FIBRING SPACES OF MAPS

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1. *Introduction*

This article summarizes briefly the role played by fibre spaces in the structure theory of various spaces and manifolds of maps. We consider both topological and differential aspects. In particular, we are concerned with conditions sufficient to insure (1) that certain maps satisfying the covering homotopy property are locally trivial; (2) that certain differentiable maps which foliate a differentiable manifold actually define a locally trivial fibration; (3) that certain locally trivial fibrations are (or are not) in fact fibre bundles associated with continuous topological (or Lie) group actions.

Special attention is drawn to three important classes of applications: (1) The evaluation map fibrations of Serre and Borsuk, in their topological framework. (2) The fibrations arising in the embedding and immersion theory of Cerf and Hirsch-Smale. (3) Applications (due to Earle-Eells) to Teichmüller theory, which produce new methods and results in the global theory of deformations of complex structures (of complex dimension 1).

Throughout, our notation and terminology will conform to that of [11] – to which we refer also for background theory.

2. *Fibrations and fibre bundles*

Let us begin with a review of the topological theory of fibre bundles; the basic references here are Cartan [2], Dold [6], Ehresmann [12], Steenrod [27]. Our viewpoint is more that of [2], [12] than of [27]. We state three fundamental theorems on which the topological theory of fibre bundles is based.

(A) Let E be a topological space and G a topological group *acting continuously on the right of* E; i.e., we have a continuous map $E \times G \to E$ written $(x,g) \to x \cdot g$, such that $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$ and $x \cdot 1 = x$ for all $x \in E$, $g_1, g_2 \in G$, where 1 denotes the neutral element of G. We let $B = E/G$ denote the orbit space and $p: E \to B$ the surjective map assigning to each $x \in E$ its orbit $p(x) \in B$ under the G-action; we give B the quotient topology (i.e., the largest topology for which p is continuous). Suppose that G acts *freely* (i.e., we have $x \cdot g = x$ only when $g = 1$); then setting $\Delta = \{(x,x') \in E \times E$: there is a (unique) $g \in G$ for which $x' = x \cdot g\}$, we have a map $\theta: \Delta \to G$ defined by $\theta(x, x \cdot g) = g$, and we say that the G-action is *proper* if θ is continuous. Finally, if every point $b \in B$ has a neighborhood V_b on which there is a continuous section ζ_b (i.e., a continuous map $\zeta_b : V_b \to E$ such that $p \cdot \zeta_b$ is the identity map on V_b), then we say that the G-action is *locally trivial*. A *principal right* G*-action* is one which is continuous, free, proper, and locally trivial. We then say that $p : E \to B$ is a *principal fibre bundle with structural group* G; or a G-*bundle*, for short.

Next, suppose that F is a topological space on which G acts continuously on the left. Then G acts principally on $E \times F$ by $(x,f) \cdot g = (x \cdot g, g^{-1} f)$; we let $W = E \times_g F$ denote its orbit space. There is a natural surjective continuous map $q: W \to B$, and we call it a *fibre bundle with structural group* G *and fibre model* F, *associated to the* G-*bundle* $p: E \to B$; or a (G,F)-*bundle*, for short.

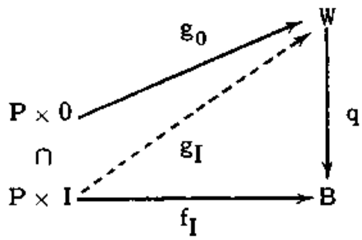
(B) A continuous surjective map $q: W \to B$ of topological spaces is *locally trivial with fibre* F if every $b \in B$ has a neighborhood V over which there is a homeomorphism θ such that the diagram

$$\begin{array}{ccc} q^{-1}(V) & \xrightarrow{\ \theta\ } & V \times F \\ & {\scriptstyle q} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & V & \end{array}$$

is commutative, where π is the indicated projection on the first factor. We will say that a locally trivial map $q:W \to B$ *defines a fibration of* W. Every (G,F)-bundle defines a fibration.

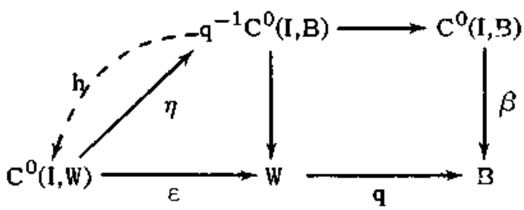
Henceforth assume that all spaces are Hausdorff.

COVERING HOMOTOPY THEOREM. *Let* $q:W \to B$ *be a fibration of* W *over the paracompact space* B. *Let* P *be a space and* $f_I:P\times I \to B$ *a continuous map. (Here* $I = [0,1]$.) *If* $g_0:P \to W$ *is a map such that* $q \cdot g_0 = f_0$, *then there is an extension* $g_I:P\times I \to W$ *such that* $q \cdot g_I = f_I$:



REMARK. An alternative reading of that theorem is valid, requiring P to be paracompact and allowing B to be arbitrary.

A beautiful proof (Hurewicz [18]) of the covering homotopy theorem is based on the existence of a Hurewicz connection: Let $C^0(I,B)$ denote the mapping space of all continuous maps of I into B, with the compact-open topology. Let $q^{-1}C^0(I,B) = \{(x,\omega)\ W\times C^0(I,B):q(x) = \omega(0)\}$:



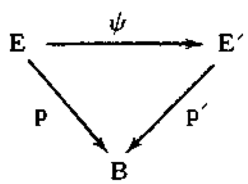
The Serre map $\beta:C^0(I,B) \to B$ is given by $\beta(\omega) = \omega(0)$; similarly for $\varepsilon:C^0(I,W) \to W$. The map η is defined by $\eta(\phi) = (\phi(0), q\cdot\phi)$. A *Hurewicz connection* for $q:W \to B$ is a continuous section h of the map η; it provides a coherent lifting of paths from B to W, with prescribed initial point. It is easy to show that the covering homotopy theorem is valid for all spaces P if and only if $q:W \to B$ admits a Hurewicz connection. Such a map is called a *Hurewicz fibration.*

(C) A map $q:W \to B$ has the *local section extension property* if every point of B has a neighborhood V such that any section of q defined on a closed subset A contained in V has an extension to a neighborhood of A in V.

SECTION EXTENSION THEOREM. *Let* $q:W \to B$ *be a fibration of* W *over the paracompact space* B, *and suppose that* q *has the local section extension property. Assume that the fibre* F *is contractible. If* B_0 *is a closed subspace of* B *and* s_0 *a section of* q *over* B_0, *then there is an extension to a section* s *over all* B.

(D) If G is a topological group, then a G-bundle $p_G:E_G \to B_G$ is G-*universal* if the space E_G is contractible. It is known [23, 6] that every topological group admits a G-universal bundle, which is sufficiently unique for topological classification purposes.

An *isomorphism* of G-bundles over B



is a homeomorphism ψ which is equivariant (i.e., $\psi(x\cdot g) = \psi(x)\cdot g$ for all $x \in E$, $g \in G$) and which induces the identity map on B.

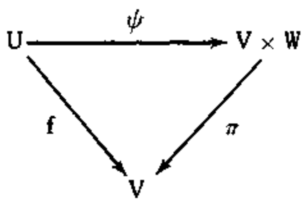
CLASSIFICATION THEOREM. *Let* G *be a topological group and* $p_G:E_G \to B_G$ *a universal* G-*bundle. Let* B *be a paracompact space. Then the isomorphism classes of* G-*bundles over* B *are in natural bijective correspondence with the homotopy classes of maps* $B \to B_G$.

The proof depends on both the covering homotopy theorem and the section extension theorem.

3. *Foliations and fibrations*

(A) Let X and Y be paracompact C^1-manifolds modeled on Banach spaces, and $f:X \to Y$ a surjective C^1-map. Suppose that for each $x \in X$ its differential $f_*(x):X(x) \to Y(f(x))$ is surjective and that its kernel $\operatorname{Ker} f_*(x)$ is a direct summand of $X(x)$. We will then say that f *foliates* X. The components of the sets $f^{-1}(y)$ (for each $y \in Y$) are called the *leaves* of the foliation. The following assertion – which is a consequence of the inverse function theorem – insures that the leaves of a foliation are closed C^1-submanifolds of X:

For each $x \in X$ there are neighborhoods U of x and V of $f(x)$, an open set W in a Banach space isomorphic to $\operatorname{Ker} f_*(x)$, and a C^1-diffeomorphism $\psi:U \to V \times W$ such that the following diagram is commutative:



where again π denotes projection on the first factor. An example of Douady [7, Ch. I, §8] shows that the direct summand condition on $\operatorname{Ker} f_*(x)$ is essential for the foliation of X by f.

Let $\operatorname{Ker} f_* = \cup\{\operatorname{Ker} f_*(x):x \in X\}$. Then $\operatorname{Ker} f_*$ is an integrable subbundle of the tangent vector bundle $T(X)$ of X, as in [11, §4F], and we have the following exact sequence

$$0 \to \operatorname{Ker} f_* \to T(X) \xrightarrow{f_*} f^{-1}T(Y) \to 0 . \tag{1}$$

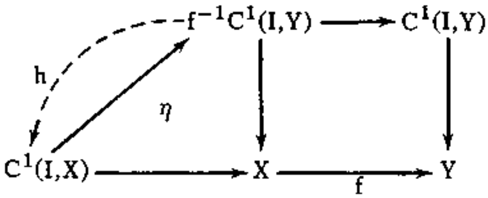
(B) Let α and β be Finsler structures for X and Y, and suppose that X is complete in the metric induced from α. Let s be a locally Lipschitz splitting of the sequence (1), viewed as a map $s:f^{-1}T(Y) \to T(X)$ which is bounded and linear on the fibres, and such that $f_* \cdot s = 1$. We will say that s *is bounded locally over* Y if for each $y_0 \in Y$ there is a number $\eta_0 > 0$ and a neighborhood V_0 of y_0 such that $\|s(x)\|_x \le \eta_0$ for all $x \in f^{-1}(V_0)$, where

$$\|s(x)\|_x = \sup\{\alpha_x(s(x)v)/\beta_{f(x)}(v) : v \neq 0 \ \text{ in } \ Y(f(x))\} .$$

The following result is the main object of [9]:

THEOREM. *Let* (X,α), (Y,β) *be Finsler* C^1*-manifolds modeled on Banach spaces, and suppose that* (X,α) *is complete. Let* $f:X \to Y$ *be a surjective* C^1*-map which foliates* X. *If there is a locally Lipschitz splitting of the sequence* (1) *which is bounded locally over* Y, *then* $f:Y \to Y$ *is a locally* C^0*-trivial fibration.*

The idea of the proof is to use the splitting and the fundamental theorem of ordinary differential equations in Banach manifolds to construct a sort of Hurewicz connection h:



EXAMPLE (Ehresmann [12]). If $f:X \to Y$ foliates X and is proper (i.e., the inverse image of every compact subset of Y is compact in X), then $f:X \to Y$ is a locally C^0-trivial fibration.

EXAMPLE (Hermann [16]). Let X be a separable complete Riemannian C^k-manifold $(1 \leq k \leq \infty)$ and $f:X \to Y$ a surjective C^k-map foliating X. For every $x \in X$ we let $K_x^{\perp}$ denote the orthogonal complement of $\operatorname{Ker} f_*(x)$. If each $f_*(x)|K_x^{\perp} \to Y(f(x))$ is an isometry, then $f:X \to Y$ is a locally C^k-trivial fibration. (The theorem admits an easy modification (applicable to this example) to take into account higher differentiability.)

(C) Let (X,α) be a Finsler C^1-manifold and G an abstract group of C^1-diffeomorphisms which are furthermore isometries of the Finsler structure α. Suppose that the G-orbits foliate X. If $Y = X/G$ is the orbit space and $f:X \to Y$ the orbit map, then the quotient topology on Y is given by the metric

$$\tau(y,y') = \inf\{\sigma(x,x') : x \in f^{-1}(y),\ x' \in f^{-1}(y')\}\ ,$$

where σ is the metric on X induced from α. Then Y is itself a Finsler C^1-manifold, and for each $y \in Y$ the quotient Finsler structure β_y is defined as the quotient norm.

The next result is an application of the theorem in §3B.

PROPOSITION. *Let* (X,α) *be a complete Finsler* C^1*-manifold, and* G *a group of* C^1*-diffeomorphisms and isometries of* X. *Suppose that the* G*-orbits foliate* X. *Then the orbit map* $f: X \to Y = X/G$ *is a locally* C^0 *-trivial fibration.*

The following illustration is a reformation and specialization of results in [8]:

EXAMPLE. Let X be the open unit disc in $L^\infty(U;C)$, where U is the upper half plane. Let α be the Finsler structure which assigns to each $\mu \in X$ the norm $\nu \to \alpha_\mu(\nu)$ given by

$$\alpha_\mu(\nu) = 2\left\| \frac{\nu}{1-|\mu|^2} \right\|_\infty$$

on $L^\infty(U;C)$. It is easily verified that α is a complete Finsler structure. Let G denote the group of quasi-conformal maps of U onto itself leaving the real axis pointwise fixed. Then G acts as a group of diffeomorphisms of the manifold X and as a group of isometries of the Finsler structure. Furthermore, G foliates X, and each leaf is a complex analytic submanifold. Therefore $f: X \to X/G = Y$ is locally C^0-trivial. (See [8] and [11, §5G] for an explicit formula for f and for its differential.) The quotient space is the *universal Teichmüller space* (for complex dimension one). A similar construction can be made for the Teichmüller space $T(\Gamma)$ of any Fuchsian group Γ.

It is natural to ask whether G can be given a topology so that its action is principal. The answer is no, because then each fibre would be homeomorphic to G, and it is known that G is not a topological group with the induced topology of $L^\infty(U;C)$.

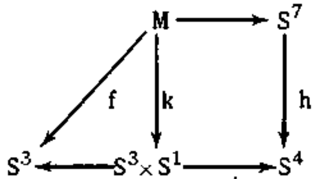
4. *Various examples of fibre bundles*

(A) Let $p:E \to B$ be a locally trivial fibration with locally compact, locally connected fibres F. If we let $\mathcal{C}(F)$ denote the group of all homeomorphisms of F on itself, with the compact-open topology, then $\mathcal{C}(F)$ is a topological group and the evaluation map $\mathcal{C}(F) \times F \to F$ defined by $(g,z) \to g(z)$ is continuous [1, Theorem 4]. In fact, the compact-open topology on $\mathcal{C}(F)$ is the smallest topology for which $\mathcal{C}(F)$ is a topological transformation group of F. Thus *we can view* $p:E \to B$ *as a* $(\mathcal{C}(F), F)$-*bundle.*

Similarly, let $f:X \to Y$ be a locally C^∞-trivial fibration of C^∞-manifolds with finite dimensional fibres with model F. If $\mathfrak{D}(F)$ denotes the group of C^∞-diffeomorphisms of F, with the compact-open topology on differentials of all orders, then $\mathfrak{D}(F)$ is a topological group and the evaluation map $\mathfrak{D}(F) \times F \to F$ is continuous. Thus *we can view* $f:X \to Y$ *as a* $(\mathfrak{D}(F), F)$-*bundle.*

There are examples of such bundles having no finite dimensional Lie structural group G; *in particular,* $\mathfrak{D}(F)$ *will then have no Lie group as deformation retract.* The following construction has been used for several different purposes [28]:

EXAMPLE (Serre). Let $h:S^7 \to S^4$ be the indicated Hopf fibration; that is an analytic principal $Sp(1) = S^3$-bundle. Let $\phi:S^3 \times S^1 \to S^4$ be a smooth map of degree 1, and $k:M \to S^3 \times S^1$ the induced $Sp(1)$-bundle:



If π denotes projection on the first factor, then the composition $f = \pi \cdot k$ is a locally C^∞-trivial fibration (being a composition of foliations of compact C^∞-manifolds). It is easily seen that its fibre is $S^3 \times S^1$, so that

$f:M \to S^3$ is a $(\mathfrak{D}(S^3 \times S^1), S^3 \times S^1)$-bundle. On the other hand, for any topological group G the isomorphism classes of G-bundles over S^3 are classified [27, §18] by the $\pi_0(G)$-classes of $\pi_2(G)$. For any finite dimensional Lie group G a theorem of E. Cartan asserts that $\pi_2(G) = 0$, so that any (G,F)-bundle over S^3 is trivial. However, $f:M \to S^3$ is not trivial, for (using the Gysin sequence) M is not cohomologically the product of S^3 and $S^3 \times S^1$.

(B) EXAMPLE [10]. Let S be a compact oriented C^∞-manifold of dimension 2, without boundary and of genus $(S) = g \geq 2$. Let $\mathfrak{D}_0(S)$ be the identity component of $\mathfrak{D}(S)$. If M(S) denotes the manifold of all C^∞-complex structures on S, with its C^∞-topology, then $\mathfrak{D}_0(S)$ *operates principally on* M(S), *and the orbit space* $T(S) = M(S)/\mathfrak{D}_0(S)$ *is canonically identified with the Teichmüller space of* S. In fact, if we represent $S = U/\Gamma$ as in the example of §3C, then $T(S) = T(\Gamma)$. Furthermore, since M(S) is contractible, we see that $M(S) \to T(S)$ *is a universal bundle for the topological group* $\mathfrak{D}_0(S)$. It is a theorem of Teichmüller that T(S) is a (3g-3)-$\dim_{\mathbb{C}}$ cell (it is moreover known to be a Stein manifold), so that *every* $\mathfrak{D}_0(S)$*-bundle over a paracompact space is topologically trivial.* Another corollary of this construction (with simple modifications to take care of the exceptional cases $g = 0$, $g = 1$), together with the fact that $\mathfrak{D}_0(S)$ is an absolute neighborhood retract, is the following: *If* $g = 0$, then $\mathfrak{D}(S) = \mathfrak{D}_0(S)$ *has* SO(3) *as deformation retract* (theorem of Smale); *if* $g = 1$, *then* $\mathfrak{D}_0(S)$ *has* $S^1 \times S^1$ *as deformation retract; if* $g \geq 2$, *then* $\mathfrak{D}_0(S)$ *is contractible.*

REMARK. The topological groups $\mathcal{C}(S)$ of homeomorphisms of S are more difficult to handle. ($\mathcal{C}(S)$ is a Baire space and locally contractible; but it is not known (to me) whether $\mathcal{C}(S)$ is an absolute neighborhood retract.) It has been shown by Hamstrom [15] that the identity components $\mathcal{C}_0(S)$ of $\mathcal{C}(S)$ have the homotopy groups of the above-mentioned deformation retracts for the $\mathfrak{D}_0(S)$ cases.

(C) If S is a finite dimensional connected manifold and we fix a point $s_0 \in S$ to define the evaluation map $p:\mathcal{C}(S) \to S$ by $p(g) = g(s_0)$, then *we have a principal* $\mathcal{C}_{s_0}(S)$*-bundle, where* $\mathcal{C}_{s_0}(S) = \{g \in \mathcal{C}(S): g(s_0) = s_0\}$; see for instance [21]. Thus S has the representation $\mathcal{C}(S)/\mathcal{C}_{s_0}(S)$ as a homogeneous space.

REMARK. In the case that S is a compact connected C^∞-manifold there is reason to hope that the analogous map $p: \mathfrak{D}(S) \to S$ will possess good differential $\mathfrak{D}_{s_0}(S)$-bundle properties in relation to the exact sequence

$$0 \to C^\infty_{s_0}(T(S)) \to C^\infty(T(S)) \overset{\overline{p}}{\to} S(s_0) \to 0$$

of vector spaces, where $C^\infty_{s_0}(T(S))$ = the space of all C^∞-vector fields on S which vanish at s_0 and $\overline{p}(u) = u(s_0)$; in particular, with respect to the exponential map $C^\infty(T(S)) \to \mathfrak{D}(S)$. As a first step in this direction, Leslie [19] has introduced a differential structure (which we will call of class WC^∞) on $\mathfrak{D}(S)$ relative to which $\mathfrak{D}(S)$ is a WC^∞-manifold and the group operations are WC^∞.

It is known that the exponential map is not locally bijective in the C^∞-topology. However, the WC^∞-structure may still be significant in treating finite dimensional and finite codimensional subspaces of $C^\infty(T(S))$.

(D) EXAMPLE. Let G be a compact group and $p: E \to B$, $p': E' \to B'$ two G-bundles. Assume that B (and hence E) is locally compact, and that B′ is paracompact. Denote by $C(B,B')$ the space of all continuous maps of B into B′, topologized by the compact-open topology; similarly, let $C_G(E,E')$ denote the space of G-equivariant maps. *There is a natural map* $\eta: C_G(E,E') \to C(B,B')$, *which is a Hurewicz fibration over its image; its fibre model is* $F = \{u: E \to G: u(x \cdot g) = g^{-1}u(x)g$ *for all* $x \in E, g \in G\}$, *with the compact-open topology*. In case $p': E' \to B'$ is a smooth bundle and E is compact we have a differentiable form of that theorem, in terms of manifolds of maps. Also, if B_0 is a closed subspace of B and $p_0: E_0 = p^{-1}(B_0) \to B_0$ is the restriction, then we have the natural map $C_G(E,E') \to C_G(E_0,E')$, which under mild conditions on (B,B_0) is a Hurewicz fibration. See I. M. James, *The space of bundle maps*. Topology 2 (1963), 45-59, where applications are made.

(D) EXAMPLE. Let G be a metrizable topological group and K a closed subgroup. A theorem of Michael [22] asserts that *if* K *is complete and locally convex, there is a local section of the coset map* $G \to G/K$; *therefore we have a homogeneous* K-*bundle*. That is particularly interesting when $G = GL(V)$, the group of linear automorphisms of an infinite dimensional Hilbert space, topologized as an open subset of the Banach algebra $L(V)$ of endomorphisms of V. For then by a theorem of Kuiper G is an absolute retract, and $G \to G/K$ is a universal K-bundle.

Similarly, if V is a separable infinite dimensional Hilbert space and $UL_s(V)$ denotes the group of unitary operators with its strong operator topology, then $UL_s(V)$ is metrizable and contractible [5, §10]. For any locally compact group K we take its Lebesgue space $L^2(V)$ using Haar measure; then K can be imbedded as a closed subgroup of $UL_s(L^2(K))$ — and whenever it has a local section we have a universal K-bundle.

REMARK. It would be interesting to know whether every compact Lie group can operate principally on every C^∞-manifold X modeled on the infinite dimensional separable Hilbert space E. It is true that G *operates principally and smoothly on* E *itself*. For G can be represented faithfully as a closed subgroup of an orthogonal group O_n, which operates principally and analytically on the Stiefel manifold $V_n(E)$ of orthonormal n-frames in E. But by a theorem of Bessaga, coupled with a remark made to me by Husemoller, $V_n(E)$ is C^∞-diffeomorphic to E.

5. *Certain fibrations of mapping spaces*

(A) That certain evaluation maps define a sort of fibration (with sufficient structure to insure the covering homotopy theorem for cells; such a map is called a *Serre fibration*) follows from Borsuk's extension theorem [13, 24]. Such fibrations of path spaces have been used extensively in the celebrated thesis of Serre.

THEOREM [24]. *Let* S *be a compact space and* A *a closed subspace. Then for any absolute neighborhood retract* M *the evaluation map* $\overline{f}: C(S,M) \to C(A,M)$ *defined by* $\overline{f}(x) = x|A$ *is a Serre fibration over the image.*

Here C(S,M) denotes the mapping space of all continuous maps of S into M, topologized by the compact-open topology.

A similar theorem with somewhat different hypotheses is the following [17]:

Let S *be a locally compact absolute neighborhood retract and* A *a closed subset which is also an absolute neighborhood retract. Then for any space* M *the evaluation map* $\overline{f}: C(S,M) \to C(A,M)$ *is a Hurewicz fibration over the image.*

Suppose that M is a C^{r+2}-manifold $(r \geq 0)$ modeled on a C^{r+2}-smooth Banach space, and that S is a compact space. Then C(S,M) and C(A,M) are C^r-manifolds; furthermore, then $\overline{f}$ *is a* C^r*-map, and is locally* C^0*-trivial.* If S is metrizable and A has separable frontier, then $\overline{f}$ is a foliation map (see [14, §11]). In that case the sequence (1) becomes

$$0 \to \operatorname{Ker} f_* \to C(S,T(M)) \to \overline{f}^{-1}C(A,T(M)) \to 0 \; ;$$

and we can construct a locally Lipschitz splitting to apply Theorem 3B, giving a differentiable interpretation of that result.

(B) If in the above theorem S and A are compact C^∞-manifolds and $f: A \to S$ a C^∞-embedding, then for any C^∞-manifold M the induced C^∞-map $\overline{f}: C^r(S,M) \to C^r(A,M)$ is a foliation map over its image $(0 \leq r < \infty)$. We then obtain [25] the

THEOREM. $\overline{f}: C^r(S,M) \to C^r(A,M)$ *is a locally* C^0*-trivial fibration over its image.*

The space $\mathrm{Em}^r(S,M)$ of C^r-embeddings of S in M is an open submanifold of $C^r(S,M)$. The following result is due to Thom [29], Cerf [3], and Palais [25]; see also [20]:

THEOREM. $\overline{f}: \mathrm{Em}^r(S,M) \to \mathrm{Em}^r(A,M)$ *is a locally* C^0*-trivial fibration over its image* $(2 \leq r \leq \infty)$.

We can suppose that S and A have boundaries; and S need not be compact.

Analogously [4], *let* $\tilde{\mathfrak{D}}$ *be an open subgroup of* $\mathfrak{D}(S)$. *Then* $\tilde{\mathfrak{D}}$ *operates principally on* $\mathrm{Em}^r(S,M)$.

EXAMPLE. Taken together with Example 4B, we find the following result, of interest in the calculus of variations: *If* S *is a closed surface of genus* (S) ≥ 2, *then the orbit map* $\mathrm{Em}^r(S,M) \to \mathrm{Em}^r(S,M)/\mathfrak{D}_0(S)$ *is a homotopy equivalence.*

(C) We fix an embedding $h: S \to M$ and thereby view S as a submanifold of M. Denote by $\mathrm{Em}^r(S,M;A)$ the totality of C^r-embeddings of S into M which induce the identity on A. Let $J^r_A\mathrm{Em}^r(S,M;A)$ denote the space of r-jets of these embeddings which are tangent through order r at every point of $A (1 \leq r \leq \infty)$.

The following result – and its many variants – play a fundamental role in the theory of Cerf [3, 4]:

THEOREM. *The canonical map*

$$\mathrm{Em}^r(S,M;A) \to J^r_A\mathrm{Em}^r(S,M;A)$$

is a locally trivial fibration.

EXAMPLE. If M is a compact m-manifold without boundary and D^p the closed p-dimensional Euclidean disc centered at 0 ($p \leq m$), then

$$\overline{f}: \mathrm{Em}^\infty(D^p,M) \to J^1_0\mathrm{Em}^\infty(D^p,M;0)$$

is a locally trivial fibration. Now we have a canonical identification of $J^1_0\mathrm{Em}^\infty(D^p,M;0)$ with the Stiefel manifold $V_{m,p}(M)$ of p-frames of M. Furthermore, $\overline{f}$ has aspherical fibres, whence *we have a homotopy equivalence* $\overline{f}: \mathrm{Em}^\infty(D^p,M) \to V_{m,p}(M)$.

(D) A corresponding theory of fibrations for immersions is more delicate and difficult.

The space $\mathrm{Im}^r(S,M)$ of C^r-immersions of S in M is also an open submanifold of $C^r(S,M)$. The following result was first established (in a slightly different form) by Smale and Thom ([29]; see [26] for further bibliography), with refinements made by Hirsch-Palais.

THEOREM. *If* $\dim S < \dim M$, *then the restriction map* $f: \mathrm{Im}^r(S,M) \to \mathrm{Im}^r(A,M)$ *is a Hurewicz fibration over its image* $(2 \leq r \leq \infty)$.

Again, we can suppose that S and A have boundaries.

A primary application of that fibration is the *fundamental classification theorem for immersions* of Hirsch-Smale [26]:

If dim S < dim M, *then the map* $\phi \to \phi_*$ (*the differential of* ϕ) *induces a bijective correspondence between the regular homotopy classes of immersions of* S *in* M *and the (monomorphism) homotopy classes of monomorphisms of* T(S) *into* T(M).

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