Challenge: What's Wrong with this Proof of the Fundamental Theorem of Algebra

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The fundamental theorem of algebra is a pretty quick consequence of the following:

Lemma: *If P is a non-constant one-variable polynomial with complex coefficients and if $P(0) \neq 0$ then there's a complex number w such that*

$$|\,P(w)\,| \;<\; |\,P(0)\,|.$$

One uses the immediate corollary that for any u such that $P(u) \neq 0$ we can find w such that $|P(w)| < |P(u)|$ (just use the lemma for the polynomial that sends z to $P(u{+}z)$) and that's enough to imply that if P has a value closest to 0 then that value is 0. But first-semester (real) analysis allows us to find that value closest to 0 once we've proven that when z is far from 0 then so is $P(z)$. [1]

So what's wrong with this proof of the lemma: start by rewriting $P(z)$ as $P(0) + z^m c\, Q(z)$ where m is a positive integer, c a complex number (necessarily not 0) and Q is a polynomial such that $Q(0) = 1$.

Chose an s so that

$$s^m \;=\; -P(0)/c. \quad [2]$$

Then for any non-negative real number r:

$$P(rs) \;=\; P(0) + r^m s^m c\, Q(rs) \;=\; P(0)\left(1 - r^m\, Q(rs)\right).$$

What we need, therefore, is to show that for sufficiently small but positive r:

$$\left|1 - r^m\, Q(rs)\right| \;<\; 1.$$

Put another way, we need $r^m\, Q(rs)$ to be in the open disk of radius 1 centered at 1. But for $0 < r < 1$ if $Q(rs)$ is in that open convex set—with 0 on its boundary—then so is $r^m\, Q(rs)$ and the continuity of Q says that there's a real $\delta > 0$ such that $Q(rs)$ is in that disk if $|rs| < \delta$. Hence $|\,P(rs)\,| < |\,P(0)\,|$ whenever r is positive but less than both 1 and $\delta/|s|$.

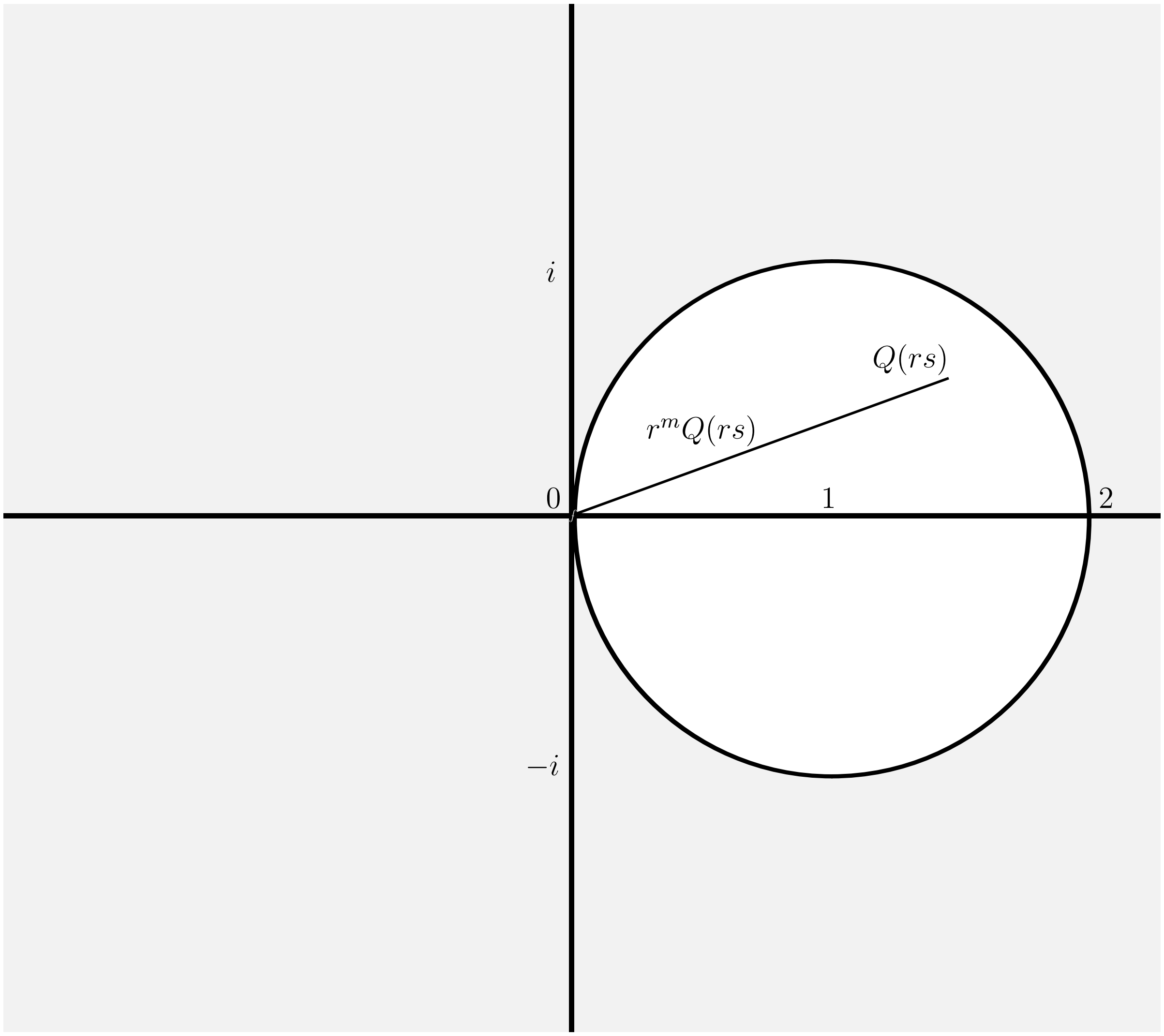
How could such an easy proof for such a big theorem be so unknown? What's wrong?

example →

Available at http://www.math.upenn.edu/~pjf/FTA.pdf
And check out http://www.math.upenn.edu/~pjf/Hamilton.pdf

[1] If the degree of P is n and its leading coefficient is C then $|z^{-n}P(z)|$ is close to $|C|$ for large z and that says there's a real K such that $|z| > K$ implies $|P(z)| > \frac{1}{2}|z^n C|$. (It's worth noting that this is the only place where we will use that P is a polynomial; the proof of the lemma—about to come—appears to work for any convergent complex power series.)

[2] Any complex number is of the form $r\left(\cos\theta + i\sin\theta\right)$ where r and θ are real numbers and r is non-negative. High-school trigonometry suffices to show that the product of $\cos\alpha + i\sin\alpha$ and $\cos\beta + i\sin\beta$ is $\cos(\alpha{+}\beta) + i\sin(\alpha{+}\beta)$. An m^{th} root of $r\,(\cos\theta + i\sin\theta)$ is thus constructable as $\sqrt[m]{r}\,\left(\cos(\theta/m) + i\sin(\theta/m)\right)$.



i
$Q(rs)$
$r^m Q(rs)$
0
1
2
$-i$