T-duality in rational homotopy theory

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January 13-20, 2018

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arXiv:1712.00758

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Introduction

A connected and simply connected space X has a canonically defined based loop space $\mathbf{\Omega}X$.

From the space $\mathbf{\Omega}X$ one can reconstruct X up to homotopy, as the classifying space for principal $\mathbf{\Omega}X$-fibrations

$$\begin{tikzcd} \mathbf{\Omega}X \arrow[r] & PX \arrow[d] \\ & X \end{tikzcd}$$

so the homotopy type of X is completely known to the ∞-group $\mathbf{\Omega}X$

By analogy with the classical Lie group/Lie algebra correspondence, it should then be possible to reconstruct at least part of the homotopical content of X from an infinitesimal version of the ∞-group $\mathbf{\Omega}X$.

One of the main result of rational homotopy theory is that this rather vague statement can be rigorously formalized, and that a considerable amount of the homotopy type of X is actually reconstructed:

the rational homotopy type of X is completely and faithfully encoded into a suitable L_∞-algebra $\mathfrak{l}X$ which one may think of as being the infinitesimal version of the loop group $\mathbf{\Omega}X$.

The *semifree* DG-algebras of rational homotopy theory are then the Chevalley-Eilenberg algebras of these L_∞-algebras.

The L_∞-algebra $\mathfrak{l}X$ can always be chosen to be concentrated in strictly negative degrees and with trivial differential, and these requirements determine $\mathfrak{l}X$ up to isomorphism.

The corresponding Chevalley-Eilenberg algebras are the *Sullivan model* DG-algebras of rational homotopy theory

$$A_X = \mathrm{CE}(\mathfrak{l}X)$$

topological space	loop ∞-group	L_∞-algebra	Sullivan model
X	$\mathbf{\Omega}X$	$\mathfrak{l}X$	$\mathrm{CE}(\mathfrak{l}X)$

The dgca $A_X = \mathrm{CE}(\mathfrak{l}X)$ is directly related to the geometry of X via the de Rham complex

$$A_X \to \Omega^\bullet(X)$$

More generally, if X is a smooth manifold and $\mathfrak{A}$ is a Lie algebroid, a $\mathfrak{A}$-valued cocycle on X is a morphism of Lie algebroids $TX \to \mathfrak{A}$, and so, equivalently, a morphism of dgcas

$$\mathrm{CE}(\mathfrak{A}) \longrightarrow \Omega^\bullet(X).$$

When $\mathfrak{A} = TY$ one usually says "Y-valued cocycles" for "TY-valued cocycles".

An Y-valued cocycle on X is

$$\varphi\colon \Omega^\bullet(Y) \to \Omega^\bullet(X)$$

These are all of the form

$$\varphi = f^*$$

for some smooth map $f\colon X \to Y$.

In other words, if Y is a smooth manifold, then Y-valued cocycles on X are precisely smooth maps $X \to Y$.

This suggests the following definition: if $\mathfrak{l}Y$ is a Sullivan model for a smooth manifold Y, a smooth map $X \to \mathfrak{l}Y$ is a dgca morphism

$$\mathrm{CE}(\mathfrak{l}Y) \longrightarrow \Omega^\bullet(X).$$

Explicitly, $\mathrm{CE}(\mathfrak{l}Y)$ is a free polynomial algebra, so

$$dx_{\alpha_i} = P_{\alpha_i}(x_{\alpha_1}, \ldots x_{\alpha_k})\ .$$

for some polynomial P_{α_i}. A smooth map $X \to \mathfrak{l}Y$ is therefore the datum of a collection of differential forms ω_{α_i} on X such that

$$d\omega_{\alpha_i} = P_{\alpha_i}(\omega_{\alpha_1}, \ldots \omega_{\alpha_k}),$$

where now d is the de Rham differential and the product is the wedge product of differential forms.

Read the other way round, every system of differential equations of the form

$$dx_{\alpha_i} = P_{\alpha_i}(x_{\alpha_1}, \dots x_{\alpha_k}) \ .$$

can be seen as a smooth map to a real Sullivan model.

In particular, a field theory whose fields are differential forms obeying equations of the above form can be interpreted as a σ-model type field theory, with target space given by a Sullivan model. All this immediately generalizes to the case of a smooth supermanifold X.

An example from M-theory

The fields usually denoted G_4 and G_7 in M-theory are the datum of a 4-form and a 7-form on a spacetime X with $dG_4 = 0$ and $dG_7 = G_4 \wedge G_4$.

This is precisely the datum of a smooth map from the smooth (super-)manifold X to $ʃ S^4$.

In particular, the superMinkowski space $\mathbb{R}^{10,1|\mathbf{32}}$ is equipped with a distinguished map

$$\mathbb{R}^{10,1|\mathbf{32}} \to ʃ S^4$$

This implies that every worldvolume in the spacetime $\mathbb{R}^{10,1|\mathbf{32}}$ is naturally equipped with a map to $ʃ S^4$, and so with M-theory fields, by restriction.

The superMinkowski space $\mathbb{R}^{10,1|\mathbf{32}}$ behaves, from the point of view of rational homotopy theory, as a principal $U(1)$-bundle over the superMinkowski space $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$.

So we have the following situation

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[r] \arrow[d] & \iota S^4 \\
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} &
\end{tikzcd}$$

We have the following geometric situation:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \\ M & \end{tikzcd}$$

The total space P is the homotopy fiber of the classifying map

$$M \to BU(1)$$

for the $U(1)$-bundle $P \to M$.

The homotopy fiber functor has a right adjoint, called "cyclification", mapping a space Y to the *twisted loop space* $\mathrm{cyc}(Y) = \mathcal{L}Y /\!/ U(1)$, given by the homotopy quotient of the free loop space of Y by the rotation of loops action.

The smooth map $P \to Y$ will, therefore, be equivalent to the datum of a smooth map $M \to \mathrm{cyc}(Y)$.

$$\begin{tikzcd} \mathrm{spaces} \arrow[rr, bend left, "\mathrm{cyc}"] & & \mathrm{spaces}/BU(1). \arrow[ll, bend left, "\mathrm{hofib}"] \end{tikzcd}$$

This topological construction is known in the Physics literature as “double dimensional reduction”.

This immediately translates to the rational homotopy theory/L_∞-algebra setting, where we find an adjunction

$$\begin{tikzcd} L_\infty\text{-algebras} \arrow[rr, bend left, "\mathrm{cyc}"] & & L_\infty\text{-algebras}/b\mathfrak{u}_1. \arrow[ll, bend left, "\mathrm{hofib}"] \end{tikzcd}$$

Example

When applied to the $\mathfrak{l}S^4$-valued cocycle on $\mathbb{R}^{10,1|\mathbf{32}}$, this produces a $\mathrm{cyc}(\mathfrak{l}S^4)$-valued cocycle on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$, which can be identified with (part of the data of) a twisted even K-theory cocycle.

In the Physics literature this is known as the double dimensional reduction from M-brane charges in 11d to string and brane charges in 10d type IIA string theory.

The superMinkowski space $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ is in turn, again from the point of view of rational homotopy theory, a principal $U(1)$-bundle over the superMinkowski space $\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}}$ and, as such, it is classified by a 2-cocycle c_2^{IIA} in the (super-)Chevalley-Eilenberg algebra of $\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}}$.

Quite remarkably, $\mathrm{CE}(\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}})$ carries also another, independent, 2-cocycle c_2^{IIB}, corresponding to the superMinkowski space $\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}$.

Moreover, the product $c_2^{\mathrm{IIA}} c_2^{\mathrm{IIB}}$ is an exact 4-cocycle with an explicit trivializing 3-cochain.

Thus, the pair of superMinkowski spaces $(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}, \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}})$ realizes in rational homotopy theory the data of a topological T-duality configuration.

As a consequence, one can bijectively transfer twisted K^0-cocycles in type IIA string theory to K^1-cocycles in type IIB string theory.

This phenomenon, known as rational topological T-duality and explicitly expressed by the Hori's formula, can be formally derived by the properties of the L_∞-algebra $b\mathfrak{tfold}$, providing the rational homotopy theory description of the classifying space for T-duality.

As we are going to see, Hori's formula is precisely a Fourier-Mukai transform in the context of twisted L_∞-algebra cohomology.

In order to prepare for the kind of construction we are going to describe in the setting of L_∞-algebras, let us first recall its classical geometric counterpart: the Fourier-Mukai transform in twisted de Rham cohomology.

Twisted de Rham cohomology and twisted FM transforms

Let X be a smooth manifold. One can twist the de Rham differential $d\colon \Omega^\bullet(X;\mathbb{R}) \xrightarrow{d} \Omega^\bullet(X;\mathbb{R})$ by a 1-form α, defining the twisted de Rham operator $d_\alpha\colon \Omega^\bullet(X;\mathbb{R}) \xrightarrow{d} \Omega^\bullet(X;\mathbb{R})$ as $d_\alpha\omega = d\omega + \alpha \wedge \omega$.

The operator d_α does not square to zero in general: d_α^2 is the multiplication by the exact 2-form $d\alpha$.

This means that precisely when α is a closed 1-form, the operator d_α is a differential, defining an α-twisted de Rham complex $(\Omega^\bullet(X), d_\alpha)$.

The cohomology of this complex is called the α-twisted de Rham cohomology of X and it will be denoted by the symbol $H^\bullet_{\mathrm{dR};\alpha}(X)$.

From a geometric point of view, the operator d_α is a connection on the trivial $\mathbb{R}$-bundle over X, which is flat precisely when α is closed. This means that for a closed 1-form α, the α-twisted de Rham cohomology of X is actually a particular instance of flat cohomology or cohomology with local coefficients.

Having identified d_α with a connection, it is natural to think of gauge transformations as the natural transformations in twisted de Rham cohomology.

Since we are in an abelian setting with a trivial $\mathbb{R}$-bundle, two connections d_{α_1} and d_{α_2} will be gauge equivalent exactly when there exists a smooth function β on X such that

$$\alpha_1 = \alpha_2 + d\beta,$$

i.e., when the two closed 1-forms α_1 and α_2 are in the same cohomology class.

When this occurs, the two twisted de Rham complexes $(\Omega^\bullet(X), d_{\alpha_1})$ and $(\Omega^\bullet(X), d_{\alpha_2})$ are isomorphic, with an explicit isomorphism of complexes given by the multiplication by the smooth function e^β.

In particular, multiplication by e^β induces an isomorphism in twisted cohomology

$$e^\beta \colon H^\bullet_{\mathrm{dR};\alpha_1}(X) \xrightarrow{\sim} H^\bullet_{\mathrm{dR};\alpha_2}(X) \, .$$

Let us investigate the functorial behavior of twisted cohomology with respect to a smooth map $\pi\colon Y \to X$. Since the pullback morphism $\pi^*\colon \Omega^\bullet(X) \to \Omega^\bullet(Y)$ is a morphism of DGCAs, it induces a morphism of complexes

$$\pi^*\colon (\Omega^\bullet(X), d_\alpha) \longrightarrow (\Omega^\bullet(Y), d_{\pi^*\alpha}) \ .$$

This gives a pullback morphism in twisted cohomology

$$\pi^*\colon H^\bullet_{\mathrm{dR};\alpha}(X) \longrightarrow H^\bullet_{\mathrm{dR};\pi^*\alpha}(Y) \ .$$

The pushforward morphism is a bit more delicate.

To begin with, given a smooth map $\pi\colon Y \to X$ we in general have no pushforward morphism of complexes $\pi_*\colon \Omega^\bullet(Y) \to \Omega^\bullet(Y)$.

However we do have such a morphism of complexes, up to a degree shift, if $Y \to X$ is not a general smooth map but it is an oriented fiber bundle with typical fiber F which is a compact closed oriented manifold.

In this case π_* is given by integration along the fiber and is a morphism of complexes $\pi_*\colon \big(\Omega^\bullet(Y), d\big) \to \big(\Omega^\bullet(X)[-\dim F], d[-\dim F]\big)$.

Yet, π_* will *not* induce a morphism $\pi_*\colon \big(\Omega^\bullet(Y), d_\alpha\big) \to \big(\Omega^\bullet(X)[-\dim F], d_{\pi_*\alpha}[-\dim F]\big)$ and actually a minute's reflection reveals that the symbol $d_{\pi_*\alpha}$ just makes no sense.

However, when α is not just a generic 1-form on Y but it is a 1-form pulled back from X, then everything works fine.

Namely, the projection formula

$$\pi_*(\pi^*\alpha \wedge \omega) = (-1)^{\deg \alpha \dim F} \alpha \wedge \pi_*\omega$$

precisely says that π_* is a morphism of chain complexes

$$\pi_* \colon (\Omega^\bullet(Y), d_{\pi^*\alpha}) \longrightarrow \big(\Omega^\bullet(X)[-\dim F], d_\alpha[-\dim F]\big)$$

and so it induces a pushforward morphism in twisted cohomology

$$\pi_* \colon H^\bullet_{\mathrm{dR};\pi^*\alpha}(Y) \longrightarrow H^{\bullet-\dim F}_{\mathrm{dR};\alpha}(X).$$

We can now define a Fourier-type transform in twisted cohomology.

Assume we are given a span of smooth manifolds

$$\begin{tikzcd} & Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2, \end{tikzcd}$$

with $Y \xrightarrow{\pi_2} X_2$ an oriented fiber bundle with compact closed oriented fibers. Let α_i be a closed 1-form on X_i, and assume that the two 1-forms $\pi_1^*\alpha_1$ and $\pi_2^*\alpha_2$ are cohomologous in Y, with $\pi_1^*\alpha_1 - \pi_2^*\alpha_2 = d\beta$.

Then we have the sequence of morphisms of chain complexes

$$(\Omega^\bullet(X_1), d_{\alpha_1}) \xrightarrow{\pi_1^*} (\Omega^\bullet(Y), d_{\pi_1^*\alpha_1}) \xrightarrow{e^\beta} (\Omega^\bullet(Y), d_{\pi_2^*\alpha_2}) \xrightarrow{\pi_{2*}} \big(\Omega^\bullet(X_2)[-\dim F_2], d_{\alpha_2}[-\dim F_2]\big)$$

whose composition defines the Fourier-Mukai transform with kernel β in twisted de Rham cohomology

$$\Phi_\beta \colon H^\bullet_{\mathrm{dR};\alpha_1}(X_1) \longrightarrow H^{\bullet-\dim F_2}_{\mathrm{dR};\alpha_2}(X_2)\ .$$

Writing "$\int_F$" for π_{2*} and writing "$\cdot$" for the right action of $\Omega^\bullet(X)$ on $\Omega^\bullet(Y)$ given by $\eta \cdot \omega = \eta \wedge \pi_1^* \omega$ makes it evident why this is a kind of Fourier transform

$$\Phi_\beta \colon \omega \longmapsto \int_{F_2} e^\beta \cdot \omega \ .$$

If also $\pi_1 \colon Y \to X_1$ is an oriented fiber bundle with compact closed oriented fibers, then we also have a Fourier-Mukai transform in the inverse direction, with kernel $-\beta$.

By evident degree reasons the transforms Φ_β and $\Phi_{-\beta}$ are *not* inverses.

A particular way of obtaining a span of oriented fiber bundles $X_1 \leftarrow Y \to X_2$ with compact closed oriented fibers is to consider a single oriented fiber bundle $Y \to Z$ with compact closed oriented fiber $F_1 \times F_2$. Then the manifolds X_1 and X_2 are given by the total spaces of the F_2-fiber bundle and F_1-fiber bundles on Z, respectively, associated with the two factors of $F_1 \times F_2$ together with the canonical projections.

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X_1 \arrow[dr] & & X_2 \arrow[dl] \\ & Z & \end{tikzcd}$$

In particular, an oriented 2-torus bundle $Y \to Z$ produces this way a span $X_1 \leftarrow Y \to X_2$ where both $\pi_i \colon Y \to X_i$ are S^1-bundles.

From 1-form twists to 3-form twists.

Assume now that α is a 3-form on X instead of a 1-form. Then we can still define the operator d_α on differential forms as $d_\alpha \omega = d\omega + \alpha \wedge \omega$, but this will no more be a homogeneous degree 1 operator.

We can heal this by adding a formal variable u with $\deg(u) = 2$ and with $du = 0$, and defining the degree 1 operator

$$d_\alpha \colon \Omega^\bullet(X)[[u^{-1}, u]] \longrightarrow \Omega^\bullet(X)[[u^{-1}, u]]$$

as the $\mathbb{R}[[u^{-1}, u]]$-linear extension of

$$d_\alpha \omega = d\omega + u^{-1} \alpha \wedge \omega.$$

Doing so, the above discussion verbatim applies, with the de Rham complex $\Omega^\bullet(X)$ replaced by the periodic de Rham complex $\Omega^\bullet(X)[[u^{-1}, u]]$.

In particular, if we have a span $X_1 \leftarrow Y \rightarrow X_2$ of oriented S^1-bundles and if α_i are 3-forms on X_i such that $\pi_1^*\alpha_1 - \pi_2^*\alpha_2 = d\beta$ for some 2-form β on Y, then we have Fourier-Mukai transforms

$$\Phi_\beta \colon H^\bullet_{\mathrm{dR};\alpha_1}(X_1; u^{-1}, u) \longrightarrow H^{\bullet-1}_{\mathrm{dR};\alpha_2}(X_2; u^{-1}, u) ,$$
$$\Phi_{-\beta} \colon H^\bullet_{\mathrm{dR};\alpha_2}(X_2; u^{-1}, u) \longrightarrow H^{\bullet-1}_{\mathrm{dR};\alpha_1}(X_1; u^{-1}, u) .$$

Having introduced the variable u, our cohomology is now endowed with a natural shift, given by the multiplication by u, and we may wonder whether the Fourier-Mukai transforms Φ_β and $\Phi_{-\beta}$ may be inverses to one another up to shift. As we are going to see, this is precisely what happens in rational T-duality configurations.

The above construction actually works for any closed differential form of odd degree, so there is apparently no point in considering 3-forms rather than 1-forms or 5-forms. There is, however, an important geometrical reason to focus on degree 3 forms:
when the coefficients are taken in a characteristic zero field, periodic de Rham cohomology is isomorphic (via the Chern character) to K-theory. Under this isomorphism, K-theory twists (which are topologically given by principal $U(1)$-gerbes) precisely become closed 3-forms.
In other words, for α_1 and α_2 closed 3-forms as above, the Fourier-Mukai transform Φ_β is to be thought as a morphism

$$\Phi_\beta \colon K^\bullet_{\mathcal{G}_1}(X_1) \otimes \mathbb{R} \longrightarrow K^{\bullet-1}_{\mathcal{G}_2}(X_2) \otimes \mathbb{R} \,.$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the twisting gerbes This is indeed the rationalization, with real coefficients, of a topological Fourier-Mukai transform

A particular situation we will be interested in is the case when the span $X_1 \leftarrow Y \rightarrow X_2$ of oriented S^1-bundles is induced by a 2-torus bundle $Y \rightarrow Z$, and so by a classifying map $Z \rightarrow B(U(1) \times U(1)) \cong BU(1) \times BU(1)$.

More specifically, we will also require that the canonical $U(1)$-2-gerbe associated with the torus bundle $Y \rightarrow Z$ is trivialized, i.e., we will be considering what is known as a topological T-duality configuration.

We will be investigating these from the point of view of rational homotopy theory, realizing the Fourier-Mukai transform as a morphism in twisted L_∞-algebra cohomology and proving that a pair of L_∞-algebras in a rational T-duality configuration comes equipped with a canonical Fourier-Mukai transform which turns out to be an isomorphism.

Basics of rational homotopy theory

The idea at the heart of rational homotopy theory is that, up to torsion, all of the homotopy type of a connected and simply connceted space with finite rank cohomology groups is encoded in its de Rham algebra with coefficients in a characteristic zero field, as a differential graded commutative algebra, up to homotopy.

With same care, the theory can be extended to a *simple space*, i.e., a connected topological space that has a homotopy type of a CW complex and whose fundamental group is abelian and acts trivially on the homotopy and homology of the universal covering space. A classical example is S^1, which we are actually going to meet several times what follows.

Moreover, since one has the freedom to replace the de Rham algebra with any homotopy equivalent DGCA, one sees that up to torsion the homotopy type of a simple space X is encoded into its so called minimal model or Sullivan algebra: a DGCA A_X such that:

- it is equipped with a quasi-isomorphism of dgca $A_X \to \Omega^\bullet(X)$
- it is semi-free, i.e., which is a free graded commutative algebra when one forgets the differential
- $A_X^1 = 0$
- the differential has no linear component

(for non simply connected simple spaces, one drops the condition $A_X^1 = 0$ and replaces it with a suitable nilpotency condition which is automatically satisfied if $A_X^1 = 0$)

In other words, A_X is a DGCA of the form $(\bigwedge^\bullet \mathfrak{l}X^*, d) = (\mathrm{Sym}^\bullet(\mathfrak{l}X[1]^*), d)$ for a suitable graded vector space $\mathfrak{l}X$ concentrated in strictly negative degrees (and finitely dimensional in each degree) and a suitable degree 1 differential d with $d(\mathfrak{l}X^*) \subseteq \bigwedge^{\geq 2} \mathfrak{l}X^*$.

Here $\mathfrak{l}X^*$ denotes the graded linear dual of $\mathfrak{l}X$, and the degree shift in the definition of $\bigwedge^\bullet$ is there in order to match the degree coming from geometry: the de Rham algebra is generated by 1-forms, which are in degree 1.

The minimal model is unique up to isomorphism and the quasi-isomorphism to the de Rham algebra is unique up to homotopy, so that one can talk of *the* minimal model of a space X.

The pair $(\bigwedge^\bullet \mathfrak{l}X^*, d)$ is what is called a *minimal L_∞-algebra* structure on $\mathfrak{l}X$ in the theory of L_∞-algebras.

Equivalently, one says that the DGCA $(\bigwedge^\bullet \mathfrak{l}X^*, d)$ is the Chevalley-Eilenberg algebra of the L_∞-algebra $\mathfrak{l}X$ and writes

$$(A_X, d_X) \cong (\mathrm{CE}(\mathfrak{l}X), d_X)$$

as the defining equation of the L_∞-algebra $\mathfrak{l}X$.

One says that the L_∞-algebra $\mathfrak{l}X$ is the rational approximation of X.

Geometrically, it can be thought of as the tangent L_∞-algebra to the ∞-group given by the based loop space of X (as X is connceted and simply connected, the choice of a basepoint is irrelevant).

A smooth map $f\colon Y \to X$ is faithfully encoded into the DGCA morphism $f^*\colon \Omega^\bullet(X) \to \Omega^\bullet(Y)$, so that the rational approximation of f is encoded into a DGCA morphism, which we will continue to denote f^*,

$$f^*\colon A_X \longrightarrow A_Y.$$

In turn (by definition) this is a morphism of L_∞-algebras $\mathfrak{l}f\colon \mathfrak{l}Y \to \mathfrak{l}X$.

Here $\mathfrak{l}X$ and $\mathfrak{l}Y$ are minimal, but up to homotopy every L_∞-algebra is equivalent to a minimal one: this is the dual statement of the fact that every (well behaved) DGCA is homotopy equivalent to a minimal DGCA.

Therefore we get the fundamental insight of rational homotopy theory:

the category of simply connected homotopy types over $\mathbb{R}$ is (equivalent to) the homotopy category of L_∞-algebras over $\mathbb{R}$ with cohomology concentrated in strictly negative degrees.

(this can actually be generalized to simple homotopy types and to an arbitrary characteristic zero field $\mathbb{K}$)

The Sullivan model of $BU(1)$

The real cohomology of $BU(1)$ is $H^\bullet(BU(1);\mathbb{R}) \cong \mathbb{R}[x_2]$, where x_2 is a degree 2 element, the universal first Chern class.
As $H^\bullet(BU(1);\mathbb{R})$ is a free polynomial algebra, we can think of it as a semifree DGCA with trivial differential.
Choosing a de Rham representative for the first Chern class defines a quasi-isomorphism

$$(\mathbb{R}[x_2], 0) \longrightarrow (\Omega^\bullet(BU(1)), d)$$

exhibiting $(\mathbb{R}[x_2], 0)$ as the Sullivan model of $BU(1)$.
The equation

$$(\mathbb{R}[x_2], 0) \cong (\mathrm{CE}(\mathfrak{l}BU(1)), d_{BU(1)})$$

characterizes $\mathfrak{l}BU(1))$ as the L_∞-algebra consisting of the cochain complex $\mathbb{R}[1]$ consisting of the vector space $\mathbb{R}$ in degree -1 and zero in all other degrees (with zero differential).
We will denote this L_∞-algebra by the symbol $b\mathfrak{u}_1$.

A principal $U(1)$-bundle $P \to X$ is classified by a map $X \to BU(1)$. The rational approximation of this map is an L_∞-morphism

$$\mathfrak{l}X \longrightarrow b\mathfrak{u}_1.$$

Equivalently, by definition, this is a DGCA morphism

$$(\mathbb{R}[x_2], 0) \longrightarrow (A_X, d_X),$$

i.e., it is a degree 2 closed element in A_X.
Composing with $(A_X, d_X) \xrightarrow{\sim} (\Omega^\bullet(X), d)$ we get a closed 2-form ω_2 on X associated to $P \to X$.
Since the quasi-isomorphism $(A_X, d_X) \xrightarrow{\sim} (\Omega^\bullet(X), d)$ is only unique up to homotopy, the 2-form ω_2 is only well defined up to an exact term so it is actually $[\omega_2]$ to be canonically associated with $P \to X$.
No surprise, $[\omega_2]$ is the image in de Rham cohomology of the first Chern class of $P \to X$.

Compact abelian Lie groups

Given a compact Lie group G, then the inclusion $\Omega^\bullet(G)^G \hookrightarrow \Omega^\bullet$ of G-invariant differential forms on G into the de Rham complex of G is a quasi-isomorphism. As a graded vector space $\Omega^\bullet(G)^G \cong \bigwedge^\bullet \mathfrak{g}^*$, where $\mathfrak{g}$ denotes the Lie algebra of G. The de Rham differential on $\Omega^\bullet(G)^G$ corresponds to the Chevalley-Eilenberg differential on $\bigwedge^\bullet \mathfrak{g}^*$. From this we see that a semifree model for G is $\mathrm{CE}(\mathfrak{g})$.

However, $\mathrm{CE}(\mathfrak{g})$ is *not* a Sullivan model for G, unless $\mathfrak{g}$ is nilpotent. This happens in particular for compact abelian Lie groups, so that, for instance $\mathrm{CE}(\mathfrak{u}_1)$ is indeed the Sullivan model of $U(1)$.

The Sullivan models of spheres

We have

$$H^\bullet(S^n;\mathbb{R}) \simeq \begin{cases} \mathbb{R}[t_n] & \text{if } n \text{ is odd} \\ \mathbb{R}[t_n]/({t_n}^2) & \text{if } n \text{ is even} \end{cases}$$

as graded commutative rings, where t_n has degree n.

In the odd case, the rational cohomology of S^n is a free graded polynomial algebra, and so it essentially coincides with its own Sullivan model, we only need to add a trivial differential to the picture:

$$\mathrm{CE}(\mathfrak{l}S^{2k+1}) = \big(\mathbb{R}[x_{2k+1}];\ dx_{2k+1} = 0\big).$$

Namely, if ω_{2k+1} is a volume form for S^{2k+1}, the map $x_{2k+1} \mapsto \omega_{2k+1}$ defines a quasi-isomorphism of dgcas

$$\big(\mathbb{R}[x_{2k+1}];\ dx_{2k+1} = 0\big) \longrightarrow \big(\Omega^\bullet(S^{2k+1};\mathbb{R});\ d_{\mathrm{dR}}\big)$$

For even $n = 2k$ we have to cure the constraint ${t_{2k}}^2 = 0$. This is done by lifting the cohomology relation ${t_{2k}}^2 = 0$ to the equation $x_{2k} \wedge x_{2k} = dx_{4k-1}$.

$$\begin{aligned}\big(\mathbb{R}[x_{2k}, x_{4k-1}];\ dx_{2k} = 0,\, dx_{4k-1} = x_{2k} \wedge x_{2k}\big) &\longrightarrow \big(\Omega^\bullet(S^{2k}; \mathbb{R});\ d_{\mathrm{dR}}\big) \\ x_{2k} &\longmapsto \omega_{2k} \\ x_{4k-1} &\longmapsto 0\end{aligned}$$

is a quasi-isomorphism of DGCAs. Moreover, $\mathbb{R}[x_{2k}, x_{4k-1}]^1 = 0$ and the differential is decomposable. In other words,

$$\mathrm{CE}(\mathfrak{l}S^{2k}) = \big(\mathbb{R}[x_{2k}, x_{4k-1}];\ dx_{2k} = 0,\, dx_{4k-1} = x_{2k} \wedge x_{2k}\big).$$

Given the identification between simple homotopy types and L_∞-algebras mentioned above, from now on we can work directly with L_∞-algebras, with no reference to the space they can be a rationalization of.

A span $X_1 \leftarrow Y \rightarrow X_2$ as in the discussion of Fourier-Mukai transforms in twisted de Rham cohomology becomes a span

$$\begin{tikzcd} & \mathfrak{h} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathfrak{g}_1 & & \mathfrak{g}_2 \end{tikzcd}$$

of L_∞-algebras.

As we want that the π_i's represent the S^1-bundles our next step is the characterization of those L_∞-morphism that correspond to principal $U(1)$-bundles.

Central extensions of L_∞-algebras

A principal $U(1)$-bundle over a smooth manifold X is encoded up to homotopy into a map $f\colon X \to BU(1)$ from X to the classifying space $U(1)$. The total space P as well as the projection $P \to X$ are recovered by f by taking its homotopy fiber, i.e., by considering the homotopy pullback

$$\begin{tikzcd} P \arrow[rr] \arrow[d] & & * \arrow[d] \\ X \arrow[r, "f"] & BU(1) \, . \end{tikzcd}$$

As rationalization commutes with homotopy pullbacks, the rational approximation of the above diagram is

$$\begin{tikzcd} \mathfrak{l}P \arrow[rr] \arrow[d] & & 0 \arrow[d] \\ \mathfrak{l}X \arrow[r, "\mathfrak{l}f"] & b\mathfrak{u}_1 \, . \end{tikzcd}$$

Dually, this means that we have a homotopy pushout of DGCAs

$$\begin{tikzcd}
(\mathbb{R}[x_2], 0) \arrow[r] \arrow[d, "f^*"'] & (\mathbb{R}, 0) \arrow[d] \\
(A_X, d_X) \arrow[r] & (A_P, d_P)\ .
\end{tikzcd}$$

This is easily computed. All we have to do is to replace the DCGA morphism $\mathbb{R}[x_2] \to \mathbb{R}$ with an equivalent cofibration.

The easiest way of doing this is to factor $\mathbb{R}[x_2] \to \mathbb{R}$ as

$$(\mathbb{R}[x_2], 0) \hookrightarrow (\mathbb{R}[y_1, x_2], dy_1 = x_2) \xrightarrow{\sim} \mathbb{R}$$

Then A_P is computed as an ordinary pushout

$$\begin{tikzcd}
(\mathbb{R}[x_2], 0) \arrow[r] \arrow[d, "f^*"'] & (\mathbb{R}[y_1, x_2], dy_1 = x_2) \arrow[d] \\
(A_X, d_X) \arrow[r] & (A_P, d_P)
\end{tikzcd} ,$$

i.e.,

$$(A_P, d_P) = (A_X[y_1], d_P\omega = d_X\omega \;\text{ for }\; \omega \in A_X,\ d_P y_1 = f^* x_2).$$

This immediately generalizes to the case of an arbitrary L_∞-morphism $f\colon \mathfrak{g} \to b\mathfrak{u}_1$. The homotopy fiber of f will be the L_∞-algebra $\hat{\mathfrak{g}}$ characterized by

$$\mathrm{CE}(\hat{\mathfrak{g}}) = \mathrm{CE}(\mathfrak{g})[y_1],$$

where y_1 ia a variable in degree 1 and where the differential in $\mathrm{CE}(\hat{\mathfrak{g}})$ extends that in $\mathrm{CE}(\mathfrak{g})$ by the rule $d_{\hat{\mathfrak{g}}} y_1 = f^*(x_2)$.

Example. If $\mathfrak{g}$ is a Lie algebra (over $\mathbb{R}$), then an L_∞-morphism $f\colon \mathfrak{g} \to b\mathfrak{u}_1$ is precisely a Lie algebra 2-cocycle on $\mathfrak{g}$ with values in $\mathbb{R}$. The L_∞-algebra $\hat{\mathfrak{g}}$ is again a Lie algebra in this case, and it is the central extension of $\mathfrak{g}$ by $\mathbb{R}$ classified by the 2-cocycle f.

The above construction admits an immediate generalization to higher degree cocycles.

Twisted L_∞-algebra cohomology

An L_∞-algebra $\mathfrak{g}$ is encoded into its Chevalley-Eilenberg algebra $(\mathrm{CE}(\mathfrak{g}), d_{\mathfrak{g}})$. The L_∞-algebra cohomology of $\mathfrak{g}$ is defined as

$$H^\bullet_{L_\infty}(\mathfrak{g};\mathbb{R}) = H^\bullet\left(\mathrm{CE}(\mathfrak{g}), d_{\mathfrak{g}}\right).$$

When $\mathfrak{g}$ is a Lie algebra this reproduces the Lie algebra cohomology of $\mathfrak{g}$ (with coefficients in the trivial $\mathfrak{g}$-module $\mathbb{R}$).

If $\mathfrak{g}$ is the L_∞-algebra representing the rational homotopy type of a simple space X, then the L_∞-algebra cohomology of $\mathfrak{g}$ computes the de Rham cohomology of X:

$$\begin{aligned} H^\bullet_{L_\infty}(\mathfrak{l}X;\mathbb{R}) &= H^\bullet\left(\mathrm{CE}(\mathfrak{l}X), d_X\right) \\ &= H^\bullet\left(A_X, d_X)\right) \cong H^\bullet\left(\Omega^\bullet(X), d)\right) = H^\bullet_{\mathrm{dR}}(X). \end{aligned}$$

This is more generally true if instead of the Sullivan model $\mathrm{CE}(\mathfrak{l}X)$ one considers an arbitrary semifree model $\mathrm{CE}(\mathfrak{g}_X)$.

Example

If $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, then one recovers the classical statement that the Lie algebra cohomology of $\mathfrak{g}$ computes the de Rham cohomology of G:

$$H^{\bullet}_{\mathrm{Lie}}(\mathfrak{g}; \mathbb{R}) \cong H^{\bullet}_{\mathrm{dR}}(G).$$

This has actually been one of the motivating examples in the definition of Lie algebra cohomology.

Exactly as we twisted de Rham cohomology, we can twist L_∞-algebra cohomology.

If a is a degree 3 cocycle on $\mathfrak{g}$ then we can consider the degree 1 differential $d_{\mathfrak{g};a} \colon x \mapsto d_{\mathfrak{g}} x + u^{-1} a\, x$ on $\mathrm{CE}(\mathfrak{g})[[u^{-1}, u]]$ and define

$$H^\bullet_{L_\infty;a}(\mathfrak{g}; \mathbb{R}[[u^{-1}, u]]) = H^\bullet \left(\mathrm{CE}(\mathfrak{g})[[u^{-1}, u]], d_{\mathfrak{g};a} \right).$$

If a_1 and a_2 are cohomologous 3-cocycles with $a_1 - a_2 = db$ then $e^{u^{-1} b}$ is a cochain complexes isomorphism between $(\mathrm{CE}(\mathfrak{g})[[u^{-1}, u]], d_{\mathfrak{g};a_1})$ and $(\mathrm{CE}(\mathfrak{g})[[u^{-1}, u]], d_{\mathfrak{g};a_2})$ and so induces an isomorphism

$$e^{u^{-1} b} \colon H^\bullet_{L_\infty;a_1}(\mathfrak{g}; \mathbb{R}[[u^{-1}, u]]) \xrightarrow{\sim} H^\bullet_{L_\infty;a_2}(\mathfrak{g}; \mathbb{R}[[u^{-1}, u]]).$$

If $f\colon \mathfrak{h} \to \mathfrak{g}$ is an L_∞ morphism, then by definition f is a DGCA morphism $f^*\colon \mathrm{CE}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{h})$ so that f^*a is a 3-cocycle on $\mathfrak{h}$ for any 3-cocycle a on $\mathfrak{g}$, and f^* is a morphism of cochain complexes between $(\mathrm{CE}(\mathfrak{g})[[u^{-1}, u]], d_{\mathfrak{g};a})$ and $(\mathrm{CE}(\mathfrak{h})[[u^{-1}, u]], d_{\mathfrak{h};f^*a})$, thus inducing a morphism between the twisted cohomologies

$$f^*\colon H^\bullet_{L_\infty;a}(\mathfrak{g}; \mathbb{R}[[u^{-1}, u]]) \longrightarrow H^\bullet_{L_\infty;f^*a}(\mathfrak{h}; \mathbb{R}[[u^{-1}, u]]).$$

We, therefore, see that in order to define Fourier-Mukai transforms at the level of twisted L_∞-algebra cohomology the only ingredient we miss is a pushforward morphism

$$\pi_*\colon (\mathrm{CE}(\hat{\mathfrak{g}}), d_{\hat{\mathfrak{g}}}) \longrightarrow \big(\mathrm{CE}(\mathfrak{g})[-1], d_{\mathfrak{g}}[-1]\big)$$

for any central extension $\pi\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ induced by a 2-cocycle $\mathfrak{g} \to b\mathfrak{u}_1$, which is a morphism of cochain complexes and which satisfies the projection formula identity.

The degree 1 element y_1 in the Chevalley-Eilenberg of the central extension $\hat{\mathfrak{g}}$

$$(\mathrm{CE}(\hat{\mathfrak{g}}), d_{\hat{\mathfrak{g}}}) = (\mathrm{CE}(\mathfrak{g})[y_1], d_{\hat{\mathfrak{g}}} y_1 = f^* x_2)$$

geometrically represents a vertical volume form.

The fiber integration morphism π_* is then

$$\pi_* \colon (\mathrm{CE}(\mathfrak{g})[y_1], d_{\hat{\mathfrak{g}}} y_1 = f^* x_2) \longrightarrow (\mathrm{CE}(\mathfrak{g})[-1], d_{\mathfrak{g}}[-1])$$
$$a + y_1\, b \longmapsto b,$$

It is immediate to see that π_* is indeed a morphism of chain complexes and that the projection formula holds:

$$\pi_*((\pi^* a)\, \omega) = (-1)^a a\, \pi_* \omega,$$

for every $\omega \in \mathrm{CE}(\hat{\mathfrak{g}})$.

Summing up, we have reproduced at the L_∞-algebra/rational homotopy theory level all of the ingredients we needed to define Fourier-Mukai transforms.

Given a span $\mathfrak{g}_1 \xleftarrow{\pi_1} \mathfrak{h} \xrightarrow{\pi_2} \mathfrak{g}_2$ of central extensions (by the abelian Lie algebra $\mathbb{R}$) of L_∞-algebras, and given a triple (a_1, a_2, b) consisting of 3-cocycles a_i on $\mathfrak{g}_i$ and of a degree 4 element b in $\mathrm{CE}(\mathfrak{h})$ such that $d_{\mathfrak{h}} b = \pi_1^* a_1 - \pi_2^* a_2$ we have Fourier-Mukai transforms

$$\Phi_b \colon H^\bullet_{L_\infty;a_1}(\mathfrak{g}_1; \mathbb{R}[[u^{-1}, u]]) \longrightarrow H^{\bullet-1}_{L_\infty;a_2}(\mathfrak{g}_2; \mathbb{R}[[u^{-1}, u]])$$
$$\Phi_{-b} \colon H^\bullet_{L_\infty;a_2}(\mathfrak{g}_2; \mathbb{R}[[u^{-1}, u]]) \longrightarrow H^{\bullet-1}_{L_\infty;a_1}(\mathfrak{g}_1; \mathbb{R}[[u^{-1}, u]])$$

given by the images in cohomology of the morphisms of complexes

$$\omega \longmapsto \pi_{2*}(e^{u^{-1}b_2}\pi_1^*\omega) \qquad \text{and} \qquad \omega \longmapsto \pi_{1*}(e^{-u^{-1}b_2}\pi_2^*\omega),$$

respectively.

We are going to see how to produce a quintuple $(\pi_1, \pi_2, a_1, a_2, b)$ inducing a Fourier-Mukai transform in tomorrow's lecture. But first let us spend a few more words on the geometric properties of the pushforward morphism π_*.

As $\pi_* \colon (\mathrm{CE}(\hat{\mathfrak{g}}), d_{\hat{\mathfrak{g}}}) \to (\mathrm{CE}(\mathfrak{g})[-1], d_{\mathfrak{g}}[-1])$ is a morphism of cochain complexes, it in particular maps degree $n+1$ cocycles in $\mathrm{CE}(\hat{\mathfrak{g}})$ to degree n cocycles in $\mathrm{CE}(\mathfrak{g})$. But, if $\mathfrak{h}$ is any L_∞-algebra, a degree k cocycle in $\mathrm{CE}(\mathfrak{h})$ is precisely an L_∞-morphism $\mathfrak{h} \to b^{k-1}\mathfrak{u}_1$.

Therefore we see that π_* induces a morphism of sets

$$\mathrm{Hom}_{L_\infty}(\hat{\mathfrak{g}}, b^n \mathfrak{u}_1) \longrightarrow \mathrm{Hom}_{L_\infty}(\mathfrak{g}, b^{n-1}\mathfrak{u}_1).$$

This is actually part of a much larger picture, to see which we need a digression on free loop spaces.

Cyclification of L_∞-algebras.

Let X be a smooth manifold, let $\pi\colon P \to X$ be a principal $U(1)$-bundle over X, and let $\varphi\colon P \to Y$ a map from P to another smooth manifold Y. Let $\gamma\colon P \times U(1) \to Y$ be the composition

$$P \times U(1) \longrightarrow P \xrightarrow{\varphi} Y$$

where the first map is the right $U(1)$-action on P.

By the multiplication by S^1/free loop space adjunction, γ is, equivalently, a $U(1)$-equivariant morphism from P to the free loop space $\mathcal{L}Y$ of Y.

Equivalently, γ is a morphism between the homotopy quotients $X = P//U(1)$ and $\mathcal{L}Y//U(1)$ over $BU(1)$:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & \mathcal{L}Y//U(1) \arrow[dl] \\ & BU(1) & \end{tikzcd}$$

Writing $\mathrm{cyc}(Y)$ for the "cyclification" $\mathcal{L}Y /\!/ U(1)$ and recalling that the total space P is the homotopy fiber of the morphism $f\colon X \to BU(1)$, we see that the above discussion can be elegantly summarized by saying that cyclification is the right adjoint to homotopy fiber,

$$\begin{tikzcd} \mathrm{spaces} \arrow[rr, bend left, "\mathrm{cyc}"] & & \mathrm{spaces}/BU(1). \arrow[ll, bend left, "\mathrm{hofib}"] \end{tikzcd}$$

The above topological construction immediately translates to the L_∞-algebra setting, where we find an adjunction

$$
\begin{tikzcd}
L_\infty\text{-algebras} \arrow[rr, bend left, "\mathrm{cyc}"] && L_\infty\text{-algebras}/b\mathfrak{u}_1 . \arrow[ll, bend left, "\mathrm{hofib}"]
\end{tikzcd}
$$

We have already seen that the homotopy fiber functor from L_∞-algebras over $b\mathfrak{u}_1$ to L_∞-algebras consists in forming the $\mathbb{R}$-central extension classified by the 2-cocycle. So we have now to complete the picture by describing the cyclification functor.

If X is 2-connected an L_∞-algebra representing the rational homotopy type of the free loop space $\mathcal{L}X$ is easily deduced from the multiplication by S^1/free loop space adjunction.
A Sullivan model for $Y \times S^1$ is $A_{Y\times S^1} = A_Y \otimes A_{S^1} = A_Y[t_1]$ with $dt_1 = 0$.
From this one gets

$$A_{\mathcal{L}X} = (\bigwedge^{\bullet}(\mathfrak{l}X^* \oplus s\mathfrak{l}X^*), d_{\mathcal{L}X})$$

where $s\mathfrak{l}X^* = \mathfrak{l}X^*[1]$ is a shifted copy of $\mathfrak{l}X^*$, with $d_{\mathcal{L}X}\big|_{A_X} = d_X$ and $[d_{\mathcal{L}X}, s] = 0$, where $s\colon A_{\mathcal{L}X} \to A_{\mathcal{L}X}$ is the shift operator $s\colon \mathfrak{l}X^* \xrightarrow{\sim} (s\mathfrak{l}X^*)[-1]$ extended as a degree -1 differential.
For an arbitrary L_∞-algebra $\mathfrak{g}$ we define $\mathfrak{L}\mathfrak{g}$ as the L_∞-algebra

$$(\mathrm{CE}(\mathcal{L}\mathfrak{g}), d_{\mathfrak{L}\mathfrak{g}}) = \big(\bigwedge^{\bullet}(\mathfrak{g}^* \oplus s\mathfrak{g}^*), d_{\mathfrak{L}\mathfrak{g}}\big|_{\mathrm{CE}(\mathfrak{g})} = d_{\mathfrak{g}}, [d_{\mathfrak{L}\mathfrak{g}}, s] = 0]\big).$$

Deriving an L_∞-algebra model for the cyclification $\mathrm{cyc}(X)$ is a bit more involved. One finds

$$A_{\mathrm{cyc}(X)} == \big(\bigwedge^{\bullet}(\mathfrak{l}X^* \oplus s\mathfrak{l}X^*)[x_2], d_{\mathrm{cyc}(X)}\big),$$

where x_2 is a degree 2 closed variable and $d_{\mathrm{cyc}(X)}$ acts on an element $a \in \mathfrak{l}X^* \oplus s\mathfrak{l}X^*$ as $d_{\mathrm{cyc}X}a = d_{\mathfrak{L}\mathfrak{g}}a + x_2 \wedge sa$.
For an arbitrary L_∞-algebra $\mathfrak{g}$ one defines the $\mathrm{cyc}(\mathfrak{g})$ as

$$\mathrm{CE}(\mathrm{cyc}(\mathfrak{g})) = \big((\bigwedge^{\bullet}(\mathfrak{g} \oplus s\mathfrak{g})^*)[x_2], d_{\mathrm{cyc}(\mathfrak{g})}\big),$$

where x_2 is a degree 2 variable with $d_{\mathrm{cyc}(\mathfrak{g})}x_2 = 0$ and $d_{\mathrm{cyc}(\mathfrak{g})}$ acts on an element $a \in \mathfrak{g}^*[-1] \oplus \mathfrak{g}^*$ as

$$d_{\mathrm{cyc}(\mathfrak{g})}a = d_{\mathfrak{L}\mathfrak{g}}a + x_2 \wedge sa.$$

Notice that there is a canonical inclusion of dgcas $\mathbb{R}[x_2] \hookrightarrow \mathrm{CE}(\mathrm{cyc}(\mathfrak{g}))$, giving a canonical 2-cocycle $\mathrm{cyc}(\mathfrak{g}) \to b\mathfrak{u}_1$.

The L_∞ algebras $b^n\mathfrak{u}_1$ have a particularly simple cyclification. $\mathrm{CE}(\mathrm{cyc}(b^n\mathfrak{u}_1))$ is obtained from $\mathrm{CE}(b^n\mathfrak{u}_1) = (\mathbb{R}[x_{n+1}], 0)$ by adding a generator $y_n = s x_{n+1}$ in degree n and a generator z_2 in degree 2, with differential

$$dx_{n+1} = z_2\, y_n; \qquad dy_n = 0; \qquad dz_2 = 0.$$

From this one sees that we have an injection $(\mathbb{R}[y_n], 0) \hookrightarrow (\mathrm{CE}(\mathrm{cyc}(b^n\mathfrak{u}_1)), d)$ and so dually a fibration

$$\mathrm{cyc}(b^n\mathfrak{u}_1) \longrightarrow b^{n-1}\mathfrak{u}_1$$

of L_∞-algebras.

Given an $\mathbb{R}$-central extension $\pi\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ we can form the composition

$$\mathrm{Hom}_{L_\infty}(\hat{\mathfrak{g}}, b^n\mathfrak{u}_1) \cong \mathrm{Hom}_{L_\infty/b\mathfrak{u}_1}(\mathfrak{g}, \mathrm{cyc}(b^n\mathfrak{u}_1)) \to \mathrm{Hom}_{L_\infty}(\mathfrak{g}, \mathrm{cyc}(b^n\mathfrak{u}_1)) \to \mathrm{Hom}_{L_\infty}(\mathfrak{g}, b^{n-1}\mathfrak{u}_1),$$

and this coincides with the fiber integration morphism

$$\pi_*\colon \mathrm{Hom}_{L_\infty}(\hat{\mathfrak{g}}, b^n\mathfrak{u}_1) \longrightarrow \mathrm{Hom}_{L_\infty}(\mathfrak{g}, b^{n-1}\mathfrak{u}_1)$$

An example from string theory/M-theory.

The Sullivan model for S^4 is

$$\mathrm{CE}(\mathfrak{l}S^4) = (\mathbb{R}[z_4, z_7],\ dz_4 = 0,\ dz_7 = {z_4}^2).$$

Therefore, the Sullivan model for $\mathcal{L}S^4//U(1)$ is

$$\mathrm{CE}(\mathrm{cyc}(\mathfrak{l}S^4)) = (\mathbb{R}[f_2, f_4, f_6, h_3, h_7],\ df_2 = 0,\ dh_3 = 0,\ df_4 = h_3 f_2,\ df$$

Therefore, a smooth cocycle $X \to \mathrm{cyc}(\mathfrak{l}S^4)$ on a smooth (super)manifold X will be the datum of a closed 3-form H_3 and of 2-, 4- and 6-forms F_2, F_4 and F_6 on X such that

$$dF_2 = 0; \qquad dF_4 = H_3 \wedge F_2; \qquad dF_6 = H_3 \wedge F_4,$$

together with a 7-form H_7 which is a potential for the closed 8-form $F_4 \wedge F_4 - 2F_2 \wedge F_6$.

The above equations for the differentials of the F_{2n}'s are precisely (a subset of) the equations for a H_3-twisted cocycle $\sum_{n=-\infty}^{\infty} F_{2n}u^n$ in $(\Omega^\bullet(X)[[u^{-1}, u]], d_{H_3})$ with $F_0 = 0$.

If $Y \to X$ is rationally a principal S^1-bundle, then a $\mathfrak{l}S^4$ cocycle on Y will induce, by the hofiber/cyclification adjunction, such a set of differential forms on X.

This is the mechanism by which the M-theory cocycle $\mathbb{R}^{10,1|\mathbf{32}} \to \mathfrak{l}S^4$ induces twisted (rational) even K-theory cocycles on on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$.

The classifying spaces of T-duality configurations

The same way as the classifying space $BU(1)$ of principal $U(1)$-bundles is a $K(\mathbb{Z}, 2)$, the classifying space $B^3U(1)$ of principal $U(1)$-3-bundles (or principal $U(1)$-2-gerbes) is a $K(\mathbb{Z}; 4)$.

This implies that the cup product map

$$\cup\colon K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \longrightarrow K(\mathbb{Z}, 4)$$

is equivalently a map

$$\cup\colon BU(1) \times BU(1) \longrightarrow B^3U(1),$$

i.e., to any pair of principal $U(1)$ bundles P_1 and P_2 on a manifold X is canonically associated a $U(1)$-2-gerbe $P_1 \cup P_2$ on X.

By definition, a topological T-duality configuration is the datum of two such principal $U(1)$-bundles together with a trivialization of their cup product.

In other words, a topological T-duality configuration on a manifold X is a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[d] & & * \arrow[d] \\ BU(1) \times BU(1) \arrow[rr, "\cup"] & & B^3U(1) \end{tikzcd} .$$

By the universal property of the homotopy pullback this is in turn equivalent to a map from X to the homotopy fiber of the cup product, which will therefore be the classifying space for topological T-duality configurations.

$$
\begin{tikzcd}
B\mathrm{Tfold} \arrow[r] \arrow[d] & * \arrow[d] \\
BU(1) \times BU(1) \arrow[r, "\cup"] & B^3U(1)\,.
\end{tikzcd}
$$

The rationalization of $B\mathrm{Tfold}$ is obtained as the L_∞-algebra $b\mathfrak{tfold}$ given by the homotopy pullback

$$
\begin{tikzcd}
b\mathfrak{tfold} \arrow[r] \arrow[d] & 0 \arrow[d] \\
b\mathfrak{u}_1 \times b\mathfrak{u}_1 \arrow[r, "\cup"] & b^3\mathfrak{u}_1
\end{tikzcd}
$$

In order to get an explicit description of it we only need to give an explicit description of the 4-cocycle $b\mathfrak{u}_1 \times b\mathfrak{u}_1 \xrightarrow{\cup} b^3\mathfrak{u}_1$.

This is easily read in the dual picture: it is the obvious morphism of CGDAs

$$\begin{aligned}(\mathbb{R}[x_4], 0) &\longrightarrow (\mathbb{R}[\check{x}_2, \tilde{x}_2], 0) \cong (\mathbb{R}[x_2], 0) \otimes (\mathbb{R}[x_2], 0) \\ x_4 &\longmapsto \check{x}_2\, \tilde{x}_2.\end{aligned}$$

The Chevalley-Eilenberg algebra of $b\mathfrak{tfold}$ is then given by the homotopy pushout

$$\begin{tikzcd}
(\mathbb{R}[x_4], 0) \arrow[r] \arrow[d, "\cup^*"'] & (\mathbb{R}, 0) \arrow[d] \\
(\mathbb{R}[\check{x}_2, \tilde{x}_2], 0) \arrow[r] & (\mathrm{CE}(b\mathfrak{tfold}), d)
\end{tikzcd} ,$$

i.e., by the pushout

$$\begin{tikzcd}
(\mathbb{R}[x_4], 0) \arrow[r] \arrow[d, "\cup^*"'] & (\mathbb{R}[y_3, x_4], dy_3 = x_4) \arrow[d] \\
(\mathbb{R}[\check{x}_2, \tilde{x}_2], 0) \arrow[r] & (\mathrm{CE}(b\mathfrak{tfold}), d)
\end{tikzcd} .$$

Explicitly, this means that

$$(\mathrm{CE}(b\mathfrak{tfold}), d) = (\mathbb{R}[\check{x}_2, \tilde{x}_2, y_3], d\check{x}_2 = 0, d\tilde{x}_2 = 0, dy_3 = \check{x}_2\,\tilde{x}_2),$$

and so an L_∞-morphism $\mathfrak{g} \to b\mathfrak{tfold}$ is precisely what we should have expected it to be: a pair of 2-cocycles on $\mathfrak{g}$ together with a trivialization of their product.

Moreover, one manifestly has an isomorphism

$$(\mathrm{CE}(b\mathfrak{tfold}), d) \cong (\mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1), d)$$

so that the $b\mathfrak{tfold}$ L_∞-algebra is isomorphic to the cyclification of $b^2\mathfrak{u}_1$.

This result actually already holds at the topological level, i.e., there is a homotopy equivalence $BTfold \cong \mathrm{cyc}(K(\mathbb{Z}, 3)) \cong \mathrm{cyc}(B^2U(1))$. Proving this equivalence beyond the rational approximation is however considerably harder.

The L_∞-algebra $b\mathfrak{tfold}$ has two independent 2-cocycles $f_1, f_2 \colon b\mathfrak{tfold} \to b\mathfrak{u}_1$ given in the dual picture by $f_1^*(x_2) = \check{x}_2$ and by $f_2^*(x_2) = \tilde{x}_2$. Let us denote by $\mathfrak{p}_1$ and $\mathfrak{p}_2$ the central extensions of $b\mathfrak{tfold}$ corresponding to f_1 and f_2, respectively.

They are clearly isomorphic as L_∞-algebras; however they are not equivalent as L_∞-algebras over $b\mathfrak{tfold}$ as the two classifying morphisms f_1 and f_2 are not homotopy equivalent.

Let us now write $\mathbb{R}[x_3]$ for the Chevalley-Eilenberg algebra $\mathrm{CE}(b^2\mathfrak{u}_1)$, so that we have

$$\mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1)) = \mathbb{R}[x_3, y_2, z_2]$$

with

$$dx_3 = z_2 y_2, \qquad dy_2 = 0 \qquad dz_2 = 0$$

The canonical 2-cocycle $\mathrm{cyc}(b^2\mathfrak{u}_1) \to b\mathfrak{u}_1$ is given by

$$\begin{aligned} f^*_{\mathrm{cyc}} \colon \mathbb{R}[x_2] &\longrightarrow \mathbb{R}[x_3, y_2, z_2] \\ x_2 &\longmapsto z_2. \end{aligned}$$

The isomorphism of L_∞-algebras $\varphi_1 \colon b\mathfrak{tfold} \to \mathrm{cyc}(b^2\mathfrak{u}_1)$ given by $x_3 \mapsto y_3$, $y_2 \mapsto \tilde{x}_2$ and $z_2 \mapsto \check{x}_2$ is such that the diagram of DGCAs

$$\begin{tikzcd} & \mathrm{CE}(b\mathfrak{u}_1) \arrow[dl, "f_{\mathrm{cyc}}^*"'] \arrow[dr, "f_1^*"] & \\ \mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1)) \arrow[rr, "\varphi_1^*"] & & \mathrm{CE}(b\mathfrak{tfold}) \end{tikzcd}$$

commutes, i.e., φ_1 is an isomorphism over $b\mathfrak{u}_1$.

By the hofiber/cyclification adjunction, it corresponds to an L_∞ morphism from the homotopy fiber of f_1 to $b^2\mathfrak{u}_1$, i.e., to a 3-cocycle $a_{3,1}$ over $\mathfrak{p}_1$.

Repeating the same reasoning for f_2 we get a canonical 3-cocycle $a_{3,2}$ over $\mathfrak{p}_2$.

Therefore, we see how some of the ingredients of a rational T-duality configuration naturally emerge form the T-fold L_∞-algebra.

The cocycles $a_{3,1}$ and $a_{3,2}$ can be easily given an explicit description, by unwinding the hofiber/cyclification adjunction in this case. Let us do this for a_1.

The homotopy fiber $\mathfrak{p}_1$ of f_1 is defined by the homotopy pushout of DGCAs

$$\begin{tikzcd}
(\mathbb{R}[x_2], 0) \arrow[r] \arrow[d, "f_1^*"'] & (\mathbb{R}, 0) \arrow[d] \\
(\mathbb{R}[\check{x}_2, \tilde{x}_2, y_3], d\check{x}_2 = d\tilde{x}_2 = 0, dy_3 = \check{x}_2\tilde{x}_2) \arrow[r] & (\mathrm{CE}(\mathfrak{p}_1), d_{\mathfrak{p}_1}) \, .
\end{tikzcd}$$

So it is given by

$(\mathrm{CE}(\mathfrak{p}_1), d_{\mathfrak{p}_1}) = (\mathbb{R}[\check{y}_1, \check{x}_2, \tilde{x}_2, y_3], d\check{y}_1 = \check{x}_2, d\check{x}_2 = d\tilde{x}_2 = 0, dy_3 = \check{x}_2\tilde{x}_2)$.

One immediately sees the relation

$$dy_3 = d(\check{y}_1\tilde{x}_2),$$

i.e., that $y_3 - \check{y}_1\tilde{x}_2$ is a 3-cocycle on $\mathfrak{p}_1$.

Under the hofiber/cyclification adjunction this 3-cocycle corresponds to the morphism of DGCAs $\mathrm{CE}(\mathrm{cyc}(b^2\mathfrak{u}_1)) \to \mathrm{CE}(b\mathfrak{tfold})$ mapping x_3 to y_3, y_2 to $\tilde{x}_2$ and z_2 to $\check{x}_2$, i.e., to the morphism φ_1. In other words,

$$a_{3,1} = y_3 - \check{y}_1 \tilde{x}_2.$$

In a perfectly similar way $a_{3,2} = y_3 - \check{x}_2 \tilde{y}_1$.

Finally, let us form the homotopy fiber product
$\mathfrak{t} = \mathfrak{p}_1 \times_{b\mathfrak{tfold}} \mathfrak{p}_2$.
It is described by the Chevalley-Eilenberg algebra

$$(\mathrm{CE}(\mathfrak{t}), d_{\mathfrak{t}}) = (\mathbb{R}[\check{y}_1, \tilde{y}_1, \check{x}_2, \tilde{x}_2, y_3], d\check{y}_1 = \check{x}_2, d\tilde{y}_1 = \tilde{x}_2, dy_3 = \check{x}_2\tilde{x}_2),$$

with the projections $\pi_i \colon \mathfrak{t} \to \mathfrak{p}_i$ given in the dual picture by the obvious inclusions.

By construction, π_1 and π_2 are $\mathbb{R}$-central extensions, classified by the 2-cocycles $\tilde{x}_2$ and $\hat{x}_2$, respectively.

One computes

$$\pi_1^* a_{3,1} - \pi_{3,2}^* a_2 = db_2,$$

where $b_2 \in \mathrm{CE}(\mathfrak{t})$ is the degree 2 element $b = \check{y}_1 \tilde{y}_1$.

Thus we see that the L_∞-algebra $b\mathfrak{tfold}$ actually contains all the data of a quintuple $(\pi_1, \pi_2, a_{3,1}, a_{3,2}, b_2)$ inducing a Fourier-Mukai transform.

Maps to $\mathfrak{btfold}$

All of the construction of the quintuple $(\pi_1, \pi_2, a_1, a_2, b)$ out of the the L_∞-algebra $b\mathfrak{tfold}$ can be pulled back along a morphism of L_∞-algebras $\mathfrak{g} \to b\mathfrak{tfold}$.

That is, given such a morphism one has two $\mathbb{R}$-central extensions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ together with 3-cocycles $a_{3,1}$ and $a_{3,2}$ on $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, and a degree 2 element b_2 on the (homotopy) fiber product L_∞-algebra $\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2$ with $\pi_1^* a_{3,1} - \pi_2^* a_{3,2} = db_2$.

Let us see in detail how this works.

To begin with, the datum of a morphism $\mathfrak{g} \to b\mathfrak{tfold}$ is precisely the datum of two 2-cocycles $\check{c}_2$ and $\tilde{c}_2$ on $\mathfrak{g}$ together with a degree 3 element $h_3 \in \mathrm{CE}(\mathfrak{g})$ such that $dh_3 = \check{c}_2 \tilde{c}_2$.

The two cocycles $\check{c}_2$ and $\tilde{c}_2$ define the two $\mathbb{R}$-central extensions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$ defined, respectively, by

$$
\begin{aligned}
(\mathrm{CE}(\mathfrak{g}_1), d_{\mathfrak{g}_1}) &= (\mathrm{CE}(\mathfrak{g})[\check{e}_1], d\check{e}_1 = \check{c}_2) \ , \\
(\mathrm{CE}(\mathfrak{g}_2), d_{\mathfrak{g}_2}) &= (\mathrm{CE}(\mathfrak{g})[\tilde{e}_1], d\tilde{e}_1 = \tilde{c}_2) \ .
\end{aligned}
$$

On the L_∞-algebra $\mathfrak{g}_1$ we have the 3-cocycle $a_{3,1} = h_3 - \check{e}_1 \tilde{c}_2$, and on the L_∞-algebra $\mathfrak{g}_2$ we have the 3-cocycle $a_{3,2} = h_3 - \check{c}_2 \tilde{e}_1$.

Finally, the homotopy fiber product $\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2$ is given by

$$\left(\mathrm{CE}(\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2), d_{\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2}\right) = \left(\mathrm{CE}(\mathfrak{g})[\check{e}_1, \tilde{e}_1];\ d\check{e}_1 = \check{c}_2, d\tilde{e}_1 = \tilde{c}_2\right),$$

and so in $\mathrm{CE}(\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2)$ we have $\pi_1^* a_{3,1} - \pi_2^* a_{3,2} = db_2$, where π_1^* and π_2^* are the obvious inclusions and $b_2 = \check{e}_1 \tilde{e}_1$.

Notice that $\mathrm{CE}(\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2)$ is built from $\mathrm{CE}(\mathfrak{g}_1)$ by adding the additional generator $\tilde{e}_1$ and from $\mathrm{CE}(\mathfrak{g}_2)$ by adding the additional generator $\check{e}_1$.

We can now make completely explicit the Fourier-Mukai transform

$$\Phi_{b_2} \colon H^{\bullet}_{L_\infty; a_{3,1}}(\mathfrak{g}_1; \mathbb{R}[[u^{-1}, u]]) \longrightarrow H^{\bullet-1}_{L_\infty; a_{3,2}}(\mathfrak{g}_2; \mathbb{R}[[u^{-1}, u]]).$$

To fix notation, let

$$\begin{tikzcd}[column sep=small] & \mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathfrak{g}_1 \arrow[dr, "p_1"'] & & \mathfrak{g}_2 \arrow[dl, "p_2"] \\ & \mathfrak{g} & \end{tikzcd}$$

be the homotopy fiber product defining $\mathfrak{g}_1 \times_{\mathfrak{g}} \mathfrak{g}_2$.

Notice that the Beck-Chevalley condition

$$p_2^* p_{1*} = \pi_{2*} \pi_1^*$$

holds.

Let us write $\omega_{2n} = \alpha_{2n} + \check{e}_1\beta_{2n-1}$ for a degree $2n$ element in $\mathrm{CE}(\mathfrak{g}_1)$ and $\omega = \sum_{n\in\mathbb{Z}} u^{k-n}\omega_{2n}$ for a degree $2k$ element in $\omega \in \mathrm{CE}(\mathfrak{g}_1)[[u^{-1}, u]]$.

The Fourier-Mukai transform Φ_{b_2} maps the element ω to $\pi_{2*}(e^{b_2}\pi_1^*\omega)$.

Since π_1^* is just the inclusion and $e^{u^{-1}b_2} = e^{u^{-1}\check{e}_1\tilde{e}_1} = 1 + u^{-1}\check{e}_1\tilde{e}_1$, we find

$$\begin{aligned}\Phi_{b_2}(\omega) &= \pi_{2*}(\omega + u^{-1}\check{e}_1\tilde{e}_1\omega)\\ &= \sum_{n\in\mathbb{Z}} u^{k-n}(\beta_{2n-1} + \tilde{e}_1\alpha_{2n-2})\ .\end{aligned}$$

Let $\tilde{\omega}_{2n-1} = \beta_{2n-1} + \tilde{e}_1\alpha_{2n-2}$ and $\tilde{\omega} = \sum_{n\in\mathbb{Z}} u^{k-n}\tilde{\omega}_{2n-1}$, so that $\tilde{\omega}$ is a degree $2k-1$ element in $\mathrm{CE}(\mathfrak{g}_2)[[u^{-1}, u]]$ and $\tilde{\omega} = \Phi_{b_2}(\omega)$.

We know from the general construction of Fourier-Mukai transforms we have been developing that if ω is an an $a_{3,1}$-twisted cocycle, then $\tilde{\omega}$ is an $a_{3,2}$-twisted cocycle. We can directly show this as follows.

The degree $2k$ cochain ω is a $a_{3,1}$-twisted degree $2k$ cocycle precisely when

$$d_{\mathfrak{g}_1}\omega + u^{-1}a_{3,1}\,\omega = 0.$$

This equation is in turn equivalent to the system of equations

$$d_{\mathfrak{g}_1}\omega_{2n} + a_{3,1}\omega_{2n-2} = 0, \qquad n \in \mathbb{Z}.$$

Writing $\omega_{2n} = \alpha_{2n} + \check{e}_1 \beta_{2n-1}$ and recalling that $a_{3,1} = h_3 - \check{e}_1 \tilde{c}_2$, this becomes

$$d_{\mathfrak{g}} \alpha_{2n} + \check{c}_2 \beta_{2n-1} - \check{e}_1 d_{\mathfrak{g}} \beta_{2n-1} + h_3 \alpha_{2n-2} - \check{e}_1 \tilde{c}_2 \alpha_{2n-2} - \check{e}_1 h_3 \beta_{2n-3} = 0,$$

i.e.,

$$\begin{cases} d_{\mathfrak{g}} \alpha_{2n} + h_3 \alpha_{2n-2} = \check{c}_2 \beta_{2n-1}, \\ d_{\mathfrak{g}} \beta_{2n-1} + h_3 \beta_{2n-3} = \tilde{c}_2 \alpha_{2n-2}. \end{cases}$$

Then we can compute

$$
\begin{aligned}
d_{\mathfrak{g}_2}\tilde{\omega}_{2n-1} &= d_{\mathfrak{g}_2}(\beta_{2n-1} + \tilde{e}_1\alpha_{2n-2}) \\
&= -a_{3,2}\tilde{\omega}_{2n-3} \;,
\end{aligned}
$$

which shows that $\tilde{\omega}$ is a degree $2k-1$ $a_{3,2}$-twisted cocycle.

Looking at the explicit formula for Φ_{b_2} we have now determined above, we see that Φ_{b_2} acts as

$$\sum_{n\in\mathbb{Z}} u^{k-n}(\alpha_{2n} + \check{e}_1\beta_{2n-1}) \longmapsto \sum_{n\in\mathbb{Z}} u^{k-n}(\beta_{2n-1} + \tilde{e}_1\alpha_{2n-2}) \,.$$

So it is manifestly a linear isomorphism between the space of degree $2k$ cochains in $\mathrm{CE}(\mathfrak{g}_1)[[u^{-1}, u]]$ and degree $2k-1$ cochains in $\mathrm{CE}(\mathfrak{g}_2)[[u^{-1}, u]]$.

Repeating verbatim the above argument one sees that Φ_{b_2} is also a linear isomorphism between degree $2k-1$ cochains in $\mathrm{CE}(\mathfrak{g}_1)[[u^{-1}, u]]$ and degree $2k-2$ cochains in $\mathrm{CE}(\mathfrak{g}_2)[[u^{-1}, u]]$. Not surprisingly, the inverse morphism is $u\Phi_{-b_2}$ in both cases.

This can be showed directly by repeating once more the argument above, or specializing to a rational T-duality configuration the general formula for the composition of two Fourier-Mukai transforms (we are going to show this in a while).

Either way, one sees that Φ_{b_2} is an isomorphism of complexes and so the Fourier-Mukai transform associated to an L_∞-morphism $\mathfrak{g} \to b\mathfrak{tfold}$ is an isomorphism

$$\Phi_{b_2} \colon H^\bullet_{L_\infty;a_{3,1}}(\mathfrak{g}_1;\mathbb{R}[[u^{-1},u]]) \xrightarrow{\sim} H^{\bullet-1}_{L_\infty;a_{3,2}}(\mathfrak{g}_2;\mathbb{R}[[u^{-1},u]])\,.$$

Compositions of Fourier-Mukai transforms

To conclude, let us describe the composition of Fourier-Mukai transforms.

To that end, we will consider a pair of quintuple $(\pi_1, \pi_2, a_{3,1}, a_{3,2}, b_2)$ and $(\tilde{\pi}_1, \tilde{\pi}_2, a_{3,2}, a_{3,3}, \tilde{b}_2)$, which induce two corresponding Fourier-Mukai transforms $\Phi_{b_2}\colon H^{\bullet}_{L_\infty;a_{3,1}}(\mathfrak{g}_1;\mathbb{R}[[u^{-1},u]]) \to H^{\bullet-1}_{L_\infty;a_{3,2}}(\mathfrak{g}_2;\mathbb{R}[[u^{-1},u]])$ and $\Phi_{\tilde{b}_2}\colon H^{\bullet}_{L_\infty;a_{3,2}}(\mathfrak{g}_2;\mathbb{R}[[u^{-1},u]]) \to H^{\bullet-1}_{L_\infty;a_{3,3}}(\mathfrak{g}_3;\mathbb{R}[[u^{-1},u]])$, respectively.

To describe the composition $\Phi_{\tilde{b}_2} \circ \Phi_{b_2}$, we form the fiber product $\mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are the L_∞ algebras appearing as "roofs" in the spans defining $\Phi_{\tilde{b}_2}$ and Φ_{b_2}, respectively. Notice that, as $\pi_2\colon \mathfrak{h}_1 \to \mathfrak{g}_2$ and $\tilde{\pi}_1\colon \mathfrak{h}_2 \to \mathfrak{g}_2$ are fibrations, $\mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2$ is actually a model for the homotopy fiber product of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ over $\mathfrak{g}_2$.

Then we have the diagram

$$
\begin{tikzcd}
 & & \mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2 \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[ddll, bend right, "p_1"'] \arrow[ddrr, bend left, "p_2"] & & \\
 & \mathfrak{h}_1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \mathfrak{h}_2 \arrow[dl, "\tilde{\pi}_1"'] \arrow[dr, "\tilde{\pi}_2"] & \\
\mathfrak{g}_1 & & \mathfrak{g}_2 & & \mathfrak{g}_3 ,
\end{tikzcd}
$$

where q_1 and q_2 are the projections, and where $p_1 = \pi_1 q_1$ and $p_2 = \tilde{\pi}_2 q_2$.

By definition of Fourier-Mukai transform and by the Beck-Chevalley condition $\tilde{\pi}_1^*\pi_{2*} = q_{2*}q_1^*$, for any ω in $\mathrm{CE}(\mathfrak{g}_1)$ we have

$$\begin{aligned}(\Phi_{\tilde{b}_2} \circ \Phi_{b_2})(\omega) &= \tilde{\pi}_{2*}(e^{u^{-1}\tilde{b}_2}\tilde{\pi}_1^*\pi_{2*}(e^{u^{-1}b_2}\pi_1^*\omega)) \\ &= \tilde{\pi}_{2*}(e^{u^{-1}\tilde{b}_2}q_{2*}(e^{u^{-1}q_1^*b_2}\, p_1^*\omega)).\end{aligned}$$

Now recall the projection formula, and use the fact that $e^{u^{-1}\tilde{b}_2}$ entirely consists of even components to get

$$q_{2*}(q_2^*(e^{u^{-1}\tilde{b}_2})\, e^{u^{-1}q_1^*b_2}\, p_1^*\omega) = e^{u^{-1}\tilde{b}_2}\, q_{2*}(e^{u^{-1}q_1^*b_2}\, p_1^*\omega).$$

Therefore,

$$\begin{aligned}(\Phi_{\tilde{b}_2} \circ \Phi_{b_2})(\omega) &= \tilde{\pi}_{2*} q_{2*}(q_2^*(e^{u^{-1}\tilde{b}_2})\, e^{u^{-1} q_1^* b_2}\, p_1^* \omega) \\ &= p_{2*}(e^{u^{-1}(q_2^* \tilde{b}_2 + q_1^* b_2)}\, p_1^* \omega).\end{aligned}$$

By definition of fiber product, the two morphisms $q_2^* \tilde{\pi}_1^*$ and $q_1^* \pi_2^*$ coincide. Therefore,

$$\begin{aligned}d_{\mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2}(q_2^* \tilde{b}_2 + q_1^* b_2) &= q_2^* d_{\mathfrak{h}_2} \tilde{b}_2 + q_1^* d_{\mathfrak{h}_1} b_2 \\ &= p_1^* a_{3,1} - p_2^* a_{3,3}.\end{aligned}$$

This shows that $\Phi_{\tilde{b}_2} \circ \Phi_{b_2}$ is indeed the Fourier-Mukai transform associated with the quintuple $(p_1, p_2, a_{3,1}, a_{3,3}, q_1^* b_2 + q_2^* \tilde{b}_2)$. We write this as

$$\Phi_{\tilde{b}_2} \circ \Phi_{b_2} = \Phi_{q_1^* b_2 + q_2^* \tilde{b}_2}.$$

Notice that $p_1\colon \mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2 \to \mathfrak{g}_1$ and $p_2\colon \mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2 \mathfrak{g}_3$ are not $\mathfrak{u}_1$-central extensions but $\mathfrak{u}_1 \times \mathfrak{u}_1$-central extensions, so the Fourier-Mukai transform $\Phi_{q_1^* b_2 + q_2^* \tilde{b}_2}$ lowers the degree by 2.

It is interesting to specialize this to the case where $(\pi_1, \pi_2, a_{3,1}, a_{3,2}, b_2)$ is the quintuple associated with a rational T-duality configuration $\mathfrak{g} \to b\mathfrak{t}\mathfrak{f}\mathfrak{o}\mathfrak{l}\mathfrak{d}$ and $(\tilde{\pi}_1, \tilde{\pi}_2, a_{3,2}, a_{3,3}, \tilde{b}_2) = (\pi_2, \pi_1, a_{3,2}, a_{3,1}, -b_2)$.

In this case

$$\left(\mathrm{CE}(\mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2), d_{\mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2}\right) = \left(\mathrm{CE}(\mathfrak{g})[\check{e}_{1,1}, \tilde{e}_1, \check{e}_{1,2}];\ d\check{e}_{1,1} = d\check{e}_{1,2} = \check{c}_2, d\tilde{e}_1 = \tilde{c}_2\right),$$

and the morphisms $q_i^*\colon \mathrm{CE}(\mathfrak{h}_i) \to \mathrm{CE}(\mathfrak{h}_1 \times_{\mathfrak{g}_2} \mathfrak{h}_2)$ are the inclusions of $\mathrm{CE}(\mathfrak{g})[\check{e}_1, \tilde{e}_1]$ into $\mathrm{CE}(\mathfrak{g})[\check{e}_{1,1}, \tilde{e}_1, \check{e}_{1,2}]$ given by $\check{e}_1 \mapsto \check{e}_{1,i}$.

Therefore, we have

$$q_1^* b_2 + q_2^*(-b_2) = (q_1^* - q_2^*)(\check{e}_1 \tilde{e}_1) = (\check{e}_{1,1} - \check{e}_{1,2})\tilde{e}_1 \ .$$

As a consequence, the Fourier-Mukai transform $\Phi_{q_1^* b_2 + q_2^*(-b_2)}$ acts on a degree $2k$ element $\omega = \sum_{n \in \mathbb{Z}} u^{k-n}(\alpha_{2n} + \check{e}_1 \beta_{2n-1})$ in $\mathrm{CE}(\mathfrak{g}_1)[[u^{-1}, u]]$ as

$$\Phi_{q_1^* b_2 + q_2^*(-b_2)}(\omega) = \sum_{n \in \mathbb{Z}} u^{k-n-1}(\alpha_{2n} + \check{e}_1 \beta_{2n-1}) = u^{-1}\omega.$$

The same holds for odd degree elements, so that $\Phi_{q_1^* b_2 + q_2^*(-b_2)} = u^{-1}\mathrm{Id}$ and so $u\Phi_{-b_2} \circ \Phi_{b_2} = \mathrm{Id}$.
The same argument shows that $\Phi_{b_2} \circ u\Phi_{-b_2} = \mathrm{Id}$, so that, finally,

$$\Phi_{b_2}^{-1} = u\Phi_{-b_2},$$

i.e., we have shown that the Fourier-Mukai transform associated with a rational T-fold configuration is indeed invertible, with inverse provided (up to a shift in degree, given by the multiplication by u) by the Fourier-Mukai transform with opposite kernel 2-cochain.

Another example from string theory

Another example from string theory.

All of the above constructions immediately generalize from L_∞-algebras to super-L_∞-algebras, and it is precisely in this more general setting that we find an interesting example from the string theory literature.

Let **16** be the unique irreducible real representation of $\mathrm{Spin}(8,1)$ and let $\{\gamma_a\}_{a=0}^{d-1}$ be the corresponding Dirac representation on $\mathbb{C}^{16}$ of the Lorentzian $d = 9$ Clifford algebra. Write $\mathbf{16} + \mathbf{16}$ for the direct sum of two copies of the representation **16**, and write $\psi = \binom{\psi_1}{\psi_2}$ with ψ_1 and ψ_2 in **16** for an element ψ in $\mathbf{16} + \mathbf{16}$.

Finally, for $a = 0, \cdots, 8$, consider the Dirac matrices

$$\Gamma^a = \begin{pmatrix} 0 & \gamma^a \\ \gamma^a & 0 \end{pmatrix}, \qquad \Gamma_9^{\mathrm{IIA}} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix},$$

$$\Gamma_9^{\mathrm{IIB}} = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}, \quad \text{and} \qquad \Gamma_{10} = \begin{pmatrix} i\mathrm{I} & 0 \\ 0 & -i\mathrm{I} \end{pmatrix},$$

where I is the identity matrix.

The super-Minkowski super Lie algebra $\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}}$ is the super Lie algebra whose dual Chevalley-Eilenberg algebra is the differential $(\mathbb{Z}, \mathbb{Z}/2)$-bigraded commutative algebra generated from elements $\{e^a\}_{a=0}^{8}$ in bidegree $(1, \mathrm{even})$ and from elements $\{\psi^\alpha\}_{\alpha=1}^{32}$ in bidegree $(1, \mathrm{odd})$ with differential given by

$$d\psi^\alpha = 0 \quad , \quad de^a = \overline{\psi}\Gamma^a\psi \, ,$$

where $\overline{\psi}\Gamma^a\psi = (C\Gamma^a)_{\alpha\beta}\, \psi^\alpha\psi^\beta$, with C the charge conjugation matrix for the real representation $\mathbf{16} + \mathbf{16}$.

Since $d\psi^\alpha = 0$ for any α, both

$$c_2^{\mathrm{IIA}} = \overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi \qquad \text{and} \qquad c_2^{\mathrm{IIB}} = \overline{\psi}\Gamma_9^{\mathrm{IIB}}$$

are degree (2,even) cocycles on $\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}}$.

The central extensions they classify are obtained by adding a new degree (1,even) generator e_A^9 or e_B^9 to $\mathrm{CE}(\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}})$ with differential

$$de_A^9 = \overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi \qquad \text{and} \qquad de_B^9 = \overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi \ ,$$

respectively.

These two central extensions are, therefore, themselves super-Minkowski super Lie algebras. Namely, the extensions classified by c_2^{IIA} and c_2^{IIB} are

$$\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \qquad \text{and} \qquad \mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}},$$

respectively.

Finally, let μ_{F1}^{IIA} be the degree (3,even) element in $\mathrm{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ given by

$$\mu_{F1}^{\mathrm{IIA}} = \mu_{F1}^{8,1} - i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi e_A^9 = -i\sum_{a=0}^{8}\overline{\psi}\Gamma_a\Gamma_{10}\psi e^a - i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi e_A^9 .$$

The element μ_{F1}^{IIA} is actually a cocycle, so that

$$d\mu_{F1}^{8,1} = (i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi)(\overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi) .$$

A simple direct computation shows $\Gamma_9^{\mathrm{IIB}} = i\,\Gamma_9^{\mathrm{IIA}}\Gamma_{10}$, so that

$$d\mu_{F1}^{8,1} = (\overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi)(\overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi) = c_2^{\mathrm{IIA}}c_2^{\mathrm{IIB}} .$$

As the element $\mu_{F1}^{8,1}$, as well as the elements c_2^{IIA} and c_2^{IIB} actually belong to the differential bigraded subalgebra $\mathrm{CE}(\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}})$ of $\mathrm{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$, the relation

$$d\mu_{F1}^{8,1} = c_2^{\mathrm{IIA}} c_2^{\mathrm{IIB}}$$

actually holds in $\mathrm{CE}(\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}})$, so that the triple $(c_2^{\mathrm{IIA}}, c_2^{\mathrm{IIB}}, \mu_{F1}^{8,1})$ defines an L_∞-morphism

$$\mathbb{R}^{8,1|\mathbf{16}+\mathbf{16}} \longrightarrow b\mathfrak{tfold}.$$

The 3-cocycles on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ and on $\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}$ associated with this L_∞-morphism are

$$\mu_{F1}^{8,1} - e_A^9 c_2^{\mathrm{IIB}} \qquad \text{and} \qquad \mu_{F1}^{8,1} - c_2^{\mathrm{IIA}} e_B^9,$$

respectively.

As $\Gamma_9^{\mathrm{IIB}} = i\,\Gamma_9^{\mathrm{IIA}}\Gamma_{10}$, we see that

$$\mu_{F1}^{8,1} - e_A^9 c_2^{\mathrm{IIB}} = \mu_{F1}^{8,1} - e_A^9 \overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi = \mu_{F1}^{8,1} - i\overline{\psi}\Gamma_9^{\mathrm{IIA}}\Gamma_{10}\psi e_A^9 = \mu_{F1}^{\mathrm{IIA}} \ .$$

We then set $\mu_{F1}^{\mathrm{IIB}} = \mu_{F1}^{8,1} - c_2^{\mathrm{IIA}} e_B^9$. An explicit expression for the $(3, \mathrm{even})$-cocycle μ_{F1}^{IIB} on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ is

$$\mu_{F1}^{\mathrm{IIB}} = \mu_{F1}^{8,1} - \overline{\psi}\Gamma_9^{\mathrm{IIA}}\psi e_B^9 = -i\sum_{a=0}^{8}\overline{\psi}\Gamma_a\Gamma_{10}\psi e^a - i\overline{\psi}\Gamma_9^{\mathrm{IIB}}\psi e_B^9 \ ,$$

where we used $\Gamma_9^{\mathrm{IIA}} = i\Gamma_9^{\mathrm{IIB}}\Gamma_{10}$.

We have therefore an explicit Fourier-Mukai isomorphism

$$\Phi_{e_A^9 e_B^9} : H^{\bullet}_{L_\infty;\mu_{F1}^{\mathrm{IIA}}}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}};\mathbb{R}[[u^{-1},u]]) \xrightarrow{\sim} H^{\bullet-1}_{L_\infty;\mu_{F1}^{\mathrm{IIB}}}(\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}};\mathbb{R}[[u^{-1},u]]) \,.$$

This isomorphism is known as Hori's formula or as the Buscher rules for RR-fields in the string theory literature. A direct computation shows that it maps the μ_{F1}^{IIA}-twisted cocycles found by Chryssomalakos-de Azcárraga-Izquierdo-Pérez Bueno on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ to the μ_{F1}^{IIB}-twisted cocycles found by Sakaguchi, on $\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}}$.