$0, 1, 2, 4$ *

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Theorem: *Any finite-dimensional associative real algebra D without zero divisors is isomorphic either to a single point, the reals, the complex numbers or the quaternions.*

The one substantive ingredient is the algebraic completeness of the complex numbers, in particular, the fact that every finite dimensional real field is isomorphic to one of the two real subfields of the complex numbers; the rest is first-semester linear algebra.

But the first step—that D has a unit—works for any finite-dimensional vector space over any field. Assuming $\dim D > 0$ let $0 \neq a \in D$. The linear self-map obtained by multiplying by a, being one-to-one, is onto, hence there's an element e such that $a = ae$. Since $a(ex - x) = 0$ we know $ex = x$, that is, any right-unit for a is a left-unit for all x. In particular, e is also a left-unit for a and, presto!, by the the mirror-image argument it's also a right-unit for all x. We will henceforth denote e as 1 and call the line it spans the *real axis*.

For any x not in the real axis the subspace generated by the powers of x (counting $x^0 = 1$ as a power) is a commutative subalgebra without zero divisors, hence a finite dimensional field over the reals, isomorphic to the complex numbers. *Every plane containing the real axis is a copy of the complex numbers.*

Assuming $\dim D > 1$ there is such a copy and we choose $i \in D$ such that $i^2 = -1$. Let $f : D \to D$ denote conjugation by i, that is, $fx = ixi^{-1}$. Since conjugating with i^2 is the identify function, f is an involution ($f^2x = x$) and we obtain a splitting of D as $D_+ \oplus D_-$ where D_+ is the set of f-fixed points ($fx = x$) and D_- the f-negated points ($fx = -x$). [1] In the case at hand we may describe the elements of D_+ as those that commute with i and of D_- as those that anti-commute (that is, $xi = -ix$). If D_+ were to have an element not in the complex plane containing i, we would obtain an even larger commutative subalgebra, hence $\dim D_+ = 2$.

Assuming $\dim D > 2$ let $0 \neq v \in D_-$. Using that f is an automorphism on the algebraic structure of D and that it is invariant—but not the identity function—on the plane spanned by two of its eigenvectors (1 and v) we obtain a copy of the complex plane on which f is its unique non-trivial automorphism. If we let j denote one of its square roots of -1 then necessarily $fj = -j$. We'll rewrite this as $ij = -ji$.

Define $k = ij$. We obtain Hamilton's celebrated equations $i^2 = j^2 = k^2 = ijk = -1$, [2] that is, D contains a copy of the quaternions.

We thus finish by showing that $\dim D \leq 4$, equivalently, that $\dim D_- \leq 2$. If a pair of elements each anti-commute with i, then their product must commute with i, [3] hence multiplication by j induces a one-to-one linear transformation $D_- \to D_+$ which, of course, forces $\dim D_- \leq \dim D_+ \leq 2$.

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* Available at `www.math.upenn.edu/~pjf/Hamilton.pdf`

I find it hard to believe that this proof of the classification of associative real division algebras could be new but for many years now I've never found a mathematician who was less than astounded by the existence of such an easy proof.

[1] Because $x = \frac{x+fx}{2} + \frac{x-fx}{2}$ for all x and the intersection $D_+ \cap D_-$ is trivial.

[2] Because k^2 and ijk are both equal to $ijij = i(-ij)j = ii^{-1}j^2 = -1$.

[3] Because $i(xy) = (-xi)y = -x(iy) = -x(-yi) = (xy)i$.