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DRAFT VERSION ONLY

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Baez-Dolan:

$$Z\colon nBord \to \mathcal{C} = Cat_{(\infty,n-1)} \text{ enriched over k-modules}$$

determined by $Z(\cdot)$.

For now, condition looks hard to verify.

We want some hint as to what will produce such $Z(\cdot)$.

The starting point: Atiyah-style TQFT
$F(S^1)$ was a Frobenius algebra.

Look at the calues on all spheres $F(S^k)$ and see what structure results.

Consider the subcategory, $k \leq n$,

$$kBord \hookrightarrow nBord$$

$$\hom_{kBord}(S^{k-1}, S^{k-1}) = \coprod_{[W^k] s.t. \partial W = S^{k-1} \coprod S^{k-1}} BDiff(W, \partial W)$$

$$B(I) = \hom(\coprod_{\alpha} S^{k-1}_{\alpha}, S^{k-1}) = \coprod_{[W^k] s.t. \partial W = S^{k-1} \coprod S^{k-1}} BDiff(W, \partial W)$$

FIGURES

The collection of spaces described in the figures form an operad.

$$kBord \xrightarrow{Z} \mathcal{C}$$

then $Z(S^{k-1})$ becomes a B-algebra

$$B(I) \to \hom_{\mathcal{C}}(F(\coprod_I S^{k-1}) F(S^{k-1})$$

This still seems hard, so we want to forget information:

- (first step): too many manifolds

  $k=3, W^3$ such that $\partial W = S^2 \coprod \cdots \coprod S^2$ restricted to $W = D^k - \coprod_{\alpha \in I} D^k_\alpha$

- (second step): $Diff(D^k - \coprod_I D^k, \partial)$ compl

FIGURES
This fiber bundle classified by this map (which I did not draw!)

Definition 1. *The* **little k-disks operad**

$$\mathcal{E}_k(I) = \textit{configuration space of } I \textit{ embedded k-disks in } D^k$$

This gives

$$Z\colon kBord \to \mathcal{C}$$

$Z(S^{k-1})$ is an $\mathcal{E}_k$-algebra.
For $Z\colon nBord \to \mathcal{C}$, then each $Z(S^k)$ becomes an $\mathcal{E}_k$-algebra for $k = o, \ldots, n$.

Example: (k=0) $V = Z(\cdot)$ is a dualizable object

$$Z(S^0) = Z(\cdot^+ \coprod \cdot^-) = V \otimes \check{V} = End(V)$$

This is an associative $\mathcal{E}_1$-algebra.
For $n = 3$,
$Z(S^1)$ an $\mathcal{E}_2$-algebra$(Cat_{(\infty,1)})$

This is a braided monoidal category.
Example: $Rep(U_q(G))$
Question: Where should such a sequence of $\mathcal{E}_k$-monoidal higher categories come from?
In particular, where do $\mathcal{E}_k$-monoidal categories come from?

Most symmetric monoidal categories ceom from algebraic geomety:
Tannakian duality

FIGURE
First example: $\mathcal{C} = RepG$, G an affine algebraic group. Can we recover G?

Not quite, we can recover BG since $QC_{BG} \cong Rep(G)$. Thn we can recover G from BG eith a choice of a point

$$* \xrightarrow{e} BG$$

Generalization: X large clas of stacks for which we can recover X from QC_X.

Theorem 2. *(Lurie)*

$$X(A) \cong Fun^{\otimes}(QC_X, Mod_A)$$

Pretheorem: This works in the derived setting.

$$\text{derived algebraic geometry} \;\leftrightarrow\; \text{symmetric monoidal } \infty-\text{categories}$$

$$\mathcal{E}_n - rings \overset{Mod}{\to} \mathcal{E}_{n-1} - monoidal \infty - categories$$
$$R \mapsto Mod_R$$

MORE TO THIS FIGURE

Is there a trivial ring which maps out of form half-boring hom-bimodules to get monoidal ∞-category structure?

Definition 3. *An $\mathcal{E}_n$* **scheme** *is a topological space X with a sheaf of |mathcalE$_n$-rings such that locally it is of the form SpecA, where A is an $\mathcal{E}_n$-algebra.*

Definition 4. *$A \in \mathcal{E}_n$-ring,*
Spec A $=(spec\pi_0 A, \mathcal{O}), f \in \pi_0 A$ *such that* $\wr(U_f) = A[f^{-1}]$.

Theorem 5. *$\mathcal{E}_n$-geometry works as well as $\mathcal{E}_\infty$-geometry 90% of the time.*

Given X an $\mathcal{E}_n$-scheme or stack, QC_X is an $\mathcal{E}_{n-1}$-monoidal ∞-category.

$$\{(\infty, k) - category/X\}$$

is an $\mathcal{E}_{n-k-1}$-monoidal $(\infty, k+1)$-category.
Applications:

- *Rep*$(U_q(G))$ where is the group?
  Take the "Tannakian dual" of this $\mathcal{E}_2$-monoidal ∞-category produces an $\mathcal{E}_3$-stack B
  A candidate for the "Classifying stack of the quantum group".
- In char 0, $\mathcal{E}_n$-schemes come from symplectic geometry, n odd.

Theorem 6. *(Tamarkin, Kontsevich) $\mathcal{E}_n$ operad is formal.*

$$\mathcal{E}_n(2) \cong Conf(2\text{points in } D^2) \cong S^{n-1}$$

$A \in \mathcal{E}_n - alg(Mod_k)$
SOME STUFF

Theorem 7. *This defines a Poisson algebra and the bracket is a Lie bracket.*

AND A LITTLE MORE...