

Twisted cohomotopy and the level quantization of the 6d Hopt-Wess-Zumino term

(based on [arXiv:1906.07417](#),
joint work with Hisham Sati
and Urs Schreiber)

The origin of the problem

In "Anomaly Matching and a Hopt-Wess-Zumino Term in 6d $N=(2,0)$ Field Theories",

Kenneth Intriligator defines a

Hopt-invariant type formula for the WZ-term for a 5-brane:

$$S[\mathbf{1}, H_3] = \int_{\Sigma^7} 2\hat{\mathbf{f}}^* G_7 + \hat{H}_3 \wedge \hat{\mathbf{f}}^* G_4$$

where G_4 is the C-field strength, $G_7 = *G_4$

$\mathbf{f}$ is a smooth map from the 6-dimensional worldsheet Σ^6 to the spacetime $X^8 \times \mathbb{R}^{1,2}$,

H_3 is some other field (more on this later),

and $\hat{\mathbf{f}}, \hat{H}_3$ are extension to Σ^7

with $\partial \Sigma^7 = \Sigma^6$

In order for $S[l, H_3]$ to be well defined, i.e. independent of the extensions $(\hat{l}, \hat{H}_3)$, at least mod $\mathbb{Z}$ (so that $\exp(2\pi i S[l, H_3])$ is well defined)

one needs suitable assumptions on the background fields (G_4, G_7) that essentially reduce to requiring that

$$G_4 = c^*(\text{vol}_{S^1})$$

for some smooth map $c: X^8 \rightarrow S^1$

One refers to this by saying that

C -field charges take values in degree 4 cohomology

This is however too a restrictive assumption from an M-theory point of view (at least so I am told). For instance, it generally fails the half-integral shift quantization condition $[C_4 + \frac{1}{4}p_1(X^8)] \in H^4(X, \mathbb{Z})$.

A consistent set-up where this condition is always satisfied while keeping the Hopt-WZ term well defined is obtained by assuming c is a map $c: X^8 \rightarrow S^1/\mathbb{Z}$, i.e. that

C -field charges take values in degree 4 G -twisted cohomology

The background (The geometric data)

X 8-dimensional (connected enough)
smooth $\text{Spin}(8)$ -manifold

∇ $\text{Spin}(8)$ connection on TX

$$G_4 \in \Omega^4(X)$$

$$G_7 \in \Omega^7(X)$$

$$\text{such that } \begin{cases} dG_4 = 0 \\ dG_7 = \frac{1}{2} \left(-G_4 \wedge G_4 + \frac{1}{4} p_1(\nabla) \wedge \frac{1}{4} p_1(\nabla) \right) \end{cases}$$

(where $p_1(\nabla) = \langle R_\nabla \wedge R_\nabla \rangle$ is the first Pontrjagin form of ∇)

Introducing the shifted form

$G_4^{\text{int}} = G_4 + \frac{1}{4} p_1(\nabla)$ the equations became

$$\begin{cases} dG_4^{\text{int}} = 0 \\ dG_7 = - \left(G_4^{\text{int}} \wedge G_4^{\text{int}} - \frac{1}{2} p_1(\nabla) \wedge G_4^{\text{int}} \right) \end{cases}$$

The reason for the shift $G_4 \rightsquigarrow G_4^{\text{int}}$ is that now we can add to our data the "shifted C-field flux quantization" condition

$$[G_4^{\text{int}}] \in H^4(X; \mathbb{Z})$$

Our worldsheet will be
 a 6-dimensional manifold Σ' (compact,
 oriented, without boundary)
 and our fields will be a pair

$$f: \Sigma' \longrightarrow X^8$$

$$H_3 \in \Omega^3(\Sigma')$$

such that

$$dH_3 = f^* \left(G_4 - \frac{1}{4} \eta_1(\nabla) \right)$$

(represents an integral class)

(equivalently, $dH_3 = f^* \left(G_4^{\text{int}} - \frac{1}{2} \eta_1(\nabla) \right)$)

The local case

Assume $X = V \cong \mathbb{R}^8$

Then there exists potentials

C_3, C_6 , with $C_i \in \Omega^i(X)$

with

$$\begin{cases} dC_3 = C_4 + \frac{1}{4}P_2(\nabla) \\ dC_6 = 2C_7 + C_3 \wedge (C_4 - \frac{1}{4}P_2(\nabla)) \end{cases}$$

(this is more generally true if
 $H^4(X; \mathbb{R}) = H^7(X; \mathbb{R}) = 0$)

We define the local 6d WZ action
as

$$S_{WZ}[b, H_3] = \frac{1}{2} \int_{\Sigma^6} -H_3 \wedge \dagger^* C_3 + 2 \dagger^* C_6$$

How can we globalize?

A toy example

X a simply connected manifold

$$\omega_2 \in \Omega^2_d(X)$$

$$f: S^1 \rightarrow X$$

If $X = U \cong \mathbb{R}^n$, then exists a

potential $C_1 \in \Omega^1(U)$ such that

$dC_1 = \omega_2$ and we can define

$$S[f] = \int_{S^1} f^* C_1$$

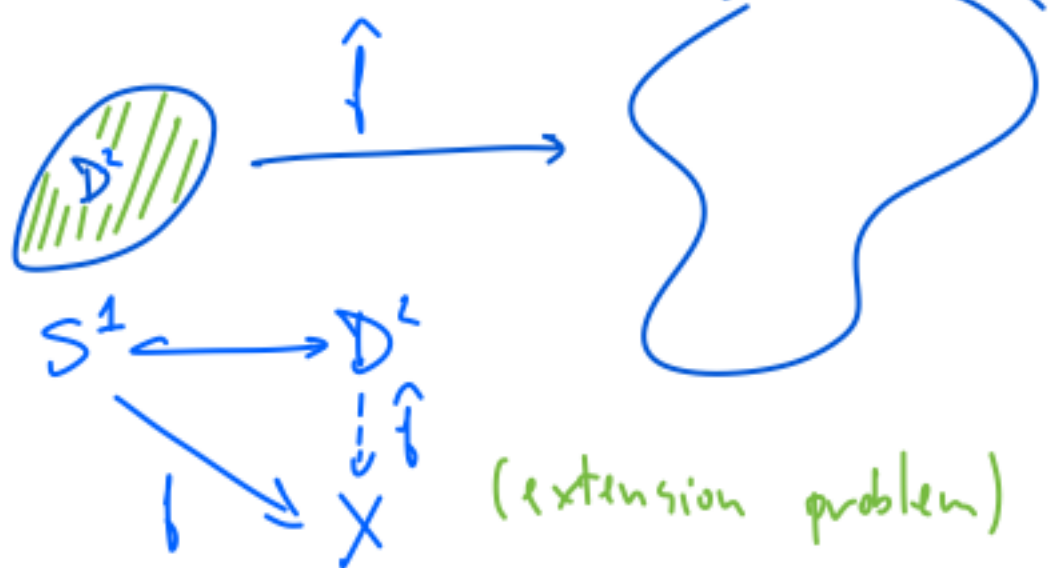
(more generally this works as
soon as $H^2(X; \mathbb{R}) = 0$)

How can we globalize?

Add the integrality condition

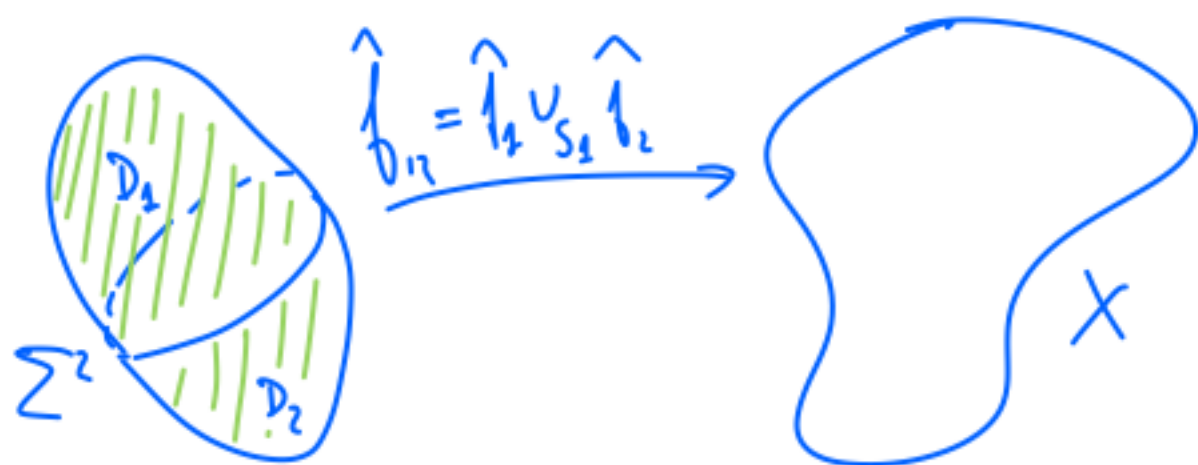
$$[\omega] \in H^2(X; \mathbb{Z})$$

$$f: S^1 \rightarrow X$$



$$S[f] = \int_D \hat{f}^* \omega \mod \mathbb{Z}$$

is well defined:



$$S[1]_{\hat{f}_1} - S[1]_{\hat{f}_2} = \int_{\Sigma^2} \hat{f}_{12}^* \omega$$

$$= \langle f_{12}^*[\omega] | [\Sigma^2] \rangle \in \mathbb{Z}$$

If $\omega_2 = dC_1$ This definition recovers the previous one mod $\mathbb{Z}$

Do we really need X to be simply connected? This can actually be relaxed:

Assume we have

$$X \xrightarrow{\phi} Y$$

with $\omega = \phi^* \eta$, $\eta \in \Omega^2_d(Y)$

$$[\eta] \in H^2(Y; \mathbb{Z})$$

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathbb{D}^2 \\ \downarrow & & \downarrow \hat{f} \\ X & \xrightarrow{\phi} & Y \end{array}$$

$$S[1] = \int_{\Sigma^2} \hat{f}^* \eta$$

(more generally, we can replace $\mathbb{D}^2$
with any oriented 2-manifold with
 $\partial \Sigma^2 = S^1$)

$$\begin{array}{ccc}
 Y & \xrightarrow{\eta} & \Omega^2_d \\
 \downarrow & & \downarrow \\
 BU(1) \cong K(\mathbb{Z}, 2) & \longrightarrow & K(\mathbb{R}, 2)
 \end{array}$$

so the universal Y is

$$\begin{array}{ccc}
 BU(1)_{\text{conn}} & \xrightarrow{\text{curv}} & \Omega^2_d \\
 \downarrow \perp & & \downarrow \\
 BU(1) & \longrightarrow & K(\mathbb{R}, 2)
 \end{array}$$

We have $\pi_1(BU(1)_{\text{conn}}) = 0$

(every $U(1)$ -connection on S^1 extends to
 a $U(1)$ -connection on $\mathbb{D}^2$)

So our background becomes

$$X \xrightarrow{\nabla} BU(1)_{\text{conn}}$$

i.e., X is equipped with a complex line bundle with $U(1)$ -connection and the action for $f: S^1 \rightarrow X$

is

$$S[f] = \int_{D^2} \tilde{f}^*(F_{\nabla})$$

where F_{∇} is the curvature of ∇ .

From this description we see X is purely auxiliary. What we are really concerned with is

$$S^1 \xrightarrow{\nabla} BU(1)_{\text{conn}}$$

i.e., a $U(1)$ -connection on S^1 .

The action becomes

$$S[\nabla] = \int_{D^2} \hat{F} \hat{\nabla}$$

$$\begin{array}{ccc} S^1 & \hookrightarrow & D^2 \\ & \searrow \nabla & \downarrow \hat{\nabla} \\ & & BU(1)_{\text{conn}} \end{array}$$

$\forall \epsilon$ have

$$\begin{array}{ccc} [D^2, BU(1)_{\text{conn}}] & \xrightarrow{\text{curv}} & [D^2, \Omega^1 d] \\ \downarrow |_{S^1} & \searrow \text{2} & \downarrow \int_{D^2} \text{mod } \mathbb{Z} \\ [S^1, BU(1)_{\text{conn}}] & \xrightarrow{\text{hol}_{S^1}} & U(1) \end{array}$$

Therefore

$$S[\nabla] = \text{hol}_{S^1}(\nabla) \quad \left(\begin{array}{l} \text{with no reference} \\ \text{to an extension} \\ \text{to } D^2 \end{array} \right)$$

Back to the 6d WZ functional

How can we globalize?

$$\text{As } \Omega_6^{SO} = 0$$

↖ (oriented cobordism ring
in dimension 6)

there exists Σ^7 with $\partial \Sigma^7 = \Sigma^6$.

Assume we can extend $(\mathbb{I}, H_3)$ to
 $(\hat{\mathbb{I}}, \hat{H}_3)$ on Σ^7 . Then we can define

$$S[\mathbb{I}, H_3] = \frac{1}{2} \int_{\Sigma^7} \hat{H}_3 \wedge \hat{\mathbb{I}}^* \left(\mathcal{L}_4 + \frac{1}{4} \eta_1(\mathbb{I}) \right) + 2 \hat{\mathbb{I}}^* \mathcal{L}_7 \text{ mod } \mathbb{Z}$$

If $X \cong U$, then this extends the previous
definition, mod $\mathbb{Z}$.

Problem: is this well defined?

From our toy example we know S_{ψ_2} will be automatically well defined if we can interpret the data of our background and extended fields the datum of a homotopy commutative diagram $\Sigma^7 \longrightarrow E_7$

$$\begin{array}{ccc} \hat{f} \downarrow & \swarrow \hat{H}_3 & \downarrow \\ X^8 & \longrightarrow & E_4 \end{array}$$

In such a way that

$$\frac{1}{2} \left(\hat{H}_3 \wedge \hat{f}^* (G_4 + \frac{1}{4} \eta_1(v)) + 2 \hat{f}^* G_7 \right)$$

is the pullback of an integral form on E_7

A digression on Sullivan models

Let X, Y be smooth manifolds.

Then a smooth map $f: X \rightarrow Y$ is completely encoded into the morphism of dgas $f^*: \Omega^*(Y) \rightarrow \Omega^*(X)$.

Moreover, every $\varphi: \Omega^*(Y) \rightarrow \Omega^*(X)$ is of the form $\varphi = f^*$ for a unique f (this is sometimes called "the Milnor exercise")

In other words, all of the geometry of X is encoded in the dga $\Omega^*(X)$.

This suggests that all of the homotopy theory of X is encoded in the homotopy class of $\Omega^*(X)$. This is almost true: it is true up to Torsion.

If A is a dgca over $\mathbb{R}$
 we can define a functor

$$\begin{aligned} \Phi_A: \text{Smooth manifolds} &\longrightarrow \text{Sets} \\ X &\longmapsto \text{Hom}_{\text{dgca}}(A, \Omega^*(X)) \end{aligned}$$

Φ_A is a smooth sheaf!

$$\coprod U_i \rightarrow X$$

$$\lim_{\substack{\parallel \\ i}} (\Phi_A(U_i)) \rightarrow \prod_i \Phi_A(U_i) \rightrightarrows \prod_{i,j} \Phi_A(U_{ij})$$

$$\lim_{\text{dgca}} \text{Hom}(A, \Omega^*(U_i)) = \text{Hom}_{\text{dgca}}(A, \lim_{\text{dgca}} \Omega^*(U_i)) = \text{Hom}_{\text{dgca}}(A, \Omega^*(X))$$

$$\parallel$$

$$\Phi_A(X)$$

Example: $A = CE(\mathbb{R}[n]) = \mathbb{R}[t_{n+1}]$

$$dy t_{n+1} = n+1, \quad dt_{n+1} = 0$$

$$\Phi_A = \Omega^{n+1} d(-)$$

If M is a connected and simply connected manifold, then $\Omega^*(M)$ has a minimal model: a dgas

$$A_M \xrightarrow{\sim} \Omega^*(M) \quad \text{(or Sullivan model)}$$

such that A_M is a polynomial algebra and the differential has the form

$$dx_i = \sum_j a_{ij} x^j x^i$$

Example: $A_{S^4}^{\text{min}} = \mathbb{R}[w_4, w_7]$

$$dw_4 = 0$$

$$dw_7 = -w_4 \wedge w_4$$

$$A_{S^7}^{\text{min}} = \mathbb{R}[w_7]; \quad dw_7 = 0$$

The quaternionic Hopf fibration

$$S^7 \longrightarrow S^4$$

is described in terms of Sullivan models by

$$\mathbb{R}[\omega_4, \omega_7] \longrightarrow \mathbb{R}[\omega_7]$$

$$\omega_4 \longmapsto 0$$

$$\omega_7 \longmapsto \omega_7$$

More generally, we can consider a Lie group action such that the quaternionic Hopf fibration is G -equivariant.

For instance, we can take

$G = Sp(2)$ with its standard action on $S^7 = S^2(\mathbb{H})$ and $S^4 = \mathbb{P}^1(\mathbb{H})$

Doing so we get the $Sp(2)$ -equivariant Hopf fibration

$$S^7 //_{Sp(2)} \longrightarrow S^4 //_{Sp(2)}$$

rationally modeled by

$$\left\{ \mathbb{R}[w_4, w_7, p_1, x_8] ; \begin{array}{l} dw_4 = 0 \\ dw_7 = -w_4 \wedge w_4 + \left(\frac{1}{4} p_1\right)^2 - x_8 \\ dp_1 = 0 ; dx_8 = 0 \end{array} \right\}$$

(2 rational model but not 2 Sullivan one, due to the linear term)

models $S^4 //_{Sp(2)}$

$$\left\{ \mathbb{R}[w_7, x_8] ; \begin{array}{l} dw_7 = -x_8 \\ dx_8 = 0 \end{array} \right\}$$

models $S^7 //_{Sp(2)}$

The relations on $A_{S^4 //_{Sp(2)}}$ are such that, denoting by $(S^4 //_{Sp(2)})_{\mathbb{R}}$ the associated sheaf, a map $X \longrightarrow (S^4 //_{Sp(2)})_{\mathbb{R}}$ is almost our background data on X .

We have the additional term X_8 , so we have to kill it by adding a new generator Θ_7 with $d\Theta_7 = X_8$

This defines a new algebra which we denote E^4_M . It is a rational model for the homotopy

$$\begin{array}{ccc} \text{fiber } E^4 & \longrightarrow & * \\ \downarrow & & \downarrow \\ S^4 // Sp(2) & \xrightarrow{X_8} & K(\mathbb{Z}; 8) \end{array}$$

The sheaf E^4_M is now such $E^4_M(X)$ consists of a triple (G_4, G_7, Θ_7) where (G_4, G_7) are as above, and $d\Theta_7 = X_8(\nabla)$

In detail, given a dgca map

$$\begin{array}{ccc}
 A_{\bar{E}^4} & \longrightarrow & \Omega^*(X) \\
 & \nwarrow \quad \nearrow & \\
 & A_{BSp(2)} & \\
 & \nearrow \quad \nwarrow & \\
 & q_1 \quad x_1 & \\
 & \nearrow \quad \nwarrow & \\
 & p_1(\sigma) \quad x_8(\sigma) &
 \end{array}$$

the differential forms $(\omega_4, \omega_7, \theta_7)$
are defined by

$$\begin{array}{ccc}
 \omega_4 & \longmapsto & \omega_4 \\
 \omega_7 & \longmapsto & 2\omega_7 - \theta_7 \\
 \theta_7 & \longmapsto & \theta_7
 \end{array}$$

and vice versa.

Similarly,

$$A_{\bar{E}^7} = \left\{ R[\omega_7, \theta_7, \chi_8]; \begin{array}{l} d\omega_7 = -\chi_8 \\ d\theta_7 = \chi_8 \\ d\chi_8 = 0 \end{array} \right\}$$

$E^7 \rightarrow E^4$ is a fibration
 (with fibre S^3), but $A_{E^4} \rightarrow A_{E^7}$
 is not a cofibration. We
 correct this by factoring $A_{E^4} \rightarrow A_{E^7}$

$$\text{as } A_{E^4} \hookrightarrow \widetilde{A}_{E^7} \simeq A_{E^7}$$

$$\uparrow$$

$$\{ \mathbb{R}[\omega_4, \omega_7, \theta_7, \rho_1, \chi_8, h_3] \}$$

$$d\omega_4 = 0$$

$$dh_3 = \omega_4 - \frac{1}{4}\rho_1$$

$$d\omega_7 = -dh_3 \wedge (\omega_4 + \frac{1}{4}\rho_1) - \chi_8$$

$$d\theta_7 = \chi_8$$

$$d\rho_1 = 0 ; \quad d\chi_8 = 0$$

In $\widetilde{A}_{E^7}$ we have a degree 7 closed
 element

$$\omega_7 + h_3 \wedge (\omega_4 + \frac{1}{4}\rho_1) + \theta_7$$

A homotopy commutative diagram

$$\begin{array}{ccc} \Sigma^7 & \longrightarrow & E^7 \\ \hat{f} \downarrow & \swarrow \hat{H}_3 & \downarrow \\ X & \longrightarrow & E^4 \end{array}$$

translates to a homotopy commutative dgca diagram

$$\begin{array}{ccc} A_{E^4} & \longrightarrow & \Omega^*(X) \\ \uparrow & \swarrow \hat{H}_3 & \downarrow \hat{f}^* \\ A_{E^7} \simeq \tilde{A}_{E^7} & \longrightarrow & \Omega^*(\Sigma^7) \end{array}$$

The dgca map $\tilde{A}_{E^4} \longrightarrow \Omega^*(\Sigma^7)$

in this diagram produces the closed 7-form

$$\begin{aligned} 2\hat{f}^*C_7 - \cancel{\hat{f}^*C_7} + \hat{H}_3 \wedge \hat{f}^*(C_4 + \frac{1}{4}P_1(\sigma)) + \cancel{\hat{f}^*C_7} \\ = 2\hat{f}^*C_7 + \hat{H}_3 \wedge \hat{f}^*(C_4 + \frac{1}{4}P_1(\sigma)) \end{aligned}$$

How can we see that

$2\hat{f}^* \hat{G}_7 + \hat{H}_3 \wedge \hat{f}^* (\hat{G}_4 + \frac{1}{4} \hat{q}_1(\sigma))$ is an integral class?

By construction, this class is a pullback of the closed element

$$\hat{f}_7 = \omega_7 + h_3 \wedge (\omega_4 + \frac{1}{4} q_1) + \theta_7$$

in $\tilde{A}_{E^7}$. So we only need to show that via

$$H^*(\tilde{A}_{E^7}) \cong H^*(A_{E^7}) \xrightarrow{\sim} H^*(E^7; \mathbb{Z})$$

The element $\hat{f}_7$ represents an element in $H^7(E^7; \mathbb{Z})$.

We do this by considering the Gysin sequence for the S^3 -fiber

$$\begin{array}{ccc} \text{sequence } S^3 & \longrightarrow & E^7 \\ & & \downarrow \\ & & E^4 \end{array}$$

i.e.,

$$\begin{aligned} \rightarrow H^7(E^7; \mathbb{Z}) &\rightarrow H^7(E^7; \mathbb{Z}) \xrightarrow{\int_{S^3}} H^4(E^4; \mathbb{Z}) \rightarrow \\ &\xrightarrow{-\cup(\Gamma_4^{\text{int}} - \frac{1}{2}\rho_1)} H^8(E^4; \mathbb{Z}) \rightarrow \end{aligned}$$

$\uparrow$
 Euler class

One has $\Gamma_4^{\text{int}} \cup (\Gamma_4^{\text{int}} - \frac{1}{2}\rho_1) = 0$

(recall that $G_4^{\text{int}} \wedge G_4^{\text{int}} - \frac{1}{2}G_4^{\text{int}} \wedge \frac{1}{2}\rho_1(\vartheta) = dZG_7$)

so there exists an integral class

$2\tilde{S} \in H^7(E^7; \mathbb{Z})$ such that

$$\int_{S^3} 2\tilde{S} = \Gamma_4^{\text{int}}$$

In the rational model $\tilde{A}_{E^7}$, the vertical volume form of the fiber S^3 is represented by the element h_3

and fiber integration is given by

$$\begin{array}{ccc} \tilde{A}_{E^7} & \xrightarrow{\int_{S^3}} & A_{E^4} \\ \alpha + h_3 \wedge \beta & \longmapsto & \beta \end{array}$$

Therefore, $\int_{S^3} \gamma_7 = \omega_4 + \frac{1}{4} p_1$

and so in real cohomology we find

$$\begin{array}{ccc} H^7(\tilde{A}_{E^7}) & \xrightarrow{\int_{S^3}} & H^4(A_{E^4}) \\ \downarrow \gamma_7 & \xrightarrow{\omega_4 + \frac{1}{4} p_1} & \downarrow \\ \downarrow \int_{S^3} & & \downarrow \\ H^7(E^7; \mathbb{R}) & \xrightarrow{\int_{S^3}} & H^4(E^4; \mathbb{R}) \end{array}$$

$\downarrow \gamma_7$ maps γ_7 to $p_7 + h_3 \wedge p_4^{int} + \theta_7$
 $\downarrow$ maps $\omega_4 + \frac{1}{4} p_1$ to p_4^{int}

i.e.,

$$2\tilde{S} - (p_7 + h_3 \wedge p_4^{int} + \theta_7) \in \text{Ker} \left(\int_{S^7} : H^7(E^7; \mathbb{R}) \rightarrow H^4(E^4; \mathbb{R}) \right)$$

By the Gysin sequence (now with real coefficients)

$$2\tilde{S} - (\pi_7 + H_3 \wedge \pi_4^{\text{int}} + \Theta_7) = \pi^* \beta$$

$$\text{with } \beta \in H^7(E^4; \mathbb{R}), \quad E^7 \xrightarrow{\pi} E^4$$

$$\text{But } H^7(E^4; \mathbb{R}) = 0!$$

Therefore

$$\pi_7 + H_3 \wedge \pi_4^{\text{int}} + \Theta_7 = 2\tilde{S} \in H^7(E^7; \mathbb{Z})$$