Dyadic spaces I (see Problems 4.5.9–4.5.11)

3.12.12. A compact space X is called a *dyadic space* (Alexandroff [1936]) if X is a continuous image of the Cantor cube $D^{\mathfrak{m}}$ for some $\mathfrak{m} \geq \aleph_0$.

(a) (Marczewski [1941], Tukey [1941]) Note that for every dyadic space X we have $c(X) = \aleph_0$ and deduce that $A(\mathfrak{m})$ is not a dyadic space if $\mathfrak{m} > \aleph_0$.

Hint. See Theorem 2.3.17.

(b) (Šanin [1948] (announcement [1946b])) Show that a dyadic space of weight $\mathfrak{m} \geq \aleph_0$ is a continuous image of $D^{\mathfrak{m}}$.

Hint (Engelking and Pełczyński [1963]). Apply Exercise 3.2.H(d).

(c) (Engelking and Pełczyński [1963]) Prove that for every continuous real-valued function $f: X \to R$ defined on a dyadic space X there exists a compact space $X_0 \subset X$ of weight $\aleph_0$ such that $f(X_0) = f(X)$.

Note that R can be replaced by any Tychonoff space of weight $\aleph_0$.

(d) (Engelking and Pełczyński [1963]) Deduce from (c) that the two arrows space is not dyadic and that if the Čech-Stone compactification of a space X is dyadic, then X is pseudocompact.

Hint. Show that βR is not a dyadic space and observe that any non-pseudocompact Tychonoff space can be continuously mapped onto a dense subset of R.

(e) (Esenin-Volpin [1949]) Show that for every dyadic space X we have $w(X) = \chi(X)$ (cf. parts (g) and (h) below).

Hint. Prove a counterpart of Problem 2.7.14(a) for a space Y whose one-point subsets are intersections of $\mathfrak{m} \geq \aleph_0$ open sets.

(f) (Šanin [1948] (announcement [1946b])) Prove that every linearly ordered dyadic space is second-countable.

Hint. Apply (e) and Problem 3.12.4(a).

(g) (Efimov [1963a]) Prove that if $\chi(x, X) \leq \mathfrak{m} \geq \aleph_0$ for every x in a dense subset of a dyadic space X, then $w(X) \leq \mathfrak{m}$.

Hint (E. Pol and R. Pol [1976]). Let $\chi(x, X) \leq \mathfrak{m}$ for every x in a set B dense in X. Consider a mapping $f: D^{\mathfrak{n}} \to X$ of a Cantor cube $D^{\mathfrak{n}} = \prod_{s\in S} D_s$ onto X and for every $a \in A = f^{-1}(B)$ choose an $S(a) \subset S$ such that $p_{S(a)}^{-1}p_{S(a)}(a) = f^{-1}f(a)$ and $|S(a)| \leq \mathfrak{m}$. Define inductively increasing sequences $S_1 \subset S_2 \subset \ldots$ and $A_1 \subset A_2 \subset \ldots$ of subsets of S and A respectively, such that $|S_i| \leq \mathfrak{m}$, $|A_i| \leq \mathfrak{m}$,

$$p_{S_i}(A) \subset \overline{p_{S_i}(A_i)} \quad \text{and} \quad S_{i+1} = S_i \cup \bigcup \{S(a) : a \in A_i\}.$$

Observe that for $S_0 = \bigcup_{i=1}^{\infty} S_i$ and $A_0 = \bigcup_{i=1}^{\infty} A_i$ we have $p_{S_0}(A) \subset \overline{p_{S_0}(A_0)}$ and $f(p_{S_0}^{-1}p_{S_0}(a)) = f(a)$ for every $a \in A$. Consider the set $A' = p_{S_0}(A) \times \prod_{s\in S\setminus S_0}\{a_s\}$, where $a_s = 0$ for $s \in S \setminus S_0$, and show that $f(\overline{A'}) = X$.

(h) (Arhangel'skiĭ and Ponomarev [1968]) Prove that for every dyadic space X we have $w(X) = \tau(X)$.

Hint (Arhangel'skiĭ [1969]). Consider a mapping $f: D^{\mathfrak{m}} \to X$ onto X and the set $\Sigma_{\mathfrak{n}} \subset D^{\mathfrak{m}}$, where $\mathfrak{n} = \tau(X)$, consisting of those points of $D^{\mathfrak{m}}$ which have at most $\mathfrak{n}$ coordinates distinct from zero; show that $f(\Sigma_{\mathfrak{n}}) = X$ and note that it suffices to prove that $d(X) \leq \mathfrak{n}$.

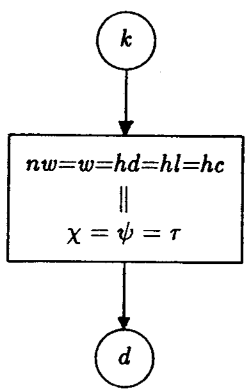
Making use of Theorem 2.3.15 observe that for every $\mathbf{k} \geq \mathbf{n}$ and a set $A \subset X$ of cardinality $\leq \exp \mathbf{k}$ there exists a set $B \subset X$ such that $A \subset \overline{B}$ and $|B| \leq \mathbf{k}$. Applying three times this observation and using Theorem 1.5.3 show that $|X| \leq \exp \exp \mathbf{n}$ (to this end, consider a set $A \subset X$ such that $|A| \leq \exp \exp \exp \mathbf{n}$), applying it twice more prove that $d(X) \leq \mathbf{n}$.

(i) (Engelking [1965]; for $\mathbf{m}$ of the form $\aleph_{\alpha+1}$, Efimov [1965a]) Prove that if X is a dyadic space and $\chi(x_0, X) = \mathbf{m} \geq \aleph_0$, then X contains a subspace M homeomorphic to $D(\mathbf{m})$ such that $M \cup \{x_0\}$ is homeomorphic to $A(\mathbf{m})$.

Hint. Consider a mapping $f: \prod_{s \in S} D_s \to X$ onto X, the fiber $A = f^{-1}(x_0)$ and the set $S_0 \subset S$ consisting of those $s \in S$ for which one can find points $a(s) \in A$ and $b(s) \notin A$ such that $p_{s'}(a(s)) = p_{s'}(b(s))$ for $s' \neq s$; for every $s \in S_0$ choose $a(s)$ and $b(s)$ with the above properties. Prove that $A = p_{S_0}(A) \times \prod_{s \in S \setminus S_0} D_s$ and deduce that $|S_0| \geq \mathbf{m}$. Observe that fibers of the function b from S_0 to $B = b(S_0)$ are finite and deduce that $|B| \geq \mathbf{m}$; verify that all accumulation points of B belong to A and let $M = f(B)$.

Remark. As proved by Hagler in [1975], Gerlits in [1976] and Efimov in [1977], every dyadic space of a regular weight $\mathbf{m}$ contains a subspace homeomorphic to $D^{\mathbf{m}}$ (the last two papers discuss also the case when the weight is non-regular). A similar result was obtained for Tychonoff cubes by Ščepin in [1979]: every Hausdorff space that is a continuous image of a Tychonoff cube and whose weight $\mathbf{m}$ is regular contains a subspace homeomorphic to $I^{\mathbf{m}}$.

(j) Verify that the following diagram (cf. Problem 1.7.12(a)) contains all equalities and inequalities between the cardinal functions appearing in it which hold in the class of all dyadic spaces (the symbol $k(X)$ denotes here the cadinality of X; from part (a) it follows that for every dyadic space X we have $c(X) = \aleph_0$, and clearly $l(X) = e(X) = \aleph_0$).



(k) (Efimov [1963a]) Prove that every hereditarily normal dyadic space is second-countable.

Hint. Consider a mapping $f: D^{\mathbf{m}} \to X$ onto X and a Σ-product $\Sigma(a) \subset D^{\mathbf{m}}$. Show that if $f(\Sigma(a)) = X$, then $w(X) \leq \aleph_0$ (cf. Problem 3.12.24(f)). To this end, observe that every separable subspace of X is second-countable and so is every subspace of cardinality $\leq \mathbf{c}$; then apply (i) and (e). In the case when $f(\Sigma(a)) \neq X$, take a point $x \in X \setminus f(\Sigma(a))$ and show – applying (i) and Theorem 3.10.21 – that the space $X \setminus \{x\}$ is not normal.

(b) (Montgomery [1935]) Show that if every point in a subset A of a metrizable space X has a neighbourhood U in the space X such that the intersection $A \cap U$ is a set of the multiplicative class $\alpha > 0$ (the additive class α) in the subspace U of X, then A is a set of the same class (cf. Problem 2.7.1).

Hint (Michael [1954]). Apply (a) and the fact that X has a σ-locally finite base.

(c) (Montgomery [1935]) Prove that if X and Y are metrizable spaces and $f: X \to Y$ is a measurable mapping of class α, then the graph $G(f)$ is a set of the multiplicative class α in the Cartesian product $X \times Y$.

Hint (Engelking [1967]). Show that for any base $\{B_s\}_{s\in S}$ for the space Y there exists a family $\{A_s\}_{s\in S}$ of open subsets of Y such that $(X \times Y) \setminus G(f) = \bigcup_{s\in S}(f^{-1}(A_s) \times B_s)$. Apply (a) and the fact that Y has a σ-locally finite base.

Dyadic spaces II (see Problem 3.12.12)

4.5.9. (a) (Sierpiński [1928]) Show that every non-empty closed subset A of the Cantor set C is a retract of C.

Hint (Halmos [1963]). Check that the metric σ on the set $D^{\aleph_0}$, defined by letting

$$\sigma(x,y) = \sum_{i=1}^{\infty} \frac{1}{10^i}|x_i - y_i| \quad \text{for} \quad x = \{x_i\},\ y = \{y_i\},$$

induces the topology of the Cartesian product. Observe that if $\sigma(x,y) = \sigma(x,z)$, then $y = z$ and deduce that for every $x \in D^{\aleph_0}$ there exists exactly one point $a \in A$ such that $\sigma(x,a) = \sigma(x,A)$.

(b) (Alexandroff [1927] (announcement [1925]), Hausdorff [1927]) Observe that from (a) and Theorem 3.2.2 it follows that every non-empty compact metrizable space is a continuous image of the Cantor set, i.e., is a dyadic space (cf. Theorem 3.2.2 and Problem 3.12.12(a)).

4.5.10 (Efimov [1963]). Show that every non-empty closed G_δ-set $F \subset D^{\mathfrak{m}}$ is a retract of $D^{\mathfrak{m}}$. Deduce that dyadicity is hereditary with respect to non-empty closed G_δ-sets.

Hint (Engelking and Pełczyński [1963]). Take a function $f: D^{\mathfrak{m}} \to R$ such that $F = f^{-1}(0)$, apply Exercise 3.2.H(a) and Problem 4.5.9(a).

4.5.11 (Efimov [1963a]). Show that every dyadic compactification cX of a metrizable space X is second-countable, i.e., is metrizable.

Hint (Engelking and Pełczyński [1963]). Observe first that the space X is separable, then apply Exercise 3.5.F and Problem 3.12.12(c).

One can also apply Problem 3.12.12(g) and Exercise 2.1.C(a).

Σ-products III (see Problems 2.7.14, 2.7.15, 3.12.24 and Exercise 3.10.D)

4.5.12. (a) (Gul'ko [1977], M. E. Rudin [1977]) Let $\Sigma(a)$ be a Σ-product of metrizable spaces $\{X_s\}_{s\in S}$, where $a = \{a_s\} \in \prod_{s\in S} X_s$. Prove that for every discrete family $\mathcal{F}$ of closed subsets of $\Sigma(a)$ there exists an open σ-locally finite cover $\mathcal{U}$ of $\Sigma(a)$ such that the closure of each member of $\mathcal{U}$ intersects at most one member of $\mathcal{F}$.

Hint. For each intersection $U = \Sigma(a) \cap \prod_{s\in S} U_s$ of $\Sigma(a)$ and a member $\prod_{s\in S} U_s$ of the canonical base $\mathcal{B}$ for the Cartesian product $\prod_{s\in S} X_s$ let $S(U) = \{s \in S : U_s \neq X_s\}$, for each