Limit-Free Identification of e and π.

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Let $f : \mathbb{R} \to \mathbb{R}$ be a function not constantly 0 such that for all x, y in the unit interval

$$((1-x+y)fx - fy)^2 \leqq (x-y)^4$$

Then

$$f1/f0 = \mathrm{e}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that for all x, y in the unit interval

$$\left|(gx-gy)(1+x^2)-x+y\right| \leqq (x-y)^2$$

Then

$$4(g1 - g0) = \pi$$

There's some conjuring going on here. One way, of course, to get such results is to impose impossible hypotheses: all sorts of conclusions are then available. (We'll get to a theorem that tells us—in some generality—that there really are f and g as required.)

As every conjurer knows, a little distraction can turn even the simplest of phenomena into magic; the two conditions have been written to distract the viewer. To bring a little transparency to the matter note first that when $x = y$ the conditions are vacuous and when $x \neq y$ you can rewrite to advantage. For f:

$$\left|\frac{fx - fy}{x - y} - fx\right| \leqq |x - y| \quad [1]$$

hence

$$\left|\lim_{y \to x} \frac{fx - fy}{x - y} - fx\right| \leqq \lim_{y \to x} |x - y| = 0$$

thus $f'x = fx$ and there's a constant A such that

$$fx = A\mathrm{e}^x$$

For g we can rewrite to obtain

$$\left|\frac{gx - gy}{x - y} - \frac{1}{1 + x^2}\right| \leqq \frac{|x - y|}{1 + x^2} \leqq |x - y|$$

[1] If $|a - bc| \leq c^2$ then $\frac{|a-bc|}{|c|} \leq \frac{|c^2|}{|c|}$ hence $\left|\frac{a}{c} - b\right| = \left|\frac{a-bc}{c}\right| = \frac{|a-bc|}{|c|} \leq \frac{|c^2|}{|c|} = |c|$.

and we can—in the same manner—conclude

$$gx \quad = \quad A + \arctan x$$

So the conditions on f and g do the trick, But we don't yet have much reason to believe that f and g satisfy the semiquations. So:

Limit-Free Characterization of Lipschitz Derivatives:

For real-valued functions h_0, h_1 *on a non-trivial real interval* I*:*

There exists K *such that* $\left|h_0x - h_0y - (x-y)h_1x\right| \leqq K(x-y)^2$ *for all* x, y

iff

h_1 *is a Lipschitz function and it is the derivative of* h_0.

(For the first example this makes it all very easy since after finding a K we can make it disappear by changing A to K.[1] For the second example, the unique inflexion point of g' in the unit interval is when $x = 1/\sqrt{3}$ and $\left|g''(1/\sqrt{3})\right|$ is easily less than 2.)

One part (as we have seen) of the theorem is very easy. Rewrite the semiquation $|h_0x - h_0y - (x-y)h_1x| \leqq K(x-y)^2$ as:

$$\left|\frac{h_0x - h_0y}{x-y} - h_1x\right| \quad \leqq \quad K|x-y|$$

then:

$$\left|\lim_{y\to x}\frac{h_0x - h_0y}{x-y} - h_1x\right| \quad \leqq \quad \lim_{y\to x} K|x-y| \quad = \quad 0$$

which yields, of course, $h_0' = h_1$.

That h_1 is Lipschitz also has (when found!) a surprisingly easy proof. Rewrite the semiquation as a pair of semiquations (which pair we'll call the **expanded semiquation**):

$$-K(x-y)^2 \quad \leqq \quad h_0x - h_0y - (x-y)h_1x \quad \leqq \quad K(x-y)^2$$

write another pair obtained by flipping the two variables:

$$-K(y-x)^2 \quad \leqq \quad h_0y - h_0x - (y-x)h_1y \quad \leqq \quad K(y-x)^2$$

and then add the two rows to obtain:

$$-2K(x-y)^2 \quad \leqq \quad (x-y)(h_1y - h_1x) \quad \leqq \quad 2K(x-y)^2$$

which—presto!—says that $2K$ works as a Lipschitz number for h_1:

$$|h_1x - h_1y| \quad \leqq \quad 2K|x-y|.$$

For the converse we replace each h_0 with h, each h_1 with h' and take K to be half of a Lipschitz number for h'. The expanded semiquation becomes:

$$-K(x-y)^2 \quad \leqq \quad hx - hy - (x-y)h'x \quad \leqq \quad K(x-y)^2.$$

We'll start with the left half of the expanded semiquation; what we need is

$$0 \quad \leqq \quad K(x-y)^2 + hx - hy - (x-y)h'x.$$

Pick an arbitrary constant $a \in I$ and define a unary function s by

$$st \;=\; K(a-t)^2 + ha - ht - (a-t)h'a.$$

What we need is that s is nonnegative for all t and, since (clearly) $sa = \dot{s}a = 0$, we need to show that a is an absolute minimal point and for that it suffices to show the pair of lemmata $(t \leqq a) \Rightarrow (\dot{s}t \leqq 0)$ and $(a \leqq t) \Rightarrow (0 \leqq \dot{s}t)$. It turns out to be easier to show the single lemma $(t_1 \leqq t_2) \Rightarrow (\dot{s}t_1 \leqq \dot{s}t_2)$. [2]

Thus the left half of the expanded semiquation follows from the lemma:

$$t_1 \leqq t_2 \qquad \Rightarrow \qquad -2K(a-t_1) - h't_1 + h'a \;\leqq\; -2K(a-t_2) - h't_2 + h'a$$

which immediately simplifies to:

$$t_1 \leqq t_2 \qquad \Rightarrow \qquad h't_2 - h't_1 \;\leqq\; 2K(t_2 - t_1)$$

a clear consequence of the choice of K. Each half of the expanded semiquation is equivalent to the other half when h is replaced with its negation, hence the right half follows from:

$$t_1 \leqq t_2 \qquad \Rightarrow \qquad -h't_2 + h't_1 \;\leqq\; 2K(t_2 - t_1)$$

an equally clear consequence. Done. [3]

All of this is just a piece of an ongoing effort to "algebratize" a lot of analysis in the special sense of reducing everything to operators and (universally quantified) equations.[4] We develop a theory with the **special property** that all of its consistent Lipschitz extensions have Archimedean models, [5] and that obviates all sorts of constructability problems and—when there's only one Archimedean model—many issues of computability.

As just one example, starting with x equal to either $\frac{e}{4}$ or $\frac{\pi}{4}$ there is a finite equational theory with the special property and for which there is only one Archimedean model. We obtain the binary expansion for x by iterating the nondeterministic procedure in which the notation $\vdash \cdots \leqq \cdots$ means that $\cdots \leqq \cdots$ is provable just using the rules of equational logic (or—as is equivalent—true in the initial model):

`If` $\vdash\ x \leqq \frac{1}{2}$ `then` `emit` "0"; `replace` x `with` $2x$	$\Vert$	`If` $\vdash\ \frac{1}{2} \leqq x$ `then` `emit` "1"; `replace` x `with` $2x-1$

This procedure is using the fact that the Archimedean value of x will never be $\frac{1}{2}$. [6]

[2] As is the case for any differential function, this condition is equivalent, of course, to the convexity of s.

[3] There's a standard way of making the theorem a corollary of one with no existential quantifier. An assertion about Lipschitz functions is equivalent to an assertion about the special case of "nonexpansive functions." So we could replace "Lipschitz" with "nonexpansive" by replacing K with 1/2. Since h is Lipschitz iff h/K is nonexpansive for some K we obtain the semiquation $2|h_0x - h_0y - (x-y)h_1x| \le K(x-y)^2$ by dividing both hs by K, applying the special case and then multiplying both sides by $2K$.

[4] This special sense of algebra is often called "universal algebra." (Some of us prefer the name "equational theories.") Note that all the $\le$s can be replaced with $=$s by using—for example—the absolute-value operator: $a \le b$ is equivalent to $b - a = |b - a|$.

[5] See 10.5 in `http://www.math.upenn.edu/~pjf/analysis.pdf`

[6] If there's only one Archimedean model and t, t' are terms such that neither t ≤ t' nor t' ≤ t is provable using the equational substitution rules then necessarily t = t' in that unique Archimedean model. My favorite anomaly is when there are terms for which it's an "iff," that is, when t = t' in the Archimedean model precisely when neither ⊢ t ≤ t' nor ⊢ t' ≤ t e.g. add a constant ε and an equation $\varepsilon^2 = 0$ (check out "dual numbers" in Wikipedia): then there's only one possible Archimedean value of $x\varepsilon$ but there are non-Archimedean models for both positive and negative values.

Such is not a problem with either $\frac{e}{4}$ or $\frac{\pi}{4}$ but it's worth pointing out that we can move to the interval $[-1,+1]$ and use "signed binary expansions":

$$\left[\begin{array}{c}\texttt{If}\\ \vdash\ x \leqq 0\\ \texttt{then}\\ \texttt{emit}\ \text{“}-1\text{”};\\ \texttt{replace}\ x\ \texttt{with}\\ 2(x+\frac{1}{2})\end{array}\right] \;\|\; \left[\begin{array}{c}\texttt{If}\\ \vdash\ -\frac{1}{2} \leqq x \leqq +\frac{1}{2}\\ \texttt{then}\\ \texttt{emit}\ \text{“}0\text{”};\\ \texttt{replace}\ x\ \texttt{with}\\ 2x\end{array}\right] \;\|\; \left[\begin{array}{c}\texttt{If}\\ \vdash\ 0 \leqq x\\ \texttt{then}\\ \texttt{emit}\ \text{“}+1\text{”};\\ \texttt{replace}\ x\ \texttt{with}\\ 2(x-\frac{1}{2})\end{array}\right]$$

The special property and the uniqueness of the Archimedean model insures for every x that at least one of the three alternatives is provable. [7][8]

Available at
https://drive.google.com/drive/folders/1rPnHnUrm1yfon0MOMFDA2uOiT8yxmwjE?usp=sharing

[7] Use the uniqueness of the Archimedean model to obtain for any term x that either $\vdash -\frac{1}{2} \leq x$ or $\vdash x \leq 0$ and, dually, either $\vdash 0 \leq x$ or $\vdash x \leq \frac{1}{2}$.

[8] The special property is rare. There is no way of creating a finitely presented equational theory for the entire real numbers that guarantees nontrivial Archimedean quotients for its finite Lipschitz extensions. In case this has not been known before consider the following rather complicated proof in which for each natural number n and n^{th} degree polynomial with integer coefficients P we create an equational theory whose nontrivial Archimedean model would decide whether $P(x_1, x_2, \ldots, x_n) = 0$ has an integral solution. We add a unary function s and $n+1$ constants $p, a_1, a_2, \ldots, a_n$ subject to the equational-theory axioms:

1 $\left|s(x) - s(y) - s(x+p)(x-y)\right| \leq (x-y)^2$;
2 $s(x+2p) = -s(x)$;
3 $1 \leq p \leq 2$ and $s(0) = 0$ and $s(p) = 1$;
4 $s(2pa_i) = 0$ each i;
5 $P(a_1, a_2, \ldots, a_n) = 0$.

Working in an Archimedean model the first line tells us that $s(x+p)$ is the derivative of $s(x)$. The equation $s(y+2p) = -s(y)$ tells us that the second derivative of s is its negation. Line 3 finishes the information for proving that s is the sine function and p is $\frac{1}{2}\pi$. (The derivative of $s^2 + s'^2$ is $2s'(s+s'') = 0$ hence $s^2 + s'^2$ is constant and the difference between s and any function satisfying all the information from the first three lines will be constantly 0.) Finally, the last two lines of equations say that a_i is an integer each i and they provide a solution for the diophantine problem.

If the polynomial equation has no integral solution then the theory is inconsistent and such can be shown by proving $x = y$. (And, of course, when it does have a solution such can be found.)

All of which would say that Hilbert was right when he stated his 10th problem.