Amplifications, Diminutions, Subscorings
for
Categories, Allegories

PETER FREYD

pjf@upenn.edu

April 20, 2020

- **p3 Diminution for [1.1] Four of the eight axioms are redundant!**
  (Thanks to Martin Knopman):

The axioms:

$$xy \ \text{ is defined iff } \ x\square = \square y,$$
$$(\square x)x = x \ \text{ and } \ x(x\square) = x,$$
$$x(yz) = (xy)z$$

- **p3 [1.13] should now start with:**

$(\square x)\square = \square x$ because $(\square x)x$ is defined, similarly $\square(x\square) = x\square$. If xy is defined then $(xy)(y\square)$ is defined (because $(xy)(y\square) = x(y(y\square))$) hence $(xy)\square = \square(y\square) = y\square$. Similarly, if xy is defined then $\square(xy) = \square x$. Using the convention for $\succcurlyeq$ [1] we have $(xy)\square \succcurlyeq y\square$ and $\square(xy) \succcurlyeq \square x$.

- **p5 [1.17] A footnote for the word "groupoid":**

The roots of category theory lie in algebraic topology and the word "groupoid" has been used in this way since at least Paul Smith's 1949 "Homotopy groups of certain algebraic systems." (Proc. Nat. Acad. Sci.) Smith cites Reidemeister's 1932 *Einführung in die kombinatorische topologie* (Braunschweig : F. Vieweg & So W.) for his usage The notion is usually credited to H Brandt, "Über eine Verallgemeinerung des Gruppengriffes" Math. Ann. 96, 360-366, 1926. Indeed, there was a time when categories were most quickly described as "Brandt semigroupoids." But, be warned: in some communities "groupoid" means a set with any binary operation without restriction.

- **p21 A one-to-one correspondence from the last paragraph of [1.364] made explicit:**

Given $0 \in S \subset A$ the Cantor-Bernstein-Schroeder theorem delivers a one-to-one correspondence from $S \times A^\star$ to $A^\star$. It can be described as follows, in which n is a natural number, $w \in A^\star$, $a \in A \setminus S$, $s \in S \setminus \{0\}$:

$$\begin{aligned} \langle 0, 0^n \rangle &\rightarrow 0^n \\ \langle 0, 0^n aw \rangle &\rightarrow 0^n aw \\ \langle 0, 0^n sw \rangle &\rightarrow 0^{n+1} sw \\ \langle s, w \rangle &\rightarrow sw \end{aligned}$$

---

[1] \begin{picture}(17,7)\qbezier(5,6.5)(6,5)(12,5)\qbezier(5,1.5)(6,3)(12,3)\end{picture}

• **p50 Add small–print comment to end of [1.461]:** (Thanks to Peter Selinger!)

The category composed of local homeomorphisms has equalizers and, rather surprisingly, it actually does have products, indeed all non-empty diagrams have limits. The construction of pullbacks and equalizers, however, is much easier than the construction of even binary products, Unlike pullbacks, products are not preserved by forgetful functors. The product of the space of rational numbers with itself, for example, is uncountably large. (And for every infinite cardinal there's a space whose product with itself has the next power-cardinal as its size.)

• **p52 [1.475] remove last sentence (see *Corrections*) and add:**

For an example of a one-valued non-special cartesian category let the objects be topological unital rings in which the topology is either discrete or the other extreme, indiscrete (the only open sets are empty or entire). The maps are continuous ring homomorphisms. The one-element ring is a strict terminator, that is, the only maps therefrom are isomorphisms. The only constant maps in this category are those targeted at the terminator, hence for any map either its source must be discrete or its target must be indiscrete. Equalizers can be obtained using the standard construction. For binary products topologize the standard ring product with the indiscrete topology when both factors are indiscrete (otherwise with the discrete topology). Then $A \times -$ is faithful iff A is indiscrete. (Instead of rings one could use semi-lattices or, for that matter, any equational theory [2] with constants 0 and 1 and a binary operation such that $0x = 0$ and $1x = x$. For a less algebraic example use posets with top and bottom.) By adding a strict coterminator to any one-valued non-special cartesian category one may obtain a two-valued example.

• **p55 Footnote for the first paragraph of [1.493 small print]:**

A more conceptual description: first lexicographically order the product of the f-targets. Then $\langle T; f_1, \ldots, f_n \rangle \in \tau$ iff the induced map from T to the product preserves order.

• **p66 Footnote for the middle paragraph of [1.4(12) small print]:**

A more conceptual description: $\langle a_1, \ldots, a_n \rangle < \langle b_1, \ldots, b_m \rangle$ iff for all sufficiently large x, $a_1 x^n + a_2 x^{n-1} + \cdots + a_n x \ < \ b_1 x^m + b_2 x^{m-1} + \cdots + b_m x$.

• **p68–69 Correction for [1.512] remove top sentence on p69 and add two new entries:**

• **p69 New entry [1.515]:** *The class of covers is closed under left cancellation (if $A \to B \to C$ is a cover, then so must be $B \to C$). In a category with pullbacks the class of covers is closed under composition.*

BECAUSE: Left cancellation is immediate For composition, suppose that $A \to B$ and $B \to C$ are covers and that a monic $D \to C$ allows their composition. Let

$$
\begin{array}{ccc}
E & \to & B \\
\downarrow & & \downarrow \\
D & \to & C
\end{array}
$$

be a pullback. Since $E \to B$ is monic and allows $A \to B$ it must be an isomorphism. We thus obtain a map from B to D which shows that $D \to C$ allows $B \to C$. Hence $D \to C$ is an isomorphism.

---

[2] Sometimes "algebraic theory." The problem with that name is the theory of fields—the quintessential theory of algebra for most mathematicians—would not be an algebraic theory.

- **p69 New entry [1.516 small print]:**

In a category without pullbacks, covers need not be closed under composition. Consider the subcategory of the category of sets with three objects named 1,2,3 each with the indicated number of elements. Allow no maps from 1 or 2 into 3 but allow all other maps. (The only maps targeted at 3 are endomorphisms.) Any endomorphism on 3 is, by default, a cover. To obtain a counterexample, compose a constant endomorphism on 3 with an onto map from 3 to 2. Each is a cover but the composition is not.

- **p74 New entry [1.536]:**

In the previous small-print exploration of τ-categories we defined a universal property for slice categories [1.4(11)6]. It's worth describing the material without the full uniqueness allowed by the τ-structure:

For an object in a cartesian category $B \in \mathbf{A}$ define the *generic point* in $\mathbf{A}/B$ to be the map $g : \Delta 1 \to \Delta B$ carried by Σ to the diagonal map $B \to B \times B$.

The slice category $\mathbf{A}/\mathrm{B}$ is the result of freely adjoining a point to the object B.

That is, given any representation of cartesian categories $T : \mathbf{A} \to \mathbf{C}$ and point $1 \xrightarrow{b} T(B)$ there is a representation, unique up to natural equivalence, $T' : \mathbf{A}/B \to C$ that carries the generic point to b and each Δf to Tf.

Given an object $a : A \to B$ construct $T'a$ by using the pullback:

$$
\begin{array}{ccc}
T'a & \to & TA \\
\downarrow & & \downarrow Ta \\
1 & \xrightarrow{b} & TB
\end{array}
$$

- **p109 Add to end of [1.641]:**

For the same reason, given any $f : A \to B$ in a boolean pre-logos, the map induced by inverse-images, $f^{\#} : Sub(B) \to Sub(A)$, preserves complements.

- **p117 [1.7] Add to very top:**

Recall that in a regular category the inverse image and direct image operations satisfy: $f(A') \subset B'$ iff $A' \subset f^{\#}(B')$ [1.51].

- **p129 Small print addition for [1.74(10)]:**

The Freyd curve may be described with a four-state automaton with states $\mathsf{E}', \mathsf{E}, \mathsf{N}, \mathsf{N}'$, and inputs "$-, \circ, +$." (we're suppressing the 1s):

Next State	E′	E	N	N′
$+$	E	N	E	N
$\circ$	E	E′	N′	N
$-$	E	N	E	N

Output	E′	E	N	N′
$+$	$+$	$+$	$-$	$-$
$\circ$				
$-$	$-$	$-$	$+$	$+$

For an open onto map any one of the states may be taken as initial. (To get the function described above start at E.) Each next state is always an adjacent state as defined by the list $\mathsf{E}', \mathsf{E}, \mathsf{N}, \mathsf{N}'$. The $\circ$ input toggles E', E and N, N'. The two non-$\circ$ inputs have the same next-state behavior: they both always target one of the two middle positions which fact together with

the fact that all next states are adjacent determines their action. In the diagram below the gray (double) arrows show the next-state behavior for the ○ input.

$$\mathsf{E}' \overset{\leftrightarrow}{\to} \mathsf{E} \, \leftrightarrow \mathsf{N} \overset{\leftrightarrow}{\rightleftarrows} \mathsf{N}'$$
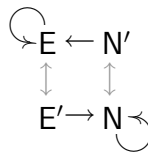
As for output: when ○ is the input there is no output; in the first two states, E', E the output echoes the non-○ input; in the last two, N, N', it negates.

The first task is to show that the resulting function is defined not just on sequences but on elements of $[-\frac{1}{2}, +\frac{1}{2}]$, that is, the same output is engendered by sequences that name the same interval element. We need to consider the output of two machines that have been fed the same initial inputs but one machine will henceforth hear ○++++⋯ and the other +−−−−⋯. We'll do better with a single machine but with two demons jumping from state to state each according to the commands issued by its appointed sequence. We feed them both the same initial sequence that brings them to the same state. For each of those four possible states the jump then commanded by the input ○+ produces the same output as that commanded by + but the demons will land in different states Each will be at one of the two center states, E, N, but not the same center state. Thereafter they will continue to exchange position forever and that means that when one machine echoes the other negates. Which is just what is needed for the output engendered by a constant sequence of +s on one machine to be the same as that engendered by a constant sequence of −s on the other machine. All of this easily dualizes for the pair ○−−−−⋯ and −++++⋯.

Another four Freyd curves may also be described with a four-state automaton:

	Next State						Output			
	E	E′	N	N′				E	E′	N	N′
+	E	N	N	E			+	+	+	−	−
○	E′	E	N′	N			○				
+	E	N	N	E			−	−	−	+	+

The ○ input toggles E, E' and N, N'. The two non-○ inputs have the same next-state behavior: E and N are stationary and the only targets. In the diagram below, the vertical gray (double) arrows show the next-state behavior for the ○ input:

$$\mathsf{E} \leftarrow \mathsf{N}'$$
$$\updownarrow \quad \updownarrow$$
$$\mathsf{E}' \rightarrow \mathsf{N}$$

As for output: when ○ is the input there is no output; in the two left-hand states, E, E', the output echoes the non-○ input; in the two right-hand states, N, N', it negates.

For this machine when we start two demons in the same state, one listening to ○++++⋯ and the other +−−−−⋯ the jump commanded by the input + again produces the same output as that commanded by ○+ but the demons will land in different states One will be in E the other N and there they'll stay forever. One of the demons will echo the input, the other will negate it, which is—again—just what is needed for the output engendered by a constant sequence of +s on one machine to be the same as that engendered by a constant sequence of −s on the other machine. All this, again, easily dualizes.

- **p118 New entry [1.715]:**

A Boolean pre-logos is a logos.

BECAUSE: $f^{\#\#}(A') = \neg f(\neg A')$. That is, $B' \subset \neg f(\neg A')$ iff $f(\neg A') \subset \neg B'$ iff $\neg A' \subset f(\neg B')$ iff $\neg A' \subset \neg f(B')$ iff $f^{\#}(B') \subset A'$, using the definition of direct image, [1.641] and the fact that complementation is order-reversing. Thus a Boolean pre-logos is automatically a BOOLEAN LOGOS, and representations of Boolean pre-logoi are automatically representations of Boolean logoi.

- **p133 Insert after first two sentences of [1.77]:**

Note that $(R^\circ)^+ = (R^+)^\circ$. (If T is transitive then so is T°, and since $R^\circ \subset (R^+)^\circ$ we have $(R^\circ)^+ \subset (R^+)^\circ$ and since $R = (R^\circ)^\circ \subset ((R^\circ)^+)^\circ$ we have $R^+ \subset ((R^\circ)^+)^\circ$ therefore $(R^+)^\circ \subset (R^\circ)^+$.) Similarly $(R^\circ)^\star = (R^\star)^\circ$.

- **p196 Add to end of [2.1]:**

See 2.113.

- **p197 Insert in [2.113 small print] after "MODULAR LATTICE":**

The Horn sentence is equivalent to the equation $\quad (A \cup X) \cap (A \cup B) = A \cup (X \cap (A \cup B))$
which is equivalent to the containment $\quad (A \cup X) \cap (A \cup B) \subset A \cup (X \cap (A \cup B))$
since in any lattice [3] $\quad A \cup (X \cap (A \cup B)) \subset (A \cup X) \cap (A \cup B)$

When catenation is to be interpreted as union the universally quantified containment in the middle line is easily equivalent with the universally quantified containment $RS \cap T \subset (R \cap TS)S$. [4]

- **p198 Add to end of [2.13]:**

The important families of endo-relations, reflexive/symmetric/transitive/coreflexive/equivalence, are each closed under intersection:

Reflexive: $\;1 \subset 1 \cap 1 \subset R \cap S$.
Symmetric: $(R \cap S)^\circ \subset R^\circ$ and $(R \cap S)^\circ \subset S^\circ$.
Transitive: $(R \cap S)^2 \subset R^2 \subset R$ and $(R \cap S)^2 \subset S^2 \subset S$.

- **p199–200 A better version of [2.135]:**

The isomorphisms in $\mathbf{A}$ and $Map(\mathbf{A})$ coincide. That is:

If R is an isomorphism then R is a map and $R^{-1} = R^\circ$.

Or, using the convention for $\succcurlyeq$:

$R^{-1} \succcurlyeq R^\circ$ *hence all isomorphisms are maps.*

This is an immediate consequence of a much better lemma [indeed, the most conspicuous omission in the published book]:

$1 \subset RS$ *and* $SR \subset 1$ *iff* R *is a map and* S *is its reciprocal* $(S = R^\circ)$.

BECAUSE: R is entire since RS is. Hence $S \subset S1 \subset SRR^\circ \subset 1R^\circ \subset R^\circ$. But we also have $1 \subset S^\circ R^\circ$ and $R^\circ S^\circ \subset 1$ and the same argument yields $R^\circ \subset S$.

---

[3] $A \subset (A \cup X)$, $A \subset (A \cup B)$, $(X \cap (A \cup B)) \subset X \subset (A \cup X)$ and $(X \cap (A \cup B)) \subset (A \cup B)$.

[4] Assuming the middle line specialize X, A, B to R, S, T for $RS \cap T = (X \cup A) \cap B \subset (A \cup X) \cap (A \cup B) \subset A \cup (X \cap (A \cup B)) = S(R \cap ST) = (R \cap TS)S$. For the converse specialize R, S, T to $X, A, (A \cup B)$ for $(A \cup X) \cap (A \cup B) = SR \cap T = RS \cap T \subset (R \cap TS)S = (X \cap ((A \cup B) \cup A)) \cup A = A \cup (X \cap (A \cup B))$.

(If an allegory is viewed as a 2-category, then the hypothesis, $1 \subset RS$ and $SR \subset 1$, says that S and R form an *adjoint pair*.)

• p200 New entry [2.137 small print]:

A VON NEUMANN ALLEGORY is an allegory that obeys the equation:

$$RR^{\circ}R \;=\; R \quad \text{[5]}$$

Since one containment is a consequence of the standard axioms it is enough to require:

$$RR^{\circ}R \;\subset\; R$$

A VON NEUMANN REGULAR CATEGORY is a regular category whose associated allegory is von Neumann.

An equational theory is a MAL′CEV THEORY if a MAL′CEV OPERATOR can be constructed, to wit, a ternary operator, traditionally denoted as t and whose values are traditionally denoted as $txyz$, that satisfies the equations:

$$txxz \;=\; z \quad \text{and} \quad txzz \;=\; x$$

Given any equational theory, the category of its models is regular and the forgetful functor to sets is a faithful representation. As we sill see, *the category of models of an equational theory is von Neumann regular iff the theory is Mal′cev.*

First, a few examples of Mal′cev theories. In any theory that contains the theory of groups we can construct $txyz$ as $xy^{-1}z$. For Heyting algebras, $txyz$ may be taken as $((x \to y) \to z) \wedge ((z \to y) \to x)$. [6] [7] For a less symmetric example take the theory of *colonies*: a single binary operation, with values denoted $x\!:\!y$, subject to just one equation $x\!:\!(y\!:\!x) = y$ (equivalently $(x\!:\!y)\!:\!x = y$ [8]) take $txyz \;=\; (x\!:\!x)\!:\!(z\!:\!(x\!:\!y))$ [9] or any of the other 23 terms of the same size that do the trick.[10] (Examples of colonial operations include $x\!:\!y = (xy)^{-1}$ in any group.)[11]

Colonies are a special case of the theory of *quasigroups*, that is, the theory of binary operations—associative or not—with unique left and right division. The equational formu-

---

[5] A von Neumann regular ring is one with an anti-involution such that $xx^*x = x$ all x.

[6] To see that $txxz = z$ note that the first half, $((x \to x) \to z)$, is equal to $1 \to z = z$ and it suffices to show the second half is at least z, but $z \le (z \to x) \to x$ is equivalent to $z \wedge (z \to x) \le x$ which, of course, is equivalent to $z \to x \le z \to x$.

[7] An *orthomodular lattice*, beloved by physicists, is a lattice with 0, 1 and an anti-involution x' such that $x \vee x' = 1$, plus one consequence of modularity, to wit, $a \vee (a' \wedge (a \vee b)) = a \vee b$ (*e.g.*, the lattice of closed subspaces of a Hilbert space). Take $txyz = (x \vee (y' \wedge (y \vee z))) \wedge (z \vee (y' \wedge (y \vee x)))$. Then $txzz = (x \vee (z' \wedge (z \vee z))) \wedge (z \vee (z' \wedge (z \vee x))) = (x \vee (z' \wedge z)) \wedge (z \vee x) = (x \vee 0) \wedge (z \vee x) = x$.

[8] If $(x\!:\!y)\!:\!x = y$ all x, y then $x\!:\!(y\!:\!x) = ((y\!:\!x)\!:\!y)\!:\!(y\!:\!x) = y$.

[9] $txxz = (x\!:\!x)\!:\!(z\!:\!(x\!:\!x)) = z$ and $txzz = (x\!:\!x)\!:\!(z\!:\!(x\!:\!z)) = (x\!:\!x)\!:\!x = (x\!:\!x)\!:\!(x\!:\!(x\!:\!x)) = x$.

[10] This is the simplest possible consistent Mal′cev theory (inconsistent means all equations are provable). The proof is an exercise in equational theories. There's no way of obtaining a ternary operator without having an n-ary operation with $n \ge 2$ and if we have any such then we have one for $n = 2$. We need, of course, at least one equation. There's no way of deducing a singular equation—that is, one in the form $s = y$ where y is a "naked" variable and s is not—from nonsingular equations, hence if we have just one equation it has to be singular. If y does not appear in s then we deduce $y = y'$ (hence all equations) from $s = y$ and $s = y'$. If y appears in the extreme right position of s then a model of the theory is one in which all terms are equal to their far right variable, that is, there are models that can not have a Mal′cev operator. Ditto for the far left side. In particular s can not have just one instance of the binary operation. Assuming just two instances, there are two ways of putting in the parentheses but symmetry says—for our present purpose—that it doesn't matter which. If the one equation is $(x\!:\!y)\!:\!z = y$ then we can deduce $x\!:\!z = y$ (since $x\!:\!z = ((x\!:\!x)\!:\!y)\!:\!z = y$) thus returning to the case $s = y$ where y is absent from s. All of which says that the only equational axioms left are $(x\!:\!y)\!:\!x = y$ and $x\!:\!(y\!:\!x) = y$.

[11] The motivating example of colonies for this writer arose from "orientable surfaces without diagonals," that is, triangulated orientable surfaces in which the 1-skeleton is the complete graph on the vertices. Define $x\!:\!y$ for distinct vertices x, y to be the vertex one reaches when one moves from x to y and then takes the "first right turn." Define $x\!:\!x = x$. The two known examples yield colonies arising from the Galois fields of order 4 and 7 with $x\!:\!y = -cx - c^2y$ where c is a primitive cube root of 1 (*i.e.*, $c^2 + c + 1 = 0$).

lation requires three binary operations and four equations, to wit, any two of the following three rows:

$$\begin{array}{rcccl} (x/y)\,y & = & x & = & (x\,y)/\,y \\ y\,(y\backslash x) & = & x & = & y\backslash(y\,x) \\ (y/x)\backslash y & = & x & = & y/(x\backslash y) \end{array}$$

<div style="text-align: right">[12]</div>

For quasigroups one may take $txyz = (x/x)\backslash((x/y)z)$. [13]

We will see yet another Mal$'$cev theory in [2.34] (below, in these amplifications). For the theory of scales,[14] take $txyz = (x|z)\overset{\vee}{\frown}(\dot{y}|\bot)$.

For the connection between the von Neumann equation on the allegory of relations and the existence of a Mal$'$cev operator, first observe that given a relation $R\colon A \to B$ between Mal$'$cev algebras, suppose that aRb, $a'Rb$, and $a'Rb'$. If we view R as a subalgebra of $A\times B$ then $t\langle a, b\rangle\langle a', b\rangle\langle a', b'\rangle = \langle taa'a', tbbb'\rangle = \langle a, b'\rangle$ must be in R, in other words, $aRR^\circ Rb'$ implies aRb'. A corollary, using [1.593], is that

Abelian categories are von Neumann regular categories.

For the reverse connection, suppose that the von Neumann equation holds in the allegory of relations between algebras of a given equational theory. Let F be the free model on two genterators, a and b. Let $R \subset F\times F$ be the submodel generated by pairs $\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle$. When we view R as a relation the von Newmann equation says that $RR^\circ R \subset R$ and since $bRbR^\circ aRa$ we obtain bRa hence $\langle b, a\rangle \in R$. But for any equational theory we obtain the subalgebra generated by three elements as the set of values of all ternary operators (primitive and derived) applied to those three elements. That is, there must be a ternary operator t such that $t\langle a.a\rangle\langle a, b\rangle\langle b, b\rangle = \langle b, a\rangle$ which says, of course, that $t\langle a.a\rangle\langle a, b\rangle\langle b, b\rangle = \langle taab, tbba\rangle = \langle b, a\rangle$, hence that t is a Mal$'$cev operator.

In a von Neumann allegory in which symmetric idempotents split we may define (up to isomorphism) an image for a relation R; because RR° and $R^\circ R$ are symmetric idempotents—one on the domain of R, one on its codomain—with R and R° yielding an inverse pair of isomorphisms between their splittings.

The von Neumann equation implies two conditions on endomorphisms: reflexive implies symmetric (**RIS**), that is, $1 \subset R$ implies $R^\circ = R$; and reflexive implies transitive (**RIT**), that is, $1 \subset R$ implies $RR \subset R$ (if $1 \subset R$ then $R^\circ \subset 1R^\circ 1 \subset RR^\circ R \subset R$ and $RR \subset R1R \subset RR^\circ R \subset R$.) [15]

Both RIS and RIT imply equivalence relations commute (**ERC**): if E and F are equivalence relations then since $1 \subset EF$ RIS implies $EF = (EF)^\circ = F^\circ E^\circ = FE$ and RIT implies $FE \subset 1FE1 \subset EFEF = (EF)^2 \subset EF$, similarly $EF \subset FE$. When $EF = FE$ we know that

[12] Colonies are quasigroups in which $xy = y\backslash x = y/x = x\colon y$.

[13] First use the 4th equation to obtain $txxz = (x/x)\backslash((x/x)z) = z$, then use the 1st, 4th and the 1st equation again to obtain $txzz = (x/x)\backslash((x/z)z) = (x/x)\backslash x = (x/x)\backslash((x/x)x) = x$. It is curious that only two of the defining equations are needed. There is a transitive group of symmetries on the six-element set of operations and their "twists" that preserve the equations and we obtain six pairs of equations each of which suffices for the construction of a Mal$'$cev operator. There are no other such pairs. There are usually an infinity of Mal$'$cev terms if there's one, *e.g.,* $tx(tx(txyz)z)z$. (There's only one $txyz$ in the free abelian group generated by x, y, z. When the word "abelian" is dropped there are infinitely many, *e.g.,* $xz^{-1}xy^{-1}zx^{-1}z$.) In the case of quasigroups if we stick to terms of the same size there are 72 versions. Heavenly.

[14] http://www.math.upenn.edu/~pjf/analysis.pdf

[15] Both RIS and RIT can be strengthened: it is enough to assume that a morphism in a von Neumann allegory contains both its domain and codomain to imply symmetry and it is enough to assume that it contains the intersection of its domain and codomain to imply transitivity: if $\mathrm{Dom}(R), \mathrm{Dom}(R^\circ) \subset R$ then $\mathrm{Dom}(R^\circ)R^\circ \mathrm{Dom}(R) \subset RR^\circ R \subset R$; if $\mathrm{Dom}(R^\circ)\cap \mathrm{Dom}(R) \subset R$ then (using 2.121) $RR \subset R\,\mathrm{Dom}(R^\circ)\mathrm{Dom}(R)R = R\big(\mathrm{Dom}(R^\circ)\cap\mathrm{Dom}(R)\big)R \subset RR^\circ R$.

EF is itself an equivalence relation, which makes it the smallest equivalence relation containing both E and F. In [2.113], particularly as amplified on page 5 of these amplifications, any lattice of pairwise commuting equivalence relations was seen to be automatically modular (for Mal′cev theories this is usually stated as the modularity of the congruence lattices).[16]
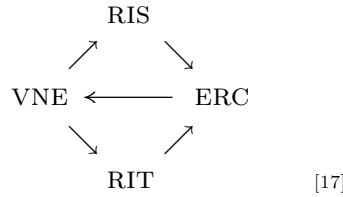
Without further conditions RIS and RIT are independent. For a one-object allegory where RIS holds but not RIT let R be a reflexive symmetric morphism in any allegory. Consider the suballegory generated by R. All of its elements remain reflexive symmetric. They are all transitive only if R was transitive. (If one starts with the allegory of relations on a three-element set and takes R to be any one of the three reflexive symmetric non-transitive relations, this construction yields a three-element allegory. If one starts with the natural numbers and takes R to be the adjacency-relation-made-reflexive then this construction yields an allegory with an infinite number of reflexive relations all of which are symmetric and only one is transitive.)

For a one-object allegory where RIT holds but not RIS let R be a reflexive transitive relation in any allegory and suppose further that R defines a total order (*e.g.,* on a 2 element set). Assuming that $R \neq 1$, the suballegory generated by R has exactly four elements all of which are reflexive transitive but only two of which are symmetric.

But for allegories arising from regular categories (or equivalently, as explained below, for tabular allegories) both RIS and RIT imply the von Neumann equation because each implies ERC and ERC implies the von Neumann equation: given a relation R that is spanned by a pair of maps, f, g, that is, such that $R = f°g$, ERC implies $RR°R = (f°g)(f°g)°(f°g) = f°(gg°)(ff°)g = f°(ff°)(gg°)g = (f°f)f°g(g°g) \subset 1f°g1 = R$. To summarize the connections, let **VNE** mean the von Neumann equation Then for all allegories:

$$
\begin{array}{ccc}
 & \text{RIS} & \\
\nearrow & & \searrow \\
\text{VNE} & & \text{ERC} \\
\searrow & & \nearrow \\
 & \text{RIT} &
\end{array}
$$
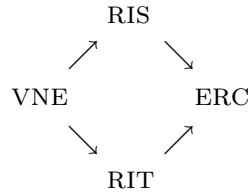
The two counterexamples above show that neither vertical arrow may be inserted (and hence none of the arrows may be reversed) in the context of *all* allegories. But—as just seen—for those allegories in which there are enough maps:

$$
\begin{array}{ccc}
 & \text{RIS} & \\
\nearrow & & \searrow \\
\text{VNE} \longleftarrow & & \text{ERC} \\
\searrow & & \nearrow \\
 & \text{RIT} &
\end{array}
$$

[17]

---

[16] ERC is equivalent to equivalence relations being closed under composition. Indeed, for equivalence relations $EF = FE$ iff EF is an equivalence relation: if $EF = FE$ then EF is immediately symmetric and imminently transitive ($EFEF = EEFF = EF$); if EF is an equivalence relation then it's symmetric and $EF = (EF)° = F°E° = FE$. Put another way: ERC could just as well have stood for *equivalence relations compose*.

[17] What does it mean if there's a Mal′cev operator in a category, in particular suppose there's a Mal′cev operator in the category of models of a Mal′cev theory. Put another way suppose that t is a Mal′cev operator that is a homomorphism with respect to t.

In general, given an m'ary operator f and an n'ary operator g then f is a homomorphism with respect to g means that for any $m \times n$ matrix of variables if we apply f to each of the n columns and then g to the resulting row, we obtain the same as if we had applied g to each of the m rows and then f to the resulting column. (Yes, it's a symmetric condition: f is a homomorphism with respect to g iff g is a homomorphism with respect to f.)

- **p205 Amplifications for [2.157 small print]:**

We say that a modular lattice whose only elements aside from the top and bottom are its atoms and co-atoms satisfies the theorem of Desargues iff for any six atoms $a_1, b_1, c_1, a_2, b_2, c_2$:

$$(a_1 \vee a_2) \wedge (b_1 \vee b_2) \;\leqq\; c_1 \vee c_2$$
$$implies$$
$$(a_1 \vee b_1) \wedge (a_2 \vee b_2) \;\leqq\; \big((a_1 \vee c_1) \wedge (a_2 \vee c_2)\big) \vee \big((c_1 \vee b_1) \wedge (c_2 \vee b_2)\big)$$

Note that every "$- \vee -$" is a co-atom.[18]

[See diagram on next page.]

When $f = g = t$ and $m = n = 3$ and the matrix of variables is:

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

we obtain the equation

$$t(tabc)(tdef)(tghi) = t(tadg)(tbeh)(tcfi)$$

This equation plus the two Mal$'$cev equations turn out to be defining what may be described as the result of removing zero from the theory of abelian groups. In imitation (but pronounceably) of Jacobson we'll call this the theory of *grups*.

Given an element a in a grup G we can obtain an abelian-group structure by defining

$$\begin{aligned} 0_a &= a \\ x +_a y &= txay \\ -_a x &= taxa \end{aligned}$$

The Mal$'$cev equations immediately yield $x +_a 0_a = x = 0_a +_a x$. Using the variables $\begin{matrix} u & a & v \\ a & a & a \\ x & a & y \end{matrix}$ we obtain the equation

$(u +_a v) +_a (x +_a y) = (u +_a x) +_a (v +_a y)$ which in [1.591] on p88 was shown to imply that $+_a$ is commutative and associative.

Use the variables $\begin{matrix} a & a & x \\ x & a & a \\ a & a & a \end{matrix}$ to obtain $-_a x +_a x = 0_a$. We'll denote the resulting abelian group as G_a.

Given a pair of elements in G the canonical isomorphism from G_a to G_b is given by $txab$. Use the variables $\begin{matrix} x & a & b \\ a & a & b \\ y & a & b \end{matrix}$ to obtain $t(x +_a y)ab = (txab) +_b (tyab)$. When $a = b$ we get the identity function. If we first apply the function named by $txab$ then the one named by $txbc$ use $\begin{matrix} x & a & b \\ a & a & b \\ c & c & c \end{matrix}$ to prove that we obtain the one named by $txac$.

An abelian group has, of course, a grup structure (as always in the presence of an abelian group structure $txyz$ is $x - y + z$). Note that translation by a, that is, the function that sends x to $x + a$ is a grup-automorphism. Given a pair of abelian groups, the grup-homomorphisms from the first to the second are precisely the group-homomorphisms followed by a translation by an element of the second. (When abelian groups are replaced by vector spaces such functions are, of course, called "affine transformations.") The resulting category just misses being equivalent to the category of grups. What we miss hitting is the empty grup. Hence add a new strict initial object to the category of abelian groups and grup-homomorphisms and we obtain a category equivalent to the category of grups.

Finally, what if we have a pair of Mal$'$cev operators, t, t' each a homomorphism with respect to the other? Just as in [1.591] we can prove $t = t'$.

[18] We're using the fact that the variables name <u>six</u> atoms.

$$
\begin{aligned}
(a_1 \vee a_2) \wedge (b_1 \vee b_2) &= p \\
(a_1 \vee b_1) \wedge (a_2 \vee b_2) &= w \\
(a_1 \vee c_1) \wedge (a_2 \vee c_2) &= u \\
(c_1 \vee b_1) \wedge (c_2 \vee b_2) &= v
\end{aligned}
$$
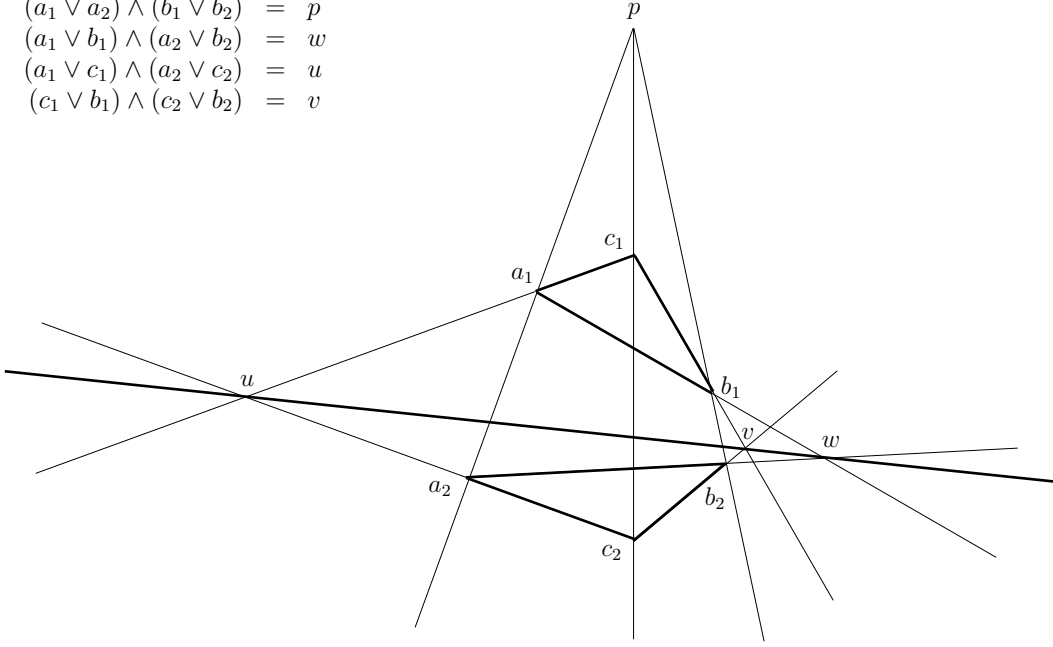
In the allegory of sets (hence for relations in any regular category)
$$(A_1 A_2 \cap B_1 B_2) \subset C_1 C_2$$
implies
$$(A_1^\circ B_1 \cap A_2 B_2^\circ) \subset (A_1^\circ C_1 \cap A_2 C_2^\circ)(C_1^\circ B_1 \cap C_2 B_2^\circ).$$

Given the hypothesis and elements x, y such that $x(A_1^\circ B_1 \cap A_2 B_2^\circ)y$
we need to show $x(A_1^\circ C_1 \cap A_2 C_2^\circ)(C_1^\circ B_1 \cap C_2 B_2^\circ)y$.
Actually we're given elements x, s_1, s_2, y satisfying:
$$xA_1^\circ s_1 B_1 y \quad \text{and} \quad xA_2 s_2 B_2^\circ y.$$
The hypothesis then says that since
$$s_1(A_1 A_2 \cap B_1 B_2)s_2 \; [\text{because } s_1 A_1 x A_2 s_2 \text{ and } s_1 B_1 y B_2 s_2]$$
there must be z such that $s_1 C_1 z C_2 s_2$. And that yields
$$x(A_1^\circ C_1 \cap A_2 C_2^\circ)z \; [\text{because } xA_1^\circ s_1 C_1 z \text{ and } xA_2 s_2 C_2^\circ z] \quad \text{and}$$
$$z(C_1^\circ B_1 \cap C_2 B_2^\circ)y \; [\text{because } zC_1^\circ s_1 B_1 y \text{ and } zC_2 s_2 B_2^\circ y] \text{ hence,}$$
finally, $x(A_1^\circ C_1 \cap A_2 C_2^\circ)z(C_1^\circ B_1 \cap C_2 B_2^\circ)y$.

● **p211 Add to end of [2.162]:**

Note that R is necessarily simple.

● **p213 A simplification for [2.216(10)] 14ᵗʰ line up:**

Define G' by $G' = G \cap FR$.

● **p214 Add to end of [2.16(11)]:**

(The word "neighbors" was chosen for its intransitivity. Let a, b, c be idempotent functions on the set $\{0, 1, 2\}$ defined by the array:

	a	b	c
0	0	0	1
1	2	0	1
2	2	2	2

Then a and b are neighbors; b and c are neighbors; a and c are not.)

● **p215 New entry [2.16(15) small print]:**

By a functor between allegories is meant a functor between their underlying categories (as opposed to a representation of allegories). A functor between allegories is said to be a 2-FUNCTOR if it preserves the partial order on morphisms (The term is dictated by the theory of 2-categories.) An immediate consequence of [2.135] (new version, above) is that a 2-functor preserves maps and their reciprocals. If the source allegory is tabular than a 2-functor preserves reciprocals of arbitrary morphisms, hence preserves such properties as entirety, simplicity, symmetry and, of course, reflexivity and transitivity.

Let F be a 2-functor between tabular allegories. If we view F as a functor between the corresponding categories of maps we obtain a functor that preserves monics and covers but not (necessarily) pullbacks. Define a NEAR-PULLBACK to be a diagram of the form:

$$
\begin{array}{ccc}
 & y & \\
A & \rightarrow & C \\
x \downarrow & & \downarrow g \\
B & \rightarrow & D \\
 & f &
\end{array}
$$

such that the induced map from A to the pullback of f, g is a cover *The restriction of a 2-functor between allegories restricts to a functor that preserves near-pullbacks between their categories of maps.* (The diagram is a near-pullback iff $x^\circ y = fg^\circ$.) Conversely: *Any functor between the corresponding categories of maps that preserves near-pullbacks extends uniquely to a 2-functor between the allegories.* Note that f is a monic/cover iff the first/second diagram below is a near-pullback:

$$
\begin{array}{ccccccc}
 & 1 & & & & f & \\
A & \rightarrow & A & & A & \rightarrow & B \\
1 \downarrow & & \downarrow f & & f \downarrow & & \downarrow 1 \\
A & \rightarrow & B & & B & \rightarrow & B \\
 & f & & & & 1 &
\end{array}
$$

Hence: *A functor between regular categories preserves near-pullbacks iff it preserves covers and carries plain pullbacks to near-pullbacks.*

● **p215 New entry [2.16(16) small print]:**

R is CO-SEMI-SIMPLE if there exist simple F and G such that $R = FG^\circ$. Co-semi-simplicity implies $RR^\circ R \subset R$ (because $(FG^\circ)(FG^\circ)^\circ(FG^\circ) = F(G^\circ G)(F^\circ F)G^\circ \subset F1^2G^\circ = FG^\circ$). If symmetric idempotents split then the converse holds: given $RR^\circ R \subset R$ we have that RR° is a symmetric idempotent and if it splits then necessarily we have simple F such that $FF^\circ = RR^\circ$. [2.162] Define $G = R^\circ F$ and verify its simplicity: $G^\circ G = F^\circ RR^\circ F = F^\circ FF^\circ F \subset 1$. Then $FG^\circ = FF^\circ R = RR^\circ R = R$. Thus *an allegory in which all symmetric idempotents split is a von Neumann allegory* [19] *iff all morphisms are co-semi-simple.* (It's worth noticing that unlike semi-simplicity which is hereditary—that is, morphisms contained in semi-simple morphisms are themselves semi-simple [2.16(10)]—in any unitary allegory all morphisms are contained in co-semi-simple morphisms.)

---

[19] As defined in the new entry [2.137] above.

- **p216 Expand [2.22]:**

In any distributive allegory reciprication establishes an isomorphism (note: not an anti-isomorphism) between the lattices (α, β) and (β, α) and that yields:

$$(0_R)^\circ = 0_{R^\circ} \qquad (R \cup S)^\circ = R^\circ \cup S^\circ$$

- **p221 New entry [2.21(11) small print]:**

In a von Neumann [20] distributive allegory all endomorphisms are symmetric: since $(R \cap 1)^\circ = R \cap 1$ in any allegory and $1 \cup R = (1 \cup R)^\circ$ in a von Neumann allegory we may use distributivity to obtain that $R = R \cap (1 \cup R) = R \cap (1 \cup R)^\circ = R \cap (1 \cup R^\circ) = (R \cap 1) \cup (R \cap R^\circ)$ is necessarily symmetric. Any map of the form $f : A \to A$ must be an involution (because by [2.134] $f = f^\circ$ only if $f = f^{-1}$).

Von Neumann pre-logoi are precisely the categories equivalent to distributive lattices: that is, identity maps are the the only idempotents (since they're involutions) hence the only left or right invertibles are isomorphisms; any diagonal map, $A \to A \times A$, being right invertible is an isomorphism, hence for any A' there's at most one $A' \to A$; thus any von Neumann pre-logos is equvalent to its (distributive) lattice of subterminators and any von Neumann logos is a Heyting algebra.

There are no *positive* von Neumann pre-logoi.

- **p224 Additional comment for [2.228 small print], at the end of the penultimate paragraph add:**

Indeed the allegory has arbitrary unions and they all distribute with composition (which forces it to have a binary partial operation as in the definition of division allegory [2.311]).

- **p224 New entry [2.23]:**

A BOOLEAN ALLEGORY is a distributive allegory in which for every pair of objects the lattice of morphisms, is a boolean algebra It may be formalized via a unary operator whose values are denoted $\neg R$ with equations:

$$
\begin{aligned}
\Box(\neg R) &= \Box R; \\
(\neg R)\Box &= R\Box; \\
R \cap \neg R &= 0; \\
S \cap (R \cup \neg R) &= (\Box R)S(R\Box).
\end{aligned}
$$

(The 4th equation is equivalent with $S \cap (R \cup \neg R) \succcurlyeq S$.)

We could be more parsimonious and define 0_R as $R \cap \neg R$ and $R \cup S$ as $\neg(\neg R \cap \neg S)$ We leave it to the reader to find the remarkably few equations needed.

A representation of distributive allegories is a representation of boolean allegories iff it preserves the complements of all 0's, that is, preserves the the top morphism for every pair of objects. The category of maps of a unitary tabular boolean allegory is a boolean pre-logos (therefore a boolean logos).

- **p226 Add to bottom of [2.313]:**

Any boolean allegory is a division allegory. Construct R/S as $\neg((\neg R)S^\circ)$. Then $T \subset \neg((\neg R)S^\circ)$ iff $T \cap (\neg R)S^\circ = 0$ iff $TS \cap (\neg R) = 0$ iff $TS \subset R$. (Note that in any distributive allegory $AB \cap C = 0$ implies $A \cap CB^\circ = 0$ since $A \cap CB^\circ \subset (AB \cap C)B^\circ$.)

---

[20] As defined in the new entry [2.137] above.

- **p234 New entry [2.36 small print]:**

The equational theory of one-object division allegories is a Mal′cev theory (as defined in the new entry [2.137] above) with a Mal′cev operator given by:

$$(1 \cap Y/X)\backslash Z \ \cap \ (1 \cap Y/Z)\backslash X$$

(Its mirror image would, of course, work as well, as would both of the possible mixtures of it and its mirror image.[21]) Heyting algebras first appeared in this work as special cases of logoi. They may also be viewed as one-object division allegories in which all morphisms are coreflexive. In that case the present construction of a Mal′cev operator specializes to the first construction $((x \to y) \to z) \wedge ((z \to y) \to x)$, previously seen in the new entry [2.137]. (For the record: the best-known generalization views Heyting algebras not as logoi or division allegories but as exponential categories.)

In the many-object case the theory of division allegories is an essentially equational theory. The term above is defined precisely when X, Y and Z are "parallel," that is, when $\square X = \square Y = \square Z$ and $X\square = Y\square = Z\square$. We thus have an example of a Mal′cev essentially equational theory but, alas, such theories do not seem to lead to a connection with von Neumann allegories.

But for a fixed set of objects one may define the regular category whose objects are division allegories with the given fixed set of objects and whose morphisms are the representations of division allegories that preserve the objects is a von Neumann regular category. (This is a special case of a regular category arising from a "many-sorted" equational theory.)

- **p238 New entry [2.419 small print]:**

We resolved the "first attempt" in [2.4] to obtain an equational definition of power allegories by moving to a division allegory. It is not the only resolution.

First, let's name the various uniqueness conditions.

PLAIN:
$$If \ \ 1 \subset SS^\circ \ \ and \ \ S^\circ S \ \subset \ 1 \ \ then \ \ S(\square \ni_R) = \Lambda(S \ni_R)$$

SIMPLE:
$$If \ \ F^\circ F \ \subset \ 1 \ \ then \ \ F(\square \ni_R) \ \subset \ \Lambda(F \ni_R)$$

FANCY:
$$\ni/\ni \ \cap (\ni/\ni)^\circ \subset 1$$

Simple implies plain (using [2.133]) and fancy implies simple (using [2.352]). Note that if coreflexives split then plain easily implies simple.

If equivalence relations split then plain implies fancy: let E be an equivalence relation such that $E\ni = \ni$ (e.g., $E = \ni/\ni \cap (\ni/\ni)^\circ$); let h be a map such that $hh^\circ = E$; then $h\Lambda(h^\circ \ni) \ni = \ni$ and plain uniqueness forces $h\Lambda(h^\circ \ni) = 1$ thus $E \subset hh^\circ \subset h\Lambda(h^\circ \ni)\Lambda^\circ(h^\circ \ni)h^\circ \subset 1$.

We will investigate equational formulations of the simple and fancy conditions in [2.447]. But first we separate the three uniqueness conditions with a pair of examples, both of which turn out to be the positive completion of a three-element one-object allegory. The objects

---

[21]
$$\begin{array}{lll} X/(Z\backslash Y \ \cap 1) & \cap & Z/(X\backslash Y \ \cap 1) \\ X/(Z\backslash Y \ \cap 1) & \cap & (1 \cap Y/X)\backslash Z \\ (1 \cap Y/Z)\backslash X & \cap & Z/(X\backslash Y \ \cap 1) \end{array}$$

may be taken as natural numbers. A morphism from m to n will be an $m \times n$ matrix with entries from the given one-object allegory.

To separate the plain from the simple we will use the three-element linearly ordered (idempotent) monoid A with elements $0 < m < 1$ viewed as a one-object allegory (of coreflexive morphisms). Intersection coincides with composition and the reciprocation operation is the identity. To separate the simple from the fancy let B be the same monoid but ordered $0 < 1 < m$. In each case the category of maps is isomorphic to the skeletal category of finite sets. And in each case $\ni_n$ may be constructed as a $3^n \times n$ matrix in which every possible row appears exactly once. Plain uniqueness is clear in both cases. In the first case, if $\ni_1$ is taken to be the column matrix $(1, m, 0)^\top$ then for $F = (m, 0, 0)$ we have $\Lambda(F \ni_0) = (0, 1, 0)$ and the simple condition fails. In the second case, the coreflexives already split in B^+ and plain existence implies simple existence. For the failure of the fancy condition note that $\ni_0$ is the unique 1×0 matrix but $\ni_0 / \ni_0 \cap (\ni_0 / \ni_0)^\circ$ is not the 1×1 identity matrix (it's the maximal 1×1 matrix).[22]

Note that both of these examples are full sub-allegories of Grothendieck topoi. In the first case, one may use the topos of presheaves on the two-element linearly ordered set and in the second case one may use the category of M-sets where M is the 4-element monoid of endofunctions on a two-element set. In the first case the example is the full allegory of finite copowers of the terminator. In the second case it is the full allegory of finite copowers of the two-element set on which M is defined as acting.

- **p246 Rewording of [2.442–2.443]:**

The proof of [2.442] shows that the law of metonymy implies the semi-simplicity of all straight morphisms and such should be explicitly stated. The argument in [2.443] proves the converse: metonymy is equivalent with the semi-simplicity of straight morphisms.

- **p248 A better version of [2.444 small print]:**

The law of metonymy is not a consequence of the other equations. Consider the full sub-allegory of the allegory of Z-sets (sets, each with a distinguished automorphism) of all those Z-sets in which no orbit has more than 3 elements For power objects start with the usual construction then remove all orbits with more than 3 elements. If A is a 5-element Z-set consisting of two orbits (necessarily one has 2 elements and the other has 3) then the $\ni$ relation from $[A]$ to A is not semi-simple (therefore, neither is the container relation on $[A]$). (This example and the three examples in [2.418] are all full allegories of full coreflective subcategories of topoi.)

- **p250 Improvements for [2.446 small print]:**

Replace the 2ⁿᵈ sentence of the 2ⁿᵈ paragraph with:

Note that R appears in an order-reversing position, hence $R \subset S$ implies $0_S \subset 0_R$.

Replace the final sentence of the same paragraph with:

Any order-reversing deflationary operation on a poset is idempotent: suppose that $f(x) \leqq x$ all x and that $x \leqq y$ implies $f(y) \leqq f(x)$; since $f(x) \leqq x$ implies $f(x) \leqq f^2(x)$ and

---

[22] The genesis of the second example was curious. The problem was easily reduced to finding a topos with a coreflective full subcategory closed under subobject formation but not such that a coreflector—the map from a coreflection—is always monic. No examples initially presented themselves. A rather painful analysis of all possible coreflective full subcategories closed under subobject formation that occur in categories of M-sets resulted in the reduction of the existence of such to an elementary condition on M. The elementary condition allowed the abstract definition of a minimal such M. It had 3 elements. The resulting one-object allegory B had five elements. Matters were simplified by adding another element to M to obtain the example above.

since f is deflationary we also have $f^2(x) \preceq f(x)$, that is, we obtain $f^2 = f$; an order-reversing function when applied to itself is order-preserving hence f is both order-preserving and order-reversing, that is, it is constant on each connected component and since it is deflationary it delivers the bottom element on each component. There is at most one order-reversing deflationary operation on a given poset.

While here, replace the word "covariant" in the sentence following the definition of union with "order-preserving."

• p250 New entry [2.447 small print]:

In the "first attempt" to define power-allegories we took $\ni$ and Λ as primitive unary operations subject to containments:

$$
\begin{aligned}
\Lambda(R)\ni_R &= R \\
1 &\subset \Lambda(R)\Lambda^\circ(R) \\
\Lambda^\circ(R)\Lambda(R) &\subset 1
\end{aligned}
$$

We resolved the uniqueness question with the "fancy" condition (as named in in the new entry [2.419] above) which required the division operation and in the presence of division we noted that Λ was not needed as a primitive, But it is, in fact, possible to make the Λ-definition equational without taking division as primitive.

First note that the "plain" condition (as named in in the new entry [2.419] above) has two immediate equational consequences:

$$
\begin{aligned}
\Lambda(\ni) &= 1 \\
\Lambda(RS) &= \Lambda(R)\Lambda(\ni_R S)
\end{aligned}
$$

In the presence of a union operation that distributes with composition these five displayed equational conditions allow a construction of division. First show that $\ni/\ni$ is obtainable as $\Lambda^\circ(\ni'\ni)\ni'$ — where $\ni'$ denotes $\ni_\ni$ — by showing that $T \subset \Lambda^\circ(\ni'\ni)\ni'$ iff $T\ni \subset \ni$.

If $T\ni \subset \ni$ then

$$T \subset T \cup 1 \subset \Lambda(T \cup 1)\ni' \subset \Lambda(T \cup 1)\Lambda(\ni'\ni)\Lambda^\circ(\ni'\ni)\ni' \subset$$

$$\Lambda((T \cup 1)\ni)\Lambda^\circ(\ni'\ni)\ni' \subset \Lambda(T\ni \cup \ni)\Lambda^\circ(\ni'\ni)\ni' \subset \Lambda(\ni)\Lambda^\circ(\ni'\ni)\ni' \subset \Lambda^\circ(\ni'\ni)\ni'$$

(The 4ᵗʰ and 7ᵗʰ containments use the new equations. The 5ᵗʰ containment uses the distributivity of composition with union. The 6ᵗʰ containment uses the hypothesis $T\ni \subset \ni$.)

If $T \subset \Lambda^\circ(\ni'\ni)\ni'.'$ then

$$T\ni \subset \Lambda^\circ(\ni'\ni)\ni'\ni \subset \Lambda^\circ(\ni'\ni)\Lambda(\ni'\ni)\ni \subset \ni$$

In any allegory, if R/S exists then $(fR)/(gS)$ may be constructed as $f(R/S)g^\circ$. Hence we may construct R/S as $\Lambda(R)\Lambda^\circ(\ni'\ni)\ni'\Lambda^\circ(S)$. (Essentially the same proof as for $f/g = fg^\circ$.)

Thus, plain uniqueness and distributive unions imply a division structure. But one needn't require unions; instead one may impose the containment (which says that $-/\ni$ is order-preserving):

$$\Lambda(R \cap S)\Lambda^\circ(\ni'\ni)\ni' \subset \Lambda(S)\Lambda^\circ(\ni'\ni)\ni'$$

The union-free proof that $T\ni \subset \ni$ implies $T \subset \Lambda^\circ(\ni'\ni)\ni'$ is then:

$$T \;\subset\; \Lambda(T)\ni' \;\subset\; \Lambda(T)\Lambda(\ni'\ni)\Lambda^{\circ}(\ni'\ni)\ni' \;\subset\; \Lambda(T\ni)\Lambda^{\circ}(\ni'\ni)\ni' \;\subset\;$$

$$\Lambda(T\ni\cap\ni)\Lambda^{\circ}(\ni'\ni)\ni' \;\subset\; \Lambda(\ni)\Lambda^{\circ}(\ni'\ni)\ni' \;\subset\; \Lambda^{\circ}(\ni'\ni)\ni'$$

Note that we are using the new containment only when $S = \ni$, hence it would suffice to impose:

$$\Lambda(R\cap\ni)\Lambda^{\circ}(\ni'\ni)\ni' \;\subset\; \Lambda(\ni)\Lambda^{\circ}(\ni'\ni)\ni'$$

We may further simplify by absorbing the equation $\Lambda(\ni) = 1$ to obtain:

$$\Lambda(R\cap\ni)\Lambda^{\circ}(\ni'\ni)\ni' \;\subset\; \Lambda^{\circ}(\ni'\ni)\ni'$$

We may now easily translate the simple and fancy uniqueness conditions to equations. Fancy uniqueness is, of course:

$$\Lambda^{\circ}(\ni'\ni)\ni' \cap \in'\Lambda(\ni'\ni) \;\subset\; 1$$

Simple uniqueness curiously simplifies. We may quantify over simple morphisms be recalling that $\frac{R}{1}$ is always simple and whenever F is already simple then $F = \frac{F}{1}$. Simple uniqueness is easily seen to be equivalent with the containment $F\cap(1/F)^{\circ} \;\subset\; \Lambda(F\ni)$. It translates to:

$$F\cap(\Lambda(1)\Lambda^{\circ}(\ni''\ni')\ni''\Lambda^{\circ}(F))^{\circ} \;\subset\; \Lambda(F\ni)$$

We may further complicate to obtain:

$$\Lambda(F)\ni' \cap \Lambda(F)\in''\Lambda(\ni''\ni')\Lambda^{\circ}(1) \;\subset\; \Lambda(F)\Lambda(\ni'\ni)$$

If we specialize F to $\ni$ we obtain

$$\ni' \cap \in''\Lambda(\ni''\ni')\Lambda^{\circ}(1) \;\subset\; \Lambda(\ni'\ni)$$

And that suffices: we may compose with $\Lambda(F)$ on both sides and use modularity to obtain the previous containment.

One final note: we do not have a proof showing that the plain uniqueness is not equivalent to a set of equations. The first example in the last section, A^{+}, satisfies an equational condition that implies plain uniqueness (but is very unlikely to be equivalent with plain uniqueness), to wit:

$$(1\cap\Lambda(1\cap F(1/F))\Lambda^{\circ}(1))F\Lambda(1) \;\subset\; \Lambda(F)$$

- **p250 New entry [2.448 small print]:**

Transitive logoi were defined in [1.77]. A TRANSITIVE ALLEGORY is one in which every endomorphism R is contained in a minimal transitive, reflexive relation, $R^{\star}$.

Power allegories are transitive allegories. Define a unary operation on endomorphisms:

$$((1\cap(\ni/(\ni R)))\ni)\backslash\ni$$

Note first that the variable, R, appears just once and in a covariant position (that is, $R \subset S$ implies $R^{\star} \subset S^{\star}$).[23] Second, the operation is inflationary, that is, $R \subset R^{\star}$ because $R \subset X\backslash Y$ iff $XR \subset Y$ and $(1\cap(\ni/(\ni R))) \ni R \subset (\ni/(\ni R))(\ni R) \subset \ni$.

---

[23] Actually it appears six times:

$$((\Box R\cap(\ni_R/(\ni_R R)))\ni_R)\backslash\ni_R$$

But if we restrict attention to Rs on a fixed object then all six appearances are, indeed, in covariant positions (because, of course, five of them are in constant positions).

Third, $R^\star$ is reflexive because it is of the form $(AS)\backslash S$ where $A \subset 1$ and transitive because for any such morphism $(AS)\backslash S = (AS)\backslash(AS)$. This last equation can be proved by noting first that $(AS)\backslash(AS) \subset (AS)\backslash S$ using the order-preservation of the "upper" variable in the division operation and then noting that $(AS)\backslash S \subset (AS)\backslash(AS)$ is equivalent with $AS((AS)\backslash S) \subset AS$ which is immediate if one uses that the leftmost A is equal to AA.

Thus $R^\star$ is a covariant inflationary operation all of whose values are reflexive transitive. We need only one further property:

If R is reflexive transitive then $R^\star \subset R$.

(It should be noted that we have not yet used any property of $\ni$ other than that its target is the object on which R is an endomorphism and that it depends only on that object.)

Suppose, therefore, that R is reflexive and transitive. Finish by proving the following five containments:

$$R^\star \subset (\Lambda^\circ(R)\Lambda(R)\ni)\backslash\ni \subset (\Lambda^\circ(R)R)\backslash\ni \subset \Lambda^\circ(R)\backslash\ni \subset \Lambda(R)\ni \subset R$$

The 1st containment will be discussed below. The 2nd containment is the first use of the thickness of $\ni$: it is a consequence of $R \subset \Lambda(R)\ni$ (we are replacing $\Lambda(R)\ni$ with R in an contravariant position). The 3rd containment is the unique use of the reflexivity of R (we are replacing R with 1 in a contravariant position). The 4th containment is the second (and last) use of the thickness of $\ni$, which property is equivalent with the entirety of $\Lambda(R)$. (For any entire S, $S^\circ\backslash T \subset SS^\circ(S^\circ\backslash T) \subset ST$.) The 5th containment follows immediately from the definition of $\Lambda(R)$ as $\frac{R}{\ni}$ (and not from any properties of $\ni$).

Now for the 1st containment We must show:

$$((1 \cap (\ni/(\ni R)))\ni)\backslash\ni \subset (\Lambda^\circ(R)\Lambda(R)\ni)\backslash\ni$$

and (because division is order-reversing in the lower variable) for that it clearly suffices to show

$$\Lambda^\circ(R)\Lambda(R)\ni \subset (1 \cap (\ni/(\ni R)))\ni$$

which, of course, is an immediate consequence of

$$\Lambda^\circ(R)\Lambda(R) \subset 1 \cap (\ni/(\ni R))$$

and that is equivalent to the two containments:

$$\Lambda^\circ(R)\Lambda(R) \subset 1 \text{ and } \Lambda^\circ(R)\Lambda(R) \subset \ni/(\ni R).$$

The left-hand containment, the simplicity of $\Lambda(R)$ is the unique use of the straightness of $\ni$. The right-hand containment is equivalent with $\Lambda^\circ(R)\Lambda(R)\ni R \subset \ni$. We need to prove (using the definition of $\Lambda(R)$):

$$\frac{\ni}{R}\frac{R}{\ni}\ni R \subset \frac{\ni}{R}RR \subset \frac{\ni}{R}R \subset \ni.$$

The 1st and 3rd of these last containments use the semi-cancellation [2.35] for symmetric division and the 2nd is the unique use of the transitivity of R.

Note that if S is symmetric then so is $S^\star$: such is easily [24] a consequence of $(R^\star)^\circ = (R^\circ)^\star$.

(The matter could be made entirely equational by recalling that R/R is always reflexive transitive for any R, endomorphism or not, and if R is already reflexive transitive then

---

[24] See addition to [1.77] in these amplifications.

$R = R/R$. [25] Thus we need the containment $(R/R)^\star \subset R/R$, equivalently, $(R/R)^\star R \subset R$. Since the reverse containment is automatic—it follows easily from the reflexivity of the values of $(-)^\star$—there must be an entirely equational proof, that is, one that starts with the term on one arbitrary variable:

$$(((\Box R \cap (\ni_R / (\ni_R (R/R)))) \ni_R) \setminus \ni_R) R$$

and using a sequence of substitutions—each of which is an instance of one of the equational axioms for power allegories—transforms it to the term R. [26])

- **p258 New entry [2.543]:**

The author of the first published paper on elementary topoi thought that its main aspect was the collective faithfulness of bicartesian representations of elementary topoi into well-pointed topoi (that is, topoi in which the terminator generates). Note that well-pointed implies boolean. The last section [2.542] says that it suffices to prove it for a boolean topos, $\mathbb{B}$, and for boolean topoi we have a stronger theorem:

The topos representations from a boolean topos into well-pointed topoi are collectively faithful.

First apply the capitalization lemma to obtain a faithful representation $\mathbb{B} \to \underline{\underline{\mathbb{B}}}$ where $\underline{\underline{\mathbb{B}}}$ is capital, then finish with the observation that $\underline{\underline{\mathbb{B}}}/\mathcal{F}$ is well-pointed for every ultra-filter $\mathcal{F} \subset Val(\underline{\underline{\mathbb{B}}})$.

- **p287–296 Add to SUBJECT INDEX:**

SUBSCORINGS

It is said that "subscoring" is short for "substitution underscorings," to wit, an array wherein the underscores indicate the sub-strings to be replaced.

---

[25] Nice: $R \subset R/R$ iff R is transitive; $R/R \subset R$ iff R is reflexive.

[26] Of course $R \backslash R$ would work as well; show that any R is equal to:

$$R(((1 \cap (\ni /(\ni (R \backslash R)))) \ni) \backslash \ni)$$

When working with relations we need to indicate whether the replacement string yields a containment or an equality. In the following a single underline indicates that the upper term is contained in the lower term, a double underline (or is it a much elongated equal sign?) indicates equality.[27]

<div align="center">

Subscorings for these *Amplifications* start on page 27.
Subscorings for *Cats & Alligators* start here.

</div>

- **p88 [1.591 small print]:**

$$\underline{\underline{(u \underset{\text{L}}{+} x)}} \underset{\text{R}}{+} \underline{\underline{(v \underset{\text{L}}{+} y)}} \qquad \underline{\underline{(u \underset{\text{R}}{+} v)}} \underset{\text{L}}{+} \underline{\underline{(x \underset{\text{R}}{+} y)}}$$

$$\underline{\underline{(1\ 1)\binom{u}{x} \underset{\text{R}}{+} (1\ 1)\binom{v}{y}}} \qquad \underline{\underline{(u\ v)\binom{1}{1} \underset{\text{L}}{+} (x\ y)\binom{1}{1}}}$$

$$(1\ 1)\left(\underline{\underline{\binom{u}{x} \underset{\text{R}}{+} \binom{v}{y}}}\right) \qquad \underline{\underline{\left((u\ v) \underset{\text{L}}{+} (x\ y)\right)}}\binom{1}{1}$$

$$(1\ 1)\left(\binom{u\ v}{x\ y}\binom{1}{1}\right) \quad = \quad \left((1\ 1)\binom{u\ v}{x\ y}\right)\binom{1}{1}$$

- **P93–94 [1.597 small print]:**

$$
\begin{aligned}
x - 0 &= x \\
x - x &= 0 \\
(u - v) - (x - y) &= (u - x) - (v - y) \\
x + y &= (x - (0 - y))
\end{aligned}
$$

$$x + 0$$
$$\underline{\underline{}}$$
$$x - (0 - 0)$$
$$\underline{\underline{}}$$
$$x - 0$$
$$\underline{\underline{}}$$
$$x$$

$$0 + x$$
$$\underline{\underline{}}$$
$$0 - (0 - x)$$
$$\underline{}$$
$$(x - x) - (0 - x)$$
$$\underline{\underline{}}$$
$$(x - 0) - (x - x)$$
$$\underline{\underline{}} \quad \underline{\underline{}}$$
$$x - 0$$
$$\underline{\underline{}}$$
$$x$$

---

[27] The macro (using the package `ulem`) `{\scor}[1]{\uuline{\rule[-7pt]{0pt}{0pt}#1}` is useful. For a single underline use `{\uline{\rule[-7.2pt]{0pt}{0pt}#1}\rule[-14.5pt]{0pt}{0pt}}`. For other examples check out the closing pages of
`www.math.upenn.edu/~pjf/combinators.pdf`   `www.math.upenn.edu/~pjf/iso-detector.pdf`
`www.math.upenn.edu/~pjf/analysis.pdf`

$$(u + v) + (x + y)$$
$$\overline{\overline{\overline{(u - (0 - v)) - (0 - (x - (0 - y)))}}}$$
$$(u - 0) - ((0 - v) - (x - (0 - y)))$$
$$(u - 0) - ((0 - x) - (v - (0 - y)))$$

$$(u + x) + (v + y)$$
$$\overline{\overline{\overline{(u - (0 - x)) - (0 - (v - (0 - y)))}}}$$
$$(u - 0) - ((0 - x) - (v - (0 - y)))$$
$$(u - 0) - ((0 - v) - (x - (0 - y)))$$

$$(u - 0) - ((0 - x) - (v - (0 - y))) \quad = \quad (u - 0) - ((0 - v) - (x - (0 - y)))$$

- **p101 [1.62]:**

$$
\begin{array}{ccccccccc}
xy^\circ & = & \bar{y}^\circ\bar{x} & 1 & \subset & ff^\circ & f^\circ f & \subset & 1 & (\bar{y}f)^\circ\bar{y}f & \subset & 1 \\
yx^\circ & = & \bar{x}^\circ\bar{y} & 1 & \subset & gg^\circ & g^\circ g & \subset & 1 & (\bar{x}g)^\circ\bar{x}g & \subset & 1
\end{array}
$$

$$\bar{x}g = \bar{y}f$$
$$x^\circ x \cup y^\circ y = 1$$
$$R = x^\circ f \cup y^\circ g$$

$$1$$
$$x^\circ 1 x \cup y^\circ 1 y$$
$$x^\circ ff^\circ x \cup y^\circ gg^\circ y$$
$$(x^\circ f \cup y^\circ g)(f^\circ x \cup g^\circ y)$$
$$RR^\circ$$

$$R^\circ R$$
$$(x^\circ f \cup y^\circ g)^\circ(x^\circ f \cup y^\circ g)$$
$$(f^\circ x \cup g^\circ y)(x^\circ f \cup y^\circ g)$$
$$f^\circ x x^\circ f \cup f^\circ x y^\circ g \cup g^\circ y x^\circ f \cup g^\circ y y^\circ g$$
$$f^\circ 1 f \cup f^\circ \bar{y}^\circ \bar{x} g \cup g^\circ \bar{x}^\circ \bar{y} f \cup g^\circ 1 g$$
$$f^\circ f \cup (\bar{y}f)^\circ \bar{y}f \cup (\bar{x}g)^\circ \bar{x}g \cup g^\circ g$$
$$1 \cup 1 \cup 1 \cup 1$$
$$1$$

$$
\begin{array}{ccc}
xR' & = & f \\
yR' & = & g
\end{array}
$$

$$R'$$
$$1R'$$
$$(x^\circ x \cup y^\circ y)R'$$
$$x^\circ x R' \cup y^\circ y R'$$
$$x^\circ f \cup y^\circ g$$
$$R$$

- **p196 [2.11]:**

$$
\begin{array}{c}
1 \\
\overline{\overline{}} \\
1^{\circ\circ} \\
\overline{\overline{\overline{}}} \\
(1^{\circ})^{\circ} \\
\overline{\overline{}} \\
(11^{\circ})^{\circ} \\
\overline{\overline{\overline{\overline{}}}} \\
1^{\circ\circ}1^{\circ} \\
\overline{\overline{}} \\
11^{\circ} \\
\overline{} \\
1^{\circ}
\end{array}
$$

- **p198 [2.122]:**

$$ A \subset 1 $$

$$ 1 \cap RR^{\circ} \subset A \qquad\qquad R \subset AR $$

$$
\begin{array}{c}
R \\
\overline{\overline{}} \\
1R \cap R \\
\hline
(1 \cap RR^{\circ})R \\
\hline
AR
\end{array}
\qquad\qquad
\begin{array}{c}
1 \cap RR^{\circ} \\
\overline{} \\
1 \cap \overline{A}RR^{\circ} \\
\hline
A(A^{\circ}1 \cap RR^{\circ}) \\
\hline
AA^{\circ} \\
\overline{} \\
A1 \\
\overline{\overline{}} \\
A
\end{array}
$$

- **p199 [2.124]:**

$$
\begin{array}{c}
1 \cap RS^{\circ} \\
\overline{\overline{}} \\
1 \cap (1 \cap (1 \cap RS^{\circ})) \\
\hline
1 \cap (1 \cap (1S^{\circ\circ} \cap R)S^{\circ}) \\
\overline{\overline{}} \\
1 \cap (1 \cap (S \cap R)S^{\circ}) \\
\hline
1 \cap (S \cap R)(1(S \cap R)^{\circ} \cap S^{\circ}) \\
\overline{} \\
1 \cap (S \cap R)((S \cap R)^{\circ} \cap S^{\circ}) \\
\overline{\overline{}} \\
1 \cap (S \cap R)((S \cap R) \cap S)^{\circ} \\
\overline{\overline{}} \\
1 \cap (S \cap R)(S \cap R)^{\circ}
\end{array}
$$

- **p199 [2.133]**:

$$f \; \subset \; g \quad (\text{hence } f^\circ \; \subset \; g^\circ)$$

$$
\begin{array}{c}
g \\
= \\
1g \\
- \\
ff^\circ g \\
- \\
fg^\circ g \\
\hline
f1 \\
= \\
f
\end{array}
$$

- **p210 [2.162]**:

$$(RS)^\circ = RS \qquad SR = 1$$

$$
\begin{array}{c}
R^\circ \\
\equiv \\
1\,R^\circ \\
= \\
SRR^\circ \\
- \\
SS^\circ SRR^\circ \\
\equiv \\
SS^\circ 1 R^\circ \\
\equiv \\
S(RS)^\circ \\
\equiv \\
SRS \\
\equiv \\
1S \\
= \\
S
\end{array}
\qquad
\begin{array}{c}
S^\circ \\
\equiv \\
S^\circ 1 \\
= \\
S^\circ SR \\
- \\
S^\circ SRR^\circ R \\
\equiv \\
S^\circ 1 R^\circ R \\
\equiv \\
(RS)^\circ R \\
\equiv \\
RSR \\
\equiv \\
R1 \\
= \\
R
\end{array}
$$

22

- **p212 [2.166]:**

$$\begin{array}{rcl} R & \subset & f^\circ g \\ ff^\circ \cap gg^\circ & = & 1 \end{array} \qquad \begin{array}{rcl} h^\circ h & = & 1 \cap fRg^\circ \\ hh^\circ & = & 1 \end{array}$$

$$\frac{\underline{\underline{(hf)^\circ(hg)}}}{\underline{\frac{f^\circ h^\circ hg}{f^\circ 1 g}}}$$
$$\underset{=}{f^\circ g}$$
$$\overline{\overline{\overline{R}}}$$

$$\frac{\overset{=}{R}}{\frac{R \cap R}{\underline{\frac{f^\circ 1 g \cap R}{\frac{(f^\circ 1 \cap Rg^\circ)g}{\underline{\underline{f^\circ(1 \cap fRg^\circ)g}}}}}}}$$
$$\underline{\underline{f^\circ h^\circ hg}}$$
$$(hf)^\circ(hg)$$

$$\frac{\underline{\underline{(hf)(hf)^\circ}} \cap \underline{\underline{(hg)(hg)^\circ}}}{\underline{\underline{hff^\circ h^\circ \cap hgg^\circ h^\circ}}}$$
$$\underline{\underline{h(ff^\circ \cap gg^\circ)h^\circ}}$$
$$\underline{h1h^\circ}$$
$$1$$

- **p214 [2.16(11) small print]:**

$$\begin{array}{rcl} ee & = & e \\ e'e' & = & e' \end{array} \quad \begin{array}{rcl} ee'e & = & e \\ e'ee' & = & e' \end{array} \quad \begin{array}{rcl} xy & = & e \\ yx & = & 1 \end{array} \quad \begin{array}{rcl} x' & = & e'x \\ y' & = & e'y \end{array}$$

$$\frac{\underset{==\ ==}{y'\,x'}}{\frac{ye'e'x}{\underline{1ye'x1}}}$$

$$\frac{\underset{==\ ==}{x'\,y'}}{\frac{e'xye'}{\underline{\underline{e'ee'}}}}$$
$$e'$$

$$\frac{\underset{=\quad =}{yxye'xyx}}{\frac{yee'ex}{\underline{yex}}}$$
$$\underset{=}{yxyx}$$
$$\underset{==\ ==}{11}$$
$$\underline{1}$$

23

$$A^2 = A \qquad RS = A \qquad SR = 1$$

$$(A \cap A^\circ)A(A \cap A^\circ)$$
$$(A \cap A^\circ)A(A \cap A^\circ) \ \cap \ (A \cap A^\circ)A(A \cap A^\circ)$$
$$AAA \ \cap \ (RS \cap S^\circ R^\circ)RS(RS \cap S^\circ R^\circ)$$
$$A \ \cap \ S^\circ(SRS \cap R^\circ)RS(RSR \cap S^\circ)R^\circ$$
$$A \ \cap \ S^\circ(SRS)RS(RSR)R^\circ$$
$$A \ \cap \ S^\circ 1111 R^\circ$$
$$A \ \cap \ A^\circ$$

$$A \cap A^\circ$$
$$(A \cap A^\circ)(A \cap A^\circ)(A \cap A^\circ)$$
$$(A \cap A^\circ)A(A \cap A^\circ)$$

$$A$$
$$RS$$
$$R1S$$
$$R(1^2 \cap 1)S$$
$$R(SRSR \cap 1)S$$
$$RS(RS \cap S^\circ R^\circ)RS$$
$$A(A \cap A^\circ)A$$

$$A(A \cap A^\circ)A$$
$$AAA$$
$$A$$

• p246–248 [2.442, 2.443]:

$$\supset \ = \ \ni/\ni \qquad \cup = \Lambda(\ni'\ni) \qquad \cap = \Lambda(\in'\backslash\ni)$$

$$\supset \Lambda^\circ(1)$$
$$\supset \left(\tfrac{1}{\ni}\right)^\circ$$
$$\supset ((1/\ni) \cap (\ni/1)^\circ)^\circ$$
$$\supset ((1/\ni)^\circ \cap (\ni/1))$$
$$\supset (\ni/1)$$
$$(\ni/\ni)\ni$$
$$\ni$$

$$\ni$$
$$\ni 1$$
$$\ni \Lambda(1)\Lambda^\circ(1)$$
$$((\ni \Lambda(1)\ni)/\ni)\Lambda^\circ(1)$$
$$((\ni 1)/\ni)\Lambda^\circ(1)$$
$$(\ni/\ni)\Lambda^\circ(1)$$
$$\supset \ \Lambda^\circ(1)$$

$$\Lambda(f \cup g)\textstyle\bigcup \ni$$
$$\overline{\overline{}}$$
$$\Lambda(f \cup g)\Lambda(\ni'\ni)\ni$$
$$\overline{\overline{}}$$
$$\Lambda(f \cup g)\ni'\ni$$
$$\overline{\overline{}}$$
$$(f \cup g)\ni$$
$$\overline{\overline{}}$$
$$f\ni \cup\, g\ni$$

$$\Lambda(f \cup g)\textstyle\bigcap \ni$$
$$\tfrac{f\cup g}{\ni'}\;\Lambda(\in'\backslash\ni)\;\ni$$
$$\tfrac{f\cup g}{\ni'}\;(\in'\backslash\ni)$$
$$(((f \cup g)/\ni') \cap (\ni'/(f \cup g))^\circ))(\ni^\circ/(\, \in')^\circ)^\circ$$
$$(\ni'/(f \cup g))^\circ(\in/\ni')^\circ$$
$$((\in/\ni')(\ni'/(f \cup g)))^\circ$$
$$(\, \in/(f \cup g))^\circ$$
$$(\, \in/f \;\cap\; \in/g)^\circ$$
$$(\in f^\circ \cap\, \in g^\circ)^\circ$$
$$f\ni \,\cap\, g\ni$$

- **p250 [2.446 small print]:**

$$0_R = (R/\ni)^\circ\backslash \ni$$

$$0_R$$
$$\overline{\overline{}}$$
$$1\,0_R$$
$$\overline{}$$
$$(R/\ni)(R/\ni)^\circ 0_R$$
$$\overline{\overline{}}$$
$$(R/\ni)(R/\ni)^\circ((R/\ni)^\circ\backslash \ni)$$
$$\overline{}$$
$$(R/\ni)\, \ni$$
$$\overline{}$$
$$R$$

25

$$R \cup S = ((\ni/R) \cap (\ni/S)) \backslash \ni$$

$$\frac{\frac{\overline{\overline{R \cup R}}}{((\ni/R) \cap (\ni/R)) \backslash \ni}}{\frac{1(\ni/R) \backslash \ni}{(R/\ni)(\ni/R)((\ni/R) \backslash \ni)}}$$

$$\frac{(R/\ni)\ni}{R}$$

$$\ell\ell^\circ = 1 = rr^\circ \qquad \ell r^\circ = 0 = r\ell^\circ \qquad R = R_1\ell \cup R_2 r \qquad S = \ell^\circ \cup r^\circ$$

$$\frac{(R_1 \cup R_2) \cap T}{(R_1 1 \cup R_1 0 \cup R_2 0 \cup R_2 1) \cap T}$$

$$\frac{(R_1\ell\ell^\circ \cup R_1\ell r^\circ \cup R_2 r\ell^\circ \cup R_2 rr^\circ) \cap T}{(R_1\ell \cup R_2 r)(\ell^\circ \cup r^\circ) \cap T}$$

$$\frac{RS \cap T}{(R \cap TS^\circ)S}$$

$$\frac{(R \cap TS^\circ)(\ell^\circ \cup r^\circ)}{}$$

$$\frac{(R \cap TS^\circ)\ell^\circ \ \cup \ (R \cap TS^\circ)r^\circ}{(R\ell^\circ \cap TS^\circ\ell^\circ) \ \cup \ (Rr^\circ \cap TS^\circ r^\circ)}$$

$$((R_1\ell \cup R_2 r)\ell^\circ \cap T(\ell \cup r)\ell^\circ) \ \cup \ ((R_1\ell \cup R_2 r)r^\circ \cap T(\ell \cup r)r^\circ)$$

$$((R_1\ell\ell^\circ \cup R_2 r\ell^\circ) \cap T(\ell\ell^\circ \cup \ell^\circ)) \ \cup \ ((R_1\ell r^\circ \cup R_2 rr^\circ) \cap T(\ell r^\circ \cup rr^\circ))$$

$$((R_1 1 \cup R_2 0) \cap T(1 \cup 0)) \ \cup \ ((R_1 0 \cup R_2 1) \cap T(0 \cup 1))$$

$$(R_1 \cap T) \ \cup \ (R_2 \cap T)$$

- **p5n in these amplifications [2.113 small print]:**

$$X = R \qquad A = S \qquad B = T$$

$$R\,S \cap T$$
$$\underset{\overline{\overline{}}}{=}\underset{=}{}\quad\underset{=}{}$$
$$(X \cup A) \cap B$$
$$(A \cup X) \cap (A \cup B)$$
$$A \cup (X \cap (A \cup B))$$
$$= \quad = \quad = \quad =$$
$$S(R \cap ST)$$
$$S(R \cap TS)$$
$$(R \cap TS)S$$

$$R = X \qquad S = A \qquad T = (A \cup B)$$

$$(A \cup X) \cap (A \cup B)$$
$$= \quad = \quad$$
$$SR \cap T$$
$$RS \cap T$$
$$(X \cap ((A \cup B) \cup A) \cup A$$
$$(X \cap (A \cup B)) \cup A$$
$$A \cup (X \cap (A \cup B))$$

- **p6—8 in these amplifications [2.137]:**

$$R = f^{\circ}g$$

$$R\,R^{\circ}R$$
$$= = =$$
$$f^{\circ}g(f^{\circ}g)^{\circ}f^{\circ}g$$
$$f^{\circ}(gg^{\circ})(ff^{\circ})g$$
$$f^{\circ}f\ f^{\circ}g\ g^{\circ}g$$
$$1f^{\circ}g1$$
$$= \quad =$$
$$f^{\circ}g$$
$$R$$

$$\forall_{x,y} \; x \vee (x' \wedge (x \vee y))) \;=\; x \vee y \qquad \forall_x \; x' \wedge x \;=\; 0$$

$$\frac{\dfrac{(x \vee (z' \wedge (z \vee z))) \wedge (z \vee (z' \wedge (z \vee x)))}{(x \vee (z' \wedge z)) \wedge (z \vee x)}}{\dfrac{(x \vee 0) \wedge (z \vee x)}{x}}$$

$$\forall_{x,y} \; (x : y) : x = y \qquad\qquad \forall_{x,y} \; x : (y : x) = y$$

$$\frac{\dfrac{x : (y : x)}{((y : x) : y) : (y : x)}}{y} \qquad\qquad \frac{\dfrac{(x : x) : (z : (x : z))}{(x : x) : x}}{\dfrac{(x : x) : (x : (x : x))}{x}}$$

• **p11 in these amplifications [2.16(16) small print]:**

$$F G^\circ = R \qquad F^\circ F \subset 1 \qquad G^\circ G \subset 1$$

$$\frac{\dfrac{R \, R^\circ R}{F G^\circ (F G^\circ)^\circ F G^\circ}}{\dfrac{F \, G^\circ G \, F^\circ F \, G^\circ}{\dfrac{F \, 1 \, 1 \, G^\circ}{\dfrac{F G^\circ}{R}}}}$$

$$F F^\circ = R R^\circ \qquad F^\circ F \subset 1 \qquad G = R^\circ F$$

$$\frac{\dfrac{G^\circ G}{(R^\circ F)^\circ R^\circ F}}{\dfrac{F^\circ R R^\circ F}{\dfrac{F^\circ F \, F^\circ F}{1}}} \qquad\qquad \frac{\dfrac{F G^\circ}{F (R^\circ F)^\circ}}{\dfrac{F F^\circ R}{\dfrac{R R^\circ R}{R}}}$$

- **p15–16 in these amplifications [2.447]:**

$$T \ni \; \subset \; \ni \qquad\qquad T \; \subset \; \Lambda^\circ(\ni'\ni)\ni'$$

$$\frac{T}{T \cup 1}$$
$$\Lambda(T \cup 1) \ni'$$
$$\Lambda(T \cup 1)1 \ni'$$
$$\Lambda(T \cup 1)\Lambda(\ni'\ni)\Lambda^\circ(\ni'\ni)\ni'$$
$$\Lambda((T \cup 1)\ni)\Lambda^\circ(\ni'\ni)\ni'$$
$$\Lambda(T\ni \cup \ni)\Lambda^\circ(\ni'\ni)\ni'$$
$$\Lambda(\ni)\Lambda^\circ(\ni'\ni)\ni'$$
$$1\Lambda^\circ(\ni'\ni)\ni'$$
$$\Lambda^\circ(\ni'\ni)\ni'$$

$$\frac{T\ni}{\Lambda^\circ(\ni'\ni)\ni'\ni}$$
$$\Lambda^\circ(\ni'\ni)\Lambda(\ni'\ni)\ni$$
$$1\ni$$
$$\ni$$

$$T\ni \; \subset \; \ni$$
$$\Lambda(R \cap \ni)\Lambda^\circ(\ni'\ni)\ni' \; \subset \; \Lambda^\circ(\ni'\ni)\ni'$$

$$T$$
$$\Lambda(T) \ni'$$
$$\Lambda(T)1 \ni'$$
$$\Lambda(T)\Lambda(\ni'\ni)\Lambda^\circ(\ni'\ni)\ni'$$
$$\Lambda(T\ni)\Lambda^\circ(\ni'\ni)\ni'$$
$$\Lambda(T\ni \cap \ni)\Lambda^\circ(\ni'\ni)\ni'$$
$$\Lambda^\circ(\ni'\ni)\ni'$$

$$\int$$