A^*, Y_*) & \textit{and} & \boldsymbol{M}(A, Y_*) \to \operatorname{diag} \boldsymbol{M}(A^*, Y_*) \end{array}$$

are weak equivalences.

Proof. This follows readily from 52.3 and

52.5. Proposition. *Let $X \in \boldsymbol{S}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ be a diagram (in $\boldsymbol{S}$) in which all the maps are weak equivalences, then the obvious map $X[0] \to \operatorname{diag} X \in \boldsymbol{S}$ is also a weak equivalence.*

Proof. It follows readily from 49.6 (i)′ that the functor $\hom(\Delta[-], -)\colon \boldsymbol{S} \to \boldsymbol{S}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ is a right Quillen functor and hence that its left adjoint $\operatorname{diag}\colon \boldsymbol{S}^{\boldsymbol{\Delta}^{\mathrm{op}}} \to \boldsymbol{S}$ is a left Quillen functor. The desired result now is an easy consequence of the fact that all objects of $\boldsymbol{S}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ are (Reedy) cofibrant.

Closely related to left and right function complexes are

52.6. Left and right homotopies. Given a model category $\boldsymbol{M}$, objects $B, X \in \boldsymbol{M}$, a cosimplicial frame B^* on B and a simplicial frame X_* on X, then the vertices of $\boldsymbol{M}(B^*, X)$ and $\boldsymbol{M}(B, X_*)$ correspond with all the maps $B \to X \in \boldsymbol{M}$ and the 1-simplices with some of the left and the right homotopies between them. If B is cofibrant (resp. X is fibrant) then these homotopies have the property that, in the notation of 45.1, the map $B \amalg B \to B'$ is a cofibration (resp. the map $X' \to X \amalg X$ is a fibration). Furthermore, given two objects $B, X \in \boldsymbol{M}$, two maps $f, g\colon B \to X \in \boldsymbol{M}$ and a left (resp. a right) homotopy between them (which for cofibrant B (resp. fibrant X) is subject to the just mentioned restriction), there exists a cosimplicial frame B^* on B (resp. a simplicial frame X_* on X) such that the given homotopy corresponds with a 1-simplex of $\boldsymbol{M}(B^*, X)$ (resp. $\boldsymbol{M}(B, X_*)$). Therefore proposition 52.3 implies

52.7. Proposition. *Given a model category $\boldsymbol{M}$, a cosimplicial frame A^* on a cofibrant object $A \in \boldsymbol{M}$ and a simplicial frame Y_* on a fibrant object $Y \in \boldsymbol{M}$, the obvious maps* (45.2)

$$\pi_0 \boldsymbol{M}(A^*, Y) \to \operatorname{Ho} \boldsymbol{M}(A, Y) \quad \text{and} \quad \pi_0 \boldsymbol{M}(A, Y_*) \to \operatorname{Ho} \boldsymbol{M}(A, Y)$$

are isomorphisms (of sets).

Next we observe that left and right function complexes are sometimes useful in deciding whether a map is a weak equivalence. In fact, a rather straightforward argument using 52.4 and 52.6 yields the following

52.8. Simplicial detection lemma. *Given a model category $\boldsymbol{M}$, a map $P \to Q \in \boldsymbol{M}$ is a weak equivalence if*

(i) *for every cofibrant object $A \in \boldsymbol{M}$, there exists a cosimplicial frame A^* on A such that the induced map $\boldsymbol{M}(A^*, P) \to \boldsymbol{M}(A^*, Q) \in \boldsymbol{S}$ is a weak equivalence and the converse holds if P and Q are assumed to be fibrant, or dually*

(ii) *for every fibrant object $Y \in \boldsymbol{M}$, there exists a simplicial frame Y_* on Y such that the induced map $\boldsymbol{M}(Q, Y_*) \to \boldsymbol{M}(P, Y_*) \in \boldsymbol{S}$ is a weak equivalence and the converse holds if P and Q are assumed to be cofibrant.*

We also note the following application to Quillen pairs

52.9. Proposition. *Let $\boldsymbol{M}$ and $\boldsymbol{M}'$ be model categories, let $g\colon \boldsymbol{M} \leftrightarrow \boldsymbol{M}' :g'$ be a Quillen pair of adjoint functors* (4.1)*, let A^* be a cosimplicial frame on a cofibrant object $A \in \boldsymbol{M}$ and let Y'_* be a simplicial frame on a fibrant object $Y' \in \boldsymbol{M}'$. then gA^* is a cosimplicial frame on the (cofibrant) object gA and $g'Y'_*$ is a simplicial frame on the (fibrant) object $g'Y'$ and hence the adjunction gives rise to isomorphisms of function complexes*

$$\boldsymbol{M}(A^*, g'Y') \approx \boldsymbol{M}(gA^*, Y') \quad \text{and} \quad \boldsymbol{M}(A, g'Y'_*) \approx \boldsymbol{M}(gA, Y'_*).$$

We end with a

52.10. Remark on naturality and composability. Given a model category, one can clearly choose cosimplicial and simplicial frames on its objects which are functorial and which therefore give rise to left and right function complexes which are *natural in both variables*, but which in general are not composable. However there are, as we will see in the remaining sections of this chapter, some very useful model categories in which one can choose these frames in such a manner that the resulting left and right function complexes are not only natural in both variables, but also *composable* and *canonically isomorphic.*

53. Framed model categories

We noted in 52.10 that a *functorial* choice of cosimplicial and simplicial frames (52.1) on the objects of a model category gives rise to functorial (although not necessarily composable) left and right function complexes between all objects. Furthermore we will see in the next chapter that this same choice makes it possible to construct homotopy colimit and limit functors which generalize the usual such functors for simplicial sets. We therefore introduce here the notion of a *framed model category* as a model category with such a functorial choice of cosimplicial and simplicial frames (which we will call a *framing*) and note that such *framings* *always exist* and are *"unique up to homotopy"*. Some model categories, such as topological spaces, simplicial sets and simplicial universal algebras, admit so-called *self adjoint framings* which have the pleasant property that the resulting left and right function complexes are canonically isomorphic (although still not necessarily composable).

53.1. Framed model categories. A *framed model category* will be a model category $\boldsymbol{M}$ together with a *framing* i.e. a functorial choice of cosimplicial and simplicial frames on its objects (52.1) or equivalently (5.10), a (not necessarily adjoint) pair of functors

$$\otimes\colon \boldsymbol{M}\times\boldsymbol{S}\to\boldsymbol{M} \qquad \text{and} \qquad \hom\colon \boldsymbol{S}^{\mathrm{op}}\times\boldsymbol{M}\to\boldsymbol{M}$$

(called a *left framing* and a *right framing*) such that, for every pair of objects $A, Y \in \boldsymbol{M}$,

(i) the diagrams (5.8) $A\otimes\Delta[-]$ and $\hom(\Delta[-],Y)$ are cosimplicial and simplicial frames on A and Y respectively, and
(ii) the restrictions $A\otimes -\colon \boldsymbol{S}\to\boldsymbol{M}$ and $\hom(-,Y)\colon \boldsymbol{S}^{\mathrm{op}}\to\boldsymbol{M}$ preserve colimits and limits respectively,

and a *map* between two such framings $(\otimes,\hom)$ and $(\otimes',\hom')$ of $\boldsymbol{M}$ will consist of a pair of natural transformations $\otimes\to\otimes'$ and $\hom\to\hom'$ such that, for every pair of objects $A, Y\in\boldsymbol{M}$, the induced maps

$$A\otimes\Delta[-]\to A\otimes'\Delta[-] \qquad \text{and} \qquad \hom'(\Delta[-],Y)\to\hom(\Delta[-],Y)$$

are maps of frames (52.1). Finally a framing $(\otimes,\hom)$ of $\boldsymbol{M}$ will be called *self adjoint* if the functors $\otimes$ and hom are partial adjoints of each other, i.e. if, for every object $K\in\boldsymbol{S}$ there is an adjunction

$$-\otimes K\colon \boldsymbol{M}\leftrightarrow\boldsymbol{M} :\hom(K,-).$$

Clearly, for every pair of objects A and Y in a *framed* model category $\boldsymbol{M}$, the simplicial sets

$$\boldsymbol{M}\big(A\otimes\Delta[-],Y\big) \qquad \text{and} \qquad \boldsymbol{M}\big(A,\hom(\Delta[-],Y)\big)$$

are *left* and *right function complexes* form A to Y (52.2) respectively, which are *natural in both variables.* We will denote them respectively by

$$\boldsymbol{M}_{\mathrm{L}}(A,Y) \qquad \text{and} \qquad \boldsymbol{M}_{\mathrm{R}}(A,Y).$$

If the framing of $\boldsymbol{M}$ is *self adjoint*, then the adjunction induces a *natural isomorphism* $\boldsymbol{M}_{\mathrm{L}}(A,Y)\approx\boldsymbol{M}_{\mathrm{R}}(A,Y)$.

A left or right framing need *not* be a left or right Quillen functor (4.1), even if the framing is self adjoint. Still one readily verifies that every framing has the following

53.2. Partial Quillen property. Let $\boldsymbol{M}$ be a framed model category. Then

(i) for every cofibrant object $A\in\boldsymbol{M}$, the pair of adjoint functors (5.10)

$$A\otimes -\colon \boldsymbol{S}\leftrightarrow\boldsymbol{M} :\boldsymbol{M}_{\mathrm{L}}(A,-)$$

is a Quillen pair (4.1) and dually

(ii) so is, for every fibrant object $Y\in\boldsymbol{M}$, the pair of adjoint functors

$$\boldsymbol{M}_{\mathrm{R}}(-,Y)\colon \boldsymbol{M}\leftrightarrow\boldsymbol{S}^{\mathrm{op}} :\hom(-,Y).$$

Furthermore a straightforward argument using 12.11 yields the following

53.3. Homotopy property. Let $\boldsymbol{M}$ be a framed model category. Then, for every object $K \in \boldsymbol{S}$,

(i) the functor $-\otimes K\colon \boldsymbol{M} \to \boldsymbol{M}$ preserves weak equivalences between cofibrant objects and dually
(ii) the functor $\hom(K,-)\colon \boldsymbol{M} \to \boldsymbol{M}$ preserves weak equivalences between fibrant objects.

53.4. Examples.

(i) For the category $\boldsymbol{S}$ of simplicial sets (6.8) the functors

$$\otimes\colon \boldsymbol{S} \times \boldsymbol{S} \to \boldsymbol{S} \qquad \text{and} \qquad \hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{S} \to \boldsymbol{S}$$

of 49.6(i) and (i)′ form a self adjoint framing. The resulting (left and right) function complex from an object A to an object Y is the simplicial set $\hom(A,Y)$.
(ii) For the category $\boldsymbol{T}$ of *compactly generated topological spaces* (6.9) the functors

$$\otimes\colon \boldsymbol{T} \times \boldsymbol{S} \to \boldsymbol{T} \qquad \text{and} \qquad \hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{T} \to \boldsymbol{T}$$

of 49.6(ii) and (ii)′ form a self adjoint framing. The resulting (left and right) function complex from an object A to an object Y is the singular complex $\operatorname{Sin} Y^A$ of the function space Y^A.
(iii) For every category $\boldsymbol{A}$ of universal algebras, the functors

$$\otimes\colon \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}} \times \boldsymbol{S} \to \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}} \qquad \text{and} \qquad \hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}} \to \boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$$

of 49.6(iii) and (iii)′ form a self adjoint framing of the category $\boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of *simplicial universal algebras.*
(iv) Every model category has a (not necessarily self adjoint) framing and such a framing is "unique up to homotopy" as one readily verifies.

53.5. Proposition. *Let $\boldsymbol{M}$ be a model category. Then the (nerve of the) category of the frames of $\boldsymbol{M}$ and the maps between them* (53.1) *is contractible* (10.6) *and hence non-empty.*

54. Cartesian $\boldsymbol{S}$-actions

In this section we investigate the composability of the left and the right function complexes in a framed (53.1) model category $\boldsymbol{M}$. We show

(i) if the left framing $\otimes\colon \boldsymbol{M}\times\boldsymbol{S} \to \boldsymbol{M}$ is a *cartesian $\boldsymbol{S}$-action*, i.e. is compatible with the (cartesian) product functor $\boldsymbol{S} \times \boldsymbol{S} \to \boldsymbol{S}$, then the left function complexes are composable,
(ii) if the opposite $\hom^{\mathrm{op}}\colon \boldsymbol{M}^{\mathrm{op}} \times \boldsymbol{S} \to \boldsymbol{M}^{\mathrm{op}}$ of the right framing $\hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{M}$ is a cartesian $\boldsymbol{S}$-action, then the right function complexes are composable, and
(iii) if in addition the framing is self adjoint, then the adjunction induces an isomorphism between the left and the right function complexes which is compatible with the composition.

Furthermore, if the framing of $\boldsymbol{M}$ is self adjoint, then the left framing is a cartesian $\boldsymbol{S}$-action iff the opposite of the right framing is so, and in this case both these cartesian $\boldsymbol{S}$-actions are *closed cartesian $\boldsymbol{S}$-actions*.

Examples are again provided by the categories of *simplicial sets*, of *topological spaces* and of *simplicial universal algebras* with the self adjoint framing of 53.4.

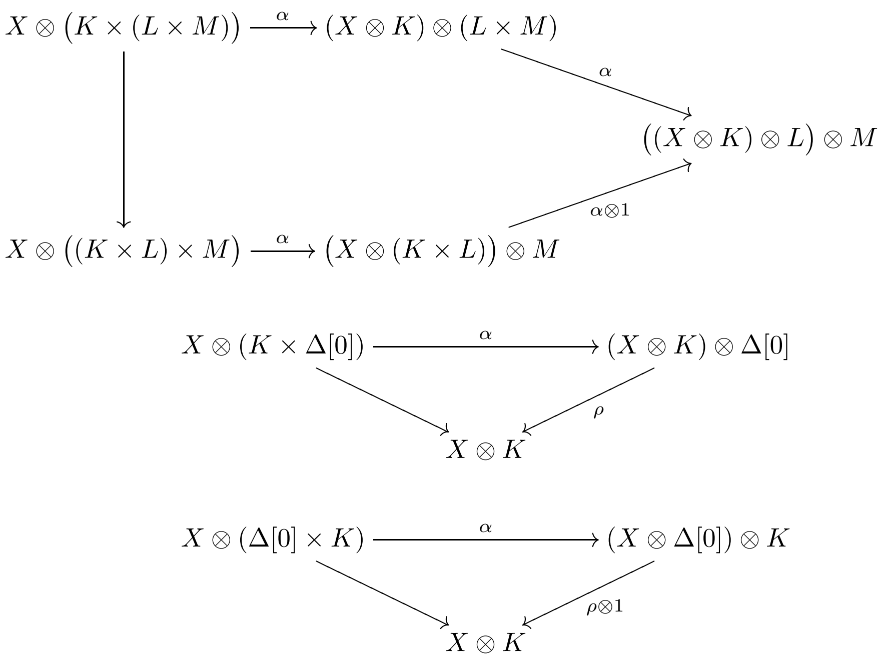
Before discussing cartesian $\boldsymbol{S}$-actions, we recall the notion of

Many $\boldsymbol{S}$-categories are obtained from an (ordinary) category $\boldsymbol{C}$ and a so-called cartesian $\boldsymbol{S}$-action thereon as follows:

54.1. Cartesian $\boldsymbol{S}$-actions. Given a category $\boldsymbol{C}$, a *cartesian $\boldsymbol{S}$-action* on $\boldsymbol{C}$ consists of a functor $\otimes\colon \boldsymbol{C}\times\boldsymbol{S}\to\boldsymbol{C}$ together with

(i) a natural isomorphism $\rho\colon X\otimes\Delta[0]\to X\in\boldsymbol{C}$ $(X\in\boldsymbol{C})$ and
(ii) a natural isomorphism $\alpha\colon X\otimes(K\times L)\to(X\otimes K)\otimes L\in\boldsymbol{C}$ $(X\in\boldsymbol{C}, K,L\in\boldsymbol{S})$

such that the following diagrams



in which the unmarked maps are the obvious ones, commute for all $X\in\boldsymbol{C}$ and $K,L,M\in\boldsymbol{S}$.

Given such a cartesian $\boldsymbol{S}$-action we next construct

54.2. The $\boldsymbol{S}$-category associated with a cartesian $\boldsymbol{S}$-action. Given a category $\boldsymbol{C}$ and a cartesian $\boldsymbol{S}$-action $\otimes\colon\boldsymbol{C}\times\boldsymbol{S}\to\boldsymbol{S}$, the *associated $\boldsymbol{S}$-category* will be the $\boldsymbol{S}$-category $\boldsymbol{C}_*$ with the objects of $\boldsymbol{C}$ as objects, in which, for every two objects $X,Y\in\boldsymbol{C}$, the simplicial set $\boldsymbol{C}_*(X,Y)$ consists of the maps $X\otimes\Delta[n]\to Y\in\boldsymbol{C}$ $(n\geq 0)$ and in which, for every three objects $X,Y,Z\in\boldsymbol{C}$, the composition of a map $f\colon X\otimes\Delta[n]\to Y\in\boldsymbol{C}$ with a map $g\colon Y\otimes\Delta[n]\to Z\in\boldsymbol{C}$ is the composition

(54.1)

$$X \otimes \Delta[n] \xrightarrow{1\otimes \mathrm{diag}} X \otimes (\Delta[n] \times \Delta[n]) \xrightarrow{\alpha} (X \otimes \Delta[n]) \otimes \Delta[n] \xrightarrow{f\otimes 1} Y \otimes \Delta[n] \xrightarrow{g} Z$$

A lengthy but straightforward calculation then yields that

(i) *$\boldsymbol{C}_*$ is indeed an $\boldsymbol{S}$-category,*
(ii) *the function $i\colon \boldsymbol{C} \to \boldsymbol{C}_*$ which is the identity on objects and which sends a map $X \to Y \in \boldsymbol{C}$ to its composition with the natural isomorphism $\rho\colon X \otimes \Delta[0] \to X \in \boldsymbol{C}$* (54.1), *is a functor which maps $\boldsymbol{C}$ isomorphically onto the 0-dimensional part of $\boldsymbol{C}_*$, and*
(iii) *the function which sends a pair of maps $X \otimes \Delta[n] \to Y \in \boldsymbol{C}$ and $K \to L \in \boldsymbol{S}$ to the composition*

$$(X \otimes K) \otimes \Delta[n] \xrightarrow{\approx} \big(X \otimes \Delta[n]\big) \otimes K \to Y \otimes L \in \boldsymbol{C}$$

is a functor $\otimes\colon \boldsymbol{C}_* \times \boldsymbol{S} \to \boldsymbol{C}_*$ which extends the $\boldsymbol{S}$-action $\otimes\colon \boldsymbol{C} \times \boldsymbol{S} \to \boldsymbol{S}$ in the sense that the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{C} \times \boldsymbol{S} & \xrightarrow{\otimes} & \boldsymbol{C} \\ \big\downarrow{\scriptstyle i\times 1} & & \big\downarrow{\scriptstyle i} \\ \boldsymbol{C}_* \times \boldsymbol{S} & \xrightarrow{\otimes} & \boldsymbol{C}_* \end{array}$$

Dually, given a functor $\hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{C} \to \boldsymbol{C}$ of which the opposite $\hom^{\mathrm{op}}\colon \boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{S} \to \boldsymbol{C}^{\mathrm{op}}$ is a cartesian $\boldsymbol{S}$-action, there is of course an associated $\boldsymbol{S}$-category $(\boldsymbol{C}^{\mathrm{op}})^{\mathrm{op}}$, in which, for every two objects $X, Y \in \boldsymbol{C}$, the simplicial set $(\boldsymbol{C}^{\mathrm{op}}_*)^{\mathrm{op}}(X,Y)$ consists of the maps $X \to \hom\big(\Delta[n], Y\big) \in \boldsymbol{C}$ $(n \geq 0)$, with similar properties.

These definitions readily imply

54.3. Proposition. *Let $\boldsymbol{M}$ be a framed model category* (53.1). *If the left framing is a cartesian $\boldsymbol{S}$-action, then, in the notation of* 53.1 *and* 54.2*,*

$$\boldsymbol{M}_{\mathrm{L}}(-,-) = \boldsymbol{M}_*(i-, i-)\colon \boldsymbol{M}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{S}$$

and dually, if the opposite of the right framing is a cartesian $\boldsymbol{S}$-action, then

$$\boldsymbol{M}_{\mathrm{R}}(-,-) = (\boldsymbol{M}^{\mathrm{op}}_*)^{\mathrm{op}}(i-, i-)\colon \boldsymbol{M}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{S}$$

If in addition the framing is self adjoint, then the adjunction induces an isomorphism $(\boldsymbol{M}_*)^{\mathrm{op}} \approx (\boldsymbol{M}^{\mathrm{op}})_*$ which is the identity on the objects.

Next we consider

54.4. Closed cartesian $\boldsymbol{S}$-actions. Given a category $\boldsymbol{C}$, a cartesian $\boldsymbol{S}$-action will be called *closed* if

(i) for every object $K \in \boldsymbol{S}$, the restriction $- \otimes K\colon \boldsymbol{C} \to \boldsymbol{C}$ has a right adjoint, which will be denoted by $\hom(K, -)\colon \boldsymbol{C} \to \boldsymbol{C}$, and
(ii) for every object $X \in \boldsymbol{C}$, the restriction $X \otimes -\colon \boldsymbol{S} \to \boldsymbol{C}$ has a right adjoint.

One then readily verifies

54.5. Proposition. *The notion of a closed cartesian $\boldsymbol{S}$-action is self dual*, i.e.

(i) if $\otimes\colon \boldsymbol{C}\times\boldsymbol{S}\to\boldsymbol{C}$ is a closed cartesian $\boldsymbol{S}$-action on $\boldsymbol{C}$, then its *dual*, the opposite $\hom^{\mathrm{op}}\colon \boldsymbol{C}^{\mathrm{op}}\times\boldsymbol{S}\to\boldsymbol{C}^{\mathrm{op}}$ of the functor $\hom\colon \boldsymbol{S}^{\mathrm{op}}\times\boldsymbol{C}\to\boldsymbol{C}$ (54.4(i)), is a closed cartesian $\boldsymbol{S}$-action on $\boldsymbol{C}^{\mathrm{op}}$,

(ii) the dual of this $\boldsymbol{S}$-action on $\boldsymbol{C}^{\mathrm{op}}$ is again the original $\boldsymbol{S}$-action on $\boldsymbol{C}$, and

(iii) the associated $\boldsymbol{S}$-categories are each isomorphic to the opposite of the other.

54.6. Corollary. *Let $\boldsymbol{M}$ be a model category with a self adjoint framing* $(\otimes,\hom)$ (53.1). *Then the following four statements are equivalent:*

(i) *the left framing $\otimes\colon \boldsymbol{M}\times\boldsymbol{S}\to\boldsymbol{M}$ is a closed cartesian $\boldsymbol{S}$-action,*

(ii) *the opposite of the right framing* $\hom^{\mathrm{op}}\colon \boldsymbol{M}^{\mathrm{op}}\times\boldsymbol{S}\to\boldsymbol{M}^{\mathrm{op}}$ *is a cartesian $\boldsymbol{S}$-action, and*

(iii) *the opposite of the right framing* $\hom^{\mathrm{op}}\colon \boldsymbol{M}^{\mathrm{op}}\times\boldsymbol{S}\to\boldsymbol{M}^{\mathrm{op}}$ *is a closed cartesian $\boldsymbol{S}$-action.*

We end with a few

54.7. Examples.

(i) The left framings and the opposites of the right framing in 53.4(i)–(iii) are all closed cartesian $\boldsymbol{S}$-actions.

(ii) For every category $\boldsymbol{C}$ with a (closed) cartesian $\boldsymbol{S}$-action $\otimes\colon \boldsymbol{C}\times\boldsymbol{S}\to\boldsymbol{C}$ and every small category $\boldsymbol{D}$, the composition (2.3)

$$\boldsymbol{C}^{\boldsymbol{D}}\times\boldsymbol{S}\xrightarrow{1\times c^*}\boldsymbol{C}^{\boldsymbol{D}}\times\boldsymbol{S}^{\boldsymbol{D}}\approx(\boldsymbol{C}\times\boldsymbol{S})^{\boldsymbol{D}}\xrightarrow{\otimes^{\boldsymbol{D}}}\boldsymbol{C}^{\boldsymbol{D}}$$

is a (closed) cartesian $\boldsymbol{S}$-action on the *diagram category* $\boldsymbol{C}^{\boldsymbol{D}}$.

(iii) More generally, for every category $\boldsymbol{C}$ with a (closed) cartesian $\boldsymbol{S}$-action and every small $\boldsymbol{S}$-category $\boldsymbol{E}$ (47.1), the composition (54.2)

$$\boldsymbol{C}_*^{\boldsymbol{E}}\times\boldsymbol{S}\xrightarrow{1\times c}\boldsymbol{C}_*^{\boldsymbol{E}}\times\boldsymbol{S}^{\boldsymbol{E}}\approx(\boldsymbol{C}_*\times\boldsymbol{S})^{\boldsymbol{E}}\xrightarrow{\otimes^{\boldsymbol{E}}}\boldsymbol{C}_*^{\boldsymbol{E}}$$

is a (closed) cartesian $\boldsymbol{S}$-action on the *simplicial diagram* category $\boldsymbol{C}^{\boldsymbol{E}}$. It is not difficult to see that an n-simplex in the simplicial hom-set between two objects U and V in the resulting $\boldsymbol{S}$-category $(\boldsymbol{C}_*^{\boldsymbol{E}})_*$ can be described as a functor $\boldsymbol{E}\times(0\xrightarrow{\Delta[n]}1)\to\boldsymbol{C}_*$ for which the restrictions to $\boldsymbol{E}\times 0$ and $\boldsymbol{E}\times 1$ are just U and V (where $(0\xrightarrow{\Delta[n]}1)$ denotes the $\boldsymbol{S}$-category with two objects 0 and 1 and $\Delta[n]$ as the simplicial set of maps from 0 to 1).

55. Simplicial model categories

In this last section we introduce *simplicial model categories*, which are model categories with some simple additional structure that ensures the existence of *functorial left and right function complexes* which are *canonically isomorphic* as well as *composable*. More precisely, we define a *simplicial model category* as a model category $\boldsymbol{M}$, together with a functor $\otimes\colon \boldsymbol{M}\times\boldsymbol{S}\to\boldsymbol{M}$ which

(i) is a *cartesian $\boldsymbol{S}$-action*, i.e. (54.1) is compatible with the (cartesian) product functor $\boldsymbol{S}\times\boldsymbol{S}\to\boldsymbol{S}$, and

(ii) is a *left Quillen functor*, i.e. (49.3) has both partial right adjoints and is in a certain precise sense compatible with the model category structures of $\boldsymbol{M}$ and $\boldsymbol{S}$, and we show that these two conditions imply that *the functor* $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ *and its partial right adjoint* $\hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{M}$ *form a self adjoint framing of* $\boldsymbol{M}$, so that (54.3) the resulting function complexes have the desired properties. We also note that for such a simplicial model category $\boldsymbol{M}$ and a small category $\boldsymbol{D}$, the *diagram category* $\boldsymbol{M}^{\boldsymbol{D}}$ inherits from $\boldsymbol{M}$ a simplicial model category structure whenever either $\boldsymbol{M}$ is *cofibrantly generated* (5.3) or $\boldsymbol{D}$ is a *Reedy category* (12.1).

Examples are again the categories of *simplicial sets*, *topological spaces* and *simplicial universal algebras* as well as their *diagram categories* and even their *simplicial diagram categories* (47.1).

55.1. Simplicial model categories. A *simplicial model category* consists of a model category $\boldsymbol{M}$ together with a functor $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ such that the following two axioms hold:

M6: $\boldsymbol{S}$*-action axiom.* The functor $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ is a cartesian $\boldsymbol{S}$-action (54.1).

M7: *Corner map axiom.* The functor $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ is a left Quillen functor (49.3).

The partial right adjoint $\boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{M}$ of the functor $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ will be denoted by $\hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{M}$.

Clearly the cartesian $\boldsymbol{S}$-action $\otimes\colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ is (49.3) a *closed* cartesian $\boldsymbol{S}$-action (54.4). Furthermore the notion of a simplicial model category is *self dual* as the opposite $\hom^{\mathrm{op}}\colon \boldsymbol{M}^{\mathrm{op}} \times \boldsymbol{S} \to \boldsymbol{M}^{\mathrm{op}}$ of the functor $\hom\colon \boldsymbol{S}^{\mathrm{op}} \times \boldsymbol{M} \to \boldsymbol{M}$ is both a left Quillen functor (49.4) and a (closed) cartesian $\boldsymbol{S}$-action (54.5).

55.2. Examples.

(i) The categories $\boldsymbol{S}$ of *simplicial sets* (6.8) and $\boldsymbol{T}$ of *compactly generated topological spaces* (6.9) and, for every category $\boldsymbol{A}$ of universal algebras (47.2) the category $\boldsymbol{A}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ of *simplicial universal algebras*, with the functors $\otimes$ and hom of 49.6 (54.7(i)).

(ii) In view of **something**, for every simplicial model category $\boldsymbol{M}$ and every *Reedy category* $\boldsymbol{B}$ (12.1), the Reedy model category structure on the diagram category $\boldsymbol{M}^{\boldsymbol{B}}$ (12.4) together with the cartesian $\boldsymbol{S}$-action of 54.7(ii) is a simplicial model category structure on $\boldsymbol{M}^{\boldsymbol{B}}$.

(iii) In view of 49.5, for every *cofibrantly generated* (5.3) simplicial model category $\boldsymbol{M}$ and every small category $\boldsymbol{D}$, the cofibrantly generated model category structure on the diagram category $\boldsymbol{M}^{\boldsymbol{D}}$ (9.6) together with the cartesian $\boldsymbol{S}$-action of 54.7(ii) is a simplicial model category structure on $\boldsymbol{M}^{\boldsymbol{D}}$.

(iv) More generally, for every cofibrantly generated simplicial model category $\boldsymbol{M}$ and every small $\boldsymbol{S}$-category $\boldsymbol{E}$ (47.1), the arguments used in proving 9.6 and 49.5 yield similar results for the *simplicial* diagram category $\boldsymbol{M}_*^{\boldsymbol{E}}$ (47.1 and 54.2) and the resulting cofibrantly generated model category structure on $\boldsymbol{M}_*^{\boldsymbol{E}}$, together with the cartesian $\boldsymbol{S}$-action of 54.7(iii) is therefore a simplicial model category structure on $\boldsymbol{M}_*^{\boldsymbol{E}}$.

Of course we still have to prove

55.3. Proposition. *Let* $\boldsymbol{M}$ *be a simplicial model category. Then the pair of functors* $(\otimes, \hom)$ *is a self adjoint framing of* $\boldsymbol{M}$*, i.e.* (53.1)

(i) *for every object* $A \in \boldsymbol{M}$, *the object* $A \otimes \Delta[-] \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ (5.8) *is a cosimplicial frame on* A (52.1) *and dually,*
(ii) *for every object* $Y \in \boldsymbol{M}$, *the object* $\hom(\Delta[-], Y) \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ *is a simplicial frame on* Y.

Proof. If A is cofibrant (or Y is fibrant) the proposition follows readily from 55.9 (ii) (or 55.9 (iv)). The rest of the proposition follows from 55.5 and 55.7 below which involve

55.4. The simplicial homotopy relation and simplicial homotopy equivalences. Given a simplicial model category $\boldsymbol{M}$ and objects $B, X \in \boldsymbol{M}$, two maps $f_0, f_1 \colon B \to X \in \boldsymbol{M}$ are said to be *simplicially homotopic* if there exists a 1-simplex $h \in \boldsymbol{M}_*(B, X)$ such that (with a slight abuse of notation) $d_0 h = f_0$ and $d_1 h = f_1$. Similarly a map $f \colon B \to X \in \boldsymbol{M}$ is called a *simplicial homotopy equivalence* if there exists a map $g \colon X \to B \in \boldsymbol{M}$ such that the compositions fg and gf are simplicially homotopic to the identity maps of X and B respectively.

Some very useful examples of simplicial homotopy equivalences are given by

55.5. Proposition. *Let* $\boldsymbol{M}$ *be a simplicial model category and let* $B, X \in \boldsymbol{M}$ *be objects. Then, for every integer* $n \geq 0$ *the obvious maps*

$$B \otimes \Delta[n] \to B \otimes \Delta[0] \approx B \quad \text{and} \quad X \approx \hom(\Delta[0], X) \to \hom(\Delta[n], X)$$

are simplicial homotopy equivalences.

Proof. A rather straightforward calculation shows that the proposition holds for the case that $\boldsymbol{M} = \boldsymbol{S}$ and $B = X = \Delta[0]$ and this readily implies the general case.

Finally we prove

55.6. Proposition. *Let* $\boldsymbol{M}$ *be a simplicial model category and let* $f_0, f_1 \colon B \to X \in \boldsymbol{M}$ *be two maps which are simplicially homotopic. Then their images in* $\operatorname{Ho}\boldsymbol{M}$ *under the canonical functor* $\boldsymbol{M} \to \operatorname{Ho}\boldsymbol{M}$ (46.1) *coincide.*

55.7. Corollary. *Let* $\boldsymbol{M}$ *be a simplicial model category and let* $f \colon B \to X \in \boldsymbol{M}$ *be a simplicial homotopy equivalence. Then* f *is a weak equivalence.*

Another immediate consequence of 55.5 and 55.6 is

55.8. Corollary. *Let* $\boldsymbol{M}$ *be a simplicial model category and let* $f_0, f_1 \colon B \to X \in \boldsymbol{M}$ *be two maps which are simplicially homotopic. Then* f_0 *and* f_1 *are both left and right homotopic* (45.1).

It thus remains to give a

Proof of 55.6. Choose weak equivalences $p \colon A \to B \in \boldsymbol{M}$ and $q \colon X \to Y \in \boldsymbol{M}$ such that A is cofibrant and Y is fibrant. Then it clearly suffices to show that the images of gf_0p and gf_1p in $\operatorname{Ho}\boldsymbol{M}$ coincide, but this is not difficult to verify, using the already proven part of 55.3.

We end with an

55.9. Alternate description of simplicial model categories. *A model category $\boldsymbol{M}$ with a closed cartesian $\boldsymbol{S}$-action $\otimes\colon \boldsymbol{M}\times\boldsymbol{S}\to\boldsymbol{M}$ (54.4) is a simplicial model category iff one and hence all of the following equivalent statements hold:*

(i) *If $f\colon A\to B\in\boldsymbol{M}$ is a cofibration, then the pushout corner maps (2.5 and 6.5)*

$$A\otimes\Delta[n]\amalg_\bullet B\otimes\dot\Delta[n]\to B\otimes\Delta[n]\in\boldsymbol{M}$$

($n\ge 0$) are cofibrations, which are trivial if f is, and the pushout corner maps

$$A\otimes\Delta[1]\amalg_\bullet B\otimes\Delta^\epsilon[1]\to B\otimes\Delta[1]\in\boldsymbol{M}$$

($\epsilon=0,1$) are trivial cofibrations.

(ii) *If $f\colon A\to B\in\boldsymbol{M}$ and $g\colon K\to L\in\boldsymbol{S}$ are cofibrations then the pushout corner map*

$$A\otimes L\amalg_\bullet B\otimes K\to B\otimes L\in\boldsymbol{M}$$

is a cofibration, which is trivial if either f or g is.

(iii) *If $f\colon A\to B\in\boldsymbol{M}$ is a cofibration and $h\colon X\to Y\in\boldsymbol{M}$ is a fibration, then the pullback corner map (2.5 and 54.2)*

$$\boldsymbol{M}_*(B,X)\to\boldsymbol{M}_*(A,X)\,\Pi_\bullet\,\boldsymbol{M}_*(B,Y)\in\boldsymbol{S}$$

is a fibration, which is trivial if either f or h is.

(iv) *If $g\colon K\to L\in\boldsymbol{S}$ is a cofibration and $h\colon X\to Y\in\boldsymbol{M}$ is a fibration, then the pullback corner map*

$$\hom(L,X)\to\hom(K,X)\,\Pi_\bullet\hom(L,Y)\in\boldsymbol{M}$$

is a fibration, which is trivial if either g or h is.

(v) *If $h\colon X\to Y\in\boldsymbol{M}$ is a fibration, then the pullback corner maps*

$$\hom(\Delta[n],X)\to\hom(\dot\Delta[n],X)\,\Pi_\bullet\hom(\Delta[n],Y)\in\boldsymbol{M}$$

($n\ge 0$) are fibrations, which are trivial if h is, and the pullback corner maps

$$\hom(\Delta[1],X)\to\hom(\Delta^\epsilon[1],X)\,\Pi_\bullet\hom(\Delta[1],Y)\in\boldsymbol{M}$$

($\epsilon=0,1$) are trivial fibrations.

Proof. The equivalence of (ii), (iii) and (iv) readily follows from 49.3 and the equivalence of (iv) and (v) can be obtained by duality from the equivalence of (i) and (ii).

It thus remains to prove the equivalence of (i) and (ii). Clearly (ii) implies (i) and the first two parts of (i) imply the first two parts of (ii). To prove the third part of (ii) or equivalently the first part of (iii), it suffices to show (6.5) that for every pair of cofibrations $f\colon A\to B\in\boldsymbol{M}$ and $g\colon K\to L\in\boldsymbol{S}$ and every fibration $h\colon X\to Y\in\boldsymbol{M}$, the pullback corner map of (iii) has the right lifting property with respect to the pushout corner maps $K\times\Delta[1]\amalg_\bullet L\times\Delta^\epsilon[1]\to L\times\Delta[1]\in\boldsymbol{S}$ ($\epsilon=0,1$) which by adjointness is equivalent to showing that the pushout corner map

$$\big(A\otimes(L\times\Delta[1])\big)\amalg_\bullet\big(B\otimes(K\times\Delta[1]\amalg_\bullet L\times\Delta^\epsilon[1])\big)\to B\otimes(L\times\Delta[1])\in\boldsymbol{M}$$

is a trivial cofibration. But this last map is isomorphic to the map

$$\big((A \otimes L \amalg_{\bullet} B \otimes K) \otimes \Delta[1]\big) \amalg_{\bullet} \big((B \otimes L) \otimes \Delta^{\epsilon}[1]\big) \to (B \otimes L) \otimes \Delta[1] \in \boldsymbol{M}$$

which, in view of the first and the last part of (i), is indeed a trivial fibration.

One final

55.10. Remark. The (cartesian) product $\boldsymbol{S} \times \boldsymbol{S} \to \boldsymbol{S}$ turns the category $\boldsymbol{S}$ into what one might call a *symmetric monoidal model category*, i.e. a model category with a symmetric monoidal structure which is compatible with the model category structure and a simplicial model category could be considered as a kind of "module" over this symmetric monoidal model category. This suggests that there might be similar useful notions such as *topological* or *spectral model categories* which would be "modules" over suitable symmetric monoidal model categories of topological spaces of spectra.

CHAPTER XIV

Homotopy colimit and limit functors

56. Introduction

56.1. Summary. Given a model category $\boldsymbol{M}$ and a small category $\boldsymbol{B}$, the functors

$$\operatorname{colim}^{\boldsymbol{B}} \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M} \quad \text{and} \quad \lim^{\boldsymbol{B}} \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

in general do *not* send objectwise weak equivalences in $\boldsymbol{M}^{\boldsymbol{B}}$ to weak equivalences in $\boldsymbol{M}$, even if one restricts oneself to objectwise weak equivalences between objectwise cofibrant or fibrant diagrams. Our aim in this chapter is to show that, given a framing (53.1) on $\boldsymbol{M}$, one can construct *homotopy variations*

$$\operatorname{hocolim}^{\boldsymbol{B}} \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M} \quad \text{and} \quad \operatorname{holim}^{\boldsymbol{B}} \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

of the functors $\operatorname{colim}^{\boldsymbol{B}}$ and $\lim^{\boldsymbol{B}}$, called *homotopy colimit and limit functors*, which do not have this problem (i.e. they *do* send objectwise weak equivalences between objectwise cofibrant or fibrant diagrams in $\boldsymbol{M}^{\boldsymbol{B}}$ to weak equivalences in $\boldsymbol{M}$) and which, for simplicial sets and topological spaces, coincide with the usual [**2**] homotopy colimit and limit functors. We of course also obtain some of their properties.

56.2. Further details. Just as colimits and limits are dual notions, so are homotopy colimits and homotopy limits and we therefore devote most of the rest of this section to some further details on homotopy colimits.

We want the homotopy colimit functor $\operatorname{hocolim}^{\boldsymbol{B}} \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ to be a *homotopy variation* of the colimit functor $\operatorname{colim}^{\boldsymbol{B}} \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$, i.e. we want these two functors

(i) to have *homotopy meaning*, i.e. to have *total left derived functors* (43.3)

$$\mathbf{L}\operatorname{hocolim}^{\boldsymbol{B}} \colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M} \quad \text{and} \quad \mathbf{L}\operatorname{colim}^{\boldsymbol{B}} \colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

where $\operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}}$ denotes the localization (42.2) of $\boldsymbol{M}^{\boldsymbol{B}}$ with respect to the objectwise weak equivalences, and

(ii) to have *the same homotopy meaning*, i.e. to come with a natural transformation $\operatorname{hocolim}^{\boldsymbol{B}} \to \operatorname{colim}^{\boldsymbol{B}}$ which induces a *natural isomorphism*

$$\mathbf{L}\operatorname{hocolim}^{\boldsymbol{B}} \approx \mathbf{L}\operatorname{colim}^{\boldsymbol{B}}.$$

Thus we first have to show that existence of the total left derived functor of $\operatorname{colim}^{\boldsymbol{B}}$, which we do by constructing a class of so-called *virtually cofibrant* diagrams in $\boldsymbol{M}^{\boldsymbol{B}}$ such that

(iii) the full subcategory $\boldsymbol{M}^{\boldsymbol{B}}_{vc} \subset \boldsymbol{M}^{\boldsymbol{B}}$ spanned by these virtually cofibrant diagrams is a *left deformation retract* (42.5) of $\boldsymbol{M}^{\boldsymbol{B}}$, and

(iv) the functor $\operatorname{colim}^{\boldsymbol{B}}$ sends objectwise weak equivalences in $\boldsymbol{M}^{\boldsymbol{B}}_{vc}$ to weak equivalences in $\boldsymbol{M}$.

It also readily follows from (iii) and (iv) that there are lots of homotopy variations of the functor $\operatorname{colim}^{\boldsymbol{B}}$ and that many of these are better behaved with respect to the objectwise weak equivalences then $\operatorname{colim}^{\boldsymbol{B}}$ itself (56.1). To obtain a particular such homotopy variation thus requires some choices.

56.3. A first choice. The obvious thing to do is to choose a *left deformation retraction* (r, s) from $\boldsymbol{M}^{\boldsymbol{B}}$ to $\boldsymbol{M}^{\boldsymbol{B}}_{vc}$ (42.5). Then the composition $\operatorname{colim}^{\boldsymbol{B}} r \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ sends *all* objectwise weak equivalences in $\boldsymbol{M}^{\boldsymbol{B}}$ to weak equivalences in $\boldsymbol{M}$. Moreover the functor $\operatorname{colim}^{\boldsymbol{B}} r$ has a total left derived functor and the natural transformation $\operatorname{colim}^{\boldsymbol{B}} s \colon \operatorname{colim}^{\boldsymbol{B}} r \to \operatorname{colim}^{\boldsymbol{B}}$ induces a natural isomorphism $\mathbf{L}(\operatorname{colim}^{\boldsymbol{B}} r) \to \mathbf{L} \operatorname{colim}^{\boldsymbol{B}}$. But a major disadvantage of choosing such a left deformation retraction is that there is *no* preferred way of doing so, even if $\boldsymbol{M}$ is a *simplicial* model category. We therefore prefer the following two-step approach which however results in a functor which sends *only* objectwise weak equivalences in $\boldsymbol{M}^{\boldsymbol{B}}$ between *objectwise cofibrant diagrams* to weak equivalences in M.

56.4. A better choice. A better choice is based on the observations that *virtually cofibrant diagrams are objectwise cofibrant* and that (42.5) *the full subcategory $\boldsymbol{M}^{\boldsymbol{B}}_c \subset \boldsymbol{M}^{\boldsymbol{B}}$ spanned by the objectwise cofibrant diagrams is a left deformation retract of $\boldsymbol{M}^{\boldsymbol{B}}$.* It thus suffices to construct a functor $r' \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{B}}$ and a natural transformation $s' \colon r' \to 1_{\boldsymbol{M}^{\boldsymbol{B}}}$ such that the restriction of the pair (r', s') to $\boldsymbol{M}^{\boldsymbol{B}}_v$ is a left deformation retraction from $\boldsymbol{M}^{\boldsymbol{B}}_v$ to construct homotopy colimit functors for diagrams in $\boldsymbol{S}$, the category of simplicial sets. That approach seems to use the *simplicial* model category structure on $\boldsymbol{S}$, but a closer examination reveals that one really needs only the resulting *left framing* (53.1). Thus the construction of homotopy colimit functors for diagrams in an arbitrary model category $\boldsymbol{M}$ requires exactly the same choice as the construction, in the previous chapter, of functorial (although not necessarily composable) left function complexes, namely the choice of a *left framing* of $\boldsymbol{M}$.

56.5. The actual construction. The above discussion suggests that we first construct the functor $r' \colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{B}}$ and then define the functor $\operatorname{hocolim}^{\boldsymbol{B}}$ as the composition $\operatorname{colim}^{\boldsymbol{B}} r'$. However the functor r' is more difficult to describe then the functor $\operatorname{hocolim}^{\boldsymbol{B}}$ itself as it turns out that, for every diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$ and object $B \in \boldsymbol{B}$, one has

$$(r'X)B = \operatorname{hocolim}^{(\boldsymbol{B}\downarrow B)} j^* X$$

where $j \colon (\boldsymbol{B} \downarrow B) \to \boldsymbol{B}$ denotes the forgetful functor.

We therefore proceed essentially as in [**2**] and, given a model category $\boldsymbol{M}$ with a *left framing* $\otimes \colon \boldsymbol{M} \times \boldsymbol{S} \to \boldsymbol{M}$ (53.1), a small category $\boldsymbol{B}$ and an object $X \in \boldsymbol{M}^{\boldsymbol{B}}$, we define $\operatorname{hocolim}^{\boldsymbol{B}} X$ directly as the object of $\boldsymbol{M}$ obtained by "attaching" to each other a copy of the object $XB_0 \otimes \Delta[n] \in \boldsymbol{M}$ for every object $B_0 \in \boldsymbol{B}$, integer $n \geq 0$ and sequence $B_0 \to \cdots \to B_n$ of maps in $\boldsymbol{B}$. If one replaces in this construction everywhere the $\Delta[n]$ $(n \geq 0)$ by $\Delta[0]$, one gets just the ordinary colimit $\operatorname{colim}^{\boldsymbol{B}} X$ and the unique maps $\Delta[n] \to \Delta[0] \in \boldsymbol{S}$ $(n \geq 0)$ thus induce a natural map $\operatorname{hocolim}^{\boldsymbol{B}} X \to \operatorname{colim}^{\boldsymbol{B}} X \in \boldsymbol{M}$. This is the desired natural transformation $\operatorname{hocolim}^{\boldsymbol{B}} \to \operatorname{colim}^{\boldsymbol{B}}$.

56.6. The relative case. Given a small functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ (2.2) there is a similar *homotopy v-colimit functor* $\operatorname{hocolim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ given by

$$(\operatorname{hocolim}^v X)D = \operatorname{hocolim}^{(c\downarrow D)} j^* X \qquad X \in \boldsymbol{M}^{\boldsymbol{B}}, D \in \boldsymbol{D}$$

where again $j\colon (v \downarrow D) \to \boldsymbol{B}$ denotes the forgetful functor. It comes with a natural transformation $\operatorname{hocolim}^v \to \operatorname{colim}^v$ induced (43.2) by the natural transformations $\operatorname{hocolim}^{(c\downarrow D)} \to \operatorname{colim}^{(v\downarrow D)}$ $(D \in \boldsymbol{D})$. One can then show that the pair

$$(\operatorname{hocolim}^{1_{\boldsymbol{B}}}, \operatorname{hocolim}^{1_{\boldsymbol{B}}} \to \operatorname{colim}^{1_{\boldsymbol{B}}} = 1_{\boldsymbol{M}^{\boldsymbol{B}}})$$

is just the pair (r', s') mentioned in 56.4.

56.7. Organization of the chapter. After fixing some terminology and notation (in 56.8) we prove (in §57 and §58) that, for every model category $\boldsymbol{M}$ and small category $\boldsymbol{B}$, the functors

$$\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M} \quad \text{and} \quad \lim^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

and, more generally, for every small functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ (2.2), the functors

$$\operatorname{colim}^{v}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}} \quad \text{and} \quad \lim^{v}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$$

have *homotopy meaning* in the sense of 56.2. Next we discuss (in §59), given a *framing* $(\otimes, \hom)$ of $\boldsymbol{M}$ (53.1) a resulting pair of functors

$$\otimes_{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \times \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}} \to \boldsymbol{M} \quad \text{and} \quad \hom_{\boldsymbol{B}}\colon (\boldsymbol{S}^{\boldsymbol{B}})^{\mathrm{op}} \times \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

which we need in §60 to efficiently describe the attaching and coattaching processes (56.5) that we use to construct homotopy colimits and limits. The relative case then is dealt with in §61, while we devote the last section (§62) to some *cofinality* results and a discussion of the *relationship between function complexes and homotopy colimits and limits.*

We end this section with a brief discussion of

56.8. Diagrams of over simplices. Given a simplicial set K, one can associate with every object $f\colon A \to K \in (\boldsymbol{S} \downarrow K)$ (2.4) a *diagram of over simplices of f* (5.9)

$$\Lambda[f] = \operatorname{colim}^{\boldsymbol{\Delta} f} (\boldsymbol{\Delta} f)^* \Delta[K] \in \boldsymbol{S}^{\boldsymbol{\Delta} K}$$

which assigns to every map $u\colon \Delta[n] \to K \in \boldsymbol{S}$ $(n \geq 0)$ the pullback $\Delta[n] \,\Pi_K\, A$ (2.5). In particular $\Lambda[1_K] = \Delta[K]$ the diagram of simplices of K. One then readily verifies that *the resulting functor*

$$\Lambda[-] = \operatorname{colim}^{\boldsymbol{\Delta} -} (\boldsymbol{\Delta} -)^* \Delta[K]\colon (\boldsymbol{S} \downarrow K) \to \boldsymbol{S}^{\boldsymbol{\Delta} K}$$

preserves cofibrations and has a right adjoint, or equivalently that *the opposite functor*

$$\Lambda^{\mathrm{op}}[-]\colon (\boldsymbol{S} \downarrow K)^{\mathrm{op}} \to (\boldsymbol{S}^{\boldsymbol{\Delta} K})^{\mathrm{op}}$$

preserves fibrations and has a left adjoint.

57. Homotopy categories of diagrams

Given a model category $\boldsymbol{M}$ and a small category $\boldsymbol{B}$, it seems that the category $\boldsymbol{M}^{\boldsymbol{B}}$ of the $\boldsymbol{B}$-diagrams in $\boldsymbol{M}$ need not admit a model category structure in which the weak equivalences are the objectwise ones, unless one imposts some restriction on either $\boldsymbol{M}$ (9.6) or $\boldsymbol{B}$ (12.4). However no such restriction is needed to prove that *the localization* $\mathrm{Ho}\,\boldsymbol{M}^{\boldsymbol{B}}$ *of* $\boldsymbol{M}^{\boldsymbol{B}}$ *with respect to the objectwise weak equivalences exists*, i.e. (42.4) has small hom-sets. We prove this by observing the equivalence of $\mathrm{Ho}\,\boldsymbol{M}^{\boldsymbol{B}}$ with the homotopy categories of certain homotopically full subcategories of the model categories of the Reedy diagrams in $\boldsymbol{M}$ (12.4) indexed by

57.1. The category of simplices $\boldsymbol{\Delta}\,\boldsymbol{B}$ of a category $\boldsymbol{B}$ and its opposite $\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}$. Given a small category $\boldsymbol{B}$, its *category of simplices* will be the category of simplices of its nerve (5.9 and 10.6), i.e. the category, denoted by $\boldsymbol{\Delta}\,\boldsymbol{B}$ instead of $\boldsymbol{\Delta}\,\mathrm{N}\boldsymbol{B}$, which has as objects the maps $\Delta[n] \to \mathrm{N}\boldsymbol{B} \in \boldsymbol{S}$ $(n \geq 0)$ and as maps the obvious commutative triangles or equivalently, has as objects the functors $\boldsymbol{n} \to \boldsymbol{B}$ $(n \geq 0)$ (10.6) and as maps $(f_1\colon \boldsymbol{n}_1 \to \boldsymbol{B}) \to (f_2\colon \boldsymbol{n}_2 \to \boldsymbol{B})$ the commutative triangles of the form

$$\begin{array}{ccc} \boldsymbol{n}_1 & \longrightarrow & \boldsymbol{n}_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & \boldsymbol{B} & \end{array}$$

and we denote by $\Delta[\boldsymbol{B}]$ instead of $\Delta[\mathrm{N}\boldsymbol{B}]$ the associated functor $\boldsymbol{\Delta}[\boldsymbol{B}] \to \boldsymbol{S}$. We also need its opposite, which we denote by $\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}$ instead of $\boldsymbol{\Delta}^{\mathrm{op}}\,\mathrm{N}\boldsymbol{B}$. Clearly $\boldsymbol{\Delta}\,\boldsymbol{0} = \boldsymbol{\Delta}$ and $\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{0} = \boldsymbol{\Delta}^{\mathrm{op}}$.

The category of simplices comes with a *terminal projection* functor $p_t\colon \boldsymbol{\Delta}\,\boldsymbol{B} \to \boldsymbol{B}$ which sends an object $(f\colon \boldsymbol{n} \to \boldsymbol{B}) \in \boldsymbol{\Delta}\,\boldsymbol{B}$ to the object $fn \in \boldsymbol{B}$ and similarly its opposite comes with an *initial projection* functor $p_i\colon \boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B} \to \boldsymbol{B}$ which sends an object $(f\colon \boldsymbol{n} \to \boldsymbol{B}) \in \boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}$ to the object $f0 \in \boldsymbol{B}$. Furthermore we denote, for every object $B \in \boldsymbol{B}$, by $p_t^{-1}B \in \boldsymbol{\Delta}\,\boldsymbol{B}$ (resp. $p_i^{-1}B \in \boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}$) the subcategory which has as objects the functors $\boldsymbol{n} \to \boldsymbol{B}$ $(n \geq 0)$ which send n (resp. 0) to B and which has as maps the commutative triangles involving only the functors $\boldsymbol{n}_1 \to \boldsymbol{n}_2$ $(n_1, n_2 \geq 0)$ which send n_1 to n_2 (resp. 0 to 0). One then readily verifies that

(i) *for every object $B \in \boldsymbol{B}$, the category $p_t^{-1}B$ has an initial object and is a terminal subcategory of $(p_t \downarrow B)$ and the category $p_i^{-1}B$ has a terminal object and is an initial subcategory of $(B \downarrow p_i)$* (10.7),

(ii) *for every functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ between small categories (e.g. the identity functor of $\boldsymbol{B}$) and every object $D \in \boldsymbol{D}$, there are obvious isomorphisms $(vp_t \downarrow D) \approx \boldsymbol{\Delta}(v \downarrow D)$ and $(D \downarrow vp_i) \approx \boldsymbol{\Delta}^{\mathrm{op}}(D \downarrow v)$, and*

(iii) *for every category $\boldsymbol{C}$* (2.5) *and every object $X \in \boldsymbol{C}^{\boldsymbol{B}}$, the adjunction maps $\mathrm{colim}^{p_t}\, p_t^* X \to X \in \boldsymbol{C}^{\boldsymbol{B}}$ and $X \to \lim^{p_i}\, p_i^* X \in \boldsymbol{C}^{\boldsymbol{B}}$ are isomorphisms.*

We also need the notions of

57.2. Restricted $\boldsymbol{\Delta}\,\boldsymbol{B}$ and $\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}$-diagrams in a model category. Given a model category $\boldsymbol{M}$ and a small category $\boldsymbol{B}$, we call a diagram $X \in \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ (resp. $\boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}}$) *restricted* if, for every object $B \in \boldsymbol{B}$ and every map $h \in p_t^{-1}B$ $(p_i^{-1}B)$ (57.1), the map $Xh \in \boldsymbol{M}$ is a weak equivalence and we denote by $\boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}_{\mathrm{res}} \subset \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$

(resp. $\boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}}_{\mathrm{res}} \subset \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}}$) the homotopically full subcategories spanned by these restricted diagrams.

Now we can state

57.3. Proposition. *For every model category $\boldsymbol{M}$ and small category $\boldsymbol{B}$ the localization* $\operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}}$ *of* $\boldsymbol{M}^{\boldsymbol{B}}$ *with respect to the objectwise weak equivalences* (42.2) *exists* (42.4)*. In fact*

(i) *the pair of adjoint functors* (57.1)

$$\operatorname{colim}^{p_t}\colon \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}} : p_t^*$$

gives rise to an adjoint pair of total derived functors

$$\mathbf{L}\operatorname{colim}^{p_t}\colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}} \leftrightarrow \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} : \mathbf{R}\,p_t^*$$

which restricts to an (inverse) pair of equivalences of categories (42.1) $\operatorname{Ho}\boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}_{\mathrm{res}} \leftrightarrow \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}}$ (57.2)*, and dually*

(ii) *the pair of adjoint functors*

$$p_i^*\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} : \lim^{p_i}$$

gives rise to an adjoint pair of total derived functors

$$\mathbf{L}\,p_i^*\colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} : \mathbf{R}\lim^{p_i}$$

which restricts to an (inverse) pair of equivalences of categories $\operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}}_{\mathrm{res}}$.

Proof. This follows readily from 43.5, 42.5 and propositions 57.4 and 57.5 below.

57.4. Proposition. *Let $\boldsymbol{M}$ be a model category and let $\boldsymbol{B}$ be a small category. Then*

(i) *the functor* $\operatorname{colim}^{p_t}\colon \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{B}}$ *sends weak equivalences between cofibrant objects* (12.4) *to objectwise weak equivalences between objectwise cofibrant diagrams, and dually*

(ii) *the functor* $\lim^{p_i}\colon \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{B}}$ *sends weak equivalences between fibrant objects to objectwise weak equivalences between objectwise fibrant diagrams.*

Proof. In view of 57.1 (ii) a cofibrant $\boldsymbol{\Delta}\,\boldsymbol{B}$-diagram in $\boldsymbol{M}$ gives, for every object $B \in \boldsymbol{B}$, rise to a $\boldsymbol{\Delta}(\boldsymbol{B}\downarrow B)$-diagram which (51.5) is also cofibrant and the desired result now follows readily from 43.2 and 12.8.

57.5. Proposition. *Let $\boldsymbol{M}$ be a model category and let $\boldsymbol{B}$ be a small category. Then*

(i) *for every cofibrant object* $Y \in \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}_{\mathrm{res}}$ *and every object* $X \in \boldsymbol{M}^{\boldsymbol{B}}$*, a map* $Y \to p_t^*X \in \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ *is a weak equivalence iff its adjoint* $\operatorname{colim}^{p_t} Y \to X \in \boldsymbol{M}^{\boldsymbol{B}}$ *is an objectwise weak equivalence, and dually*

(ii) *for every fibrant object* $Y \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}}_{\mathrm{res}}$ *and every object* $X \in \boldsymbol{M}^{\boldsymbol{B}}$*, a map* $p_i^*X \to Y \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}}$ *is a weak equivalence iff its adjoint* $X \to \lim^{p_i} Y \in \boldsymbol{M}^{\boldsymbol{B}}$ *is an objectwise weak equivalence.*

Proof. The cofibrant $\boldsymbol{\Delta}\,\boldsymbol{B}$-diagram Y gives, for every object $B \in \boldsymbol{B}$, rise to a cofibrant $\boldsymbol{\Delta}(\boldsymbol{B}\downarrow B)$-diagram (51.5) which (10.7 and 57.1 (i)) has the same colimit as its restriction to $p_t^{-1}B$. A straightforward calculation then shows that the latter diagram satisfies the conditions of 12.9 and the desired result now follows immediately from 43.2 and 57.1 (ii) and (iii).

58. Total derived functors of colim and lim

Given a model category $\boldsymbol{M}$ and a small category B, the functors

$$\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M} \quad \text{and} \quad \lim^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

do *not* in general send objectwise weak equivalences in $\boldsymbol{M}^{\boldsymbol{B}}$ to weak equivalences in $\boldsymbol{M}$, even if we restrict ourselves to objectwise weak equivalences between objectwise cofibrant or fibrant diagrams. Still these functors have some homotopy meaning in the sense that they have a total left or right derived functor. In fact we will show

58.1. Proposition. *Let $\boldsymbol{M}$ be a model category and let $\boldsymbol{B}$ be a small category. Then the pairs of adjoint functors* (2.5)

$$\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M} :c^* \quad \textit{and} \quad c^*\colon \boldsymbol{M} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}} :\lim^{\boldsymbol{B}}$$

*give rise to adjoint pairs of total derived functors (*43.3 *and* 57.3*)*

$$\mathbf{L}\operatorname{colim}^{\boldsymbol{B}}\colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \operatorname{Ho}\boldsymbol{M} :\mathbf{R}\,c^* \quad \textit{and} \quad \mathbf{L}\,c^*\colon \operatorname{Ho}\boldsymbol{M} \leftrightarrow \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} :\mathbf{R}\lim^{\boldsymbol{B}}$$

and more generally

58.2. Proposition. *Let $\boldsymbol{M}$ be a model category and let $v\colon \boldsymbol{B} \to \boldsymbol{D}$ be a small functor* (2.2)*. Then the pairs of adjoint functors* (43.2)

$$\operatorname{colim}^{v}\colon \boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \boldsymbol{M}^{\boldsymbol{D}} :v^* \quad \textit{and} \quad v^*\colon \boldsymbol{M}^{\boldsymbol{D}} \leftrightarrow \boldsymbol{M}^{\boldsymbol{B}} :\lim^{v}$$

give rise to adjoint pairs of total derived functors

$$\mathbf{L}\operatorname{colim}^{v}\colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \leftrightarrow \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{D}} :\mathbf{R}\,v^* \quad \textit{and} \quad \mathbf{L}\,v^*\colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{D}} \leftrightarrow \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} :\mathbf{R}\lim^{v}.$$

We also prove

58.3. Proposition. *Let $\boldsymbol{M}$ be a model category and let $u\colon \boldsymbol{A} \to \boldsymbol{B}$ and $v\colon \boldsymbol{B} \to \boldsymbol{D}$ be small functors* (2.2)*. Then the natural transformations* (44.2)

$$\mathbf{L}\operatorname{colim}^{v}\mathbf{L}\operatorname{colim}^{u} \xrightarrow{\bar\epsilon} \mathbf{L}\operatorname{colim}^{vu} \quad \textit{and} \quad \mathbf{R}\lim^{vu} \xrightarrow{\bar\eta} \mathbf{R}\lim^{v}\mathbf{R}\lim^{u}$$

are natural isomorphisms. Moreover, for every small functor $w\colon \boldsymbol{D} \to \boldsymbol{E}$, the following two diagrams (of functors and natural isomorphisms between them) commute.

$$\begin{array}{ccc}
\mathbf{L}\operatorname{colim}^{w}\mathbf{L}\operatorname{colim}^{v}\mathbf{L}\operatorname{colim}^{u} & \xrightarrow{\bar\epsilon\,\mathbf{L}\operatorname{colim}^{u}} & \mathbf{L}\operatorname{colim}^{wv}\mathbf{L}\operatorname{colim}^{u} \\
\Big\downarrow{\scriptstyle \mathbf{L}\operatorname{colim}^{w}\bar\epsilon} & & \Big\downarrow{\scriptstyle \bar\epsilon} \\
\mathbf{L}\operatorname{colim}^{w}\mathbf{L}\operatorname{colim}^{vu} & \xrightarrow{\bar\epsilon} & \mathbf{L}\operatorname{colim}^{wvu}
\end{array}$$

$$\begin{array}{ccc} \mathbf{R}\lim^{wvu} & \xrightarrow{\bar\eta} & \mathbf{R}\lim^{w}\mathbf{R}\lim^{vu} \\ \Big\downarrow{\scriptstyle\boldsymbol{\eta}} & & \Big\downarrow{\scriptstyle\mathbf{R}\lim^{w}\bar\eta} \\ \mathbf{R}\lim^{wv}\mathbf{R}\lim^{u} & \xrightarrow{\bar\eta\,\mathbf{R}\lim^{u}} & \mathbf{R}\lim^{w}\mathbf{R}\lim^{v}\mathbf{R}\lim^{u}\,. \end{array}$$

Proofs. To prove 58.2 (and hence 58.1) it suffices (43.5 and 42.5) to show the existence of left and right deformation retracts of $\boldsymbol{M}_c^{\boldsymbol{B}}$ and $\boldsymbol{M}_f^{\boldsymbol{B}}$ (3.4) respectively on which the functors colim^u and $\lim^u$ preserve objectwise weak equivalences. Moreover the existence of such deformation retracts also implies 58.3 (44.4).

As a first step to obtaining the just mentioned left and right deformation retracts we note the following complement of proposition 57.4.

58.4. Proposition. *Let $\boldsymbol{M}$ be a model category and let $v\colon \boldsymbol{B} \to \boldsymbol{D}$ be a small functor* (2.2). *Then*

(i) *for every two cofibrant objects $X, Y \in \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$, the functor* $\operatorname{colim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ *sends every objectwise weak equivalence* $\operatorname{colim}^{p_t} X \to \operatorname{colim}^{p_t} Y \in \boldsymbol{M}^{\boldsymbol{B}}$ *to an objectwise weak equivalence in $\boldsymbol{M}_c^{\boldsymbol{D}}$, and dually*

(ii) *for every two fibrant objects $X, Y \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}}$, the functor* $\lim^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ *sends every objectwise weak equivalence* $\lim^{p_i} X \to \lim^{p_i} Y \in \boldsymbol{M}^{\boldsymbol{B}}$ *to an objectwise weak equivalence in $\boldsymbol{M}_f^{\boldsymbol{D}}$.*

This suggests the notions of

58.5. Virtually cofibrant and fibrant diagrams. Given a model category $\boldsymbol{M}$ and a small category $\boldsymbol{B}$, we call

(i) an object $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *virtually cofibrant* if there exists an isomorphism $\operatorname{colim}^{p_t} Y \approx X \in \boldsymbol{M}^{\boldsymbol{B}}$ with $Y \in \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ cofibrant, and dually

(ii) an object $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *virtually fibrant* if there exists an isomorphism $\lim^{p_i} Y \approx X \in \boldsymbol{M}^{\boldsymbol{B}}$ with $Y \in \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ fibrant.

Clearly (57.4) *every virtually cofibrant or fibrant diagram is objectwise cofibrant or fibrant.*

If we denote by $(\boldsymbol{M}^{\boldsymbol{B}})_{vc}$ and $(\boldsymbol{M}^{\boldsymbol{B}})_{vf}$ the full subcategories of $\boldsymbol{M}^{\boldsymbol{B}}$ spanned by the virtually cofibrant and fibrant diagrams respectively, then (57.4) one can restate proposition 58.4:

58.6. Proposition. *Given a model category $\boldsymbol{M}$ and let $v\colon \boldsymbol{B} \to \boldsymbol{D}$ be a small functor* (2.2). *Then the restrictions*

$$\operatorname{colim}^v\colon (\boldsymbol{M}^{\boldsymbol{B}})_{vc} \to \boldsymbol{M}_c^{\boldsymbol{D}} \quad \textit{and} \quad \lim^v\colon (\boldsymbol{M}^{\boldsymbol{B}})_{vf} \to \boldsymbol{M}_f^{\boldsymbol{D}}$$

both preserve objectwise weak equivalences.

Finally the existence of the (see above) desired left and right deformation retracts now follows from 42.5, 58.6 and proposition 58.7 below, which is a ready consequence of 57.1 (iii) and 57.5.

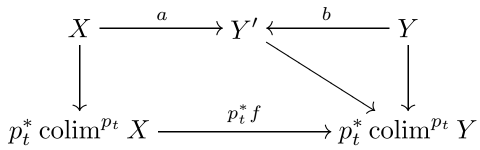
58.7. Proposition. *Given a model category* $\boldsymbol{M}$ *and a small category* $\boldsymbol{B}$*, the categories* $(\boldsymbol{M}^{\boldsymbol{B}})_{vc}$ *and* $(\boldsymbol{M}^{\boldsymbol{B}})_{vf}$ *are left and right deformation retracts of* $\boldsymbol{M}_c^{\boldsymbol{B}}$ *and* $\boldsymbol{M}_f^{\boldsymbol{B}}$ *respectively. In fact*

(i) *if* (r,s) *is a left deformation retraction* (42.5) *from* $\boldsymbol{M}_c^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ *to* $(\boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}})_c$ (3.4), *then* $(\operatorname{colim}^{p_t} rp_t^*, \operatorname{colim}^{p_t} sp_t^*)$ *is a left deformation retraction from* $\boldsymbol{M}_c^{\boldsymbol{B}}$ *to* $(\boldsymbol{M}^{\boldsymbol{B}})_{vc}$ *and dually*

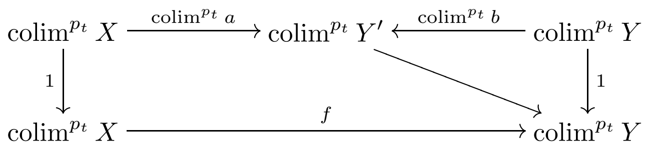
(ii) *if* (r,s) *is a right deformation retraction from* $\boldsymbol{M}_f^{\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}}$ *to* $(\boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}})_f$*, then* $(\lim^{p_i} rp_i^*, \lim^{p_i} sp_i^*)$ *is a right deformation retraction from* $\boldsymbol{M}_f^{\boldsymbol{B}}$ *to* $(\boldsymbol{M}^{\boldsymbol{B}})_{vf}$*.*

It thus remains to give a

Proof of 58.4. Given a weak equivalence $f\colon \operatorname{colim}^{p_t} X \to \operatorname{colim}^{p_t} Y \in \boldsymbol{M}^{\boldsymbol{B}}$ with $X, Y \in \boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ cofibrant, it is not difficult to form a commutative diagram in $\boldsymbol{M}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ of the form



in which b is a trivial cofibration, the slanted map is a fibration and the vertical maps are the obvious ones. Adjunction then yields a similar commutative diagram



in $\boldsymbol{M}^{\boldsymbol{B}}$ and the desired result now follows readily form 57.4 and the cofibrancy of X, Y' and Y.

59. The functors $\otimes_{\boldsymbol{B}}$ and $\hom_{\boldsymbol{B}}$

In preparation for the remaining sections of this chapter in which we construct homotopy colimit and limit functors and establish their main properties, we note in this section that a *framing* $(\otimes, \hom)$ on a model category $\boldsymbol{M}$ (53.1) induces, for every small category $\boldsymbol{B}$, a rather useful pair of functors

$$\otimes_{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \times \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}} \to \boldsymbol{M} \quad \text{and} \quad \hom_{\boldsymbol{B}}\colon (\boldsymbol{S}^{\boldsymbol{B}})^{\mathrm{op}} \times \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

with similar properties. They are defined as follows.

59.1. The functors $\otimes_{\boldsymbol{B}}$ and $\hom_{\boldsymbol{B}}$. Given a *framed model category* $\boldsymbol{M}$ (53.1) and a small category $\boldsymbol{B}$, we denote by

$$\otimes_{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \times \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}} \to \boldsymbol{M}$$

the functor which sends a pair of objects $X \in \boldsymbol{M}^{\boldsymbol{B}}$ and $K \in \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}}$ to the *coend* of the resulting object $X \otimes K \in \boldsymbol{M}^{\boldsymbol{B}\times\boldsymbol{B}^{\mathrm{op}}}$, i.e. [**12**, Ch IX] the colimit of the diagram in $\boldsymbol{M}$ which consists of

(i) for every object $B \in \boldsymbol{B}$, a copy $(X \otimes K)_B$ of $XB \otimes KB$, and

(ii) for every map $b\colon B \to B' \in \boldsymbol{B}$, a copy $(X \otimes K)_b$ of $XB \otimes KB'$ and the pair of maps

$$(X \otimes K)_B \leftarrow (X \otimes K)_b \rightarrow (X \otimes K)_{B'}$$

induced by b,

and dually we denote by

$$\hom_{\boldsymbol{B}}\colon (\boldsymbol{S}^{\boldsymbol{B}})^{\mathrm{op}} \times \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

the functor which sends a pair of objects $L \in \boldsymbol{S}^{\boldsymbol{B}}$ and $Y \in \boldsymbol{M}^{\boldsymbol{B}}$ to the *end* of the object $\hom(L,Y) \in \boldsymbol{M}^{\boldsymbol{B}^{\mathrm{op}} \times \boldsymbol{B}}$, i.e. the limit of the diagram in $\boldsymbol{M}$ which consists of

(i)′ for every object $B \in \boldsymbol{B}$, a copy $\hom(L,Y)_B$ of $\hom(: B, LY)$, and

(ii)′ for every map $b\colon B \to B' \in \boldsymbol{B}$, a copy $\hom(L,Y)_b$ of $\hom(LB, YB')$ and the pair of maps

$$\hom(L,Y)_B \rightarrow \hom(L,Y)_b \leftarrow \hom(L,Y)_{B'}$$

induced by b.

Of course one could in (ii) and (ii)′ above have restricted oneself to a *set of generators* (for the maps) of $\boldsymbol{B}$, i.e. a set of non-identity maps such that every non-identity map of $\boldsymbol{B}$ is a finite composition of maps in this set.

The functors $\otimes_{\boldsymbol{B}}$ and $\hom_{\boldsymbol{B}}$ simplify considerably if one of the variables is a constant diagram or if the diagram of simplicial sets is "freely generated by a simplicial set in a single spot". In fact one readily verifies

59.2. Proposition. *There are obvious natural isomorphisms* (2.3)

$$X \otimes_{\boldsymbol{B}} c^*H \approx \operatorname{colim}^{\boldsymbol{B}}(X \otimes H) \qquad \hom_{\boldsymbol{B}}(c^*H, Y) \approx \lim^{\boldsymbol{B}} \hom(H,Y)$$
$$c^*Z \otimes_{\boldsymbol{B}} K \approx Z \otimes \operatorname{colim}^{\boldsymbol{B}^{\mathrm{op}}} K \qquad \hom_{\boldsymbol{B}}(L, c^*Z) \approx \hom(\operatorname{colim}^{\boldsymbol{B}} L, Z)$$

where $X, Y \in \boldsymbol{M}^{\boldsymbol{B}}$*,* $Z \in \boldsymbol{M}$*,* $K \in \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}}$*,* $L \in \boldsymbol{S}^{\boldsymbol{B}}$ *and* $H \in \boldsymbol{S}$*. Moreover if the framing on* $\boldsymbol{M}$ *is self adjoint* (53.1)*, then the first line can be simplified to*

$$X \otimes_{\boldsymbol{B}} c^*H \approx \operatorname{colim}^{\boldsymbol{B}} X \otimes H \qquad \hom_{\boldsymbol{B}}(c^*H, Y) \approx \hom(H, \lim^{\boldsymbol{B}} Y).$$

59.3. Proposition. *There are obvious natural isomorphisms* (47.2)

$$X \otimes_{\boldsymbol{B}} \big(H \otimes \boldsymbol{B}^{\mathrm{op}}(B,-)\big) \approx XB \otimes H \qquad \hom_{\boldsymbol{B}}\big(H \otimes \boldsymbol{B}(B,-), Y\big) \approx \hom(H, YB)$$

where $X, Y \in \boldsymbol{M}^{\boldsymbol{B}}$*,* $H \in \boldsymbol{S}$ *and* $B \in \boldsymbol{B}$*.*

Furthermore it is not difficult to prove using 49.5 and **something**

59.4. Proposition. *If* $\boldsymbol{M}$ *is a simplicial model category* (55.1)*,* $\boldsymbol{B}$ *is a small category and either* $\boldsymbol{M}$ *is cofibrantly generated* (5.3) *or* $\boldsymbol{B}$ *is a Reedy category* (12.1)*, then (*9.6 *and* 12.4*) the functors*

$$\otimes_{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \times \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}} \to \boldsymbol{M} \quad \textit{and} \quad \hom_{\boldsymbol{B}}\colon (\boldsymbol{S}^{\boldsymbol{B}})^{\mathrm{op}} \times \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

are respectively left and right Quillen functors (49.3)*.*

However if the conditions of this proposition are not satisfied, then the functors $\otimes_{\boldsymbol{B}}$ and $\hom_{\boldsymbol{B}}$ clearly (6.10) still have the following

59.5. Partial Quillen property. *Given a framed model category $\boldsymbol{M}$ and a small category $\boldsymbol{B}$, the adjunctions of* 53.2 *induce*

(i) *for every object $X \in \boldsymbol{M}^{\boldsymbol{B}}$, a pair of adjoint functors*

$$X \otimes_{\boldsymbol{B}} -\colon \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}} \leftrightarrow \boldsymbol{M} : \boldsymbol{M}_{\mathrm{L}}(X, -)$$

which is a Quillen pair (4.1) *whenever X is objectwise cofibrant and*

(ii) *for every object $Y \in \boldsymbol{M}^{\boldsymbol{B}}$, a pair of adjoint functors*

$$\boldsymbol{M}_{\mathrm{R}}(-, Y)\colon \boldsymbol{M} \leftrightarrow (\boldsymbol{S}^{\boldsymbol{B}})^{\mathrm{op}} : \hom_{\boldsymbol{B}}(-, Y)$$

which is a Quillen pair whenever Y is objectwise cofibrant.

In view of 56.8 this implies

59.6. Corollary. *Let $\boldsymbol{M}$ be a framed model category, let K be a simplicial set and let $\boldsymbol{B}$ be a Reedy category* (12.1)*. Then*

(i) *for every objectwise cofibrant diagram $X \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}} K}$, the composition* (56.8)

$$(\boldsymbol{S} \downarrow K) \xrightarrow{\Lambda[-]} \boldsymbol{S}^{\boldsymbol{\Delta} K} \xrightarrow{X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} K} -} \boldsymbol{M}$$

preserves cofibrations and has a right adjoint and therefore preserves the Reedy cofibrancy of $\boldsymbol{B}$-diagrams, and dually

(ii) *for every objectwise fibrant diagram $Y \in \boldsymbol{M}^{\boldsymbol{\Delta} K}$, the composition*

$$(\boldsymbol{S} \downarrow K)^{\mathrm{op}} \xrightarrow{\Lambda^{\mathrm{op}}[-]} (\boldsymbol{S}^{\boldsymbol{\Delta} K})^{\mathrm{op}} \xrightarrow{\hom(-, Y)} \boldsymbol{M}$$

preserves fibrations and has a left adjoint and therefore preserves the Reedy fibrancy of $\boldsymbol{B}$-diagrams.

Furthermore it is not difficult to see (59.3) that homotopy property 53.3 generalizes to

59.7. Homotopy property. *If $\boldsymbol{M}$ is a framed model category and $\boldsymbol{B}$ is a small category, then*

(i) *for every cofibrant diagram $K \in \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}}$* (6.10)*, the functor*

$$- \otimes_{\boldsymbol{B}} K\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

preserves objectwise weak equivalences between objectwise cofibrant diagrams.

Next we note that 59.5 readily implies

59.8. Proposition. *Let $\boldsymbol{M}$ be a framed model category and let $u\colon \boldsymbol{A} \to \boldsymbol{B}$ be a small functor* (2.2)*. Then there are obvious natural isomorphisms*

$$u^* X \otimes_{\boldsymbol{A}} K' \approx X \otimes_{\boldsymbol{B}} \operatorname{colim}^{u^{\mathrm{op}}} K' \quad \text{and} \quad \hom_{\boldsymbol{B}}(\operatorname{colim}^{u} L', Y) \approx \hom_{\boldsymbol{A}}(L', u^* Y)$$

where $X, Y \in \boldsymbol{M}^{\boldsymbol{B}}$, $K' \in \boldsymbol{S}^{\boldsymbol{A}^{\mathrm{op}}}$ and $L' \in \boldsymbol{S}^{\boldsymbol{A}}$.

This in turn can be used to describe the

59.9. Naturality in $\boldsymbol{B}$. Let $\boldsymbol{B}$ be a framed model category and let $u\colon \boldsymbol{A} \to \boldsymbol{B}$ be a small functor (2.2). Then it is not difficult to see that (59.1) u induces, for every pair of objects $X \in \boldsymbol{M}^{\boldsymbol{B}}$ and $K \in \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}}$, a *natural map* $u^*X \otimes_{\boldsymbol{A}} (u^{\mathrm{op}})^*K \to X \otimes_{\boldsymbol{B}} K \in \boldsymbol{M}$ which admits a *natural factorization*

$$u^*X \otimes_{\boldsymbol{A}} (u^{\mathrm{op}})^*K \to X \otimes_{\boldsymbol{B}} \operatorname{colim}^{u^{\mathrm{op}}} (u^{\mathrm{op}})^*K \to X \otimes_{\boldsymbol{B}} K$$

in which the first map is as in 59.7 and the second is induced by the adjointness of $\operatorname{colim}^{u^{\mathrm{op}}}$ and $(u^{\mathrm{op}})^*$, and dually, for every pair of objects $L \in \boldsymbol{S}^{\boldsymbol{B}}$ and $Y \in \boldsymbol{M}^{\boldsymbol{B}}$, a *natural map* $\hom_{\boldsymbol{B}}(L, Y) \to \hom_{\boldsymbol{A}}(u^*L, u^*Y) \in \boldsymbol{M}$ which admits a similar *natural factorization*

$$\hom_{\boldsymbol{B}}(L, Y) \to \hom_{\boldsymbol{B}}(\operatorname{colim}^{u} u^*L, Y) \xrightarrow{\approx} \hom_{\boldsymbol{A}}(u^*L, u^*Y).$$

As an application we prove the following variation on 59.6.

59.10. Proposition. *Let $\boldsymbol{M}$ be a framed model category and let $K \in \boldsymbol{S}$. Then*

(i) *for every objectwise cofibrant diagram of weak equivalences* (50.6) $X \in \boldsymbol{M}^{\boldsymbol{\Delta}^{\mathrm{op}} K}$*, the composition* 59.6

$$(\boldsymbol{S} \downarrow K) \xrightarrow{\Lambda[-]} \boldsymbol{S}^{\boldsymbol{\Delta} K} \xrightarrow{X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} K} -} \boldsymbol{M}$$

is a left Quillen functor (4.1)*, and dually*

(ii) *for every objectwise fibrant diagram of weak equivalences $Y \in \boldsymbol{M}^{\boldsymbol{\Delta} K}$, the composition*

$$(\boldsymbol{S} \downarrow K)^{\mathrm{op}} \xrightarrow{\Lambda^{\mathrm{op}}[-]} (\boldsymbol{S}^{\boldsymbol{\Delta} K})^{\mathrm{op}} \xrightarrow{\hom_{\boldsymbol{K}}(-, Y)} \boldsymbol{M}$$

is a right Quillen functor.

Proof. In view of 59.6 it suffices to show that the functor $X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} K} \Lambda[-]$ preserves trivial cofibrations. To do this we note that 59.9, 59.7, 59.2 and 53.1 together with the fact that (57.1) $\boldsymbol{\Delta}\, \Delta[n] \approx \boldsymbol{\Delta}\, \boldsymbol{n}$ $(n \ge 0)$ has as terminal object the identity functor $1_{\boldsymbol{n}}\colon \boldsymbol{n} \to \boldsymbol{n}$, imply that, for every map $u\colon \Delta[n] \to K \in \boldsymbol{S}$, there are natural isomorphisms and weak equivalences in $\boldsymbol{M}$

$$X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} K} \Lambda[-] \approx u^*X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} \boldsymbol{n}} \Delta[n] \leftarrow i^*(X1_{\boldsymbol{n}}) \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} \boldsymbol{n}} \Delta[n] \approx X1_{\boldsymbol{n}} \otimes \Delta[n] \to X1_{\boldsymbol{n}}.$$

Consequently, for every map $q\colon \Delta[k] \to \Delta[n] \in \boldsymbol{S}$ $(k \ge 0)$ the induced map

$$X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} K} \Lambda[uq] \to X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}} K} \Lambda[u] \in \boldsymbol{M}$$

is a weak equivalence and the desired result now readily follows by using once again 59.6.

We end with a few examples.

59.11. The geometric realization of a simplicial set. Every simplicial set K can be considered as a simplicial discrete topological space, i.e. an object if $\boldsymbol{T}^{\boldsymbol{\Delta}^{\mathrm{op}}}$, and it is not difficult to verify that *the geometric realization $|K|$ of K* (6.1) *is naturally isomorphic to $K \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}} \Delta[-]$.*

59.12. The diagonal of a bisimplicial set. *The diagonal of a bisimplicial set* (i.e. a simplicial simplicial set) *is naturally isomorphic to* $X \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta[-]$ (5.8). To see this let, for every pair (k,n) of non-negative integers, $\Delta[k,n]$ denote the bisimplicial set freely generated by one element in bidegree (k,n). Then a straightforward calculation yields that

(i) there are obvious isomorphisms $\operatorname{diag} \Delta[k,n] \approx \Delta[k,n] \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta[-] \in \boldsymbol{S}$ $(k,n \geq 0)$ which are compatible with the maps between the $\Delta[k,n]$'s, and

(ii) every bisimplicial set X can canonically be written as the colimit of a diagram of $\Delta[k,n]$'s and maps between them, which contains a copy of $\Delta[k,n]$ for every element of X of bidegree (k,n). The desired result now follows immediately from the fact that the functors $\operatorname{diag}\colon \boldsymbol{S}^{\mathbf{\Delta}^{\mathrm{op}}} \to \boldsymbol{S}$ and $- \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta[-]\colon \boldsymbol{S}^{\mathbf{\Delta}^{\mathrm{op}}} \to \boldsymbol{S}$ both preserve colimits.

59.13. Function complexes in $\boldsymbol{S}^{\boldsymbol{B}}$. Given a small category $\boldsymbol{B}$ and diagrams $X, Y \in \boldsymbol{S}^{\boldsymbol{B}}$, *the function complex* (54.2 and 55.2)

$$\hom(X,Y) \approx (\boldsymbol{S}^{\boldsymbol{B}})(X,Y)$$

is naturally isomorphic to $\hom_{\boldsymbol{B}}(X,Y)$.

60. The functors $\operatorname{hocolim}^{\boldsymbol{B}}$ and $\operatorname{holim}^{\boldsymbol{B}}$

Given a *framed* (53.1) model category $\boldsymbol{M}$ and a small category $\boldsymbol{B}$, we now use the functors $\otimes_{\boldsymbol{B}}$ and $\hom_{\boldsymbol{B}}$ of the previous section to define *homotopy colimit* and *limit functors*, i.e. homotopy variations of the functors $\operatorname{colim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ and $\lim^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ which are better behaved with respect to the weak equivalences. More precisely we construct *functors*

$$\operatorname{hocolim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M} \quad \text{and} \quad \operatorname{holim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$$

and *natural transformations*

$$\operatorname{hocolim}^{\boldsymbol{B}} \to \operatorname{colim}^{\boldsymbol{B}} \quad \text{and} \quad \lim^{\boldsymbol{B}} \to \operatorname{holim}^{\boldsymbol{B}}$$

such that

(i) *the functors* $\operatorname{hocolim}^{\boldsymbol{B}}$ *and* $\operatorname{holim}^{\boldsymbol{B}}$ *send objectwise weak equivalences between objectwise cofibrant or fibrant diagrams to weak equivalences in* $\boldsymbol{M}$*, and*

(ii) *the natural transformations* $\operatorname{hocolim}^{\boldsymbol{B}} \to \operatorname{colim}^{\boldsymbol{B}}$ *and* $\lim^{\boldsymbol{B}} \to \operatorname{holim}^{\boldsymbol{B}}$ *send virtually cofibrant or fibrant diagrams* (58.5) *to weak equivalences in* $\boldsymbol{M}$*.*

From this we then readily deduce that

(iii) *the total derived functors* $\mathbf{L}\operatorname{hocolim}^{\boldsymbol{B}}$ *and* $\mathbf{R}\operatorname{holim}^{\boldsymbol{B}}$ *exist and are naturally isomorphic to the total derived functors* $\mathbf{L}\operatorname{colim}^{\boldsymbol{B}}$ *and* $\mathbf{R}\lim^{\boldsymbol{B}}$*, and*

(iv) *a different choice of framing of* $\boldsymbol{M}$ *would, at least on objectwise cofibrant or fibrant diagrams, have produced naturally weakly equivalent homotopy colimit and limit functors.*

We start with the

60.1. Definitions. Given a *framed* (53.1) model category $\boldsymbol{M}$, a small category $\boldsymbol{B}$ and a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$, the *homotopy colimit* of X will be the object of $\boldsymbol{M}$, denoted by $\operatorname{hocolim}^{\boldsymbol{B}} X$ or $\operatorname{hocolim} X$, which is obtained from the initial object of $\boldsymbol{M}$ by successively "attaching", for every "non-degenerate" functor $f\colon \boldsymbol{n} \to \boldsymbol{B}$, a copy of the object $X(f0) \otimes \Delta[n]$. More precisely and compactly (57.1 and 59.1)

$$\operatorname{hocolim}^{\boldsymbol{B}} X = p_i^* X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} \Delta[\boldsymbol{B}].$$

Similarly the *homotopy limit* of X will be the object of $\boldsymbol{M}$, denoted by $\operatorname{holim}^{\boldsymbol{B}} X$ or $\operatorname{holim} X$, obtained by a dual "coattaching" process, i.e.

$$\operatorname{holim}^{\boldsymbol{B}} X = \hom_{\boldsymbol{\Delta}\boldsymbol{B}}(\Delta[\boldsymbol{B}], p_t^* X).$$

These definitions are *natural* in X and the resulting functors come with *natural transformations*

$$\operatorname{hocolim}^{\boldsymbol{B}} \to \operatorname{colim}^{\boldsymbol{B}} \quad \text{and} \quad \lim^{\boldsymbol{B}} \to \operatorname{holim}^{\boldsymbol{B}}$$

which send an object $X \in \boldsymbol{M}^{\boldsymbol{B}}$ to the compositions (57.1 and 59.2)

$$p_i^* X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} \Delta[\boldsymbol{B}] \to p_i^* X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} c^*\Delta[0] \approx \operatorname{colim}^{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} p_i^* X \approx \operatorname{colim}^{\boldsymbol{B}} X$$
$$\lim^{\boldsymbol{B}} X \approx \lim^{\boldsymbol{\Delta}\boldsymbol{B}} p_t^* X \approx \hom_{\boldsymbol{\Delta}\boldsymbol{B}}(c^*\Delta[0], X) \to \hom_{\boldsymbol{\Delta}\boldsymbol{B}}(\Delta\boldsymbol{B}, X)$$

where the arrow indicates the map induced by the (unique) map $\Delta[\boldsymbol{B}] \to c^*\Delta[0] \in \boldsymbol{S}^{\boldsymbol{\Delta}\boldsymbol{B}}$ (2.3).

It is often convenient to do the above attaching or coattaching process in two steps, namely by first only attaching or coattaching to each other the objects of the form $XB \otimes \Delta[n]$ or $\hom(\Delta[n], XB)$ involving the same object $B \in \boldsymbol{B}$, which results in the formation of the objects

$$XB \otimes \mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow B) \quad \text{and} \quad \hom\bigl(\mathrm{N}(\boldsymbol{B} \downarrow B), XB\bigr)$$

respectively. In view of 59.8 this yields the following

60.2. Alternate description. *Given a framed model category $\boldsymbol{M}$, a small category $\boldsymbol{B}$ and a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$, there are natural commutative diagrams*

$$\begin{array}{ccc} \operatorname{hocolim}^{\boldsymbol{B}} X & \searrow & \\ \approx\downarrow & & \operatorname{colim}^{\boldsymbol{B}} X \\ X \otimes_{\boldsymbol{B}} \mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) & \nearrow & \end{array} \qquad \begin{array}{ccc} & \nearrow & \hom_{\boldsymbol{B}}\bigl(\mathrm{N}(\boldsymbol{B} \downarrow -), X\bigr) \\ \lim^{\boldsymbol{B}} X & & \downarrow\approx \\ & \searrow & \operatorname{holim}^{\boldsymbol{B}} X \end{array}$$

in which the vertical isomorphisms are as in 59.8*, the downward slanting maps are as in* 60.1 *and the upward slanting maps are induced by the (unique) maps* $\mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) \to c^*\Delta[0] \in \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}}$ *and* $\mathrm{N}(\boldsymbol{B} \downarrow -) \to c^*\Delta[0] \in \boldsymbol{S}^{\boldsymbol{B}}$ (2.3).

Furthermore 59.8 and 59.9 imply

60.3. Naturality in $\boldsymbol{B}$. *Given a framed model category $\boldsymbol{M}$, a small functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ (2.2) and a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$, there are natural commutative squares*

$$\begin{array}{ccc} \operatorname{hocolim}^{\boldsymbol{A}} u^*X & \longrightarrow & \operatorname{hocolim}^{\boldsymbol{B}} X \\ \approx\downarrow & & \downarrow\approx \\ X \otimes_{\boldsymbol{B}} \mathrm{N}(u^{\mathrm{op}} \downarrow -) & \longrightarrow & X \otimes_{\boldsymbol{A}} \mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) \end{array} \qquad \begin{array}{ccc} \hom_{\boldsymbol{B}}\big(\mathrm{N}(\boldsymbol{B} \downarrow -), X\big) & \longrightarrow & \hom_{\boldsymbol{A}}\big(\mathrm{N}(u \downarrow -), X\big) \\ \approx\downarrow & & \downarrow\approx \\ \operatorname{holim}^{\boldsymbol{B}} X & \longrightarrow & \operatorname{holim}^{\boldsymbol{A}} u^*X \end{array}$$

in which the vertical maps are as in 59.8*, the upper left and lower right maps are as in* 59.9 *and the other two maps are induced by u^{op} and u.*

60.4. The case $\boldsymbol{B} = \boldsymbol{n}$. If $\boldsymbol{B} = \boldsymbol{n}$ (10.6), then one readily verifies that

(i) *for every objectwise cofibrant diagram $X \in \boldsymbol{M}^{\boldsymbol{n}}$, the natural map* $\operatorname{hocolim}^{\boldsymbol{n}} X \to \operatorname{colim}^{\boldsymbol{n}} X \approx Xn \in \boldsymbol{M}$ *is a weak equivalence between cofibrant objects*, and dually

(ii) *for every objectwise fibrant diagram $X \in \boldsymbol{M}^{\boldsymbol{n}}$, the natural map* $X0 \approx \lim^{\boldsymbol{n}} X \to \operatorname{holim}^{\boldsymbol{n}} X \in \boldsymbol{M}$ *is a weak equivalence between fibrant objects.*

60.5. The case $\boldsymbol{M} = \boldsymbol{S}$. If $\boldsymbol{M} = \boldsymbol{S}$ with the self adjoint framing of 53.3, then the functors $\operatorname{hocolim}^{\boldsymbol{B}}$ and $\operatorname{holim}^{\boldsymbol{B}}$ clearly coincide with the usual [**2**, Ch. XI and Ch. XII] homotopy colimit and limit functors. In particular, for every small category $\boldsymbol{B}$ and object $K \in \boldsymbol{S}$, there is an obvious natural isomorphism (10.6 and 59.2)

$$\operatorname{hocolim}^{\boldsymbol{B}} c^*K \approx K \times \mathrm{N}\boldsymbol{B}$$

while, for every object $X \in \boldsymbol{S}^{\boldsymbol{B}}$, one has (59.13)

$$\operatorname{holim}^{\boldsymbol{B}} X \approx \hom\big(\mathrm{N}(\boldsymbol{B} \downarrow -), X\big) = (\boldsymbol{S}^{\boldsymbol{B}})_*\big(\mathrm{N}(\boldsymbol{B} \downarrow -), X\big).$$

To prove that the functors $\operatorname{hocolim}^{\boldsymbol{B}}$ and $\operatorname{holim}^{\boldsymbol{B}}$ and the natural transformations $\operatorname{hocolim}^{\boldsymbol{B}} \to \operatorname{colim}^{\boldsymbol{B}}$ and $\lim^{\boldsymbol{B}} \to \operatorname{holim}^{\boldsymbol{B}}$ have properties (i) and (ii) mentioned at the beginning of this section, one notes that there is another two step approach to the attaching and coattaching processes of 60.1, namely by first only attaching to each other those objects $X(f0) \otimes \Delta[n]$ which involve the same object $fn \in \boldsymbol{B}$ or coattaching to each other the objects $\hom\big(\Delta[n], fn\big)$ involving the same object $f0 \in \boldsymbol{B}$. This results in the rather useful

60.6. Description of $\operatorname{hocolim}^{\boldsymbol{B}}$ and $\operatorname{holim}^{\boldsymbol{B}}$ in terms of $\operatorname{colim}^{\boldsymbol{B}}$ and $\lim^{\boldsymbol{B}}$. *Given a framed model category $\boldsymbol{M}$, a small category $\boldsymbol{B}$ and a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$, the isomorphisms of* 59.8 *give rise to natural isomorphisms*

$$\begin{aligned} \operatorname{hocolim}^{\boldsymbol{B}} X &\approx \operatorname{colim}^{\boldsymbol{\Delta}\,\boldsymbol{B}} (\operatorname{hocolim} u^*X)_{u \in \boldsymbol{\Delta}\,\boldsymbol{B}} \\ &\approx \operatorname{colim}^{\boldsymbol{B}} \operatorname{colim}^{p_t} (\operatorname{hocolim} u^*X)_{u \in \boldsymbol{\Delta}\,\boldsymbol{B}} \\ \operatorname{holim}^{\boldsymbol{A}} X &\approx \lim^{\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}} (\operatorname{holim} u^*X)_{u \in \boldsymbol{\Delta}\,\boldsymbol{B}} \\ &\approx \lim^{\boldsymbol{B}} \lim^{p_i} (\operatorname{holim} u^*X)_{u \in \boldsymbol{\Delta}\,\boldsymbol{B}}. \end{aligned}$$

Proof. As $\Delta[\boldsymbol{B}] \approx \operatorname{colim}^{\boldsymbol{\Delta}\,\boldsymbol{B}}\big(\Lambda[u]\big)_{u\in\boldsymbol{\Delta}\,\boldsymbol{B}}$ (56.8), this follows readily from 59.6 and the fact that (59.8), for every object $u \in \boldsymbol{\Delta}\,\boldsymbol{B}$,

$$\operatorname{hocolim} u^*X \approx p_i^*X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}\,\boldsymbol{B}} \Lambda[u] \quad \text{and} \quad \operatorname{holim} u^*X \approx \hom_{\boldsymbol{\Delta}\,\boldsymbol{B}}\big(\Lambda[u], p_t^*X\big).$$

The usefulness of this description of the homotopy colimit and limit functors is due to

60.7. Proposition. *Let $\boldsymbol{M}$ be a framed model category, let $\boldsymbol{B}$ be a small category and let $X \in \boldsymbol{M}^{\boldsymbol{B}}$.*

(i) *If X is objectwise cofibrant, then the $\boldsymbol{B}$-diagram* $\operatorname{colim}^{p_t}(\operatorname{hocolim} u^*X)_{u\in\boldsymbol{\Delta}\,\boldsymbol{B}}$ *is virtually cofibrant* (58.5) *and the natural map*

$$\operatorname{colim}^{p_t}(\operatorname{hocolim} u^*X)_{u\in\boldsymbol{\Delta}\,\boldsymbol{B}} \to \operatorname{colim}^{p_t}(\operatorname{colim} u^*X)_{u\in\boldsymbol{\Delta}\,\boldsymbol{B}} \approx X \in \boldsymbol{M}^{\boldsymbol{B}}$$

is an objectwise weak equivalence, and dually

(ii) *if X is objectwise fibrant, then the $\boldsymbol{B}$-diagram* $\lim^{p_i}(\operatorname{holim} u^*X)_{u\in\boldsymbol{\Delta}\,\boldsymbol{B}}$ *is virtually fibrant and the natural map*

$$X \approx \lim^{p_i}(\lim u^*X)_{u\in\boldsymbol{\Delta}\,\boldsymbol{B}} \to \lim^{p_i}(\operatorname{holim} u^*X)_{u\in\boldsymbol{\Delta}\,\boldsymbol{B}} \in \boldsymbol{M}^{\boldsymbol{B}}$$

is an objectwise weak equivalence.

Proof. This follows from 60.4, 59.6 and the fact that the diagram $\Delta[\boldsymbol{B}] \in \boldsymbol{S}^{\boldsymbol{\Delta}\,\boldsymbol{B}}$ is Reedy cofibrant.

60.8. Corollary. *Let $\boldsymbol{M}$ be a framed model category and let $\boldsymbol{B}$ be a small category. Then*

(i) *the functor* $\operatorname{hocolim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ *sends objectwise weak equivalences between objectwise cofibrant diagrams to weak equivalences between cofibrant objects, and dually*

(ii) *the functor* $\operatorname{holim}^{\boldsymbol{B}}\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}$ *sends objectwise weak equivalences between objectwise fibrant diagrams to weak equivalences between fibrant objects,*

which implies the existence (43.3) *of the total derived functors*

$$\mathbf{L}\operatorname{hocolim}^{\boldsymbol{B}}\colon \operatorname{Ho}(\boldsymbol{M}^{\boldsymbol{B}}) \to \operatorname{Ho}\boldsymbol{M} \quad \textit{and} \quad \mathbf{R}\operatorname{holim}^{\boldsymbol{B}}\colon \operatorname{Ho}(\boldsymbol{M}^{\boldsymbol{B}}) \to \operatorname{Ho}\boldsymbol{M}.$$

60.9. Corollary. *Let $\boldsymbol{M}$ be a framed model category and let $\boldsymbol{B}$ be a small category. Then the natural transformations*

$$\operatorname{hocolim}^{\boldsymbol{B}} \to \operatorname{colim}^{\boldsymbol{B}} \quad \textit{and} \quad \lim^{\boldsymbol{B}} \to \operatorname{holim}^{\boldsymbol{B}}$$

send virtually cofibrant and virtually fibrant diagrams respectively to weak equivalences between cofibrant or fibrant objects in $\boldsymbol{M}$ and hence these natural transformations induce natural isomorphisms

$$\mathbf{L}\operatorname{hocolim}^{\boldsymbol{B}} \approx \mathbf{L}\operatorname{colim}^{\boldsymbol{B}} \quad \textit{and} \quad \mathbf{R}\lim^{\boldsymbol{B}} \approx \mathbf{R}\operatorname{holim}^{\boldsymbol{B}}.$$

And combining this with 53.1 one gets

60.10. Corollary. *Every map between two framings of a model category $\boldsymbol{M}$ induces natural transformations between the resulting homotopy colimit and limit functors, which send objectwise cofibrant or fibrant diagrams to weak equivalences in $\boldsymbol{M}$.*

61. The functors $\operatorname{hocolim}^v$ and holim^v

We now relativize the results of the previous section, i.e., given a framed model category $\boldsymbol{M}$ (53.1) and a small functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ (2.2), we construct *homotopy v-colimit* and *v-limit functors*

$$\operatorname{hocolim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}} \quad \text{and} \quad \operatorname{holim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$$

and natural transformations

$$\operatorname{hocolim}^v \to \operatorname{colim}^v \quad \text{and} \quad \lim^v \to \operatorname{holim}^v$$

with properties like the ones mentioned at the beginning of §60. We also note that

(i) there are natural isomorphisms

$$\operatorname{hocolim}^v \approx \operatorname{colim}^v \operatorname{hocolim}^{1_{\boldsymbol{B}}} \quad \text{and} \quad \operatorname{holim}^v \approx \lim^v \operatorname{holim}^{1_{\boldsymbol{B}}}$$

and that

(ii) the pairs, consisting of a functor and a natural transformation

$$(\operatorname{hocolim}^{1_{\boldsymbol{B}}}, \operatorname{hocolim}^{1_{\boldsymbol{B}}} \to \operatorname{colim}^{1_{\boldsymbol{B}}} = 1_{\boldsymbol{M}^{\boldsymbol{B}}}) \quad \text{and} \quad (\operatorname{holim}^{1_{\boldsymbol{B}}}, 1_{\boldsymbol{M}^{\boldsymbol{B}}} = \lim^{1_{\boldsymbol{B}}} \to \operatorname{holim}^{1_{\boldsymbol{B}}})$$

restrict to left and right deformation retractions (42.5) from $\boldsymbol{M}^{\boldsymbol{B}}_c$ to $(\boldsymbol{M}^{\boldsymbol{B}})_{vc}$ and $\boldsymbol{M}^{\boldsymbol{B}}_f$ to $(\boldsymbol{M}^{\boldsymbol{B}})_{vf}$ (3.4 and 58.5) respectively.

We start with defining

61.1. The functors $\operatorname{hocolim}^v$ and holim^v and the natural transformations $\operatorname{hocolim}^v \to \operatorname{colim}^v$ and $\lim^v \to \operatorname{holim}^v$. Given a framed model category $\boldsymbol{M}$ (53.1) and a small functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$ (2.2), we denote by $\operatorname{hocolim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ the functor which sends a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$ to its *homotopy u-colimit*, i.e. the $\boldsymbol{D}$-diagram given by

$$(\operatorname{hocolim}^v X)D = \operatorname{hocolim}^{(v\downarrow D)} j^*X \qquad (D \in \boldsymbol{D})$$

(where $j\colon (v\downarrow D) \to \boldsymbol{B}$ denotes the forgetful functor) and dually we denote by $\operatorname{holim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ the functor which sends a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$ to its *homotopy v-limit* given by

$$(\operatorname{holim}^v X)D = \operatorname{holim}^{(D\downarrow v)} j^*X \qquad (D \in \boldsymbol{D}).$$

These functors come with natural transformations

$$\operatorname{hocolim}^v \to \operatorname{colim}^v \quad \text{and} \quad \lim^v \to \operatorname{holim}^v$$

induced (43.2) by the natural transformations hocolim → colim and lim → holim of 60.1.

A simple calculation using 57.1 (ii) and 60.6 yields the following

61.2. Description of $\operatorname{hocolim}^v$ **and** holim^v **in terms of** colim^v **and** $\lim^v$**.** *Given a framed model category* $\boldsymbol{M}$*, a small functor* $v\colon \boldsymbol{B} \to \boldsymbol{D}$ *and a diagram* $X \in \boldsymbol{M}^{\boldsymbol{B}}$*, the isomorphisms of* 60.6 *give rise to natural isomorphisms*

$$\operatorname{hocolim}^v X \approx \operatorname{colim}^v \operatorname{colim}^{p_t} (\operatorname{hocolim} u^* X)_{u \in \boldsymbol{\Delta}\, \boldsymbol{B}}$$

and

$$\operatorname{holim}^v X \approx \lim^v \lim^{p_i} (\operatorname{holim} u^* X)_{u \in \boldsymbol{\Delta}\, \boldsymbol{B}}.$$

One now readily verifies the analogs of 60.8—60.10, in particular

61.3. Proposition. *Let* $\boldsymbol{M}$ *be a framed model category and let* $v\colon \boldsymbol{B} \to \boldsymbol{D}$ *be a small functor. Then*

(i) *the functor* $\operatorname{hocolim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ *preserves objectwise weak equivalences between objectwise cofibrant diagrams, and dually*
(ii) *the functor* $\operatorname{holim}^v\colon \boldsymbol{M}^{\boldsymbol{B}} \to \boldsymbol{M}^{\boldsymbol{D}}$ *preserves objectwise weak equivalences between objectwise fibrant diagrams.*

which implies the existence (43.3) *of the total derived functors*

$$\mathbf{L}\operatorname{hocolim}^v\colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \to \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{D}} \quad \text{and} \quad \mathbf{R}\operatorname{holim}^v\colon \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{B}} \to \operatorname{Ho}\boldsymbol{M}^{\boldsymbol{D}}.$$

61.4. Proposition. *Let* $\boldsymbol{M}$ *be a framed model category and let* $v\colon \boldsymbol{B} \to \boldsymbol{D}$ *be a small functor. Then the natural transformations*

$$\operatorname{hocolim}^v \to \operatorname{colim}^v \quad \text{and} \quad \lim^v \to \operatorname{holim}^v$$

send virtually cofibrant and virtually fibrant diagrams (58.5) *respectively to objectwise weak equivalences between objectwise cofibrant or fibrant diagrams and hence these natural transformations induce natural isomorphisms*

$$\mathbf{L}\operatorname{hocolim}^v \approx \mathbf{L}\operatorname{colim}^v \quad \text{and} \quad \mathbf{R}\lim^v \approx \mathbf{R}\operatorname{holim}^v.$$

We end with observing that 60.7, 61.2, 44.4 and 46.6 also imply the following results.

61.5. Proposition. *Let* $\boldsymbol{M}$ *be a framed model category and let* $u\colon \boldsymbol{A} \to \boldsymbol{B}$ *and* $v\colon \boldsymbol{B} \to \boldsymbol{D}$ *be small functors* (2.2)*. Then the natural isomorphisms of* 61.2 *induce*

(i) *a natural isomorphism*

$$\operatorname{colim}^v \operatorname{hocolim}^u \approx \operatorname{hocolim}^{vu}$$

for which the resulting composite natural transformation

$$\operatorname{hocolim}^v \operatorname{hocolim}^u \to \operatorname{colim}^v \operatorname{hocolim}^u \approx \operatorname{hocolim}^{vu}$$

sends objectwise cofibrant diagrams to objectwise weak equivalences between objectwise cofibrant diagrams, and dually

(ii) *a natural isomorphism*

$$\operatorname{holim}^{vu} \approx \lim^{v} \operatorname{holim}^{u}$$

for which the resulting composite natural transformation

$$\operatorname{holim}^{vu} \approx \lim^{v} \operatorname{holim}^{u} \to \operatorname{holim}^{v} \operatorname{holim}^{u}$$

sends objectwise fibrant diagrams to objectwise weak equivalences between objectwise fibrant diagrams.

61.6. Corollary. *Given a framed model category* $\boldsymbol{M}$ *and a small functor* $v\colon \boldsymbol{B} \to \boldsymbol{D}$, *there are natural isomorphisms*

$$\operatorname{hocolim}^{v} \approx \operatorname{colim}^{v} \operatorname{hocolim}^{1_{\boldsymbol{B}}} \quad \textit{and} \quad \operatorname{holim}^{v} \approx \lim^{v} \operatorname{holim}^{1_{\boldsymbol{B}}} .$$

61.7. Proposition. *Given a framed model category* $\boldsymbol{M}$ *and a small category* $\boldsymbol{B}$, *the pairs, consisting of a functor and a natural transformation,*

$$(\operatorname{hocolim}^{1_{\boldsymbol{B}}}, \operatorname{hocolim}^{1_{\boldsymbol{B}}} \to \operatorname{colim}^{1_{\boldsymbol{B}}} = 1_{\boldsymbol{M}^{\boldsymbol{B}}}) \quad \textit{and} \quad (\operatorname{holim}^{1_{\boldsymbol{B}}}, 1_{\boldsymbol{M}^{\boldsymbol{B}}} \approx \lim^{1_{\boldsymbol{B}}} \to \operatorname{holim}^{1_{\boldsymbol{B}}})$$

restrict to left and right deformation retractions (42.5) *from* $\boldsymbol{M}^{\boldsymbol{B}}_{c}$ *to* $(\boldsymbol{M}^{\boldsymbol{B}})_{vc}$ *and from* $\boldsymbol{M}^{\boldsymbol{B}}_{f}$ *to* $(\boldsymbol{M}^{\boldsymbol{B}})_{vf}$ *(*3.4 *and* 58.5*) respectively.*

62. Further properties of the functors hocolim and holim

In this section we

(i) show that, for a *simplicial* model category (55.1) the homotopy colimit and limit functors commute *on the nose* with the function complexes and that for a merely *framed* model category (53.1) the still do so *up to homotopy*, and

(ii) obtain two *cofinality* results which, given a framed model category $\boldsymbol{M}$, a small functor (2.2) $u\colon \boldsymbol{A} \to \boldsymbol{B}$ and a diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$, describe sufficient conditions on u and X in order that, for every small functor $v\colon \boldsymbol{B} \to \boldsymbol{D}$, the induced map

$$\operatorname{hocolim}^{vu} u^{*}X \to \operatorname{hocolim}^{v} X \quad \text{or} \quad \operatorname{holim}^{v} X \to \operatorname{holim}^{vu} u^{*}X$$

be an objectwise weak equivalence in $\boldsymbol{M}^{\boldsymbol{D}}$.

We start with a precise formulation of (i) in the following two propositions.

62.1. Proposition. *Let* $\boldsymbol{M}$ *be a simplicial model category* (55.1) *and let* $v\colon \boldsymbol{B} \to \boldsymbol{D}$ *be a small functor* (2.2). *Then the simplicial model category structure induces, for every pair of objects* $Y \in \boldsymbol{M}$ *and* $X^{\bullet} \in \boldsymbol{M}^{\boldsymbol{B}}$, *isomorphisms* (54.2) *in* $\boldsymbol{S}^{\boldsymbol{D}}$

$$\boldsymbol{M}_{*}(\operatorname{hocolim}^{v} X^{\bullet}, Y) \approx \operatorname{holim}^{v^{\mathrm{op}}} \boldsymbol{M}_{*}(X^{\bullet}, Y) \quad \textit{and} \quad \boldsymbol{M}_{*}(Y, \operatorname{holim}^{v} X^{\bullet}) \approx \operatorname{holim}^{v} \boldsymbol{M}_{*}(Y, X^{\bullet})$$

which are natural in both variables.

62.2. Proposition. *Let $\boldsymbol{M}$ be a framed model category* (53.1) *and let* $v\colon \boldsymbol{B} \to \boldsymbol{D}$ *be a small functor. Then*

(i) *for every pair of objects $Y \in \boldsymbol{M}$ and $X^\bullet \in \boldsymbol{M}^{\boldsymbol{B}}$ with Y fibrant and $X^\bullet$ objectwise cofibrant, the $\boldsymbol{D}$-diagrams in $\boldsymbol{S}$* (53.1)

$$\boldsymbol{M}_{\mathrm{L}}(\operatorname{hocolim}^v X^\bullet, Y) \quad \textit{and} \quad \operatorname{holim}^{v^{\mathrm{op}}} \boldsymbol{M}_{\mathrm{L}}(X^\bullet, Y)$$

are naturally objectwise weakly equivalent, and dually

(ii) *so are, for every pair of objects $Y \in \boldsymbol{M}$ and $X^\bullet \in \boldsymbol{M}^{\boldsymbol{B}}$ with Y cofibrant and $X^\bullet$ objectwise fibrant, the $\boldsymbol{D}$-diagrams in $\boldsymbol{S}$*

$$\boldsymbol{M}_{\mathrm{R}}(Y, \operatorname{holim}^v X^\bullet) \quad \textit{and} \quad \operatorname{holim}^v \boldsymbol{M}_{\mathrm{R}}(Y, X^\bullet).$$

Proof of 62.1 *and* 62.2. It clearly (61.1) suffices to prove (the first halves of) 62.1 and 62.2 for the case that $\boldsymbol{D} = \boldsymbol{0}$ (10.6). In this case (53.1 and 60.2) there are, for $X^\bullet \in \boldsymbol{M}^{\boldsymbol{B}}$ and $Y \in \boldsymbol{M}$, natural isomorphisms

$$\boldsymbol{M}_{\mathrm{L}}(\operatorname{hocolim}^{\boldsymbol{B}} X^\bullet, Y) \approx \boldsymbol{M}\big((X^\bullet \otimes_{\boldsymbol{B}} \mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -)) \otimes \Delta[-], Y\big)$$

and (55.2, 59.3 and 60.5)

$$\begin{aligned}\operatorname{holim}^{v^{\mathrm{op}}} \boldsymbol{M}_{\mathrm{L}}(X^\bullet, Y) &\approx \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}}\big(\mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) \times \Delta[-], \boldsymbol{M}_{\mathrm{L}}(X^\bullet, Y)\big)\\ &\approx \boldsymbol{M}\big(X^\bullet \otimes_{\boldsymbol{B}} (\mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) \times \Delta[-]), Y\big).\end{aligned}$$

If $\boldsymbol{M}$ is a simplicial model category, then

$$\big(X^\bullet \otimes_{\boldsymbol{B}} \mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -)\big) \otimes \Delta[-] \quad \text{and} \quad X^\bullet \otimes_{\boldsymbol{B}} \big(\mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) \times \Delta[-]\big)$$

are canonically isomorphic, which implies 62.1, and if $\boldsymbol{M}$ is merely a framed model category, then the objectwise cofibrancy of $X^\bullet$ implies (59.3) that these two cosimplicial objects are (52.1) weakly equivalent frames on the cofibrant object $\operatorname{hocolim}^{\boldsymbol{B}} X^\bullet \approx X^\bullet \otimes_{\boldsymbol{B}} \mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) \in \boldsymbol{M}$, so that the desired result follows from the fibrancy of Y and 52.4.

The remainder of this section is devoted to the two cofinality results mentioned in (ii) above. In the first of these, apart form of course assuming that X be objectwise cofibrant or fibrant, we require the rather *strong* property of being initial or terminal (10.7), while the second involves the *weaker* requirement that (the nerve of) u be a weak equivalence, for which we however have to compensate by assuming that all maps in X are weak equivalences in $\boldsymbol{M}$. More precisely

62.3. Strong cofinality result. *Let $\boldsymbol{M}$ be a framed model category* (53.1) *and let $v\colon \boldsymbol{B} \to \boldsymbol{D}$ be a small functor* (2.2). *Then*

(i) *for every small functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ which is terminal* (10.7) *and every objectwise cofibrant diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$, the induced map*

$$\operatorname{hocolim}^{vu} u^* X \to \operatorname{hocolim}^v X \in \boldsymbol{M}^{\boldsymbol{D}}$$

is an objectwise weak equivalence, and dually

(ii) *so is, for every small functor $u\colon \boldsymbol{A} \to \boldsymbol{B}$ which is initial and every objectwise fibrant diagram $X \in \boldsymbol{M}^{\boldsymbol{B}}$, the induced map*

$$\operatorname{holim}^v X \to \operatorname{holim}^{vu} u^* X \in \boldsymbol{M}^{\boldsymbol{D}}.$$

62.4. Corollary. *Let* $u\colon \boldsymbol{A} \to \boldsymbol{B}$ *be a small functor which is initial or terminal. Then* (60.5) *its nerve* $\mathrm{N}u\colon \mathrm{N}\boldsymbol{A} \to \mathrm{N}\boldsymbol{B} \in \boldsymbol{S}$ *is a weak equivalence.*

Proof. Again it suffices (61.1) to prove (the first half of) the case that $\boldsymbol{D} = \mathbf{0}$ (10.6), but this case is an immediate consequence of 59.3, 60.3 and the fact that the induced map $\mathrm{N}(u^{\mathrm{op}} \downarrow -) \to \mathrm{N}(\boldsymbol{B}^{\mathrm{op}} \downarrow -) \in \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}}$ is a weak equivalence between cofibrant (6.10) objects.

It remains to state and prove the

62.5. Weak cofinality result [4]**.** *Let* $\boldsymbol{M}$ *be a framed model category, let* $u\colon \boldsymbol{A} \to \boldsymbol{B}$ *and* $v\colon \boldsymbol{B} \to \boldsymbol{D}$ *be small functors and assume that the map* $\mathrm{N}u\colon \mathrm{N}\boldsymbol{A} \to \mathrm{N}\boldsymbol{B} \in \boldsymbol{S}$ *is a weak equivalence. Then*

(i) *for every objectwise diagram* $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *of weak equivalences* (50.6)*, the induced map*

$$\operatorname{hocolim}^{vu} u^* X \to \operatorname{hocolim}^{v} X \in \boldsymbol{M}^{\boldsymbol{D}}$$

is an objectwise weak equivalence, and dually

(ii) *so is, for every objectwise fibrant diagram* $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *of weak equivalences, the induced map*

$$\operatorname{holim}^{v} X \to \operatorname{holim}^{vu} u^* X \in \boldsymbol{M}^{\boldsymbol{D}}.$$

62.6. Corollary. *Let* $\boldsymbol{M}$ *be a framed model category and let* $\boldsymbol{B}$ *be a small category which is contractible* (10.6)*. Then*

(i) *for every object* $B \in \boldsymbol{B}$ *and every objectwise cofibrant diagram* $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *of weak equivalences, the obvious map* $XB \to \operatorname{hocolim}^{\boldsymbol{B}} X \in \boldsymbol{M}$ *is a weak equivalence and dually*

(ii) *so is, for every object* $B \in \boldsymbol{B}$ *and every objectwise fibrant diagram* $X \in \boldsymbol{M}^{\boldsymbol{B}}$ *of weak equivalences, the obvious map* $\operatorname{holim}^{\boldsymbol{B}} X \to XB \in \boldsymbol{M}$*.*

Proof. Again it suffices (61.1) to prove (the first half of) the case that $\boldsymbol{D} = \mathbf{0}$ (10.6), i.e. that the induced map $\operatorname{hocolim}^{\boldsymbol{A}} u^* X \to \operatorname{hocolim}^{\boldsymbol{B}} X \in \boldsymbol{M}$ is a weak equivalence or equivalently that the induced map (56.8)

$$p_i^* X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} \Lambda[n] \to p_i^* X \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}\boldsymbol{B}} \Delta[\boldsymbol{B}] \in \boldsymbol{M}$$

is so. But this we already proved in 59.10.

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