Overview of the fields in QFT (FK8017 HT15)

Real scalar field

Field: $\phi(x) \in \mathbb{R}$

Symmetry: O(1) (no continuous symmetries)

Complex scalar field

Field dofs: $\phi(x), \phi^\dagger(x) \in \mathbb{C}$

Symmetry: $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ = mass degeneracy of the real doublet

Complex field from a real doublet ϕ_1, ϕ_2:

$$\phi(x) = \tfrac{1}{\sqrt{2}}(\phi_1(x) + \mathrm{i}\phi_2(x))$$

$$\phi^\dagger(x) = \tfrac{1}{\sqrt{2}}(\phi_1(x) - \mathrm{i}\phi_2(x))$$

Spinor field

Two-component (Weyl) spinor field

Field dofs: $\chi_a(x) \in \mathbb{C}^2 + \mathrm{hc}$

Symmetry: O(1)

Bispinor field

Four-component spinor field

Field dofs: $\chi_a(x), \psi_a(x) \in \mathbb{C}^2 + \mathrm{hc}$

Symmetry: $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ = mass degeneracy of the 2-spinor doublet

Bispinor from a 2-spinor doublet χ_{1a}, χ_{2a}:

$$\chi_a(x) = \tfrac{1}{\sqrt{2}}(\chi_{1a}(x) + \mathrm{i}\chi_{2a}(x))$$

$$\psi_a(x) = \tfrac{1}{\sqrt{2}}(\chi_{1a}(x) - \mathrm{i}\chi_{2a}(x))$$

Real vector field

Field: $A_\mu(x) \in \mathbb{R}^4$

Gauge symmetry (if free)

Complex vector field

Field dofs: $A_\mu(x), A_\mu^\dagger(x) \in \mathbb{C}^4$

Gauge symmetry (if free)

Real scalar multiplet

$\phi_i(x) \in \mathbb{R},\ i = 1, ..., n$

Symmetries depend on mass degeneracies (at most O(n)).

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \in \mathbb{R}^n$$

Complex scalar multiplet

$\phi_i(x), \phi_i^\dagger(x) \in \mathbb{C},\ i = 1, ..., n$

Symmetries depend on mass degenaracies (at most U(n)).

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \in \mathbb{C}^n$$

Real vector multiplet

Fields: $A_\mu^i(x) \in \mathbb{R}^4,\ i = 1, ..., n$

To do...
(this column is not complete!)

Free theory = kinetic + opt. mass term

Symmetrized (hermitian) kinetic term:

Real scalar field — `kinetic term` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

Complex scalar field — `kinetic term` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = \partial_\mu\phi^\dagger\,\partial^\mu\phi$$

Spinor field — `kinetic term (alt)` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = \tfrac{1}{2}\left(\chi^a\,\mathrm{i}\partial_{a\dot a}\overline{\chi}^{\dot a} + \overline{\chi}_{\dot a}\,\mathrm{i}\overline{\partial}^{\dot a a}\chi_a\right)$$

Bispinor field — `kinetic term (alt)` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = \psi^a\,\mathrm{i}\partial_{a\dot a}\overline{\psi}^{\dot a} + \overline{\psi}_{\dot a}\,\mathrm{i}\overline{\partial}^{\dot a a}\psi_a + \chi^a\,\mathrm{i}\partial_{a\dot a}\overline{\chi}^{\dot a} + \overline{\chi}_{\dot a}\,\mathrm{i}\overline{\partial}^{\dot a a}\chi_a$$

Real vector field — `kinetic term` ($\mathcal{L} = \mathcal{L}$)
$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$

Complex vector field — `kinetic term` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = -\tfrac{1}{2}F_{\mu\nu}^\dagger F^{\mu\nu}$$
where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$

Real scalar multiplet — `kinetic term` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\Phi^\intercal\,\partial^\mu\Phi$$

Complex scalar multiplet — `kinetic term` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = \partial_\mu\Phi^\dagger\,\partial^\mu\Phi$$

Real vector multiplet — `kinetic term` ($\mathcal{L} = \mathcal{L}^\dagger$)
$$\mathcal{L} = -\tfrac{1}{4}G_{\mu\nu}^i G^{i\,\mu\nu}$$
$$G_{\mu\nu}^i = 2\partial_{[\mu}A_{\nu]}^i + g f^{ijk} A_\mu^j A_\nu^k$$

EoM based kinetic term:

Real scalar field — `kinetic term (alt)`
$$\mathcal{L}' = -\tfrac{1}{2}\phi\,\Box\phi$$
$$\mathcal{L}' = \mathcal{L} + \tfrac{1}{2}\partial_\mu(\phi\,\partial^\mu\phi)$$

Complex scalar field — `kinetic term (alt)` ($\mathcal{L}' \neq \mathcal{L}'^\dagger$)
$$\mathcal{L}' = -\phi^\dagger\,\Box\phi$$
$$\mathcal{L}' = \mathcal{L} + \partial_\mu(\phi^\dagger\,\partial^\mu\phi)$$

Spinor field — `kinetic term` ($\mathcal{L}' \neq \mathcal{L}'^\dagger$)
$$\mathcal{L}' = \chi^a\,\mathrm{i}\partial_{a\dot a}\overline{\chi}^{\dot a}$$

Bispinor field — `kinetic term` ($\mathcal{L}' \neq \mathcal{L}'^\dagger$)
$$\mathcal{L}' = \psi^a\,\mathrm{i}\partial_{a\dot a}\overline{\psi}^{\dot a} + \overline{\chi}_{\dot a}\,\mathrm{i}\overline{\partial}^{\dot a a}\chi_a$$

Real vector field — `kinetic term (alt)`
$$\mathcal{L}' = \tfrac{1}{2}A^\mu\big(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\big)A^\nu$$
$$\mathcal{L}' = \mathcal{L} + \tfrac{1}{2}\partial_\mu(A_\nu F^{\mu\nu})$$

Complex vector field — `kinetic term (alt)` ($\mathcal{L}' \neq \mathcal{L}'^\dagger$)
$$\mathcal{L}' = A^{\dagger\mu}\big(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\big)A^\nu$$
$$\mathcal{L}' = \mathcal{L} + \partial_\mu(A_\nu^\dagger F^{\mu\nu})$$

Real scalar multiplet — `kinetic term (alt)`
$$\mathcal{L}' = -\tfrac{1}{2}\Phi^\intercal\,\Box\Phi$$
$$\mathcal{L}' = \mathcal{L} + \tfrac{1}{2}\partial_\mu(\Phi\,\partial^\mu\Phi)$$

Complex scalar multiplet — `kinetic term` ($\mathcal{L}' \neq \mathcal{L}'^\dagger$)
$$\mathcal{L}' = -\Phi^\dagger\,\Box\Phi$$
$$\mathcal{L}' = \mathcal{L} + \partial_\mu(\Phi^\dagger\,\partial^\mu\Phi)$$

Optional quadratic terms (mass and gauge fixing):

Real scalar field — `mass term`
$$-\tfrac{1}{2}m^2\,\phi^2$$

Complex scalar field — `mass term`
$$-\,m^2\,\phi^\dagger\phi$$

Spinor field — `mass term`
$$-\tfrac{1}{2}\big(m\chi^a\chi_a + m^*\overline{\chi}_{\dot a}\overline{\chi}^{\dot a}\big)$$
$$-\tfrac{1}{2}m\big(\chi^a\chi_a + \overline{\chi}_{\dot a}\overline{\chi}^{\dot a}\big) \quad \text{for } m = m^*$$

Bispinor field — `mass term`
$$-\big(m\psi^a\chi_a + m^*\overline{\chi}_{\dot a}\overline{\psi}^{\dot a}\big)$$
$$-m\big(\psi^a\chi_a + \overline{\chi}_{\dot a}\overline{\psi}^{\dot a}\big) \quad \text{for } m = m^*$$

Real vector field — `mass term`
$$+\tfrac{1}{2}m^2\,A_\mu A^\mu$$
`gauge fixing term`
$$-\tfrac{1}{2}\zeta\,(\partial_\mu A^\mu)^2$$

Complex vector field — `mass term`
$$+\,m^2\,A_\mu^\dagger A^\mu$$
`gauge fixing term`
$$-\,\zeta\,(\partial_\mu A^{\dagger\mu})(\partial_\nu A^\nu)$$

Real scalar multiplet — `quadric term`
$$-\tfrac{1}{2}m^2\,\Phi^\intercal\Phi$$
For the general mass term,
$$-\tfrac{1}{2}\Phi^\intercal M\Phi,$$
where M is a positive definite real matrix with k distinct eigenvalues m_i each with degeneracy n_i ($k \le n = \sum n_i$), the internal symmetry group is,
$$\mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_k).$$

Complex scalar multiplet — `quadratic term`
$$-\,m^2\,\Phi^\dagger\Phi$$
For the general mass term,
$$-\Phi^\dagger M\Phi,$$
where M is a positive definite Hermitian matrix with k distinct eigenvalues m_i each with degeneracy n_i ($k \le n = \sum n_i$), the internal symmetry group is,
$$\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k).$$

Note: There can be other internal symmetries (in addition to those coming out of mass term degeneracy).

Real vector multiplet — `mass term`
$$+\tfrac{1}{2}m^2\,A_\mu^i A^{i\,\mu}$$
`gauge fixing term`
$$-\tfrac{1}{2}\zeta\,(\partial_\mu A^{i\,\mu})(\partial_\nu A^{i\,\nu})$$

Possible self-interaction terms:

Real scalar field — `cubic term`
$$-\tfrac{1}{3!}\mu\,\phi^3$$
`quartic term`
$$-\tfrac{1}{4!}\lambda\,\phi^4$$

Complex scalar field — `cubic term`
$$-\mu\,\phi^\dagger(\phi^\dagger + \phi)\phi$$
`quartic term`
$$-\,\lambda\,(\phi^\dagger\phi)^2$$

Real vector field — `mixed term`
$$-(A_\mu A^\mu)(\partial_\nu A^\nu)$$
`quartic term`
$$-\,(A_\mu A^\mu)^2$$

Complex vector field — `mixed terms (+hc)`
$$-(A_\mu^\dagger A^\mu)(\partial_\nu A^\nu)$$
`quartic term`
$$-\,(A_\mu^\dagger A^\mu)^2$$

Note: The self-interaction terms for vector fields lead to inconsistencies unless their coupling constants are precisely chosen on the basis of a special type of symmetry, which must involve several vector fields. This symmetry underlies the non-Ahelian gauge theories.

Real vector multiplet — `ghost term for QCD`
$$\partial^\mu\eta_i[\partial_\mu\tilde\eta_i + g_s f^{ijk}\tilde\eta^j A_\mu^k]$$
$$= \partial_\mu\tilde\eta^i\partial^\mu\eta^i + g_s f^{ijk}(\partial^\mu\tilde\eta^i)A_\mu^j\eta^k$$
`ghost term for QED`
$$\partial_\mu\eta\,\partial^\mu\tilde\eta$$

Spinor / Bispinor (Majorana and Dirac notation):

$$\psi_a := \chi_a$$

$$\Psi_{\mathrm{M}}(x) = \begin{pmatrix} \chi_a(x) \\ \overline{\chi}^{\dot a}(x) \end{pmatrix},\quad \Psi_{\mathrm{M}}^\dagger = \begin{pmatrix} \overline{\chi}_{\dot a} \\ \chi_a \end{pmatrix}^\intercal,$$

$$\overline{\Psi}_{\mathrm{M}} = \Psi_{\mathrm{M}}^\dagger\beta = \begin{pmatrix} \chi_a \\ \overline{\chi}_{\dot a} \end{pmatrix}^\intercal$$

$$\Psi(x) = \begin{pmatrix} \chi_a(x) \\ \overline{\psi}^{\dot a}(x) \end{pmatrix},\quad \Psi^\dagger = \begin{pmatrix} \overline{\chi}_{\dot a} \\ \psi^a \end{pmatrix}^\intercal,$$

$$\overline{\Psi} = \Psi^\dagger\beta = \begin{pmatrix} \psi^a \\ \overline{\chi}_{\dot a} \end{pmatrix}^\intercal,$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{a\dot b} \\ \overline{\sigma}^{\mu\dot a b} & 0 \end{pmatrix},\quad \beta = \begin{pmatrix} 0 & \delta_a{}^b \\ \delta^{\dot a}{}_{\dot b} & 0 \end{pmatrix}$$

Free-field examples (with propagators):

Real scalar field — `Massive neutral spin-0`
$$\mathcal{L} = \tfrac{1}{2}\phi\,(-\Box - m^2)\phi$$
`Meson propagator:`
$$\mathrm{i}\Delta_{\mathrm{F}}(k) = \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon^+}$$

Complex scalar field — `Massive charged spin-0 / KGF`
$$\mathcal{L} = \phi^\dagger(-\Box - m^2)\phi$$
`Meson propagator:`
$$\mathrm{i}\Delta_{\mathrm{F}}(k) = \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon^+}$$

Spinor field — `Majorana Lagrangian` ($\mathcal{L} \neq \mathcal{L}^\dagger$)
$$\mathcal{L} = \overline{\Psi}_{\mathrm{M}}(\mathrm{i}\gamma^\mu\partial_\mu - m)\Psi_{\mathrm{M}}$$

Bispinor field — `Dirac Lagrangian` ($\mathcal{L} \neq \mathcal{L}^\dagger$)
$$\mathcal{L} = \overline{\Psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\Psi$$
$$\mathcal{L}^\dagger = \mathrm{i}\partial_\mu\overline{\Psi}(\gamma^\mu - m)\Psi$$
$$\mathcal{L}' = \mathrm{Re}\,\mathcal{L} = \tfrac{1}{2}\overline{\Psi}\,\mathrm{i}\gamma^\mu\overleftrightarrow{\partial}_\mu\Psi - m\overline{\Psi}\Psi$$
$$= \mathcal{L} - \tfrac{1}{2}\mathrm{i}\partial_\mu(\overline{\Psi}\gamma^\mu\Psi) = \mathcal{L}'^\dagger$$
`Dirac fermion propagator:`
$$\mathrm{i}S_{\mathrm{F}}(p) = \frac{\mathrm{i}(\gamma^\mu p_\mu + m)}{p^2 - m^2 + \mathrm{i}\varepsilon^+}$$

Real vector field — `Massless neutral spin-1 (Maxwell)`
$$\mathcal{L} = \tfrac{1}{2}A^\mu\Big(g_{\mu\nu}\Box - (1 - \zeta)\partial_\mu\partial_\nu\Big)A^\nu$$
`Photon propagator:`
$$\mathrm{i}D_{\mathrm{F}}^{\mu\nu}(k) = \frac{-\mathrm{i}\big(g^{\mu\nu} - \frac{\zeta - 1}{\zeta}k^\mu k^\nu\big)}{k^2 + \mathrm{i}\varepsilon^+}$$
$\zeta = 1:$ Feynman gauge ($\xi = \zeta^{-1} = 1$)
$\zeta \to \infty:$ Landau gauge ($\xi = \zeta^{-1} = 0$)

`Massive neutral spin-1 (Proca)`
$$\mathcal{L} = \tfrac{1}{2}A^\mu\Big(g_{\mu\nu}(\Box + m^2) - \partial_\mu\partial_\nu\Big)A^\nu$$
`Massive vector propagator:`
$$\mathrm{i}D_{\mathrm{F}}^{\mu\nu}(k) = \frac{-\mathrm{i}\big(g^{\mu\nu} - \frac{1}{m^2}k^\mu k^\nu\big)}{k^2 - m^2 + \mathrm{i}\varepsilon^+}$$
`Stückelberg Lagrangian`
$$\mathcal{L} = -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(\partial_\mu\phi + mA_\mu)(\partial^\mu\phi + mA^\mu)$$
Gauge-fixing $\phi = 0$, yields the Proca action.

Legend:

$g_{\mu\nu}:$ Minkowski metric, diag$(1, -1, -1, -1)$
$\mu, \nu, ...:$ (World) tensor indices
$\alpha, \beta, ...:$ Four-component (Dirac) spinor indices
$a, b, ..., \dot a, \dot b, ...:$ Two-component (Weyl) spinor indices
$\partial_{a\dot a} := \sigma^\mu_{a\dot a}\partial_\mu,\quad \overline{\partial}^{\dot a a} := \overline{\sigma}^{\mu\,\dot a a}\partial_\mu$

Possible interaction terms:

`Yukawa coupling`
$$-g\,\overline{\Psi}\Psi\,\phi$$

`Scalar QED`
$$+e\,\phi\partial^\mu\phi^\dagger\,A_\mu$$
$$+e\,(\phi^\dagger\phi)(A_\mu A^\mu)$$

`(Spinor) QED`
$$-e\,\overline{\Psi}\gamma^\mu\Psi\,A_\mu$$

`Yukawa coupling`
$$-g\,\overline{\Psi}\Psi\,\phi$$

`(Spinor) QED`
$$-e\,\overline{\Psi}\gamma^\mu\Psi\,A_\mu$$

`Scalar QED`
$$+e\,\phi\partial^\mu\phi^\dagger\,A_\mu$$
$$+e\,(\phi^\dagger\phi)(A_\mu A^\mu)$$

Prescription for building a consistent Lagrangian:

1. The Lagrangian must be real modulo total derivative (required by CPT invariance).
2. All the added terms must be Lorentz-invariant.
3. All the added terms must be of dimension less or equal M^4.

N.b. Any interaction of higher dimension than M^4 leads to a nonrenormalizable theory. Hence, ignoring all the constants, the dimension of any term must be at most M^4 since the dimension of a Lagrangian density is M^4.

$$[S] = \mathsf{M}^0,\ [\mathcal{L}] = \mathsf{M}^4,\ [\partial_\mu] = [\phi] = [A_\mu] = \mathsf{M}^1,\ [\Psi] = [\chi_a] = \mathsf{M}^{3/2},\ [F_{\mu\nu}] = \mathsf{M}^2.$$