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CONFIGURATION SPACES

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I. Introduction.

A knowledge of the space of configurations of n points in a manifold leads to information about the topological and non-homotopy properties of the manifold, as well as the identity components of the spaces of autohomeomorphisms and diffeomorphisms of the manifold.

In the first part of this paper we give the fundamental sequence of fibrations for the study of configuration spaces, and localize the problem of computing the homotopy groups of these spaces to finding a cross section of a particular fibration.

In the second section a theorem is proved which tells of the existence or non-existence of cross sections in many situations.

The last section is devoted to some diverse applications of this investigation.

II. The Fundamental sequence of fibrations.

Let M denote an arbitrary manifold (locally Euclidean connected Hausdorff space) and $Q_m=\{q_1,\ldots,q_m\}$ a fixed set of m distinct points of M. We will assume throughout that $\dim M \geqq 2$; the case $\dim M=1$ is of little interest. Define $F_{m,n}(M) \subseteq \underbrace{M \times \ldots \times M}_{n}$ as follows:

$$F_{m,n}(M) = \{(p_1,\ldots,p_n) \mid p_i \in M - Q_m, p_i \neq p_j \text{ for } i \neq j\}.$$

When the manifold M is fixed we will designate $F_{m,n}(M)$ simply by $F_{m,n}$. Give $F_{m,n}$ the topology induced by M and note that $F_{m,n}$ is a nk dimensional manifold if $k=\dim M$. Note that $F_{m,n}$ is essentially independent of the set Q_m chosen since any manifold is m-homogeneous.

THEOREM 1. *$\pi: F_{m,n} \to M - Q_m$, where $\pi(p_1,\ldots,p_n)=p_1$ and $n>1$, is a locally trivial fiber space with fiber $F_{m+1,n+1}$. If $m \geqq 1$, π admits a cross section.*

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PROOF. a) We first show that π is locally trivial. Add another point q_{m+1} to the set Q_m to form Q_{m+1} and fix $x_0 \in M - Q_m$. Let $\alpha: M \to M$ denote a homeomorphism fixed on Q_m such that $\alpha(q_{m+1}) = x_0$. Let U denote a Euclidean neighborhood of x_0 which *avoids* Q_m. Furthermore, let

$$\theta\colon U \times \overline{U} \to \overline{U}$$

denote a map with the following properties. Setting $\theta_x(y) = \theta(x,y)$ we require

1. $\theta_x\colon \overline{U} \to \overline{U}$ is a homeomorphism having $\partial\overline{U}$ fixed.
2. $\theta_x(x) = x_0$.

θ has the obvious extension

$$\theta\colon U \times M \to M$$

by setting $\theta(x,y) = y$ for $y \notin U$. The required local product representation

$$\begin{array}{ccc} \pi^{-1}(U) & \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} & U \times F_{m+1,n-1} \\ \pi \big\downarrow & & \big\downarrow \\ & U & \end{array}$$

is obtained by setting

$$\varphi(x, p_2, \ldots, p_n) = \left(x, \theta_x^{-1}\alpha(p_2), \ldots, \theta_x^{-1}\alpha(p_n)\right),$$
$$\varphi^{-1}(x, p_2, \ldots, p_n) = \left(x, \alpha^{-1}\theta_x(p_2), \ldots, \alpha^{-1}\theta_x(p_n)\right).$$

b) Now we show that π admits a cross section if $m \geqq 1$. Let V denote a Euclidean neighborhood of q_1 whose closure avoids $q_i, i \geqq 2$. We may assume without loss that V is a spherical neighborhood of O in Euclidean space with unit radius. Let W denote the spherical neighborhood of O of radius $\frac{1}{2}$ and $y_2, \ldots, y_n$ mutually distinct points on ∂W. On $\overline{V} - q_1$ define

$$f_i(x) = \|x\| y_i, \qquad 2 \leqq i \leqq n, \quad x \in \overline{V} - q_1,$$

and extend to $M - Q_m$ by setting

$$f_i(x) = y_i, \qquad x \notin V.$$

Now, set $f_1 =$ identity and observe that for $x \in M - Q_m$

$$\left(f_1(x), f_2(x), \ldots, f_n(x)\right) \in F_{m,n}.$$

Since each f_i is a map and $f_1(x) = x$, it is clear that

$$f(x) = \left(f_1(x), \ldots, f_n(x)\right)$$

is the required cross section.

This completes the proof of the theorem.

Observation. With regard to the existence of cross sections the following remark is obvious:

$\pi: F_{m,n} \to M - Q_m$ admits a cross section if and only if there exist $n-1$ fixed point free maps $f_2, \ldots, f_n: M - Q_m \to M - Q_m$ which are non-coincident, i.e.,

$$f_i(x) \neq f_j(x), \qquad i \neq j, \quad x \in M - Q_m .$$

Therefore, what we showed above is that the manifold $M - Q_m$ has the property required for cross sections if $m \geqq 1$. Now, for $m = 0$ no cross sections need exist since the manifold may have the fixed point property, e.g. the projective plane.

The need for information concerning the existence of cross sections for the case $m = 0$ is apparent from looking at the following fundamental sequence of fibrations.

$$\begin{array}{cccccccc}
 & F_{0,n} & & F_{1,n-1} & & F_{n-3,3} & & F_{n-2,2} \\
F_{1,n-1} & \downarrow & F_{2,n-2} & \downarrow & \ldots\ F_{n-2,2} & \downarrow & F_{n-1,1} & \downarrow \\
 & M & & M - Q_1 & & M - Q_{n-3} & & M - Q_{n-2}
\end{array}$$

Since $F_{n-1,1} = M - Q_{n-1}$, we may start at the extreme right and employ the existence of cross sections at each stage except the one on the extreme left to conclude

Theorem 2. *For any manifold M,*

$$\pi_i(F_{1,n-1}) = \sum_{k=1}^{n-1} \pi_i(M - Q_k) \qquad \text{(direct sum)}$$

for $i \geqq 2$. If $\pi: F_{0,n} \to M$ (fiber $= F_{1,n-1}$) *admits a cross section then*

$$\pi_i(F_{0,n}) = \sum_{k=0}^{n-1} \pi_i(M - Q_k), \qquad i \geqq 2 .$$

Corollary 2.1. *If M is Euclidean r-space, then*

$$\pi_i(F_{0,n}) = \sum_{k=1}^{n-1} \pi_i(\underbrace{S^{r-1} \vee \ldots \vee S^{r-1}}_{k}), \qquad i \geqq 2 .$$

Corollary 2.2. *If M is a compact 2-manifold, then $F_{m,n}$ is aspherical if $m \geqq 1$, that is, $\pi_i(F_{m,n}) = 0$, $i \geqq 2$. If M is neither the projective plane P^2 nor the 2-sphere S^2, the statement remains valid for $m = 0$.*

Proof. $M - Q_m$ is aspherical as long as $m \geqq 1$. Thus, the first part of the corollary is an immediate consequence of the preceding theorem. Now, if M is not P^2 or S^2 it is aspherical. Consider the exact sequence

$$\ldots \to \pi_i(F_{1,n-1}) \to \pi_i(F_{0,n}) \to \pi_i(M) \to \ldots .$$

Since, $\pi_i(M)=0=\pi_i(F_{1,n-1})$ for $i \geqq 2$, it follows that $F_{0,n}$ is aspherical in this case.

An interpretation of the vanishing of the higher homotopy groups of $F_{0,n}(M^2)$ (Corollaries 2.1, 2.2) yields the following:

COROLLARY 2.3. *If M^2 is E^2 or a compact manifold different from S^2 or P^2, then any n coincidence-free maps of a k-sphere $(k>1)$ into M^2 may be extended to n coincidence-free maps of the $k+1$ ball into M^2.*

REMARK. We will see shortly that $\pi: F_{0,n} \to M$ (fiber $=F_{1,n-1}$) actually admits a cross section if M is a 2-manifold different from P^2 or S^2.

We mention next an extension of Theorem 1. Its proof, which is a slight modification of the proof of Theorem 1, is omitted. Let M denote a fixed manifold and consider the map

$$\pi\colon F_{m,n} \to F_{m,r}, \qquad n \geqq r, \qquad m \geqq 0$$

given by

$$\pi(p_1, \ldots, p_n) = (p_1, \ldots, p_r) .$$

THEOREM 3. *$\pi: F_{m,n} \to F_{m,r}$ is a locally trivial fiber space with fiber $F_{m+r,n-r}$.*

III. Cross Sections.

DEFINITION. A manifold M is called *suitable* if it has the following property: Let $G(M)$ denote the group of all homeomorphisms of M onto M with the compact–open topology and fix $q_1 \in M$. Then, M is called suitable if there exists a map

$$\theta\colon M \to G(M)$$

such that $\theta(x)(x)=q_1$ and $\theta(q_1)=$identity.

REMARK. Every Lie group is suitable. Among the spheres S^n, S^n is suitable only for $n=1,3,7$ since one can easily show that suitability implies the existence of an H-space structure [1].

THEOREM 4. *If M is suitable, $\pi: F_{0,n} \to M$ is actually a product, that is, $F_{0,n}=M\times F_{1,n-1}$.*

PROOF. Let $\theta: M \to G(M)$ be the map given by the suitability condition. Define $\varphi: F_{0,n} \rightleftarrows M\times F_{1,n-1}: \varphi^{-1}$ by

$$\varphi(x, p_2, \ldots, p_n) = \big(x, \theta(x)(p_2), \ldots, \theta(x)(p_n)\big)\,,$$
$$\varphi^{-1}(x, p_2, \ldots, p_n) = \big(x, \theta(x)^{-1}(p_2), \ldots, \theta(x)^{-1}(p_n)\big)\,.$$

Therefore, $F_{0,n}$ is homeomorphic to $F_{1,n-1} \times M$, preserving projections.

The following theorem gives some further insight into the existence of cross sections for the fiber map $\pi: F_{0,n} \to M$.

THEOREM 5. *Consider the fiber map $\pi: F_{0,n} \to M$ with fiber $F_{1,n-1}$ and $n \geqq 2$. Then,*

a) *There are no cross sections if M has the fixed point property.*

b) *If $M \supseteq L$ as a retract and L admits $n-1$ fixed point free, non-coincident maps, π admits a cross section.*

c) *If M is differentiable and admits a non-vanishing vector field, then π admits a cross section.*

d) *If M is an even dimensional sphere, π admits a cross section only when $n = 2$.*

PROOF. a) This part is immediate from the observation made in § II.

b) If $\varrho: M \to L$ is a retraction and $f_2, \ldots, f_n: L \to L$ are $n-1$ fixed point free non-coincident maps, then $f_i \varrho: M \to M$, $2 \leqq i \leqq n$, are $n-1$ fixed point free non-coincident maps and again the observation in § II applies.

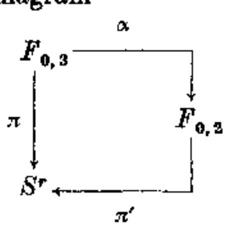
c) At each point of M take a geodesic arc of length k in the direction given by the vector field. Then, the points on the geodesic at a distance 0, $k/(n-1)$, $2k/(n-1), \ldots, k$ from any given point define a cross section for π. (This proof follows a suggestion of M. Hirsch.)

d) That $\pi: F_{0,2} \to S^r$ always admits a cross section is clear by employing the antipodal map.

Next, we show that $\pi: F_{0,3} \to S^r$ does not admit a cross section for r even. (This part can also be proved by employing Lefschetz's theory of coincidences.) We will employ the following simple lemma.

LEMMA. *Suppose (E, p, B) is a fiber space (the covering homotopy property valid for a class of spaces containing B [3]) and $\sigma: B \to E$ is a map such that $p\sigma \sim 1$, that is, $p\sigma$ is homotopic to the identity map 1. Then, p admits a cross section.*

Consider the following diagram



where π, π', α are fiber maps given by Theorems 1 and 3 in § II. We prove now that π admits a cross section if and only if α does. First, recall that π' always admits a cross section. Hence, if α does, so does π. Secondly, suppose π admits a cross section σ. Since π' is a homotopy equivalence, $\alpha(\sigma\pi') \sim 1$. The argument here is as follows. Suppose $\beta': S^r \to F_{0,2}$ is a homotopy inverse for π'. Now $\pi'\alpha\sigma = \pi'\sigma = 1$. Therefore, $\beta'\pi'\alpha\sigma = \beta' \sim \alpha\sigma$ and hence $1 \sim \beta'\pi' \sim \alpha\sigma\pi'$. Our lemma then applies. It suffices now to show that α does not admit a cross section. Let $f: S^r \to S^r$ denote the antipodal map and consider the subset A of $F_{0,2}$ consisting of pairs $(x, f(x))$, $x \in S^r$. If a cross section existed, it would generate a map $g: S^r \to S^r$ such that $g(x) \neq x, f(x)$, $x \in S^r$. Since, for every $x, g(x) \neq f(x)$ it is easy to see that $g \sim 1$ and g has degree 1 and hence fixed points which is a contradiction. Thus, $\pi: F_{0,3} \to S^r$ does not admit a cross section.

To complete the proof of part d), consider the following diagram

$$\begin{array}{ccc} F_{0,n} & \xrightarrow{\alpha} & F_{0,n-1}\,. \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S^r & = & S^r \end{array}$$

If π' fails to admit a cross section so does π and hence induction completes the proof.

COROLLARY 5.1. *If M is compact and the first Betti number of M does not vanish, $\pi: F_{0,n} \to M$ admits a cross section for every n.*

PROOF. In this case M is a multicoherent Peano continuum [5] and hence there is a 1-sphere $L \subseteq M$ which is a retract of M. Part b) of Theorem 5 now applies.

COROLLARY 5.2. *If M is an odd dimensional differentiable manifold, $\pi: F_{0,n} \to M$ admits a cross section for every n.*

COROLLARY 5.3. *If $M = S^r$ (r-sphere) with r odd, then*

$$\pi_i(F_{0,n}) = \pi_i(S^r) + \sum_{k=1}^{n-2} \pi_i(\underbrace{S^{r-1} \vee \ldots \vee S^{r-1}}_{k})$$

for $i \geqq 2$.

IV. Applications.

1. Let M denote a manifold (differentiable) and $G(M)$ $(D(M))$ denote the group of homeomorphisms (diffeomorphisms) of M onto itself which are isotopic (diffeotopic) to the identity. These groups are assumed to have the compact–open topology [4]. Furthermore, let Q_n denote a fixed set of n points in M and $G_n(M), D_n(M)$ the subgroups of $G(M)$ and $D(M)$, respectively, which leave Q_n pointwise fixed. Both $G_n(M)$ and $D_n(M)$ are, respectively, closed subgroups which admit local cross sections. Furthermore, it is easy to see that $G(M)/G_n(M)$, $D(M)/D_n(M)$ and $F_{0,n}(M) = F_{0,n}$ are homeomorphic. Thus, the fiberings

$$\begin{array}{ccc} G(M) & \qquad & D(M) \\ {\scriptstyle G_n(M)}\Big\downarrow & & \Big\downarrow{\scriptstyle D_n(M)} \\ F_{0,n} & & F_{0,n} \end{array}$$

lead to exact sequences

$$\ldots \to \pi_{i+1}(F_{0,n}) \to \pi_i\big(G_n(M)\big) \overset{j^*}{\to} \pi_i\big(G(M)\big) \to \pi_i(F_{0,n}) \to \ldots,$$
$$\ldots \to \pi_{i+1}(F_{0,n}) \to \pi_i\big(D_n(M)\big) \overset{j^*}{\to} \pi_i\big(D(M)\big) \to \pi_i(F_{0,n}) \to \ldots.$$

THEOREM 6. *Let M denote a r-manifold which is k-connected $\big(\pi_i(M) = 0$ for $i \leq k\big)$ and let $\mu = \min(r-2, k)$. Then, j^* is an isomorphism in both the above sequences for $i < \mu$ and when $i = \mu, j^*$ is epic (onto).*

PROOF. Since M is k-connected and of dimension r, $M - Q_m$ is μ-connected where Q_m is a finite set of m points and $\mu = \min(r-2, k)$. Employing the fundamental sequence of fibrations (§ II), it is then clear that $F_{0,n}$ is μ-connected and the result follows.

COROLLARY 6.1. *If $M = E^r$ or S^r, then $\pi_i\big(G_n(M)\big) \approx \pi_i\big(G(M)\big)$ for $i < r-2$ and $\pi_{r-2}\big(G(M)\big)$ is a homeomorph of $\pi_{r-2}\big(G_n(M)\big)$. The corresponding result holds for $D(M)$ and $D_n(M)$.*

2. Let M denote an r-manifold which is k-connected and as in the first application $\mu = \min(r-2, k)$. Then, if Σ^n is the permutation group on n letters, Σ^n acts freely on $F_{m,n}$. Let $B_{m,n} = F_{m,n}/\Sigma^n$. We thus have a principal bundle

$$p\colon F_{m,n} \to B_{m,n}\,.$$

If G is any finite group, there exists an n such that $G \subseteq \Sigma^n$. Thus, G generates a principal bundle [4]

$$w\colon F_{m,n} \to F_{m,n}/G\,.$$

THEOREM 7. *$w\colon F_{m,n} \to F_{m,n}/G$ is a μ-universal bundle for G, where $\mu = \min(r-2, k)$.*

PROOF. The proof is immediate since $F_{m,n}$ is μ-connected.

COROLLARY 7.1. *If $M = E^r$ or S^r, $w\colon F_{m,n} \to F_{m,n}/G$ is an $(r-2)$-universal bundle for G.*

3. Let M denote a manifold and let $B_{0,n}$ be as in the preceding section. Following R. H. Fox [2], we define the braid group $B_n(M)$ (n strings) by

$$B_n(M) = \pi_1(B_{0,n}) .$$

We recall that $F_{0,n}$ is a covering space over $B_{0,n}$ and hence

$$\pi_i(F_{0,n}) = \pi_i(B_{0,n}), \qquad i \geqq 2 .$$

The following theorem of P. Smith will be employed (see [3, p. 287]).

THEOREM (P. Smith). *Let X be a finite dimensional $K(\pi, 1)$. Then, π has no elements of finite order.*

THEOREM 8. *If M is any compact 2-manifold, except P^2 or S^2, or if $M = E^2$, then the braid groups $B_n(M)$ have no elements of finite order.*

PROOF. By previous results $F_{0,n}$ is aspherical, hence a $K(\pi, 1)$, and of finite dimension. Therefore, $B_{0,n}$ is a $K(B_n(M), 1)$ of finite dimension. The above lemma, then, completes the proof of the theorem.

REMARK. It is easy to see that $B_2(S^2) = Z_2$ (cyclic group of order 2) as follows. Consider the fibrations

$$F_{0,2} \to S^2, \text{ fiber } F_{1,1} .$$

Since $F_{1,1}$ is contractible, $F_{0,2} \sim S^2$ and hence $F_{0,2}$ is 1-connected. Therefore, $F_{0,2} \to B_{0,2}$ is a universal covering with fiber Z_2 and hence $\pi_1(B_{0,2}) = B_2(S^2) = Z_2$.

THEOREM 9. *If M is an r-manifold, $r \geqq 3$, which is 1-connected, $B_n(M) = \Sigma^n$ (symmetric group on n letters) and hence there is no braid theory on this class of manifolds.*

PROOF. By employing the fundamental sequence of fibrations (§ II) and the fact that $M - Q_i$, $0 \leqq i \leqq n-2$, is 1-connected, it follows that $F_{0,n}$ is 1-connected. Hence, $p: F_{0,n} \to B_{0,n}$ is the universal covering of $B_{0,n}$ with fiber Σ^n. The theorem is then immediate.

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