THE COHOMOLOGY OF BRST COMPLEXES

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ABSTRACT

We abstract the general cohomological features of BRST complexes: both on states and on operators. We discuss a decomposition theorem analogous to that of Hodge for the de Rham complex. This allows a characterization of the operator cohomology fully in terms of the standard BRST cohomology. In particular we infer a vanishing theorem for the operator cohomology from the similar result for BRST cohomology. The characterization of the operator cohomology allows us to prove the necessity of the vanishing theorem for consistency of the BRST quantization. Finally the decomposition theorem together with the vanishing theorem allows a reformulation of the "no-ghost" theorem which is directly amenable to computation.

§1 INTRODUCTION

In this paper we study BRST quantization from a purely cohomological point of view. The fundamental tool in our analysis is the existence of a positive definite inner product in the Fock space. Of course cohomology is an algebraic construction and as such does not depend on the particular inner product. However the extra structure makes it very easy to obtain interesting results. The inner product we consider has the added benefit that it coincides with the "true" inner product on the physical states if the theory is free of "ghosts". In fact the "no-ghost" theorem is reformulated in precisely these terms. This was first done in **[1]** for the case of the open bosonic string.

This paper is organized as follows. In §2 we introduce two BRST complexes: the one where the BRST operator acts on states and the one where it acts on operators. In §3 we discuss the analogue of the Hodge decomposition theorem for BRST cohomology. In §4 we use the decomposition theorem to characterize the operator cohomology in terms of the ordinary BRST cohomology. We prove that the vanishing theorem is necessary for a consistent BRST quantization and that a vanishing theorem in BRST cohomology implies a vanishing theorem in operator cohomology and viceversa. In §5 we use the decomposition theorem to reformulate the "no ghost" theorem in this language, and under the assumption that a vanishing theorem holds for BRST cohomology we reduce the proof of the "no-ghost" theorem to a direct computation. We also relate the partition function of the theory to a character-valued index of a Dirac-type operator made out of the BRST operator. Finally in §6 we give some concluding remarks.

In a sequel[6] to this paper we exploit the methods advocated here to obtain some interesting results for the case of the open bosonic string.

§2 BRST COHOMOLOGY

It is often difficult to treat a dynamical system while keeping only the degrees of freedom which have physical relevance and we sometimes opt to work with a larger phase space. This "covariant" phase space therefore contains redundant degrees of freedom; but, nevertheless, the physical phase space can easily be recovered via symplectic reduction.

This problem still persists in canonical quantization. It is often difficult to quantize only the functions on the physical phase space so we resort to quantizing the functions on the covariant phase space; a task which is usually straight-forward. However we are left with an added complication. We need a prescription to recover the physical quanta from the full Hilbert space of the covariantly quantized theory. Dirac[2] gave a prescription for doing just this. Roughly in Dirac quantization one quantizes in particular the functions which generate flows in the unphysical directions and imposes that the operators representing them annihilate the physical states. If the constraint algebra still closes after quantization and the dynamics still preserve the constraints, *i.e.* there are no anomalous terms in the commutators among the constraints and between the constraints and the Hamiltonian, then this prescription makes sense and can be implemented.

The BRST procedure is, in most cases, an equivalent formulation of the Dirac prescription. The covariant phase space is enlarged by the addition of new unphysical degrees of freedom: ghosts and antighosts, and the constraints are replaced by a single nilpotent object known as the BRST operator. This system is no harder to quantize than the functions on the covariant phase space and, upon quantization, the physical space is defined to be a particular cohomology group of the quantized BRST operator. Of course, this only makes sense if the quantum BRST operator is nilpotent and commutes with the Hamiltonian, which are precisely the obstructions one encounters trying to implement the Dirac prescription. There are advantages, however, in the BRST formulation. First of all it is easier computationally. But more importantly it makes direct contact with the original problem:

quantizing the functions on the physical phase space. One can show[3] that under certain circumstances the classical counterpart of the cohomology group used to define the physical states in the quantum theory consists of precisely the functions on the physical phase space.

Furthermore the BRST procedure gives us a hint as to what kind of physical systems can be quantized consistently. The obstruction to the nilpotency of the quantum BRST operator can be analyzed using Lie algebra cohomology.[3] Roughly this cohomology class is the class associated to the central extension of the quantum constraint algebra defined by the Schwinger term in the commutator of the constraints. Hence it is independent to a large extent from the particular physical system one is analyzing but depends only on the algebraic structure of the constraint algebra. The allowed physical systems are precisely those whose algebra of observables affords a projective representation of the opposite class. We are familiar with this in string theory where, for instance, in the open bosonic string, the energy momentum tensor of the reparametrization ghosts has a central term in its operator product expansion which only depends on the conformal character of the ghost fields themselves. This constraints the allowed matter fields by requiring that the central term in the operator product expansion of the combined energy momentum tensor vanishes.

We now look in detail at some general features of BRST quantized systems, Consider a physical system quantized in the BRST formalism. We assume that the quantization is consistent so that the BRST operator is indeed nilpotent, leaves the vacuum invariant and is self-adjoint with respect to the non-degenerate inner product induced by the quantization procedure. So let $\mathcal{F}$ denote our Fock space and Q the nilpotent BRST operator. It is clear that for such an operator to be non-trivial the norm of $\mathcal{F}$ must be indefinite. Otherwise for all states ψ in $\mathcal{F}$, $\|Q\psi\|^2 = \langle Q\psi, Q\psi\rangle = \langle \psi, Q^2\psi\rangle = 0$ and hence Q is identically zero. Therefore $\mathcal{F}$ must have null states (*i.e.* states of zero norm), but because the norm is non-degenerate it must also contain negative norm states. Consequently the indefinite nature of the Fock space seems inherent to the BRST formalism and independent from the fact that we may be quantizing in a Lorentz covariant fashion; although this is how indefinite Fock spaces usually enter in quantum field theory.

Because of the construction of our Fock space we can assign to every state ψ an operator $\mathcal{O}_\psi$ which creates it when acting on the Fock vacuum. This operator will be a polynomial in the creation operators. Of particular importance are the states created by monomials. These generate the entire Fock space and will hereafter be referred to as basis states.

There is a natural grading of $\mathcal{F}$ provided by the ghost number operator $\mathcal{G}$. This a self-adjoint operator with integer eigenvalues[1] which, when acting on a basis state ψ, counts the number of ghost oscillators in $\mathcal{O}_\psi$ minus the number of antighost oscillators. Therefore $\mathcal{F}$ can be decomposed as the following direct sum of vector spaces

$$\mathcal{F} = \bigoplus_g \mathcal{F}_g \ , \tag{2.1}$$

where for ψ in $\mathcal{F}_g$, $\mathcal{G}\psi = g\,\psi$.

We also assume that there are enough mutually commuting operators which in turn commute with Q and with $\mathcal{G}$ and which provide a decomposition of $\mathcal{F}$ into finite dimensional subspaces. This assumption does not seem very restrictive and will be used strongly in many of the constructions which follow. In the case of the open bosonic string, for instance, we will have the momentum of the string and the level operator in the Hamiltonian. For definiteness of notation we will assume that these eigenvalues are discrete and we will write the decomposition of $\mathcal{F}_g$ as

$$\mathcal{F}_g = \bigoplus_\lambda \mathcal{F}_g(\lambda) \qquad \text{and} \qquad \dim \mathcal{F}_g(\lambda) < \infty \ . \tag{2.2}$$

By construction the BRST operator obeys $[\mathcal{G}\,,\,Q] = Q$, so that it has ghost

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[1] We assume this for notational convenience. As is well known, in some theories the grading will be half-integral.

number 1. For convenience we shall often denote by Q_g the map

$$Q_g : \mathcal{F}_g \longrightarrow \mathcal{F}_{g+1} \ . \tag{2.3}$$

Nilpotency of Q implies that $Q_{g+1} \circ Q_g = 0$ for all g and thus the following sequence defines a differential complex known as the **BRST complex**:

$$\cdots \longrightarrow \mathcal{F}_{g-1} \xrightarrow{Q_{g-1}} \mathcal{F}_g \xrightarrow{Q_g} \mathcal{F}_{g+1} \longrightarrow \cdots \tag{2.4}$$

For every g define the following subspaces of $\mathcal{F}_g$

$$\begin{aligned} \ker Q_g &= \{\psi \in \mathcal{F}_g \mid Q\psi = 0\} \ , \\ \operatorname{im} Q_{g-1} &= \{Q\psi \mid \psi \in \mathcal{F}_{g-1}\} \ . \end{aligned} \tag{2.5}$$

Elements of $\ker Q_g$ are called **BRST cocycles** and elements of $\operatorname{im} Q_{g-1}$ are called **BRST coboundaries**. It is clear that $\operatorname{im} Q_{g-1} \subset \ker Q_g$. The obstruction to the reverse inclusion is measured by the g^{th} cohomology space

$$H^g(Q) = \frac{\ker Q_g}{\operatorname{im} Q_{g-1}} \ . \tag{2.6}$$

The cohomology of this complex is the direct sum of vector spaces

$$H(Q) = \bigoplus_g H^g(Q) \ , \tag{2.7}$$

and is called the **BRST cohomology**.

Using the decomposition in (2.2) we can decompose the BRST complex into subcomplexes indexed by $\{\lambda\}$:

$$\cdots \longrightarrow \mathcal{F}_{g-1}(\lambda) \xrightarrow{Q^\lambda_{g-1}} \mathcal{F}_g(\lambda) \xrightarrow{Q^\lambda_g} \mathcal{F}_{g+1}(\lambda) \longrightarrow \cdots \tag{2.8}$$

and we can equally well decompose the cohomology space $H^g(Q)$ as follows

$$H^g(Q) = \bigoplus_\lambda H^g_\lambda(Q) \ , \tag{2.9}$$

where $H^g_\lambda(Q) = H^g(Q^\lambda)$ is the cohomology of the restricted operator.

So far we have been discussing the action of the BRST operator on states. However the BRST operator acts on operators as well. In fact, given that the functions on the phase space are mapped via the quantization procedure to self-adjoint operators, it is the BRST action on operators that should be addressed when trying to make contact with the classical BRST operator and its cohomology[**4**],[**3**]

Let us denote by $\mathrm{End}\,\mathcal{F}$ the algebra of operators on $\mathcal{F}$. It is again graded by ghost number:

$$\mathrm{End}\,\mathcal{F} = \bigoplus_g \mathrm{End}_g\mathcal{F} \;, \tag{2.10}$$

where

$$\varphi \in \mathrm{End}_g\mathcal{F} \quad \Leftrightarrow \quad [\mathcal{G}\,,\,\varphi] = g\,\varphi \;. \tag{2.11}$$

There is a natural action of the BRST operator on $\mathrm{End}\,\mathcal{F}$ given by the graded commutator and denoted $\mathrm{ad}Q$,

$$\mathrm{ad}Q\cdot\varphi \equiv Q\circ\varphi - (-1)^g\,\varphi\circ Q \qquad \text{for } \varphi\in\mathrm{End}_g\mathcal{F} \;. \tag{2.12}$$

Since the BRST operator has ghost number 1, $\mathrm{ad}Q$ maps $\mathrm{End}_g\mathcal{F}$ to $\mathrm{End}_{g+1}\mathcal{F}$ and furthermore $(\mathrm{ad}Q)^2 = \mathrm{ad}Q^2 = 0$ giving rise to the following complex:

$$\cdots \longrightarrow \mathrm{End}_{g-1}\mathcal{F} \stackrel{\mathrm{ad}\,Q}{\longrightarrow} \mathrm{End}_g\mathcal{F} \stackrel{\mathrm{ad}\,Q}{\longrightarrow} \mathrm{End}_{g+1}\mathcal{F} \longrightarrow \cdots \tag{2.13}$$

whose cohomology $H(\mathrm{ad}Q)$ we shall refer to as the **operator BRST cohomology**.

Let us try to understand the operator cohomology in terms of the usual BRST cohomology defined in (2.6) and (2.7) . An operator $\varphi \in \mathrm{End}_g\mathcal{F}$ which (anti)-commutes with Q — *i.e.* which lies in $\ker\mathrm{ad}Q$ — stabilizes both $\ker Q$ and $\mathrm{im}\,Q$ and thus induces a map $\varphi_* : H^p(Q) \longrightarrow H^{p+g}(Q)$ in BRST cohomology. A map such as φ is called a **chain map**. On the other hand operators which lie in $\mathrm{im}\,\mathrm{ad}Q$

are said to be **chain homotopic to zero**. These operators induce the zero map in BRST cohomology. Therefore the operator cohomology groups are given by

$$H^g(\mathrm{ad}Q) = \frac{\text{chain maps in } \mathrm{End}_g\mathcal{F}}{\text{maps in } \mathrm{End}_g\mathcal{F} \text{ which are chain homotopic to zero}} \; .$$

Each class in $H^g(\mathrm{ad}Q)$ induces a well defined map in BRST cohomology via $[\varphi] \mapsto \varphi_*$. It is clearly well defined because two chain maps which are chain homotopic induce the same map in BRST cohomology. What is not completely obvious but nevertheless very interesting is that there is a converse to that statement. Namely that two chain maps inducing the same map in cohomology are chain homotopic. The proof is not difficult given the decomposition theorem of the next section, and we'll present the proof there. Therefore we can think of the operator chomology as sitting inside the algebra of operators $\mathrm{End}\, H(Q)$. Moreover we'll see that given any operator in $\mathrm{End}\, H(Q)$ there is a chain map inducing it which by the above remark is unique up to chain homotopy. Therefore the operator cohomology is precisely $\mathrm{End}\, H(Q)$.

A final comment is in order. Notice that the operator cohomology has a further algebraic structure. Namely it is a graded algebra with a multiplication

$$H^p(\mathrm{ad}\, Q) \otimes H^q(\mathrm{ad}\, Q) \longrightarrow H^{p+q}(\mathrm{ad}\, Q) \tag{2.14}$$

induced from composition of operators. To see this notice that

$$\mathrm{ad}\, Q(\varphi \circ \psi) = (\mathrm{ad}\, Q\, \varphi) \circ \psi + (-1)^g \varphi \circ (\mathrm{ad}\, Q \psi) \qquad \text{for } \varphi \in \mathrm{End}_g\mathcal{F} \; . \tag{2.15}$$

Therefore composition of operators maps

$$\begin{aligned} \ker \mathrm{ad}\, Q \otimes \ker \mathrm{ad}\, Q &\longrightarrow \ker \mathrm{ad}\, Q \\ \ker \mathrm{ad}\, Q \otimes \mathrm{im}\, \mathrm{ad}\, Q &\longrightarrow \mathrm{im}\, \mathrm{ad}\, Q \; , \end{aligned}$$

which makes the following operation well defined

$$[\varphi] \cdot [\psi] \longrightarrow [\varphi \circ \psi] \ . \tag{2.16}$$

We will see that the BRST invariant states will be created by BRST invariant operators acting on the vacuum. Therefore the BRST cohomology can be given a multiplication induced from the one on operators. Since physical states are defined (see next section) as BRST cohomology classes at zero ghost number, this may be an interesting way to define interaction vartices, thought of as maps $\mathcal{H}_{\text{phys}} \otimes \mathcal{H}_{\text{phys}} \longrightarrow \mathcal{H}_{\text{phys}}$ where $\mathcal{H}_{\text{phys}} \subseteq H(Q)$.

§3 THE DECOMPOSITION THEOREM

In this section we prove the decomposition theorem. This allows us to identify the BRST cohomology – which is a subquotient — as a particular subspace of the kernel of the BRST operator. In other words, the decomposition theorem picks out a privileged representative from each cohomology class. This theorem is very powerful and we present in the next two sections two immediate consequences. The first one is the characterization of the operator cohomology introduced in the last section in terms of the BRST cohomology. The second one is the reformulation of the "no-ghost" theorem which is reduced to the computation of two weighted traces, given the vanishing theorem for the BRST cohomology.

As we remarked earlier the BRST procedure introduces degrees of freedom which were not present in the original formulation of the theory and it is therefore necessary to define which states in $\mathcal{F}$ correspond to physical quanta. In the case of BRST quantization we define the physical states $\mathcal{H}_{\text{phys}}$ not as a subspace of $\mathcal{F}$ but as an equivalence class. In fact the physical space is defined as $H^0(Q)$, the BRST cohomology space at zero ghost number.

Notice that $H(Q)$ inherits a well defined inner product from $\mathcal{F}$ since Q is self-adjoint. Indeed pick any two classes in $H(Q)$ and define their inner product by choosing any representative from each class and evaluating their inner product in $\mathcal{F}$. This is independent of the choice of representatives because $\operatorname{im} Q$ is orthogonal to $\ker Q$: a fact that follows since Q is self-adjoint.

However cohomology is an algebraic construction that is independent of the inner product. In this sense we are free to choose a convenient inner product which, in principle, is different from the inner product induced by the quantization procedure. In particular it is very convenient, as we will now see, to have a positive definite inner product. We achieve this via the introduction of a self-adjoint involution (*cf.* [**1**]) $\mathcal{C}$ in $\mathcal{F}$. Its sole purpose is to redefine the inner product so that it be both positive definite and hermitian.

To this effect we now choose an pseudo-orthonormal basis in $\mathcal{F}$, *i.e.* a basis whose elements are mutually orthogonal and of norm ± 1. We can restrict ourselves to the finite dimensional eigenspaces of the family of commuting self-adjoint operators whose existence was assumed in §2, since eigenspaces corresponding to distinct eigenvalues will be orthogonal. In this basis the metric will be diagonal with entries equal to ± 1. We now define $\mathcal{C}$ to be the identity when restricted to the subspace of positive norm and minus the identity when restricted to its complement. $\mathcal{C}$ defined this way is unique because any two such bases will be related by a pseudo-orthogonal transformation which leaves $\mathcal{C}$ invariant. In fact, notice that in any such basis the matrices for $\mathcal{C}$ and for the inner product agree numerically. Finally, we remark that because we take the Fock vacuum to have unit norm, $\mathcal{C}$ leaves the vacuum invariant.

Equipped with such an operator we now introduce a new inner product in $\mathcal{F}$ as follows:

$$\langle \psi, \phi \rangle_{\mathcal{C}} \stackrel{\text{def}}{=} \langle \psi, \mathcal{C}\phi \rangle = \langle \mathcal{C}\psi, \phi \rangle \ , \tag{3.1}$$

for all ψ and ϕ in $\mathcal{F}$. The positive definiteness of this new inner product implies that $\mathcal{C}$ must map $\mathcal{F}_g$ to $\mathcal{F}_{-g}$, since the old inner product only coupled states of opposite ghost number. This is easily seen from the fact that the inner product in

$\mathcal{F}$ is given by the canonical supercommutation relations of the oscillators and these vanish except between oscillators of opposite ghost number. Other properties of $\mathcal{C}$ are particular to the actual theory we are quantizing. For instance, in the case of the open bosonic string, $\mathcal{C}$ will turn out to involve time reversal as well.

Under this new inner product Q is no longer self-adjoint. In fact we denote its adjoint by Q^*. That is, for any two states ψ and ϕ in $\mathcal{F}$, $\langle Q\psi,\, \phi\rangle_{\mathcal{C}} = \langle \psi,\, Q^*\phi\rangle_{\mathcal{C}}$. It is easy to give an explicit expression for Q^*. In fact let $\mathcal{O}$ be any operator, self-adjoint or not. Then,

$$\begin{aligned} \langle \psi,\, \mathcal{O}^*\phi\rangle_{\mathcal{C}} =& \langle \mathcal{O}\psi,\, \phi\rangle_{\mathcal{C}} \\ =& \langle \mathcal{O}\psi,\, \mathcal{C}\phi\rangle \\ =& \langle \psi,\, \mathcal{O}^\dagger \mathcal{C}\phi\rangle \\ =& \langle \psi,\, \mathcal{C}\mathcal{O}^\dagger \mathcal{C}\phi\rangle_{\mathcal{C}} \qquad \text{since } \mathcal{C} \text{ is an involution.} \end{aligned}$$

Therefore we see that $\mathcal{O}^* = \mathcal{C}\,\mathcal{O}^\dagger\,\mathcal{C}$; and, in particular, $Q^* = \mathcal{C}\,Q\,\mathcal{C}$.

This new operator Q^* has similar properties to Q. In particular it is nilpotent and it has ghost number -1. Therefore we have the following differential complex dual to the BRST complex:

$$\cdots \longrightarrow \mathcal{F}_{g+1} \xrightarrow{Q^*_{g+1}} \mathcal{F}_g \xrightarrow{Q^*_g} \mathcal{F}_{g-1} \longrightarrow \cdots \tag{3.2}$$

Just as we did in §2 for the BRST complex we can define its cohomology as the direct sum of vector spaces

$$H(Q^*) = \bigoplus_g H^g(Q^*)\ , \tag{3.3}$$

where the definition of $H^g(Q^*)$ parallels that of $H^g(Q)$ in §2.

These cohomologies are not unrelated. Indeed, we claim that $H^{-g}(Q^*)$ is isomorphic to $H^g(Q)$. Consider the isomorphism $\mathcal{C}\colon \mathcal{F}_g \longrightarrow \mathcal{F}_{-g}$. It follows from the explicit expression for Q^* that the squares in the following diagram commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{F}_{g-1} & \stackrel{Q_{g-1}}{\longrightarrow} & \mathcal{F}_g & \stackrel{Q_g}{\longrightarrow} & \mathcal{F}_{g+1} & \longrightarrow & \cdots \\
 & & \downarrow \mathcal{C} & & \downarrow \mathcal{C} & & \downarrow \mathcal{C} & & \\
\cdots & \longrightarrow & \mathcal{F}_{-g+1} & \stackrel{Q^*_{-g+1}}{\longrightarrow} & \mathcal{F}_{-g} & \stackrel{Q^*_g}{\longrightarrow} & \mathcal{F}_{-g-1} & \longrightarrow & \cdots
\end{array}$$

That is, $\mathcal{C}$ is a chain map and thus induces a well-defined map in cohomology sending $H^g(Q) \longrightarrow H^{-g}(Q^*)$ which, abusing the notation, will also be referred to as $\mathcal{C}$ and which is defined to map $[\psi] \mapsto [\mathcal{C}\psi]$. Since $\mathcal{C}$ is an isomorphism, the claim follows.

Now we come to the decomposition theorem. Because the new inner product is positive definite we can split $\mathcal{F}$ as the orthogonal direct sum of vector spaces[2]

$$\mathcal{F} = \operatorname{im} Q \oplus (\operatorname{im} Q)^{\perp} \ . \tag{3.4}$$

Notice, however, that $(\operatorname{im} Q)^{\perp} = \ker Q^*$ since Q and Q^* are adjoints under this inner product. Now let ψ be a state in $\ker Q$. Under the above decomposition of $\mathcal{F}$ we can write ψ uniquely as a sum of two states $\phi + Q\chi$, where ϕ is in $\ker Q^*$. Let $\mathcal{H}$ stand for the intersection $\ker Q \cap \ker Q^*$. Then $\mathcal{H}$ is a direct sum

$$\mathcal{H} = \bigoplus_g \mathcal{H}^g \ , \tag{3.5}$$

where $\mathcal{H}^g = \mathcal{H} \cap \mathcal{F}_g$. Let h denote the projection onto $\mathcal{H}$. Then $h(\psi) = \phi$, where ψ and ϕ are as above. This projection induces a map in cohomology, which we also call h, and which maps $[\psi] \mapsto h(\psi)$. It is clearly independent of the particular

[2] This decomposition can be done in steps by restricting ourselves to each $\mathcal{F}(\lambda)$, since these eigenspaces are orthogonal and hence $\mathcal{C}$ stabilizes each eigenspace. For notational convenience, however, we drop the λ dependence which will be understood.

representative we chose and moreover it is injective since $h(\psi) = 0$ if and only if ψ is cohomologous to zero. This provides us with an isomorphism between $H^g(Q)$ and $\mathcal{H}^g$.

We could have done exactly the same construction with Q^* and thus obtain an isomorphism $H^g(Q^*) \simeq \mathcal{H}^g$. This gives us an isomorphism $H^g(Q) \simeq H^g(Q^*)$ which, together with the isomorphism $H^g(Q) \simeq H^{-g}(Q^*)$ induced by $\mathcal{C}$, gives the first important result about the BRST cohomology; namely

$$H^g(Q) \simeq H^{-g}(Q) \ . \tag{3.6}$$

In analogy with the similar result for the de Rham cohomology of a compact oriented manifold we will refer to the above isomorphism as Poincaré duality.

Notice that the above construction for both Q and Q^* gives a decomposition of $\mathcal{F}_g$ as the orthogonal direct sum

$$\mathcal{F}_g = \operatorname{im} Q_{g-1} \oplus \operatorname{im} Q^*_{g+1} \oplus \mathcal{H}^g \ . \tag{3.7}$$

We may further identify the space $\mathcal{H}^g$ with the kernel of a new operator. Define the **BRST laplacian** as

$$\triangle \stackrel{\text{def}}{=} QQ^* + Q^*Q \ . \tag{3.8}$$

It is a self-adjoint operator which satisfies the following properties:

$$\triangle\, \mathcal{C} = \mathcal{C} \triangle \tag{3.9}$$

$$\triangle\, Q = Q \triangle \tag{3.10}$$

$$\triangle\, Q^* = Q^* \triangle \tag{3.11}$$

$$\mathcal{H} = \ker \triangle \tag{3.12}$$

The first three properties are trivially verified and are left as exercises for the reader. We prove the last one. Let ψ be in $\ker \triangle$. Then, in particular $\langle \triangle\psi,\, \psi\rangle_{\mathcal{C}} = 0$. But by definition, $\langle \triangle\psi, \psi\rangle_{\mathcal{C}} = \|Q\psi\|^2 + \|Q^*\psi\|^2$ which, being a sum of non-negative quantities, must vanish termwise. Therefore, since the norm is positive definite, ψ must be annihilated by both Q and Q^* and hence be an element of $\mathcal{H}$. Conversely, if $\psi \in \mathcal{H}$ it is trivially in $\ker\triangle$. This proves the assertion. States in $\mathcal{H}$ will be referred to as "harmonic", in analogy with the Hodge decomposition for de Rham cohomology. It is worth remarking that it follows from the definition of $\triangle$ that it commutes with any operator commuting with Q and Q^* or, equivalently, with Q and $\mathcal{C}$. Therefore, in particular, it maps $\mathcal{F}_g(\lambda) \longrightarrow \mathcal{F}_g(\lambda)$.

We now define the **Green's operator** to be an inverse to the BRST laplacian away from its kernel. In fact let $h : \mathcal{F} \to \mathcal{H}$ denote the projection onto the harmonic states. Then letting $\mathcal{H}^\perp$ stand for $\mathrm{im}\,Q \oplus \mathrm{im}\,Q^*$ we define the Green's operator to be a map $G : \mathcal{F} \to \mathcal{H}^\perp$ such that $G\,\psi = \omega$, where ω is the unique solution of $\triangle\,\omega = \psi - h(\psi)$ in $\mathcal{H}^\perp$. That such a solution is indeed unique is easy to verify.

The most important property of the Green's operator is that it commutes with every operator which commutes with the laplacian. In fact let T be any operator commuting with the BRST laplacian. Then T stabilizes both the image and kernel of the BRST laplacian. However the image of the BRST laplacian is just $\mathcal{H}^\perp$. Therefore let $\psi \in \mathcal{H}$. Then $G\,\psi = 0$, hence $T\,G\,\psi = 0$. But also $T\,\psi \in \mathcal{H}$ and hence $G\,T\,\psi = 0$. Now let $\psi \in \mathcal{H}^\perp$. Then by definition $G\,\psi = \omega$ where ω is the unique solution to $\triangle\,\omega = \psi$, since $h(\psi) = 0$. Therefore, $T\,G\,\psi = T\,\omega$. Now, $G\,T\,\psi = \phi$, where ϕ is the unique solution to $\triangle\,\phi = T\,\psi$, since $h(T\,\psi) = T\,h(\psi) = 0$. But $T\,\omega$ also satisfies $\triangle\,T\,\omega = T\,\triangle\,\omega = T\,\psi$. Hence by uniqueness $T\,\omega = \phi$ and G and T commute.

As a corollary of the above result we have that G commutes with $\mathcal{C}$, Q, Q^*, $\mathcal{G}$ and the family of commuting self-adjoint operators whose existence was assumed in §2. In particular G stabilizes each $\mathcal{F}_g(\lambda)$. It also stabilizes $\mathrm{im}\,Q$ and $\mathrm{im}\,Q^*$.

It is worth remarking that 3.9 together with 3.12 imply the isomorphism 3.6 and therefore, comparing this to the proof of Poincaré duality from the Hodge decomposition theorem, we see that $\mathcal{C}$ plays an analogous rôle to the Hodge duality operator.

In this language we see that the space of physical states $\mathcal{H}_{\mathrm{phys}}$ is, by definition, $\mathcal{H}^0$, the harmonic states of zero ghost number. In fact, we will see in §4 that for a consistent BRST quantization there must not be any other harmonic states. This result is known as the "vanishing theorem" for BRST cohomology. It was proven in [**5**] as a corollary of a general vanishing theorem valid for a large class of graded Lie algebras and representations.

§4 THE OPERATOR BRST COHOMOLOGY

We now come to the first application of the decomposition theorem proven in the previous section. Here we will prove that the operator cohomology $H(\mathrm{ad}\,Q)$ is isomophic to the algebra of operators $\mathrm{End}\,H(Q)$. Recall that we have a well-defined map

$$_* : H(\mathrm{ad}\,Q) \longrightarrow \mathrm{End}\,H(Q)$$

defined by

$$[\varphi] \mapsto \varphi_* \ .$$

We show that ths map is an isomorphism. That is, we show that every map in cohomology (*i.e.* every element of $\mathrm{End}\,H(Q)$) is induced by a chain map and hence by a class in $H(\mathrm{ad}\,Q)$, thus proving surjectivity. Then we show that if two chain maps induce the same map in cohomology they are necessarily chain homotopic, thus proving injectivity.

It will be very convenient for both steps to introduce an auxiliary concept. Let us denote by $\pi : H(Q) \xrightarrow{\cong} \mathcal{H}$ the isomorphism between the BRST cohomology and the BRST harmonic states which the decomposition theorem yields. Given any map $\psi \in \mathrm{End}\,H(Q)$ let's denote by $\widehat{\psi} \in \mathrm{End}\,\mathcal{F}$ the map $\pi\,\psi\,\pi^{-1}$ extended trivially to all of $\mathcal{F}$. In other words, $\pi\,\psi\,\pi^{-1}$ as it stands is a map in $\mathrm{End}\,\mathcal{H}$. The trivial extension consists in having it vanish identically in $\operatorname{im} Q \oplus \operatorname{im} Q^*$. We call this the **minimal extension** of ψ and it is easily checked that it is a chain map with respect to both Q and Q^*. Moreover it is also easy to see that $\widehat{\psi}_* = \psi$. Hence this already proves surjectivity.

To prove injectivity all we have to show that if φ is any chain map then it is chain homotopic to the minimal extension $\widehat{\varphi}_*$ of φ_*. Given the decomposition $\mathcal{F} = \mathcal{H} \oplus \operatorname{im} Q \oplus \operatorname{im} Q^*$ we find it convenient to express all endomorphisms as 3×3 matrices of endomorphisms. Thus, for example, Q is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Q \\ 0 & 0 & 0 \end{pmatrix} ; \tag{4.1}$$

and the minimal extension $\widehat{\psi}$ of ψ is represented by

$$\begin{pmatrix} \widehat{\psi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.2}$$

Now let $\varphi \in \operatorname{End}_g \mathcal{F}$ be a chain map. Because it must map $\operatorname{im} Q \to \operatorname{im} Q$ and $\ker Q \to \ker Q$ it has the following matrix representation

$$\varphi = \begin{pmatrix} F & 0 & A \\ B & C & D \\ 0 & 0 & E \end{pmatrix} . \tag{4.3}$$

First of all, it is obvious that $F : \mathcal{H} \to \mathcal{H}$ must coincide with the minimal extension of φ_*. Also because it is a chain map, $Q\,\varphi = (-1)^g\,\varphi\, Q$ and hence C and E are not independent but rather

$$Q\,E = (-1)^g\,C\,Q\ . \tag{4.4}$$

Therefore the difference between the chain map φ and the minimal extension $\widehat{\varphi}_*$

can be represented by

$$\varphi - \widehat{\varphi}_* = \begin{pmatrix} 0 & 0 & A \\ B & C & D \\ 0 & 0 & E \end{pmatrix} \ , \tag{4.5}$$

where E and C obey equation 4.4 . We proceed to show that this is chain homotopic to zero. Indeed, consider the endomorphism $\mathbb{K} \in \mathrm{End}_{g-1}\,\mathcal{F}$ given by

$$\mathbb{K} = \begin{pmatrix} 0 & W & 0 \\ 0 & 0 & 0 \\ X & Y & Z \end{pmatrix} \ , \tag{4.6}$$

where

$$\begin{aligned} W &: \mathrm{im}\, Q \to \mathcal{H} \\ X &: \mathcal{H} \to \mathrm{im}\, Q^* \\ Y &: \mathrm{im}\, Q \to \mathrm{im}\, Q^* \\ Z &: \mathrm{im}\, Q^* \to \mathrm{im}\, Q^* \ . \end{aligned}$$

After a straight-forward calculation we see that

$$Q\,\mathbb{K} + (-1)^g\,\mathbb{K}\,Q = \begin{pmatrix} 0 & 0 & (-1)^g\,W\,Q \\ & & \\ Q\,X & Q\,Y & Q\,Z \\ & & \\ 0 & 0 & (-1)^g\,Y\,Q \end{pmatrix} \ . \tag{4.7}$$

Equating this with 4.5 we find that the following identities must be satisfied

$$\begin{aligned} A &= (-1)^g\,W\,Q \quad \text{mapping} \quad \mathrm{im}\, Q \to \mathcal{H} \\ B &= Q\,X \quad \text{mapping} \quad \mathcal{H} \to \mathrm{im}\, Q \\ C &= Q\,Y \quad \text{mapping} \quad \mathrm{im}\, Q \to \mathrm{im}\, Q \end{aligned}$$

$$D = Q\,Z \quad \text{mapping} \quad \operatorname{im} Q^* \to \operatorname{im} Q$$
$$E = (-1)^g\, Y\, Q \quad \text{mapping} \quad \operatorname{im} Q^* \to \operatorname{im} Q^* \ .$$

First of all we notice that since $\ker Q \cap \operatorname{im} Q^* = \mathbb{O}$, $Q\ |_{\operatorname{im} Q^*}$ is invertible and its inverse is given by $G\,Q^*$, where G is the Green's operator. Therefore we can indeed solve for X, W, Y, and Z in terms of A, B, C, D, and E as follows

$$\begin{aligned} W &= (-1)^g\, A\, G\, Q^* \\ X &= G\, Q^*\, B \\ Y &= G\, Q^*\, C \\ Z &= G\, Q^*\, D \\ Y &= (-1)^g\, E\, G\, Q^* \ . \end{aligned}$$

We must, of course, satisfy a consistency condition: namely that the two expressions for Y are really the same. But this can be trivially seen to follow from 4.4 .

Therefore we have shown that every chain map is chain homotopic to the minimal extension of the map it induces in cohomology. But this is clearly equivalent to injectivity. For let φ and ϑ be two chain maps which induce the same map in cohomology, *i.e.* such that $\varphi_* = \vartheta_*$. This implies that $\widehat{\varphi}_* = \widehat{\vartheta}_*$. Hence φ and ϑ are both chain homotopic to $\widehat{\varphi}_* = \widehat{\vartheta}_*$, and hence they are mutually chain homotopic.

Now suppose that a vanishing theorem holds for BRST cohomology, *i.e.*

$$\mathcal{H}^{g\neq 0}(Q) = \mathbb{O} \ . \tag{4.8}$$

Then it is clear that a vanishing theorem holds for the operator BRST cohomology since the only non-trivial endomorphisms of BRST cohomology consists of maps taking $H^0(Q)$ to $H^0(Q)$.

Conversely suppose that a vanishing theorem holds for the operator cohomology. Every physical state can be obtained from the vacuum by a suitable BRST invariant operator; just think of both the vacuum and the state as harmonic states and then find an endomorphism which takes one to the other. Then we see that all physical states have the same ghost number as the vacuum which is the vanishing theorem for BRST cohomology.

With this result in mind it is easy to justify why the vanishing theorem for BRST cohomology is physically desirable. Suppose that there is a BRST harmonic state ψ with ghost number g different from zero. Then since the operator cohomology coincides with the endomorphisms on $H(Q)$ there is certainly at least one BRST invariant operator $\mathcal{O}$ which creates ψ from the vacuum. Moreover and without loss of generality we can choose $\mathcal{O}$ to be a chain map with respect Q^* as well. Let $\mathcal{O}^*$ denote its adjoint under the positive definite inner product. It follows from the definition of this inner product that $\mathcal{O}^*$ has ghost number $-g$ and moreover that it is a chain map as well with respect to both Q and Q^*.

Consider the state $\mathcal{O}^* \psi$. This state cannot be zero because of positivity of the inner product: just take the inner product with the vacuum; and, furthermore, it is a BRST harmonic state of zero ghost number, *i.e.* a physical state! But this operator has ghost and anti-ghost excitations and hence would not be present in the spectrum of the theory had we quantized the physical phase space directly, however non-covariantly, without the introduction of the ghost and anti-ghost degrees of freedom. Hence the quantization procedure would be inconsistent. Therefore we conclude that a consistent quantization necessitates the vanishing theorem. Of course, the vanishing theorem is not sufficient, for there could be harmonic states of zero ghost number but which contain ghost and antighost excitations.

§5 THE REFORMULATION OF THE "NO-GHOST" THEOREM

In order to reformulate the no-ghost theorem we will analyze how $\mathcal{C}$ acts on the physical space. Recall that the physical space $\mathcal{H}_{\rm phys}$ is defined to be the harmonic states at zero ghost number $\mathcal{H}^0$. Because $\mathcal{C}$ maps $\mathcal{H}^g$ isomorphically to $\mathcal{H}^{-g}$, we see that it leaves $\mathcal{H}^0$ invariant and because $\mathcal{C}^2 = \mathrm{Id}$, we can break $\mathcal{H}^0$ into eigenspaces corresponding to its eigenvalues ± 1. We denote by $\mathcal{H}^0_\pm$ the subspaces of $\mathcal{H}^0$ on which $\mathcal{C}$ acts as $\pm\mathrm{Id}$. The definition of $\mathcal{C}$ was such that it was the identity when restricted to the positive definite subspace of $\mathcal{F}$ and minus the identity when restricted to the negative definite one. If the physical subspace is to be free of negative norm states then $\mathcal{C}$ must be the identity when restricted to it. That is, the "no-ghost" theorem is true if $\mathcal{H}^0 = \mathcal{H}^0_+$. Notice, however, that[3]

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}^0}\, C &= \dim\, \mathcal{H}^0_+ - \dim\, \mathcal{H}^0_- \\ &\leq \dim\, \mathcal{H}^0 \ . \end{aligned}$$

Thus it is precisely when this bound is saturated that the physical space is free of negative norm states. Let us define the signature $\tau(Q)$ of the BRST complex as $\mathrm{Tr}_{\mathcal{H}^0}\, \mathcal{C}$.

In practice the computation of $\mathrm{Tr}_{\mathcal{H}^0}\, C$ can be quite non-trivial, because the definition of $\mathcal{H}^0$ is not directly amenable to computations. However we can use the decomposition theorem to make this calculation easier. Recall that from the decomposition theorem of §3, $\mathcal{F}_0$ breaks up as

$$\mathcal{F}_0 = \mathrm{im}\, Q_{-1} \oplus \mathrm{im}\, Q_1^* \oplus \mathcal{H}^0 \ , \tag{5.1}$$

and that $\mathcal{C}$ maps $\mathrm{im}\, Q_{-1}$ isomorphically to $\mathrm{im}\, Q_1^*$ because it is a chain map. Therefore if we took the trace of $\mathcal{C}$ over all of $\mathcal{F}_0$ it would only pick a contribution from

[3] From now on traces over infinite dimensional spaces are to be interpreted as weighted with a regularizing parameter over each of the finite dimensional subspaces indexed by $\{\lambda\}$.

$\mathcal{H}^0$. Therefore we have

$$\mathrm{Tr}_{\mathcal{H}^0}\, \mathcal{C} = \mathrm{Tr}_{\mathcal{F}_0}\, \mathcal{C}\ . \tag{5.2}$$

In fact, since $\mathcal{C}$ takes $\mathcal{F}_g$ to $\mathcal{F}_{-g}$ we may extend the trace to the whole Fock space $\mathcal{F}$.

Now assume that the vanishing theorem for BRST cohomology holds, that is, $H^{g\neq 0}(Q) = 0$. Then we have the following equality

$$\dim\, \mathcal{H}^0 = \sum_g (-1)^g\, \dim\, \mathcal{H}^g\ . \tag{5.3}$$

The right hand side of this equation is the Euler characteristic of this differential complex and will be denoted by $\chi(Q)$. Again the calculation of $\chi(Q)$ may be non-trivial to perform. We rewrite it in a suitable way using the following standard fact from linear algebra:

$$\mathcal{F}_g \simeq \ker Q_g \oplus \mathrm{im}\, Q_g\ . \tag{5.4}$$

However $\ker Q_g$ splits into $\mathcal{H}^g \oplus \mathrm{im}\, Q_{g-1}$ which implies the following

$$\dim\, \mathcal{H}^g = \dim\, \mathcal{F}_g - \dim\, \mathrm{im}\, Q_g - \dim\, \mathrm{im}\, Q_{g-1}\ . \tag{5.5}$$

Performing the alternating sum we see that the last two terms of the right hand side cancel pairwise and we are left with the identity known in homological algebra as the Euler-Poincaré principle

$$\chi(Q) = \sum_g (-1)^g\, \dim\, \mathcal{F}_g = \mathrm{Tr}_{\mathcal{F}}\, (-1)^{\mathcal{G}}\ . \tag{5.6}$$

Therefore we can express succintly the condition for the absence of negative norm states from our physical space —under the assumption that the vanishing theorem holds— as

$$\chi(Q) = \tau(Q) \ . \tag{5.7}$$

However, for computational purposes, the identity to check is the following

$$\mathrm{Tr}_{\mathcal{F}}\,(-1)^{\mathcal{G}} = \mathrm{Tr}_{\mathcal{F}_0}\,\mathcal{C} \ . \tag{5.8}$$

We can rewrite these results in terms of an index theorem in much the same way that the Euler characteristic of a compact manifold can be expressed as the index of a suitable elliptic operator acting on the space of differential forms.

To this end we introduce a new grading in the BRST complex. We define the following "even" and "odd" subspaces

$$\mathcal{F}^e = \bigoplus_n \mathcal{F}_{2n} \qquad\qquad \mathcal{F}^o = \bigoplus_n \mathcal{F}_{2n+1} \ .$$

Then define the operator $D = Q + Q^*$ mapping $\mathcal{F}^e \to \mathcal{F}^o$. It is formally self adjoint except for the domain of definition. In fact its adjoint is $D^* = Q + Q^*$ mapping $\mathcal{F}^o \to \mathcal{F}^e$. In this way we turn the BRST complex into a two-space complex. We can furthermore grade each space as follows:

$$\mathcal{F}^e = \bigoplus_\lambda \mathcal{F}^e(\lambda) \qquad\qquad \mathcal{F}^o = \bigoplus_\lambda \mathcal{F}^o(\lambda) \ .$$

We can think of the family of cmmuting operators as providing a toral action on the Fock space. This action commutes with the operator D and its adjoint and therefore one can define its character-valued index. Therefore we define

$$\mathrm{ind}_q D = \sum_\lambda q^\lambda \,\mathrm{ind}_\lambda D \ ,$$

where q^λ is shorthand for $\prod_{i=1}^N q_i^{\lambda_i}$ (N is the number of mutually commuting operators providing this decomposition) an $\mathrm{ind}_\lambda D$ is the index of the operator D restricted to the eigenspace with eigenvalue λ. This index is finite by assumption because of the finite dimensionality of the eigenspaces $\mathcal{F}_g(\lambda)$.

Fixing an eigenvalue λ and restricting ourselves to $\mathcal{F}(\lambda)$ we can compute $\mathrm{ind}_\lambda D$ very easily using the relation between the cohomology classes and the harmonic states provided by Hodge decomposition. First of all notice that because of the positive definiteness of the inner product $D^*\,D\,\psi = 0 \Leftrightarrow D\,\psi = 0$ and $D\,D^*\,\psi = 0 \Leftrightarrow D^*\,\psi = 0$, or equivalently, $\ker D^*\,D = \ker D$ and $\ker D\,D^* = \ker D^*$. Notice also that $D^*\,D$ is nothing but the BRST laplacian restricted to the "even" subspace: $\triangle_e$, and $D\,D^*$ is the BRST laplacian restricted to the "odd" subspace: $\triangle_o$. Therefore,

$$\begin{aligned}\mathrm{ind}_\lambda D &= \dim\,\ker D - \dim\,\ker D^* \\ &= \dim\,\ker D^*\,D - \dim\,\ker D\,D^* \\ &= \dim\,\ker\,\triangle_e - \dim\,ker\,\triangle_o \\ &= \sum_g (-1)^g\,\dim\,H^g_\lambda(Q) \\ &= \chi_\lambda(Q)\ .\end{aligned} \tag{5.9}$$

And therefore the character-valued index is nothing but the weighted trace of the Euler characteristic of the BRST complex

$$\begin{aligned}\mathrm{ind}_q D &= \chi_q(Q) \\ &= \sum_\lambda q^\lambda\,\chi_\lambda(Q)\ .\end{aligned} \tag{5.10}$$

The "no-ghost" theorem for the open bosonic string was proven this way in [**1**]. The same can be done for the NSR string with and without GSO projection.

§6 CONCLUSIONS

In this paper we have examined the cohomological aspects of BRST quantization. It is remarkable that quite a lot of general interesting results follow from the consideration of some extra structure: namely a positive definite inner product. This is reminiscent of the interplay between geometry and topology in a differentiable manifold. There one uses geometric properties in order to probe the topology of the manifold while at the same time the topology restricts the kind of metrics that the manifold may admit.

It is our hope that a self-contained proof of the vanishing theorem for, say, the open bosonic string or even some general class of theories may be obtained using the methods advocated in this paper. In a sequel to this paper[**6**] we use these methods to prove a number of interesting results in the BRST cohomology of the open bosonic string.

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