M2-branes, ADE and Lie superalgebras

José Figueroa-O'Farrill



UTokyo hep-th Seminar 28 September 2009

http://www.maths.ed.ac.uk/~jmf/CV/Seminars/Hongo.pdf

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This talk is based on

- arXiv:0809.1086 [hep-th] (with PDM, EME, PR)
- arXiv:0908.2125 [hep-th] (with PDM, EME)
- arXiv:0909.1063 [hep-th] (with PDM, SG, EME)

where

PDM = Paul de Medeiros

SG = Sunil Gadhia

EME = Elena Méndez-Escobar

PR = Patricia Ritter

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Motivation

After more than 15 years we still have not answered this:

Main question

What is M-theory?

- not a theory of strings!
- a theory of membranes?
- maybe, but quantising membranes is difficult.
- AdS/CET: try to at least understand dual theory

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• We have a fairly good proposal for the 3d CFTs dual to M2-branes

Bagger+Lambert (2006,2007) Gustavsson (2007)

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- In this talk we will learn some of these words:
 - we will classify U/C> /4 M2-brane geometries in terms of U/CP /4 M2-brane geometrie
 - . we will classify 3/2 & superconformal
 - Chem-Simons+matter theories in terms of metric Lie
 - superalgebras, or if you prefer, metric triple systems
- Just like with natural languages (but for different reasons!) it is too naive to expect a bijection between these two sets of words, but it's a departure point for a more systematic study

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- 2 M2-brane geometries and ADE
- Superconformal Chern–Simons theories
- Triple systems and Lie superalgebras

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2 M2-brane geometries and ADE

- 3 Superconformal Chern–Simons theories
- 4 Triple systems and Lie superalgebras

The M2-brane solution

Definition

The elementary M2-brane:

$$g = H^{-\frac{2}{3}} ds^{2}(\mathbb{R}^{2,1}) + H^{\frac{1}{3}} ds^{2}(\mathbb{R}^{8}$$
$$F = dvol(\mathbb{R}^{2,1}) \wedge dH^{-1},$$

where

$$\mathsf{H} = \alpha + \frac{\beta}{r^6} \; ,$$

for $\alpha, \beta \in \mathbb{R}$ not both equal to zero.

It is half-supersymmetric for .

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It is half-supersymmetric for $\alpha\beta \neq 0$.

Asymptotia

• $\beta \rightarrow 0 \text{ (or } r \rightarrow \infty)$:

 $(g,F) \to (ds^2(\mathbb{R}^{10,1}),0)$

:: Minkowski vacuum

• $\alpha \rightarrow 0$ (or $r \rightarrow 0$):

 $H^{\frac{1}{2}} as^{2}(\mathbb{R}^{6}) \rightarrow H^{\frac{1}{2}}(as^{2} + s^{2}as^{2}(S^{2})) \rightarrow g^{\frac{1}{2}} \frac{dr^{2}}{r^{2}} \rightarrow g^{\frac{1}{2}}as^{2}(S^{2})$

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Killing superalgebra

 Every supersymmetric supergravity background has an associated Lie superalgebra, generated by the Killing spinors: the Killing superalgebra

FO (1999), FO+MEESSEN+PHILIP (20)

• For $AdS_4 \times S^7$ it is $\mathfrak{osp}(8|4)$

• The even subalgebra is

 $\mathfrak{so}(8)\oplus\mathfrak{sp}(4,\mathbb{R})\cong\mathfrak{so}(8)\oplus\mathfrak{so}(3,2),$

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Conformal superalgebra

- The Killing superalgebra is isomorphic to the conformal superalgebra of the dual theory
- Now so(3, 2) is the conformal algebra of ℝ^{2,1} and so(8) is the R-symmetry algebra
- In general, three-dimensional conformal field theories admit realisations of the conformal superalgebras osp(𝒴|4), with R-symmetry so(𝒴), for 𝒴 ≤ 8.
- Some of these theories are dual to M2-brane geometries with conical singularities.

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Generalised M2-brane solution

• Replace the S^7 with M^7 :

$$\begin{split} g &= H^{-\frac{2}{3}} \, ds^2(\mathbb{R}^{2,1}) + H^{\frac{1}{3}}(dr^2 + r^2 ds^2(M^7)) \\ F &= dvol(\mathbb{R}^{2,1}) \wedge dH^{-1}, \end{split}$$

- field equations → M is Einstein
- supersymmetry $\implies M$ admits (real) Killing spinors:

$$\nabla_{\mathbf{m}}\varepsilon = \frac{1}{2}\Gamma_{\mathbf{m}}\varepsilon$$

(Notice: here supersymmetry \implies field equations)

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Bär's cone construction

Question

Which manifolds admit real Killing spinors?

- The metric cone of a riemannian manifold (M, g_M) is the manifold $C = \mathbb{R}^+ \times M$ with metric $g_C = dr^2 + r^2 g_M$ e.g., the metric cone of the round sphere S^n is $\mathbb{R}^{n+1} \setminus \{0\}$
- (M, g_M) admits real Killing spinors if and only if (C, g_C) admits parallel spinors
 BÄR (1993)
- If *M* is complete, then *C* is either irreducible or flat

GALLOT (1979)

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Irreducible holonomies

Simply-connected 8-manifolds with parallel spinors:

N	Cone holonomy	7-dimensional geometry
8	{1}	sphere
3	Sp(2)	3-Sasaki
2	SU(4)	Sasaki-Einstein
1	Spin(7)	weak G ₂ holonomy

M. WANG (1989)

So generalised supersymmetric M2-brane solutions describe M2 branes at a conical singularity in an 8-manifold with special holonomy.

$\mathcal{N} > 3$ and sphere quotients

 To obtain 8 > N > 3 we need to consider quotients S⁷/Γ, for Γ ⊂ SO(8) such that

- Γ acts freely on S⁷ (so that S⁷/Γ is smooth).
- If lifts to Spin(8) (for S⁷/If to be spin).
- If leaves some chiral spinors invariant (for supersymmetry)

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ADE subgroups of Sp(1)

Dynkin diagram	Label	Name	Order
••-•	An	cyclic	n + 1
· · · · · · · · · · · · · · · · · · ·	$D_{n \geqslant 4}$	binary dihedral	4(n - 2)
•••••	E ₆	binary tetrahedral	24
••••••	E ₇	binary octahedral	48
••••••	E ₈	binary icosahedral	120

... and the twist

Let Γ ⊂ Sp(1) be one of the ADE subgroups
Let τ ∈ Aut(Γ) be an automorphism
Let us embed Γ → SO(8) via

 $\mathbf{u} \cdot (\mathbf{x}, \mathbf{y}) = (\mathbf{u}\mathbf{x}, \mathbf{\tau}(\mathbf{u})\mathbf{y}) ,$

for $x, y \in \mathbb{H}$ and $u \in Sp(1) \subset \mathbb{H}$

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The $\mathcal{N} > 3$ classification

The backgrounds $AdS_4 \times M^7$ with $\mathscr{N} > 3$ are those with $M = S^7/\Gamma$ with $\Gamma \subset SO(8)$ given by pairs (ADE, τ):

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If $\tau = 1$ we don't write it and ν is the unique nontrivial outer automorphism of E_{7.8}. (The ones in red were not known.)

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2 M2-brane geometries and ADE

- Superconformal Chern–Simons theories
- 4 Triple systems and Lie superalgebras

(Supersymmetric) M2-brane degrees of freedom



 X, ψ are in a unitary representation m of the metric Lie algebra g, that is in has an invariant inder product

(all fields in $\mathbb{R}^{2,1}$)

- X: real scalars corresponding to transverse excitations
- ψ: real 2-component spinors
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Superconformal theories in 3 dimensions

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 - so do the fermions () (but of opposite chirality, if 0// is even)
 - The gauge fields /Lare inert, since supersymmetry is rigid.
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Spinor representations

N	1	2		3			4
$\mathfrak{so}(\mathscr{N})$ spinors	$\mathbb R$	u(1) ℂ		sp(' Ⅲ	1) sp($ \begin{array}{l} 1) \oplus \mathfrak{sp}(1) \\ \mathbb{H} \oplus \mathbb{H} \end{array} $
•					I	_	
N	5)		6		7	8
$\mathfrak{so}(\mathscr{N})$ spinors	sp(Ⅲ	2) 2	51 (ι(4) C ⁴	sc I	₀(7) ℝ ⁸	$\mathfrak{so}(8)\ \mathbb{R}^{8}\oplus\mathbb{R}^{8}$

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Matter representations

- The degrees of freedom of any physical theory are fundamentally real
- This determines the type (i.e., ℝ, C or ℍ) of the matter g-representation 𝔐 in terms of the type of the R-symmetry representation:

if N = 1, 7, 8, then 𝔐 is real, written 𝔐 ∈ Rep(𝑔, 𝔅)
 if 𝔐 = 2, 6, then 𝔐 is complex, written 𝔐 ∈ Rep(𝑔, 𝔅)
 if 𝔐 = 3, 4, 5, then 𝔐 is

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 - if $\mathcal{N} = 1, 7, 8$, then \mathfrak{M} is real, written $\mathfrak{M} \in \operatorname{Rep}(\mathfrak{g}, \mathbb{R})$
 - if $\mathscr{N} = 2, 6$, then \mathfrak{M} is complex, written $\mathfrak{M} \in \mathsf{Rep}(\mathfrak{g}, \mathbb{C})$
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- If *N* = 3, 4, 5 we have to impose a symplectic reality condition on the fields

$\mathcal{N} \leqslant$ 3 theories

For $\mathcal{N} \leq 3$ theories, the matter representation \mathfrak{M} is not constrained beyond its type:

- *N*=1 theory = 𝔅 ∈ Rep(𝔅, ℝ) + quartic 𝔅-invariant superpotential
- 𝒩=2 theory = 𝔅 ∈ Rep(𝔅, 𝔅) + quartic 𝔅-invariant F-term superpotential
- $\mathcal{N}=3$ theory = $\mathfrak{M} \in \operatorname{Rep}(\mathfrak{g}, \mathbb{H})!$
- Rigidity of *N*=3 theories agrees with the geometric rigidity of complete 3-Sasakian manifolds

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Matter representation	

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N	Matter representation	Remarks
4	$\Delta^{(4)}_{\pm} \otimes W_1 \oplus \Delta^{(4)}_{\mp} \otimes W_2$	$W_{1,2} \in \operatorname{Rep}(\mathfrak{g}, \mathbb{H})_{\operatorname{aLTS}}$
5	$\Delta^{(5)}\otimes W$	$W \in Irr(\mathfrak{g},\mathbb{H})_{aLTS}$
6	$\Delta^{({f 6})}_{\pm}\otimes { m V}\oplus\Delta^{({f 6})}_{\mp}\otimes \overline{{ m V}}$	$V\in {\textnormal{Irr}}({\mathfrak{g}},{\mathbb{C}})_{\textnormal{aJTS}}$
8	$\Delta_{\pm}^{(\boldsymbol{8})}\otimes U$	$U\in \text{Irr}(\mathfrak{g},\mathbb{R})_{\text{3LA}}$

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Some remarks

- For $\mathcal{N} \ge 5$, irreducible representations decouple
- $\mathcal{N} = 7$ theories are automatically $\mathcal{N} = 8$
- Representation theory uniquely determines $\mathcal{N} \ge 3$ theories
- *N* ≥ 4 theories can be defined in terms of 3-algebras or triple systems, as in the original BLG model
- As in ABJM, we can now show this language is not necessary
- As in GAIOTTO+WITTEN (2008), we may adopt the more familiar language of Lie superalgebras
- **Question**: Can the theories be reformulated solely in terms of Lie superalgebras?

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M2-branes and AdS/CFT

2 M2-brane geometries and ADE

3 Superconformal Chern–Simons theories

Triple systems and Lie superalgebras

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Superalgebras from representations

Slogan

When a Lie algebra admits an invariant inner product, its unitary representations give rise to superalgebras.

• The superalgebra consists of two subspaces

- A unitary representation V in degree 1
- and three products
 - $\bigcirc g \times g \to g$ is the Lie bracket on g
 - $O_{-g} \times V \rightarrow V$ is the action of g on V.
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 - product on the superalgebra is invariant; equivalently, it is
- If the Jacobi identity holds, V is Lie-embeddable
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 - $\bigcirc g \times V \rightarrow V$ is the action of g on V.
 - $\mathbb{O} : \mathbb{V} \times \mathbb{V} \to \mathfrak{g}$ (or $\to \mathfrak{g}_{\mathbb{C}})$ is defined so that the natural inner
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Triple systems from superalgebras

Slogan

The odd subspace of a superalgebra is a triple system.

- $\mathfrak{g} \oplus V$ a superalgebra
- We define a 3-bracket on V by

$[\mathfrak{u},\mathfrak{v},\mathfrak{w}]:=[[\mathfrak{u},\mathfrak{v}],\mathfrak{w}]$

where the RHS brackets come from the superalgebra

- For the $\mathcal{N} \ge 4$ theories, the relevant triple systems are

 - metric anti-Jordan triple systems for .//
 - symplectic anti-Lie triple systems for 2/ = 4,5

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 - symplectic anti-Lie triple systems for $\mathcal{N} = 4, 5$

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Triple systems from superalgebras

Slogan

The odd subspace of a superalgebra is a triple system.

- $\mathfrak{g} \oplus V$ a superalgebra
- We define a 3-bracket on V by

 $[\mathfrak{u},\mathfrak{v},\mathfrak{w}]:=[[\mathfrak{u},\mathfrak{v}],\mathfrak{w}]$

where the RHS brackets come from the superalgebra

- For the $\mathcal{N} \ge 4$ theories, the relevant triple systems are
 - metric 3-Lie algebras for $\mathcal{N} > 6$
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Metric Lie algebras

- Let g be a Lie algebra with an invariant inner product (-,-), not necessarily positive-definite.
- Let (X_i) be a basis for g:

 $[\mathbf{X}_{i}, \mathbf{X}_{j}] = f_{ij}^{k} \mathbf{X}_{k}$ and $(\mathbf{X}_{i}, \mathbf{X}_{j}) = \kappa_{ij}$

- Invariance means that f_{ijk} = f_{ik}^ℓκ_{ℓk} is totally skewsymmetric
- Not all Lie algebras are metric: reductive (=semisimple + abelian) are,...

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Real unitary representations of a metric Lie algebra

- Let U ∈ Rep(g, ℝ) be a real unitary representation with g-invariant inner product ⟨−, −⟩
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Lie-embeddable real representations

 Let U_± = U but with degree ±1 and define on U_− ⊕ g ⊕ U₊ the following Lie brackets and inner product

$$\begin{split} [e_{a}^{+}, e_{b}^{-}] &= \kappa^{ij} T_{i\,ab} X_{j}, \quad [X_{i}, e_{a}^{\pm}] = T_{i}{}^{b}{}_{a} e_{b}^{\pm}, \quad [X_{i}, X_{j}] = f_{ij}{}^{k} X_{k} \\ (e_{a}^{+}, e_{b}^{-}) &= g_{ab} = -(e_{a}^{-}, e_{b}^{+}) \qquad (X_{i}, X_{j}) = \kappa_{ij} \end{split}$$

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Classification

Theorem

There is precisely one (up to scale) irreducible positive-definite metric 3-Lie algebra.

(Conjectured in FO+Papadopoulos (2003)) NAGY (2007) Papadopoulos (2008) GAUNTLETT+GUTOWSKI (2008) DE MEDEIROS+FO+MÉNDEZ-ESCOBAR (2008) bllows from the classification of simple 3-Lie algebras LING (1993) CANTARINI+KAC (2009)

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The metric 3-Lie algebra

- $\mathfrak{g} = \mathfrak{so}(4) \cong \mathfrak{sp}(1)_{-k} \oplus \mathfrak{sp}(1)_k$, with subscripts indicating the multiple of the Killing form
- A priori $k \in \mathbb{R}$ but it is quantised in the quantum theory
- U = ℝ⁴ of so(4), or U = ℍ with sp(1) ⊕ sp(1) acting by leftand right-multiplications
- $F_{abcd} = k^{-1} \varepsilon_{abcd}$
- The resulting superconformal Chern–Simons theory is the original BLG model

Bagger+Lambert (2006,2007) Gustavsson (2007)

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Complex unitary representations

Let V ∈ Rep(g, C) with hermitian inner product g_{αb}
 On the 3-graded superspace V ⊕ g_C ⊕ V define

 $[\boldsymbol{e}_{a}, \boldsymbol{e}_{\overline{b}}] = \kappa^{ij} \mathsf{T}_{i\overline{b}a} X_{j}, \quad [\mathbf{X}_{i}, \boldsymbol{e}_{a}] = \mathsf{T}_{i}{}^{b}{}_{a} \boldsymbol{e}_{b}, \quad [\mathbf{X}_{i}, X_{j}] = \mathsf{f}_{ij}{}^{k} X_{k}$ $(\boldsymbol{e}_{a}, \boldsymbol{e}_{\overline{b}}) = \mathfrak{g}_{a\overline{b}} = -(\boldsymbol{e}_{\overline{b}}, \boldsymbol{e}_{a}), \quad (\mathbf{X}_{i}, X_{j}) = \kappa_{ij}$

- Jacobi $\iff F_{a\overline{b}c}{}^d := \kappa^{ij} T_{i\overline{b}c} T_{j}{}^d{}_a = -F_{c\overline{b}a}{}^d$
- The sesquibilinear 3-bracket $V \times \overline{V} \times V \rightarrow V$

$$[e_{a}, e_{\overline{b}}, e_{c}] := [[e_{a}, e_{\overline{b}}], e_{c}]$$

turns V into anti-Jordan triple system

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- Jacobi $\iff F_{a\overline{b}c}{}^{d} := \kappa^{ij} T_{i\overline{b}c} T_{j}{}^{d}{}_{a} = -F_{c\overline{b}a}{}^{d}$ $\iff \mathscr{N} = 6 \text{ supersymmetry!}$
- The sesquibilinear 3-bracket $V \times \overline{V} \times V \rightarrow V$

$$[e_{a}, e_{\overline{b}}, e_{c}] := [[e_{a}, e_{\overline{b}}], e_{c}]$$

turns V into a metric anti-Jordan triple system FAULKNER+FERRAR (1980) BAGGER+LAMBERT (2008)

Classification

 Indecomposable N = 6 theories ↔ V ∈ Irr(g, C)_{aJTS}
 V irreducible ↔ embedding Lie superalgebra is simple PALMKVIST, FO (2009)

 Two classes of simple 3-graded complex (metric) Lie superalgebras: A(m, n) and C(n + 1)
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Symplectic anti-Lie triple systems

• On $\mathfrak{g}_{\mathbb{C}} \oplus W$ we define

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• Indecomposable $\mathscr{N} = 5$ theories $\iff W \in Irr(\mathfrak{g}, \mathbb{H})_{aLTS}$

- The simple complex (metric) Lie superalgebras with quaternionic odd subspace are: A(m, n), B(m, n), C(n + 1), D(m, n), $D(2, 1; \alpha)$, F(4) and G(3) KAC (1977)

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A comment on the $\mathcal{N} = 4$ theories

- Indecomposable $\mathcal{N} = 4$ theories of GAIOTTO+WITTEN (2008) — those with matter representations $\Delta_{\pm}^{(4)} \otimes W$ are also classified by irreducible symplectic anti-Lie triple systems; equivalently by simple metric complex Lie superalgebras with quaternionic odd subspace
- The more general $\mathscr{N} = 4$ theories of HOSOMICHI+3LEE+PARK (2008) — those with matter representations $\Delta_{\pm}^{(4)} \otimes W_1 \oplus \Delta_{\mp}^{(4)} \otimes W_2$ — are classified in terms of dominoes whose tiles are the above objects and two files are said to match if they have a metric Lie subalgebra in common

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Summary

- We classified complete, smooth $\mathcal{N} \ge 4$ Freund–Rubin backgrounds $AdS_4 \times M^7$ by
 - showing that M = S⁷/F for some freely acting F ⊂ SO(8)
 classifying such F in terms of pairs (ADE, a) consisting of an ADE subgroup of Sp(1) together with an automorphism
- We showed that superconformal Chern–Simons+matter theories are governed by the representation theory of metric Lie algebras
- We classified the *N* ≥ 4 theories in terms of metric Lie superalgebras, equivalently, in terms of certain metric triple systems
- **Important**: the inner product on the Lie superalgebra is a crucial part of the data

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Further results

- arXiv:0908.2125 [hep-th] contains more results!
- The representation theory also explains naturally the mechanism of supersymmetry enhancements:
 - $0 \cdot 1 = 4 \text{ to } 1 = 5$
 - Ø ...// = 5 to ...// = 6
 - Ø ∪ / = 6 to ∪ / = 8, in particular why ∪ / = 7 implies ∪ / = 8
- In the paper we also construct the theories starting from the representation-theoretic data; in particular, we give explicit expressions for the superpotentials
- That in itself is worth another seminar, and luckily PAUL and ELENA will be visiting IPMU from October (^_^)

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