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THE BRAID GROUPS

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1. Introduction.

The braid groups B_n, $n = 1, 2, 3, \ldots$, were introduced in 1926 by E. Artin [1] and have been the subject of numerous investigations. Although there is a well-known presentation of B_n that has been derived several times the derivations that appear in the literature e.g. [1], [2] are all, in one way or another, somewhat devious. Our principal object is to give a straightforward derivation of this presentation, based on the previously unnoted remark that B_n may be considered as the fundamental group of the space $\tilde{E}^{2n}$ of configurations of n undifferentiated points in the plane.

Our derivation uses a method of computation that has never been published, although knowledge of it is probably widely distributed. It is proposed to publish the details of this method in a later paper; however the ideas involved are transparent enough to be believably communicated very briefly, and this we do in § 2 of the present paper.

By examining a certain covering of $\tilde{E}^{2n}$ and using the results of [3] it is shown that $\tilde{E}^{2n}$ is aspherical, and certain consequences of this fact are noted. In particular it follows immediately that B_n has no elements of finite order; we believe that this was not previously known.

2. Computation of π_1.

If X is a regular cell-complex, then we consider mappings of X onto X/R where R is a relation obtained from a family Φ of identifications of the cells of X. Φ is required to satisfy the following conditions:

0) Each φ in Φ is a homeomorphism with domain a closed cell of X.
i) If U is a cell of X, $\varphi: \overline{U} \to \overline{U}$ is in Φ if and only if φ is the identity.
ii) If $\varphi \in \Phi$, $\varphi: \overline{U}_1 \to \overline{U}_2$ then $\varphi^{-1}: \overline{U}_2 \to \overline{U}_1$ is in Φ.
iii) If $\varphi: \overline{U}_1 \to \overline{U}_2$ and $\varphi^1: \overline{U}_2 \to \overline{U}_3$ are in Φ, so is $\varphi^1\varphi: \overline{U}_1 \to \overline{U}_3$.
iv) If $\varphi: \overline{U}_1 \to \overline{U}_2$ is in Φ and V_1 is a cell contained in $\overline{U}_1$ then $V_2 = \varphi(V_1)$ is also a cell, and $\varphi|\overline{V}_1: \overline{V}_1 \to \overline{V}_2$ is in Φ.

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In what follows X and X/R will be manifolds of dimension n, and we shall compute π_1 of the complement of an $(n-2)$-dimensional subcomplex K of X/R.

The algorithm for the computation is roughly as follows: Select in X/R a maximal n-dimensional "cave" $\mathscr{C}$ of n-dimensional, and oriented $(n-1)$-dimensional cells (this will be dual to a maximal tree in a dual cell decomposition). To each oriented $(n-1)$-cell not in the cave will correspond a generator of π_1. This generator is represented by a loop that penetrates the $(n-1)$ cell once with intersection number 1 but otherwise lies entirely in $\mathscr{C}$. To each $(n-2)$-cell of X/R that does not belong to K will correspond a relation, obtained from the "non-abelian coboundary" of the $(n-2)$-cell in question. More precisely, a small loop about an $(n-2)$-cell σ will intersect, in a certain order and sense, all the $(n-1)$-cells having σ on their boundary. Joining this loop to the base point will give a representative of an element of the fundamental group of the union of the n, and $(n-1)$-cells of $X/R-K$. In this way a set of elements of the free group generated by the $(n-1)$-cells not in $\mathscr{C}$ is defined. This set of elements, one for each $(n-2)$-cell not in K, will be a complete set of relations for $\pi_1(X/R-K)$.

3. A cellular decomposition of S^{2n}.

An ordered n-tuple $(p_1, \ldots, p_n)$ of points of the plane E^2 may be considered to be a point p of $2n$-dimensional space E^{2n}. If the coordinates of p_i are x_i, y_i, the coordinates of the corresponding point p are

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n .$$

Let us write $i_1 < i_2$ whenever the abscissa of p_{i_1} is smaller than the abscissa of p_{i_2}, $i_1 \veebar i_2$ whenever p_{i_1} and p_{i_2} have the same abscissa, and the ordinate of p_{i_1} is smaller than the ordinate of p_{i_2}, and $i_1 = i_2$ whenever p_{i_1} coincides with p_{i_2}. Information of this sort can be condensed into a single symbol, θ, describing a point set in E^{2n}. Thus, for example, the symbol $(3<5=1<6 \veebar 4 \veebar 2=7)$ denotes the set of all points $(x_1, y_1, \ldots, x_7, y_7)$ in E^{14} such that

$$x_3 < x_5 = x_1 < x_6 = x_4 = x_2 = x_7 ,$$
$$y_5 = y_1, \quad y_6 < y_4 < y_2 = y_7 .$$

(Of course the same information is indicated by each of the symbols

$$(3 < 1 = 5 < 6 \veebar 4 \veebar 2 = 7) ,$$
$$(3 < 5 = 1 < 6 \veebar 4 \veebar 7 = 2) ,$$
$$(3 < 1 = 5 < 6 \veebar 4 \veebar 7 = 2) ;$$

we shall not distinguish between such equivalent symbols). The same symbol θ will be used to denote the set of all those points p satisfying the indicated conditions.

It is easy to see that each θ is a convex subset of E^{2n} and that, together with the point at infinity, they are the (open) cells of a regular cell-subdivision of the $2n$-dimensional sphere $S^{2n}=E^{2n}\cup\infty$. The dimension of the cell θ is obviously equal to $2n$ minus the sum of the number of occurences of $\veebar$ and twice the number of occurrences of $=$. The lower dimensional cells that are on the boundary of θ are obtained by replacing instances of $i_1<i_2$ by $i_1\veebar i_2$ or $i_2\veebar i_1$ and/or replacing instances of $j_1\veebar j_2$ by $j_1=j_2$ (or $j_2=j_1$). For example the boundary of the 5-dimensional cell $(1<2\veebar 3)$ consists of the 4-dimensional cells $(1\veebar 2\veebar 3)$, $(2\veebar 1\veebar 3)$, $(1<2=3)$, the 3-dimensional cells $(1=2\veebar 3)$, $(1\veebar 2=3)$, $(2\veebar 1=3)$, the 2-dimensional cell $(1=2=3)$, and the vertex ∞.

In what follows we shall be concerned especially with the cells of dimension $\geq 2n-2$. There are $n!$ cells of dimension $2n$. One of them is $(1<2<\ldots<n)$, and the others may be obtained from this by permuting the indices $1,2,\ldots,n$. The $(2n-1)$-cells on the boundary of

$$(1 < 2 < \ldots < n)$$

are

$$\begin{aligned}
&(1 \veebar 2 < 3 < \ldots < n)\,, \\
&(2 \veebar 1 < 3 < \ldots < n)\,, \\
&(1 < 2 \veebar 3 < \ldots < n)\,, \\
&(1 < 3 \veebar 2 < \ldots < 2n) \quad \text{etc.},
\end{aligned}$$

and the $(2n-2)$-cells on the boundary of, say, $(1\veebar 2<3<\ldots<n)$ are

$$\begin{aligned}
&(1 = 2 < 3 < \ldots < n)\,, \\
&(1 \veebar 2 \veebar 3 < \ldots < n)\,, \\
&(1 \veebar 3 \veebar 2 < \ldots < n)\,, \\
&(3 \veebar 1 \veebar 2 < \ldots < n)\,, \\
&(1 \veebar 2 < 3 \veebar 4 < \ldots < n)\,, \\
&(1 \veebar 2 < 4 \veebar 3 < \ldots < n) \quad \text{etc.}
\end{aligned}$$

4. The action of Σ_n on S^{2n}.

To the permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix}$$

associate the autohomeomorphism of S^{2n} that maps an arbitrary point

$(p_1, p_2, \ldots, p_n)$ of E^{2n} into the point $(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$; thus an action on S^{2n} of the symmetric group Σ_n of permutations of n symbols $1, 2, \ldots, n$ is defined. Denote the collapsed space by $\hat{S}^{2n}$, and the image of E^{2n} under the collapsing Λ by $\hat{E}^{2n}$. Each of the autohomeomorphisms of S^{2n} considered maps ∞ into ∞ and permutes the cells θ; the collapsing Λ maps one or more m-cells σ upon an m-cell τ of $\hat{E}^{2n}$, (not necessarily homeomorphically). The cells τ, together with the image of the point at ∞, constitute a regular cell-subdivision with identifications of $\hat{S}^{2n}$. A symbolic designation of the cells τ is readily derived. For example the cells of $\hat{S}^6$ are $\{1<2<3\}$, $\{1<2\veebar 3\}$, $\{1\veebar 2<3\}$, $\{1\veebar 2\veebar 3\}$, $\{1<2=3\}$, $\{1=2<3\}$, $\{1\veebar 2=3\}$, $\{1=2\veebar 3\}$, $\{1=2=3\}$, and ∞.

5. The subcomplex Δ.

The points $p_1, \ldots, p_n$ of E^2 are distinct if and only if, for each $i<j$, $(x_i-x_j)^2+(y_i-y_j)^2>0$. Accordingly we consider the collection Δ of those cells θ of our decomposition of E^{2n} in whose symbols the sign $=$ occurs at least once. Since boundaries are obtained by changing $<$ to $\veebar$ or $\veebar$ to $=$, it is clear that Δ and ∞ together form a $(2n-2)$-dimensional subcomplex of the cell complex S^{2n}. Furthermore the points $p_1, \ldots, p_n$ of E^{2n} are distinct if and only if p lies in $E^{2n}-\Delta$. Let $\hat{\Delta}$ denote the image of Δ under the collapsing Λ of S^{2n} to $\hat{S}^{2n}$. Then $\hat{\Delta} \cup \infty$ is a subcomplex of the cell complex $\hat{S}^{2n}$, and $p_1, \ldots, p_n$ are distinct if and only if $\hat{p} \in \hat{E}^{2n}-\hat{\Delta}$. Note that the point $\hat{p}$ may be considered to be an unordered n-tuple of points $p_1, \ldots, p_n$ of E^2. Let $\hat{E}^{2n} = \hat{E}^{2n} - \hat{\Delta}$.

6. The Braid group.

Let $\mathscr{B}_n$ denote the braid group on n strings, φ the well-known homomorphism of $\mathscr{B}^n$ upon Σ^n, and $\mathscr{I}^n$ the kernel of φ. If we look at the plane cross sections of a braid $\mathscr{b}$, we see that it may be described kinematically as a motion of n distinct points in the plane that ends with these points back in their original position but permuted as indicated by the permutation $\varphi(\mathscr{b})$. In particular $\mathscr{b}$ belongs to $\mathscr{I}_n$ if and only if the motion described returns each point to its original position. From these remarks it should be clear that the fundamental group of $E^{2n}-\Delta$ is $\mathscr{I}_n$, the fundamental group of $\hat{E}^{2n}-\hat{\Delta}$ is $\mathscr{B}_n$, and that $E^{2n}-\Delta$ is the unbranched covering space of $\hat{E}^{2n}-\hat{\Delta}$ that belongs to the subgroup $\mathscr{I}_n$ of $\mathscr{B}_n$.

7. A presentation of $\mathscr{B}_n$.

To calculate $\pi_1(\hat{E}^{2n}-\hat{\Delta})$ choose the base point in the interior of the $2n$-cell $\lambda^{2n} = \{1<2<\ldots<n\}$. Since this is the only $2n$-cell of $\hat{S}^{2n}$, there is a generator σ_j corresponding to each $(2n-1)$-cell

$$\lambda_j^{2n-1} = \{\ldots < j \veebar j+1 < \ldots\};$$

it is represented by a loop in $\lambda^{2n} \cup \lambda_j^{2n-1}$ that cuts λ_j^{2n-1} exactly once. Let us suppose that λ_j^{2n-1} is so oriented that the motion of $p_1 \cup \ldots \cup p_n$ in E^2 described by a loop representative of σ_j causes the points p_j and p_{j+1} to interchange places (and names) by circling one another in a counterclockwise direction. The motion for σ_j^{-1} is shown in Figure 1.

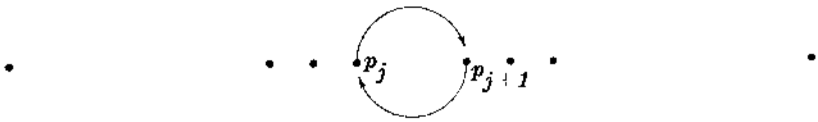

Fig. 1.

The braid σ_j^{-1} is shown in Figure 2.

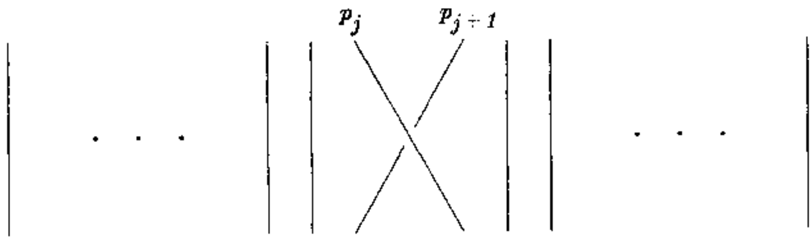

Fig. 2.

According to the general theory, a complete set of relations can be found in one to one correspondence with the cells of $\hat{E}^{2n} - \hat{\Delta}$ of dimension $2n-2$. These are of two sorts:

$$\lambda_{i,k} = \{\ldots < i \veebar i+1 < \ldots < k \veebar k+1 < \ldots\}, \qquad i+1 < k,$$
$$\lambda_{i,i+1} = \{\ldots < i \veebar i+1 \veebar i+2 < \ldots\}.$$

Now $\lambda_{i,k}$ is on the boundary of just the $(2n-1)$-cells λ_i and λ_k. Figure 3 shows a local cross section of E^{2n} by a plane perpendicular to the $(2n-2)$-cell

$$(1 < \ldots < i \veebar i+1 < \ldots < k \veebar k+1 < \ldots < n).$$

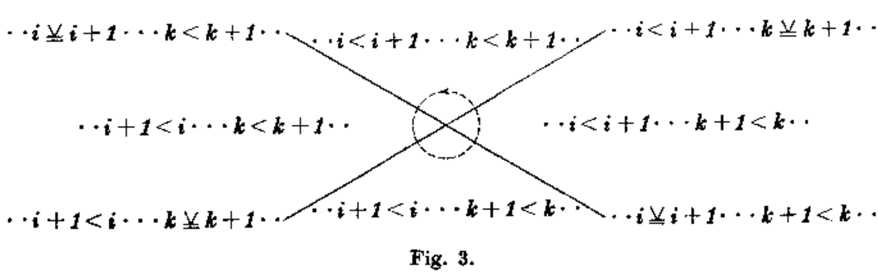

Fig. 3.

The relation $r_{i,k}$ corresponding to the cell $\lambda_{i,k}$ in $\hat{E}^{2n}$ is read as a "non-abelian coboundary" of $\lambda_{i,k}$. It is

$$r_{i,k} = \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}$$

as may be seen by traversing the dotted loop in Figure 3. The motion of $(p_1, \ldots, p_n)$ in E^2 described by $r_{i,k}$ is shown in Figure 4 and its interpretation as a braid in Figure 5.

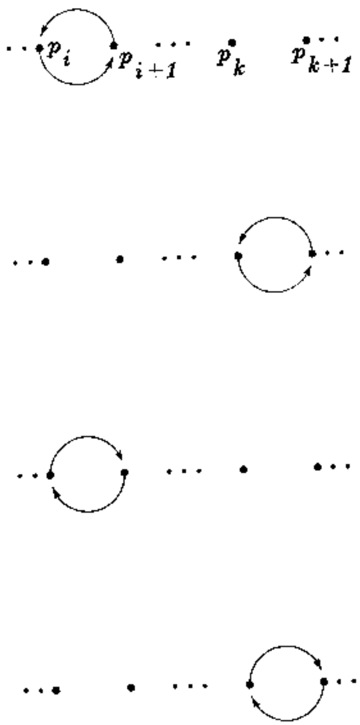

Fig. 4.

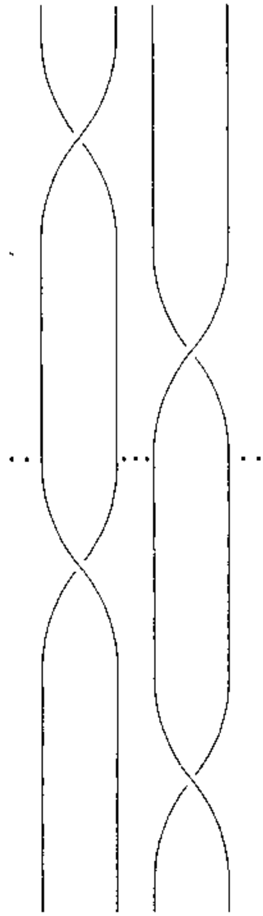

Fig. 5.

As for $\lambda_{i,i+1}$, it is on the boundary of λ_i and λ_{i+1}. A local cross section of E^{2n} by a plane perpendicular to the $(2n-2)$-cell

$$(1 < \ldots < i \veebar i+1 \veebar i+2 < \ldots < n)$$

is shown in Figure 6.

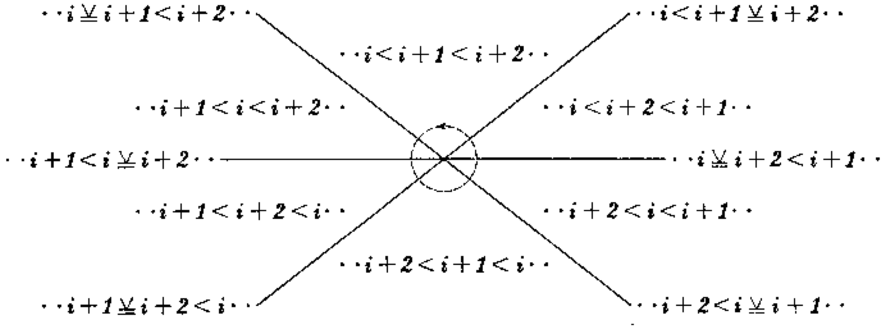

Fig. 6.

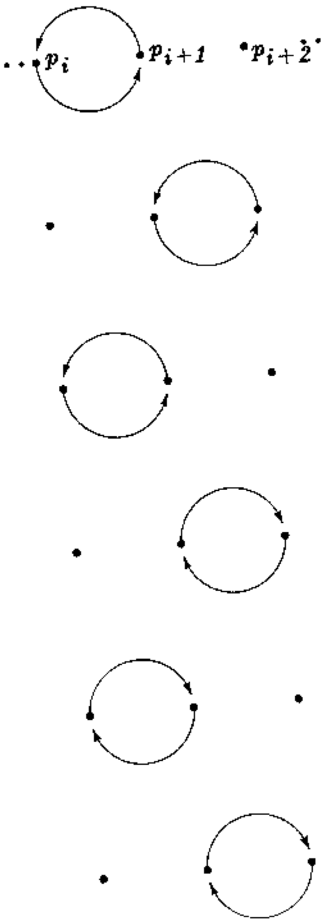

Fig. 7.

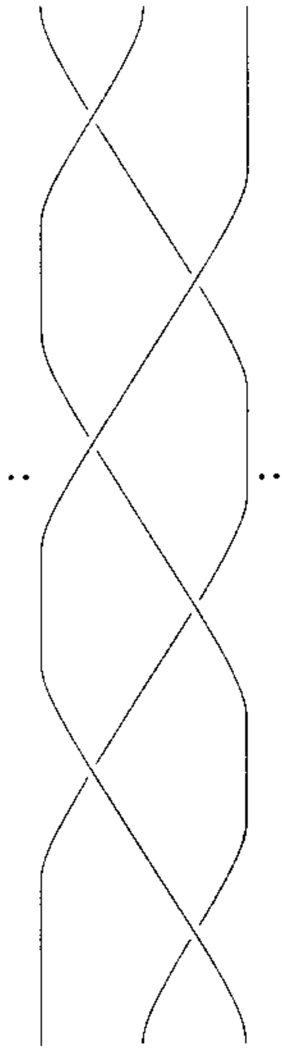

Fig. 8.

The corresponding relation $r_{i,i+1}$ is

$$r_{i,i+1} = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$$

as may be seen by traversing the dotted loop in Figure 6. The motion of $(p_1, \ldots, p_n)$ in E^2 thereby described is shown in Figure 7, and its interpretation as a braid in Figure 8.

Thus we have derived anew the well-known presentation

$$\mathscr{B}_n = (\sigma_1, \ldots, \sigma_{n-1} : r_{1,2}, r_{1,3}, \ldots, r_{n-2,n-1}).$$

REMARK. The same method could be used to find a presentation of $\mathscr{I}_n$, but the result could just as well be obtained by applying the Reidemeister–Schreier theorem.

8. Corollaries.

The covering of $\tilde{E}^{2n}$ corresponding to the representation of $\mathscr{B}_n$ on Σ_n (symmetric group of degree n) is just the space $F_{0,n}{}^2$ of [3], hence according to [3] has trivial homotopy groups above dimension 1. It follows then that *$\tilde{E}^{2n}$ is aspherical.* As an immediate corollary we have:

COROLLARY 1. *$\mathscr{B}_n$ has no elements of finite order.*

PROOF. $\tilde{E}^{2n}$ is a finite dimensional $K(\mathscr{B}_n, 1)$ space, hence every subgroup of $\mathscr{B}_n$ must be of finite geometric, hence finite cohomological dimension, but an element of finite order would generate a subgroup of infinite cohomological dimension.

Clearly $\tilde{E}^{2n}$ is an open $2n$-dimensional manifold so we have:

COROLLARY 2. *$\mathscr{B}_n$ has the homology groups of an open $2n$-dimensional manifold.*

REMARK. It seems reasonable to expect that the homology groups of $\tilde{E}^{2n}$, which by virtue of the asphericity of $\tilde{E}^{2n}$ are those of $\mathscr{B}_n$, may be calculated from the cellular decomposition of $\tilde{E}^{2n}$ which we have given.

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