Anomalies and the Atiyah-Singer Index Theorem

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Contents

1 Introduction and Overview

Since their first appearance in the physics literature in the mid 60's [1, 2] anomalies have continued to played a very important role in the understanding of quantum field theories. Understanding anomalies provided the basis for understanding of the π^0 lifetime and later the $U(1)$ axial problem in QCD. Anomalies provided (and of course still do) important guidelines form model building. In the late 70's it was realized [3, 4, 5] that the original global $U(1)$ axial anomaly [1, 2] can be viewed as a consequence of the Atiyah-Singer index theorem proved in 1968 [6]. In the mid 80's it became clear that non-abelian and gravitational anomalies could be understood in terms of the index theorem for families of elliptic operators [7].

In this paper we will try to illuminate some of the relations of anomalies to index theorems. Since the use of the word anomaly seems to be slightly different in mathematics and physics it seems worthwhile to begin by establishing a relation between the various definitions used in the literature. Even in the physics community the meaning of the term anomaly seems to vary from person to person. Probably the most common use of the term in physics is to say that a theory has an anomaly if a symmetry that is present at the classical level is absent at the quantum mechanical level. It is of crucial importance for the consistency of the theory whether the anomalous symmetry is a global or a local symmetry of the theory. While the existence of an anomalous global symmetry may be useful, as for example in the case of the π^0 decay, the existence of an anomalous local symmetry is fatal because it spoils the gauge invariance and hence the renormalizability of the theory. In other words the presence of an anomalous local symmetry renders the theory inconsistent.[1] What might sound bad at first turns out to be one of the exciting features of anomalies. They provide at least some guide in physical model building. While the restrictions imposed by anomalies are not very severe in four dimensions the requirements become more and more stringent in higher dimensions. It ten dimensions there are for example only two theories which are chiral, supersymmetric and anomaly free [8].

In the mathematical literature the term anomaly seems to be used mostly for the case when a local symmetry is broken. As we will see in section 3 the anomaly then is an obstruction to even defining the quantum theory – the theory does not exist. In the case of an anomalous global symmetry the theory does exist, there simply is no symmetry. The terms global and local anomalies in the mathematical literature both refer to an anomaly in a local symmetry. A local anomaly is then an obstruction to defining the theory that is local in field space while the global anomaly is an obstruction global in field space.

[1] It might be interesting to note at this point that in physics a theory is assumed to exist until proven otherwise.

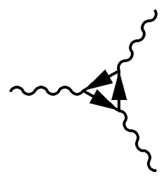
In section 2 we will study some examples from the viewpoint of physics trying, as far as possible, to specify what the physical objects are from a mathematical perspective. Anomalies occur in many different areas in physics and we obviously will not try to give a case by case study. Instead we will focus on two main examples in the standard model of particle physics. The first anomaly to be considered will be the global $U(1)$ axial anomaly that led to an understanding of the decay of the pion or rather its lifetime. Before we give a path integral derivation of the anomaly, we will try to provide some background to make the discussion more or less self-contained. The second type of anomaly we will discuss will be the gauge anomaly in the Glashow-Salam-Weinberg model of the weak interaction and its cancellation. We will for example not be able to talk about the famous Green-Schwarz anomaly cancellation [8].

In the first part of section 3 we will establish a relation between the global $U(1)$ axial anomaly and the Atiyah-Singer index theorem by studying the compactification of the theory studied in section 2 on a sphere following [3]. In the second part of this section we will explain how anomalies can be understood as an obstruction to defining the theory [9]. We will see that the theory will only exist if a certain line bundle over the space of physical field configurations is trivial. In the case of both gauge and gravitational anomalies the line bundle will be the determinant line bundle [9, 10, 11, 12] corresponding to the Dirac-operator of the fermions of the theory.

2 Anomalies from a Physical Perspective

2.1 The Global Abelian Anomaly

In this section we will try to provide some insight into the global $U(1)$ axial anomaly which was first discovered in [1, 2]. In its original form the anomaly was obtained by a careful calculation from the following Feynman diagram:



This is the first non-vanishing diagram in the perturbative expansion for the divergence of the axial current, *i.e.* the current corresponding to the axial symmetry of the action of a massless fermion in Minkowski space coupled to an external gauge field. In the first part of this section we will try to motivate why this anomaly contributes to the decay rate of the neutral pion. Our description will be far from complete and for a more detailed account we refer the reader for instance to [13]. We still think it seems worthwhile to know what we are calculating. In the second part we will write down a simpler model that exhibits the same type of anomaly and give a derivation using Fujikawa's method [14]

2.1.1 Some Background

As a starting point we will consider quantum chromodynamics with two flavors, which is defined by the action[2]:

$$S = \int d^4x \; \bar{u} i \not{D} u + \bar{d} i \not{D} d - m_u \bar{u} u - m_d \bar{d} d \tag{1}$$

For definiteness we are now working in Minkowski space with signature $(-,+,+,+)$ and $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}$. We will indicate this by using $\not{D} = \gamma^\mu(\partial_\mu - igA^a_\mu T^a)$ as notation for the Dirac operator in Minkowski signature while we will denote it D in Euclidean signature. The constant g is a coupling constant and is a parameter of the theory. From a physical point of view both $u(x)$ and $d(x)$ are Dirac-spinors transforming in the fundamental representation of $SU(3)$ and represent

[2]We will ignore the kinetic term for the gluons for now

the up- and the $down$-quark. The real vector fields $A_\mu^a(x)$ with $a = 1 \dots 8$ transform in the adjoint representation of $SU(3)$ and correspond to the eight gluons. Let us now briefly sketch the mathematical setup. We have a principal $SU(3)$-bundle with connection over spacetime together with a trivialization. What we called gluons before is (up to a factor i) the local representative of the connection. The quarks correspond to a section in the direct product $S \otimes E$ where S carries a complex representation of $Spin(3,1)$ and E carries the fundamental representation of $SU(3)$. In principle the quarks should be massless if this really were the Lagrangian for quantum chromodynamics. We know, however, that they will acquire a mass from the electro-weak sector in the standard model and we will include the masses at this point explicitly. In the absence of these mass terms the above action would posess a $SU(2)_V \times SU(2)_A$ symmetry, where the subscripts V and A stand for vector and axial respectively: [3]

$$\begin{pmatrix} u \\ d \end{pmatrix} = e^{i\vec{\alpha}_V \cdot \vec{\tau} + i\gamma^5 \vec{\alpha}_A \cdot \vec{\tau}} \begin{pmatrix} u \\ d \end{pmatrix}, \tag{2}$$

with $\vec{\alpha}_V, \vec{\alpha}_A \in \mathbb{R}^3$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

A different way of saying this is that in the absence of these terms we could write the action in a nicer way using only a single spinor field $Q(x)$ that is a section of $S \otimes E \otimes V$ where V is a trivial bundle with fiber $\mathbb{C}^2$ carrying the fundamental representation of $SU(2)$. The conserved currents would take the form:

$$J^{\mu a} = \bar{Q}\gamma^\mu \tau^a Q \quad \text{and} \quad J^{\mu a 5} = \bar{Q}\gamma^\mu\gamma^5 \tau Q \tag{4}$$

In the presence of the mass terms this symmetry is only an approximate symmetry. It is assumed that the axial part is spontaneously broken [15] due to composite fields acquiring a nonzero vacuum expectation value. It is known that spontaneously broken global symmetries lead to massless particles. In our case the symmetry that is broken is only approximate and we expect light particles, one for each broken generator. These light particles will be the pions. We expect that the composite operators $J^{\mu\pm5} = J^{\mu 15} \pm iJ^{\mu 25}$ and $J^{\mu 35}$ will be able to create the $\pi^\pm$ and the π^0 respectively.

So far none of our particles have an electromagnetic charge. We will hence couple them to the Maxwell field by modifying the covariant derivatives in the above action to $\not{D}_u = \gamma^\mu(\partial_\mu - igA_\mu^a T^a - iq_u A_\mu)$ and $\not{D}_d = \gamma^\mu(\partial_\mu - igA_\mu^a T^a - iq_d A_\mu)$, where $q_u = 2/3e$ and $q_d = -1/3e$

[3] In principle we even have a $U(2)_V \times U(2)_A$ symmetry. The $U(1)$ of the first factor represents baryon number and is a good symmetry. The $U(1)$ of the second factor is anomalous. To avoid this complication at this point we ignore both $U(1)$ factors.

are the couplings to the photon, *i.e.* the electromagnetic charges. In other words we now have a $SU(3) \times U(1)$ principle bundle and the quarks transform in the corresponding associated bundles. We see that this explicitly breaks the $SU(2)$'s down to the $U(1)$'s generated by τ_3. As mentioned above the currents corresponding to the spontaneously broken generators will create on-shell pions from the vacuum. So if we would like to calculate the decay rate of a π^0 into two photons, we would at some point have to calculate the matrix element $\langle p, q | \partial_\mu J^{\mu 35} | 0 \rangle$. The leading order contribution is given exactly by the Feynman diagram shown above. A naive calculation predicts a lifetime that is too long by several orders of magnitude. Something 'anomalous' must be going on. The resolution to this is that the remaining global $U(1)_A$ is anomalous and the anomaly we derive later yields the leading correction to the pion lifetime.

2.1.2 Path Integral Derivation of the Anomaly

We now have enough background knowledge to finally give a physical derivation for the global $U(1)$ axial anomaly. Instead of using the above action it seems convenient to consider what we might call a toy model that exhibits all the structure we need. The theory we shall consider for this purpose is a theory in four dimensional Minkowski space defined by the following action:

$$S[A, \Psi, \bar{\Psi}] = \int d^4x \; \bar{\Psi} i \not{D} \Psi \,, \tag{5}$$

where the Dirac operator takes the following explicit form:

$$\not{D} = \gamma^\mu (\partial_\mu - iqA_\mu) \tag{6}$$

and

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \tag{7}$$

The effective action for the gauge field is obtained by integrating out the fermions:

$$e^{iS_{eff}[A]} = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{iS[A,\Psi,\bar{\Psi}]} \tag{8}$$

It is well known even to physicists that the path integral in Minkowski space is not well-defined. To give a more rigorous derivation we should study the one point compactification of the Euclidean theory on a sphere. We will do so in the next section to establish a relation between the anomaly and the index theorem derived in class. For now let us work with the Minkowski theory and treat the integral as some formal object. The above action is invariant under a global transformation of the spinor of the form:

$$\Psi(x)' = e^{ia\gamma^5} \Psi(x) \quad \text{which implies} \quad \bar{\Psi}(x)' = \bar{\Psi}(x) e^{ia\gamma^5} \tag{9}$$

So we would naively conclude that the corresponding current $J^{\mu 5}$ is conserved. Let us now perform a coordinate change in the path integral. Since we integrate over all of field space the result will be unchanged *i.e.*:

$$e^{iS_{eff}[A]} = \int \mathcal{D}\Psi' \mathcal{D}\bar{\Psi}' e^{iS[A,\Psi',\bar{\Psi}']} \tag{10}$$

Let us now take the new field $\Psi'(x)$ to be of the form:

$$\Psi(x)' = e^{ia(x)\gamma^5}\Psi(x) \quad \text{and} \quad \bar{\Psi}(x)' = \bar{\Psi}(x)e^{ia(x)\gamma^5} \tag{11}$$

Since we now allow for a local transformation, the action will not be invariant. It will change according to:

$$S[A,\Psi',\bar{\Psi}'] = S[A,\Psi,\bar{\Psi}] - \int d^4x J^{\mu 5}\partial_\mu a(x) \tag{12}$$

with $J^{\mu 5} = \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x)$. The measure will also change. In fact this is the point that leads to the anomaly from the present point of view. If the measure were invariant we would simply find that the current is indeed conserved. It changes according to:

$$\mathcal{D}\Psi'\mathcal{D}\bar{\Psi}' = \mathcal{D}\Psi\mathcal{D}\bar{\Psi} Det\left(e^{-2ia(x)\gamma^5}\right) \tag{13}$$

Combining these two results and using $Det\left(e^M\right) = e^{Tr(M)}$ we find:

$$e^{iS_{eff}[A]} = \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS[A,\Psi,\bar{\Psi}]} e^{-i\int d^4x J^{\mu 5}\partial_\mu a(x) - 2i\int d^4x a(x) tr(\gamma^5)\delta(0)} \tag{14}$$

where the trace runs over the spinor indices. Since all we did was a change of integration variables this has to be independent of $a(x)$. Taking the functional derivative with respect to $a(x)$ and setting it equal to zero we conclude:[4]

$$\langle \partial_\mu J^{\mu 5}\rangle = 2tr(\gamma^5)\delta(0) \tag{15}$$

The right hand side of this equation is often called the anomaly. We see at this point that it is essentially what we called $Tr_S(1)$ in class. As it stands it has the unfortunate form $0 \times \infty$ which means we need to regulate it somehow. A convenient way of doing this is essentially the physicists version of the heat-kernel proof of the Atiyah-Singer index theorem given in class:

$$Tr(\gamma^5)\delta(0) \to \lim_{M^2\to 0}\lim_{y\to x} Tr(\gamma^5 e^{-\left(\frac{i\not{D}}{M}\right)^2}\delta(x-y)) \tag{16}$$

This is typically evaluated by using the plane wave representation of the delta function:

$$\delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)}\,, \tag{17}$$

[4]Recall that $\langle O\rangle$ is defined as $\langle O\rangle = \int O e^{iS}$

and appropriately rescaling the momentum k by factors of M,[5]. The trace over the γ matrices will then be proportional to the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ and the indices will be contracted with the covariant derivative. Due to the antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$ and the fact that the commutator of two covariant derivatives just gives the corresponding curvature this will yield the following final result:

$$Tr_S(1) = -\frac{q^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,, \tag{18}$$

which is indeed the local form of the Atiyah-Singer index theorem derived in class for the present case. So in the quantum theory the axial current is not conserved but instead satisfies:

$$\langle \partial_\mu J^{\mu 5}\rangle = -\frac{q^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \tag{19}$$

Using the language of differential forms we could write this more neatly as:

$$\langle d\star J^5\rangle = -\frac{q^2}{4\pi^2}F\wedge F \tag{20}$$

If we now look back at the action for two our two quarks couple to an electromagnetic field, we see that the global $U(1)$ axial symmetry is indeed anomalous. Looking back at the derivation we also see that we could easily have done the same for a non-abelian global symmetry and non-abelian gauge fields. In this case the anomaly would have taken the form:

$$\langle \partial_\mu J^{\mu i 5}\rangle = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu a}F^{\rho\sigma b}tr(T^iT^aT^b) \tag{21}$$

In the case where $T^i = 1$ this anomaly is called the singlet anomaly and can be written nicely in terms of forms:

$$\langle d\star J^5\rangle = -\frac{1}{4\pi^2}tr(F\wedge F) = -\frac{1}{4\pi^2}d\,tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{22}$$

This is also relevant for the QCD Lagrangian (not coupled to the electromagnetic field) because it tells us that there is a global $U(1)$ axial anomaly even in pure QCD. This observation lead to the solution of the famous $U(1)$ axial problem. As we shall see in the next part, different from anomalies in local symmetries these anomalies are not harmful for the consistency of the theory. Their understanding is however of crucial importance because as we shall see the gauge as well as gravitational anomalies can be understood in terms of global anomalies of the same theory in a higher dimensional space.

[5]This rescaling could be viewed as the analog of the Getzler rescaling

2.2 Gauge Anomalies

2.2.1 Some Words on Current Conservation

Before we go ahead and derive what is called the consistent anomaly in a path integral approach, let us first recall why anomalies in local currents ruins the consistency of the theory. To do this let us ask what happens to the effective action if we perform a infinitesimal gauge transformation $A'_\mu = A_\mu + \delta A_\mu$ with:

$$\delta A_\mu = D_\mu v = \partial_\mu v + [A_\mu, v] \tag{23}$$

On the one hand we can write the resulting change in the effective action as:

$$\delta S_{eff} = \int d^4x D_\mu v \frac{\delta S_{eff}}{\delta A_\mu} = -\int d^4x v D_\mu \langle J^\mu \rangle \tag{24}$$

In the last step we have used that we define the current as what couples to the gauge field and we have integrated by parts. This is justified because we are free to choose what v should be. In particular we can choose it to vanish outside some region of spacetime.

On the other hand we know from our definition of the effective action via the path integral that:

$$e^{i\delta S_{eff}[A]} = \frac{\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS[A',\Psi,\bar{\Psi}]}}{\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS[A,\Psi,\bar{\Psi}]}} \tag{25}$$

We can "undo" the transformation by changing variables of integration. Using the gauge invariance of the classical action and taking into account that under a change of the fermions:

$$\Psi' = U\Psi \quad \text{and} \quad \bar{\Psi}' = \bar{\Psi}\bar{U} \tag{26}$$

the measure changes we find:

$$e^{i\delta S_{eff}[A]} = \frac{\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} Det^{-1}(U\bar{U})(e^{iS[A,\Psi,\bar{\Psi}]}}{\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS[A,\Psi,\bar{\Psi}]}} \tag{27}$$

and finally in a somewhat unfortunate notation:

$$\langle (D_\mu J^\mu)^a \rangle = -i\, Tr \frac{\delta \ln (U\bar{U})}{\delta v^a}\bigg|_{v=0} =: \mathcal{A}^a(x)\,, \tag{28}$$

So if the fermion measure is invariant the right hand side vanishes and the current is covariantly conserved. If the measure is not invariant, gauge invariance is broken, the theory becomes non-renormalizable, and hence inconsistent. We also see that the global $U(1)$ axial anomaly or the singlet anomaly do not affect the gauge invariance of the theory because a change in the gauge field can still be absorbed by a change in the fermions without picking up the anomaly.

It is important to note that, if we have an effective theory containing the composit particle corresponding to the current of the broken symmetry, we have to include a term that cancels the transformation of that field. From the point of view of an effective theory containing pions this is the term that gives rise to the corrections in the pion lifetime.

2.2.2 Path-Integral Derivation of the Consistent Anomaly

Since the Glashow-Salam-Weinberg model of the weak interaction is certainly one of the most important models, some basic introduction and motivation would certainly be useful. Since it can hardly be described in a few sentences, we do, however, refer the reader to the literature [16]. We will at this point only use one crucial feature that makes it so different from what we have seen so far and hence potentially dangerous. The left-handed and right-handed particles have completely different couplings to the gauge fields. In other words the gauge fields couple to chiral currents. The toy model action we will study for these purposes is given by:

$$S[A,\Psi,\bar{\Psi}] = \int d^{2n}x\, \bar{\Psi} i \tilde{\not{D}} \Psi \,, \tag{29}$$

where $\tilde{\not{D}} = \gamma^\mu(\partial_\mu + A_\mu P_+)$ with $A_\mu = -iT^a A^a_\mu$ and $P_\pm = \frac{1}{2}(1 \pm \bar{\gamma})$ where $\bar{\gamma}^2 = 1$.[6] This immediately shows us that only the positive chirality part of the spinor couples to the gauge field or in other words that the gauge field couples to a chiral current. The effective action is then given by:

$$e^{iS_{eff}[A]} = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{iS[A,\Psi,\bar{\Psi}]} \tag{30}$$

This theory is often interpreted as a way to define what is meant by the effective action for a positive chirality spinor but we could just as well take this to be our theory. Let us just mention that by adding a local counterterm to regularize the theory it is possible to obtain a different form of the anomaly. “Our” choice seems rather natural an leads us directly to the relation between the consistent non-abelian anomaly and a global anomaly in the same theory in two dimensions higher [17]. According to our result from the last subsection the anomaly is given in terms of the transformation of the fermions needed to undo a gauge transformation of the gauge field. This transformation in the present case takes the form:

$$\Psi' = e^{iv} P_+ \Psi + P_- \Psi \quad \text{and hence} \quad \bar{\Psi}' = \bar{\Psi} P_- e^{-iv} + \bar{\Psi} P_+ \,, \tag{31}$$

[6] The derivation is independent of the number dimensionality of our spacetime so we will generalize at this point to arbitrary even dimensional spacetime.

or more conveniently:

$$U = e^{ivP_+} \quad \text{and} \quad \bar{U} = e^{-ivP_-} \,. \tag{32}$$

According to equation (28) this leads to:

$$\langle (D_\mu J^\mu)^a \rangle = Tr(T^a \bar{\gamma}) \,, \tag{33}$$

where we use capital letters for the trace if we still have to trace (or rather integrate) over spacetime. We will now regulate this trace using the new Dirac operator $\hat{\not{D}}$ and again writing the delta-function in terms of plane waves. We obtain:

$$\begin{aligned} Tr(T^a\gamma^5) &= \lim_{M\to 0} \int \frac{d^4}{(2\pi)^4} e^{-ikx} tr\left(T^a e^{-(\frac{i\hat{\not{D}}}{M})^2}\right) e^{ikx} = \\ &\lim_{M\to 0} \int \frac{d^4}{(2\pi)^4} e^{-ikx} \left(tr\left(T^a P_+ e^{\frac{\not{\partial}\not{D}}{M^2}}\right) - tr\left(T^a P_- e^{\frac{\not{D}\not{\partial}}{M^2}}\right)\right) e^{ikx} \,, \end{aligned} \tag{34}$$

or

$$Tr(T^a\gamma^5) = \lim_{M\to 0} \int \frac{d^4}{(2\pi)^4} \left(tr\left(T^a P_+ e^{\frac{(i\not{k}+\not{\partial})(i\not{k}+\not{D})}{M^2}}\right) - tr\left(T^a P_- e^{\frac{(i\not{k}+\not{D})(\not{k}+\not{\partial})}{M^2}}\right)\right) \,, \tag{35}$$

where the remaining derivatives are only nonvanishing when they act on the gauge potential. What remains is to rescale the momentum by a factor of M and a fair amount of traces and integrals. The result in four dimensions finally takes the form:

$$\langle (D_\mu J^\mu)^a \rangle = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu tr\left(T^a A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma\right) \,. \tag{36}$$

This is the result for a positive chirality spinor. We know that there was no anomaly in the case of a Dirac spinor. So we can conclude that the result for a negative chirality spinor must be the same expression with the opposite sign. In the language of forms this is usually written in the following way [17]:

$$\delta_v S_{eff} = \pm \frac{1}{24\pi^2} \int d^4x \, tr\left(vd(A \wedge dA + \frac{1}{2} A \wedge A \wedge A\right) \,. \tag{37}$$

This form of the anomaly is called the consistent anomaly because it satisfies the so-called Wess-Zumino consistency conditions [18]. Since these conditions have a nice physical meaning let us briefly sketch where these conditions come from and what they look like. Recall that we defined the anomaly as:

$$\frac{\delta S_{eff}}{\delta v^a} = -D_\mu \frac{\delta S_{eff}}{\delta A^a_\mu} = \mathcal{A}^a \tag{38}$$

The action of the gauge group on field space gives rise to the following induced vector fields $\mathcal{X}^a$ [19]:

$$\mathcal{X}^a = -D_\mu \frac{\delta}{\delta A^a_\mu} \tag{39}$$

In terms of these vector fields the anomaly can then simply be written as:

$$\mathcal{A}^a = \mathcal{X}^a S_{eff} \tag{40}$$

We know that the map $-iT^a \to \mathcal{X}^a$ is a homomorphism of the Lie-algebra of the gauge group G into the space of vector fields on field space. As can be checked by explicit calculation the induced vector fields hence satisfy the following commutation relations:

$$[\mathcal{X}^a(x), \mathcal{X}^b(y)] = f^{abc}\mathcal{X}^c(x)\delta(x-y) \tag{41}$$

The consistency conditions are obtained by applying this commutator to the effective action:

$$\mathcal{X}^a(x)\mathcal{A}^b(y) - \mathcal{X}^b(y)\mathcal{A}^a(x) = f^{abc}\mathcal{A}^c(x)\delta(x-y) \tag{42}$$

This equation is non-linear in the gauge potential implying that the anomaly can be determined completely once the leading order piece is known. It is interesting to note that this has a very nice physical interpretation. We saw that the leading order piece can be related to the $\pi^0 \to 2\gamma$ decay. Similarly, the remainder can be related to the processes $\gamma \to 2\pi$ and $2\gamma \to 3\pi$. This implies that the rates for the latter two processes can be determined once the pion decay is known.
The equation is, however, linear in the anomaly implying that the normalization cannot be fixed from the above equation. Let us now go back and compare the global $U(1)$ axial anomaly with the consistent anomaly. The former was of the form:

$$\mathcal{A}(x) \sim d\,tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) , \tag{43}$$

while the latter was given by:

$$\mathcal{A}(x) \sim tr\left(vd(A\wedge dA + \frac{1}{2}A\wedge A\wedge A\right) . \tag{44}$$

At first sight the two expressions look very similar and seem to differ only by the factor of $\frac{2}{3}$ vs. $\frac{1}{2}$. Their origin, however, is completely different. As we saw above and as we shall see later in more detail, the global $U(1)$ axial anomaly is given directly in terms of the Atiyah-Singer index theorem.
It was first discovered by Zumino et al. [19] that there was a formal algebraic relation between the

form of the consistent anomaly in $2n$ dimensions and the form of the global axial anomaly of the same theory in $2n+2$ dimensions as follows. The global axial anomaly in $2n+2$ dimensions is given by the $2n+2$ nd Chern character:

$$\mathcal{A} \sim trF^{n+1} =: \Omega_{2n+2}(A) \tag{45}$$

(We only saw the derivation for $n = 2$ but on symmetry grounds alone it is not hard to believe that this will generalize. The anomaly has to be a $2n+2$ form that is gauge invariant, and that reverses its sign under a parity transformation due to the fact that the current does.) This $2n+2$ form is closed and can be written locally as an exact form:

$$\mathcal{A} \sim d\omega^0_{2n+1}\,, \tag{46}$$

Noting that the global axial anomaly is gauge invariant we also conclude that the gauge transform of this $2n+1$ form is a closed form:

$$d\delta_v \omega^0_{2n+1} = 0\,, \tag{47}$$

and can again be written locally as an exact form:

$$\delta_v \omega^0_{2n+1} = d\omega^1_{2n} \tag{48}$$

It was shown in [19] that this $2n$ form, ω^1_{2n}, satisfies the Wess-Zumino consistency condition [18] and hence up to the normalization provides the consistent anomaly. L. Alvarez-Gaumé and P. Ginsparg showed [17] that this was not just a formal coincidence but that the consistent anomaly in $2n$ dimensions could indeed be understood in terms of a global anomaly in $2n+2$ dimensions. Soon after that it was realized that this could be put into a very nice mathematical form using the index theorem for families of elliptic operators.

To sketch the pedestrian way of understanding the anomaly let us consider the Euclidean version of the theory studied so far and let us assume that we have compactified $\mathbb{R}^{2n}$ to $\mathbb{S}^{2n}$.[7] The effective action for the gauge field is then defined by:

$$e^{-S_{eff}[A]} = \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{-\int d^{2n}x\, \bar{\Psi} i\hat{D}\Psi} \tag{49}$$

[7] The usual argument given when studying instantons as to why this should not affect the theory goes as follows. Requiring the Euclidean action to be finite forces the gauge field to tend to pure gauge at infinity. The physics will hence not be affected by compactifying to S^{2n}. In our case this is less clear because we would only require the total action, *i.e.* gauge field plus fermions to be finite. So for the compactification to make sense we have to restrict ourselves to the case where the action of the gauge field and the fermions are separately finite.

where $\hat{D} = \not{\partial} + \not{A}P_+$. We should note two things at this point. The first one is that the Euclidean action in general is not real and in fact the imaginary part of the effective action will play a crucial role later. The second one is that this choice of Dirac operator is not self-adjoint. It is, however an elliptic operator, has a well-defined eigenvalue problem and has discrete spectrum, which is all we shall need from it. This also implies that the eigenvalues are not gauge invariant. Since we can think of the determinant as the product of the eigenvalues (in some regularized way), we conclude that the determinant $deti\hat{D}$ and hence the effective action will not be gauge invariant in general, which is of course what allows for an anomaly to occur. We do know however, that the absolute value of the determinant is gauge invariant because:

$$|det(i\hat{D})|^2 = det(iD)det(\partial^2)\,, \tag{50}$$

where $D = \not{\partial} + \not{A}$ is the ordinary Dirac operator.

Before we go on let us for definiteness specify the setup in more detail. We are considering a chiral spinor on a $2n$-sphere transforming in some representation of a compact, simply connected, semi-simple Lie group G . That is we have a trivial principal G-bundle over the base manifold S^{2n} with a connection, which we specify as usual by giving a Lie-algebra valued one-form A on the base manifold. Let us now consider a one parameter family of connections defined by:

$$A^\theta = g^{-1}(\theta)Ag(\theta) + g^{-1}(\theta)d_x g(\theta)\,, \tag{51}$$

where d_x is the standard exterior derivative on the sphere and the reason for the subscript will become clear later. $g(x,\theta)$ is a gauge function satisfying periodic boundary conditions $g(x,0) = g(x,2\pi) = 1$:

$$g: \;\; S^1 \times S^{2n} \to G \tag{52}$$

It is easy to see that this one-parameter family describes a circle in the space of gauge conections. We saw above that the anomaly was given by the failure of the effective action to be gauge invariant. In terms of this parameter θ the anomaly can hence be written in a compact form:

$$\frac{dS_{eff}[A^\theta]}{d\theta} = -\mathcal{A}\,, \tag{53}$$

or using that the absolute value of the effective action is gauge invariant and writing $S_{eff}[A^\theta] = |det(iD\hat{(}A)|e^{iw(\theta,A)}$ we find:

$$i\frac{dw(\theta,A)}{d\theta} = \mathcal{A}\,. \tag{54}$$

Let us now extend to a two parameter family of gauge connections (not necessarily related by gauge transformations) as follows:

$$A^{t,\theta} = tA^{\theta} = tg^{-1}(\theta)(A + d_x)g(\theta)\,, \tag{55}$$

where $0 \leq t \leq 1$ (Note that this step requires a trivial principal bundle since otherwise we could in general not continously deform to $A = 0$.) This gives rise to a complex valued function $det(iD^{t,\theta})$ on the disc in the space of gauge connections parametrized by t, θ. [8] Restricted to the boundary, *i.e.* for $t = 1$, this map clearly gives rise to a map $e^{iw}: \ S^1 \to S^1$ which will be characterized by its winding number n defined as:[9]

$$n = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{dw(\theta, A)}{d\theta} \tag{56}$$

A non-zero winding number then implies that the effective action is not gauge invariant and hence that we have an anomaly. The goal will now be to define a Dirac operator on the $2n+2$ dimensional space parametrized by x, t, θ, where x are the coordinates on the sphere, such that its index equals the winding number. If we can do that, the local version of the Atiyah-Singer index theorem determines the anomaly as can be seen by comparing equations (54) and (56). Before we do so, let us briefly discuss the properties of $det(i\hat{D}^{t,\theta})$ for $t \neq 1$. For a generic gauge field we will have isolated zeros inside the disc at the points where one (or more) eigenvalues become zero. We can now imagine restricting our function to small circle enclosing only a single zero. This again defines a map from an S^1 to an S^1 and will be characterized by a winding number. In this way we can assign an index to each of the zeros. If we continuously (without enclosing another zero) deform the contour the winding number cannot change because it is discrete. Similarly if we consider a contour that encloses several zeros, its winding number will be given by summing the indices of all zeros inside the contour. If we now recall that the determinant is defined as the regularized product of the eigenvalues of the Dirac operator, and that a zero of the determinant corresponds to one eigenvalues going to zero, we see that we can associate the winding number to the single eigenvalues and we only have to analyze the smallest eigenvalues near the zero to determine the index of a given zero. So to determine the winding number of the function restricted to the boundary circle can be reduced to analyzing and calculating the behavior of the smallest

[8] In general this should be considered as a section in some line bundle, the determinant line bundle, but since the base space is a disc and hence contractible it must be trivial. Determinant line bundles will play a crucial role for a more general understanding of anomalies.

[9] We can without loss of generality choose a gauge connection such that the determinant is non-vanishing on the circle.

eigenvalues near the zeros. "Near the zeros" means that we are looking at small changes in the gauge field configuration and it suggests that winding number can be obtained by studying the behavior of the lowest lying eigenvalues in (degenerate) perturbation theory. This is indeed what is used in [17] to show that index of the following Dirac operator indeed reproduces these winding numbers:

$$i\not{D}_{2n+2} = i\Gamma^a(\partial_a + \mathscr{A}_a^+) \tag{57}$$

where the index $a = 1 \dots 2n+2$, $1 \dots 2n$ corresponds to the $2n$ spacetime coordinates, $2n+1$ corresponds to the coordinate t, and $2n+2$ corresponds to the coordinate θ.

The gauge Lie-algebra valued one-form $\mathscr{A}^+$ on the $2n+2$ dimensional manifold is defined as $\mathscr{A}^+(x,t,\theta) = A^{t,\theta} = tg^{-1}(A + d_x)g$. So far we cannot use the familiar Atiyah-Singer index theorem because our manifold has a boundary. We would instead have to use a generalization to manifolds with boundary [7]. We can avoid this complication by thinking of our disc in the space of gauge fields as the northern hemisphere of a two-sphere and we will call it H^+ from now on. We can the define a second disc which will be the southern hemisphere of our disc and we will call that H^-. To define a Lie-algebra valued one-form and hence a gauge field on the entire manifold $S^2 \times S^{2n}$ we will have to specify a Lie-algebra valued one form $\mathscr{A}^-$ on the lower hemisphere (or rather $H^- \times S^{2n}$) that is related to $\mathscr{A}^+$ by a gauge transformation on the overlap. We see that a possible choice is given by:

$$\mathscr{A}^-(x,s,\theta) = A - s(d_\theta g)g^{-1} \tag{58}$$

It is straightforward to see that this is indeed related to $\mathscr{A}^+$ on the overlap, *i.e.* $t = s = 1$, by the transition function $g(x,\theta)$. [10] So the gauge connection $\mathscr{A}^+$, $\mathscr{A}^-$ indeed defines a connection in a trivial principal G-bundle over the base manifold $S^2 \times S^{2n}$.

Before we go on let us give a brief sketch of what our construction looks like so far. We have constructed a (trivial) fiber bundle over the base manifold S^2, a sphere in the space of gauge connections, with fibers given by our spacetime S^{2n} including a Dirac operator on the total space. Furthermore we have constructed a principal G-bundle with connection whose base space is given by the total space, $S^2 \times S^{2n}$ of our fiber bundle over the S^2. We would like to remark at this point that this is almost all the data needed to define a family of Dirac operators.

We can now make use of the index theorem for this Dirac operator and we find:

$$\text{ind} i\not{D}_{2n+2} = \frac{i^{n+1}}{(2\pi)^{n+1}(n+1)!}\int_{S^2\times S^{2n}} tr\mathscr{F}^{n+1}\,, \tag{59}$$

[10] We note that due to the condition $g(0,x) = g(2\pi,x) = 1$ and the fact that we only consider simply connected G these maps are classified by $\pi_{2n+1}(G)$.

where $\mathscr{F} = d\mathscr{A} + \mathscr{A} \wedge \mathscr{A}$. To recover the consistent anomaly from this formula we will have to do two things. We have to rewrite this in terms of the local version of the index theorem because we saw above that that is what will give us the consistent anomaly, and we have to express the result in terms of the spacetime fields A and F.

Given the above definitions for the gauge connection on the total space it is straightforward but somewhat tedious to show that:

$$\int_0^1 dt\, tr\mathscr{F}^n = n(n+1)\int_0^1 dt(1-t)Str\left(g^{-1}d_\theta g d_x (A^\theta F^{t,\theta})^{n-1}\right) \\ - (n+1)tr((d_\theta g)g^{-1}F^n) \tag{60}$$

where Str denotes the symmetrized trace. Similarly, we obtain for the southern hemisphere:

$$\int_0^1 ds, tr\mathscr{F}^n = (n+1)tr((d_\theta g)g^{-1}F^n) \tag{61}$$

We see that this just has the effect of cancelling the last term in the result for the northern hemisphere and we conclude that [17]:

$$d_\theta w = \frac{i^{n+1}}{(2\pi)^n(n-1)!}\int_0^1 dt(1-t)Str\left(g^{-1}d_\theta g d_x (A^\theta F^{t,\theta})^{n-1}\right) \tag{62}$$

or using the notation used above:

$$\mathcal{A} = i\frac{dw}{d\theta}\bigg|_0 = \frac{i^{n+2}}{(2\pi)^n(n-1)!}\omega^1_{2n}\,, \tag{63}$$

which in four dimensions becomes:

$$\mathcal{A} = i\frac{dw}{d\theta}\bigg|_0 = \frac{1}{24\pi^2}tr(vd_x(A \wedge dA + \frac{1}{2}A \wedge A \wedge A)\,, \tag{64}$$

with $v = g^{-1}\left.\frac{dg}{d\theta}\right|_0$ in agreement with the path integral derivation.

This concludes our discussion of anomalies from a physical point of view.

3 Anomalies from a Mathematical Perspective

In this second part we will now try to take a more rigorous and more general approach towards anomalies.

We saw in the first part that the understanding of what we called the singlet anomaly is of crucial importance both because of its physical implications for the theory in $2n$ dimensions and because it provided some insight into what we called the consistent anomaly. In the next subsection we will hence give a more rigorous (and more general) derivation of the relation of the singlet anomaly to the index theorem.

We will then sketch, how anomalies in currents corresponding to local symmetries can be understood from a mathematical point of view. We would like to apologize in advance because even what we call a "mathematical point of view" will make use of the general idea of the path integral and hence lack mathematical rigor. We will still be able to gain a lot of insight into the general structure of anomalies.

3.1 The Abelian Anomaly and the Atiyah-Singer Index Theorem

To relate the singlet anomaly to the Atiyah-Singer theorem, we will start with the following setup. Let (X, g) be an (orientable) even-dimensional compact Riemannian manifold and let $(M, \nabla^M, < \cdot, \cdot >) \to X$ be a $Cliff(X)$-module with admissible connection and compatible metric over the base manifold X. That is we have a smooth map: $Cliff(X) \otimes M \to M$ that makes M_x into a $Cl_n(T_x^* X)$ module for all $x \in X$. We will write this map as $(\alpha, m) \mapsto \gamma(\alpha)m = \alpha \cdot m$. The connection should satisfy:

$$\nabla^M(\alpha \cdot m) = (\nabla^X \alpha)m + \alpha \cdot \nabla^M m \,, \tag{65}$$

where ∇^X is the connection on $Cliff(X)$ induced by the Levi-Civita connection. The metric should satisfy:

$$\langle m_1, \gamma(\alpha)m_2 \rangle = -\langle \gamma(\alpha)m_1, m_2 \rangle \tag{66}$$

for all vectors α, which is the infinitesimal version of requiring a Pin-invariant metric.

We define the Dirac operator acting on sections of M as the composition of covariant differentiation and Clifford-multiplication $D = \gamma \circ \nabla$. It will be relevant later that this operator is formally self-adjoint. For sections $\psi : X \to M$ we can then define the following action functional:

$$S = \langle \bar{\psi}, D\psi \rangle_{L^2} = \int_X \langle \bar{\psi}, D\psi \rangle \,, \tag{67}$$

giving rise to the classical equations of motion:

$$D\psi = 0 \quad \text{and} \quad D\bar{\psi} = 0 \tag{68}$$

In the classical theory we can then define the following one-form:[11]

$$J^5 = \langle \bar{\psi}, \gamma()\bar{\gamma}\psi \rangle \,, \tag{69}$$

where $\bar{\gamma} = \gamma(\omega)$ is the grading homomorphism on M, $\bar{\gamma}$ acts as ± 1 on $M^{\pm}$. This then acts on a vector field ξ as:

$$J^5(\xi) = \langle \bar{\psi}, \gamma(\hat{\xi})\bar{\gamma}\psi \rangle \,, \tag{70}$$

where $\hat{\xi}$ is the one-form defined by $\hat{\xi}(\eta) = g(\xi, \eta)$ for all vector fields η.
It is easy to check that this current is conserved in the sense that:

$$d^* J^5 = \langle D\bar{\psi}, \bar{\gamma}\psi \rangle + \langle \bar{\psi}, \bar{\gamma} D\psi \rangle \,, \tag{71}$$

which clearly vanishes if we impose the equations of motion.
Since we are working on a compact Riemannian manifold, we know how to diagonalize the Dirac operator:

$$\begin{aligned} D\psi_n &= \lambda_n \psi_n \\ D\bar{\psi}_n &= \lambda_n \bar{\psi}_n \,, \end{aligned} \tag{72}$$

where the eigenvalues λ_n are discrete, the eigenspaces to a given eigenvalue are finite dimensional, and the eigenfunctions are actually smooth sections. We can then define the following Green's function:

$$G'(x, y) = \sum_n{}' \frac{\psi_n(x) \otimes \bar{\psi}_n(y)}{\lambda_n} \,, \tag{73}$$

where the $'$ denotes that we exclude the zero modes from the sum. This satisfies:

$$DG'(x, y) = \delta(x, y) - \sum_\alpha \psi^0_\alpha(x) \otimes \bar{\psi}^0_\alpha(y) \,, \tag{74}$$

where in the second term we sum over the zero modes. Note that since $G'(x, y) \in \text{Hom}(M_y, M_x)$ the first term should be multiplied by the identity operator on M_x, which we have suppressed in our notation.
In the quantum theory we are now interested not in the current J^5 itself but rather in its vacuum

[11] The superscript 5 does not indicate that we are working on a four dimensional. It simply indicates that this is the analog of the axial current considered in the first part

expectation value $\langle J^5 \rangle$,[12] which we shall define now. Naively, we would like to define a one-form as follows:

$$\langle J^5_x \rangle = tr(\gamma()\bar{\gamma} G'(x,x))\,, \tag{75}$$

for all $x \in X$. Again this acts on a vector field ξ on X according to:

$$\langle J^5_x(\xi) \rangle = tr(\gamma(\hat{\xi})\bar{\gamma} G'(x,x))\,, \tag{76}$$

and the trace is taken over the endomorphism from M_x to M_x defined by $\gamma(\hat{\xi})\bar{\gamma}G'(x,x)$. (In terms of physics we might want to say we trace over Dirac and color indices.)

If we formally calculate the divergence of this one-form using our definition of $G'(x,x)$ we find that it vanishes:

$$d^* \langle J^5 \rangle = 2 \sum_n{}' \langle \bar{\psi}_n, \bar{\gamma}\psi_n \rangle = 0 \tag{77}$$

The second equality of course holds because we know that the asymmetry of the spectrum is concentrated on the space of zero modes which we have excluded from the sum. There is, however, clearly a problem with the above expression because we know that the Green's function is singular in the limit $y \to x$. Probably the most common way in physics to "regulate" this singularity is the point splitting method due to Schwinger. For the above expression the rough idea would be define the regulated current as a limit:

$$\langle \tilde{J}^5_x(\xi) \rangle = \tilde{\lim_{y \to x}} tr(\gamma(\hat{\xi})\bar{\gamma} G'(x,y) P(x,y)\,, \tag{78}$$

where $P(x,y)$ is the parallel transport map from the fiber at M_y to the fiber at M_x which is included to maintain gauge invariance. (Or simply to make sense of the trace.) The tilde indicates that the limit should be made symmetric in x and y. This way to define the current can indeed be used to give an independent derivation of the Atiyah-Singer index theorem which can be found in [3, 4, 5]. This is rather tedious and we will instead define a one-form that will lead us directly to the heat-kernel proof of the Atiyah index theorem given in class.

Let us first define a one-parameter family of Green's functions as follows:

$$G'_t(x,y) = \sum_n{}' \frac{\psi_n(x) \otimes \bar{\psi}_n(y)}{\lambda_n} e^{-t\lambda_n^2}\,. \tag{79}$$

This notation is intentionally chosen as to remind the reader of the heat-kernel defined in class:

$$p_t(x,y) = \sum_n \psi_n(x) \otimes \bar{\psi}_n(y) e^{-t\lambda_n^2}\,. \tag{80}$$

[12] We hope that the use of $\langle\rangle$ for both the metric on M and here for the vacuum expectation value will not cause confusion. The only place it is used to denote the vacuum expectation value is when a J^5 appears between the brackets

We saw that the C_k norm of $p_t(\cdot,\cdot)$ is finite for all finite k as long as $t > 0$ and hence that $p_t(x,y)$ is smooth for all $t \in \mathbb{R}^{>0}$ and all $x, y \in X$. The steps of the proof can be repeated to show that $G'_t(x,y)$ is also smooth in $t \in \mathbb{R}^{>0}$ and $x, y \in X$. For the details we refer the reader to the lecture notes. This naturally suggests that we should define a one-parameter family of one-forms:

$$\langle J^5_x \rangle_t = tr(\gamma()\bar{\gamma}G'_t(x,x))\,, \tag{81}$$

which will be well-behaved as long as $t > 0$. Using $d^* = -\iota \circ \nabla$, the definition of $G'_t(x,x)$, $D\psi_n = \lambda_n \psi_n$, and $D\psi_n = \lambda_n \psi_n$ it is straightforward to calculate the divergence of this one-form:

$$d^* \langle J^5_x \rangle_t = 2{\sum_n}' tr(\bar{\gamma}\psi_n(x) \otimes \bar{\psi}_n(x)) e^{-t\lambda_n^2}\,. \tag{82}$$

The right hand side is almost twice the supertrace of the heat-kernel. The important difference we have to note is that the above sum does not include the zero modes while the heat kernel was defined including the zero modes. This tells us that we can write the divergence as:

$$d^* \langle J^5_x \rangle_t = 2tr(\bar{\gamma}p_t(x,x)) - 2\sum_\alpha \langle \bar{\psi}^0_\alpha(x), \bar{\gamma}\psi^0_\alpha(x)\rangle\,,. \tag{83}$$

Let us now recall that $\bar{\gamma}$ was used to grade our Clifford module. We can hence choose a basis for the zero modes that respects the grading in the sense that $\psi^0_{\alpha^\pm}$ are sections of $M^\pm$,*i.e.* $\bar{\gamma}\psi^0_{\alpha^\pm} = \pm\psi^0_{\alpha^\pm}$. With this choice of basis the divergence of our one-form becomes:

$$d^* \langle J^5_x \rangle_t = 2tr(\bar{\gamma}p_t(x,x)) - 2\left(\sum_{\alpha^+} \langle \bar{\psi}^0_{\alpha^+}(x), \psi^0_{\alpha^+}(x)\rangle - \sum_{\alpha^-} \langle \bar{\psi}^0_{\alpha^-}(x), \psi^0_{\alpha^-}(x)\rangle\right)\,,. \tag{84}$$

We should note that, since $\bar{\gamma}$ is our grading homomorphism, the first term could written in terms of the supertrace tr_S of $p_t(x,x)$. We can identify this term as what we called the anomaly in the last section. That is we have:

$$\mathcal{A}(x) = 2tr_S(p_t(x,x))\,, \tag{85}$$

which is exactly what we found there. So we have recovered our previous result. Not only have we recovered the previous result but we have derived something that is far more general as should be clear immediately. We were working on an arbitrary even-dimensional compact Riemannian manifold. In particular we were not working on a flat space and our result knows about the anomaly in the current due to the coupling of the theory to gravity. (Of course only once we know the index theorem.)

Let us now consider the second term in the divergence of our one-form. This is often referred to as the index density for reasons that will become clear after the next few manipulations. The main ingredient will be that on a compact manifold the divergence of any one-form integrated over the manifold will vanish. This can easily be seen using the identity $d^* = (-1)^{np+n+1} * d*$ when acting on a p-form. If n is even as in the present case this simplifies to $d^* = - * d*$. So if $\alpha \in \Omega^1(X)$ we find:

$$\int_X d^*\alpha d\mu_g = -\int_X *d * \alpha d\mu_g = -\int_X d * \alpha = 0\,, \tag{86}$$

where $d\mu_g$ is the volume form on X induced by the metric g. If we normalize our zero modes such that $\langle \bar{\psi}^0_{\alpha^\pm}, \psi^0_{\alpha^\pm}\rangle_{L^2} = 1$, which is of course always possible on a compact manifold, and integrate the divergence of our current one-form, we obtain the global version of the Atiyah-Singer index theorem:

$$\text{ind}D = \text{dim ker}D|_{M^+} - \text{dim ker}D|_{M^-} = n_+ - n_- = \int_X tr_S(p_t(x,x))\,. \tag{87}$$

We can of course also write this in terms of the anomaly:

$$\text{ind}D = \frac{1}{2}\int_X *\mathcal{A}\,. \tag{88}$$

Conversely we can of course also use the local version of the Atiyah-Singer index theorem derived in class to determine what the anomaly is. Recall that any Clifford module M can (at least locally) be written[13] as $\mathbb{S}\otimes V \oplus \mathbb{S}'\otimes V'$ where $\mathbb{S}$ and $\mathbb{S}'$ are the two inequivalent $\mathbb{Z}/2\mathbb{Z}$-graded irreducible $Cliff(X)$-modules and V and V' are some vector bundles. In terms of these we found:[14]

$$\lim_{t\to 0} tr_S(p_t(x,x)) = \hat{A}(R)(ch(F) - ch(F')) \tag{89}$$

So the anomaly is given by:

$$\mathcal{A}(x) = 2\hat{A}(R)(ch(F) - ch(F'))\,, \tag{90}$$

where R is the Riemann curvature and F and F' are the curvatures associated with the connections in V and V' respectively. So we see that the anomaly is entirely determined by topology.

3.2 Gauge Anomalies – the General Idea

In this last subsection we will finally try to understand the more “dangerous” anomalies associated with currents corresponding to local symmetries from a more mathematical point of view following mostly reference [9]. The discussion is kept rather general. We will realize that the consistent

[13]In the examples considered in the fi rst part of the paper it was indeed of the form $\mathbb{S}\otimes V$

[14]Recall that even though we took the limit $t \to 0$ to evaluate the expression but that the result was independent of t.

anomaly is just one of many examples all of which have the same underlying mathematical structure. In what follows we shall limit ourselves to Euclidean quantum field theories.

Before we begin it seems useful to recall what we mean by a Euclidean quantum field theory and what we would like to compute. Our field theory will be defined on an n-dimensional Riemannian manifold (X, g). The fields will in general be maps from X to some fixed space as would be the case for example in bosonic string theory or sections in bundles over X as in most of the examples we saw in the first part. We will denote the space of all possible field configurations $C(X)$ following the notation in [9]. Usually we will limit ourselves to compact manifolds to avoid having to worry about boundary conditions on the fields which would have to be specified on non-compact manifolds. This might seem like a restriction from a mathematical point of view. From a physical point of view let us just note that we can, at least if we think of X as spacetime, think of it as a compact manifold. The compactness would be reflected in some way by observing a discrete spectrum of some sort, *i.e.* of momenta. If we make X large enough we will not be able to detect whether the spectrum is in fact discrete or not due limited resolution in measurements.

As we saw in the last section, where one of our fields was a gauge connection, we will in general have a symmetry group acting on the space of all possible field configurations, which in that example was given by the automorphism group of the underlying principle bundle, which is often denoted $\mathcal{G}$. These symmetry transformations will relate physically equivalent fields. So the space of physical fields will be the set of equivalence classes $\overline{C(X)}$. This space will of course in general not be a nice manifold.

We will define a given field theory by its action S, if we wish a complex valued "function" on field space.[15] To "solve" the quantum field theory we have to calculate the partition function Z defined as:

$$Z = \int_{\overline{C(X)}} \mathcal{D}\phi e^{-S(\phi)} \tag{91}$$

In principle what we would like to know is not the partition function per se but rather the correlation or n-point functions of operators $\mathcal{O}$ (functions on field space):

$$\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle = \int \mathcal{D}\phi e^{-S(\phi)} \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \tag{92}$$

which are some kind of weighted averages where the weight is given by the exponentiated action. The reason why one typically considers the field theory as solved once the partition function is known is that it can be used as a generating functional for n-point functions and somehow contains

[15]The reason why we have to allow for a complex action is of course that we consider Euclidean quantum field theories. The action of the field theory on Minkowski space has to be real but the reality is in general lost due to the Wick-rotation.

all the information about the theory. If we look at the expression for the partition function, there is an immediate problem. The space of fields will in general be infinite dimensional and it is not clear how to define a measure on this space. Essentially the only exceptions are matrix models, or 0-dimensional field theories, in which case the space of fields is usually finite dimensional. For our discussion we assume that we do have some measure $\mathcal{D}\phi$ on the space of fields and we will see that we will encounter problems, or anomalies, even before we get to integrate.[16] The problem is that in general the exponentiated action will not be a complex-valued function on the space of physical fields but rather a section in a complex line bundle over field space. We could have encountered the problem when we considered the relation of the consistent anomaly in $2n$-dimensional space to the singlet anomaly in $2n+2$ dimensions. We noted that the effective action for the gauge field was given by the determinant of the Dirac-operator. We should hence have considered it as a section in the determinant line bundle over the space of physical gauge fields. We avoided that by working instead on the space of all gauge fields which was contractible implying that the bundle had to be trivial. In general it is, however, more appropriate to think of the exponentiated action as a section of a complex line bundle over the space of physical fields. That is we should think of it as [9]:

$$e^{-S} = \prod_i e^{-S_i}\,, \tag{93}$$

where every factor e^{-S_i} is a section of a geometric line-bundle over the space of physical fields $L_i \to \overline{C(X)}$. The term geometric refers to the fact that we have a metric and a compatible unitary connection. The entire exponentiated action is then a section in the direct product of all the line bundles:

$$e^{-S} : \overline{C(X)} \to L = \bigotimes_i L_i\,. \tag{94}$$

This immediately tells us that the integration we should do to calculate the partition function does not necessarily make sense. We need to specify a trivialization of the line bundle L, which we will denote $\mathbf{1}$, to be able to give a meaningful definition of the partition function as follows:

$$Z = \int_{\overline{C(X)}} \mathcal{D}\phi \frac{e^{-S}}{\mathbf{1}}(\phi)\,, \tag{95}$$

where we of course divide the sections point by point.

Indeed we require the trivialization to be a geometric trivialization, by which we mean it should satisfy:

$$\begin{aligned} |\mathbf{1}| &= 1 \\ \nabla \mathbf{1} &= 0 \end{aligned} \tag{96}$$

[16]It is not clear why $\mathcal{D}\phi$ and the exponentiated action should indeed exist separately. We might think that it should be enough if $\mathcal{D}\phi e^{-S(\phi)}$ exists as our measure. Unfortunately, not even that is typically the case.

In general there may be obstructions to finding such a trivialization. This would imply that we would not even be able to define the partition function of the theory. This obstruction is what we would call the anomaly from our present point of view. We know that line bundles over a given base space B are classified by the second integer cohomology class of the base space $H^2(B, \mathbb{Z})$. In analogy to that we can define $\check{H}^2(\overline{C(X)})$ to be the set of equivalence classes of geometric line bundles over the space of physical field configurations. We can then think of the anomaly as the equivalence class of the line bundle L in $\check{H}^2(\overline{C(X)})$.

We can even assign an anomaly to each of the factors e^{-S_i} as the equivalence class of the corresponding line bundle L_i, $[L_i] \in \check{H}^2(\overline{C(X)})$. We would then say that the anomaly cancels if the $[L_i]'s$ sum to zero implying that the direct product admits a trivialization and that the partition function can be defined.

The equivalence class of a given geometric bundle L is determined by the holonomy around any smooth loop in the base space. We know that there is a map from $\check{H}^2(\overline{C(X)})$ onto $H^2(\overline{C(X)}, \mathbb{Z})$ obtained by simply forgetting about the geometric data. So clearly a line bundle will only admit a trivialization if it is trivial as a topological line bundle. We also know that the curvature of the line bundle determines the holonomy around contractible loops. So we can use the curvature of the line bundles and its topological class as approximations to the anomaly [9].[17]

This tells us that we might have an anomaly either due to the local geometry of our field space or due to its topology. These two cases are typically refered to as local and global anomalies in the mathematical literature, respectively. Our example of the consistent anomaly is an example for a local anomaly and indeed what we calculated was the curvature or the first Chern class of the determinant line bundle. Examples of global gravitational anomalies were for example studied in [20] Since the space of physical fields is in general not even a manifold it seems useful to "probe" it by nicer spaces, for example by spheres. We can map a space T into the space of fields and pull back the line bundle. If it turns out to be non-trivial over the space T then we know that it must have come from a non-trivial bundle over $\overline{C(X)}$, and we know that there is an anomaly.

This unfortunately already concludes our brief discussion of anomalies from a mathematical point of view. There would be a good deal more to say in particular about determinant line bundles and the index theorem for families of elliptic operators, but at this point we can only refer the reader to the literature.

[17] By approximation we mean that these are suffi cient conditions for the presence of an anomaly but not necessary. An interesting question should be whether "higher order" obstructions appear in some physical setup.

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