Reflecting Invertibility

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May 31, 2018

An old paper of mine, *On the concreteness of certain categories*,[1] contained the theorem that every locally small category has a set-valued sharp functor that reflects invertibility. The suggested construction initially describes the functor's values as classes each with an equivalence relation. An argument is needed to show that for each object the "number of classes is small" followed by a replacement of each class with a subset. (The axiom of choice can be, however, avoided: each equivalence-class is identifiable knowing just its members of minimal foundational rank.)

Herein we give a cleaner construction that avoids these steps, indeed replacing them with (footnoted) sequences of mindless substitutions (easier to read when you use the "subscoring" appendix). We'll say that a functor is **sharp** if it reflects the existence of both left- and right inverses.[2] A quick preview of the sharp set-valued functor we're going to construct is that it sends an object to a set of equivalence-types of its idempotents plus a "bottom" point. The equivalence relation is the one defined by $e \equiv e'$ iff $ee' = e$ and $e'e = e'$. [3]) Given a locally small category we first define a pre-functor [4] into the category composed of relations between sets: for each object A take TA to be the set of idempotents on A (this unique use of well-powering, note, says that we need only the condition that each object has only a set of idempotents); for a map $u : A \to B$ the relation $Tu : TA \to TB$ is defined for $a \in TA,\ b \in TB$ by:

$$a(Tu)b$$
$$\textit{iff there exists a map } \hat{u} : B \to A \textit{ such that}$$
$$a u \hat{u} = a \qquad \& \qquad \hat{u} a u = b.$$

Such describes a pre-functor: given another map $v : B \to C$ it's easy to check, first, that the composition of Tu and Tv is contained in the relation $T(uv)$: that is, given a, b, c such that $a(Tu)b$ and $b(Tv)c$ let $\hat{u}, \hat{v}$ be such that $au\hat{u} = a$, $\hat{u}au = b$, $bv\hat{v} = b$ and $\hat{v}bv = c$. Now simply define $\widehat{uv} = \hat{v}\hat{u}$.[5] For the other direction suppose $a(Tuv)c$, that is, suppose there's a map $\widehat{uv}$ such that $auv(\widehat{uv}) = a$ and $(\widehat{uv})auv = c$. Define $b = v(\widehat{uv})au$, $\hat{u} = v(\widehat{uv})$ and $\hat{v} = (\widehat{uv})au$.[6]

Since T is a pre-functor we know for any identity map 1 that $T1$ is an idempotent relation. In fact it is the equivalence relation defined in the second paragraph.[7]

Given $u : A \to B$ the fact that $1_A u 1_B = u$ says that T may be regarded as a relation between the sets of $\equiv$-classes, that is, if we replace TA with the set of $\equiv$-classes we have not just a pre-functor but a

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[1] *1970 Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69) pp. 431-456 Academic Press, London.* For an online version see footnote number 11 on page 4 of `http://www.math.upenn.edu/~pjf/homotopy.pdf`

[2] Each of which—by itself— implies the reflection of isomorphisms. If T reflects right-invertibility and if Tu is an isomorphism then there's v such that $uv = 1$. But Tv being a right-inverse of an isomorphism must itself have a right inverse, hence there's h such that $vh = 1$. So $vu = vu1 = vuvh = v1h = 1$.

[3] In the category of sets–and using the diagramatic order–this can be restated as e and e' have the same image.

[4] Drop the condition about preserving identity maps.

[5] Then $auv(\widehat{uv}) = a^2uv\hat{v}\hat{u} = (au\hat{u})auv\hat{v}\hat{u} = au(\hat{u}au)v\hat{v}\hat{u} = aubv\hat{v}\hat{u} = aub\hat{u} = au(\hat{u}au)\hat{u} = (au\hat{u})(au\hat{u}) = a^2 = a$ and $(\widehat{uv})auv = \hat{v}\hat{u}auv = \hat{v}bv = c$.

[6] Then $b^2 = v(\widehat{uv})auv(\widehat{uv})au = v(\widehat{uv})a^2u = v(\widehat{uv})au = b$, $au\hat{u} = auv(\widehat{uv}) = a$, $\hat{u}au = v(\widehat{uv})au = b$, $bv\hat{v} = (v(\widehat{uv})au)v((\widehat{uv})u) = v(\widehat{uv})(auv(\widehat{uv}))u = v(\widehat{uv})au = b$ and, finally, $\hat{v}bv = ((\widehat{uv})au)(v(\widehat{uv})au)v = ((\widehat{uv})auv)((\widehat{uv})auv) = c^2 = c$.

[7] Given $\hat{1}$ such that $a1\hat{1} = a$ and $\hat{1}a1 = a'$ we have $aa' = a(\hat{1}a) = (a\hat{1})a = a^2 = a$ and $a'a = (\hat{1}a)a = \hat{1}a^2 = \hat{1}a = a'$; for the other direction, given $a \equiv a'$ take $\hat{1} = a'$ to obtain $a\hat{1}1 = a$ and $\hat{1}a1 = a'$

full-fledged identity-map–preserving functor. And, presto, its values on maps are—if not full-fledged maps between sets—at least partial maps. That is, if $a(Tu)b_1$ and $a(Tu)b_2$ then $b_1 \equiv b_2$.[8][9]

All of which yields a functor into the category composed of partial maps between sets. That category, in turn, has a well-known functor to the ordinary category of sets, to wit, the "lifting" functor, the one that adjoins to each set X a new set denoted $X_\perp$, the result of adjoining a "bottom" $\perp$ to A. Given a partial map $f\colon X \to Y$ we understand:

$$\underset{\perp}{f}x \;=\; \begin{cases} y & \textit{if} \quad fx = y \\ \perp & \textit{if} \quad fx \;\; \textit{is undefined} \\ \perp & \textit{if} \quad x = \perp \end{cases}$$

The verification that this resulting set-valued functor is sharp is as follows. Given $u\colon A \to B$ such that Tu has a right inverse, it suffices to note that necessarily there is $b \in TB$ such that $1(Tu)b$, that is, there is a map $\hat{u} : B \to A$ such that $1u\hat{u} = 1$ and $\hat{u}1u = b$. But, of course, $\hat{u}$ is a right inverse for u. If Tu has a left inverse then there is $a \in TA$ such that $a(Tu)1$, hence a map $\hat{u}$ such that $au\hat{u} = a$ and $\hat{u}au = 1$. But, of course, $\hat{u}a$ is a left inverse for u.[10]

We have not, alas, found a minimal sharp functor. For an example of just how far we are from such take any non-empty set and turn it into a one-object category by choosing an identity element, 1, and defining $xy = \textit{if} \;\; y = 1 \;\; \textit{then} \;\; x \;\; \textit{else} \;\; y$. Then no two elements are equivalent but the functor that collapses all elements save the identity to a point reflects invertibility (that is, the functor that maps the category to the Sierpinski monoid, $\{0, 1\}$ under multiplication).

But we can, at least, replace T with a somewhat smaller functor. We don't need all the idempotents, just those that split. If $u : A \to B$ and $a(Tu)b$, then a splits iff b does: given $a\hat{u}u = a$, $\hat{u}au = b$ and a splitting p_a, i_a for a we obtain a splitting for b by taking $p_b = \hat{u}p_a$ and $i_b = i_a u$: [11] and given a splitting p_b, i_b for b we obtain a splitting p_a, i_a for a by taking $p_a = aup_b$ and $i_a = i_b\hat{u}$.[12]

The smaller functor above can be obtained in just this way (only one of its idempotents splits). But if we take the Karoubi envelope of both categories the resulting functor still reflects both left and right invertibility but is (infinitely) smaller than that produced by the general construction (no two of the idempotents are equivalent)

And—best of all—by restricting to split idempotents we can now recognize the functor constructed here to be equivalent to that described in *On the concreteness of certain categories*. The equivalence-class of a split idempotent may be taken as a definition of a "split subobject." [13]

Appendix: Subscoring

It is said that "subscoring" is short for "substitution underscorings," to wit, a one-column array wherein the underscores indicate the sub-strings to be altered.[14]

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[8] Given $\hat{u}_1, \hat{u}_2 : B \to A$ such that $au\hat{u}_1 = a$, $au\hat{u}_2 = a$, $\hat{u}_1au = b_1$ and $\hat{u}_2au = b_2$ then $b_1b_2 = (\hat{u}_1au)(\hat{u}_2au) = \hat{u}_1(au\hat{u}_2)au = \hat{u}_1a^2u = \hat{u}_1au = b$ and, similarly, $b_2b_1 = b_2$.

[9] We are, of course, using a "Karoubi envelope" of the category of relations. The best-known universal property for Karoubi envelopes is to serve as the reflections into the full subcategory of categories in which all idempotents split: given a functor $u\colon\mathbb{A} \to \mathbb{B}$ where all idempotents split in $\mathbb{B}$, there is—unique up to natural equivalence—a factorization through the Karoubi envelope of A. Less known: in a category composed of pre-functors the lluf subcategory of functors is coreflective and the coreflection is none other than the Karoubi envelope, the coreflector is the forgetful operation from the envelope back to the category and the uniqueness condition does not require anything about up-to-natural-equivalence.

[10] Note that even without the axiom of choice it suffices for fu to be an onto function.

[11] $p_bi_b = \hat{u}p_ai_au = \hat{u}au = b$ and $i_bp_b = i_au\hat{u}p_a = i_ap_ai_au\hat{u}p_a = i_aau\hat{u}p_a = i_aap_a = i_ap_ai_ap_a = 1$.

[12] $p_ai_a = aup_bi_b\hat{u} = aub\hat{u} = au(\hat{u}au)\hat{u} = (au\hat{u})(au\hat{u}) = a^2 = a$ and $i_ap_a = i_b\hat{u}aup_b = i_bbp_b = i_bp_bi_bp_b = !_B^2 = 1_B$.

[13] If $(p_ai_a)(p_bi_b) = p_ai_a$ and $(p_bi_b)(p_ai_a) = p_bi_b$ then i_ap_b and i_bp_a are inverse isomorphisms (because $(i_ap_b)(i_bp_a) = (i_ap_a)(i_ap_b)(i_bp_a) = i_a(p_ai_a)(p_bi_b)p_a = i_a(p_ai_a)p_a = (i_ap_a)(i_ap_a) = 1^2 = 1$ and similarly $(i_ap_b)(i_bp_a) = 1$.)

[14] For other examples check out the end pages of

`www.math.upenn.edu/~pjf/amplifications.pdf`

and

`www.math.upenn.edu/~pjf/analysis.pdf`

[5]:

$$\begin{array}{rcl} au\hat{u} & = & a \\ \hat{u}au & = & b \\ bv\hat{v} & = & b \\ \hat{v}bv & = & c \\ \widehat{uv} & = & \hat{v}\hat{u} \end{array}$$

$$\begin{array}{c} a(uv)(\widehat{uv}) \\ = \\ aauv\hat{v}\hat{u} \\ = \\ au\hat{u}auv\hat{v}\hat{u} \\ = \\ aubv\hat{v}\hat{u} \\ = \\ aub\hat{u} \\ = \\ au\hat{u}\; au\hat{u} \\ = \\ aa \\ = \\ a \end{array} \qquad \begin{array}{c} (\widehat{uv})a(uv) \\ = \\ \hat{v}\hat{u}auv \\ = \\ \hat{v}bv \\ = \\ c \end{array}$$

[6]:

$$\begin{array}{rcl} a & = & auv\widehat{uv} \\ c & = & \widehat{uv}auv \\ b & = & v\widehat{uv}au \\ \hat{u} & = & v\widehat{uv} \\ \hat{v} & = & \widehat{uv}au \end{array}$$

$$\begin{array}{c} bb \\ = \\ v\widehat{uv}auv\widehat{uv}au \\ = \\ v\widehat{uv}aau \\ = \\ v\widehat{uv}au \\ = \\ b \end{array} \qquad \begin{array}{c} \hat{u}au \\ = \\ v\widehat{uv}au \\ = \\ b \end{array}$$

$$\begin{array}{c} au\hat{u} \\ = \\ auv\widehat{uv} \\ = \\ a \end{array} \qquad \begin{array}{c} bv\hat{v} \\ = \\ v\widehat{uv}auv\widehat{uv}u \\ = \\ v\widehat{uv}au \\ = \\ b \end{array}$$

$$\begin{array}{c} \hat{v}\; b\; v \\ = \\ \widehat{uv}auv\; \widehat{uv}auv \\ = \\ cc \\ = \\ c \end{array}$$

[7]:

$$\begin{array}{rcl} a1\hat{1} & = & a \\ \hat{1}a1 & = & a' \end{array} \qquad \begin{array}{rcl} aa' & = & a \\ a'a & = & a' \\ \hat{1} & = & a' \end{array}$$

$$\begin{array}{c} aa' \\ = \\ a\hat{1}a \\ = \\ aa \\ = \\ a \end{array} \quad \begin{array}{c} a'a \\ = \\ \hat{1}aa \\ = \\ \hat{1}a \\ = \\ a' \end{array} \qquad \begin{array}{c} a\hat{1}1 \\ = \\ aa'1 \\ = \\ aa' \\ = \\ a \end{array} \quad \begin{array}{c} \hat{1}a1 \\ = \\ a'a1 \\ = \\ a'a \\ = \\ a' \end{array}$$

[8]:

$$\begin{array}{rcl} a & = & au\hat{u}_1 \\ a & = & au\hat{u}_2 \\ b_1 & = & \hat{u}_1au \\ b_2 & = & \hat{u}_2au \end{array}$$

$$\begin{array}{c} b_1\; b_2 \\ = \\ \hat{u}_1au\hat{u}_2au \\ = \\ \hat{u}_1aau \\ = \\ \hat{u}_1au \\ = \\ b_1 \end{array} \qquad \begin{array}{c} b_2\; b_1 \\ = \\ \hat{u}_2au\hat{u}_1au \\ = \\ \hat{u}_2aau \\ = \\ \hat{u}_2au \\ = \\ b_2 \end{array}$$

continue→

[11, 12]:

$$\begin{array}{rcl} a &=& a\hat{u}u \\ b &=& \hat{u}au \end{array}$$

$$\begin{array}{rcl} a &=& p_a i_a \\ 1_A &=& i_a p_a \\ p_b &=& \hat{u}p_a \\ i_b &=& i_a u \end{array} \qquad \begin{array}{rcl} b &=& p_b i_b \\ 1_B &=& i_b p_b \\ p_a &=& aup_b \\ i_a &=& i_b\hat{u} \end{array}$$

$$\begin{array}{c} i_b\; p_b \\ \hline\hline 1i_au\hat{u}p_a \\ = \\ i_ap_ai_au\hat{u}p_a \\ \hline\hline i_aau\hat{u}p_a \\ \hline\hline i_aap_a \\ = \\ i_ap_a\; i_ap_a \\ \hline\hline 11 \\ \hline\hline 1_A \end{array} \qquad \begin{array}{c} i_a\; p_a \\ \hline\hline i_b\hat{u}aup_b \\ \hline\hline i_bbp_b \\ = \\ i_bp_b\; i_bp_b \\ \hline\hline 11 \\ \hline\hline 1_B \end{array}$$

$$\begin{array}{c} p_b\; i_b \\ \hline\hline \hat{u}p_ai_au \\ \hline\hline \hat{u}au \\ \hline\hline a \end{array} \qquad \begin{array}{c} p_a\; i_a \\ \hline\hline aup_bi_b\hat{u} \\ \hline\hline aub\hat{u} \\ = \\ au\hat{u}\; au\hat{u} \\ \hline\hline aa \\ \hline\hline a \end{array}$$

[13]:

$$\begin{array}{rcl} (p_ai_a)(p_bi_b) &=& p_ai_a \\ (p_bi_b)(p_ai_a) &=& p_bi_b \end{array}$$

$$\begin{array}{c} (i_ap_b)(i_bp_a) \\ \hline\hline 1i_ap_bi_bp_a \\ = \\ i_ap_ai_ap_bi_bp_a \\ \hline\hline i_ap_a\; i_ap_a \\ \hline\hline 11 \\ \hline\hline 1_A \end{array} \qquad \begin{array}{c} (i_bp_a)(i_ap_b) \\ \hline\hline 1i_bp_ai_ap_b \\ = \\ i_ap_bi_bp_ai_ap_b \\ \hline\hline i_bp_b\; i_bp_b \\ \hline\hline 11 \\ \hline\hline 1_B \end{array}$$

Available at `http://www.math.upenn.edu/~pjf/iso-detector.pdf`