FULL COMPLETENESS FOR MODELS OF LINEAR LOGIC

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This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration. The work contained herein is, except where stated otherwise, to the best of my knowledge original.

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Contents

Abstract

This thesis covers two aspects of Linear Logic. The major part is devoted to proving full completeness results for various categorical models of Multiplicative Linear Logic (MLL). A proof net system for Multiplicative Intuitionistic Linear Logic (MILL), including the units, is also presented.

Here, by full completeness, we mean a correspondence between dinatural transformations in the categorical model and canonical morphisms in some free category with a suitable monoidal structure. The theme running through this thesis is that for suitable categorical models of MLL, it is possible to prove full completeness using a parallel result in an underlying compact closed structure. Three of the models of MLL considered in this thesis are derived from a general "double glueing" construction on a compact closed category. The first model is constructed from the category **Rel** of sets and binary relations, the second model from the category **FDVec** of finite dimensional vector spaces and linear maps, the third from a category of Conway games. The fourth model of MLL, another category of vector spaces, is different in nature to the other models – this time, we observe a compact closed subcategory, namely finite dimensional vector spaces. In all cases, we prove full completeness by extending a full completeness result for the underlying compact closed category. (In fact, full completeness in **FDVec** is an already well-known result (Schur-Weyl duality) in Invariant Theory.) In terms of Girard's proof nets, the compact closed category is where we determine the placement of the axiom links for a proof structure associated with each dinatural transformation. Of further interest, a category of coherence spaces exists as a full subcategory of the construction on **Rel**, and a full completeness result for coherence spaces is also proved.

The chapter on MILL proof nets can be viewed as a means for proving full completeness for denotational models of MILL. The system works on a basic input-output scheme. The combinatorial characterisation of an empire of a node will be crucial in resolving the problem with the units, first observed by Kelly and Mac Lane. An attractive feature of this system is the normalisation procedure, where the rewrites are "global", as opposed to "one-step".

Acknowledgements

I could not begin without thanking my supervisor, Martin Hyland. When he first met me, I had lost all self-confidence and did not believe I could write a Ph.D. thesis. Evidently, he has proved me wrong, and that is thanks to his support and his belief in me. Martin's infectious enthusiasm for mathematics has never failed to inspire me and I consider myself very fortunate to have worked under a mathematician of his calibre. I take this opportunity to thank him for his guidance and constant encouragement.

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Continuing along the lines of thesis preparation, I acknowledge those who have generously shared their computing expertise with the mathematical community, thus enabling us to typeset mathematics with considerable ease. This thesis was typeset using Leslie Lamport's LaTeX 2_{ε} document preparation system. All commutative diagrams were drawn using Paul Taylor's diagrams macros. All proof nets were drawn using xfig, whose list of contributors is too long for me to mention here.

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Finally, words cannot express the thanks I owe my mother and father for giving me every opportunity to pursue the goals I have had in life so far. I can at least try by dedicating this thesis to them both, with love.

Introduction

0.1 $*$-Autonomous categories and compact closed categories

It is widely accepted today that the correct categorical model of the multiplicative fragment of Linear Logic is a $*$-autonomous category. However, Barr's study [Bar79] of $*$-autonomous categories long preceded Girard's publication on Linear Logic [Gir87]. Moreover, Chu's appendix to [Bar79] – a construction on an autonomous category from which we derive a $*$-autonomous category – could hardly have been noted as an effective means to construct a model of Linear Logic. The inspiration behind the Chu construction was drawn from topological vector spaces, and we will in fact demonstrate the use of this particular example in Chapter 5.

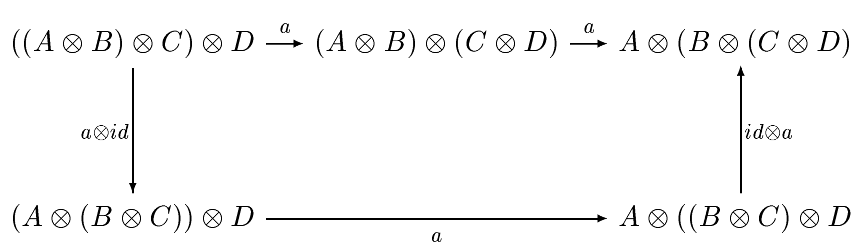
In the meantime, we begin with the definitions which supply us with a $*$-autonomous category. They are widely known, though not always presented in full form, so we provide them here for the sake of completeness.

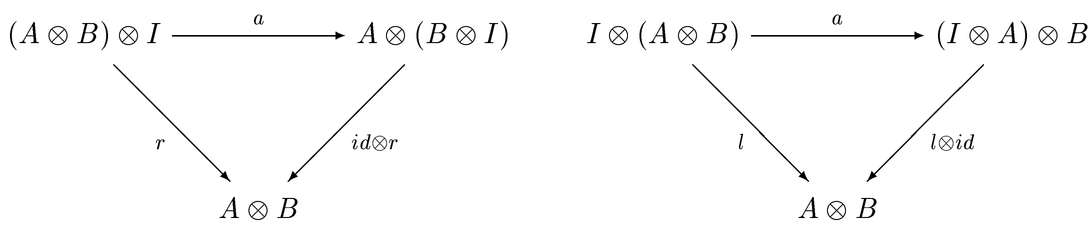
Given a category $\mathbb{C}$, and objects A and B in $\mathbb{C}$, we denote the collection of morphisms from A to B by $\mathbb{C}(A, B)$.

Definition 0.1.1 Let $\mathbb{C}$ be a category. We say that $\mathbb{C}$ is *monoidal* if it is equipped with a bifunctor $- \otimes - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ (called the *tensor product*), an object I in $\mathbb{C}$ (called the *unit*), and isomorphisms

$$\begin{aligned} a_{ABC} : (A \otimes B) \otimes C &\xrightarrow{\simeq} A \otimes (B \otimes C); \\ r_A : A \otimes I &\xrightarrow{\simeq} A; \\ l_A : I \otimes A &\xrightarrow{\simeq} A, \end{aligned}$$

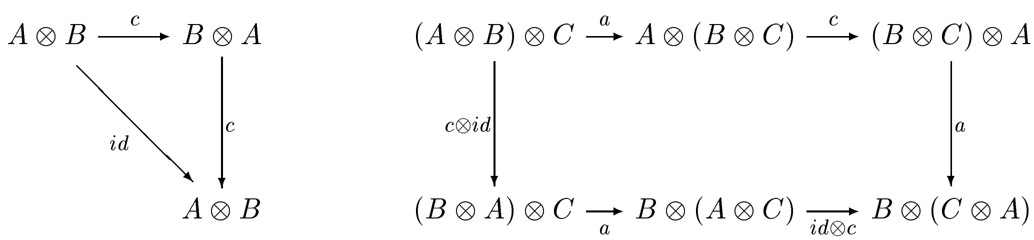
defined and natural over all objects A, B and C in $\mathbb{C}$, such that the following "coherence" diagrams commute.





In essence, the coherence diagrams above ensure that all diagrams, whose arrows are constructed from a, l, r and their inverses by means of the tensor product, commute.

Definition 0.1.2 Let $\mathbb{C} = (\mathbb{C}, \otimes, I)$ be a monoidal category. We say that $\mathbb{C}$ is *symmetric* if there exist isomorphisms $c_{AB} : A \otimes B \to B \otimes A$, defined and natural over all objects A and B in $\mathbb{C}$, such that the following coherence diagrams commute.



Definition 0.1.3 Let $\mathbb{C} = (\mathbb{C}, \otimes, I)$ be a monoidal category. We say that $\mathbb{C}$ is *closed* if there exists a bifunctor $- \multimap - : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$, such that for all objects B in $\mathbb{C}$, the functor $- \otimes B$ has a right adjoint $B \multimap -$, i.e. there is a bijection between $\mathbb{C}(A \otimes B, C)$ and $\mathbb{C}(A, B \multimap C)$, natural in A and C.

Remarks A symmetric monoidal closed (SMC) category is sometimes referred to in the literature as an *autonomous* category. Hence the term $*$-autonomous will appear to be a sensible choice.

Given objects A and B in a SMC category $\mathbb{C}$, we refer to the object $A \multimap B$ as the *internal hom* of A and B.

Definition 0.1.4 Let $\mathbb{C} = (\mathbb{C}, \otimes, I)$ be a symmetric monoidal category. Then $\mathbb{C}$ is a $*$-*autonomous category* if there exists a full and faithful functor $(-)^{\perp} : \mathbb{C}^{op} \to \mathbb{C}$ such that there exist isomorphisms $\mathbb{C}(A \otimes B, C^{\perp}) \to \mathbb{C}(A, (B \otimes C)^{\perp})$, defined and natural for all objects A, B and C in $\mathbb{C}$.

This definition is not Barr's original definition, which contained superfluous data, but the two were immediately proven to be equivalent in [Bar79].

Examples. 1. The category **Rel**, whose objects are sets and whose morphisms are binary relations (i.e. a morphism between A and B is a subset of the cartesian product $A \times B$), is $*$-autonomous. It is evidently symmetric monoidal when we take the tensor product to be the usual cartesian product, and the unit to be the one element set. A set can be regarded as "self-dualising", i.e. we take $A^* = A$. Then a subset of $A \times B$ is in bijective correspondence with a subset of $B^* \times A^* = B \times A$ and vice versa, so we have a full and faithful functor $(-)^* : \mathbf{Rel}^{op} \to \mathbf{Rel}$. Finally, $\mathbf{Rel}(A \times B, C^*) = \mathbf{Rel}(A \times B, C)$ and $\mathbf{Rel}(A, (B \times C)^*) = \mathbf{Rel}(A, B \times C)$ are clearly isomorphic – both are the collection of subsets of $A \times B \times C$.

2. The category $\mathbf{FDVec} = \mathbf{FDVec}_k$, whose objects are finite dimensional vector spaces over a fixed field k and whose morphisms are linear maps, is $*$-autonomous. It is naturally symmetric monoidal when we take the tensor product to be the usual algebraic tensor product and the unit to be k. The algebraic dual (i.e. V^* is the space of linear functionals on V) provides us with a full and faithful functor $(-)^* : \mathbf{FDVec}^{op} \to \mathbf{FDVec}$. In fact, $V \cong V^{**}$ if and only if V is finite dimensional. Other properties of finite-dimensional vector spaces are

$$\mathbf{FDVec}(U \otimes V, W) \cong \mathbf{FDVec}(U, \mathbf{FDVec}(V, W));$$

(in fact this holds in $\mathbf{Vec}_k$, the category of vector spaces over k)

$$\mathbf{FDVec}(V, W) \cong V^* \otimes W;$$

$$V^* \otimes W^* \cong (V \otimes W)^*.$$

From these facts, we can deduce that $\mathbf{FDVec}(U \otimes V, W^*) \cong \mathbf{FDVec}(U, (V \otimes W)^*)$, so $\mathbf{FDVec}$ is $*$-autonomous.

Proposition 0.1.5 *Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^\perp)$ be a $*$-autonomous category. Then $A \cong A^{\perp\perp}$ for all objects A in $\mathbb{C}$.*

Proof We have $\mathbb{C}(A \otimes B, I^\perp) \cong \mathbb{C}(A, (B \otimes I)^\perp) \cong \mathbb{C}(A, B^\perp)$. By symmetry, we have $\mathbb{C}(A \otimes B, I^\perp) \cong \mathbb{C}(B \otimes A, I^\perp) \cong \mathbb{C}(B, A^\perp)$. Thus $\mathbb{C}(A, B^\perp) \cong \mathbb{C}(B, A^\perp)$.

Since $(-)^\perp$ is full and faithful, we have a bijective correspondence between $\mathbb{C}(A, B)$ and $\mathbb{C}(B^\perp, A^\perp)$. So we have a bijective correspondence between $\mathbb{C}(A, B)$ and $\mathbb{C}(A, B^{\perp\perp})$, natural in A. Therefore $B \cong B^{\perp\perp}$. ■

Corollary 0.1.6 *Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^\perp)$ be a $*$-autonomous category. Then $\mathbb{C}(A, B) \cong \mathbb{C}(A, B^{\perp\perp}) \cong \mathbb{C}(B^\perp, A^\perp)$.*

Corollary 0.1.7 *Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^\perp)$ be $*$-autonomous. Then $\mathbb{C}$ is closed (and hence autonomous) with $\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, (B \otimes C^\perp)^\perp)$, i.e. $B \multimap C \cong (B \otimes C^\perp)^\perp$.*

The following theorem provides us with an alternative characterisation of $*$-autonomous categories. It has become the preferred characterisation for its more apparent connection with Linear Logic.

Theorem 0.1.8 *A $*$-autonomous category may be equivalently specified as a symmetric monoidal closed category $(\mathbb{C}, \otimes, I, \multimap)$ with an object $\perp$ (called the* dualising object*) such that the canonical morphism $A \to (A \multimap \perp) \multimap \perp$ is an isomorphism, for all objects A in $\mathbb{C}$.*

Proof Suppose that $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^\perp)$ is $*$-autonomous in the sense of Definition 0.1.4. Define $\perp = I^\perp$. By Corollary 0.1.7,

$$A \multimap \perp \cong (A \otimes I^{\perp\perp})^\perp \cong A^\perp,$$

so $(A \multimap \perp) \multimap \perp \cong A^{\perp\perp}$. By Proposition 0.1.5, we know that $A \to A^{\perp\perp}$ is an isomorphism.

Conversely, suppose that $\mathbb{C} = (\mathbb{C}, \otimes, I, \multimap)$ is symmetric monoidal closed and that we have a dualising object. Define $A^\perp = A \multimap \perp$. Given a morphism $f : A \to B$, we can construct a morphism $f^\perp : B^\perp \to A^\perp$ via

$$\frac{\dfrac{A \xrightarrow{f} B \to (B \multimap \perp) \multimap \perp}{A \otimes (B \multimap \perp) \to \perp}}{B \multimap \perp \to A \multimap \perp}$$

and since $B \to B^{\perp\perp}$ is an isomorphism, this procedure is entirely reversible. Therefore we have a full and faithful functor $(-)^\perp : \mathbb{C}^{op} \to \mathbb{C}$. (Identities and composition are preserved since everything in the above procedure is natural or canonical.) Finally,

$$\begin{aligned} & \mathbb{C}(A \otimes B, C^\perp) \\ = \; & \mathbb{C}(A \otimes B, C \multimap \perp) \\ \cong \; & \mathbb{C}(A \otimes B \otimes C, \perp) \\ \cong \; & \mathbb{C}(A, (B \otimes C) \multimap \perp) \\ = \; & \mathbb{C}(A, (B \otimes C)^\perp). \end{aligned}$$

Therefore $\mathbb{C}$ is $*$-autonomous in the sense of Definition 0.1.4. ■

Examples (ctd). 3. The category $\mathbb{L}$ of complete sup semilattices (i.e. objects are complete lattices and morphisms are order and sup preserving maps) is $*$-autonomous. Given objects X and Y, the hom-set $\mathbb{L}(X, Y)$ is itself an object of $\mathbb{L}$ with pointwise sup. Let $\mathbf{2}$ be the two element lattice, $0 < 1$. Then any $f_\alpha : X \to \mathbf{2}$ satisfies

$$f_\alpha(x) \;=\; \begin{cases} 0 & \text{if } x \leq x_\alpha \\ 1 & \text{otherwise} \end{cases}$$

for some unique element x_α in X (namely $x_\alpha = sup\{x \mid f_\alpha(x) = 0\}$). Thus there is a bijection between elements x_α in X and elements f_α in $\mathbb{L}(X, \mathbf{2})$. Furthermore, if $x_\alpha \leq x_\beta$ then $f_\beta \leq f_\alpha$. Therefore, $\mathbb{L}(X, \mathbf{2}) = X^{op}$, the lattice with the same elements as X and the order reversed. It follows that $\mathbb{L}$ is $*$-autonomous with (autonomous structure induced by the hom functor and) dualising object $\mathbf{2}$.

It can be shown [Mac71, p.138] that $\mathbb{L}$ is the Eilenberg-Moore category of $\mathcal{P}$-algebras of the monad $\langle \mathcal{P}, \eta, \mu \rangle$ where $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ is the covariant power set functor, each $\eta_A : A \to \mathcal{P}(A)$ sends every element $a \in A$ to the singleton subset $\{a\}$, and each $\mu_A : \mathcal{P}^2(A) \to \mathcal{P}(A)$ sends every set of subsets of A to its union. This fact will be of interest to us later.

Also in anticipation of our work on Linear Logic, we introduce the so-called "par" product.

Theorem 0.1.9 *Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^\perp)$ be a $*$-autonomous category. Then there exists another bifunctor $- \parr - : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, called the par product, with unit $\perp$, which is dual to the tensor product in the sense of de Morgan. That is, there exist isomorphisms*

$$(A \otimes B)^\perp \to A^\perp \parr B^\perp \tag{0.1}$$

or equivalently

$$(A \parr B)^\perp \to A^\perp \otimes B^\perp, \tag{0.2}$$

defined and natural over all objects A and B in $\mathbb{C}$.

Proof Define $A \parr B = (A^\perp \otimes B^\perp)^\perp$. This defines a bifunctor, since it is the composition

$$\mathbb{C} \times \mathbb{C} \hookrightarrow (\mathbb{C}^{op})^{op} \times (\mathbb{C}^{op})^{op} \xrightarrow{\otimes} \mathbb{C}^{op} \xrightarrow{(\ )^\perp} \mathbb{C}.$$

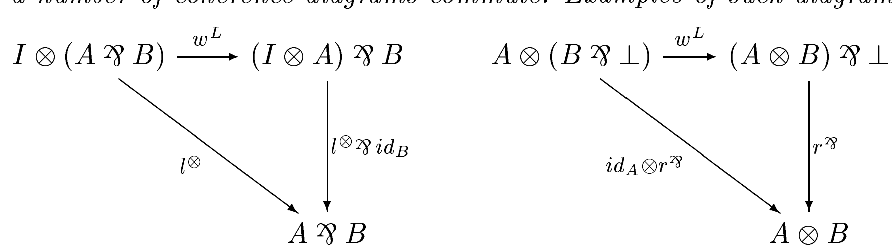
Equations (0.1) and (0.2) follow immediately. Furthermore, $A \parr \perp = (A^\perp \otimes I^{\perp\perp})^\perp \cong A$. Symmetry follows from symmetry of the tensor product, so $\perp$ is the unit for par. ■

Corollary 0.1.10 *Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^\perp)$ be a $*$-autonomous category. Then $A \parr B \cong A^\perp \multimap B$ for all objects A, B in $\mathbb{C}$.*

Theorem 0.1.11 *Let $\mathbb{C}$ be a $*$-autonomous category. Then $\mathbb{C}$ possesses* weak distributivity *natural transformations*

$$w^L_{ABC} : A \otimes (B \parr C) \to (A \otimes B) \parr C$$
$$w^R_{ABC} : A \otimes (B \parr C) \to (A \otimes C) \parr B$$

such that a number of coherence diagrams commute. Examples of such diagrams are



$$
\begin{array}{ccccc}
(A \otimes B) \otimes (C \parr D) & \xrightarrow{a^{\otimes}} & A \otimes (B \otimes (C \parr D)) & \xrightarrow{id_A \otimes w^L} & A \otimes ((B \otimes C) \parr D) \\
\Big\downarrow{\scriptstyle w^L} & & & & \Big\downarrow{\scriptstyle w^L} \\
((A \otimes B) \otimes C) \parr D & & \xrightarrow[a^{\otimes} \parr id_D]{\qquad\qquad\qquad} & & (A \otimes (B \otimes C)) \parr D
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes ((B \parr C) \parr D) & \xrightarrow{id_A \otimes a^{\parr}} & A \otimes (B \parr (C \parr D)) \\
\Big\downarrow{\scriptstyle w^L} & & \Big\downarrow{\scriptstyle w^R} \\
(A \otimes (B \parr C)) \parr D & & (A \otimes (C \parr D)) \parr B \\
\Big\downarrow{\scriptstyle w^R \parr id_D} & & \Big\downarrow{\scriptstyle w^L \parr id_B} \\
((A \otimes C) \parr B) \parr D & & ((A \otimes C) \parr D) \parr B \\
\Big\downarrow{\scriptstyle a^{\parr}} & & \Big\downarrow{\scriptstyle a^{\parr}} \\
(A \otimes C) \parr (B \parr D) & \xrightarrow[id_{A \otimes C} \parr c^{\parr}]{} & (A \otimes C) \parr (D \parr B)
\end{array}
$$

where $a^{\otimes}$ and $a^{\parr}$ are the associativity isomorphisms for tensor and par, respectively, $l_A^{\otimes} : I \otimes A \to A$ is the left-hand unit isomorphism for tensor, $r_A^{\parr} : A \parr \bot \to A$ is the right-hand unit isomorphism for par, and $c^{\parr}$ is the symmetry isomorphism for par. The other diagrams are similar to the above, and obtained by symmetry.

A discussion on the notion of a weakly distributive category can be found in [CS97a]. It comprises of two sets of associativity and unit morphisms, so that both tensor and par are monoidal, together with two weak distribution natural transformation subject to a number of coherence equations. In the non-symmetric case, the second weak distributive morphism is written $(A \otimes B) \parr C \to B \parr (A \otimes C)$, but since both tensor and par are symmetric here, we can make our modification without concern. The theorem of interest to us here is

Theorem 0.1.12 *(Cockett and Seely) The notions of symmetric weakly distributive categories with negation and $*$-autonomous categories coincide.*

The term "with negation" means the addition of two families of maps

$$\eta_A : I \to A^{\bot} \parr A; \quad \epsilon_A : A \otimes A^{\bot} \to \bot,$$

subject to a number of coherence equations, for example,

$$\begin{array}{ccccc}
A & \xrightarrow{(r^{\otimes})^{-1}} & A \otimes I & \xrightarrow{\eta_A} & A \otimes (A^{\perp} \parr A) \\
{\scriptstyle id}\downarrow & & & & \downarrow{\scriptstyle w^L} \\
A & \xleftarrow[r^{\parr}]{} & \perp \parr A & \xleftarrow[\epsilon_A]{} & (A \otimes A^{\perp}) \parr A
\end{array}
\qquad
\begin{array}{ccccc}
A^{\perp} & \xrightarrow{(l^{\otimes})^{-1}} & I \otimes A^{\perp} & \xrightarrow{\eta_A} & (A^{\perp} \parr A) \otimes A^{\perp} \\
{\scriptstyle id}\downarrow & & & & \downarrow{\scriptstyle w^R} \\
A^{\perp} & \xleftarrow[l^{\parr}]{} & A^{\perp} \parr \perp & \xleftarrow[\epsilon_A]{} & A^{\perp} \parr (A \otimes A^{\perp})
\end{array}$$

and similar diagrams obtained by symmetries. We will derive w^L for some inkling of justification of Theorem 0.1.11. Applying $(-)^{\perp}$ to

$$\frac{\mathbb{C}(A \otimes B, C)}{\mathbb{C}(A, B \multimap C)}$$

gives

$$\frac{\mathbb{C}(C^{\perp}, (A \otimes B)^{\perp})}{\mathbb{C}(B \otimes C^{\perp}, A^{\perp})}\,.$$

So we obtain canonical morphisms

$$\frac{(A \otimes B)^{\perp} \xrightarrow{\simeq} (B \otimes A)^{\perp}}{A \otimes (A \otimes B)^{\perp} \to B^{\perp}}$$

and

$$\frac{C^{\perp} \otimes B^{\perp} \xrightarrow{\simeq} (B^{\perp} \otimes C^{\perp})^{\perp\perp}}{B^{\perp} \to (B^{\perp} \multimap C) \multimap C}\,.$$

and hence the canonical morphism $A \otimes (A \otimes B)^{\perp} \to (B^{\perp} \multimap C) \multimap C$. We also have the bijection

$$\frac{A \otimes X \to Y \multimap C}{A \otimes Y \to X \multimap C}\,.$$

Therefore, we have

$$\frac{A \otimes (A \otimes B)^{\perp} \to (B^{\perp} \multimap C) \multimap C}{A \otimes (B^{\perp} \multimap C) \to (A \otimes B)^{\perp} \multimap C}$$

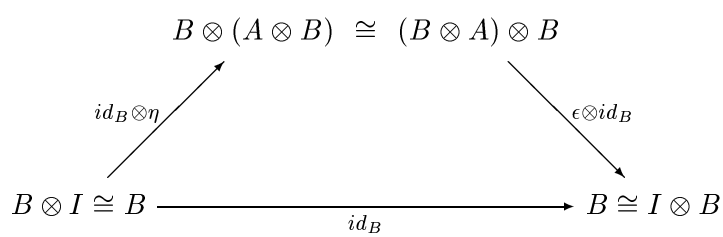
or $w^L_{ABC} : A \otimes (B \parr C) \to (A \otimes B) \parr C$, as required.

0.1.1 Compact closed categories

The definition of a compact closed category provided here is based on Kelly and Laplaza [KL80].

Definition 0.1.13 Let $\mathbb{C} = (\mathbb{C}, \otimes, I)$ be monoidal category. We say that (B, η, ϵ) is a *left adjoint of the object* A and write $B \dashv A$ if there exists a *unit* $\eta : I \to A \otimes B$ and a *counit* $\epsilon : B \otimes A \to I$ such that the following triangles commute.

$$\begin{array}{ccccc}
 & & (A \otimes B) \otimes A \;\cong\; A \otimes (B \otimes A) & & \\
 & {\scriptstyle \eta \otimes id_A}\nearrow & & \searrow{\scriptstyle id_A \otimes \epsilon} & \\
I \otimes A \cong A & & \xrightarrow[id_A]{\hspace{6em}} & & A \cong A \otimes I
\end{array}$$



Note that adjoints are unique up to isomorphism.

Definition 0.1.14 Let $\mathbb{C} = (\mathbb{C}, \otimes, I)$ be a symmetric monoidal category. We say that $\mathbb{C}$ is *compact closed* when each object A in $\mathbb{C}$ has a left adjoint, written $(A^*, \eta_A, \epsilon_A)$.

In particular, the unit and counit induce natural isomorphisms $\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, C \otimes B^*)$ and $\mathbb{C}(A, B) \cong \mathbb{C}(B^*, A^*)$. For example, given $f : A \to B$, we form $f^* : B^* \to A^*$ via

$$B^* \cong B^* \otimes I \xrightarrow{id_{B^*} \otimes \eta_A} B^* \otimes (A \otimes A^*) \xrightarrow{id_{B^*} \otimes f \otimes id_{A^*}} (B^* \otimes B) \otimes A^* \xrightarrow{\epsilon_B \otimes id_{A^*}} I \otimes A^* \cong A^*,$$

etc. So whilst there is no explicit mention of closure in the above definition, it is inherent, with $B \multimap C = C \otimes B^*$.

Examples 1. The category **Rel** is compact closed. As noted before, it is symmetric monoidal with cartesian product and unit the one element set. A set A is left adjoint to itself. Both the unit and counit of the adjunction can be viewed as subsets of $A \times A$, and in particular they are the identity relation on A, i.e. the set of ordered pairs $\{(a, a) \mid a \in A\}$.

Moreover, it can be shown that **Rel** is the Kleisli category of the same monad $\langle \mathcal{P}, \eta, \mu \rangle$ whose category of $\mathcal{P}$-algebras is the category $\mathbb{L}$ of complete sup semilattices. (See Example 3 of $*$-autonomous categories above.) So **Rel** is a compact closed full subcategory of $\mathbb{L}$.

2. The category **FDVec** is compact closed. Given a finite dimensional vector space V, with basis $\{e_i\}_{i=1}^n$ and dual basis $\{\varepsilon_j\}_{j=1}^n$ for V^* (i.e. $\varepsilon_j(e_i) = 1$ if $i = j$, 0 otherwise), we construct the unit $\eta_V : k \to V \otimes V^*$ via

$$\eta_V(1) = \sum_{i=1}^{n} e_i \otimes \varepsilon_i.$$

The counit $\epsilon_V : V^* \otimes V \to k$ is given by the evaluation map, i.e. generated by

$$\epsilon_V(\theta \otimes v) = \theta(v).$$

The fact that these two examples also happen to be $*$-autonomous categories is no coincidence. We now show that compact closed categories are a special class of $*$-autonomous category.

Lemma 0.1.15 $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^*)$ *is compact closed if and only if* $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^*)$ *is* $*$*-autonomous and* $(B \otimes C)^* \cong B^* \otimes C^*$ *for all objects* B *and* C *in* $\mathbb{C}$.

Proof Suppose that $\mathbb{C}$ is compact closed, with $A^* \dashv A$ for all objects A in $\mathbb{C}$. As remarked above, we have $\mathbb{C}(A, B) \cong \mathbb{C}(B^*, A^*)$ so this induces a full and faithful functor $(-)^* : \mathbb{C}^{op} \to \mathbb{C}$. Using the units and counits associated with B^* and C^*, we can show that $B^* \otimes C^*$ is a left adjoint of $B \otimes C$, and therefore $(B \otimes C)^* \cong B^* \otimes C^*$. So $\mathbb{C}$ is $*$-autonomous.

Conversely, suppose that $\mathbb{C}$ is $*$-autonomous with $(B \otimes C)^* \cong B^* \otimes C^*$ for all objects B and C in $\mathbb{C}$. Then manipulation of the identity morphism on A via

$$\cfrac{\cfrac{\mathbb{C}(A, A) \cong \mathbb{C}(I \otimes A, A)}{\mathbb{C}(I, (A \otimes A^*)^*) \cong \mathbb{C}(I, A^* \otimes A^{**})}}{\mathbb{C}(I, A^* \otimes A) \cong \mathbb{C}(I, A \otimes A^*)}$$

and similar manipulation on the identity morphism on A^* yield a unit and counit for the adjunction $A^* \dashv A$. Therefore $\mathbb{C}$ is compact closed. ■

Corollary 0.1.16 *Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^*)$ be a compact closed category. Then $A \parr B \cong A \otimes B$ for all objects A, B in $\mathbb{C}$.*

Corollary 0.1.17 *Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^*)$ be a compact closed category. Then $I^* \cong I$.*

Remark While **Rel** and **FDVec** can be seen directly to be compact closed, it is worth noting that they both satisfy the equivalence $(B \otimes C)^\perp \cong B^\perp \otimes C^\perp$. The category of complete sup semilattices does not satisfy this equivalence, although it does satisfy $I^\perp \cong I$. (Recall that the dualising object is the two element lattice.) See [Bar79]. This demonstrates that $I^\perp \cong I$ is not sufficient for compact closure.

0.2 Multiplicative Linear Logic

Girard intended Linear Logic [Gir87] to be used to study more carefully the properties of classical and intuitionistic logic, but it has since gained interest as a logic in its own right. The easiest way to gain insight into Linear Logic is to observe the connective $\multimap$, known as *linear implication*. The formula $A \multimap B$ is interpreted as applying the premise A precisely once to obtain B. To maintain the power of intuitionistic logic, an *exponential* ! is introduced, where $!A$ is interpreted as storage of A, i.e. we can apply A as many times as we require. Thus, the usual implication $\Rightarrow$ is expanded as

$$A \Rightarrow B = (!A) \multimap B.$$

For this reason, Linear Logic can be thought of as a resource-conscious logic.

The fragment of interest to us in this thesis is the multiplicative fragment without exponentials, which we will call Multiplicative Linear Logic (MLL). The sequent calculus shall be stated in terms of the multiplicative connectives, $\otimes$ and $\parr$. The conjunction $\otimes$ ("times" or "tensor") is denoted such because it behaves like a tensor in algebra. The formula $A \otimes B$ expresses the use of both A and B. The connective $\parr$ ("par") is more subtle. It is often referred to as the "dual tensor", because we can define the operation *linear negation* via

$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}; \quad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp},$$

and various other duality identities. The formula $A \parr B$ is perhaps best expressed as $A^{\perp} \multimap B = B^{\perp} \multimap A$.

The full sequent calculus for Linear Logic is stated clearly in [Gir95]. We provide the multiplicative fragment only in Figure 0.1. Formulae in MLL, denoted by $A, B, \ldots,$

$$\frac{}{\vdash \alpha^{\perp}, \alpha}\,(Id)$$

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,(Exchange)$$

$$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(Cut)$$

$$\frac{}{\vdash \mathbf{1}}\,(\mathbf{1}) \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,(\perp)$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,(\otimes) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(\parr)$$

Figure 0.1: Sequent calculus for MLL

are built from *propositional atoms*, $\alpha, \beta, \ldots,$ their linear negations, $\alpha^{\perp}, \beta^{\perp}, \ldots$ and the constants (or *units*) $\mathbf{1}$ and $\perp$ by the connectives $\otimes$ and $\parr$. Finite sequences of formulae are denoted by Greek capitals, $\Gamma, \Delta, \ldots$. A *literal* is either a propositional atom or the negation of a propositional atom.

Our format is one-sided, since this suits best our purposes. A two-sided format can be derived by rewriting $\vdash A, \Gamma$ as $A^{\perp} \vdash \Gamma$ etc. In defining linear negation and linear implication, we assert that

$$\alpha^{\perp\perp} = \alpha;$$
$$\mathbf{1}^{\perp} = \perp; \qquad\qquad \perp^{\perp} = \mathbf{1};$$
$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}; \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp};$$
$$A \multimap B = A^{\perp} \parr B.$$

The Exchange rule is of no importance to this discussion – it merely states that permuting the order of the premises does not affect the conclusion. We hereby remove the

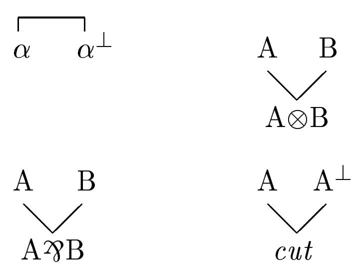
Exchange rule and assume that $\Gamma, \Delta, \ldots$ denote finite *multisets* of formulae, instead of finite sequences of formulae.

There is a cut elimination procedure which is confluent and strongly normalising. In particular, any proof in MLL can be reduced to a unique cut-free proof of the same sequent, and the (Cut) rule is thus made redundant.

0.2.1 Proof Nets

Girard's proof nets [Gir87] are graphical representations of proofs in the sequent calculus of MLL (without units). Proof nets were defined through the intermediary notion of a proof structure, satisfying a soundness condition which was to be simplified later by Danos and Regnier [DR89]. An important result about proof nets is that there is a cut elimination procedure which is confluent and strongly normalising. Thus an attractive feature about proof nets is that sequent calculus proofs which are deemed equivalent by so-called "commuting conversions" are translated to proof nets with identical form.

We begin with the definition of a proof structure. A *proof structure* is a graph with formula occurrences as vertices. The edges are built via links of the form



The link in the top left-hand corner is called an *axiom link*, and similarly *tensor, par* and *cut links* correspond to the obvious links.

Intuitively, one can mimic the formation of a proof in MLL by starting with a collection of axiom links (corresponding to identity axioms) and building a graph using the tensor, par and cut links (corresponding to the respective rules in MLL).

Girard's notion of a "trip" is the visualisation of information flowing through the proof structure. There are various ways which an information particle could flow, each possibility indicated by a "switching". Each switching gives rise to either a *long trip* (every vertex is visited twice) or a *short trip* (when the long trip condition fails). A *proof net* is a proof structure which admits no short trip. Girard proved :

Theorem 0.2.1 *(Girard) If π is a proof of $\vdash \Gamma$ in the sequent calculus for MLL, then we can naturally associate with it a proof net $N(\pi)$ whose multiset of terminal formulae is precisely the multiset Γ.*

Theorem 0.2.2 *(Girard) If β is a proof net, then there exists a proof π in the sequent calculus of MLL such that $N(\pi) = \beta$.*

As mentioned above, there is a cut elimination procedure for proof nets which is confluent and strongly normalising. Thus, every proof net has a unique cut-free normal form.

It is clear that a cut-free proof structure associated with a multiset Γ is dependent only on its axiom links. The nodes at the top of the graph are simply the literals that make up Γ, and we build the lower part of the graph in the only way possible to obtain terminal nodes which correspond to the formulae in Γ. Thus the only choice we have in determining the proof structure is in pairing up the literals to create the axiom links. This observation will be crucial in proving full completeness results.

Definitions 0.2.3 A sequent Γ is *balanced* if each propositional atom α occurs the same number of times as does its linear negation $\alpha^\perp$.

A balanced sequent Γ is *binary* if every propositional atom α in Γ and its linear negation $\alpha^\perp$ each occur precisely once.

The *length* of the multiset Γ is the number of occurrences of literals in Γ.

If Γ has length p, then we can speak of each literal occurring in a precise position, numbered 1 to p. If Γ is balanced, and consequently p is even, then we can specify the axiom links of a cut-free proof structure associated with Γ by a map $\phi : \{1, \ldots, p\} \to \{1, \ldots, p\}$ such that

- $\phi(i) \neq i$ for all i;
- if a propositional atom α occurs in position i, then there is an occurrence of $\alpha^\perp$ in position $\phi(i)$;
- $\phi^2(i) = i$ for all i.

Thus a cut-free proof structure can be specified as (Γ, ϕ), where Γ is a balanced multiset with length p and ϕ is a fixed-point-free involution on $\{1, \ldots, p\}$ indicating the placement of the axiom links.

Danos-Regnier condition

Danos and Regnier [DR89] simplified the correctness criterion for a proof net, avoiding altogether the notion of a trip.

A *DR-switching* for a cut-free proof structure (Γ, ϕ) is the assignment of [left] or [right] to each par link. In effect, we are selecting one of the two "par-edges" that make up a par link, for every par link.

Given a DR-switching S for (Γ, ϕ), we define the associated *DR-graph* $DR(\Gamma, \phi, S)$ to be the deletion of the par-edges *not* selected by the switching S.

Danos and Regnier proved the following equivalent (and our preferred) correctness criterion for proof nets.

Theorem 0.2.4 *(Danos and Regnier) A cut-free proof structure (Γ, ϕ) is a proof net if and only if for every DR-switching, the associated DR-graph $DR(\Gamma, \phi, S)$ is acyclic and connected.*

Before proceeding, we remark that since the (Cut) rule can be eliminated from proofs in MLL, there is no need to consider proof nets with cut links. We henceforth assume that all MLL proof nets are *cut-free*.

0.2.2 The Mix rule

An interesting variation to pure MLL is the addition of the Mix rule,

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\,(Mix).$$

The Mix rule can be specified in a number of equivalent forms, two of which are presented in the following proposition.

Proposition 0.2.5 *Equivalent forms of the Mix rule are $\bot \vdash \mathbf{1}$ and $A \otimes B \vdash A \parr B$.*

Proof Proof of the equivalence with $\bot \vdash \mathbf{1}$ can be found in [CS97b]. The Mix rule also implies

$$\cfrac{\cfrac{\cfrac{\vdash A^\perp, A \quad \vdash B^\perp, B}{\vdash A^\perp, B^\perp, A, B}\,(Mix)}{\vdash A^\perp \parr B^\perp, A \parr B}\,(\parr)[\text{twice}]}{A \otimes B \vdash A \parr B}\,.$$

Conversely, $A \otimes B \vdash A \parr B$ implies $\bot \vdash \mathbf{1}$ by substituting $A = \mathbf{1}$ and $B = \bot$. ∎

The addition of the Mix rule is not particularly desirable from a logical point of view – the resulting calculus does not appear as resource conscious as pure MLL. Yet it appears that many categorical models of MLL are in fact models of MLL+Mix. We shall discuss later what it means for a categorical model to support the Mix rule.

Fleury and Retoré [FR94] found an appropriate notion of proof net for MLL+Mix. Simply, the acyclicity condition remains but the connectedness condition is omitted. This ties in nicely with the intuitive idea that a MLL proof net associated with $\vdash \Gamma$ and a MLL proof net associated with $\vdash \Delta$ should be able to be juxtaposed to produce a MLL+Mix proof net associated with $\vdash \Gamma, \Delta$.

0.2.3 Categorical Interpretation of MLL

It is not difficult to see that a $*$-autonomous category forms a categorical model of MLL. In essence, tensor and par correspond with their namesakes, linear negation corresponds with the involution on $\mathbb{C}$, and a multiset Γ is interpreted as a par product $\parr\Gamma$. A sequent $\vdash \Gamma$ is interpreted as a morphism $I \to \parr\Gamma$ in the category.

More formally, assume that propositional atoms in the logic are interpreted as objects in a $*$-autonomous category $\mathbb{C}$. The *interpretation* function $[\![\ ]\!]$ sending formulae in the logic to objects in $\mathbb{C}$ is defined inductively, via

$$[\![\mathbf{1}]\!] = I; \quad [\![\bot]\!] = \bot; [\![\alpha^\bot]\!] = [\![\alpha]\!]^\bot;$$
$$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!];$$
$$[\![A, B]\!] = [\![A \parr B]\!] = [\![A]\!] \parr [\![B]\!].$$

Thus it is convenient to identify formulae in MLL with objects in $\mathbb{C}$ using the same labelling.

The identity sequents $\vdash \alpha^\bot, \alpha$ are interpreted as morphisms $I \to X^\bot \parr X$, obtained canonically from the identity morphisms in $\mathbb{C}$, via

$$\frac{\mathbb{C}(X, X) \cong \mathbb{C}(I \otimes X, X)}{\mathbb{C}(I, X \multimap X) \cong \mathbb{C}(I, X^\bot \parr X)}.$$

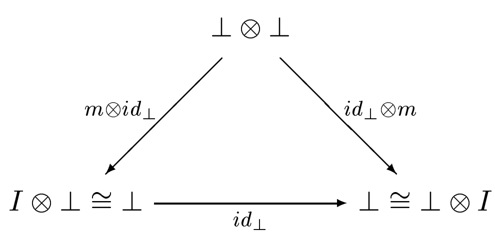
The (Cut) rule is expressed through composition of morphism. A morphism $I \to (\parr\Gamma) \parr A$ is equivalent to a morphism $(\parr\Gamma)^\bot \to A$ and a morphism $I \to A^\bot \parr (\parr\Delta)$ is equivalent to a morphism $A \to (\parr\Delta)$. Composing these two morphisms gives a morphism $(\parr\Gamma)^\bot \to (\parr\Delta)$, equivalent to $I \to (\parr\Gamma) \parr (\parr\Delta) = \parr(\Gamma, \Delta)$.

The (Exchange) rule is expressed through the symmetry and associativity of the par product in $\mathbb{C}$. The ($\mathbf{1}$) rule is simply the identity morphism on I, while the ($\bot$) rule is simply composition with the (reverse) unit isomorphism for par.

The ($\otimes$) rule is essentially taking the tensor product of two morphisms $(\parr\Gamma)^\bot \to A$ and $(\parr\Delta)^\bot \to B$. By our interpretation of multisets, the ($\parr$) rule does nothing.

0.2.4 Categorical Interpretation of Mix

Motivated by [CS97b], we will say that a categorical model of MLL *supports the Mix rule*, if there exists a *unary Mix morphism* $m : \bot \to I$ such that the following diagram commutes.



Write $\tilde{m} : \bot \otimes \bot \to \bot$ for the morphism described above. Then given morphisms $f : X \to \bot$ and $g : Y \to \bot$, we can construct a morphism $X \otimes Y \xrightarrow{f \otimes g} \bot \otimes \bot \xrightarrow{\tilde{m}} \bot$. Furthermore, m induces the *binary Mix morphisms* $z_{AB} : A \otimes B \to A \parr B$, natural in A and B, via categorical interpretation of the "binary" Mix rule :

$$\cfrac{\cfrac{\cfrac{\vdash A^{\bot}, A \quad \vdash B^{\bot}, B}{\vdash A^{\bot}, B^{\bot}, A, B}\,(Mix)}{\vdash A^{\bot} \parr B^{\bot}, A \parr B}\,(\parr)[\text{twice}]}{A \otimes B \vdash A \parr B}\,.$$

0.3 Coherence and free monoidal categories

The study of providing a description of morphisms in a specified free category is generally known as a "coherence" problem. Here, we will discuss the free compact closed category and the free $*$-autonomous category on a set of objects. We search for a minimal collection of "central" morphisms such that all diagrams built from the arrows of central morphisms commute.

0.3.1 The free compact closed category

Kelly and Laplaza [KL80] provided a description of the free compact closed category on a given category. For our purposes here, we need only describe the free compact closed category $\mathcal{F}(\mathcal{A})$ on a collection of objects $\mathcal{A}$.

Objects of $\mathcal{F}(\mathcal{A})$ are defined by the following.

- All objects in $\mathcal{A}$ are objects in $\mathcal{F}(\mathcal{A})$;
- I is an object in $\mathcal{F}(\mathcal{A})$;
- To each object A in $\mathcal{F}(\mathcal{A})$, there exists an object A^* in $\mathcal{F}(\mathcal{A})$;
- To objects A and B in $\mathcal{F}(\mathcal{A})$, there exists an object $A \otimes B$ in $\mathcal{F}(\mathcal{A})$.

We require associativity morphisms,

$$a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C); \quad a^{-1}_{ABC} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C;$$

morphisms for the unit I,

$$r_A : A \otimes I \to A; \quad r^{-1}_A : A \to A \otimes I;$$

a symmetry morphism,

$$c_{AB} : A \otimes B \to B \otimes A;$$

and units and counits,

$$\eta_A : I \to A \otimes A^*; \quad \epsilon_A : A^* \otimes A \to I;$$

over all objects A, B and C in $\mathcal{F}(\mathcal{A})$, and complete the collection of morphisms by stating that if $f : X \to Y$ and $g : Z \to W$ are morphisms in $\mathcal{F}(\mathcal{A})$ then there is a morphism $f \otimes g : X \otimes Z \to Y \otimes W$ in $\mathcal{F}(\mathcal{A})$. (Thus, we are asserting that the tensor product is functorial.) Finally, we assert all naturality and coherence equations necessary for compact closure. Evidently, we have constructed the free compact closed category on $\mathcal{A}$.

Definition 0.3.1 For any object A in a compact closed category $\mathbb{C}$, we define the *dimension* of A to be the endomorphism

$$dim(A) : I \xrightarrow{\eta_A} A \otimes A^* \xrightarrow{c_{A,A^*}} A^* \otimes A \xrightarrow{\epsilon_A} I.$$

Consider our two standard examples. In **Rel**, there are only two endomorphisms on the one element set, I. In particular, $dim(\varnothing) = 0_I$, the empty relation, and $dim(A) = id_I$ for all $A \neq \varnothing$. In $\mathbf{FDVec}_k$, $dim(V)$ is the dimension of the vector space V in the usual sense if we assume that the field k has characteristic zero.

We shall now describe syntactically the morphisms of the free compact closed category $\mathcal{F}_1(\mathcal{A})$ on the set of objects $\mathcal{A} = \{A_1, \ldots, A_n\}$ with "trivial" dimension. (A graphical description of a morphism $dim(A)$ is a "loop", passing through A and A^*, so $\mathcal{F}_1(\mathcal{A})$ has trivial dimension in the sense that all morphisms have no additional loops.) From there, we can easily provide a syntactic description of the morphisms of $\mathcal{F}(\mathcal{A})$.

Suppose that $G(A_1, \ldots, A_n)$ and $F(A_1, \ldots, A_n)$ are arbitrary objects in $\mathcal{F}_1(\mathcal{A})$, built from objects $A_1, \ldots, A_n$ and $A_1^*, \ldots, A_n^*$, (which we shall call *literals* in keeping with our terminology in a logic setting) by the connective $\otimes$. To describe a morphism of the form $G(A_1, \ldots, A_n) \to F(A_1, \ldots, A_n)$ in $\mathcal{F}_1(\mathcal{A})$ is to identify pairs of occurrences of literals in the formulae G and F, in the following manner.

- each literal occurence is paired with precisely one other literal occurrence;
- an occurrence of A_i (in G, say) may be paired with either an occurrence of A_i^* in the *same* formula (G), or with another occurrence of A_i in the distinct formula (F);
- an occurrence of A_i^* may be paired with either an occurrence of A_i in the *same* formula, or with another occurrence of A_i^* in the distinct formula.

Identifying an occurrence of A_i with an occurrence of A_i^* in the source formula G implies that we precompose with a counit $\epsilon_{A_i} : A_i^* \otimes A_i \to I$ (and a possible symmetry). Identifying an occurrence of A_i with an occurrence of A_i^* in the target formula F implies that we postcompose with a unit $\eta_{A_i} : I \to A_i \otimes A_i^*$ (and a possible symmetry). Identifying a literal occurrence with a similar occurrence in distinct formulae implies that we tensor with an identity morphism. E.g. two distinct morphisms with source and target $A^* \otimes A \otimes B^* \otimes B$ are

- $A^* \otimes A \otimes B^* \otimes B \xrightarrow{id_{A^*} \otimes id_A \otimes id_{B^*} \otimes id_B} A^* \otimes A \otimes B^* \otimes B$
- $A^* \otimes A \otimes B^* \otimes B \xrightarrow{\epsilon_A \otimes \epsilon_B} I \otimes I \xrightarrow{\eta_A \otimes \eta_B} A \otimes A^* \otimes B \otimes B^* \xrightarrow{c_{A,A^*} \otimes c_{B,B^*}} A^* \otimes A \otimes B^* \otimes B.$

In particular, if $F(\mathbf{A}, \mathbf{A})$ is a "unit-free" object built from $A_1, \ldots, A_n$, $A_1^*, \ldots, A_n^*$ by the tensor product, then a morphism $I \to F(\mathbf{A}, \mathbf{A})$ in $\mathcal{F}_1(\mathcal{A})$ is canonically equivalent to a morphism $F^-(\mathbf{A}) \to F^+(\mathbf{A})$ in the free compact closed category, where $F^-(\mathbf{A})$ and $F^+(\mathbf{A})$ are tensor products of the A_i. Such a morphism must establish a bijective correspondence between occurrences of A_i in $F^-(\mathbf{A})$ and occurrences of A_i in $F^+(\mathbf{A})$ – in other words, F^+ and F^- are tensor products of the same length, and the morphism permutes the tensor factors in $F^-(\mathbf{A})$ to produce $F^+(\mathbf{A})$. Therefore, the original morphism $I \to F(\mathbf{A}, \mathbf{A})$ is essentially a tensor product of units η_{A_i} and can be described by pairing up occurrences of A_i with occurrences of A_i^*. In a manner similar to defining axiom links for a proof structure (see previous section), we can specify this morphism by a fixed-point-free involution on $\{1, \ldots, p\}$.

We can now retrieve the free compact closed category on $\mathcal{A}$, by adding the loops which represent morphisms $dim(A) : I \to I$. A morphism $G(A_1, \ldots, A_n) \to F(A_1, \ldots, A_n)$ in $\mathcal{F}(\mathcal{A})$ is a morphism in $\mathcal{F}_1(\mathcal{A})$, as described above, tensored with a finite collection of maps of the form $dim(A_i) : I \to I$.

0.3.2 The free $*$-autonomous category

The description of the free $*$-autonomous category $\mathcal{F}^*(\mathcal{A})$ on a set of objects $\mathcal{A}$ is more subtle than that for the free compact closed category. Linear Logic has proved a useful tool in solving this particular coherence problem. We therefore choose to formulate the free $*$-autonomous category in a tensor-par setting for clarity.

Given a set of objects $\mathcal{A}$, we define the objects of $\mathcal{F}^*(\mathcal{A})$ by the following.

- All objects in $\mathcal{A}$ are objects in $\mathcal{F}^*(\mathcal{A})$.
- I is an object in $\mathcal{F}^*(\mathcal{A})$.
- To each object A in $\mathcal{F}^*(\mathcal{A})$, there exists an object $A^\perp$ in $\mathcal{F}^*(\mathcal{A})$.
- To objects A and B in $\mathcal{F}^*(\mathcal{A})$, there exist objects $A \otimes B$ and $A \parr B$ in $\mathcal{F}^*(\mathcal{A})$.

Perhaps the simplest way to describe the free $*$-autonomous category is to use Theorem 0.1.11. We thus require the usual associativity, unit and symmetry morphisms to make both tensor and par symmetric monoidal, the two weak distribution natural transformations w^L and w^R, and the negation families η and ϵ. We close this collection of morphisms by asserting the functoriality of tensor and par, and assert all naturality and coherence conditions necessary for a $*$-autonomous structure.

A partial solution to this coherence problem has been in existence for some time (though rarely acknowledged in the literature as such), namely in the form of MLL proof nets. Suppose that $\mathcal{A} = \{A_1, \ldots, A_n\}$ is a set of n objects, and consider an arbitrary "unit-free" object $F(\mathbf{A}) = F(A_1, \ldots, A_n)$, built from $A_1, \ldots, A_n$, $A_1^\perp, \ldots, A_n^\perp$ by the connectives $\otimes$ and $\parr$. A proof of $\vdash F(\mathbf{A})$ in MLL has a categorical interpretation in $\mathcal{F}^*(\mathcal{A})$ as a morphism $I \to F(\mathbf{A})$ and conversely, a morphism $I \to F(\mathbf{A})$ in $\mathcal{F}^*(\mathcal{A})$ is always the categorical interpretation of some proof of $\vdash F(\mathbf{A})$ in MLL. Therefore, MLL proof nets can be regarded as a graphical description of morphisms in the free $*$-autonomous category.

However, Girard's proof nets are insufficient to model the units of MLL, and it has taken some time for this problem to be resolved. We remark that the proof net system in [BCST96] provides a full solution to coherence for weakly distributive categories *including* the units for tensor and par, and therefore a full solution for $*$-autonomous categories. Their work was an extension of Trimble's work on the free symmetric monoidal closed category [Tri95], where the equivalent existing problem with the unit for tensor was first resolved. However, Chapter 6 of this thesis provides an independent and arguably an improved alternative to Trimble's characterisation.

Before moving on to discuss the notion of Full Completeness, we make a final remark that the free $*$-autonomous category supporting the Mix rule merely requires the addition of the unary Mix morphism $m : \perp \to I$ and the necessary coherence equations.

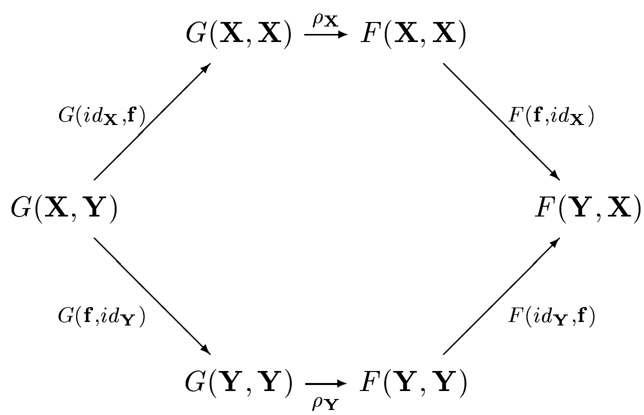
0.4 Full Completeness

The term "full completeness" was coined by Abramsky and Jagadeesan [AJ94]. In their own words, with full completeness, one has the tightest possible connection between syntax and semantics. In view of the previous section, we investigate this notion further and place it in more of a categorical setting. We emphasise at the start that we will only be concerned with full completeness for *unit-free* formulae.

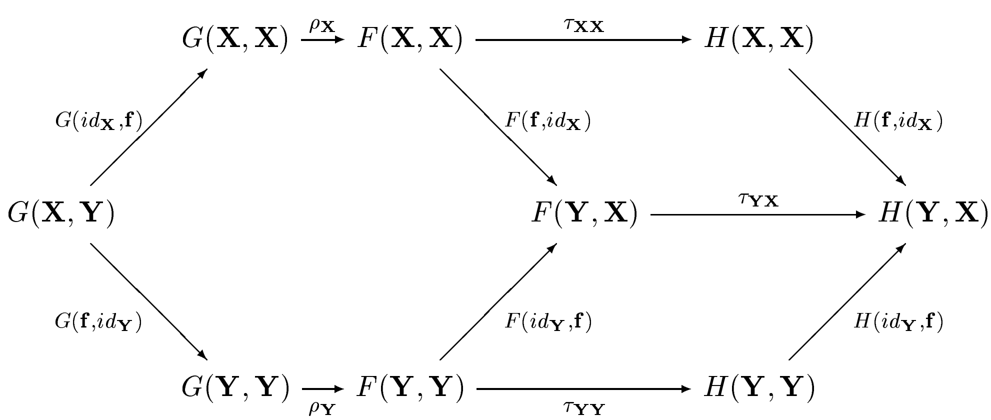
Suppose that $\mathcal{A} = \{\mathbf{X}\} = \{X_1, \ldots, X_n\}$ is a set of n objects and that $\mathcal{F}(\mathcal{A})$ is a free category of type L on $\mathcal{A}$. Typically, L will be a monoidal structure such as compact closed or $*$-autonomous, so that the functor $(-)^\perp$ is contravariant. Suppose that $F(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is a formula built from $X_1, \ldots, X_n$, $Y_1^\perp, \ldots, Y_n^\perp$. Then $F(\mathbf{X}, \mathbf{X})$ can be regarded as a unit-free object in $\mathcal{F}(\mathcal{A})$, and therefore induces a multivariant functor $[\![F]\!] : \mathbb{C}^n \times (\mathbb{C}^{op})^n \to \mathbb{C}$, whenever $\mathbb{C}$ is a category of type L.

It is reasonable to expect a morphism $I \to F(\mathbf{X}, \mathbf{X})$ in $\mathcal{F}(\mathcal{A})$ to induce a collection of morphisms $I \to [\![F]\!](\mathbf{A}, \mathbf{A})$ in $\mathbb{C}$ satisfying some naturality condition. However, the multivariance of $[\![F]\!]$ forces us to extend our notion of naturality to "dinaturality".

Definition 0.4.1 Let $\mathbb{C}$ be a category, and $G, F : \mathbb{C}^n \times (\mathbb{C}^{op})^n \to \mathbb{C}$ be multivariant functors. Write $\mathbf{X}$ for the list $X_1, \ldots, X_n$ etc. We say that a collection of morphisms $\rho = \rho_{\mathbf{X}} : G(\mathbf{X}, \mathbf{X}) \to F(\mathbf{X}, \mathbf{X})$ is a *dinatural transformation* in $\mathbb{C}$ if for all morphisms $f_i : X_i \to Y_i$ in $\mathbb{C}$, the following diagram commutes.



Thus, an object $F(\mathbf{X}, \mathbf{X})$ in $\mathcal{F}(\mathcal{A})$ induces a functor $[\![F]\!] : \mathbb{C}^n \times (\mathbb{C}^{op})^n \to \mathbb{C}$, and a morphism $I \to F(\mathbf{X}, \mathbf{X})$ in $\mathcal{F}(\mathcal{A})$ induces a collection of morphisms $I \to F(\mathbf{A}, \mathbf{A})$ in $\mathbb{C}$ which form a dinatural transformation $\rho : \mathfrak{K}_I \to F$, where $\mathfrak{K}_I$ denotes the constant functor with value I. It is essential to note that two dinatural transformations do *not*, in general, compose with one another to give a dinatural transformation. This is of no concern to us here, because all families of "base" morphisms (i.e. associativity, symmetry etc.) in the free category form natural transformations, and dinatural transformations *do* compose on either side with natural transformations to give dinatural transformations. For example, suppose that $\rho : G \to F$ is a dinatural transformation and $\tau : F \to H$ is natural in every variable. Then the following diagram commutes.



Therefore $\tau\rho : G \to H$ is a dinatural transformation.

We are now ready to state the definition of full completeness.

Definition 0.4.2 (Full Completeness) Let $\mathbb{C}$ be a category with a specified structure L. Then $\mathbb{C}$ satisfies *full completeness with respect to* L or *"L full completeness"* if every dinatural transformation $\mathfrak{K}_I \to [\![F]\!]$ (with $[\![F]\!] : \mathbb{C}^n \times (\mathbb{C}^{op})^n \to \mathbb{C}$), is induced by a morphism $I \to F(\mathbf{X}, \mathbf{X})$ in the free category of type L on n objects.

Abramsky and Jagadeesan's definition established a connection between morphisms in a categorical model and proofs in a logic. Their demonstrable semantics was a category of games ([AJ94]), where they proved that every uniform history-free winning strategy was the denotation of a proof in MLL+Mix. In this setting, it was possible to use a very basic form of naturality to achieve full completeness with respect to MLL+Mix. The notion of *embedding* meant that if a game A could be embedded into a game B, then the dual game $A^\perp$ (in which the Player's and Opponent's moves are swapped) could be embedded into the game $B^\perp$. Thus, formulae were suitably modelled by functors, and the correct denotation of a proof in MLL+Mix was essentially a transformation between functors, natural with respect to embeddings.

However, in most categorical models, natural transformations will not be sufficient. In [AJ94], it was good fortune that the functor $(-)^\perp$ could be regarded as *covariant* in the appropriate categorical setting. But often, there may be no useful notion of embedding to hand, and one will be stuck with $(-)^\perp$ as a contravariant functor, so any formula with positive and negative occurrences of literals will not be functorial. For this reason, we instead work with multivariant functors and dinatural transformations.

We now make a very useful observation about dinatural transformations in compact closed categories.

Lemma 0.4.3 *Let $\mathbb{C}$ be a compact closed category, let $F : \mathbb{C}^n \otimes (\mathbb{C}^{op})^n \to \mathbb{C}$ be a multivariant functor such that*

$$F(\mathbf{A}, \mathbf{A}) \cong A_{\mu_1} \otimes \cdots \otimes A_{\mu_l} \otimes A^*_{\lambda_1} \otimes \cdots \otimes A^*_{\lambda_m},$$

(where $\mu_i, \lambda_i \in \{1, \ldots, n\}$ for all i) and let σ be a collection of morphisms $\sigma_{\mathbf{A}} : I \to F(\mathbf{A}, \mathbf{A})$ in $\mathbb{C}$. Define $F^-(\mathbf{A}) = A_{\lambda_1} \otimes \cdots \otimes A_{\lambda_m}$ and $F^+(\mathbf{A}) = A_{\mu_1} \otimes \cdots \otimes A_{\mu_l}$, so that each $\sigma_{\mathbf{A}}$ is canonically isomorphic to a morphism $\tilde{\sigma}_{\mathbf{A}} : F^-(\mathbf{A}) \to F^+(\mathbf{A})$.

Then σ is a dinatural transformation in $\mathbb{C}$ if and only if $\tilde{\sigma}$ is a natural transformation in $\mathbb{C}$.

Proof The result is trivial when $m = 0$ or $l = 0$. We will prove the result when $m = l = 1$, with the general case being an extension of this argument.

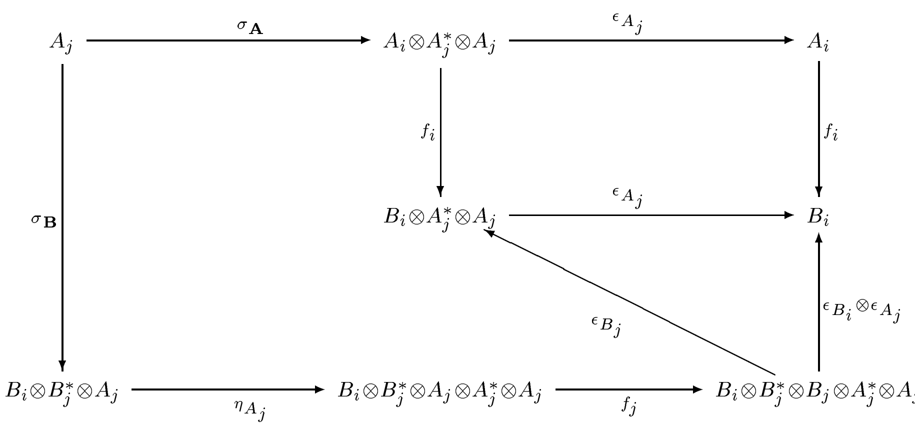
Suppose that $\sigma_{\mathbf{A}} : I \to A_i \otimes A_j^*$. The canonical map $\tilde{\sigma}_{\mathbf{A}} : A_j \to A_i$ can be expanded as

$$A_j \cong I \otimes A_j \xrightarrow{\sigma_{\mathbf{A}} \otimes A_j} A_i \otimes A_j^* \otimes A_j \xrightarrow{A_i \otimes \epsilon_{A_j}} A_i \otimes I \cong A_i.$$

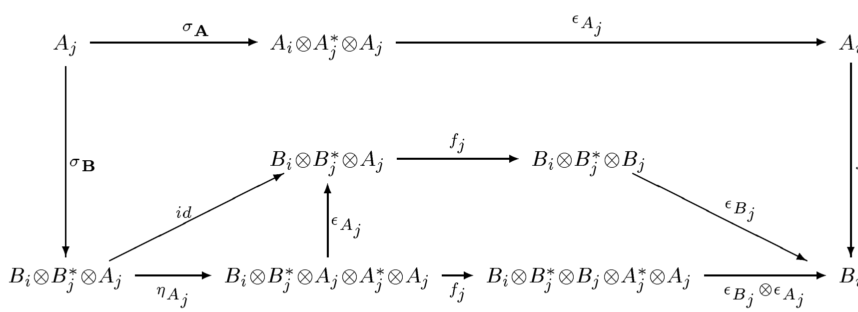
Given $f_j : A_j \to B_j$, the morphism $f_j^* : B_j^* \to A_j^*$ can be expanded as

$$B_j^* \cong B_j^* \otimes I \xrightarrow{B_j^* \otimes \eta_{A_j}} B_j^* \otimes A_j \otimes A_j^* \xrightarrow{B_j^* \otimes f_j \otimes A_j^*} B_j^* \otimes B_j \otimes A_j^* \xrightarrow{\epsilon_{B_j} \otimes A_j^*} I \otimes A_j^* \cong A_j^*.$$

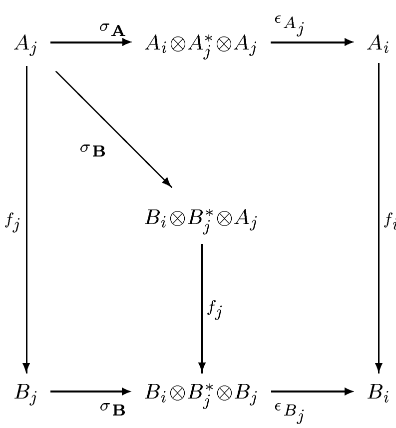
Therefore, the following diagram commutes, (with isomorphisms $A \otimes I \cong A$ omitted)



where the left hexagon commutes by dinaturality. The outer box can be filled in, to form



whence the following diagram commutes.

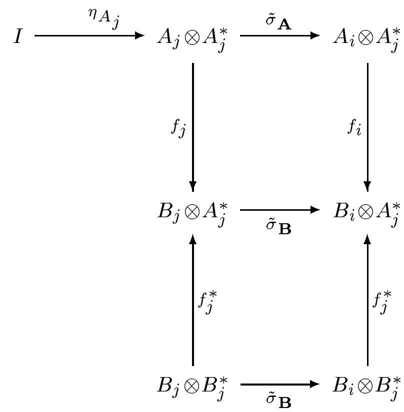


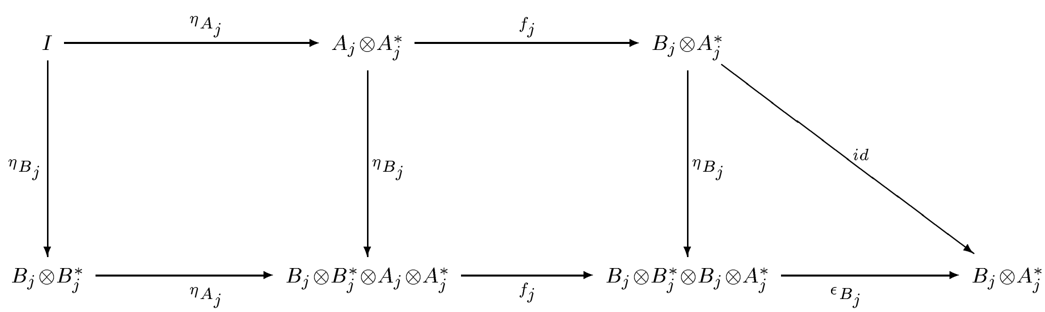
Therefore the morphisms $\tilde{\sigma}_{\mathbf{A}} : A_j \to A_i$ form a natural transformation.

Conversely, suppose that the morphisms $\tilde{\sigma}_{\mathbf{A}} : A_j \to A_i$ for a natural transformation. The morphisms $\sigma_{\mathbf{A}} : I \to A_i \otimes A_j^*$ can be expanded as

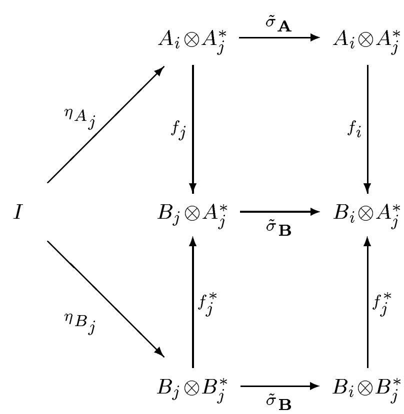
$$I \xrightarrow{\eta_{A_j}} A_j \otimes A_j^* \xrightarrow{\tilde{\sigma}_{\mathbf{A}} \otimes A_j^*} A_i \otimes A_j^*.$$

The following two diagrams commute.





Therefore, the following diagram commutes,



i.e. σ is a dinatural transformation. ■

Thus, we can restate full completeness in a compact closed setting with the following.

> A compact closed category $\mathbb{C}$ satisfies *compact closed full completeness* if every natural transformation $[\![F^-]\!] \to [\![F^+]\!]$ (with $[\![F^-]\!], [\![F^+]\!] : \mathbb{C}^n \to \mathbb{C}$), is induced by a morphism $F^-(\mathbf{X}) \to F^+(\mathbf{X})$ in the free compact closed category on n objects".

At this point, we mention that the compact closed full completeness results we will obtain in this thesis are with respect to the free compact closed category on n objects with trivial dimension. However, it is possible to remove the restriction on dimension – in two cases, this is easily done because the categorical model concerned has trivial dimension; in one case, this is done by absorbing the dimensions into an enrichment which reflects the categorical model.

0.4.1 Approaching Full Completeness

Suppose that $\mathbb{C}$ is a $*$-autonomous category, and that we are given a dinatural transformation ρ from $\mathfrak{K}_I$, the constant functor with value I, to the multivariant functor $[\![F]\!] : \mathbb{C}^n \times (\mathbb{C}^{op})^n \to \mathbb{C}$. Suppose that we wish to prove that ρ is induced by a morphism $I \to F(\mathbf{X}, \mathbf{X})$ in the free $*$-autonomous category on n objects. Since we are working with unit-free formulae, a valid specification of such a morphism is via MLL proof nets, and we will indeed use MLL proof nets to accomplish the task at hand. In particular, we will exploit the dinaturality of ρ to generate a set of axiom links for a proof structure built from the formula F. As mentioned earlier, a proof structure is uniquely determined by its axiom links, so we will have determined the complete proof structure. If we can show that this proof structure is a proof net, then we will have shown that F is provable, and that ρ is the interpretation of a "generic" proof of $\vdash F$. (By a generic proof, we mean a collection of proofs, all of the same form, obtained by varying over all propositional atoms.) Equivalently, we will have shown that ρ is the interpretation of a morphism $I \to F(\mathbf{X}, \mathbf{X})$ in the free $*$-autonomous category on n objects.

For most of this thesis, we will consider $*$-autonomous categories $\mathbb{C}^*$ with a (forgetful) functor U from $\mathbb{C}^*$ to some compact closed category $\mathbb{C}$ which preserves the $*$-autonomous structure of $\mathbb{C}^*$. Our approach to proving full completeness will rely heavily on exploiting the underlying compact closed structure in $\mathbb{C}$. (There is an example with a difference – a $*$-autonomous category with a compact closed full subcategory – but the approach there will be similar.) The basic idea is that if $\rho : \mathfrak{K}_{\mathbf{1}} \to F$ is a dinatural transformation in $\mathbb{C}^*$ then U induces a dinatural transformation $\sigma : \mathfrak{K}_I \to F$ in $\mathbb{C}$. Note that in $\mathbb{C}$, the formula F reduces to a straight tensor product.

Now suppose that we can prove compact closed full completeness of $\mathbb{C}$. By Lemma 0.4.3, a dinatural transformation can be regarded as a natural transformation $\tilde{\sigma}$ from F^- to F^+ where $F^+(\mathbf{A})$ and $F^-(\mathbf{A})$ are tensor products of A_i's. So we are asking that F^- and F^+ be formulae of the same length, and that $\tilde{\sigma}$ is a permutation on the tensor factors. Then σ is essentially (meaning "modulo composition with symmetry morphisms") a tensor product of unit morphisms $\eta_{A_i} : I \to A_i \otimes A_i^*$ obtained canonically from identity morphisms.

The canonical morphisms $\mathbf{1} \to X^\perp \mathbin{⅋} X$ are categorical interpretations of the identity axioms $\vdash \alpha^\perp, \alpha$, or the axiom links of a proof structure. Moreover, U sends these morphisms to unit morphisms η_A in $\mathbb{C}$. Thus if ρ is the interpretation of a proof in MLL, then the fixed-point-free involution on $\{1, \ldots, p\}$, specifying the axiom links for the corresponding proof structure is the same fixed-point-free involution, linking occurrences of A_i with A_i^*, which determined σ. Thus the proof structure associated with ρ in $\mathbb{C}^*$ is in fact determined by σ in $\mathbb{C}$.

If we can go on to prove that all such proof structures are indeed proof nets, then proving full completeness in $\mathbb{C}^*$ can be seen as a lifting of full completeness in $\mathbb{C}$.

0.5 Outline of Thesis

Chapter 1 describes a construction on a compact closed category which provides us with a $*$-autonomous category, and also discusses precisely when this category supports the Mix rule. The construction was first brought to my attention by J.M.E. Hyland, who observed this abstraction of the work of R. Loader. We will show that it is possible to view full completeness results in the constructed category as a lifting of full completeness in the original compact closed category. The next three chapters are specific examples of proving full completeness in this way. Chapter 2 works with **Rel** – it provides an alternative view of Loader's category of Linear Logical Predicates [Loa94b]. The results are then easily adapted to prove a full completeness result for coherence spaces, the original denotational model of Linear Logic, presented in [Gir87]. Chapter 3 works with **FDVec** – the full completeness result we achieve in **FDVec** turns out to be equivalent to an already well-known result in Invariant Theory. In Chapter 4, we introduce a compact closed category of Conway games. Existing semantics for games has been somewhat vague. We will attempt to make precise what is meant by a "position" and a "move". When we apply the glueing construction to this model, we obtain a model of MLL+Mix, rather than pure MLL.

Chapter 5 is another example of using full completeness in **FDVec** to prove full completeness in a $*$-autonomous category which models MLL+Mix. We will show that this category is an alternative characterisation of a model considered by Blute and Scott [BS96]. Moreover, we will remove their restrictions to binary formulae and "diadditive" dinatural transformations.

Chapter 6, as it stands, is an independent chapter, but it could be viewed as a means to prove full completeness in an alternative model of Linear Logic. We consider the multiplicative fragment of Intuitionistic Linear Logic and present a system of proof nets which provide a graphical description of terms in an assignment motivated by the free symmetric monoidal closed category. The proof nets handle the unit for tensor, and therefore provide a description of all morphisms in the free symmetric monoidal closed category. An attractive characteristic of this system is that the rewrites are global (as opposed to "one-step"), but the normalisation process still remains confluent and strongly normalising.

The final chapter is a summary of the thesis and a discussion on further research.

Chapter 1

A double glueing construction

Recall (Lemma 0.1.15) that a compact closed category has a collapsed $*$-autonomous structure. Consequently, it is of little interest to us as a categorical model of MLL since the tensor and par connectives are indistinguishable. In this chapter, we provide a construction on a compact closed category that produces a new $*$-autonomous category which does make the distinction between tensor and par.

As a model of linear logic, the glueing construction is very easily interpreted. An object of the new category $\mathbf{G}\mathbb{C}$ can be regarded as an object A of the original category $\mathbb{C}$ together with a set of "proofs" of A and a set of "disproofs" of A. While linear negation is defined naturally, the tensor product is more subtle. From this, the rest of the MLL structure follows immediately.

1.1 The construction

Let $\mathbb{C} = (\mathbb{C}, \otimes, I, (-)^*)$ be a compact closed category. Let H be the functor from $\mathbb{C}$ to **Set**,

$$\begin{array}{rcrcl} H(f : A \to B) & = & Hf : \mathbb{C}(I, A) & \to & \mathbb{C}(I, B) \\ & & \alpha & \mapsto & f\alpha, \end{array}$$

and let K be the functor from $\mathbb{C}^{op}$ to **Set**,

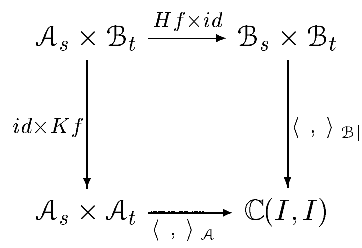
$$\begin{array}{rcrcl} K(f : A \to B) & = & Kf : \mathbb{C}(I, A^*) & \to & \mathbb{C}(I, B^*) \\ & & \alpha & \mapsto & f^*\alpha, \end{array}$$

Define a new category, $\mathbf{G}\mathbb{C}$, whose objects are triples $\mathcal{A} = (|\mathcal{A}|, \mathcal{A}_s, \mathcal{A}_t)$ where

$$\begin{array}{l} |\mathcal{A}| \text{ is an object of } \mathbb{C}, \\ \mathcal{A}_s \subseteq H(|\mathcal{A}|) = \mathbb{C}(I, |\mathcal{A}|), \text{ and} \\ \mathcal{A}_t \subseteq K(|\mathcal{A}|) = \mathbb{C}(I, |\mathcal{A}|^*) \cong \mathbb{C}(|\mathcal{A}|, I). \end{array}$$

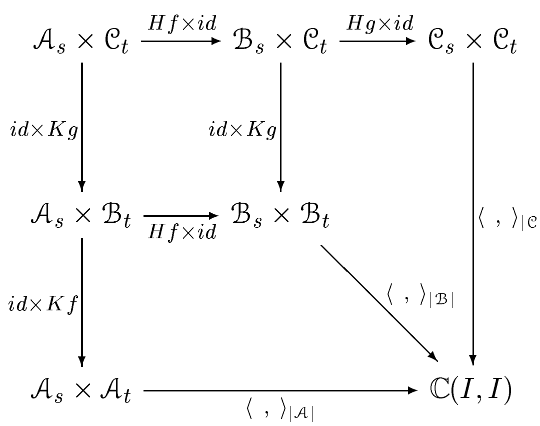
A morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{G}\mathbb{C}$ is a morphism $f : |\mathcal{A}| \to |\mathcal{B}|$ in $\mathbb{C}$ such that for every $\alpha \in \mathcal{A}_s$, the composite $(Hf)\alpha = f\alpha$ belongs to $\mathcal{B}_s$ and for every $\beta \in \mathcal{B}_t$, the composite $(Kf)\beta = f^*\beta$ belongs to $\mathcal{A}_t$. In other words, Hf maps $\mathcal{A}_s$ into $\mathcal{B}_s$ and Kf maps $\mathcal{B}_t$ into $\mathcal{A}_t$.

If $f : \mathcal{A} \to \mathcal{B}$ is a morphism in $\mathbf{G}\mathbb{C}$, the following square commutes.



(1.1)

(where $\langle \alpha, \tau \rangle_{|\mathcal{A}|} = \tau\alpha$ etc.) Thus, if $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ are morphisms in $\mathbf{G}\mathbb{C}$, then the following diagram commutes.



This shows that the composite $gf : |\mathcal{A}| \to |\mathcal{C}|$ in $\mathbb{C}$ is a morphism $gf : \mathcal{A} \to \mathcal{C}$ in $\mathbf{G}\mathbb{C}$.

The identity morphism on $\mathcal{A}$ is simply the identity morphism on $|\mathcal{A}|$ in $\mathbb{C}$, and associativity of composition of morphisms in $\mathbf{G}\mathbb{C}$ follows from that in $\mathbb{C}$. So $\mathbf{G}\mathbb{C}$ is a category.

We will show that $\mathbf{G}\mathbb{C}$ is $*$-autonomous with the following structure.

Tensor. Given objects $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{G}\mathbb{C}$, define the object $\mathcal{A} \otimes \mathcal{B}$ by

$$\begin{array}{rcl} |\mathcal{A} \otimes \mathcal{B}| & = & |\mathcal{A}| \otimes |\mathcal{B}|; \\ (\mathcal{A} \otimes \mathcal{B})_s & = & \{\sigma \otimes \tau \mid \sigma \in \mathcal{A}_s, \tau \in \mathcal{B}_s\}; \\ (\mathcal{A} \otimes \mathcal{B})_t & = & \mathbf{G}\mathbb{C}(\mathcal{A}, \mathcal{B}^{\perp}). \end{array}$$

Observe that our definition is consistent, since $(\mathcal{A} \otimes \mathcal{B})_s \subseteq \mathbb{C}(I, |\mathcal{A}| \otimes |\mathcal{B}|)$ and $(\mathcal{A} \otimes \mathcal{B})_t = \mathbf{G}\mathbb{C}(\mathcal{A}, \mathcal{B}^{\perp}) \subseteq \mathbb{C}(|\mathcal{A}|, |\mathcal{B}|^*) \cong \mathbb{C}(|\mathcal{A}| \otimes |\mathcal{B}|, I)$.

We also define the *unit* object, $\mathbf{1} = (I, \{id_I\}, \mathbb{C}(I, I))$.

Proposition 1.1.1 *$\mathbf{G}\mathbb{C}$ is a symmetric monoidal category with tensor $\otimes$ and unit $\mathbf{1} = (I, \{id_I\}, \mathbb{C}(I, I))$.*

Proof Suppose that $f : \mathcal{A} \to \mathcal{A}'$ is a morphism in $\mathbf{G}\mathbb{C}$. Write $|\mathcal{A}| = A, |\mathcal{A}|' = A'$. We want to show that given an object $\mathcal{B}$ in $\mathbf{G}\mathbb{C}$ with $|\mathcal{B}| = B$, the morphism $f \otimes B : A \otimes B \to A' \otimes B$ in $\mathbb{C}$ lifts to a morphism $f \otimes \mathcal{B} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}$.

Suppose that $\delta \in (\mathcal{A} \otimes \mathcal{B})_s$, $\delta = \delta_1 \otimes \delta_2$ where $\delta_1 \in \mathcal{A}_s$ and $\delta_2 \in \mathcal{B}_s$. Then $f\delta_1 \in \mathcal{A}'_s$ and therefore

$$(f \otimes B)(\delta_1 \otimes \delta_2) = (f\delta_1) \otimes (\delta_2) \in (\mathcal{A}' \otimes \mathcal{B})_s.$$

So $H(f \otimes B)$ maps $(\mathcal{A} \otimes \mathcal{B})_s$ into $(\mathcal{A}' \otimes \mathcal{B})_s$. On the other hand, suppose that $\gamma \in (\mathcal{A}' \otimes \mathcal{B})_t$, so that $\gamma : A' \to B^*$ in $\mathbb{C}$. We want to show that

$$A \otimes B \xrightarrow{f \otimes B} A' \otimes B \xrightarrow{\tilde{\gamma}} I \in (\mathcal{A} \otimes \mathcal{B})_t$$

or equivalently $A \xrightarrow{f} A' \xrightarrow{\gamma} B^* \in (\mathcal{A} \otimes \mathcal{B})_t$. If $\alpha \in \mathcal{A}_s$, then $f\alpha \in \mathcal{A}'_s$, so that $(\gamma f)\alpha = \gamma(f\alpha) \in \mathcal{B}_t$. If $\beta \in \mathcal{B}_s$, then $\gamma^*\beta \in \mathcal{A}'_t$, so that $(\gamma f)^*\beta = f^*(\gamma^*\beta) \in \mathcal{A}_t$. So $\gamma f \in (\mathcal{A} \otimes \mathcal{B})_t$, i.e. $K(f \otimes B)$ maps $(\mathcal{A}' \otimes \mathcal{B})_t$ into $(\mathcal{A} \otimes \mathcal{B})_t$. Therefore $f \otimes B$ lifts to a morphism $f \otimes \mathcal{B} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}$ in $\mathbf{G}\mathbb{C}$.

It is easy to see that the definition of tensor is entirely symmetric, after noting that $\mathbf{G}\mathbb{C}(\mathcal{A}, \mathcal{B}^\perp) \cong \mathbf{G}\mathbb{C}(\mathcal{B}, \mathcal{A}^\perp)$. Thus $- \otimes - : \mathbf{G}\mathbb{C} \times \mathbf{G}\mathbb{C} \to \mathbf{G}\mathbb{C}$ is a symmetric bifunctor.

It is clear that morphisms $(\alpha \otimes \beta) \otimes \gamma$ in $((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C})_s$ are naturally isomorphic to morphisms $\alpha \otimes (\beta \otimes \gamma)$ in $(\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}))_s$. Conversely, suppose that $\delta \in (\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}))_t$. Write $A = |\mathcal{A}|$, $B = |\mathcal{B}|$ and $C = |\mathcal{C}|$. Then

(i) given $\alpha \in \mathcal{A}_s$ and $\beta \in \mathcal{B}_s$, the composite $I \xrightarrow{\alpha \otimes \beta} A \otimes B \xrightarrow{\delta} C^*$ belongs to $\mathcal{C}_t$;

(ii) given $\alpha \in \mathcal{A}_s$ and $\gamma \in \mathcal{C}_s$, the composite $I \xrightarrow{\alpha \otimes \gamma} A \otimes C \xrightarrow{\delta} B^*$ belongs to $\mathcal{B}_t$;

(iii) given $\beta \in \mathcal{B}_s$ and $\gamma \in \mathcal{C}_s$, the composite $A \xrightarrow{\delta} B^* \otimes C^* \xrightarrow{\beta \otimes \gamma} I$ belongs to $\mathcal{A}_t$.

This proves that $\delta : A \otimes B \to C^*$ in $\mathbb{C}$ lifts to a morphism in $((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C})_t$. Statement (i) shows that $H\delta$ maps $(\mathcal{A} \otimes \mathcal{B})_s$ into $\mathcal{C}_t$, while statements (ii) and (iii) shows that $K\delta$ maps $\mathcal{C}_s$ into $(\mathcal{A} \otimes \mathcal{B})_t$. Therefore, we have natural isomorphisms $(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \cong \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$.

Finally, observe that

$$(\mathcal{A} \otimes \mathbf{1})_s = \{\alpha \otimes id_I : \alpha \in \mathcal{A}_s\} \cong \mathcal{A}_s$$

and if $f \in \mathcal{A}_t$, then f satisfies

$$\text{if } \alpha \in \mathcal{A}_s \text{ then } f\alpha : I \to A \to I \text{ belongs to } \mathbf{1}_t = \mathbb{C}(I, I).$$

Furthermore,

$$\text{if } \beta \in \mathbf{1}_s \text{, then } \beta = id_I \text{, so } f\beta^* = f \in \mathcal{A}_t \text{ if and only if } f \in \mathcal{A}_t.$$

So $f \in \mathcal{A}_t$ if and only if $f \in (\mathcal{A} \otimes \mathbf{1})_t$, i.e. $(\mathcal{A} \otimes \mathbf{1})_t \cong \mathcal{A}_t$. Therefore, we have natural isomorphisms $\mathcal{A} \otimes \mathbf{1} \cong \mathcal{A}$. ■

Linear negation. Given an object $\mathcal{A}$ in $\mathbf{G}\mathbb{C}$, define its *dual* object $\mathcal{A}^\perp = (|\mathcal{A}|^*, \mathcal{A}_t, \mathcal{A}_s)$. Suppose that $\mathcal{B}$ is another object in $\mathbf{G}\mathbb{C}$ and we have a morphism $f : \mathcal{A} \to \mathcal{B}$. Then $f^* : |\mathcal{B}|^* \to |\mathcal{A}|^*$ in $\mathbb{C}$ satisfies

$$(Hf^*)\beta = f^*\beta = (Kf)\beta \quad\Longrightarrow\quad Hf^* = Kf : \mathcal{B}_t \to \mathcal{A}_t$$

and

$$(Kf^*)\alpha = f\alpha = (Hf)\alpha \quad\Longrightarrow\quad Kf^* = Hf : \mathcal{A}_s \to \mathcal{B}_s$$

for all $\beta \in \mathcal{B}_s^\perp = \mathcal{B}_t$ and $\alpha \in \mathcal{A}_t^\perp = \mathcal{A}_s$. Therefore, f^* is in actual fact a morphism $f^\perp : \mathcal{B}^\perp \to \mathcal{A}^\perp$ in $\mathbf{G}\mathbb{C}$. This induces a covariant functor $(-)^\perp$ over $\mathbf{G}\mathbb{C}$.

Theorem 1.1.2 *The covariant functor $(-)^\perp$ is full and faithful, with the natural equivalence $\mathbf{G}\mathbb{C}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}^\perp) \cong \mathbf{G}\mathbb{C}(\mathcal{A}, (\mathcal{B} \otimes \mathcal{C})^\perp)$. Consequently, $\mathbf{G}\mathbb{C}$ is a $*$-autonomous category and a categorical model of MLL.*

Proof That $(-)^\perp$ is full and faithful in $\mathbf{G}\mathbb{C}$ can be deduced from the fact that $(-)^*$ is full and faithful in $\mathbb{C}$.

Suppose that $f : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}^\perp$ in $\mathbf{G}\mathbb{C}$, and put $A = |\mathcal{A}|$, $B = |\mathcal{B}|$ and $C = |\mathcal{C}|$. Then

(i) given $\alpha \in \mathcal{A}_s$ and $\beta \in \mathcal{B}_s$, the composite $I \xrightarrow{\alpha\otimes\beta} A \otimes B \xrightarrow{f} C^*$ belongs to $\mathcal{C}_s^\perp$;

(ii) given $\gamma \in \mathcal{C}_t^\perp$ and $\alpha \in \mathcal{A}_s$, the composite $I \xrightarrow{\alpha} A \xrightarrow{f} B^* \otimes C^* \xrightarrow{B^*\otimes\gamma^*} B^* \otimes I \cong B^*$ belongs to $\mathcal{B}_t$;

(iii) given $\gamma \in \mathcal{C}_t^\perp$ and $\beta \in \mathcal{B}_s$, the composite $A \xrightarrow{f} B^* \otimes C^* \xrightarrow{\beta^*\otimes\gamma^*} I$ belongs to $\mathcal{A}_t$.

From this, we deduce that f lifts to a morphism $\mathcal{A} \to (\mathcal{B} \otimes \mathcal{C})^\perp$ in $\mathbf{G}\mathbb{C}$. Statements (i) and (ii) show that Hf maps $\mathcal{A}_s$ into $(\mathcal{B} \otimes \mathcal{C})_s^\perp$, while statement (iii) shows that Kf maps $(\mathcal{B} \otimes \mathcal{C})_t^\perp$ into $\mathcal{A}_t$. The proof of the reverse implication is similar. ■

Linear Implication and Par. Given objects $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{G}\mathbb{C}$, we define $\mathcal{A} \multimap \mathcal{B} = (\mathcal{A} \otimes \mathcal{B}^\perp)^\perp$ and $\mathcal{A} \parr \mathcal{B} = (\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$. Then the unit for $\parr$ is $\perp = \mathbf{1}^\perp$.

We end this section with two important observations.

$\mathbf{G}\mathbb{C}$ is a non-trivial categorical model of MLL. That is, the tensor and par connectives are always distinct. For example, $(I, \varnothing, \varnothing) \otimes (I, \varnothing, \varnothing) = (I, \varnothing, \mathbb{C}(I, I))$ while $(I, \varnothing, \varnothing) \parr (I, \varnothing, \varnothing) = (I, \mathbb{C}(I, I), \varnothing)$.

$\mathbf{G}\mathbb{C}$ supports the Mix rule if and only if $\mathbb{C}(I, I)$ consists of only the identity morphism on I. If $\mathbb{C}(I, I) = \{id_I\}$, then $\perp = \mathbf{1} = (I, \{id_I\}, \{id_I\})$, so id_I lifts to the unary Mix morphism $m : \perp \to \mathbf{1}$. Conversely, suppose that there exists a morphism $m : \perp \to \mathbf{1}$. Then for all $\alpha \in \perp_s = \mathbb{C}(I, I)$, the composite $m\alpha$ belongs to $\mathbf{1}_s = \{id_I\}$. In particular, putting $\alpha = id_I$, we have $id_I = m\alpha = m$. Therefore $m = id_I$, in which case this forces all $\alpha \in \mathbb{C}(I, I)$ to be the identity morphism on I.

1.2 Logical motivation

We make the following trivial but useful observation.

Proposition 1.2.1 *Let $\mathcal{A}$ be an object in $\mathbf{G}\mathbb{C}$. Then*

$$\mathbf{G}\mathbb{C}(\mathbf{1}, \mathcal{A}) \cong \mathcal{A}_s; \quad \mathbf{G}\mathbb{C}(\mathcal{A}, \perp) \cong \mathcal{A}_t.$$

Proof A morphism $f : \mathbf{1} \to \mathcal{A}$ is identifiable with an element of $\mathcal{A}_s$ since it has to satisfy $f\alpha \in \mathcal{A}_s$ for all $\alpha \in \mathbf{1}_s$. Since $\alpha = id_I$ is the only element of $\mathbf{1}_s$, we deduce that $f \in \mathcal{A}_s$. Conversely, suppose that $f : I \to |\mathcal{A}|$ is a morphism in $\mathbb{C}$ that belongs to $\mathcal{A}_s$. If $\alpha \in \mathbf{1}_s$, then $\alpha = id_I$, so $f\alpha = f \in \mathcal{A}_s$. If $\beta \in \mathcal{A}_t$, then $f^*\beta \in \mathbb{C}(I, I) = \mathbf{1}_t$. So $f : \mathbf{1} \to \mathcal{A}_s$.

The dual result is similar. ∎

In a logic setting, we can thus think of each object $\mathcal{A}$ as an object $A = |\mathcal{A}|$ in $\mathbb{C}$ together with a selection of "proofs" of A (i.e. the collection $\mathcal{A}_s$), and a selection of "disproofs" of A (i.e. the collection $\mathcal{A}_t$). The definition of tensor product is now fairly intuitive. Proofs in $(\mathcal{A} \otimes \mathcal{B})_s$ are the tensor products of proofs in $\mathcal{A}_s$ and proofs in $\mathcal{B}_s$. Proofs in $(\mathcal{A} \otimes \mathcal{B})_t$ are those disproofs of $|\mathcal{A}| \otimes |\mathcal{B}|$ which, when cut with a proof of $|\mathcal{A}|$, yield a disproof of $|\mathcal{B}|$, or when cut with a proof of $|\mathcal{B}|$, yield a disproof of $|\mathcal{A}|$. The definition of linear negation is even more intuitive – a proof of $|\mathcal{A}|$ is a disproof of $|\mathcal{A}|^*$ and vice versa.

1.3 Proving full completeness of $\mathbf{G}\mathbb{C}$

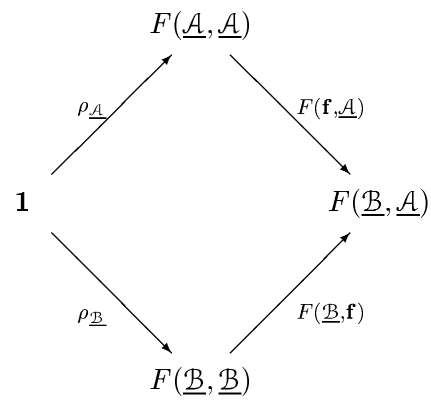
We outline the basic theme of proving full completeness of $\mathbf{G}\mathbb{C}$. The following results will show that an attempt to prove full completeness in $\mathbf{G}\mathbb{C}$ is an attempt to lift a full completeness result in the underlying compact closed category $\mathbb{C}$.

Lemma 1.3.1 *The forgetful functor $U : \mathbf{G}\mathbb{C} \to \mathbb{C}$ preserves the $*$-autonomous structure of $\mathbf{G}\mathbb{C}$. Furthermore, it has a right adjoint $R : \mathbb{C} \to \mathbf{G}\mathbb{C}$, specified by $RA = (A, \mathbb{C}(I, A), \varnothing)$ and a left adjoint $L : \mathbb{C} \to \mathbf{G}\mathbb{C}$, specified by $LA = (A, \varnothing, \mathbb{C}(I, A^*))$.*

Proof We have $U(\mathcal{A}^\perp) = |\mathcal{A}|^* = (U\mathcal{A})^*$ and $U(\mathcal{A} \otimes \mathcal{B}) = |\mathcal{A}| \otimes |\mathcal{B}| = U\mathcal{A} \otimes U\mathcal{B}$ for all objects $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{G}\mathbb{C}$. Furthermore, $U(f^\perp) = (Uf)^*$ and $U(f \otimes g) = Uf \otimes Ug$ for all morphisms f and g in $\mathbf{G}\mathbb{C}$. Also, $U(\mathbf{1}) = I$, so the $*$-autonomous structure is preserved.

To show that R is a right adjoint for U, put $|\mathcal{A}| = A$ and $\mathcal{B} = RB$. It suffices to prove that any morphism $f : A \to B$ in $\mathbb{C}$ lifts to a morphism $\mathcal{A} \to \mathcal{B}$. Given $\alpha \in \mathcal{A}$, the composite $I \xrightarrow{\alpha} A \xrightarrow{f} B$ belongs to $\mathbb{C}(I, B) = \mathcal{B}_s$. Since $\mathcal{B}_t = \varnothing$, we have proved that $f : \mathcal{A} \to \mathcal{B}$. The proof that L is a left adjoint for U is similar. ■

Theorem 1.3.2 *Suppose that we have a multivariant functor* $F : (\mathbf{G}\mathbb{C})^n \times (\mathbf{G}\mathbb{C}^{op})^n \to \mathbf{G}\mathbb{C}$, *and that* $\rho : \mathfrak{K}_\mathbf{1} \to F$ *is a dinatural transformation in* $\mathbf{G}\mathbb{C}$, *i.e. a collection of morphisms* $\rho_{\underline{\mathcal{A}}} : \mathbf{1} \to F(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ *such that for all morphisms* $f_i : \mathcal{A}_i \to \mathcal{B}_i$ *in* $\mathbf{G}\mathbb{C}$, *the following diagram commutes, (writing* $\underline{\mathcal{A}}$ *for the list* $\mathcal{A}_1, \ldots, \mathcal{A}_n$, $\mathbf{f}$ *for the list* $f_1, \ldots, f_n$*).*



If $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}_1, \ldots, \mathcal{B}_n$ *are objects in* $\mathbf{G}\mathbb{C}$ *such that* $U\mathcal{A}_i = U\mathcal{B}_i$ *for all* i, *then* $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{B}}}$.

Proof Put $\mathcal{A}'_i = RU\mathcal{A}_i$ and $f_i = id_{|\mathcal{A}_i|}$ all i. Then each f_i lifts to a morphism from $\mathcal{A}_i$ to $\mathcal{A}'_i$ in $\mathbf{G}\mathbb{C}$. By dinaturality,

$$F(\mathbf{f}, id_{\underline{\mathcal{A}}})\rho_{\underline{\mathcal{A}}} = F(id_{\underline{\mathcal{A}'}}, \mathbf{f})\rho_{\underline{\mathcal{A}'}}.$$

But since $UF(\mathbf{f}, id_{\underline{\mathcal{A}}})$ is the identity morphism on $UF(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ and $UF(id_{\underline{\mathcal{A}'}}, \mathbf{f})$ is the identity morphism on $UF(\underline{\mathcal{A}'}, \underline{\mathcal{A}'}) = UF(\underline{\mathcal{A}}, \underline{\mathcal{A}})$, we have $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{A}'}} = U\rho_{RU\underline{\mathcal{A}}}$.

It now follows that $U\rho_{\underline{\mathcal{A}}} = U\rho_{RU\underline{\mathcal{A}}} = U\rho_{RU\underline{\mathcal{B}}} = U\rho_{\underline{\mathcal{B}}}$. ■

Theorem 1.3.2 tells us that if ρ is a dinatural transformation then *all* morphisms $\rho_{\underline{\mathcal{A}}}$ with identical objects $U\mathcal{A}_i$ are completely determined by the same underlying arrow in $\mathbb{C}$, independent of the choice of sets $(\mathcal{A}_i)_s$ and $(\mathcal{A}_i)_t$. So the dinatural transformation ρ in $\mathbf{G}\mathbb{C}$ is completely determined by the dinatural transformation $U\rho$ in $\mathbb{C}$.

As discussed in §0.4.1, we will aim to prove full completeness in $\mathbf{G}\mathbb{C}$ by showing that every dinatural transformation determines a MLL or MLL+Mix proof net (and hence a description of a morphism in the relevant free category). A proof structure is uniquely determined by its axiom links, and the axiom links will be determined by a compact closed full completeness result in the underlying category $\mathbb{C}$. Lifting this result up to $\mathbf{G}\mathbb{C}$, we will then prove that the proof structure we have obtained is indeed a proof net, and hence we will have full completeness in $\mathbf{G}\mathbb{C}$.

We now put this theory to practice in the next few sections, starting with **Rel**, the category of sets and relations.

Chapter 2

The category GRel

Our first example is to apply the glueing construction to **Rel** to obtain **GRel**. This category was first brought to light by Loader [Loa94b] as the category of Linear Logical Predicates. It is a particularly useful model of MLL because, unlike many other models, the Mix rule does *not* hold. The proofs in this chapter can be seen as an abstraction of Loader's results, and a better understanding of how they work. (For example, the "test" object he used to prove acyclicity (different to the one used here) was provided without foundation.) While his work can be appreciated for its specificity to LLP and **Rel**, here we aim to capture the full generality of the full completeness result and this will enable us to extend it to other categories.

Recall from §0.1.1 that the category **Rel** of sets and binary relations is a compact closed category. The tensor product is the usual cartesian product, which is clearly associative and symmetric. The unit I for tensor is the one element set, which clearly gives us the isomorphism $A \times I \cong A$. We take all sets to be self-dual, so that $\mathbf{Rel}(A \times B, C) \cong \mathbf{Rel}(A, B^* \times C)$. Finally, recall from §0.3.1 that $dim(\varnothing) = 0_I$ and $dim(A) = id_I$ for all $A \neq \varnothing$.

We also remark that **Rel** is enriched over the category $\mathbb{L}$ of complete sup semilattices. In particular, given two morphisms $R, S : A \to B$ in **Rel**, i.e. two subsets of $A \times B$, we can form the union of these subsets to form another morphism $R \cup S : A \to B$ in **Rel**. Thus, the full completeness result we will obtain will be with respect to a free compact closed category enriched over $\mathbb{L}$.

It is helpful to consider the glued category **GRel** in a framework more specific than the general construction. Since a relation between sets I and A is a subset of $I \times A \cong A$, we can regard the objects of **GRel** as triples $(|\mathcal{A}|, \mathcal{A}_s, \mathcal{A}_t)$ where $|\mathcal{A}|$ is a set, and $\mathcal{A}_s$ and $\mathcal{A}_t$ are collections of subsets of $|\mathcal{A}|$. (Recall that sets are self-dual in this context.)

There are precisely two relations from I to itself, namely the identity relation id_I and the empty relation 0_I. Thus the unit for tensor in **GRel** is $\mathbf{1} = (I, \{id_I\}, \{id_I, 0_I\})$, and the unit for par in **GRel** is $\perp = (I, \{id_I, 0_I\}, \{id_I\})$.

Finally, since $\mathbf{Rel}(I, I) \neq \{id_I\}$, we deduce from the final remark at the end of §1.1 that **GRel** does not support the Mix rule, and is a model of pure MLL.

2.1 Full Completeness in Rel

In this section, we assume that $F(X_1, \dots, X_n, Y_1, \dots, Y_n)$ is a formula of length p (i.e. there are precisely p literal occurrences in the formula) built from $X_1, \dots, X_n, Y_1^*, \dots, Y_n^*$ by the cartesian product $\times$. Then F induces a multivariant functor $[\![F]\!] : \mathbf{Rel}^n \times (\mathbf{Rel}^{op})^n \to \mathbf{Rel}$, which by abuse of notation we will also refer to as F. We will write

$$F(\mathbf{A}, \mathbf{A}) = A_{\xi_1}^{\zeta_1} \times \cdots \times A_{\xi_p}^{\zeta_p} \tag{2.1}$$

where each $\xi_i \in \{1, \dots, n\}$ and each $\zeta_i \in \{1, *\}$. (A_j^1 is read as A_j.) We will also make reference to the sets

$$\begin{array}{ll} N = \{i : \zeta_i = *\} & Q = \{i : \zeta_i = 1\} \\ N_j = \{i \in N : \xi_i = j\} & Q_j = \{i \in Q : \xi_i = j\} \quad (1 \le j \le n). \end{array} \tag{2.2}$$

Note that $i \in N_{\xi_i}$ for all $i \in N$. A remark on notation – we write a boldface $\mathbf{A}$ for the list $A_1, \dots, A_n$, a boldface $\mathbf{N}$ (not to be confused with the set N) for the list $N_1, \dots, N_n$ etc.

Recall that in a compact closed category, dinatural transformations can be viewed as natural transformations (Lemma 0.4.3). We will assume that morphisms $\sigma_{\mathbf{A}} : I \to F(\mathbf{A}, \mathbf{A})$ can be canonically transformed to morphisms $\tilde{\sigma}_{\mathbf{A}} : F^-(\mathbf{A}) \to F^+(\mathbf{A})$ where

$$F^-(\mathbf{A}) = A_{\lambda_1} \otimes \cdots \otimes A_{\lambda_m}; \quad F^+(\mathbf{A}) = A_{\mu_1} \otimes \cdots \otimes A_{\mu_l}, \tag{2.3}$$

with all $\lambda_k, \mu_k \in \{1, \dots, n\}$, so that N has cardinality m and Q has cardinality l. We will say that a natural transformation $\tilde{\sigma} : F^- \to F^+$ is *trivial* if every $\tilde{\sigma}_{\mathbf{A}}$ is the empty relation.

It is thought that the following result is something of a common folklore, but I was unable to find a suitable reference. The proof supplied here is by J.R.B. Cockett, which extends a proof I had written for the case when we restrict each $\tilde{\sigma}_{\mathbf{A}}$ to a single isomorphism.

Theorem 2.1.1 *Suppose that $\tilde{\sigma} : F^- \to F^+$ is a natural transformation in* **Rel**. *That is, given relations $R_j : A_j \to B_j$, the following diagram commutes.*

$$\begin{array}{ccc} F^-(\mathbf{A}) & \xrightarrow{\tilde{\sigma}_{\mathbf{A}}} & F^+(\mathbf{A}) \\ {\scriptstyle F^-(\mathbf{R})}\downarrow & & \downarrow{\scriptstyle F^+(\mathbf{R})} \\ F^-(\mathbf{B}) & \xrightarrow{\tilde{\sigma}_{\mathbf{B}}} & F^+(\mathbf{B}) \end{array}$$

Then $m = l$ and $\tilde{\sigma}$ is a union of "permutations on the tensor factors", i.e. a union of isomorphisms

$$(x_1, \dots, x_m) \mapsto (x_{\delta(1)}, \dots, x_{\delta(m)}), \quad x_i \in A_{\lambda_i},$$

where $\delta \in S_m$ satisfies $\lambda_{\delta(i)} = \mu_i$ for all $i = 1, \dots, m$.

Proof Since $\mathbf{Rel} \cong \mathbf{Rel}^{op}$, we can invert all arrows in the above diagram and deduce that $\tilde{\sigma}^{op} : F^+ \to F^-$ is also a natural transformation. We will make use of this fact later.

Choose "minimal sets" $A_1, \ldots, A_n$ such that there exists an m-tuple $\mathbf{x} = (x_1, \ldots, x_m) \in F^-(\mathbf{A})$ with all x_j distinct, and such that $\cup_{i=1}^n A_i = \{x_1, \ldots, x_m\}$.

If there is no $\mathbf{y} = (y_1, \ldots, y_l) \in F^+(\mathbf{A})$ such that $\tilde{\sigma}_{\mathbf{A}}$ sends $\mathbf{x}$ to $\mathbf{y}$, then $\tilde{\sigma}$ is a trivial natural transformation. To see this, consider arbitrary sets $B_1, \ldots, B_n$ and the "total" relations $R_i : A_i \to B_i$, $R_i = \{(a,b) \mid a \in A_i, b \in B_i\}$. By naturality, we deduce that $\tilde{\sigma}_{\mathbf{B}} F^-(\mathbf{R}) = F^+(\mathbf{R})\tilde{\sigma}_{\mathbf{A}}$ sends $\mathbf{x} \in F^-(\mathbf{A})$ to no l-tuple in $F^+(\mathbf{B})$. But $F^-(\mathbf{R}) : F^-(\mathbf{A}) \to F^-(\mathbf{B})$ sends $\mathbf{x}$ to every m-tuple in $F^-(\mathbf{B})$, and therefore $\tilde{\sigma}_{\mathbf{B}}$ must be an empty relation. Hence $\tilde{\sigma}$ is trivial.

Now suppose that there does exist $\mathbf{y} \in F^+(\mathbf{A})$ such that $\tilde{\sigma}$ sends $\mathbf{x}$ to $\mathbf{y}$. Note that $\cup_{i=1}^n A_i = \{x_1, \ldots, x_m\}$ so $\mathbf{y}$ is an l-tuple built from $x_1, \ldots, x_m$.

Suppose that there exists $x_i \in A_{\lambda_i}$ which does *not* occur as an entry in $\mathbf{y}$. Consider the relations $T_j : A_j \to A_j$, $j = 1, \ldots, n$,

$$T_j = \begin{cases} id_{A_j} & \text{if } j \neq \lambda_i; \\ \{(a,a) \mid a \neq x_i\} & \text{if } j = \lambda_i. \end{cases}$$

Then $F^+(\mathbf{T})\tilde{\sigma}_{\mathbf{A}}$ sends $\mathbf{x}$ to $\mathbf{y}$. But since $\mathbf{x}$ contains an occurrence of x_i, there is no l-tuple in $F^+(\mathbf{A})$ which is related to $\mathbf{x}$ by $F^-(\mathbf{T})$, so $\tilde{\sigma}_{\mathbf{A}} F^-(\mathbf{T})$ cannot send $\mathbf{x}$ to $\mathbf{y}$. This contradicts the naturality of $\tilde{\sigma}$. Therefore x_i must occur in the l-tuple $\mathbf{y}$. Since x_i was arbitrary, we have shown that every x_i occurs somewhere in $\mathbf{y}$, and hence $m \leq l$.

By duality, we can prove the same result for $\tilde{\sigma}^{op}$. Thus every entry y_i in the l-tuple $\mathbf{y}$ occurs somewhere in the m-tuple $\mathbf{x}$, i.e. $m \geq l$.

Therefore we have $m = l$, and $\mathbf{y}$ is some permutation of the entries in $\mathbf{x}$, i.e. there exists $\delta \in S_m$ such that $\tilde{\sigma}_{\mathbf{A}}$ sends $(x_1, \ldots, x_m) \in F^-(\mathbf{A})$ to $(x_{\delta(1)}, \ldots, x_{\delta(m)}) \in F^+(\mathbf{A})$. Then $x_{\delta(i)} \in A_{\mu_i}$, i.e. $\lambda_{\delta(i)} = \mu_i$ for all $i = 1, \ldots, m$.

Now consider arbitrary sets $B_1, \ldots, B_n$ and arbitrary points $b_i \in B_{\lambda_i}$, $i = 1, \ldots, m$. Consider the relations $U_j : A_j \to B_j$, $U_j = \{(x_i, b_i) \mid \lambda_i = j\}$, $j = 1, \ldots, n$. By naturality, we have $F^+(\mathbf{U})\tilde{\sigma}_{\mathbf{A}} = \tilde{\sigma}_{\mathbf{B}} F^-(\mathbf{U})$. In particular, $F^+(\mathbf{U})\tilde{\sigma}_{\mathbf{A}}$ sends $(x_1, \ldots, x_m) \in F^-(\mathbf{A})$ to $(b_{\delta(1)}, \ldots, b_{\delta(m)}) \in F^+(\mathbf{B})$. Since $F^-(\mathbf{U})$ sends $(x_1, \ldots, x_m)$ to $(b_1, \ldots, b_m)$, and *only* $(b_1, \ldots, b_m)$, we deduce that $\tilde{\sigma}_{\mathbf{B}}$ must send $(b_1, \ldots, b_m)$ to $(b_{\delta(1)}, \ldots, b_{\delta(m)})$. This completes the proof. ■

Corollary 2.1.2 *(Full Completeness in* **Rel***) Every non-trivial natural transformation $\tilde{\sigma} : F^- \to F^+$ in* **Rel** *is induced by a unique morphism $F^-(\mathbf{X}) \to F^+(\mathbf{X})$ in the free compact closed category on a set of n objects $X_1, \ldots, X_n$ with trivial dimension, enriched over the category $\mathbb{L}$ of complete sup semilattices.*

Observe that we can in fact remove the restriction on the dimension of the objects in the free category, because the dimensions can be absorbed into the morphisms in **Rel**. **Rel** interprets dimension zero as the empty relation on I, and all non-zero dimensions as the identity relation on I – tensoring with this map is a trivial operation. Thus we have,

Corollary 2.1.3 *(Full Completeness in* **Rel***) Every natural transformation $\tilde{\sigma} : F^- \to F^+$ in* **Rel** *is induced by a morphism $F^-(\mathbf{X}) \to F^+(\mathbf{X})$ in the free compact closed category on a set of n objects $X_1, \ldots, X_n$, enriched over the category $\mathbb{L}$ of complete sup semilattices.*

Note that this form of full completeness is weaker than Corollary 2.1.2 since we no longer have uniqueness.

We now translate Corollary 2.1.2 into a form which will be useful to us in **GRel**.

Theorem 2.1.4 *Suppose that σ is a non-trivial dinatural transformation in* **Rel** *from the constant functor $\mathfrak{K}_I$ to the multivariant functor F. Then there exist fixed-point-free involutions $\phi_1, \ldots, \phi_T$ on $\{1, \ldots, p\}$ such that $\xi_{\phi_h(i)} = \xi_i$, $\zeta_{\phi_h(i)} \neq \zeta_i$ for all h, and*

$$\sigma_{\mathbf{A}} = \bigcup_{h=1}^{T} \{(x_1, \ldots, x_p) \in F(\mathbf{A}, \mathbf{A}) \mid x_{\phi_h(i)} = x_i \text{ for all } i = 1, \ldots, p\}.$$

Proof Each permutation δ associated with $\tilde{\sigma}$ identifies occurrences of A_{ξ_i} in $F^-(\mathbf{A})$ with occurrences of A_{ξ_i} in $F^+(\mathbf{A})$. Thus to σ we can associate fixed-point-free involutions $\phi_h : \{1, \ldots, p\} \to \{1, \ldots, p\}$, linking occurrences of A_{ξ_i} with occurrences of $A^*_{\xi_i}$ in the formula $F(\mathbf{A}, \mathbf{A})$, as we shall now describe.

Choose one permutation $\delta \in S_m$ associated with $\tilde{\sigma}$. By the previous theorem, we know that F is a balanced formula. Write $N = \{i_1, \ldots, i_m\}$ and $Q = \{j_1, \ldots, j_m\}$ and assume that $\xi_{i_k} = \lambda_k$, $\xi_{j_k} = \mu_k$, $k = 1, \ldots, m$.

Given elements $x_k \in A_{\xi_{i_k}}$, $k = 1, \ldots, m$, we know that

$$\tilde{\sigma}_{\mathbf{A}} \text{ maps } (x_1, \ldots, x_m) \text{ to } (x_{\delta(1)}, \ldots, x_{\delta(m)}),$$

from which we deduce that each $x_{\delta(k)}$ belongs to the set $A_{\xi_{j_k}}$. Hence

$$\xi_{j_k} = \xi_{i_{\delta(k)}}$$

and similarly

$$\xi_{i_k} = \xi_{j_{\delta^{-1}(k)}}$$

for each $k = 1, \ldots, m$.

Put $\phi(i_k) = j_{\delta^{-1}(k)}$ and $\phi(j_k) = i_{\delta(k)}$ for all $k = 1, \ldots, m$. Then

$$\phi^2(i_k) = \phi(j_{\delta^{-1}(k)}) = i_k; \quad \phi^2(j_k) = \phi(i_{\delta(k)}) = j_k,$$

for each k, so ϕ is an involution. Moreover, ϕ maps N to Q and Q to N, so $\zeta_{\phi(i)} \neq \zeta_i$. Finally,

$$\xi_{\phi(i_k)} = \xi_{j_{\delta^{-1}(k)}} = \xi_{i_k}; \quad \xi_{\phi(j_k)} = \xi_{i_{\delta(k)}} = \xi_{j_k},$$

for each $k = 1, \ldots, m$.

Now, if $(x_1, \ldots, x_p) \in \sigma_{\mathbf{A}}$ then $\tilde{\sigma}_{\mathbf{A}}$ sends $(x_{i_1}, \ldots, x_{i_m})$ to $(x_{j_1}, \ldots, x_{j_m})$. So $x_{i_{\delta(k)}} = x_{j_k}$ for each $k = 1, \ldots, m$. To each $r \in \{1, \ldots, m\}$, there exists k such that $\delta(k) = r$. Then $x_{j_k} = x_{i_{\delta(k)}} = x_{i_r}$. Thus $x_{\phi(i_r)} = x_{j_{\delta^{-1}(r)}} = x_{j_k} = x_{i_r}$. Similarly, we can show that $x_{\phi(j_k)} = x_{j_k}$ for each $k \in \{1, \ldots, m\}$. The proof is now complete. ■

2.2 Full Completeness in GRel

We now assume that $F(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ ($\underline{\mathcal{X}} = \mathcal{X}_1, \ldots, \mathcal{X}_n$ etc.) is a formula of length p built from $\mathcal{X}_1, \ldots, \mathcal{X}_n$, $\mathcal{Y}_1^\perp, \ldots, \mathcal{Y}_n^\perp$ by the connectives $\otimes$ and ⅋. Then F induces a multivariant functor $[\![F]\!] : (\mathbf{GRel})^n \times (\mathbf{GRel}^{op})^n \to \mathbf{GRel}$, which by abuse of notation we will also refer to as F.

Before proving any results, we ask the question : "What is a morphism $\rho_{\underline{\mathcal{A}}} : \mathbf{1} \to F(\underline{\mathcal{A}}, \underline{\mathcal{A}})$?" We can assume that $UF(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ takes the form of the formula $F(\mathbf{A}, \mathbf{A})$ in equation (2.1), with $|\mathcal{A}_i| = A_i$ for all i. We have previously observed that $\rho_{\underline{\mathcal{A}}}$ is an element of $F(\underline{\mathcal{A}}, \underline{\mathcal{A}})_s$, and $F(\underline{\mathcal{A}}, \underline{\mathcal{A}})_s$ is a subset of $\mathbf{Rel}(I, UF(\underline{\mathcal{A}}, \underline{\mathcal{A}}))$. So $\rho_{\underline{\mathcal{A}}}$ is essentially a subset of $UF(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = F(\mathbf{A}, \mathbf{A})$.

In **GRel**, we will say that a dinatural transformation $\rho : \mathfrak{K}_{\mathbf{1}} \to F$ is *non-trivial* if $U\rho$ is a non-trivial dinatural transformation in **Rel**, i.e. each $U\rho_{\underline{\mathcal{A}}}$ is a non-empty subset of $F(\mathbf{A}, \mathbf{A})$.

While **GRel** does not support the Mix rule, there is a full subcategory of **GRel** which does. We describe this subcategory for the general category $\mathbf{G}\mathbb{C}$.

Definition 2.2.1 Let $\mathbb{S}$ denote the full subcategory of $\mathbf{G}\mathbb{C}$ restricted to objects $\mathcal{A}$ of $\mathbf{G}\mathbb{C}$ such that given two morphisms $\alpha : \mathbf{1} \to \mathcal{A}$ and $\sigma : \mathcal{A} \to \perp$ (i.e. $\alpha \in \mathcal{A}_s$ and $\sigma \in \mathcal{A}_t$), the composite

$$U(\sigma\alpha) : I \xrightarrow{U\alpha} |\mathcal{A}| \xrightarrow{U\sigma} I$$

is the identity morphism on I in $\mathbb{C}$.

In **GRel**, $\mathbb{S}$ consists of objects $\mathcal{A}$ such that whenever $\alpha \in \mathcal{A}_s$ and $\beta \in \mathcal{A}_t$ (so that α and β are subsets of $|\mathcal{A}|$), their intersection $\alpha \cap \beta$ is non-empty.

Lemma 2.2.2 *$\mathbb{S}$ is closed under the tensor product and linear negation, and hence under the par product of* $\mathbf{G}\mathbb{C}$.

Proof It is clear that $\mathbb{S}$ is closed under the involution $(-)^\perp$.

If $\mathcal{A}$ and $\mathcal{B}$ are objects in $\mathbb{S}$, then so is $\mathcal{A}\otimes\mathcal{B}$. To see this, suppose that $\gamma : \mathbf{1}\to\mathcal{A}\otimes\mathcal{B}$ and $\nu : \mathcal{A}\otimes\mathcal{B}\to\bot$. Then $U\gamma = U\gamma_1\otimes U\gamma_2$ for some $\gamma_1 : \mathbf{1}\to\mathcal{A}$ and $\gamma_2 : \mathbf{1}\to\mathcal{B}$, and $U\nu$ is equivalently a morphism from $|\mathcal{A}|$ to $|\mathcal{B}|^*$. Therefore,

$$\begin{aligned} U(\mathbf{1}\xrightarrow{\gamma}\mathcal{A}\otimes\mathcal{B}\xrightarrow{\nu}\bot) &= I\xrightarrow{U\gamma_1}|\mathcal{A}|\xrightarrow{U\nu}|\mathcal{B}|^*\xrightarrow{U\gamma_2} I \\ &= U(\mathbf{1}\xrightarrow{\gamma_1}\mathcal{A}\xrightarrow{\nu}\mathcal{B}^\perp\xrightarrow{\gamma_2}\bot). \end{aligned}$$

But since $\mathcal{A}$ is an object in $\mathbb{S}$, $U((\gamma_2\nu)\gamma_1) = id_I$. Therefore $\mathcal{A}\otimes\mathcal{B}$ is also an object in $\mathbb{S}$.

Since $\mathbb{S}$ is closed under linear negation and tensor product, it is also closed under the par product. ■

Note that if $\mathbb{C}(I,I)\neq\{id_I\}$, then the unit $\mathbf{1}$ for tensor in $\mathbf{G}\mathbb{C}$ does not belong to $\mathbb{S}$. But $\mathbb{S}$ does have a $*$-autonomous structure when we introduce a new unit $\mathbf{1}_\mathbb{S} = (I,\{id_I\},\{id_I\})$ for the tensor in $\mathbb{S}$.

Corollary 2.2.3 $(\mathbb{S},\otimes,\mathbf{1}_\mathbb{S},(-)^\perp)$ *is a $*$-autonomous category with unit for tensor* $\mathbf{1}_\mathbb{S} = (I,\{id_I\},\{id_I\})$.

Proof Suppose that $\mathcal{A}$ is an object in $\mathbb{S}$. Clearly, $(\mathcal{A}\otimes\mathbf{1}_\mathbb{S})_s\cong\mathcal{A}_s$. Suppose that $f\in\mathcal{A}_t$. Then f satisfies

$$\text{if } \alpha\in\mathcal{A}_s \text{ then } f\alpha : I\to|\mathcal{A}|\to I = id_I\in(\mathbf{1}_\mathbb{S})_t.$$

Furthermore,

$$\text{if } \beta\in(\mathbf{1}_\mathbb{S})_s, \text{ then } \beta = id_I, \text{ so } f\beta^* = f\in\mathcal{A}_t \text{ if and only if } f\in\mathcal{A}_t.$$

So $f\in\mathcal{A}_t$ if and only if $f\in(\mathcal{A}\otimes\mathbf{1}_\mathbb{S})_t$, i.e. $(\mathcal{A}\otimes\mathbf{1}_\mathbb{S})_t\cong\mathcal{A}_t$. The coherence conditions are clear, so the result now follows. ■

Consequently, the unit for tensor and the unit for par in $\mathbb{S}$ are identical, and the identity morphism id_I in $\mathbb{C}$ lifts to a unary Mix morphism $m : \bot_\mathbb{S}\to\mathbf{1}_\mathbb{S}$, as described in §0.2.4. Therefore $\mathbb{S}$ supports the Mix rule. In particular, we have the following lemma.

Lemma 2.2.4 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are objects in $\mathbb{S}$ and that $\sigma : \mathcal{A}\to\bot$ and $\tau : \mathcal{B}\to\bot$ are morphisms in $\mathbf{G}\mathbb{C}$. Then $U\sigma\times U\tau : |\mathcal{A}|\times|\mathcal{B}|\to I$ lifts to a morphism in $\mathbf{G}\mathbb{C}$ from $\mathcal{A}\otimes\mathcal{B}$ to $\bot$.*

Proof Observe that $U\sigma\times U\tau$ is equivalent to the morphism $U\tau^*U\sigma : |\mathcal{A}|\to|\mathcal{B}|^*$. It suffices to show that $U\tau^*U\sigma\in(\mathcal{A}\otimes\mathcal{B})_t = \mathbf{G}\mathbb{C}(\mathcal{A},\mathcal{B}^\perp)$.

Suppose that $\alpha\in\mathcal{A}_s$, $\alpha : \top\to\mathcal{A}$. Then $U(\sigma\alpha) = id_I$, so

$$(U\tau^*U\sigma)U\alpha = U\tau^*(U(\sigma\alpha)) = U\tau^*\in\mathcal{B}_t.$$

Now suppose that $\beta \in \mathcal{B}_s$, $\beta : \mathbf{1} \to \mathcal{B}$. Then $U(\tau\beta) = id_I$, so

$$(U\tau^* U\sigma)^* U\beta = U\sigma^*(U(\tau\beta)) = U\sigma^* \in \mathcal{A}_t.$$

Therefore $U\tau^* U\sigma$ belongs to $\mathbf{G}\mathbb{C}(\mathcal{A}, \mathcal{B}^\perp)$. ∎

A consequence of this result is that for objects in $\mathbb{S}$, it is easy to construct a morphism $F(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \perp$. The tensor product of two morphisms $|\mathcal{A}| \to I$ and $|\mathcal{B}| \to I$ always lifts to a morphism $\mathcal{A} \parr \mathcal{B} \to \perp$. Thus, we can form an appropriate tensor product of morphisms $|\mathcal{A}_i| \to I$ to produce a map $F(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \perp$ in $\mathbf{G}\mathbb{C}$.

In Lemma 2.2.6 only, we need a further restriction on objects in **GRel**. Define the full subcategory $\mathbb{S}^1$ of $\mathbb{S}$ restricted to objects $\mathcal{A}$ in $\mathbb{S}$ such that whenever $\alpha \in \mathcal{A}_s$ and $\beta \in \mathcal{A}_t$ (so that α and β are subsets of $|\mathcal{A}|$,) their intersection $\alpha \cap \beta$ is a singleton.

Proposition 2.2.5 *The full subcategory $\mathbb{S}^1$ is closed under the $*$-autonomous structure of $\mathbb{S}$.*

Proof It is clear that $\mathbf{1}_{\mathbb{S}}$ is an object in $\mathbb{S}^1$, and that $\mathbb{S}^1$ is closed under $(-)^\perp$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are objects in $\mathbb{S}^1$, that $r_1 \times r_2 \in (\mathcal{A} \otimes \mathcal{B})_s$ and $t \in (\mathcal{A} \otimes \mathcal{B})_t$, and that $(a^*, b^*) \in (r_1 \times r_2) \cap t$. Then $a^* \in r_1 \in \mathcal{A}_s$ implies that

$$b^* \in \{b \in B \mid (a, b) \in t \text{ for some } a \in r_1\} \in \mathcal{B}_t.$$

But $b^* \in r_2 \in \mathcal{B}_s$, so b^* must be unique. A symmetrical argument proves that a^* is also unique. ∎

Lemma 2.2.6 *Suppose that ρ is a non-trivial dinatural transformation in* **GRel** *from the constant functor $\mathfrak{K}_{\mathbf{1}}$ to the multivariant functor F. Then there exists a unique fixed-point-free involution $\phi : \{1, \ldots, p\} \to \{1, \ldots, p\}$ such that $\xi_{\phi(i)} = \xi_i$ and $\zeta_{\phi(i)} \neq \zeta_i$ for all i, and*

$$\rho_{\underline{\mathcal{A}}} = \{(x_1, \ldots, x_p) \in F(\mathbf{A}, \mathbf{A}) \mid x_{\phi(i)} = x_i \text{ for all } i = 1, \ldots, p\}, \tag{2.4}$$

where $|\mathcal{A}_i| = A_i$ for all i and $F(\mathbf{A}, \mathbf{A})$ has the form in equation (2.1), thus determining a unique set of axiom links for a proof structure of the MLL formula F.

Proof Given arbitrary sets $A_1, \ldots, A_n$, define objects in **GRel**,

$$\mathcal{A}_i = (A_i, Sing(A_i), \{A_i\}), \qquad i = 1, \ldots, n,$$

where $Sing(A_i) = \{\{x\} \mid x \in A_i\}$ denotes the set of singleton subsets of A_i. Each $\mathcal{A}_i$ belongs to $\mathbb{S}^1$, so $F(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \in \mathbb{S}^1$. Therefore, given elements $x_i \in A_{\xi_i}$, $i \in N$, we can construct a map $F(\underline{\mathcal{A}},\underline{\mathcal{A}}) \to \bot$, namely $\bar{t} = t_1 \times \cdots \times t_p$, where

$$t_i = \begin{cases} \{x_i\} & \text{if } i \in N; \\ A_{\xi_i} & \text{if } i \in P. \end{cases}$$

Since $F(\underline{\mathcal{A}},\underline{\mathcal{A}}) \in \mathbb{S}^1$, $\rho_{\underline{\mathcal{A}}} \cap \bar{t}$ is a singleton. By choice of $\bar{t}$, this implies there exists a unique $(x_1, \ldots, x_p) \in U\rho_{\underline{\mathcal{A}}}$ determined by the elements x_i, $i \in N$. Similarly, by considering objects $\mathcal{B}_i = \mathcal{A}_i^{\perp}$, we deduce that there exists a unique $(x_1, \ldots, x_p) \in \rho_{\underline{\mathcal{B}}}$ determined by elements $x_i \in A_{\xi_i}$, $i \in P$.

But $U\mathcal{A}_i = U\mathcal{B}_i$ for all i, so $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{B}}} = \sigma_{\mathbf{A}}$, say. Thus the canonical natural transformation $\tilde{\sigma} : F^- \to F^+$ is a collection of *isomorphisms.*

By Theorem 2.1.1, $\tilde{\sigma}$ is a union of permutations on the tensor factors. Since each $\tilde{\sigma}_{\mathbf{A}}$ is an isomorphism, we deduce that $\tilde{\sigma}$ is precisely *one* permutation on the tensor factors. Thus, by Theorem 2.1.4, there exists a unique fixed-point-free involution ϕ on $\{1, \ldots, p\}$ such that

$$\sigma_{\mathbf{A}} = \{(x_1, \ldots, x_p) \in F(\mathbf{A}, \mathbf{A}) \mid x_{\phi(i)} = x_i \text{ for all } i = 1, \ldots, p\}.$$

Since $U\rho_{\underline{\mathcal{A}}} = \sigma_{\mathbf{A}}$, we have equation (2.4). ■

Loader called the objects of $\mathbb{S}^1$ "unitary LLPs". His proof of the existence of ϕ was set entirely in **LLP**. Here, we have shown that ϕ is in fact predetermined in **Rel**.

We have now characterised the morphisms of a non-trivial dinatural transformation in **GRel**, and in particular, we can associate to each non-trivial dinatural transformation a unique MLL proof structure. We will now prove that such a proof structure must be a proof net. Of course, the proof comes in two parts. Firstly, we show that for every DR-switching, the associated DR-graph is acyclic. Then, we show that the DR-graph is connected.

Consider a proof structure for a formula F such that

$$UF(\underline{\mathcal{A}},\underline{\mathcal{A}}) = A_{\xi_1}^{\zeta_1} \otimes \cdots \otimes A_{\xi_p}^{\zeta_p},$$

where $|\mathcal{A}_i| = A_i$ for all $i = 1, \ldots, n$. Given a DR-graph associated with a certain DR-switching, we say a pair (a, b) is *lower connected* if there is a path from $A_{\xi_a}^{\zeta_a}$ to $A_{\xi_b}^{\zeta_b}$ in the lower part of the DR-graph, that is, that part of the graph without axiom links.

Proposition 2.2.7 *Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are objects in $\mathbb{S}$ and that we have a formula*

$$\tilde{F}(\underline{\mathcal{A}},\underline{\mathcal{A}}) = \Gamma_{\underline{\mathcal{A}}} \mathbin{⅋} (\mathcal{A}_{\xi_{a_1}}^{\zeta_{a_1}} \otimes \mathcal{A}_{\xi_{b_1}}^{\zeta_{b_1}}) \mathbin{⅋} \cdots \mathbin{⅋} (\mathcal{A}_{\xi_{a_r}}^{\zeta_{a_r}} \otimes \mathcal{A}_{\xi_{b_r}}^{\zeta_{b_r}}) \qquad (2.5)$$

Suppose that we have the following morphisms in **G**$\mathbb{C}$*:*

$$\begin{array}{ll} \tau_i : \mathcal{A}_{\xi_{a_i}}^{\zeta_{a_i}} \otimes \mathcal{A}_{\xi_{b_i}}^{\zeta_{b_i}} \to \bot, & i = 1, \ldots, r; \\ \tau_j^+ : \mathcal{A}_j \to \bot, & j = 1, \ldots, n; \\ \tau_j^- : \mathcal{A}_j^{\perp} \to \bot, & j = 1, \ldots, n. \end{array}$$

Then we can construct a morphism $\tau_{\underline{\mathcal{A}}} : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \bot$ which is the lifting of the tensor product in $\mathbb{C}$ of a combination of $U\tau_j^+$ and $U\tau_j^-$, and the morphisms $U\tau_1, \dots, U\tau_r$.

Proof The tensor product in $\mathbb{C}$ of morphisms $|\mathcal{A}| \to I$ and $|\mathcal{B}| \to I$ always lifts to a morphism $\mathcal{A} \parr \mathcal{B} \to \bot$ in $\mathbf{G}\mathbb{C}$. If, in addition, $\mathcal{A}$ and $\mathcal{B}$ are objects in $\mathbb{S}$, then by Lemma 2.2.4 it also lifts to a morphism $\mathcal{A} \otimes \mathcal{B} \to \bot$. Consequently, we may form the tensor product of a suitable combination of $U\tau_j^+$ and $U\tau_j^-$ to form a morphism $\Gamma_{\underline{\mathcal{A}}} \to \bot$. Now take the tensor product of this morphism with $U\tau_1, \dots, U\tau_r$, and we have a morphism $\tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \bot$. ■

Theorem 2.2.8 *Suppose that ρ is a non-trivial dinatural transformation in* **GRel** *from the constant functor $\mathfrak{K}_\mathbf{1}$ to the multivariant functor F. Consider the unique MLL proof structure for F associated with ρ. Then for any DR-switching, the associated DR-graph is acyclic.*

Proof By Lemma 2.2.6, we know that there exists a unique fixed-point-free involution ϕ specifying a unique set of axiom links for a proof structure for F.

Suppose that for a certain DR-switching, the associated DR-graph G contains a cycle. Consider the shortest cycle, and express it as lower connected pairs $(a_1, b_1), \dots, (a_r, b_r)$ such that $\phi(b_i) = a_{i+1}$ for all $i \in \mathbb{Z}_r$. There are two parts to this proof. Firstly, we obtain a new non-trivial dinatural transformation $\tilde{\rho} : \mathfrak{K}_\mathbf{1} \to \tilde{F}$, where $\tilde{F}$ is a formula of the form (2.5), in such a way that the cycle in G is preserved in the proof structure associated with $\tilde{\rho}$. (This reduction already appears in the context of embeddings in [AJ94].) Then we show that $\tilde{\rho}$ should not exist.

Every $*$-autonomous category has the weak distributivity morphisms (Theorem 0.1.11)

$$\begin{aligned} w^L_{ABC} &: A \otimes (B \parr C) \to (A \otimes B) \parr C; \\ w^R_{ABC} &: A \otimes (B \parr C) \to (A \otimes C) \parr B. \end{aligned}$$

In $\mathbb{S}$, we also have the binary Mix morphisms

$$z_{AB} : A \otimes B \to A \parr B.$$

These morphisms are natural in A, B and C. Recall that dinaturality is preserved when composing with natural transformations. (See §0.4.) Therefore, it is possible (using also the natural commutativity and associativity morphisms of both tensor and par) to derive from ρ a non-trivial dinatural collection $\tilde{\rho}$ of morphisms $\mathbf{1} \to \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ *restricted to objects* $\tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ *in* $\mathbb{S}$, if we adopt the following procedure.

- If a fragment of F has the form $A \otimes (B \parr C)$, and A and B are lower connected, then the switching must have assigned [left] to the $\parr$-link in question. In this case, compose ρ with a natural transformation built from w^L_{ABC};

- If, on the other hand, A and C are lower connected, then the switching must have assigned [right] to the ⅋-link in question. In this case, compose with a natural transformation built from w^R_{ABC};
- Apply Mix, commutativity and associativity, whenever necessary, to separate out each lower connected pair.

In particular, we will preserve the lower connected pairs of the original DR-graph, and so the proof structure of $\tilde{F}$ will be cyclic.

Now we prove that $\tilde{\rho}$ should not exist. Choose a "test" object $\mathcal{A} = (A, \mathcal{A}_s, \mathcal{A}_t)$ in $\mathbb{S}$ such that $\mathcal{A}^\perp = \mathcal{A}$, $A \in \mathcal{A}_s$, and $A \in \mathcal{A}_t$, and such that $\mathcal{A}$ possesses a morphism $\alpha : \mathcal{A} \otimes \mathcal{A}^\perp \to \perp$ with "trace zero", i.e. the composition

$$I \xrightarrow{\eta_A} A \otimes A^* \xrightarrow{U\alpha} I$$

is the *zero* morphism on I. (In **Rel**, we mean a subset $\alpha \subseteq A \times A$ with no ordered pair of the form (x, x). An example of such an object is $\mathcal{A} = (A, \{A\}, \{A\})$ with $A = \{1, 2\}$, and $\alpha \subseteq A \times A$, $\alpha = \{(1,2), (2,1)\}$.)

Put $\mathcal{A}_i = \mathcal{A}$ for all $i = 1, \ldots, n$. Then in **Rel**, $U\tilde{\rho}$ can be decomposed into $f_1 \otimes f_2$ where

$$f_1 : I \to \Gamma_{\mathbf{A}}; \quad f_2 : I \to (A^{\zeta_{a_1}} \otimes A^{\zeta_{b_1}}) \otimes \cdots \otimes (A^{\zeta_{a_r}} \otimes A^{\zeta_{b_r}}).$$

Choose $\tau^+ : \mathcal{A} \to \perp$, $\tau^+ = A$ and $\tau^- : \mathcal{A}^\perp \to \perp$, $\tau^- = A$. Note that $id_A : A \to A$ lifts to a morphism $\mathcal{A} \to \mathcal{A}$ or equivalently $\mathcal{A} \otimes \mathcal{A}^\perp \to \perp$, but since $\mathcal{A}$ is self-dual, id_A also lifts to morphisms $\mathcal{A} \otimes \mathcal{A} \to \perp$, $\mathcal{A}^\perp \otimes \mathcal{A} \to \perp$ and $\mathcal{A}^\perp \otimes \mathcal{A}^\perp \to \perp$. Therefore, we can set $\tau_i : \mathcal{A}^{\zeta_{a_i}} \otimes \mathcal{A}^{\zeta_{b_i}} \to \perp$ to be the lifting of id_A for all $i = 1, \ldots, r-1$ and put $\tau_r = \alpha : \mathcal{A} \otimes \mathcal{A}^\perp \to \perp$. Then by the Proposition 2.2.7, we can construct a morphism $\tau_{\underline{\mathcal{A}}} : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \perp$. In particular, $U\tau_{\underline{\mathcal{A}}}$ can be decomposed into $g_1 \otimes g_2$ where

$$g_1 : \Gamma_{\mathbf{A}} \to I; \quad g_2 : (A^{\zeta_{a_1}} \otimes A^{\zeta_{b_1}}) \otimes \cdots \otimes (A^{\zeta_{a_r}} \otimes A^{\zeta_{b_r}}) \to I.$$

Moreover, since $\tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ is an object in $\mathbb{S}$, we have

$$U(\tau_{\underline{\mathcal{A}}} \tilde{\rho}_{\underline{\mathcal{A}}}) = id_I. \tag{2.6}$$

So $(U\tau_{\underline{\mathcal{A}}})(U\tilde{\rho}_{\underline{\mathcal{A}}}) = (g_1 \otimes g_2)(f_1 \otimes f_2) = (g_1 f_1) \otimes (g_2 f_2) = id_I$. By construction, the map $g_1 : \Gamma_{\mathbf{A}} \to I$ is the set $A \times \cdots \times A$, so f_1 obviously has an intersection with g_1, i.e. $g_1 f_1 = id_I$. But by construction, the composition $g_2 f_2$ is isomorphic to the map

$$I \xrightarrow{\eta_A} A \otimes A^* \xrightarrow{U\alpha} I$$

which by choice of α is the *zero* morphism on I. Therefore $(g_1 f_1) \otimes (g_2 f_2)$ cannot be the identity morphism on I, contradicting equation (2.6). Therefore $\tilde{\rho}$ cannot exist. This completes the proof. ■

To prove the connectedness of the DR-graphs, we draw attention to another full subcategory of $\mathbf{G}\mathbb{C}$. We remark that Loader's treatment of this part of full completeness contained some inaccuracies, but we will show that the result itself still holds true.

Definition 2.2.9 Suppose that $\mathbb{C}$ has a null object. Denote by $\mathbb{T}$ the full subcategory of $\mathbf{G}\mathbb{C}$ restricted to objects $\mathcal{A}$ in $\mathbf{G}\mathbb{C}$ such that given two morphisms $\alpha : \mathbf{1} \to \mathcal{A}$ and $\sigma : \mathcal{A} \to \bot$ (i.e. $\alpha \in \mathcal{A}_s$ and $\sigma \in \mathcal{A}_t$), the composite

$$U(\sigma\alpha) : I \xrightarrow{U\alpha} |\mathcal{A}| \xrightarrow{U\sigma} I$$

is the zero morphism on I in $\mathbb{C}$.

In **Rel**, the null object is of course the empty set, so we do have a zero morphism on I, namely the empty relation. Therefore in **GRel**, $\mathbb{T}$ is the collection of objects $\mathcal{A}$ such that the intersection of any set in $\mathcal{A}_s$ and any set in $\mathcal{A}_t$ is empty.

Lemma 2.2.10 *$\mathbb{T}$ is closed under the tensor product and linear negation, and hence under the par product of* $\mathbf{G}\mathbb{C}$.

Proof It is clear that $\mathbb{T}$ is closed under the involution $(-)^\perp$.

If $\mathcal{A}$ and $\mathcal{B}$ are objects in $\mathbb{T}$, then so is $\mathcal{A} \otimes \mathcal{B}$. To see this, suppose that $\gamma : \mathbf{1} \to \mathcal{A} \otimes \mathcal{B}$ and $\nu : \mathcal{A} \otimes \mathcal{B} \to \bot$. Then $U\gamma = U\gamma_1 \otimes U\gamma_2$ for some $\gamma_1 : \mathbf{1} \to \mathcal{A}$ and $\gamma_2 : \mathbf{1} \to \mathcal{B}$, and $U\nu$ is equivalently a morphism from $|\mathcal{A}|$ to $|\mathcal{B}|^*$. Therefore,

$$\begin{aligned} U(\mathbf{1} \xrightarrow{\gamma} \mathcal{A} \otimes \mathcal{B} \xrightarrow{\nu} \bot) &= I \xrightarrow{U\gamma_1} |\mathcal{A}| \xrightarrow{U\nu} |\mathcal{B}|^* \xrightarrow{U\gamma_2} I \\ &= U(\mathbf{1} \xrightarrow{\gamma_1} \mathcal{A} \xrightarrow{\nu} \mathcal{B}^\perp \xrightarrow{\gamma_2} \bot). \end{aligned}$$

But since $\mathcal{A}$ is an object in $\mathbb{T}$, $U((\gamma_2\nu)\gamma_1) = 0_I$. Therefore $\mathcal{A} \otimes \mathcal{B}$ is also an object in $\mathbb{T}$.

Since **T** is closed under linear negation and tensor product, it is also closed under the par product. ■

Note that the unit $\mathbf{1}$ for tensor in $\mathbf{G}\mathbb{C}$ only belongs to $\mathbb{T}$ in the totally degenerate case when I is also the null object of $\mathbb{C}$. (Then $A \cong A \otimes I = I$ for all A, i.e. $\mathbb{C}$ is the category with one object, one morphism.) In any case, we can show that $\mathbb{T}$ has a $*$-autonomous structure when we introduce a new unit $\mathbf{1}_\mathbb{T} = (I, \{id_I\}, \{0_I\})$ for the tensor in $\mathbb{T}$.

Corollary 2.2.11 *If $\mathbb{C}$ has a null object, then $(\mathbb{T}, \otimes, \mathbf{1}_\mathbb{T}, (-)^\perp)$ is a $*$-autonomous category with unit for tensor $\mathbf{1}_\mathbb{T} = (I, \{id_I\}, \{0_I\})$.*

Proof Suppose that $\mathcal{A}$ is an object in $\mathbb{T}$. Clearly $(\mathcal{A} \otimes \mathbf{1}_{\mathbb{T}})_s \cong \mathcal{A}_s$. Suppose that $f \in \mathcal{A}_t$. Then

$$\text{if } \alpha \in \mathcal{A}_s \text{ then } f\alpha = 0_I \in (\mathbf{1}_{\mathbb{T}})_t.$$

Furthermore,

$$\text{if } \beta \in (\mathbf{1}_{\mathbb{T}})_s \text{ then } \beta = id_I, \text{ so } f\beta^* = f \in \mathcal{A}_t \text{ if and only if } f \in \mathcal{A}_t.$$

Therefore $f \in \mathcal{A}_t$ if and only if $f \in (\mathcal{A} \otimes \mathbf{1}_{\mathbb{T}})_t$, i.e. $(\mathcal{A} \otimes \mathbf{1}_{\mathbb{T}})_t \cong \mathcal{A}_t$. The coherence conditions are clear, so the result now follows. ■

Corollary 2.2.12 *The full subcategory $\mathbb{T}_0$ of $\mathbb{T}$ restricted to objects $\mathcal{A}$ of $\mathbb{T}$ such that*

- *$\mathcal{A}_s$ and $\mathcal{A}_t$ are non-empty;*
- *neither $\mathcal{A}_s$ nor $\mathcal{A}_t$ contain a zero map,*

is closed under linear negation, the tensor product and hence the par product of $\mathbb{T}$.

Theorem 2.2.13 *Suppose that ρ is a non-trivial dinatural transformation in* **GRel** *from the constant functor $\mathfrak{K}_\mathbf{1}$ to the multivariant functor F. Consider the unique proof structure for F associated with ρ. Then for any DR-switching, the associated DR-graph is connected.*

Proof Suppose that for a certain DR-switching, the DR-graph is not connected. By applying the weak distributivity morphisms to F, we can obtain a new formula of the form $F_1 \parr \cdots \parr F_q$ where each F_k does not contain a par connective, and such that the components of the original DR-graph are preserved in a particular switching for the proof structure of the new formula, as we now explain.

- If a fragment of F takes the form $A \otimes (B \parr C)$ and the switching assigns [left] to this $\parr$-link, then compose with w^L_{ABC} and assign [left] to the new $\parr$-link.
- If, on the other hand, the switching assigns [right] to the original $\parr$-link, then compose with w^R_{ABC} and *still assign* [left] to the new $\parr$-link. (This is because B and C have now interchanged positions.)

Group the F_i into a maximal number of components $C_1, \ldots, C_q$ such that any axiom links from vertices in C_j are completely contained within C_j. To suppose that the DR-graph is not connected is to suppose that $q > 1$. Put $\hat{F} = C_1 \parr \cdots \parr C_q$. Then $F \cong \hat{F}$, and given the non-trivial dinatural collection of morphisms $\rho_{\underline{\mathcal{A}}} : \mathbf{1} \to F(\underline{\mathcal{A}}, \underline{\mathcal{A}})$, the corresponding morphisms $\hat{\rho}_{\underline{\mathcal{A}}} : \mathbf{1} \to \hat{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ will also be non-trivial and dinatural, since the symmetry morphisms for par are natural and dinaturality is preserved when composing with natural transformations.

We now prove that any C_j is itself a *provable* formula in MLL. Recall that $C_j = F_{j_1} \parr \cdots \parr F_{j_s}$ where each F_{j_k} does not contain a $\parr$ connective. Consider the proof structure for C_j obtained by using the axiom links induced by $\hat{\rho}$. Any DR-switching induces a DR-graph which is connected by design, since C_j was chosen as a component. Furthermore, any such DR-graph is also acyclic. The form of C_j implies that all $\parr$-links lie at the bottom of the proof structure. If a cycle did exist, it would exist in the part of the proof structure which remains constant over all DR-switchings. But then this would imply that all DR-graphs associated with $\hat{\rho}$ contain this cycle, contradicting Theorem 2.2.8. Therefore we have a MLL proof net for the formula C_j, and C_j is provable.

The dinatural $\hat{\rho}$ can be interpreted as a collection of morphisms

$$\hat{\rho}_{\underline{\mathcal{A}}} : C_1^{\perp}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to (C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}).$$

More importantly,

$$U\hat{\rho}_{\underline{\mathcal{A}}} : UC_1^{\perp}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \xrightarrow{\mu_{\underline{\mathcal{A}}}} I \xrightarrow{\kappa_{\underline{\mathcal{A}}}} U(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}})$$

factors through I (since each C_j forms a component), and because C_1 is provable, we know that the morphisms $\mu_{\underline{\mathcal{A}}}$ lift to morphisms $C_1^{\perp}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \perp$ in **GRel**.

Instantiate each $\mathcal{A}_i$ at a "test" object $\mathcal{A}$ in $\mathbb{T}_0$. (An example of such an object is $\mathcal{A} = (\{1,2\}, \{\{1\}\}, \{\{2\}\})$.) Then given $\alpha_{\underline{\mathcal{A}}} : \mathbf{1} \to C_1^{\perp}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$, the definition of morphism in **GRel** implies that $\hat{\rho}_{\underline{\mathcal{A}}}\alpha_{\underline{\mathcal{A}}} \in ((C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}))_s$. But $C_1^{\perp}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ (being unit-free) is an object of $\mathbb{T}_0$, so

$$\begin{aligned} U(\hat{\rho}_{\underline{\mathcal{A}}}\alpha_{\underline{\mathcal{A}}}) &= I \xrightarrow{U\alpha_{\underline{\mathcal{A}}}} UC_1^{\perp}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \xrightarrow{\mu_{|\underline{\mathcal{A}}|}} I \xrightarrow{\kappa_{|\underline{\mathcal{A}}|}} U(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \\ &= I \xrightarrow{\qquad\qquad 0_I \qquad\qquad} I \xrightarrow{\kappa_{|\underline{\mathcal{A}}|}} U(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \end{aligned}$$

forms an empty map in $(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}))_s$. That $(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}))_s$ contains an empty map is, by choice of $\mathcal{A}$ in $\mathbb{T}_0$, a contradiction, so the original DR-graph must be connected. ■

Theorem 2.2.14 *(Full Completeness in* **GRel***) Every non-trivial dinatural transformation in* **GRel** *from the constant functor $\mathfrak{K}_\mathbf{1}$ to the multivariant functor F is the denotation of a unique proof in MLL of the formula F, and is therefore induced by a unique morphism $I \to F(\mathbf{X}, \mathbf{X})$ in the free $*$-autonomous category on n objects $X_1, \ldots, X_n$.*

2.3 Full Completeness of Coherence Spaces

The category **GRel** is rich in structure and contains full subcategories of independent interest. We have already observed one such category, $\mathbb{S}$, whose objects $\mathcal{A} = (A, \mathcal{A}_s, \mathcal{A}_t)$ are such that for every $\alpha \in \mathcal{A}_s$ and $\beta \in \mathcal{A}_t$, their intersection $\alpha \cap \beta$ is non-empty.

Recall that $\mathbb{S}$ supports the Mix rule. Theorems 2.2.6 and 2.2.8 could have been proved entirely within the category $\mathbb{S}$, so they also prove a full completeness result for $\mathbb{S}$. In particular, we have the following theorem.

Theorem 2.3.1 *Suppose that $\rho : \mathfrak{K}_{\mathbf{1}} \to F$ is a non-trivial dinatural transformation in $\mathbb{S}$. Then there exists a unique set of axiom links determining a proof structure for the formula F associated with ρ. Furthermore, for any DR-switching, the associated DR-graph is acyclic.*

Corollary 2.3.2 *(Full Completeness in $\mathbb{S}$) Every non-trivial dinatural transformation $\rho : \mathfrak{K}_{\mathbf{1}} \to F$ in $\mathbb{S}$ is the denotation of a unique proof in MLL+Mix of the formula F, and is therefore induced by a unique morphism $I \to F(\mathbf{X}, \mathbf{X})$ in the free $*$-autonomous category supporting the Mix rule, on n objects $X_1, \ldots, X_n$.*

Another full subcategory with a $*$-autonomous structure is the category **Tot** of totality spaces. A *totality space* is an object $\mathcal{A}$ in **GRel** for which

- $\mathcal{A}_s$ is the set of all subsets $\alpha \subseteq |\mathcal{A}|$ such that $\alpha \cap \beta$ is a singleton, whenever $\beta \in \mathcal{A}_t$;
- $\mathcal{A}_t$ is the set of all subsets $\beta \subseteq |\mathcal{A}|$ such that $\alpha \cap \beta$ is a singleton, whenever $\alpha \in \mathcal{A}_s$;
- $|\mathcal{A}| = \bigcup \mathcal{A}_s = \bigcup \mathcal{A}_t$.

Loader proved a full completeness result for **Tot** in [Loa94a], as a categorical model of MLL+Mix. However, his treatment of **Tot** is different to our treatment of **GRel**, and bears more resemblance to the full completeness result in [AJ94], with the use of embeddings and natural transformations.

With little extra effort, we will show that **GRel** contains a category **Coh** of coherence spaces, and that the full completeness result we have proved for **Rel** also lifts to a full completeness result in **Coh**. This work followed from a suggestion made by J.R.B. Cockett. A result similar in nature can be found in [Ret]. G. Plotkin also has a full completeness result for coherence spaces, currently in preparation.

Let **Coh** denote the full subcategory of **GRel** restricted to objects $\mathcal{A}$ of **GRel** such that

- $\mathcal{A}_s$ is the set of all subsets $\alpha \subseteq |\mathcal{A}|$ such that $\alpha \cap \beta$ contains at most one element, whenever $\beta \in \mathcal{A}_t$;
- $\mathcal{A}_t$ is the set of all subsets $\beta \subseteq |\mathcal{A}|$ such that $\alpha \cap \beta$ contains at most one element, whenever $\alpha \in \mathcal{A}_s$;
- $|\mathcal{A}| = \bigcup \mathcal{A}_s = \bigcup \mathcal{A}_t$.

To prove that the objects in **Coh** are indeed coherence spaces, we will show that each object in **Coh** can be regarded as a graph $\mathcal{A}$ on the set of vertices $|\mathcal{A}|$, whose cliques are determined by $\mathcal{A}_s$ and whose cocliques are determined by $\mathcal{A}_t$. (Recall that a coherence space is essentially a graph specified by a set of cliques. See [Gir87].) Recall that a *clique* of a graph G is a subset C of vertices of G such that every two distinct points in C form an edge in G. A *coclique* of G is a subset D of vertices in G such that no two distinct points in D form an edge in G. Note that all singleton subsets $\{x\}$ (where x is a vertex) are both cliques and cocliques of G.

Consider an object $\mathcal{A} = (A, \mathcal{A}_s, \mathcal{A}_t)$ in **Coh**. Define a graph $G(A)$ on A via the equivalence

$$\{x, y\} \in \mathcal{A}_s \text{ if and only if } x \text{ and } y \text{ form an edge in } G(A).$$

Observe that $\mathcal{A}_s$ is *downward closed*, i.e. if $\alpha \in \mathcal{A}_s$ and $\alpha' \subseteq \alpha$ then $\alpha' \in \mathcal{A}_s$. (Similarly $\mathcal{A}_t$ is downward closed.) Therefore all sets in $\mathcal{A}_s$ are cliques of $G(A)$.

By definition, any $\beta \in \mathcal{A}_t$ intersects with any $\alpha \in \mathcal{A}_s$ at at most one point, so no two distinct points in β can form an edge in $G(A)$. Therefore all sets in $\mathcal{A}_t$ are cocliques of $G(A)$.

Note that cliques and cocliques observe the same property as $\mathcal{A}_s$ and $\mathcal{A}_t$, namely, any clique of $G(A)$ and any coclique of $G(A)$ intersect at at most one point. Therefore we can dualise the statement that $\mathcal{A}_t$ is contained in the set of all cocliques of $G(A)$, and deduce that $\mathcal{A}_s$ contains the set of all cliques of $G(A)$. Combining with the our previous observation that all sets in $\mathcal{A}_s$ are cliques of $G(A)$, we now have that $\mathcal{A}_s$ is precisely the set of cliques of $G(A)$ and therefore $\mathcal{A}_t$ is precisely the set of cocliques of $G(A)$.

Conversely, a graph $G(A)$ easily induces an object $\mathcal{A}$ of **Coh**. We take $|\mathcal{A}| = A$, $\mathcal{A}_s$ to be the set of cliques of $G(A)$ and $\mathcal{A}_t$ to be the set of cocliques of $G(A)$.

Therefore, **Coh** is indeed a category of coherence spaces. Recall from our construction of **GRel** that $f : \mathcal{A} \to \mathcal{B}$ is a morphism in **Coh** if it is a relation $f : |\mathcal{A}| \to |\mathcal{B}|$ in **Rel** such that given a clique $\alpha \in \mathcal{A}_s$, $\alpha : I \to |\mathcal{A}|$, the composite $I \xrightarrow{\alpha} |\mathcal{A}| \xrightarrow{f} |\mathcal{B}|$ is a clique in $\mathcal{B}_s$, and given a coclique $\beta \in \mathcal{B}_t$, $\beta : |\mathcal{B}| \to I$, the composite $|\mathcal{A}| \xrightarrow{f} |\mathcal{B}| \xrightarrow{\beta} I$ is a coclique in $\mathcal{A}_t$.

We now put a $*$-autonomous structure on **Coh**.

Linear negation. The linear negation operation in **GRel** clearly closed in **Coh**, i.e. we take $\mathcal{A}^\perp = (|\mathcal{A}|, \mathcal{A}_t, \mathcal{A}_s)$, i.e. the cograph of $\mathcal{A}$. We then have the isomorphism $\mathcal{A} \cong \mathcal{A}^{\perp\perp}$.

Tensor. **Coh** is *not* closed under the tensor product of **GRel**. If we defined $(\mathcal{A} \otimes \mathcal{B})_s$ to be the set of products of cliques in $\mathcal{A}_s$ and $\mathcal{B}_s$, we would be lacking the necessary downward closure property, i.e. all subsets of such product sets would be missing.

In effect, we do define the tensor product in **Coh** to be the downward closure of the tensor product in **GRel**. It is perhaps simpler to regard two objects $\mathcal{A}$ and $\mathcal{B}$ as graphs with cliques determined by $\mathcal{A}_s$ and $\mathcal{B}_s$ respectively, and define the tensor product of these two objects to be the *product graph* $\mathcal{A} \times \mathcal{B}$. The set of vertices of $\mathcal{A} \times \mathcal{B}$ is the cartesian product $|\mathcal{A}| \times |\mathcal{B}|$, and (a_1, b_1) and (a_2, b_2) form an edge in $\mathcal{A} \times \mathcal{B}$ if and only if a_1, a_2 form an edge in $\mathcal{A}$ and b_1, b_2 form an edge in $\mathcal{B}$. To avoid confusion with the tensor product in **GRel**, we will use the product symbol $\times$ for the tensor product in **Coh**.

Lemma 2.3.3 *The category* **Coh** *is a $*$-autonomous category supporting the Mix rule, with tensor product equal to the graph product, and linear negation equal to the cograph operation. The unit for tensor and par is* $\mathbf{1}_{\mathbf{Coh}} = \perp_{\mathbf{Coh}} = (I, \{id_I, 0_I\}, \{id_I, 0_I\})$, *i.e. it is the unique graph on one vertex.*

Proof Write $A = |\mathcal{A}|$, $B = |\mathcal{B}|$, and $C = |\mathcal{C}|$. Suppose that $f : \mathcal{A} \times \mathcal{B} \to \mathcal{C}^{\perp}$ in **Coh**. Then f is a subset of $A \times B \times C$ in **Rel**. We wish to prove that f is also identifiable as a morphism $\mathcal{A} \to (\mathcal{B} \times \mathcal{C})^{\perp}$ in **Coh**. So given $\alpha \in \mathcal{A}_s$, we wish to prove that

$$X = \{(b, c) \mid (a, b, c) \in f \text{ for some } a \in \alpha\}$$

is a coclique of $\mathcal{B} \times \mathcal{C}$, and given $\beta \times \gamma \in (\mathcal{B} \times \mathcal{C})^{\perp}_t = (\mathcal{B} \times \mathcal{C})_s$, we wish to prove that

$$Y = \{a \mid (a, b, c) \in f \text{ for some } (b, c) \in \beta \times \gamma\}$$

is a coclique of $\mathcal{A}$.

Suppose that $(b_1, c_1), (b_2, c_2) \in X$. Then there exists $a_1, a_2 \in \alpha$ (hence $\{a_1, a_2\} \in \mathcal{A}_s$) such that $(a_1, b_1, c_1), (a_2, b_2, c_2) \in f$. If b_1 and b_2 form an edge in $\mathcal{B}$, then we have $\{a_1, a_2\} \in \mathcal{A}_s$ and $\{b_1, b_2\} \in \mathcal{B}_s$, so that

$$W = \{c \mid (a_i, b_j, c) \in f,\ i, j = 1, 2\} \text{ belongs to } \mathcal{C}^{\perp}_s = \mathcal{C}_t.$$

In particular, $c_1, c_2 \in W$, so these points do not form an edge in $\mathcal{C}$. On the other hand, if c_1 and c_2 do form an edge in $\mathcal{C}$, then $\{c_1, c_2\} \in \mathcal{C}^{\perp}_t$, so

$$Z = \{(a, b) \mid (a, b, c_i) \in f,\ i = 1, 2\} \text{ belongs to } (\mathcal{A} \times \mathcal{B})_t. \tag{2.7}$$

In particular, $(a_1, b_1), (a_2, b_2) \in Z$, and since $\{a_1, a_2\} \in \mathcal{A}_s$, we deduce that b_1 and b_2 belong to a coclique of $\mathcal{B}$. Therefore (b_1, c_1) and (b_2, c_2) never form an edge in $\mathcal{B} \times \mathcal{C}$, i.e. X is a coclique of $\mathcal{B} \times \mathcal{C}$.

Now suppose that $a_1, a_2 \in Y$. Then there exists $(b_1, c_1), (b_2, c_2) \in \beta \times \gamma$ such that $(a_1, b_1, c_1), (a_2, b_2, c_2) \in f$. Since $\gamma \in \mathcal{C}_s$, we have $\{c_1, c_2\} \in \mathcal{C}_s = \mathcal{C}^{\perp}_t$, so that (2.7) holds again. In particular, $(a_1, b_1), (a_2, b_2) \in Z$, and since $\{b_1, b_2\} \in \mathcal{B}_s$, we deduce that a_1 and a_2 belong to a coclique of $\mathcal{A}$. Therefore Y is a coclique of $\mathcal{A}$.

Trivially, we can see that $\mathbf{1}_{\mathbf{Coh}}$ is the unit for the tensor product in **Coh**. Since this element is self-dual, the identity morphism on I lifts to the unary Mix morphism $m : \bot_{\mathbf{Coh}} \to \mathbf{1}_{\mathbf{Coh}}$. ■

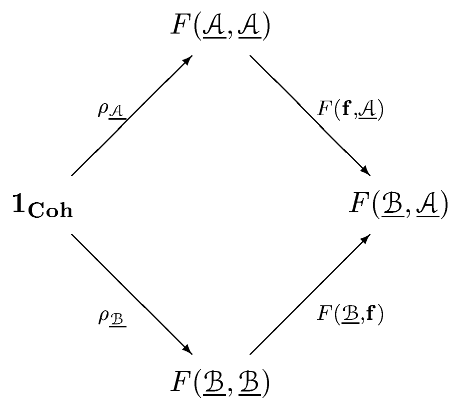
Once again, we make the trivial observation that a morphism $\mathbf{1}_{\mathbf{Coh}} \to \mathcal{A}$ is an element in $\mathcal{A}_s$, i.e. a clique of the graph $\mathcal{A}$. Similarly, a morphism $\mathcal{A} \to \bot_{\mathbf{Coh}}$ is a coclique of the graph $\mathcal{A}$.

We again assume that $F(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ ($\underline{\mathcal{X}} = \mathcal{X}_1, \ldots, \mathcal{X}_n$ etc.) is a formula of length p built from $\mathcal{X}_1, \ldots, \mathcal{X}_n$, $\mathcal{Y}_1^\bot, \ldots, \mathcal{Y}_n^\bot$ by the connectives $\times$ and $⅋$, such that

$$UF(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = A_{\xi_1}^{\zeta_1} \times \cdots \times A_{\xi_p}^{\zeta_p},$$

where $|\mathcal{A}_i| = A_i$ for all $i = 1, \ldots, n$. Then F induces a multivariant functor $[\![F]\!] : (\mathbf{Coh})^n \times (\mathbf{Coh}^{op})^n \to \mathbf{Coh}$, which by abuse of notation we will also refer to as F.

Theorem 2.3.4 *Suppose that ρ is a dinatural transformation from the constant functor $\mathfrak{K}_{\mathbf{1}_{\mathbf{Coh}}}$ to the multivariant functor $F : (\mathbf{Coh})^n \times (\mathbf{Coh}^{op})^n \to \mathbf{Coh}$, i.e. a collection of morphisms $\rho_{\underline{\mathcal{A}}} : \mathbf{1}_{\mathbf{Coh}} \to F(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ such that for all morphisms $f_i : \mathcal{A}_i \to \mathcal{B}_i$ in **Coh**, the following diagram commutes, (writing $\underline{\mathcal{A}}$ for the list $\mathcal{A}_1, \ldots, \mathcal{A}_n$, $\mathbf{f}$ for the list $f_1, \ldots, f_n$).*



*If $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}_1, \ldots, \mathcal{B}_n$ are objects in **Coh** such that $U\mathcal{A}_i = U\mathcal{B}_i$ for all i, then $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{B}}}$.*

Proof For each $i = 1, \ldots, n$, write $A_i = U\mathcal{A}_i$, and define

$$\mathcal{A}_i^* = (A_i, \mathcal{P}(A_i), Sing(A_i) \cup \{\varnothing\}),$$

where $\mathcal{P}(A_i)$ is the power set of A_i, and $Sing(A_i)$ is the set of singleton subsets of A_i. In other words, $\mathcal{A}_i^*$ is the complete graph on the set of vertices A_i.

It suffices to show that $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{A}}^*}$. The identity morphism on A_i in **Rel** lifts to a morphism $\mathcal{A}_i \to \mathcal{A}_i^*$ in **Coh**, since any clique in $(\mathcal{A}_i)_s$ belongs to $\mathcal{P}(A_i) = (\mathcal{A}_i^*)_s$ and any singleton subset in $Sing(A)$ belongs to $\mathcal{A}_t$. Put $f_i = id_{A_i}$. By dinaturality,

$$F(\mathbf{f}, id_{\underline{\mathcal{A}}})\rho_{\underline{\mathcal{A}}} = F(id_{\underline{\mathcal{A}}^*}, \mathbf{f})\rho_{\underline{\mathcal{A}}^*}.$$

But since $UF(\mathbf{f}, id_{\underline{\mathcal{A}}})$ is the identity morphism on $UF(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ and $UF(id_{\underline{\mathcal{A}}^*}, \mathbf{f})$ is the identity morphism on $UF(\underline{\mathcal{A}}^*, \underline{\mathcal{A}}^*) = UF(\underline{\mathcal{A}}, \underline{\mathcal{A}})$, we have $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{A}}^*}$.

It now follows that $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{B}}}$, since $\mathcal{A}_i^* = \mathcal{B}_i^*$ for all i. ∎

We have proved a similar statement to Theorem 1.3.2. This tells us that a dinatural transformation ρ in **Coh** is completely determined by an underlying dinatural transformation in **Rel**. We can therefore lift the full completeness result obtained in **Rel**, and completely characterise the non-trivial dinatural transformations in **Coh**.

Lemma 2.3.5 *Suppose that ρ is a non-trivial dinatural transformation in* **Coh** *from the constant functor $\mathfrak{K}_{\mathbf{1}_{\mathbf{Coh}}}$ to the multivariant functor F. Then there exists a unique fixed-point-free involution $\phi : \{1, \ldots, p\} \to \{1, \ldots, p\}$ such that $\xi_{\phi(i)} = \xi_i$ and $\zeta_{\phi(i)} \neq \zeta_i$ for all i, and*

$$\rho_{\underline{\mathcal{A}}} = \{(x_1, \ldots, x_p) \in F(\mathbf{A}, \mathbf{A}) \mid x_{\phi(i)} = x_i \textit{ for all } i = 1, \ldots, p\}, \tag{2.8}$$

where $|\mathcal{A}_i| = A_i$ for all i and $F(\mathbf{A}, \mathbf{A})$ has the form in equation (2.1), thus determining a unique set of axiom links for a proof structure for the MLL formula F.

Proof Given arbitrary sets $A_1, \ldots, A_n$, define objects in **Coh**,

$$\mathcal{A}_i = (A_i, Sing(A_i) \cup \{\varnothing\}, \mathcal{P}(A_i)), \qquad i = 1, \ldots, n.$$

In other words, each $\mathcal{A}_i$ is the empty graph on A_i. Since **Coh** supports the Mix rule, given elements $x_i \in A_{\xi_i}$, $i \in N$, we can construct a map $F(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \bot_{\mathbf{Coh}}$, i.e. a coclique of $F(\underline{\mathcal{A}}, \underline{\mathcal{A}})$, namely $\bar{t} = t_1 \times \cdots \times t_p$, where

$$t_i = \begin{cases} \{x_i\} & \text{if } i \in N; \\ A_{\xi_i} & \text{if } i \in Q. \end{cases}$$

Since $\rho_{\underline{\mathcal{A}}} : \mathbf{1}_{\mathbf{Coh}} \to F(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ is a clique of $F(\underline{\mathcal{A}}, \underline{\mathcal{A}})$, $\rho_{\underline{\mathcal{A}}}$ and $\bar{t}$ intersect at at most one point. Thus, if there exists $(x_1, \ldots, x_p) \in \rho_{\underline{\mathcal{A}}}$ then this p-tuple is unique. Similarly, by considering objects $\mathcal{B}_i = \mathcal{A}_i^{\perp}$, we deduce that given elements $x_i \in A_{\xi_i}$, $i \in Q$, if there exists $(x_1, \ldots, x_p) \in \rho_{\underline{\mathcal{A}}}$ then this p-tuple is unique.

But $U\mathcal{A}_i = U\mathcal{B}_i$ for all i, so $U\rho_{\underline{\mathcal{A}}} = U\rho_{\underline{\mathcal{B}}} = \sigma_{\mathbf{A}}$, say. Thus the canonical natural transformation $\tilde{\sigma} : F^- \to F^+$ is a collection of non-trivial partial isomorphisms.

By Theorem 2.1.1, $\tilde{\sigma}$ (being non-trivial) is a union of permutations on the tensor factors. From this, we deduce that each $\tilde{\sigma}_{\mathbf{A}}$ is in fact a full isomorphism, and $\tilde{\sigma}$ is precisely *one* permutation on the tensor factors. Thus, by Theorem 2.1.4, there exists a unique fixed-point-free involution ϕ on $\{1, \ldots, p\}$ such that

$$\sigma_{\mathbf{A}} = \{(x_1, \ldots, x_p) \in F(\mathbf{A}, \mathbf{A}) \mid x_{\phi(i)} = x_i \text{ for all } i = 1, \ldots, p\}.$$

Since $U\rho_{\underline{\mathcal{A}}} = \sigma_{\mathbf{A}}$, we have equation (2.8). ■

Thus, we can associate to each non-trivial dinatural transformation in **Coh** a unique MLL proof structure. We will now show that such a proof structure must be a MLL+Mix proof net. That is, for every DR-switching, the associated DR-graph is acyclic.

Proposition 2.3.6 *Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are objects in* **Coh** *and that we have a formula*

$$\tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = \Gamma_{\underline{\mathcal{A}}} \parr (\mathcal{A}^{\zeta_{a_1}}_{\xi_{a_1}} \times \mathcal{A}^{\zeta_{b_1}}_{\xi_{b_1}}) \parr \cdots \parr (\mathcal{A}^{\zeta_{a_r}}_{\xi_{a_r}} \times \mathcal{A}^{\zeta_{b_r}}_{\xi_{b_r}}) \tag{2.9}$$

Suppose that we have the following morphisms in **Coh***:*

$$\begin{array}{ll} \tau_i : \mathcal{A}^{\zeta_{a_i}}_{\xi_{a_i}} \times \mathcal{A}^{\zeta_{b_i}}_{\xi_{b_i}} \to \bot_{\mathbf{Coh}}, & i = 1, \ldots, r-1; \\ \tau_j^+ : \mathcal{A}_j \to \bot_{\mathbf{Coh}}, & j = 1, \ldots, n; \\ \tau_j^- : \mathcal{A}_j^{\bot} \to \bot_{\mathbf{Coh}}, & j = 1, \ldots, n. \end{array}$$

Then we can construct a morphism $\tau_{\underline{\mathcal{A}}} : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \mathcal{A}^{\zeta_{a_r}}_{\xi_{a_r}} \times \mathcal{A}^{\zeta_{b_r}}_{\xi_{b_r}}$, where $U\tau_{\underline{\mathcal{A}}} \subseteq \tilde{F}(\mathbf{A}, \mathbf{A}) \times A^{\zeta_{a_r}}_{\xi_{a_r}} \times A^{\zeta_{b_r}}_{\xi_{b_r}}$ is a cartesian product of a combination of sets $U\tau_j^+$ and $U\tau_j^-$, the sets $U\tau_1, \ldots, U\tau_{r-1}$ and the identity relation on $A^{\zeta_{a_r}}_{\xi_{a_r}} \times A^{\zeta_{b_r}}_{\xi_{b_r}}$.

Proof Put

$$G(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = \Gamma_{\underline{\mathcal{A}}} \parr (\mathcal{A}^{\zeta_{a_1}}_{\xi_{a_1}} \times \mathcal{A}^{\zeta_{b_1}}_{\xi_{b_1}}) \parr \cdots \parr (\mathcal{A}^{\zeta_{a_{r-1}}}_{\xi_{a_{r-1}}} \times \mathcal{A}^{\zeta_{b_{r-1}}}_{\xi_{b_{r-1}}}),$$

so that $F(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = G(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \parr (A^{\zeta_{a_r}}_{\xi_{a_r}} \times A^{\zeta_{b_r}}_{\xi_{b_r}})$. Then an argument parallel to Proposition 2.2.7 shows that we can construct a morphism $\tau' : G(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \bot_{\mathbf{Coh}}$ in **Coh**, where $U\tau'$ is a cartesian product of a combination of sets $U\tau_j^+$ and $U\tau_j^-$ (which lifts to a morphism $\Gamma_{\underline{\mathcal{A}}} \to \bot_{\mathbf{Coh}}$) and the sets $U\tau_1, \ldots, U\tau_{r-1}$. Now take $\tau = \tau' \parr (A^{\zeta_{a_r}}_{\xi_{a_r}} \times A^{\zeta_{b_r}}_{\xi_{b_r}}) : F(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to A^{\zeta_{a_r}}_{\xi_{a_r}} \times A^{\zeta_{b_r}}_{\xi_{b_r}}$. ■

Before proving the main result, we make the following useful observation.

Lemma 2.3.7 *Suppose that $\mathcal{A}$ is an object in* **Coh** *such that $|\mathcal{A}|$ contains more than one point. If $\omega_{\mathcal{A}} : \mathbf{1}_{\mathbf{Coh}} \to \mathcal{A} \times \mathcal{A}^{\bot}$ is a morphism in* **Coh** *then $U\omega_{\mathcal{A}} : I \to A \times A$ is not the identity relation on A.*

Proof Recall that $\omega_{\mathcal{A}}$ is a clique of the graph $\mathcal{A} \times \mathcal{A}^{\bot}$, so we are claiming that distinct points (a, a) and (a', a') belong to a clique of $\mathcal{A} \times \mathcal{A}^{\bot}$. But this implies that a and a' form an edge in both $\mathcal{A}$ and $\mathcal{A}^{\bot}$. This is clearly a contradiction. ■

Theorem 2.3.8 *Suppose that ρ is a non-trivial dinatural transformation in* **Coh** *from the constant functor $\Re_{\mathbf{1}_{\mathbf{Coh}}}$ to the multivariant functor F. Consider the unique proof structure for F associated with ρ. Then for any DR-switching, the associated DR-graph is acyclic.*

Proof By Lemma 2.3.5, we know that there exists a unique fixed-point-free involution ϕ specifying a unique set of axiom links for a proof structure for F.

Suppose that for a certain DR-switching, the associated DR-graph G contains a cycle. Consider the shortest cycle, and express it as lower connected pairs $(a_1, b_1), \ldots, (a_r, b_r)$ such that $\phi(b_i) = a_{i+1}$ for all $i \in \mathbb{Z}_r$. Following a similar argument to that in Theorem 2.2.8, we may assume that we have simplified F to a formula of the form (2.9), and we have obtained from ρ a new non-trivial dinatural transformation $\tilde{\rho} : \Re_{\mathbf{1}_{\mathbf{Coh}}} \to \tilde{F}$ in **Coh**. (We can work in the whole of **Coh** since it supports the Mix rule.)

Now we prove that $\tilde{\rho}$ should not exist. Choose a test object $\mathcal{A} = (A, \mathcal{A}_s, \mathcal{A}_t)$ such that $\mathcal{A} \cong \mathcal{A}^\perp$. An example of such an object is

$$\begin{aligned} |\mathcal{A}| &= \{1,2,3,4\}, \\ \mathcal{A}_s &= \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{2,3\},\{3,4\}\}, \\ \mathcal{A}_t &= \{\{1\},\{2\},\{3\},\{4\},\{1,3\},\{2,4\},\{1,4\}\}, \end{aligned}$$

with isomorphism $\psi : \mathcal{A} \to \mathcal{A}^\perp$, $\psi = \{13, 21, 34, 42\} \subseteq A \times A$. We can identify ψ with a morphism $\mathcal{A} \times \mathcal{A} \to \perp_{\mathbf{Coh}}$. Similarly, $\psi^{-1} : \mathcal{A}^\perp \to \mathcal{A}$, $\psi^{-1} = \{12, 24, 31, 43\}$ can be identified with a morphism $\mathcal{A}^\perp \times \mathcal{A}^\perp \to \perp_{\mathbf{Coh}}$.

Put $\mathcal{A}_i = \mathcal{A}$ for all $i = 1, \ldots, n$. Choose $\tau_j^+ : \mathcal{A} \to \perp_{\mathbf{Coh}}$ and $\tau_j^- : \mathcal{A}^\perp \to \perp_{\mathbf{Coh}}$, $\tau_j^+ = \tau_j^- = \{a\}$ for some fixed point $a \in A$. Put

$$\tau_i = \begin{cases} id_A & \text{if } \zeta_{a_i} \neq \zeta_{b_i}; \\ \psi & \text{if } \zeta_{a_i} = \zeta_{b_i} = 1; \\ \psi^{-1} & \text{if } \zeta_{a_i} = \zeta_{b_i} = \perp \end{cases}$$

for each $i = 1, \ldots, r-1$ (where we identify $\psi : \mathcal{A} \to \mathcal{A}^\perp$ as a morphism $\mathcal{A} \times \mathcal{A} \to \perp_{\mathbf{Coh}}$ etc.) Then by Proposition 2.3.6 we can construct a morphism $\tau_{\underline{\mathcal{A}}} : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \mathcal{A}^{\zeta_{a_r}} \times \mathcal{A}^{\zeta_{b_r}}$. Now compose $\tau_{\underline{\mathcal{A}}}$ with $\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1}_{\mathbf{Coh}} \to \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ to form the morphism $\tau_{\underline{\mathcal{A}}} \tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1}_{\mathbf{Coh}} \to \mathcal{A}^{\zeta_{a_r}} \times \mathcal{A}^{\zeta_{b_r}}$. Note that $U\tilde{\rho}_{\underline{\mathcal{A}}} = f_1 \times f_2$ where

$$f_1 : I \to \Gamma_{\mathbf{A}}; \quad f_2 : I \to (A \times A)^r,$$

(where f_1 lifts to a morphism $\mathbf{1}_{\mathbf{Coh}} \to \Gamma_{\underline{\mathcal{A}}}$ and $f_2 = \{(a_1, a_2, a_2, \ldots, a_{r-2}, a_{r-2}, a_1) \mid a_i \in A\}$) and $U\tau_{\underline{\mathcal{A}}} = g_1 \times g_2$ where

$$g_1 : \Gamma_{\mathbf{A}} \to I; \quad g_2 : (A \times A)^r \to A \times A.$$

So $U(\tau_{\underline{\mathcal{A}}} \tilde{\rho}_{\underline{\mathcal{A}}}) = U\tau_{\underline{\mathcal{A}}} U\tilde{\rho}_{\underline{\mathcal{A}}} = (g_1 f_1) \times (g_2 f_2)$. Since we chose the coclique $g_1 = \{(a, \ldots, a)\}$ of $\Gamma_{\underline{\mathcal{A}}}$ to be a singleton subset, it will have a nonempty intersection with the clique f_1 of $\Gamma_{\underline{\mathcal{A}}}$. Hence $g_1 f_1 = id_I$ and $U(\tau_{\underline{\mathcal{A}}} \tilde{\rho}_{\underline{\mathcal{A}}}) \cong g_2 f_2$.

Examining the cycle a little closer, we see that there are an even number, $2k$ say, of occurrences of i for which $\zeta_{a_i} = \zeta_{b_i}$, k of which are $\zeta_{a_i} = \zeta_{b_i} = 1$ and k of which are $\zeta_{a_i} = \zeta_{b_i} = \bot$.

If $\zeta_{a_r} \neq \zeta_{b_r}$, then τ was built from the same number of ψ and ψ^{-1}, and therefore we can compose g_2 and f_2 and deduce that $g_2 f_2$ is the identity relation on A. So U maps $\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1_{Coh}} \to \mathcal{A} \times \mathcal{A}^\bot$ to the identity relation on A, which by Lemma 2.3.7 is a contradiction.

If $\zeta_{a_r} = \zeta_{b_r} = 1$, then τ was built from $k-1$ occurrences of ψ and k occurrences of ψ^{-1}. Therefore we can deduce that $g_2 f_2$ is the relation ψ^{-1}. Therefore U sends $\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1_{Coh}} \to \mathcal{A} \times \mathcal{A}$ to the relation ψ^{-1}. Now, U sends the composite

$$\mathbf{1_{Coh}} \xrightarrow{\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}}} \mathcal{A} \times \mathcal{A} \xrightarrow{\psi \times \mathcal{A}} \mathcal{A}^\bot \times \mathcal{A}$$

to the identity relation on A, which by Lemma 2.3.7 is a contradiction.

If $\zeta_{a_r} = \zeta_{b_r} = \bot$, then τ was built from k occurrences of ψ and $k-1$ occurrences of ψ^{-1}. Therefore we can deduce that $g_2 f_2$ is the relation ψ. Therefore U sends $\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1_{Coh}} \to \mathcal{A}^\bot \times \mathcal{A}^\bot$ to the relation ψ. Now, U sends the composite

$$\mathbf{1_{Coh}} \xrightarrow{\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}}} \mathcal{A}^\bot \times \mathcal{A}^\bot \xrightarrow{\psi^{-1} \times \mathcal{A}^\bot} \mathcal{A} \times \mathcal{A}^\bot$$

to the identity relation on A, which by Lemma 2.3.7 is a contradiction.

So in all three cases, we derived from $\tilde{\rho}_{\underline{\mathcal{A}}}$ a morphism in **Coh** which cannot exist. Therefore, $\tilde{\rho}$ cannot exist, and the original DR-graph must have been acyclic. ∎

Corollary 2.3.9 *(Full Completeness in* **Coh***) Every non-trivial dinatural transformation $\mathfrak{K}_{\mathbf{1_{Coh}}} \to F$ is the denotation of a unique proof in MLL+Mix of the formula F, and is therefore induced by a unique morphism $I \to F(\mathbf{X}, \mathbf{X})$ in the free $*$-autonomous category supporting the Mix rule, on n objects $X_1, \dots, X_n$.*

Chapter 3

The category GFDVec

Our second example is to apply the glueing construction to **FDVec** to obtain **GFDVec**. Recall that the category $\mathbf{FDVec} = \mathbf{FDVec}_k$ of finite dimensional vector spaces over k and linear maps is a compact closed category with the usual tensor product and the usual algebraic dual. We also remark that **FDVec** is enriched over itself. In particular, a linear combination of morphisms $V \to W$ in **FDVec** is itself a morphism $V \to W$ in **FDVec**.

3.1 Preliminaries

Let V be a finite dimensional vector space over k, let V^* denote the algebraic dual space of V, i.e. the space of linear functionals on V, let $End(V)$ denote the space of all endomorphisms on V, and let $\mathbf{GL}(V) \subset End(V)$ denote the collection of automorphisms on V.

Definition 3.1.1 Let $A \subseteq End(V)$. The *commutant* of A is defined to be

$$A' = \{g \in End(V) \mid gf = fg \text{ for all } f \in A\}.$$

Consider the m-fold tensor product space $V^{\otimes m}$ and the following actions on $V^{\otimes m}$. The first action is $End(V)$ acting on the diagonal, i.e. $f \in End(V)$ induces the map $f^{\otimes m} : V^{\otimes m} \to V^{\otimes m}$, generated by

$$f \cdot (v_1 \otimes \cdots \otimes v_m) = f^{\otimes m}(v_1 \otimes \cdots \otimes v_m) = f(v_1) \otimes \cdots \otimes f(v_m),$$

for all $v_1, \ldots, v_m \in V$. Write A for the linear span $\langle f^{\otimes m} \mid f \in End(V)\rangle$.

Let S_m denote the symmetric group on m objects. Then every $\delta \in S_m$ acts on $V^{\otimes m}$, generated by

$$\delta \cdot (v_1 \otimes \cdots \otimes v_m) = v_{\delta(1)} \otimes \cdots \otimes v_{\delta(m)},$$

and extended by linearity. Let B denote the linear span of S_m. This is otherwise known as the group algebra $k[S_m]$.

For the rest of this chapter, assume that the field k *has characteristic zero*. We now state the following theorem.

Theorem 3.1.2 *The subalgebras A and B of $End(V^{\otimes m})$ are mutual commutants of each other. In particular, given $\sigma \in A'$, $\sigma \not\equiv 0$, there exist permutations $\delta_1, \dots, \delta_T \in S_m$ and scalars $\gamma_1, \dots, \gamma_T \in k$ such that*

$$\sigma(v_1 \otimes \cdots \otimes v_m) = \sum_{h=1}^{T} \gamma_h (v_{\delta_h(1)} \otimes \cdots \otimes v_{\delta_h(m)})$$

for all $v_i \in V$, $i = 1, \dots, m$.

Before proving Theorem 3.1.2, we remark that the statement remains true when we take $A = \langle f^{\otimes m} \mid f \in \mathbf{GL}(V) \rangle$. This result is known as Schur-Weyl duality. The proof of Schur-Weyl duality is almost identical to the proof we will provide for Theorem 3.1.2. It can be viewed as an application of the Double Commutant Theorem :

Theorem 3.1.3 *(Double Commutant Theorem) If A is a semisimple subalgebra of $End(V)$ then $A'' = A$.*

Proof (This is a classical result; this proof is based on lecture notes written by A.J. Wassermann [Was91].) It is clear that $A \subseteq A''$. We prove that $A'' \subset A$. Firstly,

$$\text{if } S \in A'' \text{ and } v \in V \text{ then there exists } T \in A \text{ such that } Sv = Tv. \tag{3.1}$$

Let $W = Av \subseteq V$. Then W is an A-submodule of V. Since A is semisimple, we can write $V = W \oplus W^{\perp}$ as a direct sum of A-submodules. Thus, the orthogonal projection E of V onto W commutes with A, i.e. $E \in A'$.

But $S \in A''$, so $SE = ES$. In other words, S leaves W (and $W^{\perp}$) invariant. Now, $v \in W$, so $Sv \in W$, i.e. there exists $T \in A$ such that $Sv = Tv$. This is (3.1).

Now we prove that given $S \in A''$, there exists $T \in A$ such that $S = T$. Consider the n-fold direct sum $V^{\oplus n}$, with A acting on the diagonal, i.e. $T(v_1, \dots, v_n) = (Tv_1, \dots, Tv_n)$ for all $T \in A$. It is convenient to identify elements of $V^{\oplus n}$ as column vectors with entries in V, and elements $T \in A$ with an $n \times n$ diagonal matrix $\pi(T)$ with diagonal entries T.

Now, $\pi(A)'$ is the set of all $n \times n$ matrices with entries belonging to A', i.e. $\pi(A)' = M_n(A')$. Therefore, $\pi(A)''$ is the set of all diagonal $n \times n$ matrices with diagonal entries in A'', i.e. $\pi(A)'' = \pi(A'')$.

Set $n = dim(V)$, and let $v = (e_1, \dots, e_n)^t$ where $\{e_1, \dots, e_n\}$ is a basis for V. By (3.1), we have $\pi(A)v = \pi(A)''v$, but we also have $\pi(A)'' = \pi(A'')$. So given $S \in A''$, there exists $T \in A$ such that $\pi(S)v = \pi(T)v$. Hence $Se_i = Te_i$ for all $i = 1, \dots, n$. Since $\{e_1, \dots, e_n\}$ is a basis for V, we have $S = T$. ■

We will also call on Maschke's Theorem [Pas91]. If G is a finite group and k is a field, then the group ring $k[G]$ is semisimple whenever the characteristic of k does not divide the order of the group G. The following proof is based on lecture notes by C. Procesi [Pro82].

Proof of Theorem 3.1.2 Since we are assuming that k has characteristic zero, Maschke's Theorem implies that $B = k[S_m]$ is semisimple. Hence $B'' = B$. It is therefore sufficient to prove that $B' = A$, for then we can deduce that $B'' = A'$, i.e. $B = A'$.

We also observe that because V is finite dimensional, $End(V^{\otimes m}) \cong End(V)^{\otimes m}$. (In fact $V^{\otimes m} \multimap V^{\otimes m} \cong (V \multimap V)^{\otimes m}$ in any compact closed category.)

There are two parts to proving that $B' = k[S_m]' = \langle f^{\otimes m} \mid f \in End(V)\rangle = A$. We will firstly prove that $k[S_m]'$ can be identified with $[End(V)^{\otimes m}]^{S_m}$, the elements of $End(V)^{\otimes m}$ which are fixed by the action of S_m. Then we will prove that $[End(V)^{\otimes m}]^{S_m} = \langle f^{\otimes m} \mid f \in End(V)\rangle$.

Suppose that $\{f_1, \ldots, f_r\}$ is a basis for $End(V)$. Then a basis for $End(V)^{\otimes m}$ is $\{f_{\mathbf{i}}\}$, where $f_{\mathbf{i}} = f_{i_1} \otimes \cdots \otimes f_{i_m}$, ranging over all $\mathbf{i} = (i_1, \ldots, i_m)$, $i_1, \ldots, i_m \in \{1, \ldots, r\}$.

Suppose that $T \in End(V^{\otimes m}) \cong End(V)^{\otimes m}$, $T = \sum_{\mathbf{i}} \alpha_{\mathbf{i}} f_{\mathbf{i}}$. Then $T \in k[S_m]'$ if and only if, for all $\delta \in S_m$, $v_1, \ldots, v_m \in V$,

$$\begin{array}{rccc} & T\delta(v_1 \otimes \cdots \otimes v_m) & = & \delta T(v_1 \otimes \cdots \otimes v_m) \\ \Leftrightarrow & T(v_{\delta(1)} \otimes \cdots \otimes v_{\delta(m)}) & = & \delta \sum_{\mathbf{i}} \alpha_{\mathbf{i}}(f_{i_1}(v_1) \otimes \cdots \otimes f_{i_m}(v_m)) \\ \Leftrightarrow & \sum_{\mathbf{i}} \alpha_{\mathbf{i}}(f_{i_1}(v_{\delta(1)}) \otimes f_{i_m}(v_{\delta(m)})) & = & \sum_{\mathbf{i}} \alpha_{\mathbf{i}}(f_{i_{\delta(1)}}(v_{\delta(1)}) \otimes f_{i_{\delta(m)}}(v_{\delta(m)})) \\ \Leftrightarrow & \sum_{\mathbf{i}} \alpha_{\mathbf{i}} f_{\mathbf{i}} & = & \sum_{\mathbf{i}} \alpha_{\mathbf{i}} f_{\delta(\mathbf{i})} \end{array}$$

where $\delta(i_1, \ldots, i_m) = (i_{\delta(1)}, \ldots, i_{\delta(m)})$. Therefore $T \in k[S_m]'$ if and only if T is fixed by the action of S_m, i.e. $T \in [End(V)^{\otimes m}]^{S_m}$.

We now form a basis for $[End(V)^{\otimes m}]^{S_m}$. We have just observed that $\sum_{\mathbf{i}} \alpha_{\mathbf{i}} f_{\mathbf{i}}$ belongs to $[End(V)^{\otimes m}]^{S_m}$ if and only if $\alpha_{\mathbf{i}} = \alpha_{\mathbf{j}}$ whenever $\mathbf{i}$ and $\mathbf{j}$ lie in the same orbit under the action of S_m. Such orbits are determined by r-tuples $\mathbf{h} = (h_1, \ldots, h_r)$ defining a partition $h_1 + \cdots + h_r = m$ where h_j is the number of occurrences of j in a tuple $(i_1, \ldots, i_m)$.

Write $e_{\mathbf{h}}$ for the sum of all basis elements $f_{\mathbf{i}}$ such that $\mathbf{i}$ lies in the orbit determined by $\mathbf{h}$. Then $\{e_{\mathbf{h}}\}$ is a basis for $[End(V)^{\otimes m}]^{S_m}$.

Finally, we show that $[End(V)^{\otimes m}]^{S_m} = \langle f^{\otimes m} \mid f \in End(V)\rangle = A$. It is clear that $A \subseteq [End(V)^{\otimes m}]^{S_m}$. Thus, it suffices to show that if a linear functional $\psi : [End(V)^{\otimes m}]^{S_m} \to k$ is zero on A then it is zero on the whole of $[End(V)^{\otimes m}]^{S_m}$.

Write $f = \sum \beta_i f_i \in End(V)$. Then $f^{\otimes m} = \sum_{\mathbf{h}} \beta_1^{h_1} \cdots \beta_r^{h_r} e_{\mathbf{h}}$, and hence

$$\psi(f^{\otimes m}) = \sum_{\mathbf{h}} \beta_1^{h_1} \cdots \beta_r^{h_r} \psi(e_{\mathbf{h}}).$$

Thus ψ on A forms a polynomial in $\beta_1, \ldots, \beta_r$ with coefficients $\psi(e_{\mathbf{h}})$. Therefore, if $\psi(A) \equiv 0$ then $\psi(e_{\mathbf{h}}) = 0$ for all $\mathbf{h}$. Since $\{e_{\mathbf{h}}\}$ is a basis for $[End(V)^{\otimes m}]^{S_m}$, we have the result. ■

3.2 Full Completeness in FDVec

We now prove a form of compact closed full completeness in **FDVec**. It has obvious parallels with the full completeness result proved in **Rel**. Consider the functors $F^-, F^+ : \mathbf{FDVec}^n \to \mathbf{FDVec}$, where

$$F^-(\mathbf{V}) = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}; \quad F^+(\mathbf{V}) = V_{\mu_1} \otimes \cdots \otimes V_{\mu_l},$$

with all $\lambda_k, \mu_j \in \{1, \ldots, n\}$.

Note that if $f_1 \otimes f_2 = g_1 \otimes g_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$, where f_1, f_2 are not identically zero, then f_1 and g_1 are scalar multiples of each other, as are f_2 and g_2.

We will say that natural transformation $\tilde{\sigma} : F^- \to F^+$ is a *non-trivial* if each $\tilde{\sigma}_{\mathbf{V}} : F^-(\mathbf{V}) \to F^+(\mathbf{V})$ is a non-zero morphism.

Lemma 3.2.1 *Suppose that $\tilde{\sigma} : F^- \to F^+$ is a non-trivial natural transformation, where F^- and F^+ are given by (2.3). Then (assuming the field k has characteristic zero,) $m = l$.*

Proof Put $V_i = V$ for all $i = 1, \ldots, n$. Choose a scalar $\gamma \neq 0$ and put $f_i : V \to V$, $f_i(v) = \gamma v$ for all v. Suppose that $\tilde{\sigma}_{\mathbf{V}}(v_1 \otimes \cdots \otimes v_m) = w_1 \otimes \cdots \otimes w_m$. Then $F^+(\mathbf{f})\tilde{\sigma}_{\mathbf{V}}$ sends $v_1 \otimes \cdots \otimes v_m$ to $\gamma^l(w_1 \otimes \cdots \otimes w_m)$, while $\tilde{\sigma}_{\mathbf{V}} F^-(\mathbf{f})$ sends $v_1 \otimes \cdots \otimes v_m$ to $\gamma^m(w_1 \otimes \cdots \otimes w_m)$. By naturality, we must have $\gamma^l = \gamma^m$, and when k has characteristic zero, this implies that $l = m$. ∎

Theorem 3.2.2 *If $\tilde{\sigma} : F^- \to F^+$ is a non-trivial natural transformation, then $\tilde{\sigma}$ is a linear combination of permutations on the tensor factors, i.e. there exist permutations $\delta_1, \ldots, \delta_T \in S_m$ and scalars $\gamma_1, \ldots, \gamma_T \in k$ such that*

$$\tilde{\sigma}_{\mathbf{V}}(v_1 \otimes \cdots \otimes v_m) = \sum_{h=1}^{T} \gamma_h (v_{\delta_h(1)} \otimes \cdots \otimes v_{\delta_h(m)}) \tag{3.2}$$

for all $v_i \in V_{\lambda_i}$, $i = 1, \ldots, m$.

Proof By Lemma 3.2.1, we know that $m = l$. Write $N = \{i \mid \zeta_i = *\} = \{i_1, \ldots, i_m\}$, $Q = \{i \mid \zeta_i = 1\} = \{j_1, \ldots, j_m\}$ and assume that $\lambda_l = \xi_{i_l}$, $\mu_k = \xi_{j_l}$, $l = 1, \ldots, m$. Let W be a finite dimensional vector space with dimension at least 2, and put $V_i = W$ for all i. Then

$$\begin{array}{ccc} W^{\otimes m} & \xrightarrow{\tilde{\sigma}_{\mathbf{W}}} & W^{\otimes m} \\ {\scriptstyle f^{\otimes m}}\downarrow & & \downarrow{\scriptstyle f^{\otimes m}} \\ W^{\otimes m} & \xrightarrow[\tilde{\sigma}_{\mathbf{W}}]{} & W^{\otimes m} \end{array}$$

commutes over all $f \in End(W)$. Thus $\tilde{\sigma}_{\mathbf{W}} \in \langle f^{\otimes m} \mid f \in End(W) \rangle'$. By Theorem 3.1.2, $\tilde{\sigma}_{\mathbf{W}}$ is a linear combination of permutations on the tensor factors, i.e. there exist permutations $\delta_1, \ldots, \delta_T \in S_m$ and scalars $\gamma_1, \ldots, \gamma_T \in k$ such that equation (3.2) holds whenever $V_i = W$ for all i.

For each $h \in \{1, \ldots, T\}$, put

$$\tilde{\sigma}^h_{\mathbf{W}}(w_1 \otimes \cdots \otimes w_m) = w_{\delta_h(1)} \otimes \cdots \otimes w_{\delta_h(m)},$$

so that $\tilde{\sigma}_{\mathbf{W}} = \sum_{h=1}^{T} \gamma_h \ \tilde{\sigma}^h_{\mathbf{W}}$. We wish to show that equation (3.2) holds for arbitrary finite dimensional vector spaces $V_1, \ldots, V_n$. We will prove this here, just in the case when $\tilde{\sigma}_{\mathbf{W}} = \tilde{\sigma}^h_{\mathbf{W}}$ for a fixed h. The argument can be easily extended to the general case.

Fix h and write $\delta = \delta_h$. Firstly, we prove that $\lambda_{\delta(j)} = \mu_j$ for all $j = 1, \ldots, m$. Consider arbitrary vectors $w_1, \ldots, w_m \in W$, and arbitrary linearly independent non-zero endomorphisms $f_1, \ldots, f_n : W \to W$. By naturality, we deduce that

$$\begin{array}{rrcl} & F^+(\mathbf{f})\tilde{\sigma}^h_{\mathbf{W}}(w_1 \otimes \cdots \otimes w_m) & = & \tilde{\sigma}^h_{\mathbf{W}} F^-(\mathbf{f})(w_1 \otimes \cdots \otimes w_m) \\ \Leftrightarrow & f_{\mu_1}(w_{\delta(1)}) \otimes \cdots \otimes f_{\mu_m}(w_{\delta(m)}) & = & f_{\lambda_{\delta(1)}}(w_{\delta(1)}) \otimes \cdots \otimes f_{\lambda_{\delta(m)}}(w_{\delta(m)}) \\ \Leftrightarrow & f_{\mu_1} \otimes \cdots \otimes f_{\mu_m} & = & f_{\lambda_{\delta(1)}} \otimes \cdots \otimes f_{\lambda_{\delta(m)}}. \end{array}$$

Since $w_1, \ldots, w_m$ are arbitrary, we deduce that for each $j \in \{1, \ldots, m\}$, $f_{\lambda_{\delta(j)}}$ and f_{μ_j} are scalar multiples of each other. By the linear independence of the f_i, this implies that $\lambda_{\delta(j)} = \mu_j$ (and $f_{\lambda_{\delta(j)}} = f_{\mu_j}$).

Now consider arbitrary vectors $v_j \in V_{\lambda_j}$, $j = 1, \ldots, m$, and arbitrary linear maps $g_i : V_i \to W$, $i = 1, \ldots, n$. By naturality, the following diagram commutes.

$$\begin{array}{ccc} V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m} & \xrightarrow{\tilde{\sigma}^h_{\mathbf{V}}} & V_{\mu_1} \otimes \cdots \otimes V_{\mu_m} \\ \downarrow{\scriptstyle F^-(\mathbf{g})} & & \downarrow{\scriptstyle F^+(\mathbf{g})} \\ W \otimes \cdots \otimes W & \xrightarrow[\tilde{\sigma}^h_{\mathbf{W}}]{} & W \otimes \cdots \otimes W \end{array}$$

In particular,

$$\begin{array}{rcl} \tilde{\sigma}^h_{\mathbf{W}} F^-(\mathbf{g})(v_1 \otimes \cdots \otimes v_m) & = & g_{\lambda_{\delta(1)}}(v_{\delta(1)}) \otimes \cdots \otimes g_{\lambda_{\delta(m)}}(v_{\delta(m)}) \\ & = & g_{\mu_1}(v_{\delta(1)}) \otimes \cdots \otimes g_{\mu_m}(v_{\delta(m)}) \\ & = & F^+(\mathbf{g})(v_{\delta(1)} \otimes \cdots \otimes v_{\delta(m)}), \end{array}$$

from which we deduce that $\tilde{\sigma}^h_{\mathbf{V}}(v_1 \otimes \cdots \otimes v_m) = v_{\delta(1)} \otimes \cdots \otimes v_{\delta(m)}$. ■

Corollary 3.2.3 *(Full Completeness in* **FDVec***) Every non-trivial natural transformation $\tilde{\sigma} : F^- \to F^+$ in* **FDVec** *is induced by a unique morphism $F^-(\mathbf{X}) \to F^+(\mathbf{X})$ in the free compact closed category on n objects $X_1, \ldots, X_n$ with trivial dimension, enriched over* **FDVec**.

Once again, we can remove the restriction on the dimension of the objects in the free category. **FDVec** interprets all dimensions as field elements, so the dimensions are absorbed into the enrichment of **FDVec** over itself, as scalar multiples of the morphisms. Thus we have the weaker result,

Corollary 3.2.4 *(Full Completeness in* **FDVec***) Every natural transformation $\tilde{\sigma} : F^- \to F^+$ in* **FDVec** *is induced by a morphism $F^-(\mathbf{X}) \to F^+(\mathbf{X})$ in the free compact closed category on n objects $X_1, \ldots, X_n$, enriched over* **FDVec**.

3.3 Full Completeness in GFDVec

Our approach to the proof of $*$-autonomous full completeness in **GFDVec** is along the same lines as the approach taken in **GRel**. In this specific setting, objects in **GFDVec** consist of triples $(|\mathcal{A}|, \mathcal{A}_s, \mathcal{A}_t)$ where $|\mathcal{A}| = V$ is a finite dimensional vector space over a fixed field k (with characteristic zero), $\mathcal{A}_s$ is a subset of vectors of V, and $\mathcal{A}_t$ is a subset of linear functionals on V.

The unit for tensor is $\mathbf{1} = (k, \{id_k\}, \{id_k, 0_k\})$, and in particular, $\mathbf{FDVec}(k,k) \neq \{id_k\}$, so **GFDVec** does not support the Mix rule and is a model of pure MLL. The collection $\mathbb{S}$ is the collection of objects $\mathcal{A}$ for which $\theta(v) \neq 0$ for all $v \in \mathcal{A}_s$ and $\theta \in \mathcal{A}_t$. There does exist a zero map on k in **FDVec**, so $\mathbb{T}$ is the non-trivial collection of objects $\mathcal{A}$ for which $\theta(v) = 0$ for all $v \in \mathcal{A}_s$ and $\theta \in \mathcal{A}_t$.

We assume that $F(\underline{\mathcal{X}}, \underline{\mathcal{Y}})$ is a formula built from $\mathcal{X}_1, \ldots, \mathcal{X}_n, \mathcal{Y}_1^\perp, \ldots, \mathcal{Y}_n^\perp$ by the connectives $\otimes$ and $⅋$. Then F induces a multivariant functor $[\![F]\!] : \mathbf{GFDVec}^n \times (\mathbf{GFDVec}^{op})^n \to \mathbf{GFDVec}$ which by abuse of notation we will also refer to as F.

In **GFDVec**, we will say a dinatural transformation ρ is *non-trivial* if $U\rho$ is a non-trivial dinatural transformation in **FDVec**, i.e. each morphism in $U\rho$ is non-zero in **FDVec**.

Lemma 3.3.1 *Suppose that ρ is a non-trivial dinatural transformation in* **GFDVec** *from the constant functor $\mathfrak{K}_\mathbf{1}$ to the multivariant functor F. Then ρ is characterised by the non-trivial dinatural transformation $U\rho = \sigma$ in* **FDVec** *– a linear combination of "formal" dinatural maps specified by fixed-point-free involutions $\phi_1, \ldots, \phi_T$ on $\{1, \ldots, p\}$. This enables us to form T proof structures for the MLL formula F, each ϕ_h specifying a set of axiom links for a proof structure.*

Proof By Lemma 0.4.3, the non-trivial dinatural transformation $U\rho = \sigma$ can be viewed as the natural transformation $\tilde{\sigma} : F^- \to F^+$ in the previous section. By full completeness in **FDVec**, we know that $\tilde{\sigma}$ is a linear combination of permutations on the tensor factors. Each permutation δ_h on the tensor factors induces a fixed-point-free involution ϕ_h on $\{1, \ldots, p\}$ in the same manner described in Theorem 2.1.4, so that σ is a linear combination of "formal" dinatural maps specified by each ϕ_h. ■

At this point, ideally we would like to imitate the approach taken in Chapter 2, i.e. show that ρ is a linear combination of precisely *one* formal dinatural, and that we can associate with ρ a unique MLL proof structure. In **GRel**, we appealed to a subcategory $\mathbb{S}^1$ to prove this fact, but there appears to be no obvious interpretation of this subcategory in **GFDVec**. Nevertheless, I believe this fact to still be true in **GFDVec**, having tried a few simple examples. At the time of submitting this thesis, the general proof has yet to be established. We therefore remain with ρ as a linear combination of formal dinaturals.

We now proceed in two steps. Firstly, we will suppose that ρ *is* precisely one formal dinatural, and show that the corresponding MLL proof structure is a proof net. This follows closely along the lines of Chapter 2. Secondly, I will present a sketch, suggested by J.M.E. Hyland, of how to extend the argument so as to deal with non-trivial linear combinations of formal dinaturals. The detailed analysis is complicated and has not yet been fully resolved.

Theorem 3.3.2 *Suppose that $\rho : \mathfrak{K}_\mathbf{1} \to F$ is a non-trivial formal dinatural transformation in* **GFDVec**. *Consider the unique proof structure for F associated with ρ, specified by a fixed-point-free involution ϕ. Then for any DR-switching, the associated DR-graph is acyclic.*

Proof Suppose that for a certain DR-switching, the associated DR-graph contains a cycle. Consider the shortest cycle, and express it as lower connected pairs $(a_1, b_1), \ldots, (a_r, b_r)$ such that $\phi(b_i) = a_{i+1}$ for all $i \in \mathbb{Z}_r$. Following the same argument as presented in Theorem 2.2.8, we may assume that F has been simplified to a formula $\tilde{F}$ of the form in equation (2.5) and that we have derived from ρ a new dinatural transformation, *restricted to objects and morphisms in* $\mathbb{S}$, from $\mathfrak{K}_\mathbf{1}$ to $\tilde{F}$ in such a way that the corresponding proof structure preserves the cyclic structure.

The argument of Theorem 2.2.8 is completely transferable. A "test" object in $\mathbb{S}$ in **GFDVec** is some $\mathcal{A}$ such that $\mathcal{A} \cong \mathcal{A}^\perp$, with $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \in \mathcal{A}_s$, $\frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \in \mathcal{A}_t$ (where $\{e_i\}_{i=1}^n$ is a basis for $|\mathcal{A}| = V$ and $\{\varepsilon_i\}_{i=1}^n$ is the dual basis for V^*), and such that there exists $\alpha : \mathcal{A} \otimes \mathcal{A}^\perp$ with trace zero, i.e.

$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{U\alpha} k = 0_k.$$

For example, take the object

$$\mathcal{A} = \left(\mathbb{R}^2, \left\{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \left\{\frac{1}{\sqrt{2}}(1,1)\right\}\right)$$

with $\alpha : \mathcal{A} \otimes \mathcal{A}^{\perp} \to \perp$, $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Evidently, $\mathcal{A} \cong \mathcal{A}^{\perp}$, and the matrix α clearly lifts to a morphism $\mathcal{A} \to \mathcal{A}$ or equivalently $\mathcal{A} \otimes \mathcal{A}^{\perp} \to \perp$ or $\mathcal{A}^{\perp} \otimes \mathcal{A} \to \perp$. In addition,

$$(1,0)\ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 = (0,1)\ \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which shows that $\mathbb{R} \xrightarrow{\eta} \mathbb{R}^2 \otimes (\mathbb{R}^2)^* \xrightarrow{\alpha} \mathbb{R}$ is identically zero. (Indeed α has trace zero in the usual matrix sense.)

Put $\mathcal{A}_i = \mathcal{A}$ for all $i = 1, \ldots, n$. Then in **FDVec**, $U\tilde{\rho}$ can be decomposed into $f_1 \otimes f_2$ where

$$f_1 : k \to \Gamma_{\mathbf{A}}; \quad f_2 : k \to (V^{\zeta_{a_1}} \otimes V^{\zeta_{b_1}}) \otimes \cdots \otimes (V^{\zeta_{a_r}} \otimes V^{\zeta_{b_r}}).$$

Choose $\tau^+ : \mathcal{A} \to \perp$, $\tau^+ = \frac{1}{\sqrt{n}} \sum \varepsilon_i$ and $\tau^- : \mathcal{A}^{\perp} \to \perp$, $\tau^- = \frac{1}{\sqrt{n}} \sum e_i$. Let $\psi : \mathcal{A} \to \mathcal{A}^{\perp}$ be the isomorphism $e_i \mapsto \varepsilon_i$. Put

$$\tau_i = \begin{cases} id_{\mathcal{A}} & \text{if } \zeta_{a_i} \neq \zeta_{b_i}; \\ \psi & \text{if } \zeta_{a_i} = \zeta_{b_i} = 1; \\ \psi^{-1} & \text{if } \zeta_{a_i} = \zeta_{b_i} = \perp \end{cases}$$

for each $i = 1, \ldots, r-1$, (where we identify ψ with a map $\mathcal{A} \otimes \mathcal{A}^{\perp} \to \perp$ etc.) and put

$$\tau_r = \begin{cases} \alpha & \text{if } \zeta_{a_r} \neq \zeta_{b_r}; \\ \alpha(id_{\mathcal{A}} \otimes \psi) & \text{if } \zeta_{a_i} = \zeta_{b_i} = 1; \\ \alpha((\psi^{-1} \otimes id_{\mathcal{A}^{\perp}})) & \text{if } \zeta_{a_i} = \zeta_{b_i} = \perp \end{cases}$$

Then by Proposition 2.2.7, we can construct a morphism $\tau_{\underline{\mathcal{A}}} : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \perp$. In particular, $U\tau_{\underline{\mathcal{A}}}$ can be decomposed into $g_1 \otimes g_2$ where

$$g_1 : \Gamma_{\mathbf{A}} \to k; \quad g_2 : (V^{\zeta_{a_1}} \otimes V^{\zeta_{b_1}}) \otimes \cdots \otimes (V^{\zeta_{a_r}} \otimes V^{\zeta_{b_r}}) \to k.$$

Moreover, since $\tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ is an object in $\mathbb{S}$, we have

$$U(\tau_{\underline{\mathcal{A}}} \tilde{\rho}_{\underline{\mathcal{A}}}) = id_k. \tag{3.3}$$

So $(U\tau_{\underline{\mathcal{A}}})(U\tilde{\rho}_{\underline{\mathcal{A}}}) = (g_1 \otimes g_2)(f_1 \otimes f_2) = (g_1 f_1) \otimes (g_2 f_2) = id_I$.

By construction, $g_1 f_1 = id_k$. For example, if $\Gamma_V = V \otimes V^*$ with $V = \langle e_1, e_2 \rangle$, then

$$f_1(1) = \sum_{i=1,2} e_i \otimes \varepsilon_i = e_1 \otimes \varepsilon_1 + e_2 \otimes \varepsilon_2,$$

while

$$g_1 = \frac{1}{2} \sum_{j,l=1,2} \varepsilon_j \otimes e_l = \frac{1}{2}(\varepsilon_1 \otimes e_1 + \varepsilon_1 \otimes e_2 + \varepsilon_2 \otimes e_1 + \varepsilon_2 \otimes e_2).$$

Therefore

$$g_1 f_1(1) = \frac{1}{2} \sum_{i,j,l=1,2} \varepsilon_j(e_i)\varepsilon_i(e_l) = \frac{1}{2} + \frac{1}{2} = 1.$$

Examining the cycle a little closer, we see that there are an even number, $2s$ say, of occurrences of i for which $\zeta_{a_i} = \zeta_{b_i}$, s of which are $\zeta_{a_i} = \zeta_{b_i} = 1$ and s of which are $\zeta_{a_i} = \zeta_{b_i} = \bot$. Therefore τ was built from the same number of ψ and ψ^{-1}, and so the composition $g_2 f_2$ is equivalent to the map

$$k \xrightarrow{\eta_V} V \otimes V^* \xrightarrow{U\alpha} k$$

which by choice of α is the *zero* morphism on k. Therefore $(g_1 f_1) \otimes (g_2 f_2)$ cannot be the identity morphism on I, contradicting equation (3.3). Therefore $\tilde{\rho}$ cannot exist, and the original DR-graph must have been acyclic. ■

Theorem 3.3.3 *Suppose that $\rho : \mathfrak{K}_\mathbf{1} \to F$ is a non-trivial formal dinatural transformation in* **GFDVec**. *Consider the unique proof structure for F associated with ρ specified by a fixed-point-free involution ϕ. Then for any DR-switching, the associated DR-graph is connected.*

Proof Since **FDVec**(k,k) does have a zero morphism, the collections $\mathbb{T}$ (Definition 2.2.9) and $\mathbb{T}_0$ (Proposition 2.2.12) are non-trivial. Consequently, the argument of Theorem 2.2.13 is completely transferable. We suppose that there is a DR-graph which is not connected, and simplify F to a formula $\hat{F} = C_1 \parr \cdots \parr C_q$ where the C_k are representative of the components in the original DR-graph, and they are themselves provable formulae in MLL. The non-trivial dinatural ρ induces a non-trivial dinatural $\hat{\rho} : \mathfrak{K}_\mathbf{1} \to \hat{F}$ or equivalently,

$$\hat{\rho}_{\underline{\mathcal{A}}} : C_1^\perp(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to (C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}).$$

such that $U\hat{\rho}$ factors through k, i.e.

$$U\hat{\rho}_{\underline{\mathcal{A}}} : UC_1^\perp(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \xrightarrow{\mu_{|\underline{\mathcal{A}}|}} k \xrightarrow{\kappa_{|\underline{\mathcal{A}}|}} U(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}).$$

Instantiate each $\mathcal{A}_i$ at a "test" object in $\mathbb{T}_0$. An example of such an object in **GFDVec** is

$$\mathcal{A} = \left(\mathbb{R}^2, \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \{(0,1)\}\right),$$

since $(0,1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$.

Then given $\alpha_{\underline{\mathcal{A}}} : \mathbf{1} \to C_1^\perp(\underline{\mathcal{A}}, \underline{\mathcal{A}})$, the definition of morphism in **GFDVec** implies that $\hat{\rho}_{\underline{\mathcal{A}}}\alpha_{\underline{\mathcal{A}}} \in ((C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}))_s$. But $C_1^\perp(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ is an object of $\mathbb{T}_0$, so

$$\begin{aligned} U(\hat{\rho}_{\underline{\mathcal{A}}}\alpha_{\underline{\mathcal{A}}}) &= k \xrightarrow{U\alpha_{\underline{\mathcal{A}}}} UC_1^\perp(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \xrightarrow{\mu_{|\underline{\mathcal{A}}|}} k \xrightarrow{\kappa_{|\underline{\mathcal{A}}|}} U(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \\ &= k \xrightarrow{\qquad\qquad 0_k \qquad\qquad} k \xrightarrow{\kappa_{|\underline{\mathcal{A}}|}} U(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \end{aligned}$$

forms a zero map in $(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}},\underline{\mathcal{A}}))_s$. That $(C_2 \parr \cdots \parr C_q)(\underline{\mathcal{A}},\underline{\mathcal{A}}))_s$ contains a zero map is, by choice of $\mathcal{A}$ in $\mathbb{T}_0$, a contradiction, so the original DR-graph must be connected.

■

Now, I will describe Hyland's sketch of the extension of Theorem 3.3.2 to the case when ρ is a linear combination of formal dinaturals. It is thought that Theorem 3.3.3 could be extended in a similar way.

Given a non-trivial dinatural transformation $\rho = \sum_{h=1}^{T} \gamma_h \rho^h$, choose one formal dinatural ρ^h occurring as a summand (with non-zero coefficient) in ρ, and the associated fixed-point-free involution ϕ_h. It is our aim to choose a test object and maps τ^+, τ^- $\tau_1, \ldots, \tau_r$ (along the lines of Theorem 3.3.2) which will form a map τ, composable with ρ, such that when we do compose with ρ, we *annihilate* all other formal dinaturals, thus reducing the argument to the case we have already considered in Theorem 3.3.2.

Recall that $U\rho = \sigma = \sum_{h=1}^{T} \gamma_h \sigma^h$ is a dinatural transformation in **FDVec** which can be identified as a natural transformation $\tilde{\sigma} = \sum_{h=1}^{T} \gamma_h \tilde{\sigma}^h : F^- \to F^+$ where

$$F^-(\mathbf{V}) = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}; \quad F^+(\mathbf{V}) = V_{\mu_1} \otimes \cdots \otimes V_{\mu_m},$$

and each formal natural transformation $\tilde{\sigma}^h$ is a permutation (given by $\delta_h \in S_m$, say) on the tensor factors.

Consider the test object $\mathcal{A}$ in Theorem 3.3.2 associated with F. Consider an m-fold direct sum $\mathcal{A}^{\oplus m}$, whereby we associate each direct summand with an axiom link given by ϕ_h. In fact, since each λ_i is associated with a distinct occurrence of $\mathcal{A}^\perp$ in the formula $F(\underline{\mathcal{A}},\underline{\mathcal{A}})$, and hence a distinct axiom link, we will write

$$\mathcal{A}^{\oplus m} = \mathcal{A}^1 \oplus \cdots \oplus \mathcal{A}^m, \text{ with } \mathcal{A}^i = \mathcal{A} \text{ for all } i = 1, \ldots, m,$$

and associate each λ_i with the summand $\mathcal{A}^i$, $i = 1, \ldots, m$. Note that $(\mathcal{A}^{\oplus m})^\perp = (\mathcal{A}^\perp)^{\oplus m}$.

We now form the "τ" maps. For each occurrence of $\mathcal{A}^\perp_{\lambda_i}$ and $\mathcal{A}_{\mu_{\delta_h(i)}}$ in $\Gamma_{\underline{\mathcal{A}}}$, define

$$\begin{aligned} \overline{\tau}_i{}^+ : &\quad \mathcal{A}^{\oplus m} \xrightarrow{P_i} \mathcal{A}^i \xrightarrow{\tau^+} \perp; \\ \overline{\tau}_i{}^- : &\quad (\mathcal{A}^{\oplus m})^\perp \xrightarrow{P_i} (\mathcal{A}^i)^\perp \xrightarrow{\tau^-} \perp, \end{aligned}$$

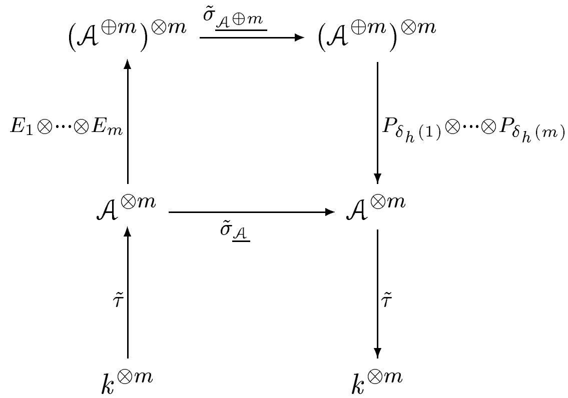
where $P_i : \mathcal{A}^{\oplus m} \to \mathcal{A}^i$ denotes the projection onto the i-th direct summand. Next for each $i = 1, \ldots, r$, choose the unique λ_j and λ_l such that

$$\lambda_j = \begin{cases} \xi_{a_i} & \text{if } \zeta_{a_i} = \perp; \\ \xi_{b_{i-1}} & \text{if } \zeta_{a_i} = 1; \end{cases} \qquad \lambda_l = \begin{cases} \xi_{b_i} & \text{if } \zeta_{b_i} = \perp; \\ \xi_{a_{i+1}} & \text{if } \zeta_{b_i} = 1. \end{cases}$$

(In other words, choose the λ_j and λ_l which associate with the same axiom links as ξ_{a_i} and ξ_{b_i}, respectively.) Define

$$\overline{\tau}_i : (\mathcal{A}^{\oplus m})^{\zeta_{a_i}} \otimes (\mathcal{A}^{\oplus m})^{\zeta_{b_i}} \xrightarrow{P_j \otimes P_l} (\mathcal{A}^j)^{\zeta_{a_i}} \otimes (\mathcal{A}^l)^{\zeta_{b_i}} \xrightarrow{\tau_i} \perp.$$

With these maps, we can form a morphism $\overline{\tau} : \tilde{F}(\underline{\mathcal{A}^{\oplus m}}, \underline{\mathcal{A}^{\oplus m}}) \to \bot$. In particular, when composing with $\rho_{\underline{\mathcal{A}^{\oplus m}}}$, we *annihilate* all other formal dinaturals specified by fixed-point-free involutions $\phi_s \neq \phi_h$. This is most easily seen by demonstrating the parallel action on the natural transformation $\tilde{\sigma}$ in **FDVec**.



From the above diagram, we can see how we are extending the action of τ on $\tilde{\sigma}_{\underline{\mathcal{A}}}$ to an action on $\tilde{\sigma}_{\underline{\mathcal{A}^{\oplus m}}}$, by embedding the ith tensor factor of $\mathcal{A}^{\otimes m}$ into the ith direct summand in $\mathcal{A}^{\oplus m}$ for all i, applying $\tilde{\sigma}_{\underline{(\mathcal{A}^{\oplus m})}}$, and then projecting the ith tensor factor in $(\mathcal{A}^{\oplus m})^{\otimes m}$ into the $\delta_h(i)$-th direct summand $\mathcal{A}^{\delta_h(i)} = \mathcal{A}$ for all i. It is clear that this composite will be the identity morphism $id_{\mathcal{A}^{\otimes m}}$ only on the formal dinatural $\tilde{\sigma}^h$ associated with the permutation $\delta_h \in S_m$, and it will be zero on all other formal dinaturals $\tilde{\sigma}^s$, $s \neq h$. Therefore $\overline{\tau}$ annihilates all other formal dinaturals ρ^s, $s \neq h$, and we are back to the simple case dealt with in Theorem 3.3.2.

Theorem 3.3.4 *(Full Completeness in* **GFDVec***) Every non-trivial dinatural transformation in* **GFDVec** *from the constant functor* $\mathfrak{K}_{\mathbf{1}}$ *to the multivariant functor F is a linear combination of formal dinatural transformations ρ^h, where each ρ^h is the denotation of a unique proof in MLL of the formula F, and is therefore induced by a unique morphism $I \to F(\mathbf{X}, \mathbf{X})$ in the free $*$-autonomous category on n objects $X_1, \ldots, X_n$.*

Chapter 4

Double glueing on Conway games

4.1 Conway Games

4.1.1 Preliminaries

In this section, we present a games semantics similar to that presented by Abramsky and Jagadeesan [AJ94], and Hyland and Ong [HO93]. (It is also different from Blass' semantics [Bla92], which fails to form a category.) However, we will be much more precise about positions and moves, and hence what it means for a strategy to be history-free.

We will present two compact closed categories of games. The first is Joyal's category of Conway games and winning strategies [Joy77], which we will denote by **Con**. However, to achieve full completeness, it is necessary to consider a subcategory $\mathbf{Con}_{hf}$, whose objects are Conway games equipped with a decomposition, and whose morphisms are winning strategies which are history-free in some sense.

We start with some basic ground rules for a "game". It is a game of moves for two players, whom we call Player and Opponent, or simply P and O. Usually, our sympathies lie with Player, and we always assume that when a game is being *played*, it is Opponent who makes the first move. Thereafter, moves alternate between Player and Opponent. Our games are finite – there are only a finite number of positions and moves in any game. Our games are deterministic – one player must lose, and therefore the other player must win. We deem a player to be the loser if he is unable to make a legal move in the game.

Before defining a game, recall that a *tree* is an acyclic connected graph. A *rooted tree* is a tree with a distinguished vertex, which we call the *root*. It is then convenient to think of the vertices of a rooted tree arranged in levels. The root is at *level 0*, and those vertices adjacent to it are at *level 1*. For $k \geq 2$, vertices are at *level k* if they are adjacent to some vertex at level $k-1$, and have not already been assigned to level $k-2$.

Definition 4.1.1 A *(Conway) game* $A = (P_A, M_A, \lambda_A)$ consists of the following data :

- a finite set P_A of *positions*, with an *initial position*, usually denoted by 0 or 0_A;
- a set of *moves* from positions to positions, which may be thought of as a relation $M_A : P_A \to P_A$;

- a labelling function $\lambda_A : M_A \to \{P, O\}$, to indicate whether a move is a Player's move or an Opponent's move;

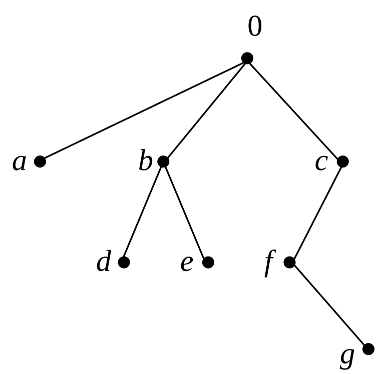
subject to the following condition. Let T_A be the graph whose vertices are the positions of A and whose edges are the moves in A. Then T_A must be a rooted tree with root 0, whose levels are consistent with the moves of A. That is, if there exists a move in A from position a to position b, then the vertex b in T_A is at level one greater than the vertex a.

We shall write $M_A^+ = \lambda^{-1}(\{P\})$ and $M_A^- = \lambda^{-1}(\{O\})$ for the sets of Player's moves (P-moves) and Opponent's moves (O-moves), respectively. We will often write a move m from position a to position b as an arrow $m : a \to b$, placing a P or an O above the arrow to indicate whether it is a P-move or an O-move. We call a and b the *source position* and *target position* of m, respectively. If $a = 0$, then we call m an *initial* move.

It is convenient to visualise a game as a rooted tree, placing O-moves pointing to the right, say, and P-moves pointing to the left. (See [Con76]). For example, the game

$$\begin{array}{rcl} P_A & = & \{0, a, b, c, d, e, f, g\} \\ M_A & = & \{0 \overset{P}{\to} a, 0 \overset{P}{\to} b, 0 \overset{O}{\to} c, \\ & & b \overset{P}{\to} d, b \overset{O}{\to} e, c \overset{P}{\to} f, f \overset{O}{\to} g\} \end{array}$$

can be pictured as



A possible scenario of play would be O moves from 0 to c, P from c to f and O from f to g. At this point, P is unable to make a move, so he loses the game. On the other hand, if P were to start, then he might foolishly choose to move from 0 to b, in which case O would have to move from b to e and then P is left without a move, so he loses again. A more sensible choice for P would be to move from 0 to a.

Definitions 4.1.2 Let A be a Conway game. A *valid play* in A is a finite sequence $s_1 \cdot \ldots \cdot s_k$ of moves in A such that the target position of s_i is the source position of s_{i+1}, for all $i = 1, \ldots, k-1$. A *legal play* in A is a valid play in which the moves strictly alternate between Player and Opponent in sequence, i.e. we have the additional condition,

$$\lambda_A(s_i) \neq \lambda_A(s_{i+1}) \text{ for each } i = 1, \ldots, k-1.$$

Denote by V_A the set of all valid plays in A, and denote by L_A the set of all legal plays in A.

Writing moves as arrows, a legal play could then, for example, be written as

$$a \xrightarrow{O} b \xrightarrow{P} c \xrightarrow{O} d.$$

The set V_A may appear a little odd, as it includes plays in which Player makes the first move, and plays in which the moves do not alternate between Player and Opponent. While it may be perfectly possible for Player to start, or either player to make 2 consecutive moves, such plays are not considered as actual game plays. However, it is essential that we consider both sets of plays to enable us to consider multiple games.

Duality. The game $A^\perp$ is described by

- $P_{A^\perp} = P_A$ with initial position 0_A;
- $M_{A^\perp} = M_A$;
- $\lambda_{A^\perp} = \overline{\lambda_A}$ where $\overline{P} = O$ and $\overline{O} = P$.

So a move in A is also a move in $A^\perp$ and vice versa. We will not make any distinction between moves in either game, for we wish to remain free to "toggle" our perception of a move whenever necessary. For example, an O-move in A may need to be considered as a P-move in $A^\perp$.

Tensor. Given games A and B, we form the tensor product game $A \otimes B$ as follows.

- $P_{A\otimes B} = P_A \times P_B$ with initial position $(0_A, 0_B)$;
- $M_{A\otimes B} = \{(a_1, b) \to (a_2, b) \mid (a_1 \to a_2) \in M_A\} \cup \{(a, b_1) \to (a, b_2) \mid (b_1 \to b_2) \in M_B\}$;
- $\lambda_{A\otimes B} : M_{A\otimes B} \to \{P, O\}$
  $(a_1, b) \to (a_2, b) \mapsto \lambda_A(a_1 \to a_2)$
  $(a, b_1) \to (a, b_2) \mapsto \lambda_B(b_1 \to b_2)$;

(This is called the *direct sum* $A+B$ in [Con76].) It is easy to check that the tensor product is associative, and we henceforth omit bracketing in multiple (tensor) games, unless essential. Furthermore, the tensor product is commutative with unit $I = (\{\cdot\}, \varnothing, -)$, i.e. it is the game in which neither player can make a move.

Note that our definition of tensor product does not put any restriction on either player switching games. It is entirely legal for either player to switch games, and he does not require his opponent to have moved immediately beforehand in the game to which he wishes to switch. This means that a subgame may observe consecutive moves by the same player, and for this reason, we must distinguish "legal" plays from "valid" plays. Thus a legal play $s \in L_{A\otimes B}$ in the game $A \otimes B$ need only satisfy $s{\downarrow_A} \in V_A$ and $s{\downarrow_B} \in V_B$. (See definition of $\downarrow$ below.)

We now provide some notation for identifying, localising and globalising moves in a multiple (tensor) game. Let $\overline{A} = A_1 \otimes \cdots \otimes A_p$ be a multiple game and let $\mathbf{A} = A_1, \ldots, A_p$ be the ordered list of subgames of $\overline{A}$. (We say that $\mathbf{A}$ is a *decomposition* of $\overline{A}$.) Suppose that $m : a \to a'$ is a move in $\overline{A}$ and write $a = (a_1, \ldots, a_p)$ and $a' = (a'_1, \ldots, a'_p)$. Then there exists a unique $i \in \{1, \ldots, p\}$ such that $a_j = a'_j$ for all $j \neq i$ and $a_i \to a'_i$ is a move in the subgame A_i. We write $[m]_{\mathbf{A}}$ for this unique i, and $m^*_{\mathbf{A}}$ for the move $a_i \to a'_i$. We can think of m as being "active" in the subgame A_i when we decompose $\overline{A}$ into the tensor product $A_1 \otimes \cdots \otimes A_p$.

Conversely, suppose that $\mathbf{A}' = A_{i_1}, \ldots, A_{i_q}$ is a sublist of $\mathbf{A}$, and suppose that $n : (a_{i_1}, \ldots, a_{i_q}) \to (a'_{i_1}, \ldots, a'_{i_q})$ is a move in the game $A_{i_1} \otimes \cdots \otimes A_{i_q}$. If $a = (a_1, \ldots, a_p)$ is a position in $\overline{A}$, then we can globalise n to a move in $\overline{A}$ by fixing all positions a_j for $j \notin \{i_1, \ldots, i_q\}$. Specifically, $n{\uparrow}^{\mathbf{A}}_{\mathbf{A}';a}$ is the move from position a to position $a' = (a'_1, \ldots, a'_p)$ where $a_j = a'_j$ for all $j \notin \{i_1, \ldots, i_q\}$. In the case when $q = 1$, we have a move $n{\uparrow}^{\mathbf{A}}_{A_i;a}$ such that

$$[n{\uparrow}^{\mathbf{A}}_{A_i;a}]_{\mathbf{A}} = i \text{ and } (n{\uparrow}^{\mathbf{A}}_{A_i;a})^*_{\mathbf{A}} = n.$$

Given a sequence s of moves in $\overline{A}$, and a sublist $\mathbf{A}' = A_{i_1}, \ldots, A_{i_q}$ of subgames of $\mathbf{A}$, we write $s{\downarrow_{A_{i_1},\ldots,A_{i_q}}}$ for the localisation of s to the game $A_{i_1} \otimes \cdots \otimes A_{i_q}$. More specifically, it is the deletion of moves m in $\overline{A}$ such that $[m]_{\mathbf{A}} \notin \{i_1, \ldots, i_q\}$ (i.e. m was not active in any of the subgames $A_{i_1}, \ldots, A_{i_q}$), and the restriction of all remaining moves to moves in the game $A_{i_1} \otimes \cdots \otimes A_{i_q}$.

4.1.2 Strategies

Definition 4.1.3 A *strategy* σ for a non-trivial game A is a non-empty prefix-closed subset of L_A satisfying the following three conditions.

(s1) If $m \cdot s \in \sigma$, then $\lambda_A(m) = O$.
(s2) If $s \cdot m, s \cdot n \in \sigma$, with $\lambda_A(m) = \lambda_A(n) = P$, then $m = n$.
(s3) If $s \in \sigma$, and $s \cdot m \in \mathrm{L}_A$ with $\lambda_A(m) = O$, then $s \cdot m \in \sigma$.

Condition (s1) says that Opponent must start. Our definition of strategy is *partial*, i.e. σ is not required to respond to every O-move. However, (s2) states that if σ does respond to a particular O-move, then that response is unique.

Note that σ induces a partial function $\hat{\sigma} : L_A \to M_A^+$ satisfying

$$s \cdot m \in \sigma, \lambda_A(m) = P \Leftrightarrow \hat{\sigma}(s) \text{ is defined and } \hat{\sigma}(s) = m,$$

which can be thought of as the function which returns a response for Player (if indeed such a response exists) to a move by Opponent. We call $\hat{\sigma}$ the *strategic response function of* σ.

There is only one "strategy" for the trivial game I, namely the trivial strategy. P is automatically the winner, since O must start and he is unable to make any move.

4.1.3 Winning strategies

A *winning play for player* P in a game A is a legal play $s_1, \ldots, s_k \in L_A$ such that $\lambda_A(s_k) = P$ and there is no legal play $s_1, \ldots, s_k, s_{k+1} \in L_A$. In other words, O is unable to respond to P's move s_k, and therefore he loses. Naturally, there is a similar definition for a winning play for player O.

A *counter-strategy* is a strategy for O, defined by interchanging P and O in (s2) and (s3) in the Definition 4.1.3. We can now define a winning strategy. Intuitively, we expect it to be a strategy which, when played against any counterstrategy, produces a winning play.

More formally, given a strategy σ (for P) and a counter-strategy τ (for O), we may pitch these strategies against each other by observing the following play:

$$\sigma \mid \tau = \bigsqcup \sigma \cap \tau.$$

This is well defined because $\sigma \cap \tau$ is a (possibly empty) prefix-closed chain, and such a chain has a least upper bound by the finiteness of the sequences in L_A. We say that σ *defeats* τ if the resultant play is non-empty and is a winning play for P, i.e. it ends with a P-move, and τ is unable to respond with any move for O. We say that σ is a *winning strategy* if it defeats all counter-strategies. Note that a winning strategy σ will always have a unique response to all initial O-moves and any O-moves arising from P-moves by σ.

4.2 The category of Conway games

We are now ready to define the category **Con** of Conway games, first considered by Joyal [Joy77]. The objects of **Con** are Conway games, and we define $\sigma : A \to B$ to be a morphism of **Con** if σ is a winning strategy for $A^{\perp} \otimes B$.

Our first non-trivial task is to prove that morphisms do compose. Given games $A_1, \ldots, A_p$, put $\overline{A} = A_1 \otimes \cdots \otimes A_p$ and $\mathbf{A} = A_1, \ldots, A_p$, and define $\mathcal{L}(\mathbf{A})$ to be the set of valid plays in the game $\overline{A}$ such that consecutive moves are active in identical or cyclicly adjacent subgames, i.e.

$$\mathcal{L}(\mathbf{A}) = \{ s_1 \cdot \ldots \cdot s_k \in V_{\overline{A}} \mid \text{for each } i = 1, \ldots, k-1, \\ |[s_i]_{\mathbf{A}} - [s_{i+1}]_{\mathbf{A}}| \leq 1 \text{ or } = p-1 \}.$$

Given strategies σ for $A^\perp \otimes B$ and τ for $B^\perp \otimes C$, we define the *composition* $\sigma;\tau$ *of* σ *and* τ to be

$$\sigma;\tau = \{\overline{s}\downarrow_{A,C} \mid \overline{s} \in \mathcal{L}(A,B,C), \overline{s}\downarrow_{A,B} \in \sigma \text{ and } \overline{s}\downarrow_{B,C} \in \tau\}$$

or

$$\sigma;\tau = \{\overline{s}\downarrow_{A,C} \mid \overline{s} \in S\}$$

where $S = \{\overline{s} \in \mathcal{L}(A,B,C) \mid \overline{s}\downarrow_{A,B} \in \sigma \text{ and } \overline{s}\downarrow_{B,C} \in \tau\}$.

Proposition 4.2.1 *Suppose that* $\sigma : A \to B$ *is a winning strategy for* $A^\perp \otimes B$ *and* $\tau : B \to C$ *is a winning strategy for* $B^\perp \otimes C$*. Then any sequence* $\overline{s} \in S$ *starts with a move which restricts to an* O*-move in* $A^\perp \otimes C$*.*

Proof Suppose that $\overline{s} = \overline{m} \cdot \overline{s}' \in S$. Then $\overline{m}^* = \overline{m}^*_{A,B,C}$ is not a move in B, for otherwise, $(s1)$ for σ would imply that $\overline{m}^*$ is an O-move in B, while $(s1)$ for τ would imply that $\overline{m}^*$ is an O-move in $B^\perp$. This is impossible.

So $\overline{m}^*$ is either a move in $A^\perp$ or a move in C. Assume the former, and we have $\overline{s}\downarrow_{A,B} = \overline{m}\downarrow_{A,B} \cdot \overline{s}'\downarrow_{A,B} \in \sigma$, so that $\overline{m}\downarrow_{A,B}$ is an O-move in $A^\perp \otimes B$ and therefore $\overline{m}^*$ is an O-move in $A^\perp$. Assume the latter, and we have $\overline{s}\downarrow_{B,C} = \overline{m}\downarrow_{B,C} \cdot \overline{s}'\downarrow_{B,C} \in \tau$, so that $\overline{m}\downarrow_{B,C}$ is an O-move in $B^\perp \otimes C$, and therefore $\overline{m}^*$ is an O-move in C.

Therefore, in either case, $\overline{m}\downarrow_{A,C}$ is an O-move in $A^\perp \otimes C$. ■

Proposition 4.2.2 *Suppose that* $\sigma : A \to B$ *is a winning strategy for* $A^\perp \otimes B$ *and* $\tau : B \to C$ *is a winning strategy for* $B^\perp \otimes C$*. Fix an integer* $k \geq 1$*, and suppose that* $\overline{s} \in S$ *takes the form*

$$\overline{s} = \overline{x_1} \cdot \overrightarrow{b_1} \cdot \overline{y_1} \cdot \ldots \cdot \overline{x_{k-1}} \cdot \overrightarrow{b_{k-1}} \cdot \overline{y_{k-1}} \tag{4.1}$$

where each $\overline{x_i}\downarrow_{A,C}$ *is an* O*-move in* $A^\perp \otimes C$*, each* $\overline{y_i}\downarrow_{A,C}$ *is a* P*-move in* $A^\perp \otimes C$*, and each* $\overrightarrow{b_i}$ *is a (possibly empty) sequence of moves, all active in* B*, such that if* $\overline{y_i}$ *is active in* $A^\perp$*, then the last move in* $\overrightarrow{b_i}$ *restricts to an* O*-move in* B*, and if* $\overline{y_i}$ *is active in* C*, then the last move in* $\overrightarrow{b_i}$ *restricts to an* O*-move in* $B^\perp$*. (We assume that* $\overline{s}$ *is an empty sequence if* $k = 1$*.)*

Suppose that $\overline{x_k}$ *is a move which restricts to an* O*-move in* $A^\perp \otimes C$ *such that* $(\overline{s} \cdot \overline{x_k})\downarrow_{A,C} \in L_{A^\perp \otimes C}$*. Then*

(i) $\overline{s} \cdot \overline{x_k} \in S$*;*

(ii) furthermore, there exists unique $\overrightarrow{b_k}$ *and* $\overline{y_k}$ *such that* $\overline{s} \cdot \overline{x_k} \cdot \overrightarrow{b_k} \cdot \overline{y_k} \in S$*, where* $\overline{y_k}\downarrow_{A,C}$ *is a* P*-move in* $A^\perp \otimes C$*, and* $\overrightarrow{b_k}$ *is a (possibly empty) sequence of moves, all active in* B*, such that if* $\overline{y_k}$ *is active in* $A^\perp$*, then the last move in* $\overrightarrow{b_k}$ *restricts to*

an O-move in B, and if $\overline{y_k}$ is active in C, then the last move in $\overrightarrow{b_k}$ restricts to an O-move in $B^\perp$.

Proof When $k=1$, *(i)* is trivial. The proof of *(ii)* is just a simplified version of what lies below, in the sense that there is nothing to check if $\overline{s}$ is empty. So assume that $k>1$.

We first remark that $\overline{s}{\downarrow_{A,B}}$ always ends with a P-move in $A^\perp \otimes B$ and $\overline{s}{\downarrow_{B,C}}$ always ends with a P-move in $B^\perp \otimes C$. Suppose $(\overline{y_{k-1}})^*_{A,B,C}$ is a P-move in $A^\perp$. Then $\overline{s}{\downarrow_{A,B}}$ ends with a P-move in $A^\perp \otimes B$. By assumption, the last move in $\overrightarrow{b_{k-1}}$ restricts to an O-move in B, i.e. a P-move in $B^\perp$, so $\overline{s}{\downarrow_{B,C}}$ ends with a P-move in $B^\perp \otimes C$. The proof when $(\overline{y_{k-1}})^*_{A,B,C}$ is a P-move in C is similar.

W.l.o.g. assume that $(\overline{x_k})^*_{A,B,C}$ is an O-move in $A^\perp$. (The case when it is an O-move in C is similar.) Then $(\overline{s}\cdot\overline{x_k}){\downarrow_{A,C}}\in L_{A^\perp\otimes C}$ implies that $(\overline{s}\cdot\overline{x_k}){\downarrow_A}\in V_{A^\perp}$. Since $\overline{s}{\downarrow_{A,B}}$ ends with a P-move in $A^\perp\otimes B$ and $\overline{x_k}{\downarrow_{A,B}}$ is an O-move in $A^\perp\otimes B$, it follows that $(\overline{s}\cdot\overline{x_k}){\downarrow_{A,B}}$ is a legal play. Furthermore, since $\overline{s}{\downarrow_{A,B}}\in\sigma$, we have $(\overline{s}\cdot\overline{x_k}){\downarrow_{A,B}}\in\sigma$. Also, $(\overline{s}\cdot\overline{x_k}){\downarrow_{B,C}}=\overline{s}{\downarrow_{B,C}}\in\tau$. Therefore, $\overline{s}\cdot\overline{x_k}\in S$. This proves *(i)*.

Since $(\overline{s}\cdot\overline{x_k}){\downarrow_{A,B}}\in\sigma$ ends with an O-move in $A^\perp\otimes B$ and σ is a winning strategy, there exists a unique P-move l_1 in $A^\perp\otimes B$, such that $(\overline{s}\cdot\overline{x_k}){\downarrow_{A,B}}\cdot l_1\in\sigma$. Now globalise l_1 to a move in $A\otimes B\otimes C$. Put $\overline{m_1}=l_1{\uparrow^{A,B,C}_{A,B;a}}$, where a is the target position of $\overline{x_k}$. Then

$$(\overline{s}\cdot\overline{x_k}\cdot\overline{m_1}){\downarrow_{A,B}}\in\sigma.$$

Either

$$\text{Case 1: } (\overline{m_1})^*_{A,B,C}\in M^+_{A^\perp} \quad\text{or}\quad \text{Case 2: } (\overline{m_1})^*_{A,B,C}\in M^+_B.$$

In Case 1, we then have $(\overline{s}\cdot\overline{x_k}\cdot\overline{m_1}){\downarrow_{B,C}}=\overline{s}{\downarrow_{B,C}}\in\tau$, so $\overline{s}\cdot\overline{x_k}\cdot\overline{m_1}\in S$. We remark that in this case, $\overrightarrow{b_k}$ is empty, and $\overline{x_k}$ and $\overline{m_1}$ are active in the same subgame.

In Case 2, since $(\overline{s}\cdot\overline{x_k}){\downarrow_{B,C}}\in\tau$ ends with a P-move in $B^\perp\otimes C$ and $\overline{m_1}{\downarrow_{B,C}}$ is an O-move in $B^\perp\otimes C$, we have $(\overline{s}\cdot\overline{x_k}\cdot\overline{m_1}){\downarrow_{B,C}}\in\tau$. Since τ is winning, there exists a unique P-move l_2 in $B^\perp\otimes C$ such that $(\overline{s}\cdot\overline{x_k}\cdot\overline{m_1}){\downarrow_{B,C}}\cdot l_2\in\tau$. Again, we can globalise l_2 to a move $\overline{m_2}$ in $A\otimes B\otimes C$ such that

$$(\overline{s}\cdot\overline{x_k}\cdot\overline{m_1}\cdot\overline{m_2}){\downarrow_{B,C}}\in\tau.$$

Since the games are finite, we can continue this process and eventually obtain either a move $\overline{u}$ or a move $\overline{v}$ and a sequence $\overrightarrow{b_k}$ such that either

$$(\overline{s}\cdot\overline{x_k}\cdot\overrightarrow{b_k}\cdot\overline{u}){\downarrow_{A,B}}\in\sigma \quad\text{or}\quad (\overline{s}\cdot\overline{x_k}\cdot\overrightarrow{b_k}\cdot\overline{v}){\downarrow_{B,C}}\in\tau,$$

where $\overline{u}^*_{A,B,C}$ is a P-move in $A^\perp$ or $\overline{v}^*_{A,B,C}$ is an P-move in C. In the former case, $\overrightarrow{b_k}$ will be a sequence of moves, all active in B, its last move restricting to an O-move in B, and we put $\overline{y_k}=\overline{u}$. In the latter case, the last move of $\overrightarrow{b_k}$ restricts to an O-move in $B^\perp$, and we put $\overline{y_k} = \overline{v}$. This completes the proof of *(ii)*. ■

We can now prove that **Con** is closed under composition of morphisms.

Lemma 4.2.3 *If $\sigma : A \to B$ is a winning strategy for $A^\perp \otimes B$ and $\tau : B \to C$ is a winning strategy for $B^\perp \otimes C$, then $\sigma;\tau$ is a winning strategy for $A^\perp \otimes C$, i.e. $\sigma;\tau : A \to C$.*

Proof We first prove (s1). Suppose that $s \in \sigma;\tau$. Then there exists $\overline{s} \in S$ such that $\overline{s}\downarrow_{A,C}= s$. By Proposition 4.2.1, we know that the first move in $\overline{s}$ restricts to an O-move in $A^\perp \otimes C$. Therefore s starts with an O-move in $A^\perp \otimes C$.

To prove (s2) and (s3), observe that (s1) starts off an induction process, since it is essentially *(i)* of Proposition 4.2.2 when $k = 1$, whence we obtain *(ii)* for $k = 1$. Then Proposition 4.2.2 can be applied inductively on k, statement *(i)* thereby proving (s3) and statement *(ii)* proving a strong version of (s2). (We take $\overline{y_k}\downarrow_{A,C}$ as the unique P-move in response to the O-move $\overline{x_k}\downarrow_{A,C}$.)

It remains to prove that *all* sequences in S do indeed restrict to legal plays in $A^\perp \otimes C$ (as opposed to just valid plays). Proposition 4.2.2 *(ii)* shows us that an O-move in $A^\perp \otimes C$ will always be followed by a uniquely determined P-move in $A^\perp \otimes C$, so we need only check that a P-move in $A^\perp \otimes C$ cannot be followed by another P-move.

Suppose that $\overline{s} \in S$ has the form given in equation (4.1) and that $\overline{s} \cdot \overline{m} \in S$. Since $\overline{s}\downarrow_{A,B}$ ends with a P-move in $A^\perp \otimes B$, $(\overline{s} \cdot \overline{m})\downarrow_{A,B}\in \sigma$ implies that $\overline{m}\downarrow_{A,B}$ is an O-move in $A^\perp \otimes B$. Similarly, since $\overline{s}\downarrow_{B,C}$ ends with a P-move in $B^\perp \otimes C$, $(\overline{s} \cdot \overline{m})\downarrow_{B,C}\in \tau$ implies that $\overline{m}\downarrow_{B,C}$ is an O-move in $B^\perp \otimes C$. It follows that $\overline{m}$ cannot be active in B, for it cannot restrict to both an O-move in B and an O-move in $B^\perp$. Therefore, $\overline{m}$ restricts to either an O-move in $A^\perp$ or an O-move in C. So all sequences in S restrict to legal plays in $A^\perp \otimes C$.

Therefore, $\sigma;\tau$ is a strategy for $A^\perp \otimes C$, and since there is a unique response for every legal O-move, $\sigma;\tau$ is in fact a winning strategy. ■

Theorem 4.2.4 **Con** *is a category.*

Proof We define the identity morphism $id_A : A \to A$ by

$$id_A = \{s = s_1 \cdot \ldots \cdot s_k \in L_{A^\perp \otimes A} \mid \lambda_{A^\perp \otimes A}(s_1) = O \text{ and for } i \text{ even},$$
$$(s_i)^*_{A^\perp,A} = (s_{i-1})^*_{A^\perp,A} \text{ but } [s_i]_{A^\perp,A} \neq [s_{i-1}]_{A^\perp,A}\}.$$

In other words, P's response to an O-move in one subgame is to play the same move as a P-move in the other subgame. This is known as the *copy-cat strategy*.

It remains to prove the associativity of composition of morphisms. Suppose that

$\sigma : A \to B, \tau : B \to C$, and $\nu : C \to D$. Write

$$(\sigma;\tau);\nu = \{t{\downarrow_{A,D}} \mid t \in \mathcal{L}(A,C,D), t{\downarrow_{C,D}} \in \nu, \text{ and there exists } \bar{t} \in \mathcal{L}(A,B,C) \\ \text{such that } \bar{t}{\downarrow_{A,C}} = t{\downarrow_{A,C}}, \bar{t}{\downarrow_{A,B}} \in \sigma, \text{ and } \bar{t}{\downarrow_{B,C}} \in \tau\},$$

and define

$$T = \{s{\downarrow_{A,D}} \mid s \in \mathcal{L}(A,B,C,D), s{\downarrow_{A,B}} \in \sigma, s{\downarrow_{B,C}} \in \tau, \text{ and } s{\downarrow_{C,D}} \in \nu\}.$$

We prove here that $T = (\sigma;\tau);\nu$. A symmetrical argument shows that $T = \sigma;(\tau;\nu)$.

Clearly $T \subseteq (\sigma;\tau);\nu$. Given $s \in \mathcal{L}(A,B,C,D)$ satisfying the conditions of T, the sequences $t = s{\downarrow_{A,C,D}}$ and $\bar{t} = s{\downarrow_{A,B,C}}$ satisfy the conditions of $(\sigma;\tau);\nu$, with $t{\downarrow_{A,D}} = s{\downarrow_{A,D}}$.

Conversely, suppose that $t \in \mathcal{L}(A,C,D)$ and $\bar{t} \in \mathcal{L}(A,B,C)$ satisfy the conditions of $(\sigma;\tau);\nu$. Write $t = t_1 \cdot \ldots \cdot t_n$, and write (a,c,d) for the source position of t_1. By Proposition 4.2.1, t_1 restricts to an O-move in $A^\perp \otimes D$, so $t_1^* = (t_1)^*_{A,C,D}$ is either an O-move in $A^\perp$ or an O-move in D.

Suppose that t_1^* is an O-move in $A^\perp$. Then t_1 restricts to an O-move in $A^\perp \otimes C$, so the first term $\overline{t_1}$ in the sequence $\bar{t}$ satisfies $t_1{\downarrow_{A,C}} = \overline{t_1}{\downarrow_{A,C}}$. Then $\overline{t_1}$ has source position (a,b,c) for some position b in B. Since $\bar{t} \in S$, Proposition 4.2.2 shows us that $\bar{t}$ will provide us with as many terms following $\overline{t_1}$ as $\overrightarrow{b_1}$ and $\overline{y_1}$, where $\overrightarrow{b_1}$ is a (possibly empty) sequence of moves, active in B, and $\overline{y_1}{\downarrow_{A,C}}$ is a P-move in $A^\perp \otimes C$. If $\overline{y_1}$ does exists, then we must have $\overline{y_1}{\downarrow_{A,C}} = \overline{t_2}{\downarrow_{A,C}}$. (The proof of Proposition 4.2.2 shows that it is impossible for t_2 to be active in D.) Put

$$s_1 = \begin{cases} (t_1 \cdot \overrightarrow{b_1} \cdot \overline{y_1}){\uparrow^{A,B,C,D}_{(a,b,c,d)}} & \text{if } \overline{y_1} \text{ exists} \\ t_1{\uparrow^{A,B,C,D}_{(a,b,c,d)}} & \text{otherwise} \end{cases}$$

where we are assuming that $(t_1 \cdot \overrightarrow{b_1} \cdot \overline{y_1}){\uparrow^{A,B,C,D}_{(a,b,c,d)}}$ is a valid play starting from source position (a,b,c,d), constructed from suitable globalisations of t_1, $\overrightarrow{b_1}$ and $\overline{y_1}$ to moves in $A \otimes B \otimes C \otimes D$, etc. Then by construction, s_1 satisfies the conditions of T.

On the other hand, suppose that t_1^* is an O-move in D. If t_2 exists, then $t_2{\downarrow_{C,D}}$ is the unique response to $t_1{\downarrow_{C,D}}$ by ν. Put

$$s_1 = \begin{cases} (t_1 \cdot t_2){\uparrow^{A,B,C,D}_{(a,b,c,d)}} & \text{if } t_2 \text{ exists} \\ t_1{\uparrow^{A,B,C,D}_{(a,b,c,d)}} & \text{otherwise.} \end{cases}$$

Then by construction, s_1 satisfies the conditions of T.

If t_2 did exist, then follow on with t_3, if it exists. Note that either $t_3{\downarrow_{A,C}}$ is an O-move in $A^\perp \otimes C$ or $t_3{\downarrow_{C,D}}$ is an O-move in $C^\perp \otimes D$. In the former case, we deal with t_3 in exactly the same way as t_1 when t_1^* was an O-move in $A^\perp$. In the latter case, we deal with

t_3 in exactly the same way as t_1 when t_1^* was an O-move in D. We thus obtain a sequence s_3 such that $s_1 \cdot s_3$ satisfies the conditions of T.

Continue this process until we have exhausted the length of the sequence t, and put $s = s_1 \cdot s_3 \cdot \ldots \cdot s_{2j-1}$ where $2j - 1 = n$ if n is odd, $2j - 1 = n - 1$ if n is even. Then s satisfies the conditions of T, with $s{\downarrow_{A,D}} = t{\downarrow_{A,D}}$. ■

Lemma 4.2.5 **Con** *is a compact closed category.*

Proof Suppose that $\sigma : A \to B$ and $\tau : C \to D$ are morphisms in **Con**. Then we have a strategy on $A^\perp \otimes C^\perp \otimes B \otimes D$ specified by

$$\widehat{\sigma \otimes \tau}(s{\cdot}m) = n \Leftrightarrow \begin{cases} n{\downarrow_{A^\perp,B}} = \hat{\sigma}((s \cdot m){\downarrow_{A^\perp,B}}) & \text{if } m \text{ restricts to an } O\text{-move in } A^\perp \otimes B \\ n{\downarrow_{C^\perp,D}} = \hat{\tau}((s \cdot m){\downarrow_{C^\perp,D}}) & \text{if } m \text{ restricts to an } O\text{-move in } C^\perp \otimes D \end{cases}$$

Since σ and τ are both winning strategies, we will have a unique response to all legal O-moves, so $\sigma \otimes \tau$ is winning. It follows from this result that $- \otimes -$ is a bifunctor from **Con** $\times$ **Con** to **Con**.

Also, $\sigma : A \to B$ naturally defines a dual morphism $\sigma^\perp : B^\perp \to A^\perp$, since we have the isomorphism $A^\perp \otimes B \cong B \otimes A^\perp$. We are doing no more than interchanging the placement of the games $A^\perp$ and B and $\sigma^\perp$ is just the corresponding representation of σ. This defines a contravariant functor over **Con** sending A to $A^\perp$ and σ to $\sigma^\perp$. It is easy to check that $(id_A)^\perp = id_{A^\perp}$ and that $(\sigma;\tau)^\perp = \tau^\perp;\sigma^\perp$.

It is clear that $(A \otimes B)^\perp$ and $A^\perp \otimes B^\perp$ are naturally isomorphic, because we can create strategies for $(A \otimes B) \otimes (A^\perp \otimes B^\perp)$ and $(A^\perp \otimes B^\perp)^\perp \otimes (A \otimes B)^\perp$ which are essentially copy-cat strategies. Therefore, a morphism $A \otimes B \to C$ is a winning strategy for $(A \otimes B)^\perp \otimes C \cong A^\perp \otimes B^\perp \otimes C$ which naturally induces a morphism $A \to B^\perp \otimes C$, and vice versa. Consequently, **Con**$(A \otimes B, C)$ and **Con**$(A, B^\perp \otimes C)$ are naturally isomorphic and **Con** is a compact closed category. ■

4.2.1 History-free strategies

Definition 4.2.6 Let $\overline{A} = A_1 \otimes \cdots \otimes A_p$ be a multiple game and write $\mathbf{A} = A_1, \ldots, A_p$. A strategy σ for the game $\overline{A}$ is *history-free with respect to the decomposition* $A_1, \ldots, A_p$ if there exists a partial function $f : \bigcup_{i=1}^p M_{A_i}^- \times \{i\} \to \bigcup_{i=1}^p M_{A_i}^+ \times \{i\}$ such that

$$\hat{\sigma}(s \cdot a) = m \implies f(a^*_{\mathbf{A}}, [a]_{\mathbf{A}}) = (m^*_{\mathbf{A}}, [m]_{\mathbf{A}}).$$

It is clear that, in this case, there is a least such partial function, which we denote by $\hat{\sigma}_{\mathbf{A}}$.

This definition is more specific than in [AJ94] or [HO93], principally because our notation for positions and moves is more specific. In a multiple game, a strategy may provide distinct responses to the same O-move in the same subgame, depending on when the move was played.

For example, let A be the game given by the following data.

$$\begin{array}{rcl} P_A & = & \{0, a\}; \\ M_A & = & \{0 \xrightarrow{P} a\}. \end{array}$$

Now consider the strategy σ for the multiple game $\overline{A} = A \otimes A \otimes A^\perp \otimes A^\perp$ which gives rise to the following strategic response function,

$$\begin{array}{llcl} \hat{\sigma}: & (0,0,0,0) \xrightarrow{O} (0,0,0,a) & \mapsto & (0,0,0,a) \xrightarrow{P} (a,0,0,a) \\ & (0,0,0,0) \xrightarrow{O} (0,0,0,a) \xrightarrow{P} (a,0,0,a) \xrightarrow{O} (a,0,a,a) & \mapsto & (a,0,a,a) \xrightarrow{P} (a,a,a,a) \\ & (0,0,0,0) \xrightarrow{O} (0,0,a,0) & \mapsto & (0,0,a,0) \xrightarrow{P} (a,0,a,0) \\ & (0,0,0,0) \xrightarrow{O} (0,0,a,0) \xrightarrow{P} (a,0,a,0) \xrightarrow{O} (a,0,a,a) & \mapsto & (a,0,a,a) \xrightarrow{P} (a,a,a,a) \end{array}$$

Then σ is *not* history-free with respect to the decomposition $A, A, A^\perp, A^\perp$ because, for example, the response to the move $0 \xrightarrow{O} a$ in the fourth game is either $0 \xrightarrow{P} a$ in the first game or $0 \xrightarrow{P} a$ in the second game, depending on whether $0 \xrightarrow{O} a$ was O's first or second move.

However, σ *is* history-free when we consider the moves as moves in $\overline{A}$. The move $(0,0,0,0) \xrightarrow{O} (0,0,0,a)$ is an entirely different move to $(a,0,a,0) \xrightarrow{O} (a,0,a,a)$, so we hold no restriction on how P responds to these moves.

In fact, a strategy for a game $\overline{A}$ will always be history-free with respect to $\overline{A}$, assuming all labels for positions and moves are distinct. Since $T_{\overline{A}}$ is a tree, there is a unique path to any O-move, hence a unique "history of events". Consequently, there are no alternative histories for a strategic response to depend on.

We now define a new category $\mathbf{Con}_{hf}$. An object $\dot{A}$ of $\mathbf{Con}_{hf}$ is a pair $(A; \underline{A})$ where A is a Conway game and $\underline{A}$ is a decomposition of A. Given objects $\dot{A}$ and $\dot{B}$ in $\mathbf{Con}_{hf}$, a morphism $\sigma : \dot{A} \to \dot{B}$ is a winning strategy for the game $A^\perp \otimes B$ which is history-free with respect to the decomposition $\underline{A}^\perp, \underline{B}$. (Given $\underline{A} = A_1, \ldots, A_p$, we write $\underline{A}^\perp$ for the list $A_1^\perp, \ldots, A_p^\perp$.)

Lemma 4.2.7 $\mathbf{Con}_{hf}$ *is a category with a compact closed structure which is preserved by the forgetful functor* $J : \mathbf{Con}_{hf} \to \mathbf{Con}$.

Proof To show that $\mathbf{Con}_{hf}$ is a category, it suffices to check that $\mathbf{Con}_{hf}$ is closed under composition of morphism, and that identity morphisms in $\mathbf{Con}$ lift to identity morphisms in $\mathbf{Con}_{hf}$.

Suppose that $\sigma : \dot{A} \to \dot{B}$ and $\tau : \dot{B} \to \dot{C}$ are morphisms in $\mathbf{Con}_{hf}$. We already know that $J\sigma; J\tau$ is a winning strategy for $A^\perp \otimes C$ in $\mathbf{Con}$. It remains to check that it is history-free with respect to the decomposition $\underline{A}^\perp, \underline{C}$.

By definition, σ is history-free at least with respect to the decomposition $\{\underline{A}^\perp, B\}$ and τ is history-free at least with respect to the decomposition $\{B^\perp, \underline{C}\}$. Write $X = \{\underline{A}^\perp, B\} = \{A_1^\perp, \ldots, A_p^\perp, B\}$, $Y = \{B^\perp, \underline{C}\} = \{B^\perp, C_1, \ldots, C_q\}$ and $Z = \{\underline{A}^\perp, \underline{C}\} = \{A_1^\perp, \ldots, A_p^\perp, C_1, \ldots, C_q\}$.

Suppose that $\overline{x}$ is an O-move in $A^\perp \otimes C$ and w.l.o.g. assume that $x = \overline{x}_Z^*$ is an O-move in $A_i^\perp$. Then $\hat{\sigma}_X$ sends x in $A_i^\perp$ to a P-move y (if it exists) in either some $A_j^\perp$ or B. In the former case, we define $h(x, i) = (y, j)$. In the latter case, we regard y as an O-move in $B^\perp$ and apply $\hat{\tau}_Y$. This sends y in $B^\perp$ to a P-move w (if it exists) in either some C_k or $B^\perp$. In the former case, we define $h(x, i) = (w, p+k)$, and in the latter case, we regard w as an O-move in B and apply $\hat{\sigma}_X$ again etc. Thus it is clear how to define a partial map from O-moves in games in Z to P-moves in games in Z. From Proposition 4.2.2 *(ii)*, we can see that h is the partial function $(\widehat{\sigma;\tau})_Z$ which ensures that $\sigma;\tau$ is history-free with respect to the decomposition $\underline{A}^\perp, \underline{C}$. Therefore $J\sigma; J\tau : A \to C$ lifts to a morphism $\sigma;\tau : \dot{A} \to \dot{C}$ in $\mathbf{Con}_{hf}$.

Furthermore, if A is a game with decomposition $\underline{A} = A_1, \ldots, A_p$, then $id_A = id_{A_1} \otimes \cdots \otimes id_{A_p}$ in $\mathbf{Con}$, which is history-free with respect to the decomposition $\underline{A}^\perp, \underline{A}$. Therefore we have identity morphisms $id_{\dot{A}} : \dot{A} \to \dot{A}$ for all objects in $\mathbf{Con}_{hf}$.

Therefore $\mathbf{Con}_{hf}$ is a category. We now prove that $\mathbf{Con}_{hf}$ has a compact closed structure.

Suppose that $\sigma : \dot{A} \to \dot{B}$ and $\nu : \dot{C} \to \dot{D}$. By Lemma 4.2.5, $J\sigma \otimes J\nu : A \otimes C \to B \otimes D$ in $\mathbf{Con}$ is a winning strategy, and by choice of σ and ν, it is history-free with respect to the decomposition $\underline{A}^\perp, \underline{C}^\perp, \underline{B}, \underline{D}$. Therefore $J\sigma \otimes J\nu$ lifts to a morphism in $\mathbf{Con}_{hf}$, denoted $\sigma \otimes \nu : \dot{A} \otimes \dot{C} \to \dot{B} \otimes \dot{D}$ when we define

$$\dot{A} \otimes \dot{B} = (A; \underline{A}) \otimes (B; \underline{B}) = (A \otimes B; \underline{A}, \underline{B})$$

for all objects $\dot{A}, \dot{B}$ in $\mathbf{Con}_{hf}$. The unit for tensor in $\mathbf{Con}_{hf}$ is clearly $\dot{I} = (I; I)$.

Also, $J\sigma : A \to B$ is history-free with respect to $\underline{A}^\perp, \underline{B}$ if and only if $(J\sigma)^\perp : B^\perp \to A^\perp$ is history-free with respect to $\underline{B}, \underline{A}^\perp$. Thus, $(J\sigma)^\perp$ lifts to a morphism in $\mathbf{Con}_{hf}$, denoted $\sigma^\perp : \dot{B}^\perp \to \dot{A}^\perp$ when we define

$$\dot{A}^\perp = (A; \underline{A})^\perp = (A^\perp; \underline{A}^\perp)$$

for all objects $\dot{A}$ in $\mathbf{Con}_{hf}$.

Finally, a morphism $\dot{A} \otimes \dot{B} \to \dot{C}$ in $\mathbf{Con}_{hf}$ is a winning strategy, history-free with respect to the decomposition $\underline{A}^\perp, \underline{B}^\perp, \underline{C}$, so it naturally induces a morphism $\dot{A} \to \dot{B}^\perp \otimes \dot{C}$, and vice versa. Therefore $\mathbf{Con}_{hf}$ is a compact closed category whose structure is preserved by $J : \mathbf{Con}_{hf} \to \mathbf{Con}$. ■

4.3 Modelling MLL – Uniform strategies and dinaturality

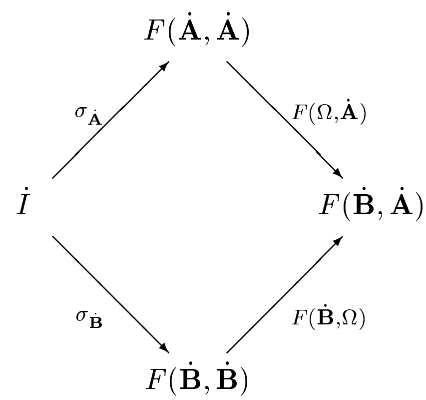
In Linear Logic, our ultimate interest is in the proofs of generic sequents in the logic, rather than proofs of one particular instance of a sequent. Thus, to model MLL in a category of games, we need to find a way to describe a notion of uniformity between winning strategies.

In this and the following section, we assume that $F(\dot{\mathbf{X}}, \dot{\mathbf{Y}}) = F(\dot{X}_1, \ldots, \dot{X}_n, \dot{Y}_1, \ldots, \dot{Y}_n)$ is a formula of length p built from literals $\dot{X}_1, \ldots, \dot{X}_n$, $\dot{Y}_1^{\perp}, \ldots, \dot{Y}_n^{\perp}$ by the connective $\otimes$. The formula F induces a multivariant functor $[\![F]\!] : (\mathbf{Con}_{hf})^n \times (\mathbf{Con}_{hf}^{op})^n \to \mathbf{Con}_{hf}$, which by abuse of notation we will also refer to as F. Furthermore, we assume that $F(\dot{\mathbf{A}}, \dot{\mathbf{A}})$ takes the form

$$F(\dot{\mathbf{A}}, \dot{\mathbf{A}}) = \dot{A}_{\xi_1}^{\zeta_1} \otimes \ldots \otimes \dot{A}_{\xi_p}^{\zeta_p}, \tag{4.2}$$

where each $\xi_i \in \{1, \ldots, n\}$, and each $\zeta_i \in \{1, \perp\}$.

Definitions 4.3.1 Let σ be a collection morphisms $\sigma_{\dot{\mathbf{A}}} : I \to F(\dot{\mathbf{A}}, \dot{\mathbf{A}})$ in $\mathbf{Con}_{hf}$. We say that σ is *uniform* or *uniformly winning (by dinaturality)* if σ is a dinatural transformation from the constant functor with value $\dot{I}$ to the multivariant functor $F : (\mathbf{Con}_{hf})^n \times (\mathbf{Con}_{hf}^{op})^n \to \mathbf{Con}_{hf}$. That is, for all winning strategies $\omega_i : \dot{A}_i \to \dot{B}_i$ in $\mathbf{Con}_{hf}$, the following diagram commutes,



where $\Omega = \omega_1, \ldots, \omega_n$.

Remarks 1. Later on, we will discuss another notion of uniformity, namely that by embeddings, which is why the word dinaturality is stressed above. But for now, the reader may safely assume the above definition is the sole definition of uniformity.

2. Write $A_i = J\dot{A}_i$, $i = 1, \ldots, n$. Note that each $\sigma_{\dot{\mathbf{A}}}$ is a winning strategy for the game $F(\mathbf{A}, \mathbf{A})$ which is history-free (at least) with respect to the decomposition

$$A_{\xi_1}^{\zeta_1}, \ldots, A_{\xi_p}^{\zeta_p}. \tag{4.3}$$

By abuse of notation, we will denote the strategic response partial function with respect to this decomposition as $\hat{\sigma}_{\dot{\mathbf{A}}}$.

3. It appears that there is no substantial advantage in regarding σ as a natural transformation from F^- to F^+, as discussed in §0.4. Naturality arguments are easy only in the case when the response to an O-move in a game in F^- is a P-move in a game in F^+. We therefore continue to work with dinatural transformations throughout this chapter.

4.4 Full Completeness in $\mathbf{Con}_{hf}$

This section provides a compact closed full completeness result for Conway games. It states that the only history-free uniformly winning strategies in $\mathbf{Con}_{hf}$ are copy-cat strategies.

The following results will be useful to us.

Lemma 4.4.1 *Suppose that σ is a uniform collection of morphisms $\sigma_{\dot{\mathbf{A}}} : \dot{I} \to F(\dot{\mathbf{A}}, \dot{\mathbf{A}})$ in $\mathbf{Con}_{hf}$. Then the formula F is balanced, i.e. each atom $\dot{A}_i$ occurs precisely the same number of times positively as it does negatively.*

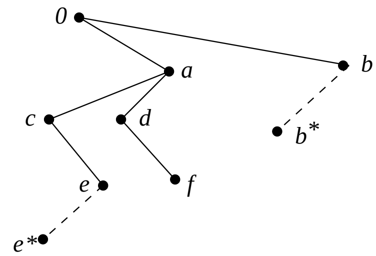
Proof Observe the winning strategies $J\sigma_{\dot{\mathbf{A}}} : I \to F(\mathbf{A}, \mathbf{A})$ in $\mathbf{Con}$. Fix $k \in \{1, \ldots, n\}$, and instantiate A_k at a game with exactly one O-move and no P-moves, and all others the trivial game. Since σ is winning, there must always be a response to any O-move. However, there can only be at most one move in any literal subgame, and therefore, there are at least as many games $A_k^\perp$ as there are games A_k. On the other hand, if we instantiated A_k at a game with exactly one P-move and no O-moves, then we can deduce that there are at least as many games A_k as there are $A_k^\perp$. Since this is true for all k, we have deduced that F is balanced. ■

Proposition 4.4.2 *Given a game B, there exists an extended game B^* (from B), for which there exists a winning strategy.*

Proof Consider the tree T_B. An attempt to construct a winning strategy for B is to attempt to form a collection of "zig-zag" paths in T_B, each starting with an initial O-move and ending with a terminal P-move. At any point where this fails, we can "graft" an extra P-move, so that P *does* have a response to every legal O-move. This creates the extended game B^*. See Figure 4.1. ■

Proposition 4.4.3 *Given a game B with decomposition $B_1, \ldots, B_t$, there exists an extended game B^* for which there exists a winning strategy, history-free with respect to the decomposition $B_1^*, \ldots, B_t^*$, where each B_i^* is a game extended from B_i.*

Proof By the previous proposition, each B_k can be extended to a game B_k^* for which there exists a winning strategy τ_k. Then $\tau_1 \otimes \cdots \otimes \tau_t$ will be a winning strategy for the



$$\begin{array}{rlcl} \hat{\sigma}_{B^*}: & 0 \stackrel{O}{\to} a & \mapsto & a \stackrel{P}{\to} c \\ & 0 \stackrel{O}{\to} b & \mapsto & b \stackrel{P}{\to} b^* \\ & c \stackrel{O}{\to} e & \mapsto & e \stackrel{P}{\to} e^* \end{array}$$

Figure 4.1: An extended game B^* with a winning strategy

game $B^* = B_1^* \otimes \cdots \otimes B_t^*$, history-free with respect to this decomposition. ■

We will write $\dot{B}^* = (B^*; B_1^*, \ldots, B_t^*)$ for the *decomposition preserving extension* of $\dot{B} = (B; B_1, \ldots, B_t)$, described in Proposition 4.4.3.

Lemma 4.4.4 *Suppose that σ is a uniform collection of morphisms $\sigma_{\dot{\mathbf{A}}} : \dot{I} \to F(\dot{\mathbf{A}}, \dot{\mathbf{A}})$ in $\mathbf{Con}_{hf}$ and that for each $i = 1, \ldots, n$, $\dot{B}_i^*$ is a decomposition preserving extension of $\dot{B}_i$. If m is an O-move in $B_{\xi_i}^{\zeta_i}$ and $\hat{\sigma}_{\dot{\mathbf{B}}}(m, i)$ exists, then $\hat{\sigma}_{\dot{\mathbf{B}}}(m, i) = \hat{\sigma}_{\dot{\mathbf{B}}^*}(m, i)$.*

Proof By construction, each id_{B_k} in $\mathbf{Con}$ lifts to a morphism $\dot{B}_k \to \dot{B}_k^*$ in $\mathbf{Con}_{hf}$. Then uniformity says that $\sigma_{\dot{\mathbf{B}}}; F(\mathbf{id_B}, \dot{\mathbf{B}}) = \sigma_{\dot{\mathbf{B}}^*}; F(\dot{\mathbf{B}}^*, \mathbf{id_B})$. In particular, for a move such as m which globalises to a move in both $F(\mathbf{B}, \mathbf{B})$ and $F(\mathbf{B}^*, \mathbf{B}^*)$, we have $\hat{\sigma}_{\dot{\mathbf{B}}}(m, i) = \hat{\sigma}_{\dot{\mathbf{B}}^*}(m, i)$. ■

Theorem 4.4.5 *Suppose that σ is a uniform collection of morphisms $\sigma_{\dot{\mathbf{A}}} : \dot{I} \to F(\dot{\mathbf{A}}, \dot{\mathbf{A}})$ in $\mathbf{Con}_{hf}$. Then σ is a copy-cat strategy, specified by a unique fixed-point free involution ϕ on $\{1, \ldots, p\}$ such that $\xi_{\phi(i)} = \xi_i$ and $\zeta_{\phi(i)} \neq \zeta_i$. That is, if O plays a move m in the game $A_{\xi_i}^{\zeta_i}$, then $\sigma_{\dot{\mathbf{A}}}$'s response is to play the same move m in the game $A_{\xi_{\phi(i)}}^{\zeta_{\phi(i)}}$.*

Proof Fix $i \in \{1, \ldots, p\}$ and put $K = \xi_i$. We will assume here that $\zeta_i = 1$, the case when $\zeta_i = \perp$ being similar.

Consider the game S_i:

$$\begin{array}{rcl} P_{S_i} & = & \{0, i\}; \\ M_{S_i} & = & \{0 \stackrel{O}{\to} i\}, \end{array}$$

and put

$$\dot{A}_k = \begin{cases} (S_i; S_i) & \text{if } k = \xi_i \\ \dot{I} & \text{otherwise} \end{cases}$$

for each $k = 1, \dots, n$. (Recall that I is the game in which neither player can make a move.) Then $A_{\xi_i}^{\zeta_i}$ is a game in which only O can make the single move $0 \to i$. Since $\sigma_{\dot{\mathbf{A}}}$ is a winning strategy, every O-move has a unique response. The only moves available are of the form $0 \to i$ (and Lemma 4.4.1 ensures that there is at least one such move), so we must have

$$\hat{\sigma}_{\dot{\mathbf{A}}}(0 \xrightarrow{O} i, i) = (0 \xrightarrow{P} i, j)$$

for some unique j satisfying $\xi_i = \xi_j$, $\zeta_i \neq \zeta_j$. Define $\phi(i) = j$.

Our aim is to prove the following :

For any objects $\dot{B}_1, \dots, \dot{B}_n$, if O plays m in $B_{\xi_i}^{\zeta_i}$, then $\sigma_{\dot{\mathbf{B}}}$'s response is to play the same move m in the game $B_{\xi_{\phi(i)}}^{\zeta_{\phi(i)}}$.

We will first prove the result for an *initial* O-move m, and will later prove that all O-moves may be treated as initial O-moves.

In order to exploit uniformity, we need to construct morphisms $\dot{A}_k \to \dot{B}_k$ in $\mathbf{Con}_{hf}$. In particular, we need winning strategies $A_k \to B_k$ in $\mathbf{Con}$, but such morphisms may not always exist. In such cases, we need to extend the $\dot{B}_k$ and instead prove the result for the games $\dot{B}_k^*$. Lemma 4.4.4 shows that we will immediately obtain the result for the original $\dot{B}_k$.

For each $k \neq K$, $\dot{A}_k = \dot{I}$, so there exists a decomposition preserving extension $\dot{B}_k^*$ of $\dot{B}_k$ for which there exists a morphism $\omega_k : \dot{A}_k \to \dot{B}_k^*$ in $\mathbf{Con}_{hf}$, by Proposition 4.4.3.

Now, we construct a morphism $\omega_K : \dot{A}_K \to \dot{B}_K^*$ for an extension $\dot{B}_K^*$ of $\dot{B}_K$, which sends the initial O-move m in B_K^* to the P-move $0 \xrightarrow{P} i$ in $A_K^\perp$.

Suppose that $\dot{B}_K = (B_K; B_{K1}, \dots, B_{Kt})$. For simplicity, assume that m is active in B_{K1} and write $m^* = m^*_{B_{K1},\dots,B_{Kt}} = 0 \to a$. By Proposition 4.4.2, there exist extended games $B_{K2}^*, \dots, B_{Kt}^*$ for which there exist winning strategies $\tau_2, \dots, \tau_t$ respectively. Now we attempt to create a winning strategy τ_1 for $A_K^\perp \otimes B_{K1}$, history-free with respect to $\{A_K^\perp, B_{K1}\}$, which sends m^* in B_1 to $0 \xrightarrow{P} i$ in $A_K^\perp$. We need to ensure that the subtree of $T_{B_{K1}}$ with root a represents a game with a winning strategy. If this fails, graft on extra P-moves as necessary, thus obtaining an extended game B_{K1}^*.

Therefore, $\omega_K = \tau_1 \otimes \cdots \otimes \tau_t$ lifts to a morphism $\dot{A}_K \to \dot{B}_K^*$ in $\mathbf{Con}_{hf}$ which sends m in B_K^* to $0 \xrightarrow{P} i$ in $A_K^\perp$, where

$$\dot{B}_K^* = (B_{K1}^* \otimes \cdots \otimes B_{Kt}^*; B_{K1}^*, \dots, B_{Kt}^*)$$

is a decomposition preserving extension of $\dot{B}_K$.

To prove that $\hat{\sigma}_{\dot{\mathbf{B}}^*}(m, i) = (m, \phi(i))$, we observe responses for the two strategies for $F(\dot{\mathbf{B}}^*, \dot{\mathbf{A}})$, namely $\sigma_{\dot{\mathbf{A}}}; F(\Omega, \dot{\mathbf{A}})$ and $\sigma_{\dot{\mathbf{B}}^*}; F(\dot{\mathbf{B}}^*, \Omega)$.

Suppose that O plays m in $B_{\xi_i}^*$. Then $\sigma_{\dot{\mathbf{A}}}; F(\Omega, \dot{\mathbf{A}})$'s response is traced as follows. The response to this move by ω_{ξ_i} is to play $0 \xrightarrow{P} i$ in the game $A_{\xi_i}^\perp$. This is equivalent to $0 \xrightarrow{O} i$

in A_{ξ_i}, and $\sigma_{\dot{\mathbf{A}}}$ responds to this move with $0 \xrightarrow{P} i$ in $A^{\perp}_{\xi_{\phi(i)}}$. This is $\sigma_{\dot{\mathbf{A}}}; F(\Omega, \dot{\mathbf{A}})$'s response to O's original move.

Now trace $\sigma_{\dot{\mathbf{B}}^*}; F(\dot{\mathbf{B}}^*, \Omega)$. By dinaturality, we know $(0 \xrightarrow{P} i, \phi(i))$ is also $\sigma_{\dot{\mathbf{B}}^*}; F(\dot{\mathbf{B}}^*, \Omega)$'s response to m in $B^*_{\xi_i}$. From here, we can deduce that $\sigma_{\dot{\mathbf{B}}^*}$'s response to m in $B^*_{\xi_i}$ is at least some P-move n in $(B^*_{\xi_{\phi(i)}})^{\perp}$. Moreover, this P-move is equivalent to an O-move in $B^*_{\xi_{\phi(i)}}$, and we know that $\omega^{\perp}_{\xi_{\phi(i)}} = \omega^{\perp}_{\xi_i}$ responds to this move with $0 \xrightarrow{P} i$ in $A^{\perp}_{\xi_{\phi(i)}}$. There is only one possibility, and that is $n = m$. See Figure 4.2 for a picture of the trace.

Therefore, $\hat{\sigma}_{\dot{\mathbf{B}}^*}(m, i) = (m, \phi(i))$, and by Lemma 4.4.4, we have $\hat{\sigma}_{\dot{\mathbf{B}}}(m, i) = (m, \phi(i))$.

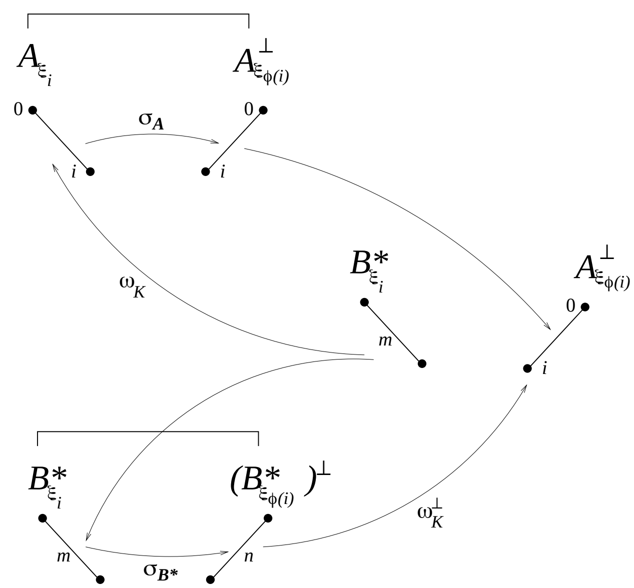


Figure 4.2: Tracing the response to the O-move m in $(B^*_{\xi_i})^{\zeta_i}$ when $\zeta_i = 1$

This completes the claim that σ copies an initial O-move in $B^{\zeta_i}_{\xi_i}$ to the initial P-move in $B^{\zeta_{\phi(i)}}_{\xi_{\phi(i)}}$.

(If instead we had assumed that $\zeta_i = \perp$, then we would have constructed S_i with one initial P-move, and then a morphism $\omega : \dot{A}_K \to \dot{B}^*_K$ which sends $0 \xrightarrow{O} i$ in $A^{\perp}_K$ to some initial P-move m in B^*_K.)

We have now established that σ copies initial O-moves in the ith subgame to initial P-moves in the $\phi(i)$th subgame, for $i = 1, \ldots, p$. The result for a general O-move follows from an inductive deletion of moves already played, so that we may regard the O-move of interest as an initial move.

Suppose that n is an O-move in $B^{\zeta_i}_{\xi_i}$ and that $\hat{\sigma}_{\dot{\mathbf{B}}}(n, i)$ exists. Suppose that $s = m_1, \ldots, m_q$ starts with an initial move in $B^{\zeta_i}_{\xi_i}$ such that $s \cdot n \in V_{B^{\zeta_i}_{\xi_i}}$. Put $K = \xi_i$ and w.l.o.g. assume that $\zeta_i = 1$. Since σ is history-free with respect to the decomposition (4.3), the value of $\hat{\sigma}_{\dot{\mathbf{B}}}(n, i)$ is not altered if we assume that all moves $m_1 : 0 \to a$ have been played in *all* games of the form B_K and $B^{\perp}_K$, either as an O-move, or as P's response. This assumption is no different to replacing B_K by the game whose tree is the subtree of T_{B_K} with root a. We can continue this procedure until we have a game with initial O-move n, and then we apply the proof above. This completes the proof that σ copies all legal O-moves.

We now show that ϕ is an involution. Fix i and consider the game T:

$$\begin{aligned} P_T &= \{0, a, b\}; \\ M_T &= \{0 \xrightarrow{O} a, a \xrightarrow{P} b\}. \end{aligned}$$

Put

$$\dot{C}_k = \begin{cases} (T^{\zeta_i}; T^{\zeta_i}) & \text{if } k = \xi_i \\ \dot{I} & \text{otherwise} \end{cases}$$

for each $k = 1, \ldots, n$. Then

$$\hat{\sigma}_{\dot{\mathbf{C}}}(0 \xrightarrow{O} a, i) = (0 \xrightarrow{P} a, \phi(i))$$

and

$$\hat{\sigma}_{\dot{\mathbf{C}}}(a \xrightarrow{O} b, \phi(i)) = (a \xrightarrow{P} b, \phi^2(i)).$$

If $0 \xrightarrow{O} a$ in $C^{\zeta_i}_{\xi_i}$, $0 \xrightarrow{P} a$ in $C^{\zeta_{\phi(i)}}_{\xi_{\phi(i)}}$, and $a \xrightarrow{O} b$ in $C^{\zeta_{\phi(i)}}_{\xi_{\phi(i)}}$ are the only moves that have been played, then the only place where P can play a move $a \xrightarrow{P} b$ is in the game $C^{\zeta_i}_{\xi_i}$. So we must have $\phi^2(i) = i$. This completes the proof. ■

Corollary 4.4.6 *(Full Completeness in* $\mathbf{Con}_{hf}$*) Every dinatural transformation* $\sigma : \mathfrak{K}_{\dot{I}} \to F$ *in* $\mathbf{Con}_{hf}$ *is induced by a unique morphism* $I \to F(\mathbf{X}, \mathbf{X})$ *in the free compact closed category on* n *objects* $X_1, \ldots, X_n$ *with trivial dimension.*

Again, it is possible to remove the restriction on the dimension of the objects in the free category. Since there is only one endomorphism on I in **Con**, and hence only one endomorphism on $\dot{I}$ in $\mathbf{Con}_{hf}$, $\mathbf{Con}_{hf}$ has trivial dimension. Therefore $\mathbf{Con}_{hf}$ interprets all dimensions as the unique trivial strategy on the trivial game $\dot{I}$. We thus have the weaker result,

Corollary 4.4.7 *(Full Completeness in* $\mathbf{Con}_{hf}$*) Every dinatural transformation* $\sigma : \mathfrak{K}_{\dot{I}} \to F$ *in* $\mathbf{Con}_{hf}$ *is induced by a morphism* $I \to F(\mathbf{X}, \mathbf{X})$ *in the free compact closed category on* n *objects* $X_1, \ldots, X_n$*.*

4.5 Double-glueing on $\mathbf{Con}_{hf}$

We now apply the glueing construction of Chapter 1 to the compact closed category $\mathbf{Con}_{hf}$. Since there is only one endomorphism on $\dot{I}$, the units for tensor and par will collapse to the same object, and consequently we obtain a model of MLL+Mix. We will not bother with the general construction again, and instead we simply present the structure of **GCon** in a form which is useful to us here.

Objects. The objects of $\mathbf{GCon}_{hf}$ are triples $\mathcal{A} = (|\mathcal{A}|, \mathcal{A}_s, \mathcal{A}_t)$ where $|\mathcal{A}| = \dot{A}$ is a Conway game A equipped with a decomposition $\underline{A}$, $\mathcal{A}_s \subseteq \mathbf{Con}_{hf}(\dot{I}, \dot{A})$ (i.e. a collection of winning strategies for A, history-free with respect to $\underline{A}$), and $\mathcal{A}_t \subseteq \mathbf{Con}_{hf}(\dot{I}, \dot{A}^{\perp})$ (i.e. a collection of winning strategies for $A^{\perp}$, history-free with respect to $\underline{A}^{\perp}$).

Morphisms. A morphism $\sigma : \dot{A} \to \dot{B}$ in $\mathbf{Con}_{hf}$ is a morphism $\sigma : \mathcal{A} \to \mathcal{B}$ in $\mathbf{GCon}_{hf}$ if it satisfies the following conditions:

- if $\tau \in \mathcal{A}_s$ then $\tau; \sigma \in \mathcal{B}_s$;
- if $\nu \in \mathcal{B}_t$ then $\nu; \sigma^{\perp} \in \mathcal{A}_t$.

Tensor. Given two objects $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{GCon}_{hf}$, define the tensor product $\mathcal{A} \otimes \mathcal{B}$ by

$$\begin{array}{lcl} |\mathcal{A} \otimes \mathcal{B}| & = & \dot{A} \otimes \dot{B}; \\ (\mathcal{A} \otimes \mathcal{B})_s & = & \{\sigma = \sigma_1 \otimes \sigma_2 : \dot{I} \to \dot{A} \otimes \dot{B} \mid \sigma{\downarrow_A} = \sigma_1 \in \mathcal{A}_s \text{ and } \sigma{\downarrow_B} = \sigma_2 \in \mathcal{B}_s\}; \\ (\mathcal{A} \otimes \mathcal{B})_t & = & \{\kappa : \dot{A} \to \dot{B}^{\perp} \mid \text{if } \tau \in \mathcal{A}_s \text{ then } \tau; \kappa \in \mathcal{B}_t \text{ and} \\ & & \qquad\qquad\qquad \text{if } \nu \in \mathcal{B}_s \text{ then } \nu; \kappa^{\perp} \in \mathcal{A}_t\}. \end{array}$$

Observe that $\sigma \in (\mathcal{A} \otimes \mathcal{B})_s$ is necessarily a strategy for $A \otimes B$ such that if O moves in A then P's strategic response must be a move in A, and similarly for B. To see this, suppose that $t_1 \cdot \ldots \cdot t_r \in \sigma$ and write $t_i^* = (t_i)^*_{A,B}$ for all i. W.l.o.g. assume that $t_1^* \in M_A^-$. Then t_2^*, if it exists, cannot be a P-move in B, for then $\sigma{\downarrow_B}$ would start with a P-move and would therefore not be a strategy for B. Thus $t_2^* \in M_A^+$. If t_3 exists, suppose first that $t_3^* \in M_A^-$. Then t_4^*, if it exists, cannot be a P-move in B for similar reasons. Thus $t_4^* \in M_A^+$. On the other hand, if we suppose that $t_3^* \in M_B^-$, then we cannot have $t_4^* \in M_A^+$ for then $\sigma{\downarrow_A}$ would not be a sequence of moves in A by alternating players. We can continue this argument inductively.

The unit for $\otimes$ is $\mathbf{1} = (\dot{I}, \{id_{\dot{I}}\}, \{id_{\dot{I}}\})$, where $id_{\dot{I}}$ is the trivial strategy for $\dot{I}$.

Linear negation. We have an involution $(-)^{\perp} : \mathbf{GCon}_{hf} \to \mathbf{GCon}_{hf}$, sending $\mathcal{A}$ to $\mathcal{A}^{\perp} = (\dot{A}^{\perp}, \mathcal{A}_t, \mathcal{A}_s)$ and σ to $\sigma^{\perp}$. Clearly, if $\sigma : \mathcal{A} \to \mathcal{B}$ then $\sigma^{\perp}$ will be a morphism $\mathcal{B}^{\perp} \to \mathcal{A}^{\perp}$.

Linear implication and Par. Define $\mathcal{B} \multimap \mathcal{C} = (\mathcal{B} \otimes \mathcal{C}^{\perp})^{\perp}$ and $\mathcal{A} \parr \mathcal{B} = (\mathcal{A}^{\perp} \otimes \mathcal{B}^{\perp})^{\perp}$. With this structure, $\mathbf{GCon}_{hf}$ is a $*$-autonomous category and is therefore a model of MLL. The unit for par is $\perp = \mathbf{1}^{\perp} = \mathbf{1}$, so the trivial strategy for $\dot{I}$ lifts to the unary Mix morphism $m : \bot \to \mathbf{1}$. Therefore $\mathbf{GCon}_{hf}$ is in fact a model of MLL+Mix.

We once again stress that a morphism $\mathbf{1} \to \mathcal{A}$ is essentially an element of $\mathcal{A}_s$, and dually, a morphism $\mathcal{A} \to \bot$ is essentially an element of $\mathcal{A}_t$. Thus, in the context of $\mathbf{GCon}_{hf}$, a morphism $\mathbf{1} \to \mathcal{A}$ is a winning strategy for A, history-free with respect to $\underline{A}$, and a morphism $\mathcal{A} \to \bot$ is a winning strategy for $A^\perp$, history-free with respect to $\underline{A}^\perp$.

4.6 Full Completeness in $\mathbf{GCon}_{hf}$

Suppose that $F(\underline{\mathcal{X}}, \underline{\mathcal{Y}}) = F(\mathcal{X}_1, \ldots, \mathcal{X}_n, \mathcal{Y}_1, \ldots, \mathcal{Y}_n)$ is now a formula of length p built from $X_1, \ldots, X_n, Y_1^\perp, \ldots, Y_n^\perp$ by the connectives $\otimes$ and $⅋$, such that $UF(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ has the form in (4.2) whenever $|\mathcal{A}_i| = \dot{A}_i$ for all i. Then F induces a multivariant functor $[\![F]\!] : (\mathbf{GCon}_{hf})^n \times (\mathbf{GCon}_{hf}^{op})^n \to \mathbf{GCon}_{hf}$ which by abuse of notation we will also refer to as F.

Lemma 4.6.1 *Suppose that ρ is a dinatural transformation in $\mathbf{GCon}_{hf}$ from the constant functor $\mathfrak{K}_\mathbf{1}$ to the multivariant functor F. Then there exists a unique fixed-point free involution ϕ on $\{1, \ldots, p\}$ such that $\xi_{\phi(i)} = \xi_i$ and $\zeta_{\phi(i)} \neq \zeta_i$ for all i, thus determining a unique set of axiom links for a proof structure of the formula F.*

Proof The collection ρ induces a uniform collection $U\rho$ of winning strategies for $UF(\mathbf{A}, \mathbf{A})$, history-free with respect to the decomposition (4.3) where $U : \mathbf{GCon}_{hf} \to \mathbf{Con}_{hf}$ is the forgetful functor. By Theorem 4.4.5, $U\rho$ is a copy-cat strategy specified by an involution ϕ. By Theorem 1.3.2, this characterises ρ over all objects in $\mathbf{GCon}_{hf}$. ■

Before proving full completeness, we make a useful observation.

Lemma 4.6.2 *For any object $\mathcal{A}$ such that $|\mathcal{A}| = \dot{A}$ is a non-trivial game, there is no morphism $\omega_\mathcal{A} : \mathbf{1} \to \mathcal{A} \otimes \mathcal{A}^\perp$ in $\mathbf{GCon}_{hf}$ such that $U\omega$ is the copy-cat strategy on $\dot{A} \otimes \dot{A}^\perp$.*

Proof Recall that $\omega_\mathcal{A}$ belongs to $(\mathcal{A} \otimes \mathcal{A}^\perp)_s$, i.e. it is a winning strategy for $A \otimes A^\perp$ such that if O moves in A then P is forced to move in A also. In other words, P cannot copy O's move in the game $A^\perp$. ■

Proposition 4.6.3 *Suppose that we have a formula*

$$\tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = \Gamma_{\underline{\mathcal{A}}} ⅋ (\mathcal{A}^{\zeta_{a_1}}_{\xi_{a_1}} \otimes \mathcal{A}^{\zeta_{b_1}}_{\xi_{b_1}}) ⅋ \cdots ⅋ (\mathcal{A}^{\zeta_{a_r}}_{\xi_{a_r}} \otimes \mathcal{A}^{\zeta_{b_r}}_{\xi_{b_r}}) \tag{4.4}$$

and suppose that we have the following morphisms in $\mathbf{GCon}_{hf}$*:*

$$\begin{array}{lll}\tau_i : (\mathcal{A}^{\zeta_{a_i}}_{\xi_{a_i}} \otimes \mathcal{A}^{\zeta_{b_i}}_{\xi_{b_i}}) \to \bot, & (U\tau_i : \dot{I} \to (\dot{A}^{\zeta_{a_i}}_{\xi_{a_i}})^\bot \otimes (\dot{A}^{\zeta_{b_i}}_{\xi_{b_i}})^\bot) & i = 1, \ldots, r-1; \\ \tau_j^+ : \mathcal{A}_j \to \bot, & (U\tau_j^+ : \dot{I} \to \dot{A}_j^\bot) & j = 1, \ldots, n; \\ \tau_j^- : \mathcal{A}_j^\bot \to \bot, & (U\tau_j^- : \dot{I} \to \dot{A}_j) & j = 1, \ldots, n;\end{array}$$

(with $|\mathcal{A}_j| = \dot{A}_j$ for all $j = 1, \ldots, n$.)
With this collection of data, we can construct a morphism $\tau_{\underline{\mathcal{A}}} : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \mathcal{A}^{\zeta_{a_r}}_{\xi_{a_r}} \otimes \mathcal{A}^{\zeta_{b_r}}_{\xi_{b_r}}$ such that $U\tau_{\underline{\mathcal{A}}}$ is a winning strategy for the game $U\tilde{F}(\mathbf{A}, \mathbf{A})^\bot \otimes A^{\zeta_{a_r}}_{\xi_{a_r}} \otimes A^{\zeta_{b_r}}_{\xi_{b_r}}$, where

- *$\tau_{\underline{\mathcal{A}}}$ plays according to $U\tau_i$ for moves in the games $(A^{\zeta_{a_i}}_{\xi_{a_i}})^\bot$ and $(A^{\zeta_{b_i}}_{\xi_{b_i}})^\bot$, $i = 1, \ldots, r-1$;*
- *$\tau_{\underline{\mathcal{A}}}$ plays a copy cat strategy on the fragment $(A^{\zeta_{a_r}}_{\xi_{a_r}})^\bot \otimes (A^{\zeta_{b_r}}_{\xi_{b_r}})^\bot \otimes A^{\zeta_{a_r}}_{\xi_{a_r}} \otimes A^{\zeta_{b_r}}_{\xi_{b_r}}$;*
- *$\tau_{\underline{\mathcal{A}}}$ plays according to $U\tau_j^+$ on each subgame in $U\Gamma^\bot_{\underline{\mathcal{A}}}$ of the form $A_j^\bot$;*
- *$\tau_{\underline{\mathcal{A}}}$ plays according to $U\tau_j^-$ on each game in $U\Gamma^\bot_{\underline{\mathcal{A}}}$ of the form A_j.*

Proof Write

$$G(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = \Gamma_{\underline{\mathcal{A}}} \parr (\mathcal{A}^{\zeta_{a_1}}_{\xi_{a_1}} \otimes \mathcal{A}^{\zeta_{b_1}}_{\xi_{b_1}}) \parr \cdots \parr (\mathcal{A}^{\zeta_{a_{r-1}}}_{\xi_{a_{r-1}}} \otimes \mathcal{A}^{\zeta_{b_{r-1}}}_{\xi_{b_{r-1}}})$$

so that $\tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) = G(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \parr (\mathcal{A}^{\zeta_{a_r}}_{\xi_{a_r}} \otimes \mathcal{A}^{\zeta_{b_r}}_{\xi_{b_r}})$. Since $\mathbf{GCon}_{hf}$ supports the Mix rule, we can form a suitable tensor product of morphisms $U\tau_j^+$ and $U\tau_j^-$ in $\mathbf{Con}_{hf}$ to create a morphism which lifts to a morphism $\Gamma_{\underline{\mathcal{A}}} \to \bot$ in $\mathbf{GCon}_{hf}$. Now tensor this morphism with $U\tau_1, \ldots, U\tau_{r-1}$ to create a morphism which lifts to $\tau' : G(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \bot$ in $\mathbf{GCon}_{hf}$. Finally, let $\tau = \tau' \parr (\mathcal{A}^{\zeta_{a_r}}_{\xi_{a_r}} \otimes \mathcal{A}^{\zeta_{b_r}}_{\xi_{b_r}}) : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \mathcal{A}^{\zeta_{a_r}}_{\xi_{a_r}} \otimes \mathcal{A}^{\zeta_{b_r}}_{\xi_{b_r}}$. ■

We are now ready to prove full completeness in $\mathbf{GCon}_{hf}$. Lemma 4.6.1 showed that we can associate to each dinatural transformation ρ a unique MLL proof structure. We now prove that for any DR-switching, the associated DR-graph is acyclic.

Theorem 4.6.4 *Suppose that $\rho : \mathfrak{K}_\mathbf{1} \to F$ is a dinatural transformation in $\mathbf{GCon}_{hf}$. Consider the proof structure for F associated with ρ. Then for any DR-switching, the associated DR-graph is acyclic.*

Proof By Lemma 4.6.1, $U\rho$ is a copy-cat strategy inducing a unique fixed-point free involution ϕ on $\{1, \ldots, p\}$, thus determining a unique MLL proof structure for the formula F.

Suppose that for a certain DR-switching, the associated DR-graph contains a cycle. Consider the shortest cycle, and express it as lower connected pairs $(a_1, b_1), \ldots, (a_r, b_r)$ such that $\phi(b_i) = a_{i+1}$ for all $i \in \mathbb{Z}_r$. Following the same argument as presented in Theorem 2.2.8, we may assume that we have simplified F to a formula of the form (4.4) and that we have obtained a new dinatural collection of morphisms $\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1} \to \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ in such a way that associated proof structure preserves the cycle.

We now prove that $\tilde{\rho}$ should not exist. Let $\mathcal{A} = (\dot{A}, \mathcal{A}_s, \mathcal{A}_t)$ be an object of $\mathbf{GCon}_{hf}$ such that $\dot{A}$ is not the trivial game $\dot{I}$, $\mathcal{A}_s \neq \varnothing \neq \mathcal{A}_t$, and such that there exists an isomorphism $\psi : \mathcal{A} \to \mathcal{A}^{\perp}$.

Put $\mathcal{A}_i = \mathcal{A}$ for all $i = 1, \dots, n$. Choose $\tau^+ : \mathcal{A} \to \perp$ and $\tau^- : \mathcal{A}^{\perp} \to \perp$. Put

$$\tau_i \quad = \quad \begin{cases} id_{\dot{A}} & \text{if } \zeta_{a_i} \neq \zeta_{b_i}; \\ \psi & \text{if } \zeta_{a_i} = \zeta_{b_i} = 1; \\ \psi^{-1} & \text{if } \zeta_{a_i} = \zeta_{b_i} = \perp \end{cases}$$

for each $i = 1, \dots, r-1$ (where we identify $\psi : \mathcal{A} \to \mathcal{A}^{\perp}$ as a morphism $\mathcal{A} \otimes \mathcal{A} \to \perp$ etc.) Then by Proposition 4.6.3 we can construct a morphism $\tau_{\underline{\mathcal{A}}} : \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}}) \to \mathcal{A}^{\zeta_{a_r}} \times \mathcal{A}^{\zeta_{b_r}}$. Now compose $\tau_{\underline{\mathcal{A}}}$ with $\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1} \to \tilde{F}(\underline{\mathcal{A}}, \underline{\mathcal{A}})$ to form the morphism $\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1} \to \mathcal{A}^{\zeta_{a_r}} \times \mathcal{A}^{\zeta_{b_r}}$. Note that $U\tilde{\rho}_{\underline{\mathcal{A}}} = f_1 \otimes f_2$ where

$$f_1 : \dot{I} \to \Gamma_{\dot{\mathbf{A}}},$$

(where f_1 lifts to a morphism $\mathbf{1} \to \Gamma_{\underline{\mathcal{A}}}$), and

$$f_2 : \dot{I} \to (\dot{A}^{\zeta_{a_1}}_{\xi_{a_1}} \otimes \dot{A}^{\zeta_{b_1}}_{\xi_{b_1}}) \otimes \cdots \otimes (\dot{A}^{\zeta_{a_1}}_{\xi_{a_r}} \otimes \dot{A}^{\zeta_{b_r}}_{\xi_{b_r}}).$$

Meanwhile, $U\tau_{\underline{\mathcal{A}}} = g_1 \otimes g_2$ where

$$g_1 : \Gamma_{\dot{\mathbf{A}}} \to \dot{I},$$

and

$$g_2 : (\dot{A}^{\zeta_{a_1}}_{\xi_{a_1}} \otimes \dot{A}^{\zeta_{b_1}}_{\xi_{b_1}}) \otimes \cdots \otimes (\dot{A}^{\zeta_{a_1}}_{\xi_{a_r}} \otimes \dot{A}^{\zeta_{b_r}}_{\xi_{b_r}}) \to (\dot{A}^{\zeta_{a_1}}_{\xi_{a_r}} \otimes \dot{A}^{\zeta_{b_r}}_{\xi_{b_r}}).$$

So $U(\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}}) = U\tau_{\underline{\mathcal{A}}}U\tilde{\rho}_{\underline{\mathcal{A}}} = (g_1 f_1) \otimes (g_2 f_2)$. We always have $g_1 f_1 = id_{\dot{I}}$ since there is only one morphism on $\dot{I}$, so $U(\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}}) \cong g_2 f_2$.

Examining the cycle a little closer, we see that there are an even number, $2k$ say, of occurrences of i for which $\zeta_{a_i} = \zeta_{b_i}$, k of which are $\zeta_{a_i} = \zeta_{b_i} = 1$ and k of which are $\zeta_{a_i} = \zeta_{b_i} = \perp$.

If $\zeta_{a_r} \neq \zeta_{b_r}$, then τ was built from the same number of ψ and ψ^{-1}, and therefore we can compose g_2 and f_2 and deduce that $g_2 f_2$ is the copy-cat strategy on $\dot{A} \otimes \dot{A}^{\perp}$. So U maps $\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1} \to \mathcal{A} \otimes \mathcal{A}^{\perp}$ to the copy-cat strategy on $\dot{A} \otimes \dot{A}^{\perp}$, which by Lemma 4.6.2 is a contradiction.

If $\zeta_{a_r} = \zeta_{b_r} = 1$, then τ was built from $k-1$ occurrences of ψ and k occurrences of ψ^{-1}. Therefore we can deduce that $g_2 f_2$ is the strategy $U\psi^{-1}$. Therefore U sends $\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1} \to \mathcal{A} \otimes \mathcal{A}$ to the strategy $U\psi^{-1}$. Now, U sends the composite

$$\mathbf{1} \xrightarrow{\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\psi \otimes \mathcal{A}} \mathcal{A}^{\perp} \otimes \mathcal{A}$$

to the copy-cat strategy on $\dot{A}^{\perp} \otimes \dot{A}$, which by Lemma 4.6.2 is a contradiction.

If $\zeta_{a_r} = \zeta_{b_r} = \bot$, then τ was built from k occurrences of ψ and $k-1$ occurrences of ψ^{-1}. Therefore we can deduce that $g_2 f_2$ is the strategy $U\psi$. Therefore U sends $\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}} : \mathbf{1} \to \mathcal{A}^\bot \otimes \mathcal{A}^\bot$ to the strategy $U\psi$. Now, U sends the composite

$$\mathbf{1} \xrightarrow{\tau_{\underline{\mathcal{A}}}\tilde{\rho}_{\underline{\mathcal{A}}}} \mathcal{A}^\bot \otimes \mathcal{A}^\bot \xrightarrow{\psi^{-1}\otimes\mathcal{A}^\bot} \mathcal{A} \otimes \mathcal{A}^\bot$$

to the copy-cat strategy on $\dot{A} \otimes \dot{A}^\bot$, which by Lemma 4.6.2 is a contradiction.

So in all three cases, we derived from $\tilde{\rho}_{\underline{\mathcal{A}}}$ a morphism in $\mathbf{GCon}_{hf}$ which cannot exist. Therefore, $\tilde{\rho}$ cannot exist, and the original DR-graph must have been acyclic. ■

Corollary 4.6.5 *(Full Completeness in* $\mathbf{GCon}_{hf}$*) Every dinatural transformation* $\mathfrak{K}_\mathbf{1} \to F$ *in* $\mathbf{GCon}_{hf}$ *is the denotation of a unique proof in MLL+Mix of the formula* F*, and is therefore induced by a unique morphism* $I \to F(\mathbf{X}, \mathbf{X})$ *in the free* $*$*-autonomous category supporting the Mix rule, on* n *objects* $X_1, \ldots, X_n$*.*

4.7 Dinaturality versus embeddings

This section is to discuss the alternative notion of uniformity, used in the work of previous authors. While this notion will prove to be equivalent in $\mathbf{Con}_{hf}$, there is apparently no extendable notion for $\mathbf{GCon}_{hf}$.

Definition 4.7.1 Let A and B be Conway games. We say that a function $e : P_A \to P_B$ is an *embedding of* A *to* B, written $e : A \hookrightarrow B$, if and only if

- $e : P_A \to P_B$ is injective;
- given a move $a_1 \to a_2$ in M_A, there exists a move $e(a_1) \to e(a_2)$ in M_B. By abuse of notation, denote the induced injection from M_A to M_B also by e;
- $\lambda_B \circ e = \lambda_A$;
- $\tilde{e}(V_A) \subseteq V_B$;
- $\tilde{e}(L_A) \subseteq L_B$,

where $\tilde{e}$ is the obvious extension of e to finite sequences of moves in A.

Lemma 4.7.2 *There is evidently a category* $\mathbf{Con}_e$ *of games and embeddings. Tensor and linear negation extend to covariant functors over* $\mathbf{Con}_e$*.*

It is worth noticing the difference in nature between winning strategies and embeddings. A winning strategy $A \to B$ will invert itself to give another winning strategy $B^\bot \to A^\bot$, whereas an embedding $A \hookrightarrow B$ preserves the direction of the arrow and yields $A^{\perp} \hookrightarrow B^{\perp}$. Thus, when discussing uniformity in terms of embeddings, it is no longer necessary to observe multivariance.

Let $F(X_1, \ldots, X_n) = F(\mathbf{X})$ be a formula built from literals $X_1, \ldots, X_n$, $X_1^{\perp}, \ldots, X_n^{\perp}$ by the connective $\otimes$. Write $F(\mathbf{X}) = X_{\xi_1}^{\zeta_1} \otimes \cdots \otimes X_{\xi_p}^{\zeta_p}$ and suppose that σ is a collection of winning strategies $\sigma_{\mathbf{A}}$ for $F(\mathbf{A})$, history free with respect to the decomposition $A_{\xi_1}^{\zeta_1}, \ldots, A_{\xi_p}^{\zeta_p}$. Then σ induces a collection of partial functions

$$\hat{\sigma}_{\mathbf{A}} : \mathcal{M}_{\mathbf{A}}^{-} \to \mathcal{M}_{\mathbf{A}}^{+}$$

where $\mathcal{M}^{\pm}$ are functors from $(\mathbf{Con}_e)^n$ to **Set** sending

$$\begin{array}{rcl} \mathbf{A} & \text{to} & \bigcup_{i=1}^{p} M^{\pm}_{A_{\xi_i}^{\zeta_i}} \times \{i\}; \\ \mathbf{e} : \mathbf{A} \to \mathbf{B} & \text{to} & \bigcup_{i=1}^{p} M^{\pm}_{e_{\xi_i}^{\zeta_i}} \times \{i\}. \end{array}$$

We say that σ is *uniform by embedding* if and only if $\{\hat{\sigma}_{\mathbf{A}}\}$ is a natural transformation, i.e. for all embeddings $\mathbf{e} : \mathbf{A} \hookrightarrow \mathbf{B}$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}_{\mathbf{A}}^{-} & \xrightarrow{\hat{\sigma}_{\mathbf{A}}} & \mathcal{M}_{\mathbf{A}}^{+} \\ \downarrow{\scriptstyle \mathcal{M}_{\mathbf{e}}^{-}} & & \downarrow{\scriptstyle \mathcal{M}_{\mathbf{e}}^{+}} \\ \mathcal{M}_{\mathbf{B}}^{-} & \xrightarrow{\hat{\sigma}_{\mathbf{B}}} & \mathcal{M}_{\mathbf{B}}^{+} \end{array}$$

The following theorem gives us the same result as Theorem 4.4.5. The proof is noticeably simpler.

Theorem 4.7.3 *Let $F(\mathbf{A}) = A_{\xi_1}^{\zeta_1} \otimes \cdots \otimes A_{\xi_p}^{\zeta_p}$, and suppose that σ is a collection of winning strategies for $F(\mathbf{A})$, history-free with respect to the decomposition $A_{\xi_1}^{\zeta_1}, \ldots, A_{\xi_p}^{\zeta_p}$, and uniform by embedding. Then σ is a copy-cat strategy, specified by a unique fixed-point free involution ϕ on $\{1, \ldots, p\}$ such that $\xi_{\phi(i)} = \xi_i$ and $\zeta_{\phi(i)} \neq \zeta_i$. That is, if O plays a move m in the game $A_{\xi_i}^{\zeta_i}$, then $\sigma_{\mathbf{A}}$'s response is to play the same move m in the game $A_{\xi_{\phi(i)}}^{\zeta_{\phi(i)}}$.*

Proof The proof that F is balanced is identical to that in Lemma 4.4.1.

For each $i = 1, \ldots, m$ consider the game S_i:

$$\begin{array}{rcl} P_{S_i} & = & \{0, i\}; \\ M_{S_i} & = & \{0 \to i\}; \\ \lambda_{S_i}(0 \to i) & = & \left\{ \begin{array}{ll} O & \text{if } \zeta_i = 1 \\ P & \text{if } \zeta_i = \perp; \end{array} \right. \end{array}$$

Then $S_i^{\zeta_i}$ is a game in which only O can make the single move $0 \to i$. Put

$$A_k = \left\{ \begin{array}{ll} S_i & \text{if } k = \xi_i \\ I & \text{otherwise} \end{array} \right.$$

for each $k = 1, \ldots, n$. (Recall that I is the game in which neither player can make a move.) Since $\sigma_{\mathbf{A}}$ is a winning strategy, every O-move has a unique response. The only moves available are of the form $0 \to i$ (and Lemma 4.4.1 ensures that there is at least one such move), so we must have

$$\hat{\sigma}_{\mathbf{A}}(0 \stackrel{O}{\to} i, i) = (0 \stackrel{P}{\to} i, j)$$

for some unique j satisfying $\xi_i = \xi_j$, $\zeta_i \neq \zeta_j$. Define $\phi(i) = j$.

> **Claim.** For any games $B_1, \ldots, B_n$, if O plays $m = m_1 \stackrel{O}{\to} m_2$ in $B_{\xi_i}^{\zeta_i}$, then $\sigma_{\mathbf{B}}$'s response is to play the same move m in the game $B_{\xi_{\phi(i)}}^{\zeta_{\phi(i)}}$.
>
> Consider the embedding
>
> $$\begin{array}{rccc} e_{\xi_i} : & S_i^{\zeta_i} & \hookrightarrow & B_{\xi_i}^{\zeta_i} \\ & 0 \stackrel{O}{\to} i & \mapsto & m_1 \stackrel{O}{\to} m_2 \end{array}$$
>
> and let $e_k : I \hookrightarrow B_k$ be any (trivial) embedding, for all $k \neq \xi_i$. Then we have
>
> $$\begin{array}{rcl} \hat{\sigma}_{\mathbf{B}} e_{\xi_i}(0 \stackrel{O}{\to} i, i) & = & e_{\xi_j} \hat{\sigma}_{\mathbf{A}}(0 \stackrel{O}{\to} i, i) \\ \Longrightarrow \hat{\sigma}_{\mathbf{B}}(m, i) & = & (m, \phi(i)). \end{array}$$

We now show that ϕ is an involution. Consider the game T:

$$\begin{array}{rcl} P_T & = & \{0, a, b\}; \\ M_T & = & \{0 \stackrel{O}{\to} a, a \stackrel{P}{\to} b\}. \end{array}$$

Put

$$C_k = \begin{cases} T^{\zeta_i} & \text{if } k = \xi_i \\ I & \text{otherwise} \end{cases}$$

for each $k = 1, \ldots, n$. Then

$$\hat{\sigma}_{\mathbf{C}}(0 \stackrel{O}{\to} a, i) = (0 \stackrel{P}{\to} a, \phi(i))$$

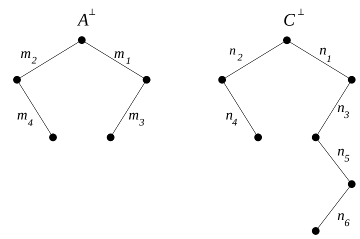
and

$$\hat{\sigma}_{\mathbf{C}}(a \stackrel{O}{\to} b, \phi(i)) = (a \stackrel{P}{\to} b, \phi^2(i)).$$

If $0 \stackrel{O}{\to} a$ in $C_{\xi_i}^{\zeta_i}$, $0 \stackrel{P}{\to} a$ in $C_{\xi_{\phi(i)}}^{\zeta_{\phi(i)}}$, and $a \stackrel{O}{\to} b$ in $C_{\xi_{\phi(i)}}^{\zeta_{\phi(i)}}$ are the only moves that have been played, then the only way P can play a move $a \stackrel{P}{\to} b$ is in the game $C_{\xi_i}^{\zeta_i}$. So we must have $\phi^2(i) = i$. ∎

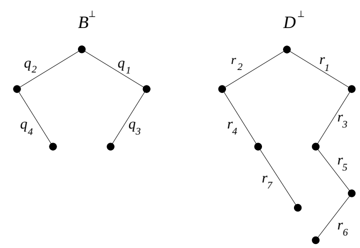
4.7.1 Extending embeddings to $\mathbf{GCon}_{hf}$

It is reasonable to expect an embedding $e : \mathcal{A} \hookrightarrow \mathcal{B}$ in $\mathbf{GCon}_{hf}$ to be some sort of embedding $e : |\mathcal{A}| \hookrightarrow |\mathcal{B}|$ in $\mathbf{Con}_{hf}$ which also embeds $\mathcal{A}_s$ into $\mathcal{B}_s$ and $\mathcal{A}_t$ into $\mathcal{B}_t$. However, there seems to be no easy way to embed tensored strategies. Let $|\mathcal{A}|$ and $|\mathcal{C}|$ be two games, their duals pictured below. Let each of $\mathcal{A}_s, \mathcal{A}_t, \mathcal{C}_s, \mathcal{C}_t$ contain its one and only possible strategy. Then we have a strategy σ for $|\mathcal{A}|^\perp \otimes |\mathcal{C}|^\perp$ which belongs to $(\mathcal{A} \otimes \mathcal{C})_t$.



$$\hat{\sigma}_{|\mathcal{A}|^\perp, |\mathcal{C}|} : \begin{cases} (n_1, 2) & \mapsto & (m_2, 1) \\ (m_4, 1) & \mapsto & (n_3, 2) \\ (n_5, 2) & \mapsto & (n_6, 2) \end{cases} \quad \begin{cases} (m_1, 1) & \mapsto & (n_2, 2) \\ (n_4, 2) & \mapsto & (m_3, 1) \end{cases}$$

However, while $\mathcal{A}$ embeds into $\mathcal{B}$ and $\mathcal{C}$ embeds into $\mathcal{D}$, as shown below, σ will not embed itself into a strategy in $(\mathcal{B} \otimes \mathcal{D})_t$, for we will be without a response if O plays r_7.



Thus it appears that embeddings, while useful to us in certain contexts, do not provide us with a general notion of uniformity. Though conceptually more difficult to grasp, dinatural transformations currently remain the most apparent and general way to describe uniformity.

Chapter 5

A Chu construction on vector spaces

The purpose of this chapter is to present a full completeness result for a category of topological vector spaces. While this work was motivated by the work of Blute and Scott [BS96], our approach to proving full completeness (and indeed our perception of the category concerned) is different. Blute and Scott presented a $*$-autonomous category **RTVec** of vector spaces equipped with a topology and proved a full completeness result with respect to MLL+Mix. We will show that this category is equivalent to a Chu construction on vector spaces (taking all "exact" pairings) and we will establish full completeness by exploiting the full completeness result proved in **FDVec** in Chapter 3.

5.1 The model

Denote by $\mathbf{Vec} = \mathbf{Vec}_k$ the category whose objects are vector spaces over a (fixed) field k and whose morphisms are linear maps between objects. As in Chapter 3, we will assume that k has *characteristic zero*. Then **Vec** is a symmetric monoidal closed category, with the usual algebraic tensor product $\otimes$, and the adjunction $- \otimes V \vdash V \multimap -$, where $V \multimap W = \mathbf{Vec}(V, W)$. For finite dimensional vector spaces V and W, we have the isomorphism $V \multimap W \cong V^* \otimes W$, so the full subcategory **FDVec** of finite dimensional vector spaces is compact closed.

5.1.1 The category RTVec

Definition 5.1.1 Let V be a vector space over a field k, and τ a topology defined on V. We say that V_τ (or (V, τ)) is a *topological vector space* if τ is Hausdorff, and vector addition and scalar multiplication are thus continuous when k is topologized discretely.

Definition 5.1.2 Let V be a vector space over k. We say that a topology τ defined on V is *linear* if there exists a basis $\mathcal{N}_\tau$ of neighbourhoods of $0 \in V$ consisting of linear subspaces of V, and neighbourhoods of $v \in V$ are obtained by taking translates $v + U$ of neighbourhoods U of 0. Thus, a linear topology can be specified completely by the collection $\mathcal{N}_\tau$. We say that V_τ is a *linearly topologized vector space* if V_τ is a topological vector space and τ is a linear topology.

Let $\mathbf{TVec} = \mathbf{TVec}_k$ denote the category whose objects are linearly topologized vector spaces over a fixed field k, and whose morphisms are linear continuous maps. This category is autonomous when equipped with the following structure.

Tensor. Let U_μ and V_τ be objects in $\mathbf{TVec}$. We wish to define $U_\mu \otimes V_\tau$. We make the natural choice that the underlying vector space should be the usual algebraic tensor product $U \otimes V$, and it is therefore a question of how to define appropriately a topology $\mu \otimes \tau$ in such a way that, given a bilinear bicontinuous morphism $b : U \times V \to W$, (W is another object with topology ν), the induced morphism $\overline{b} : U \otimes V \to W$ is linear continuous.

$$\begin{array}{ccc} U \times V & \xrightarrow{\quad b \quad} & W \\ \Big\downarrow & \overline{b} \nearrow & \\ U \otimes V & & \end{array}$$

Observe that given $Z \in \mathcal{N}_\nu$, for all $u_0 \in U$ and $v_0 \in V$, we have

$$\{v \in V : b(u_0, v) \in Z\} \in \mathcal{N}_\tau \text{ and } \{u \in U : b(u, v_0) \in Z\} \in \mathcal{N}_\mu.$$

From this, we deduce that we should define

$$\begin{aligned} \mathcal{N}_{\mu\otimes\tau} \stackrel{df}{=} \{A \mid\ & A \text{ is a subspace of } U \otimes V, \text{ and } \forall\ u_0 \in U, v_0 \in V, \\ & \{v \in V \mid u_0 \otimes v \in A\} \in \mathcal{N}_\tau, \text{ and } \{u \in U \mid u \otimes v_0 \in A\} \in \mathcal{N}_\mu\}. \end{aligned}$$

For now $\overline{b}^{-1}(Z) \in \mathcal{N}_{\mu\otimes\tau}$, since it is a subspace of $U \otimes V$, and for all $u_0 \in U$ and $v_0 \in V$, we have

$$\{v \in V \mid u_0 \otimes v \in \overline{b}^{-1}(Z)\} = \{v \in V \mid b(u_0, v) \in Z\} \in \mathcal{N}_\tau$$

and

$$\{u \in U \mid u \otimes v_0 \in \overline{b}^{-1}(Z)\} = \{u \in U \mid b(u, v_0) \in Z\} \in \mathcal{N}_\mu.$$

Furthermore, if V_τ is the *unit* for the tensor product, then the definition of $\mathcal{N}_{\mu\otimes\tau}$ shows that we must have $V = k$ and $\mathcal{N}_\tau = \{\{0\}, k\}$. This is the only linear topology on k we will consider, so we henceforth write k without its topology.

Linear implication. Recall that in $\mathbf{Vec}$, we have the natural equivalence

$$\frac{U \otimes V \xrightarrow{f} W}{U \xrightarrow{\tilde{f}} V \multimap W}$$

where $\tilde{f}(u)(v) = f(u, v)$. We wish to define $\tau \multimap \nu$ on $V \multimap W$ so that we have a corresponding equivalence in $\mathbf{TVec}$.

Suppose that $f : U \otimes V \to W$ is linear continuous. Then given $Z \in \mathcal{N}_\nu$, for all $u_0 \in U$ and $v_0 \in V$, we have

$$\{v \in V \mid f(u_0, v) \in Z\} \in \mathcal{N}_\tau \tag{5.1}$$

and

$$\{u \in U \mid f(u, v_0) \in Z\} \in \mathcal{N}_\mu. \tag{5.2}$$

Equation (5.1) says that $\tilde{f}(u) : V \to W$ is continuous, while equation (5.2) says that $\tilde{f}^{-1}([v_0, Z]) \in \mathcal{N}_\mu$ where $[v, Z] = \{g \in V \multimap W \mid g(v) \in Z\}$. From this, we deduce that we should define

$$\mathcal{N}_{\tau\multimap\nu} \stackrel{df}{=} \langle [F, Z] \mid F \text{ is a finite dimensional subspace of } V \text{ and } Z \in \mathcal{N}_\nu \rangle$$

where $[F, Z] = \{g \in V \multimap W \mid g(F) \subseteq Z\}$, for now we have

$$\begin{aligned} \tilde{f}^{-1}([F, Z]) &= \{u \in U \mid \tilde{f}(u) \in [F, Z]\} \\ &= \textstyle\bigcap_{v \in F} \{u \in U \mid \tilde{f}(u) \in [v, Z]\} \\ &= \textstyle\bigcap_{i=1}^n \tilde{f}^{-1}([v_i, Z]) \in \mathcal{N}_\mu, \end{aligned}$$

whenever F is finite dimensional and $\{v_1, \ldots, v_n\}$ is a basis for F.

Lemma 5.1.3 **TVec** *is a symmetric monoidal closed category with tensor product* $\otimes$*, unit* k*, and the adjunction* $- \otimes V \vdash V \multimap -$.

Linear negation. Given a linear map $\theta : V \to k$, write $ker(\theta) = \{v \in V \mid \theta(v) = 0\}$ for the *kernel* of θ. We now define the *dual of V with respect to τ* to be $(V_\tau)^\perp = V_\tau \multimap k$, i.e. $(V_\tau)^\perp = (V_\tau^\perp, \tau^\perp)$, where

$$\begin{aligned} V_\tau^\perp &= \{\theta : V \to k \mid \theta^{-1}(Z) \in \mathcal{N}_\tau \ \forall\ Z \in \mathcal{N}_k\} \\ &= \{\theta : V \to k \mid ker(\theta) \in \mathcal{N}_\tau\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_{\tau^\perp} &= \left\langle \{\theta \in V_\tau^\perp \mid \theta(F) \subseteq Z\} \,\middle|\, F \leq V \text{ is finite dim'l}\,, Z \in \mathcal{N}_k \right\rangle \\ &= \{\{\theta : V \to k \mid F \subseteq ker(\theta) \in \mathcal{N}_\tau\} \mid \text{finite dim'l } F \leq V\}\,. \end{aligned}$$

Continuing this theme, we get $(V_\tau)^{\perp\perp} = (V_\tau \multimap k) \multimap k$, i.e. $(V_\tau)^{\perp\perp} = (V_\tau^{\perp\perp}, \tau^{\perp\perp})$, where

$$V_\tau^{\perp\perp} = \{\xi : V_\tau^\perp \to k \mid ker(\xi) \in \mathcal{N}_{\tau^\perp}\}$$

and

$$\mathcal{N}_{\tau^{\perp\perp}} = \left\{ \{\xi : V_\tau^\perp \to k \mid G \subseteq ker(\xi) \in \mathcal{N}_{\tau^\perp}\} \,\middle|\, \text{finite dim'l } G \leq V_\tau^\perp \right\}.$$

An object $V_\mathcal{T}$ is *reflexive* if there is an isomorphism $V_\mathcal{T} \cong (V_\mathcal{T})^{\perp\perp}$. For such an object, a suitable collection of subspaces would be

$$\mathcal{N}_\mathcal{T} = \left\{ \{v \in V \mid G \subseteq ker(\xi(v))\} \,\middle|\, \text{finite dim'l } G \leq V_\mathcal{T}^{\perp} \right\},$$

where $\xi(v) : V_\mathcal{T}^{\perp} \to k$, $\xi(v)(\theta) = \theta(v)$. In other words, V is equipped with a neighbourhood basis of subspaces with *finite codimension.*

This yields us the $*$-autonomous category **RTVec**, whose objects are reflexive objects in **TVec** and whose morphisms are linear continuous maps. The fact that **RTVec** is $*$-autonomous is non-trivial. This category is an example of a technique in [Bar79] which converts a "pre-$*$-autonomous situation" to a $*$-autonomous one, by making a restriction to reflexive objects satisfying completeness properties with respect to a given class of "dense embeddings". In the case of **TVec**/**RTVec**, the completeness is with regard to finite dimensional vector spaces, which all objects satisfy, and we will in fact use something of this property later, to prove full completeness.

5.1.2 The category Chu

We recall the Chu construction on vector spaces, presented in an Appendix [Chu79] to [Bar79]. Objects of $\mathbf{Chu} = \mathbf{Chu}_k$ are triples $A = (U, X, \alpha)$ where U and X are objects in $\mathbf{Vec}_k$, and $\alpha : U \otimes X \to k$ is a linear map. In the sequel, we will mostly use the underlying bilinear map $U \times X \to k$, which by abuse of notation we will also call α.

Given two objects, $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$ in $\mathbf{Chu}_k$, morphisms $(f, g) : A \to B$ are pairs of linear maps $f : U \to V$ and $g : Y \to X$ such that the following diagram commutes.

$$\begin{array}{ccc} U \otimes Y & \xrightarrow{f \otimes Y} & V \otimes Y \\ {\scriptstyle U \otimes g}\downarrow & & \downarrow{\scriptstyle \beta} \\ U \otimes X & \xrightarrow[\alpha]{} & k \end{array}$$

i.e. for all $u \in U$ and $y \in Y$, we have $\alpha(u, g(y)) = \beta(f(u), y)$.

Linear implication. Given two objects $B = (V, Y, \beta)$ and $C = (W, Z, \gamma)$, define the object $B \multimap C$ by

$$B \multimap C = (\mathbf{Chu}(B, C), V \otimes Z, \delta),$$

where $\delta((g, h), v \otimes z) = \beta(v, h(z)) = \gamma(g(v), z)$. Note that $\mathbf{Chu}(B, C)$ naturally forms a vector space over k with componentwise addition and scalar multiplication. Specifically,

$\mathbf{Chu}(B,C)$ is the object such that the following diagram is a pullback in $\mathbf{Vec}$,

$$\begin{array}{ccc} \mathbf{Chu}(B,C) & \xrightarrow{p_1} & \mathbf{Vec}(V,W) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle w_1} \\ \mathbf{Vec}(Z,Y) & \xrightarrow[w_2]{} & \mathbf{Vec}(V\otimes Z,k) \end{array}$$

where w_1 sends $f: V \to W$ to the map

$$V \otimes Z \xrightarrow{f\otimes Z} W \otimes Z \xrightarrow{\gamma} k,$$

and w_2 sends $g: Z \to Y$ to the map

$$V \otimes Z \xrightarrow{V\otimes g} V \otimes Y \xrightarrow{\beta} k.$$

In cases such as these, it is clear how to define the associated bilinear map, and we will often omit the bilinear map when specifying an object in $\mathbf{Chu}$ if it is already clear. In fact, we will eventually show that given an object (U,X,α), with α non-degenerate, the vector space X can be viewed as the dual of U with respect to some linear topology, and therefore the bilinear map α can always be viewed as an evaluation map.

With this definition, we obtain a bifunctor $- \multimap - : \mathbf{Chu}^{op} \times \mathbf{Chu} \to \mathbf{Chu}$. This was proved in [Chu79].

Linear negation. Given an object $A = (U,X,\alpha)$, the *dual object* is defined to be $A^\perp = (X,U,\alpha \circ c_{XU})$, where c is the symmetry isomorphism. Clearly this induces an involution $(-)^\perp : \mathbf{Chu}^{op} \to \mathbf{Chu}$.

Tensor and par. With the notions of linear implication and negation, we can now induce a $*$-autonomous structure in $\mathbf{Chu}$. It is not difficult to see that

$$\mathbf{Chu}(A,B) \cong \mathbf{Chu}(B^\perp, A^\perp),$$

from which we can deduce that

$$A \multimap B \cong B^\perp \multimap A^\perp,$$

for all objects A, B in $\mathbf{Chu}$. We also have the following proposition.

Proposition 5.1.4 *(Chu) For all objects A, B, C in* $\mathbf{Chu}$*,*

$$\mathbf{Chu}(A, B \multimap C^\perp) \cong \mathbf{Chu}(C, B \multimap A^\perp).$$

From this, we have the isomorphism $\mathbf{Chu}(A, B \multimap C) \cong \mathbf{Chu}((A \multimap B^{\perp})^{\perp}, C)$. We therefore have the isomorphism $\mathbf{Chu}(A, B \multimap C) \cong \mathbf{Chu}(A \otimes B, C)$, when we define the object $A \otimes B = (A \multimap B^{\perp})^{\perp}$, i.e.

$$A \otimes B = (U \otimes V, \mathbf{Chu}(A, B^{\perp})).$$

Proposition 5.1.5 *(Chu) The map* $- \otimes -$ *is a bifunctor from* **Chu** $\times$ **Chu** *to* **Chu**, *and is a tensor product for* **Chu** *with unit* $\mathbf{1} = (k, k, \cdot)$.

Corollary 5.1.6 *(Chu) The category* **Chu** *is a* $*$*-autonomous category.*

We can also define the par product $A \parr B = A^{\perp} \multimap B \cong (A^{\perp} \otimes B^{\perp})^{\perp}$, thus defining a bifunctor dual to the tensor product in the sense of de Morgan.

Observe that when $A = (U, U^*)$ and $B = (V, V^*)$, we have

$$A \multimap B = (\mathbf{Vec}(V, W), V \otimes W^*) \text{ and } A \parr B = (\mathbf{Vec}(V^*, W), V^* \otimes W^*)$$

so it makes sense to write $\mathbf{Vec}(V, W) = V \multimap W$ and $\mathbf{Vec}(V^*, W) = V \parr W$.

5.1.3 The category ExChu

Let $\mathbf{ExChu} = \mathbf{ExChu}_k$ be the full subcategory of $\mathbf{Chu}_k$, restricted to objects (U, X, α) with α a non-degenerate (bi)linear map (so-called *exact pairings*), i.e. $\alpha(u, x) = 0$ for all x implies $u = 0$ and $\alpha(u, x) = 0$ for all u implies $x = 0$.

Lemma 5.1.7 *The category* **ExChu** *is a* $*$*-autonomous subcategory of* **Chu**.

Proof See [Bar91]. ∎

We will prove that **ExChu** is $*$-autonomously equivalent to **RTVec**. Suppose that (U, X, α) is an object in **ExChu**. Put

$$\mathcal{N}_{\mu} = \langle \{u \in U \mid \alpha(u \otimes x_i) = 0 \ \forall\, i = 1, \ldots, n\} \mid x_i \in X, n \in \mathbb{N} \rangle \tag{5.3}$$

and

$$\mathcal{N}_{\tau} = \langle \{x \in X \mid \alpha(u_i \otimes x) = 0 \ \forall\, i = 1, \ldots, n\} \mid u_i \in U, n \in \mathbb{N} \rangle. \tag{5.4}$$

These are collections of finite-codimensional subspaces which form linear topologies for U and X, respectively, so U_{μ} and X_{τ} are objects in **RTVec**.

We will show that there is an isomorphism $\Gamma : X_{\tau} \to (U_{\mu})^{\perp}$. Given $x \in X$, define $\Gamma(x) \in U_{\mu}^{\perp}$ by $\Gamma(x)(u) = \alpha(u \otimes x)$. Then Γ is injective because α is non-degenerate.

Furthermore, Γ is surjective. Suppose that $\theta \in U_\mu^\perp$. Then $ker(\theta) \in \mathcal{N}_\mu$, i.e. there exists linearly independent $x_1, \ldots, x_n \in X$ such that

$$\{u \in U \mid \theta(u) = 0\} = \{u \in U \mid \alpha(u \otimes x_i) = 0 \ \forall\ i = 1, \ldots, n\}.$$

It can then be shown that $\theta(u) = \sum_{i=1}^n \lambda_i \cdot \alpha(u \otimes x_i)$ for all u, for some scalars $\lambda_1, \ldots, \lambda_n$, and so $\Gamma(\sum_{i=1}^n \lambda_i x_i) = \theta$. (See [Köt69], §10, 4.(3).)

Finally, Γ is linear continuous. Suppose that $Z \in \mathcal{N}_{\mu^\perp}$. Then there exists a finite dimensional subspace F of U such that

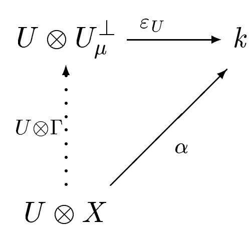
$$Z = \{\theta \in U_\mu^\perp \mid F \subseteq ker(\theta)\}.$$

Then

$$\begin{aligned} \Gamma^{-1}(Z) &= \{x \in X \mid \Gamma(x) \in Z\} \\ &= \{x \in X \mid F \subseteq ker(\Gamma(x))\} \\ &= \{x \in X \mid \Gamma(x)(u) = 0 \ \forall\ u \in F\} \\ &= \{x \in X \mid \alpha(u \otimes x) = 0 \ \forall\ u \in F\} \in \mathcal{N}_\tau, \end{aligned}$$

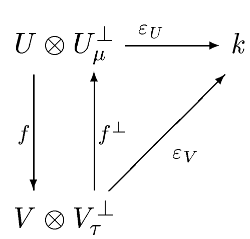
since F is finite dimensional.

Therefore, Γ is an isomorphism, and moreover, the following diagram commutes.



where $\varepsilon_U : U \otimes U_\mu^\perp \to k$ is generated by the evaluation map. So an object in **ExChu** is in fact specifying a vector space and its dual, with respect to some reflexive linear topology, i.e. an object in **RTVec**. Conversely, an object V_τ in **RTVec** gives rise to an object $(V, V_\tau^\perp, \varepsilon_V)$ in **ExChu**.

Now suppose that (U, X) and (V, Y) are objects in **ExChu**, and between them is a morphism $(f : U \to V, g : Y \to X)$. As before, we can equip U and V with linear topologies μ and τ, respectively, specified by equations (5.3) and (5.4). We naturally send $g : Y \to X$ to $\Gamma(g) : V_\tau^\perp \to U_\mu^\perp$, via $\Gamma(g)(\theta)(u) = \theta(f(u))$. This is precisely the dual map of f, $f^\perp : V_\tau^\perp \to U_\mu^\perp$. Thus the following diagram commutes.



We prove that f is continuous. Suppose that $Z \in \mathcal{N}_\tau$. W.l.o.g., assume that Z is a subspace of V with codimension 1. Then Z is the kernel of some continuous linear map $\theta : V \to k$. So

$$\begin{aligned} f^{-1}(Z) &= \{u \in U \mid \theta(f(u)) = 0\} \\ &= \{u \in U \mid \varepsilon_V(f(u) \otimes \theta) = 0\} \\ &= \{u \in U \mid \varepsilon_U(u \otimes f^\perp(\theta)) = 0\}, \end{aligned}$$

which belongs to $\mathcal{N}_\mu$ by equation (5.3). Therefore, morphisms in **ExChu** specify morphisms in **RTVec**. Conversely, a morphism $f : U_\mu \to V_\tau$ in **RTVec** gives rise to the morphism $(f, f^\perp) : (U, U_\tau^\perp) \to (V, V_\tau^\perp)$ in **ExChu**.

Theorem 5.1.8 **RTVec** *and* **ExChu** *are equivalent categories, preserving tensor, duality, and consequently their $*$-autonomous structures.*

Proof We define the functors F : **RTVec** $\to$ **ExChu** and G : **ExChu** $\to$ **RTVec** as follows. F sends objects U_μ to $(U, U_\mu^\perp, \varepsilon_U)$ and morphisms $f : U_\mu \to V_\tau$ to $(f, f^\perp)$. G sends objects $(U, X, \alpha) \cong (U, U_\mu^\perp, \varepsilon_U)$ to U_μ and objects $(f, g) \cong (f, f^\perp)$ to f. Then $FG \cong id_{\mathbf{ExChu}}$ and $GF = id_{\mathbf{RTVec}}$.

Next, we show that F and G preserve the tensor products. Suppose that U_μ and V_τ are objects in **RTVec**. Then

$$F(U_\mu \otimes V_\tau) = (U \otimes V, (U \otimes V)^\perp_{\mu\otimes\tau})$$

and

$$F(U_\mu) \otimes F(V_\tau) = (U \otimes V, \mathbf{ExChu}(F(U_\mu), F(V_\tau)^\perp)).$$

We prove that $\mathbf{ExChu}(F(U_\mu), F(V_\tau)^\perp) \cong (U \otimes V)^\perp_{\mu\otimes\tau}$. Suppose that $\theta \in (U \otimes V)^\perp_{\mu\otimes\tau}$. Then there exist maps $f : U \to V_\tau^\perp$ and $g : V \to U_\mu^\perp$ defined by

$$f(u)(v) = g(v)(u) = \theta(u \otimes v) \text{ for all } u \in U, v \in V, \tag{5.5}$$

so that $(f, g) \in \mathbf{ExChu}(F(U_\mu), F(V_\tau)^\perp)$. Conversely, given $(f, g) \in \mathbf{ExChu}(F(U_\mu), F(V_\tau)^\perp)$, we can define a map $\theta : U \otimes V \to k$ by equation (5.5). This map is continuous with respect to $\mu \otimes \tau$ because for all $u \in U$, the subspace

$$\{v \in V \mid \theta(u \otimes v) = 0\} = \{v \in V \mid f(u)(v) = 0\}$$

has codimension 1 and belongs to $\mathcal{N}_\tau$, since $f(u) \in V_\tau^\perp$, and for all $v \in V$, the subspace

$$\{u \in U \mid \theta(u \otimes v) = 0\} = \{u \in U \mid g(v)(u) = 0\}$$

has codimension 1 and belongs to $\mathcal{N}_\mu$, since $g(v) \in U_\mu^\perp$. Therefore, $ker(\theta) \in \mathcal{N}_{\mu\otimes\tau}$.

So we have a bijection $\mathbf{ExChu}(F(U_\mu), F(V_\tau)^\perp) \leftrightarrow (U \otimes V)^\perp_{\mu\otimes\tau}$. It is straightforward to check that this bijection is natural in U_μ and V_τ and also that $F(f \otimes g) \cong F(f) \otimes F(g)$ for all morphisms f and g in **ExChu**. Therefore F preserves the tensor product of **RTVec**.

Now consider objects $A = (U, X) \cong (U, U_\mu^\perp)$ and $B = (V, Y) \cong (V, V_\tau^\perp)$ in **ExChu**. Then

$$G((U, U_\mu^\perp) \otimes (V, V_\tau^\perp)) = G((U \otimes V, \mathbf{ExChu}(A, B^\perp))$$

and

$$G(U, U_\mu^\perp) \otimes G(V, V_\tau^\perp) = (U \otimes V)_{\mu \otimes \tau}.$$

We proved above that $\mathbf{ExChu}(A, B^\perp)$ is naturally isomorphic to the dual of $U \otimes V$ with respect to $\mu \otimes \tau$, so $G((U, U_\mu^\perp) \otimes (V, V_\tau^\perp)) \cong G(U, U_\mu^\perp) \otimes G(V, V_\tau^\perp)$. It is straightforward to check that $G((f, f^\perp) \otimes (g, g^\perp)) \cong G(f, f^\perp) \otimes G(g, g^\perp)$ for all morphisms $(f, f^\perp), (g, g^\perp)$ in **ExChu**. Therefore G preserves the tensor product of **ExChu**.

Finally, it is straightforward to check that F and G both preserve linear negation. Therefore F and G preserve the $*$-autonomous structures of **RTVec** and **ExChu** respectively. ■

We henceforth write objects in **ExChu** in the form $(V, V^\perp)$, assuming that $V^\perp$ is the dual of V with respect to some suitable linear topology on V, and that the associated bilinear map is the evaluation map.

We end this section by noting that **ExChu** supports the Mix rule. We have $\perp = \mathbf{1}^\perp = \mathbf{1}$ in **ExChu**, so the identity morphism on $\mathbf{1}$ serves as the unary Mix morphism $m : \perp \to \mathbf{1}$. Also, the following observation will be of use to us in the next section.

Lemma 5.1.9 *Let $A = (Y, Y^*, \varepsilon_Y)$ and $B = (V, V^\perp, \beta)$ be objects in* **ExChu**, *where Y is a finite dimensional vector space, Y^* is its algebraic dual space. Then a morphism $(f, g) : A \to B$ is completely characterised by its covariant map $J(f, g) = f : Y \to V$ in* **Vec**, *where $J :$* **ExChu** $\to$ **Vec** *denotes the forgetful functor, sending (f, g) to f.*

Proof We want $(f, g) : A \to B$ to satisfy $\beta(f(y), \theta) = \varepsilon_Y(y, g(\theta)) = g(\theta)(y)$ for all $y \in Y$ and $\theta \in V^\perp$. So g is completely determined by f. (In fact, when $V^\perp$ is the algebraic dual space of V, g is the dual map $f^* : V^* \to Y^*$.) ■

5.2 Full Completeness in ExChu

Blute and Scott's full completeness result required restrictions on the dinatural transformations they considered. The dinatural transformations had to be uniform or equivariant under the action of the additive group of integers. (These $\mathbb{Z}$-actions prompted a description of their work as a linear analogue of Läuchli's semantics for intuitionistic logic.) More importantly, they restricted themselves to linear combinations of dinatural transformations associated with *binary* formulae only (calling them diadditive dinatural transformations), the justification being that proofs of balanced sequents are substitution instances of proofs on binary sequents. This restriction is not entirely satisfactory, as it evades the issue of properly determining any axiom links for a proof structure, since if the sequent is binary, then the axiom links are uniquely determined.

We will remove all such restrictions here. Thus, it would appear that all dinatural transformations are diadditive, and the $\mathbb{Z}$-actions play no significant role. To specify a $\mathbb{Z}$-action is to specify an automorphism, so the insignificance of these actions is a mirror of the situation in **FDVec**, where Schur-Weyl duality characterises the commutant of $\langle f^{\otimes m} \mid f \in \mathbf{GL}(V)\rangle$ but we obtain the same commutant for $\langle f^{\otimes m} \mid f \in End(V)\rangle$.

Whilst we cannot replicate the "lifting" of compact closed full completeness, used in previous chapters, we can still show that compact full completeness in **FDVec**, does induce full completeness in **ExChu**. In particular, we can use **FDVec** to determine the behaviour of a dinatural transformation and hence the sets of axiom links associated with it. The key lies in the following observation.

Lemma 5.2.1 **FDVec** *is a full compact closed subcategory of* **ExChu***.*

Proof For every finite dimensional vector space W, the pair (W, W^*) is an exact pairing with the usual evaluation map. Furthermore, any morphism $(f, g) : (W, W^*) \to (Z, Z^*)$ satisfies $g = f^*$, so **FDVec** exists as a full subcategory of **ExChu**.

It is clear that **FDVec** is closed under linear negation, since $W^{**} \cong W$. Also, $\mathbf{ExChu}((W, W^*), (Z^*, Z)) \cong \mathbf{FDVec}(W, Z^*) \cong W^* \otimes Z^*$, so that

$$(W, W^*) \otimes (Z, Z^*) = (W \otimes Z, W^* \otimes Z^*) \cong (W \otimes Z, (W \otimes Z)^*).$$

Therefore the compact closed structure of **FDVec** is preserved in **ExChu**. ■

As usual, we assume $F(\mathbf{X}, \mathbf{Y})$ to be a formula of length p built from $X_1, \ldots, X_n$, $Y_1^\perp, \ldots, Y_n^\perp$ by the connectives $\otimes$ and $⅋$. Then F induces a multivariant functor from $[\![F]\!] : (\mathbf{ExChu})^n \times (\mathbf{ExChu}^{op})^n \to \mathbf{ExChu}$ which by abuse of notation we will also refer to as F. In **ExChu**, we will say that a dinatural transformation ρ is *non-trivial* if each morphism in $J\rho$ is non-zero in **Vec**.

From Lemma 5.2.1, we now know that a non-trivial dinatural transformation $\rho : \mathfrak{K}_\mathbf{1} \to F$ in **ExChu** restricts to a non-trivial dinatural transformation in **FDVec**. By full completeness in **FDVec**, we can associate to the restricted dinatural transformation at least one set of axiom links. (Recall that a natural transformation $W^{\otimes m} \to W^{\otimes m}$ in **FDVec** is a linear combination of permutations on the tensor factors, hence the possibility of more than one set of axiom links.) To prove that these axiom links remain valid when considering the dinatural transformation on all objects in **ExChu**, we will apply a density argument, which appeals to Barr's original characterisation of **RTVec**. We will now show that whenever we start with a collection of finite dimensional subspaces A_i^- and A_i^+ "densely approximating" the literal objects A_i, then there are alway sufficiently many objects $F(\mathbf{A}^-, \mathbf{A}^+)$ and $F(\mathbf{A}^+, \mathbf{A}^-)$ to approximate $F(\mathbf{A}, \mathbf{A})$ in the same way.

Let $A = (U, U^\perp, \alpha)$ be an object in **ExChu**. Denote by $A^<$ the collection of objects (X, X^*), with X a finite dimensional subspace of U, and denote by $A^>$ the collection of objects (Z, Z^*), with Z^* a finite dimensional subspace of $U^\perp$. Recall from Lemma 5.1.9 that we can always find maps $(f_X^-, g_X^-) : (X, X^*) \to (U, U^\perp)$ and $(f_Z^+, g_Z^+) : (U, U^\perp) \to (Z, Z^*)$ in **ExChu**, determined by $f_X^- : X \to U$ and $g_Z^+ : Z^* \to U^\perp$. Furthermore, we can find non-vanishing maps, satisfying $f_X^-(x) = 0$ if and only if $x = 0$ and $g_Z^+(\theta) = 0$ if and only if $\theta = 0$. We say that $A^<$ and $A^>$ *densely approximate* A, in the sense that

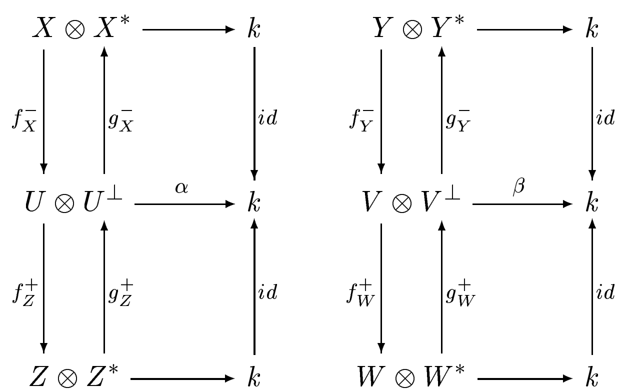
$$\begin{aligned} u \in U, u = 0 \quad &\Leftrightarrow \quad \alpha(u, g_Z^+(\psi)) = 0 \text{ for all } \psi \in Z^* \text{ and } Z^* \in A^>; \\ \theta \in U^\perp, \theta = 0 \quad &\Leftrightarrow \quad \alpha(f_X^-(x), \theta) = 0 \text{ for all } x \in X \text{ and } X \in A^<. \end{aligned}$$

Lemma 5.2.2 *The sets of objects*

$$\{F(\mathbf{Y}, \mathbf{Z}) \mid (Y_i, Y_i^*) \in B_i^<, (Z_i, Z_i^*) \in A_i^>\}$$
$$\{F(\mathbf{W}, \mathbf{X}) \mid (W_i, W_i^*) \in B_i^>, (X_i, X_i^*) \in A_i^<\}$$

in **ExChu** *densely approximate* $F(\mathbf{B}, \mathbf{A})$.

Proof Clearly, $A^>$ and $A^<$ densely approximate $A^\perp$. It suffices to prove the lemma for the formula $F(B, A) = A \multimap B$. Thus, suppose we have dense approximants to $A = (U, U^\perp, \alpha)$ and $B = (V, V^\perp, \beta)$:



We will prove that we have dense approximants :

$$\{(Z^* \otimes Y, Z \otimes Y^*) \mid (Y, Y^*) \in B^\leq, (Z, Z^*) \in A^\geq\},$$
$$\{(X^* \otimes W, X \otimes W^*) \mid (W, W^*) \in B^\geq, (X, X^*) \in A^\leq\}$$

$$\begin{array}{cccc} (Z^*\otimes Y) & \otimes\,(Z\otimes Y^*) & \longrightarrow & k \\ \downarrow f^-_{Z^*\otimes Y} & \uparrow & & \downarrow id \\ \mathbf{ExChu}(A,B) & \otimes\,(U\otimes V^\perp) & \xrightarrow{\ \delta\ } & k \\ \downarrow & \uparrow g^+_{X\otimes W^*} & & \uparrow id \\ (X^*\otimes W) & \otimes\,(X\otimes W^*) & \longrightarrow & k \end{array} \qquad (5.6)$$

to the object $F(B,A) = (\mathbf{ExChu}(A,B), U\otimes V^\perp, \delta)$. (Recall that $(Z,Z^*) \multimap (Y,Y^*) \cong (\mathbf{FDVec}(Z,Y), Z\otimes Y^*) \cong (Z^*\otimes Y, Z\otimes Y^*)$ etc.)

Given $(Z,Z^*) \in A^>$ and $(Y,Y^*) \in B^<$, define for all elements $\psi\otimes y \in Z^*\otimes Y$, $f^-_{Z^*\otimes Y}(\psi\otimes y) = (s,t)$ where

$$\begin{array}{rl} s : U \to V, & s(u) = \alpha(u, g^+_Z(\psi)) f^-_Y(y); \\ t : V^\perp \to U^\perp, & t(\theta) = \beta(f^-_Y(y),\theta) g^+_Z(\psi). \end{array}$$

Then $(s,t) : A \to B$ in **ExChu**, since for all $u \in U$, $\theta \in V^\perp$, we have

$$\begin{array}{rcl} \beta(s(u),\theta) & = & \beta(\alpha(u, g^+_Z(\psi)) f^-_Y(y), \theta) \\ & = & \alpha(u, \beta(f^-_Y(y),\theta) g^+_Z(\psi)) \\ & = & \alpha(u, t(\theta)). \end{array}$$

Furthermore, if $\psi \neq 0$ and $y \neq 0$, then $s(u) = \alpha(u, g^+_Z(\psi))y = 0$ if and only if $u = 0$ and $t(\theta) = \beta(f^-_Y(y),\theta)\psi = 0$ if and only if $\theta = 0$. Hence $(s,t) \neq (0,0)$. So the map $f^-_{Z^*\otimes Y} : Z^*\otimes Y \to \mathbf{ExChu}(A,B)$ is non-vanishing.

Also, given $(X,X^*) \in A^<$ and $(W,W^*) \in B^>$, we can define a map $g^+_{X\otimes W^*} : X\otimes W^* \to U\otimes V^\perp$, $x\otimes\theta \mapsto f^-_X(x)\otimes g^+_W(\theta)$. Clearly $g^+_{X\otimes W^*}$ is non-vanishing whenever f^-_X and g^+_W are non-vanishing.

Suppose that $(s,t) \in \mathbf{ExChu}(A,B)$. Then $s : U \to V$, $t : V^\perp \to U^\perp$ and $\beta(s(u),\theta) = \alpha(u,t(\theta))$ for all $u \in U, \theta \in V^\perp$. Suppose that $(s,t) \neq (0,0)$. Then there exists $u \in U$ such that $s(u) \neq 0$. Since $s(u) \in V$, there exists $\theta \in W^*$ such that $\beta(s(u), g^+_W(\theta)) \neq 0$. By definition of (s,t), we have $\alpha(u, t(g^+_W(\theta))) \neq 0$, i.e. $t(g^+_W(\theta)) \neq 0$. Since $t(g^+_W(\theta)) \in U^\perp$, there exists $x \in X$ such that $\alpha(f^-_X(x), t(g^+_W(\theta))) \neq 0$. In other words, $x\otimes\theta \in X\otimes W^*$ satisfies

$$\begin{array}{rcl} \delta((s,t), g^+_{X\otimes W^*}(x\otimes\theta)) & = & \delta((s,t), f^-_X(x)\otimes g^+_W(\theta)) \\ & = & \alpha(f^-_X(x), t(g^+_W(\theta))) \neq 0. \end{array}$$

Now suppose that $0 \neq u\otimes\theta \in U\otimes V^\perp$. Since $u \in U$, there exists $\psi \in Z^*$ such that $\alpha(u, g^+_Z(\psi)) \neq 0$, and since $\theta \in V^\perp$, there exists $y \in Y$ such that $\beta(f^-_Y(y),\theta) \neq 0$. Thus

$\alpha(u, g_Z^+(\psi))\beta(f_Y^-(y), \theta) \neq 0$, i.e. $\psi \otimes y \in Z^* \otimes Y$ satisfies

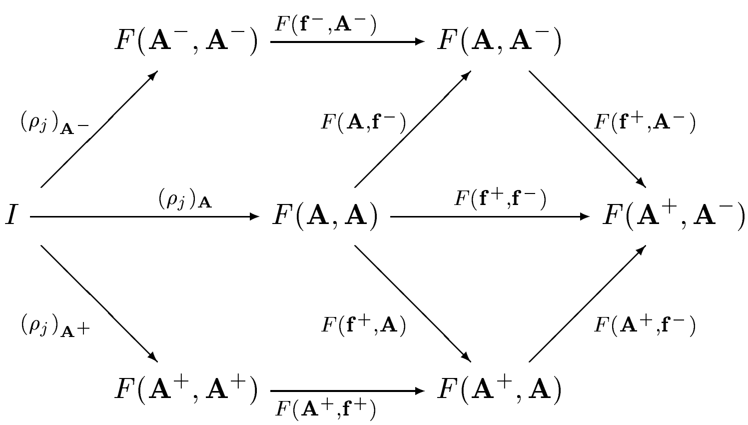
$$\begin{aligned}\delta(f_{Z^*\otimes Y}^-(\psi \otimes y), u \otimes \theta) &= \alpha(u, t(\theta)) = \alpha(u, \beta(f_Y^-(y), \theta) g_Z^+(\psi)) \\ &= \alpha(u, g_Z^+(\psi))\beta(f_Y^-(y), \theta) \neq 0.\end{aligned}$$

Therefore we have dense approximants as prescribed in (5.6). ■

With the above lemma, we can now prove :

Lemma 5.2.3 *Suppose that ρ is a non-trivial dinatural transformation in* **ExChu** *from the constant functor $\mathfrak{K}_1$ to the multivariant functor F. Then ρ is completely determined by its behaviour on the subcategory* **FDVec**.

Proof Suppose that there exist *two* distinct dinatural transformations $\rho_1, \rho_2 : \mathfrak{K}_1 \to F$ such that $\rho_1 = \rho_2 = \rho$ on objects in **FDVec**. Given objects $A_i^- \in A_i^<$ and $A_i^+ \in A_i^>$, and maps $f_i^- : A_i^- \to A_i$, $f_i^+ : A_i \to A_i^+$, the following diagram commutes, for $j = 1, 2$.



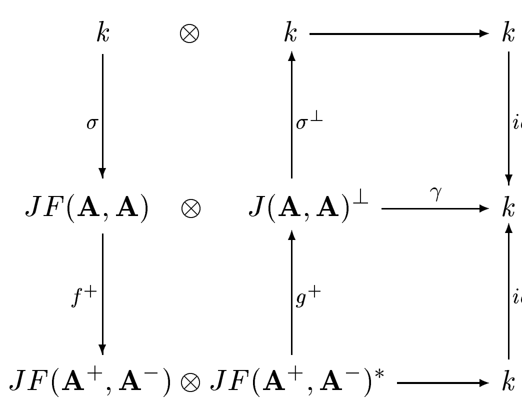
In particular,

$$I \xrightarrow{(\rho_1 - \rho_2)_{\mathbf{A}^-}} F(\mathbf{A}^-, \mathbf{A}^-) \xrightarrow{F(\mathbf{f}^-, \mathbf{A}^-)} F(\mathbf{A}, \mathbf{A}^-) \xrightarrow{F(\mathbf{f}^+, \mathbf{A}^-)} F(\mathbf{A}^+, \mathbf{A}^-)$$

is the zero map, so

$$I \xrightarrow{(\rho_1 - \rho_2)_{\mathbf{A}}} F(\mathbf{A}, \mathbf{A}) \xrightarrow{F(\mathbf{f}^+, \mathbf{f}^-)} F(\mathbf{A}^+, \mathbf{A}^-)$$

is the zero map. Thus, we have maps $(\rho_1 - \rho_2)_{\mathbf{A}} = (\sigma, \sigma^\perp)$ and $F(\mathbf{f}^+, \mathbf{f}^-) = (f^+, g^+)$,



such that $f^+\sigma \equiv 0$ and $\sigma^\perp g^+ \equiv 0$ in **Vec**.

Suppose that $(\rho_1 - \rho_2)_{\mathbf{A}} \neq 0$. Then $\sigma(1) \neq 0$, so there exists $\theta \in JF(\mathbf{A}^+, \mathbf{A}^-)^*$ such that $\gamma(\sigma(1), g^+(\theta)) \neq 0$. But by definition of $(\sigma, \sigma^\perp)$, we have $1 \cdot \sigma^\perp(g^+(\theta)) \neq 0$, hence $\sigma^\perp g^+ \not\equiv 0$. This is a contradiction, so we must have $(\rho_1 - \rho_2)_{\mathbf{A}} = 0$. Therefore, ρ is uniquely determined by its behaviour on objects in **FDVec**. ■

Corollary 5.2.4 *The non-trivial dinatural transformation $\rho : \mathfrak{K}_{\mathbf{1}} \to F$ is a linear combination of formal dinaturals $\rho_1, \ldots, \rho_T : \mathfrak{K}_{\mathbf{1}} \to F$, specified by a fixed-point free involutions $\phi_1, \ldots, \phi_T$ on $\{1, \ldots, p\}$ such that $\xi_{\phi_h(i)} = \xi_i$ and $\zeta_{\phi_h(i)} \neq \zeta_i$ for all $i = 1, \ldots, p$ and $h = 1, \ldots, T$. Therefore, we can form T proof structures for the MLL formula F, each ϕ_h specifying a set of axiom links for a proof structure.*

Proof This follows immediately from full completeness in **FDVec** (Theorem 3.2.2). ■

We now adopt the same tactic as in Chapter 3. We shall suppose that ρ is *one* formal dinatural transformation, and prove that the unique MLL proof structure associated with ρ is a proof net. We can then extend the result to cover all linear combinations of formal dinaturals in the same way as in Chapter 3, since the annihilation of formal dinaturals actually occurs in the underlying category **FDVec**. The details are thus omitted.

Theorem 5.2.5 *Suppose that $\rho : \mathfrak{K}_{\mathbf{1}} \to F$ is a non-trivial formal dinatural transformation in* **ExChu**. *Consider the unique MLL proof structure for F associated with ρ, specified by the involution ϕ. Then for any DR-switching, the associated DR-graph is acyclic.*

Proof Suppose that for a certain DR-switching, the associated DR-graph contains a cycle. Consider the shortest cycle. We can express this as lower links $(a_1, b_1), \ldots, (a_r, b_r)$ such that $\phi(b_i) = a_{i+1}$ for all $i \in \mathbb{Z}_r$. By an argument similar to that in Theorem 2.2.8, we may assume that F has been reduced to a formula $\tilde{F}$ of the form

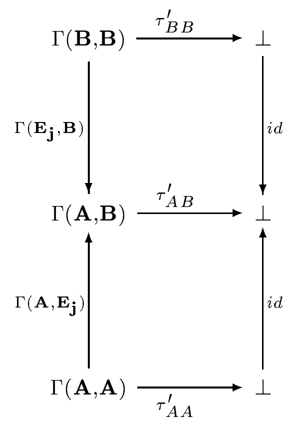
$$\tilde{F}(\mathbf{A}, \mathbf{A}) = \Gamma(\mathbf{A}, \mathbf{A}) \parr (A^{\zeta_{a_1}}_{\xi_{a_1}} \otimes A^{\zeta_{b_1}}_{\xi_{b_1}}) \parr \cdots \parr (A^{\zeta_{a_r}}_{\xi_{a_r}} \otimes A^{\zeta_{b_r}}_{\xi_{b_r}})$$

where $\Gamma(\mathbf{A}, \mathbf{A})$ is a *par* product of literals, and that we have derived from ρ a new dinatural transformation $\tilde{\rho}$ from $\mathfrak{K}_{\mathbf{1}}$ to $\tilde{F}$. Since **ExChu** supports the Mix Rule, $\tilde{\rho}$ is dinatural over *all* objects and morphisms in **ExChu**.

Choose an object A in **ExChu** such that JA is an infinite dimensional vector space V with a countable basis $\{e_i\}$, and $JA^\perp$ is the infinite dimensional vector space $V^\perp$ with countable basis $\{\varepsilon_i\}$, satisfying $\varepsilon_i(e_j) = 1$ if $i = j$, and 0 otherwise. For any $j \in \mathbb{N}$, write $V_j = \langle e_i \rangle_{i=1}^{j}$, $V_j^* = \langle \varepsilon_i \rangle_{i=1}^{j}$. Then there is an obvious isomorphism $\psi_A : A \to A^\perp$ determined by the isomorphism $V \to V^\perp, e_i \mapsto \varepsilon_i$. Similarly, we have isomorphisms $\psi_B : B \to B^\perp$ for each $B = (V_j, V_j^*)$. (From now on, we understand B to denote an object

(V_j, V_j^*).) Furthermore, the embedding $E_j : V_j \hookrightarrow V$ induces the morphism $E_j : B \to A$. (In fact, $E_j^{\perp}$ is equivalently determined by the projection $P_j : V^{\perp} \to V_j^*$.)

Put $A_i = A$ for all $i = 1, \dots, n$. Fix $j \in \mathbb{N}$, and put $B_i = B = (V_j, V_j^*)$ for all $i = 1, \dots, n$. The map $\varepsilon_1 : V_j \hookrightarrow V \to k$, determines maps $\tau_A^+ : A \to \perp$ and $\tau_B^+ : B \to \perp$. Similarly, the map $e_1 : V_j^{\perp} \hookrightarrow V^{\perp} \to k$ determines maps $\tau_A^- : A^{\perp} \to \perp$ and $\tau_B^- : B^{\perp} \to \perp$. Since Γ is a par product of literals, we can form a map $\tau'_{AA} : \Gamma(\mathbf{A}, \mathbf{A}) \to \perp$ built from the maps τ_A^+, τ_A^-. Similarly, we obtain $\tau'_{BB} : \Gamma(\mathbf{B}, \mathbf{B}) \to \perp$. Equally naturally, we obtain $\tau'_{AB} : \Gamma(\mathbf{A}, \mathbf{B}) \to \perp$, and therefore the following diagram commutes.



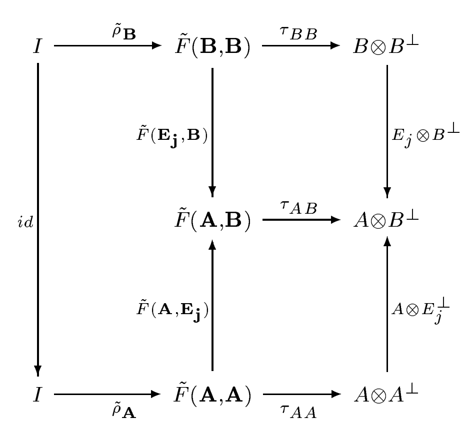
Put

$$(\tau_i)_A \quad = \quad \begin{cases} id_A & \text{if } \zeta_{a_i} \neq \zeta_{b_i}; \\ \psi_A & \text{if } \zeta_{a_i} = \zeta_{b_i} = 1; \\ \psi_A^{-1} & \text{if } \zeta_{a_i} = \zeta_{b_i} = \perp \end{cases}$$

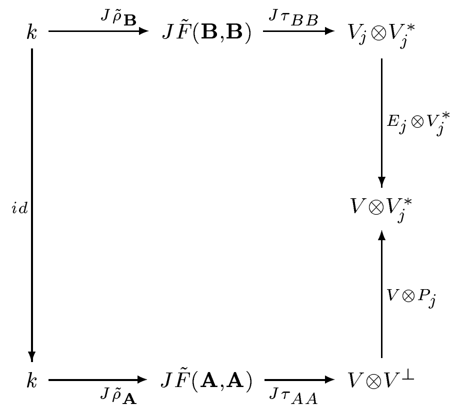
for all $i = 1, \dots, r-1$ (where we identify $\psi_A : A \to A^{\perp}$ with a morphism $A \otimes A \to \perp$ etc.) Then we can use $(\tau_1)_A, \dots, (\tau_{r-1})_A$ to construct a map

$$\tau''_{AA} : (A^{\zeta_{a_1}} \otimes A^{\zeta_{b_1}}) \parr \cdots \parr (A^{\zeta_{a_{r-1}}} \otimes A^{\zeta_{b_{r-1}}}) \to \perp.$$

These maps, together with the identity morphism on $A^{\zeta_{a_r}} \otimes A^{\zeta_{b_r}}$ form a map τ_{AA} : $\tilde{F}(\mathbf{A}, \mathbf{A}) \to A^{\zeta_{a_r}} \otimes A^{\zeta_{b_r}}$. In a natural way, we can also form τ_{BB} and τ_{AB}, so that (in the case when $\zeta_{a_r} = 1$ and $\zeta_{b_r} = \perp$ – the other cases are similar) the following diagram commutes.



Furthermore, in **Vec** we have



(5.7)

Since V_j is finite dimensional, we have $J\tilde{\rho}_{\mathbf{B}} = f_1 \otimes f_2$ where

$$f_1 : k \to \Gamma(V_j, V_j); \quad f_2 : k \to (V_j^{\zeta_{a_1}} \otimes V_j^{\zeta_{b_1}}) \otimes \cdots \otimes (V_j \otimes V_j^*),$$

and also $J\tau_{BB} = g_1 \otimes g_2$ where

$$g_1 : \Gamma(V_j, V_j) \to k; \quad g_2 : (V_j^{\zeta_{a_1}} \otimes V_j^{\zeta_{b_1}}) \otimes \cdots \otimes (V_j \otimes V_j^*) \to V_j \otimes V_j^*.$$

Therefore $J\tau_{BB} J\tilde{\rho}_{\mathbf{B}} = (g_1 \otimes g_2)(f_1 \otimes f_2) = (g_1 f_1) \otimes (g_2 f_2)$.

Observe that $J\tau_B^+$ and $J\tau_B^-$ induce a map $\beta : V_j \otimes V_j^* \to k$, where $v \otimes \theta$ is mapped to $\varepsilon_1(v)\theta(e_1)$. In particular, $e_i \otimes \varepsilon_i$ is mapped to 1 if $i = 1$ and 0 otherwise. From this, we deduce that

$$k \xrightarrow{\eta_{V_j}} V_j \otimes V_j^* \xrightarrow{\beta} k = id_k.$$

Up to symmetry, g_1 is a tensor product of η_{V_j} maps, so we deduce that $g_1 f_1 = id_k$.

Examining the cycle a little closer, we see that there are an even number, $2s$ say, of occurrences of i for which $\zeta_{a_i} = \zeta_{b_i}$, s of which are $\zeta_{a_i} = \zeta_{b_i} = 1$ and s of which are $\zeta_{a_i} = \zeta_{b_i} = \perp$. Since $\zeta_{a_r} \neq \zeta_{b_r}$, τ was built from the same number of ψ as ψ^{-1}, so we deduce that $g_2 f_2 : k \to V_j \otimes V_j^*$ is the unit map η_{V_j}.

Therefore $J\tau_{BB} J\tilde{\rho}_{\mathbf{B}} = \eta_{V_j}$. Since $j \in \mathbb{N}$ was chosen arbitrarily, diagram (5.7) now tells us that $J\tau_{AA} J\tilde{\rho}_{\mathbf{A}} : k \to V \otimes V^{\perp}$ projects onto the map η_{V_j} for all $j \in \mathbb{N}$. In particular, $J\tau_{AA} J\tilde{\rho}_{\mathbf{A}}(1)$ projects onto the element $\sum_{i=1}^{j} e_i \otimes \varepsilon_i$ for all $j \in \mathbb{N}$. However, there can be no such element because $\{e_i \otimes \varepsilon_j\}$ is a basis for $V \otimes V^{\perp}$ but we are unable to express $J\tau_{AA} J\tilde{\rho}_{\mathbf{A}}(1)$ as a *finite* linear combination of these basis vectors.

We have shown in previous chapters how to deal with the case $\zeta_{a_r} = \zeta_{b_r}$, so the details are omitted here. In summary, we deduce that τ was built from $s - 1$ occurrences of $\psi^{\zeta_{a_r}}$ and s occurrences of $\psi^{\zeta_{b_r}}$ (where we read $\psi^{\perp} = \psi^{-1}$). We postcompose each τ with a morphism $\psi^{\zeta_{a_r}} \otimes id$, and then the argument is the same as above. ∎

Again, we remark that it is possible to extend the above result to the case when ρ is a linear combination of formal dinaturals, using the same approach taken in Chapter 3. Thus we have,

Theorem 5.2.6 *(Full Completeness in* **ExChu***) Every non-trivial dinatural transformation in* **ExChu** *from the constant functor* $\mathfrak{K}_{\mathbf{1}}$ *to the multivariant functor* F *is a linear combination of formal dinatural transformations* ρ^h*, where each* ρ^h *is the denotation of a unique proof in MLL+Mix of the formula* F*, and is therefore induced by a unique morphism* $I \to F(\mathbf{X}, \mathbf{X})$ *in the free* $*$*-autonomous category supporting the Mix rule, on* n *objects* $X_1, \ldots, X_n$*.*

Chapter 6

Proof Nets for MILL

Intuitionistic Linear Logic (ILL) is a sublogic of Linear Logic, where we restrict ourselves to sequents with precisely one conclusion. In this chapter, we consider only the multiplicative fragment of Intuitionistic Linear Logic without exponentials (MILL). It is widely accepted that the correct categorical model for MILL is a symmetric monoidal closed category.

The purpose of this chapter is to provide a proof net system for MILL, which fully captures the essence of describing proofs in the sequent calculus. We will show there is a direct correspondence between equivalence classes of proof nets and equivalence classes of terms in an assignment motivated by categorical considerations. In particular, MILL proof nets will provide us with a graphical description of canonical morphisms in a free symmetric monoidal closed category, where equivalent maps in the category give rise to MILL proof nets which are equivalent in some sense. This chapter can therefore be regarded as a solution to the coherence problem in [KM71], where the difficulty in handling the units was unresolved. We remark on the existence of an independent Preliminary Report by F. Lamarche [Lam94] – a more ambitious treatment of proof nets for ILL, including the additives. Indeed, some of his ideas are evident in this work. (Particularly, the tree ordering $\geq$ in §6.3). However, in the time available, I have not been able to make a clear comparison with Lamarche's report, especially with regard to the net condition. We will conclude this chapter with comparisons with the works of Trimble [Tri95] and Cockett and Seely [CS97b].

An in-depth discussion on ILL can be found in Bierman's thesis [Bie94]. We provide here a brief restricted overview of the multiplicative fragment only.

6.1 Sequent calculus for MILL

The sequent calculus for MILL is presented in Figure 6.1. As before, finite sequences of formulae are denoted by capital Greek letters, $\Gamma, \Delta, \ldots$, while formulae $A, B, \ldots$ are built from propositional atoms $\alpha, \beta, \ldots$ by the connectives $\otimes$ and $\multimap$.

As with MLL, we lose nothing if we remove the Exchange rule and assume that $\Gamma, \Delta, \ldots$ denote finite multisets of formulae instead of finite sequences of formulae. We assume this henceforth.

$$\frac{}{\alpha \vdash \alpha}\,(Id)$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\,(Exchange)$$

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\,(Cut)$$

$$\frac{\Gamma \vdash A}{\Gamma, I \vdash A}\,(I_{\mathcal{L}}) \qquad \frac{}{\vdash I}\,(I_{\mathcal{R}})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\,(\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap_{\mathcal{R}})$$

Figure 6.1: Sequent calculus for MILL

Note that we have restricted the usual (Id) rule $A \vdash A$ to just those on propositional atoms. This does not affect the logic, and will make easier our formulation of proof nets.

6.1.1 Cut elimination for sequent calculus

There is a cut elimination procedure which removes applications of the (Cut) rule in a sequent calculus proof to produce a cut-free proof of the same sequent.

Cut elimination in the sequent calculus is a form of proof normalisation. (Indeed it corresponds to proof normalisation in linear natural deduction, but we omit discussion of this alternative system). Thus two desirable properties are that it is strongly normalising (there is no infinite reduction) and that it is confluent (a proof normalises to a unique cut-free proof, independent of the order of elimination of applications of (Cut).)

Axiom reductions

AR1.

$$\frac{\alpha \vdash \alpha \quad \alpha, \Delta \overset{\pi'}{\vdash} B}{\alpha, \Delta \vdash B}\,(Cut) \quad \text{reduces to} \quad \alpha, \Delta \overset{\pi'}{\vdash} B.$$

AR2.

$$\dfrac{\dfrac{}{\vdash I}\,(I_{\mathcal{R}}) \qquad \dfrac{\Gamma \overset{\pi'}{\vdash} A}{I, \Gamma \vdash A}\,(I_{\mathcal{L}})}{\Gamma \vdash A}\,(Cut) \qquad \text{reduces to} \qquad \Gamma \overset{\pi'}{\vdash} A.$$

Symmetric reductions

SR1.

$$\dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad \Delta \overset{\pi_2}{\vdash} B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes_{\mathcal{R}}) \qquad \dfrac{A, B, \Theta \overset{\pi_3}{\vdash} C}{A \otimes B, \Theta \vdash C}\,(\otimes_{\mathcal{L}})}{\Gamma, \Delta, \Theta \vdash C}\,(Cut)$$

reduces to

$$\dfrac{\Delta \overset{\pi_2}{\vdash} B \qquad \dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad A, B, \Theta \overset{\pi_3}{\vdash} C}{B, \Gamma, \Theta \vdash C}\,(Cut)}{\Gamma, \Delta, \Theta \vdash C}\,(Cut).$$

SR2.

$$\dfrac{\dfrac{\Gamma, A \overset{\pi_1}{\vdash} B}{\Gamma \vdash A \multimap B}\,(\multimap_{\mathcal{R}}) \qquad \dfrac{\Delta \overset{\pi_2}{\vdash} A \quad B, \Theta \overset{\pi_3}{\vdash} C}{A \multimap B, \Delta, \Theta \vdash C}\,(\multimap_{\mathcal{L}})}{\Gamma, \Delta, \Theta \vdash C}\,(Cut)$$

reduces to

$$\dfrac{\dfrac{\Delta \overset{\pi_2}{\vdash} A \quad \Gamma, A \overset{\pi_1}{\vdash} B}{\Gamma, \Delta \vdash B}\,(Cut) \qquad B, \Theta \overset{\pi_3}{\vdash} C}{\Gamma, \Delta, \Theta \vdash C}\,(Cut).$$

Commutative reductions

CR1.

$$\dfrac{\Gamma \overset{\pi_1}{\vdash} A \qquad \dfrac{A, \Delta \overset{\pi_2}{\vdash} B}{A, \Delta, I \vdash B}\,(I_{\mathcal{L}})}{\Gamma, \Delta, I \vdash B}\,(Cut) \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad A, \Delta \overset{\pi_2}{\vdash} B}{\Gamma, \Delta \vdash B}\,(Cut)}{\Gamma, \Delta, I \vdash B}\,(I_{\mathcal{L}})$$

There is a similar reduction when the left hand proof ends with $(I_{\mathcal{L}})$.

CR2.

$$\dfrac{\Gamma \overset{\pi_1}{\vdash} A \qquad \dfrac{A, B, C, \Delta \overset{\pi_2}{\vdash} D}{A, B \otimes C, \Delta \vdash D}\,(\otimes_{\mathcal{L}})}{\Gamma, B \otimes C, \Delta \vdash D}\,(Cut) \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad A, B, C, \Delta \overset{\pi_2}{\vdash} D}{\Gamma, B, C, \Delta \vdash D}\,(Cut)}{\Gamma, B \otimes C, \Delta \vdash D}\,(\otimes_{\mathcal{L}}).$$

There is a similar reduction when the left hand proof ends with $(\otimes_L)$.

CR3.

$$\dfrac{\Gamma \overset{\pi_1}{\vdash} A \qquad \dfrac{A, \Delta \overset{\pi_2}{\vdash} C \quad \Theta \overset{\pi_3}{\vdash} D}{A, \Delta, \Theta \vdash C \otimes D}\,(\otimes_{\mathcal{R}})}{\Gamma, \Delta, \Theta \vdash C \otimes D}\,(Cut)$$

reduces to

$$\dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad A,\Delta \overset{\pi_2}{\vdash} C}{\Gamma,\Delta \vdash C}\,(Cut) \qquad \Theta \overset{\pi_3}{\vdash} D}{\Gamma,\Delta,\Theta \vdash C \otimes D}\,(\otimes_{\mathcal{R}})$$

There is a similar reduction when A is a left premise of π_3.

CR4.

$$\dfrac{\Gamma \overset{\pi_1}{\vdash} A \qquad \dfrac{A,\Delta \overset{\pi_2}{\vdash} B \quad C,\Theta \overset{\pi_3}{\vdash} D}{A,\Delta,B \multimap C,\Theta \vdash D}\,(\multimap_{\mathcal{L}})}{\Gamma,\Delta,B \multimap C,\Theta \vdash D}\,(Cut)$$

reduces to

$$\dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad A,\Delta \overset{\pi_2}{\vdash} B}{\Gamma,\Delta \vdash B}\,(Cut) \qquad C,\Theta \overset{\pi_3}{\vdash} D}{\Gamma,\Delta,B \multimap C,\Theta \vdash D}\,(\multimap_{\mathcal{L}}).$$

There is a similar reduction when A is a left premise in π_3. There is also a similar reduction when the left hand proof ends with $(\multimap_{\mathcal{L}})$.

CR5.

$$\dfrac{\Gamma \overset{\pi_1}{\vdash} A \qquad \dfrac{A,\Delta,B \overset{\pi_2}{\vdash} C}{A,\Delta \vdash B \multimap C}\,(\multimap_{\mathcal{R}})}{\Gamma,\Delta \vdash B \multimap C}\,(Cut) \quad \text{reduces to} \quad \dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad A,\Delta,B \overset{\pi_2}{\vdash} C}{\Gamma,\Delta,B \vdash C}\,(Cut)}{\Gamma,\Delta \vdash B \multimap C}\,(\multimap_{\mathcal{R}}).$$

Bierman [Bie94] defined a "cut rank" on sequent calculus proofs, whereby proofs were cut-free if and only if they had cut rank 0. He proved that given a proof π of $\Gamma \vdash A$ with cut rank $c(\pi) > 0$, one could reduce π (using the cut reductions above) to a proof π' of $\Gamma \vdash A$ with cut rank strictly less than $c(\pi)$. Thus any sequent calculus proof reduces to a cut-free proof of the same sequent.

Any technical discussion about cut elimination requires some measures on formulae and proofs. We will make use of the following measures.

Definitions 6.1.1 Define the *rank* $|A|$ of a formula A inductively by

$$\begin{aligned} |\alpha| &= 1; \\ |I| &= 1; \\ |A \otimes B| &= |A| + |B| + 1; \\ |A \multimap B| &= |A| + |B| + 1. \end{aligned}$$

(NB : Bierman [Bie94] defined $|\alpha| = |I| = 0$.)

Define the *depth* $d(\pi)$ of a proof π inductively by

$$d(\pi) \quad = \quad \begin{cases} 0 & \text{if } \pi = (Id) \text{ or } (I_{\mathcal{R}}) \\ d(\pi') + 1 & \text{if } \pi \text{ is a unary rule applied to } \pi' \\ max\{d(\pi_1), d(\pi_2)\} + 1 & \text{if } \pi \text{ is a binary rule applied to } \pi_1, \pi_2. \end{cases}$$

Define the *cut size* $s(\pi)$ of a proof π inductively by

$$s(\pi) \quad = \quad \begin{cases} 1 & \text{if } \pi = (Id) \text{ or } (I_{\mathcal{R}}) \\ s(\pi') + 1 & \text{if } \pi \text{ is a unary rule applied to } \pi' \\ s(\pi_1) + s(\pi_2) & \text{if } \pi \text{ is a binary rule, not } (Cut)\text{, applied to } \pi_1, \pi_2. \\ s(\pi_1) + s(\pi_2) + 1 & \text{if } \pi \text{ is } (Cut) \text{ applied to } \pi_1, \pi_2. \end{cases}$$

Strong normalisation and confluence

We will prove strong normalisation by using a multiset ordering technique, popularised by Dershowitz and Manna. See [Der82]. Suppose that we are given a partial order $>$ on a set S. If X and Y are multisets of elements in S, we define the partial order $\gg$ by

$X \gg Y$ if and only if whenever x occurs more times in Y than in X, there exists $y > x$ such that y occurs more times in X than in Y.

A partial order $>$ is *well-founded* if there is no infinite ordered sequence $x_1 \underset{\neq}{>} x_2 \underset{\neq}{>} \dots$ of elements. It can be shown that $\gg$ is well-founded if and only if $>$ is well-founded.

Observe that $X \gg X'$ if X' is the multiset obtained from X, either by deleting an element of X, or by replacing an element y in X with a finite set of elements $x_1, \dots, x_k$, where $y > x_i$ for all $i = 1, \dots, k$.

Let π be a proof with n (Cut)s, labelled 1 to n. Let π_i be the proof within π ending with the ith (Cut), with cut formula A_i. Assign to the ith (Cut) the ordered pair $(|A_i|, s(\pi_i))$. Define the *set* $C(\pi)$ *of cut values of* π by

$$C(\pi) = \{(|A_i|, s(\pi_i)) \mid i = 1, \dots, n\}.$$

We will use the partial order $\gg$ induced by the partial order $>$ on $\mathbb{N} \times \mathbb{N}$ defined by

$$(m, n) > (m', n') \text{ if and only if } \begin{cases} m > m', \text{ or} \\ m = m' \text{ and } n > n'. \end{cases}$$

Note that $>$ is well-founded in $\mathbb{N} \times \mathbb{N}$, and therefore $\gg$ is also well-founded.

Theorem 6.1.2 *Cut elimination is strongly normalising.*

Proof It suffices to show that if π' is the result of applying a cut reduction to a proof π, then $C(\pi) \underset{\neq}{\gg} C(\pi')$. Under AR1 and AR2, an element in $C(\pi)$ is deleted. Under SR1 and

SR2, an element in $C(\pi)$, namely $y = (|A| + |B| + 1, s(\pi_1) + s(\pi_2) + s(\pi_3) + 2)$, is replaced by two elements, $x_1 = (|A|, s(\pi_1) + s(\pi_3) + 1)$ and $x_2 = (|B|, s(\pi_2) + s(\pi_1) + s(\pi_3) + 2)$, for which $y > x_1, x_2$. Under CR1–CR5, an element y in $C(\pi)$ is replaced by an element $x < y$. Therefore, $C(\pi) \underset{\neq}{\gg} C(\pi')$ under all cut reductions.

Since $\gg$ is well founded, there is no infinite reduction sequence. Therefore cut elimination is strongly normalising. ■

Theorem 6.1.3 *Cut elimination is confluent.*

Proof This result follows from strong normalisation and weak confluence. That is, if π' and π'' are two distinct proofs obtained by a single cut reduction on π, then π' and π'' both reduce to some proof σ. Indeed, we can immediately verify that permuting two cut reductions applied to distinct (Cut)s yields the same result. ■

6.2 Term assignment for the sequent calculus

The term assignment system presented in Figure 6.2 is due to Benton et al [BBPH92]. It was motivated by categorical considerations, anticipating that we wish to model MILL using a symmetric monoidal closed category $\mathbb{C} = (\mathbb{C}, \otimes, I, \multimap)$.

The principal idea is that a proof of the term $A_1, \ldots, A_n \vdash f : C$ is interpreted as a morphism $f : A_1 \otimes \cdots \otimes A_n \to C$ in $\mathbb{C}$. We will write $f : \Gamma \to C$ for this morphism if Γ is the multiset $A_1, \ldots, A_n$. Note that the Exchange rule is immediately absorbed since the monoidal structure is symmetric.

An application of (Cut) corresponds to composition of morphism. So if $f : \Gamma \to A$ and $g : A \otimes \Delta \to B$ are morphisms denoting proofs of $\Gamma \vdash f : A$ and $x : A, \Delta \vdash g : B$, then the composite

$$\Gamma \otimes \Delta \xrightarrow{f \otimes \Delta} A \otimes \Delta \xrightarrow{g} B$$

corresponds to the operation

$$\frac{\Gamma \vdash f : A \quad x : A, \Delta \vdash g : B}{\Gamma, \Delta \vdash g[f/x] : B}\,(Cut).$$

Clearly, we wish to identify the tensor in the logic with the tensor $\otimes$ in $\mathbb{C}$, the unit in the logic with the unit for tensor I in $\mathbb{C}$, and linear implication in the logic with $\multimap$ in $\mathbb{C}$. Naturally, if two distinct terms give rise to two canonical maps which are equivalent in $\mathbb{C}$, we want these terms to be equivalent as well. This gives rise to the β-equations (6.1)–(6.3), η-equations (6.4)–(6.6), and naturality equations (6.7), (6.8) listed in Figure 6.3.

$$x : \alpha \vdash x : \alpha$$

$$\frac{\Gamma \vdash e : A \quad x : A, \Delta \vdash f : B}{\Gamma, \Delta \vdash f[e/x] : B}\,(Cut)$$

$$\frac{\Gamma \vdash e : A}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\,(I_{\mathcal{L}}) \qquad \frac{}{\vdash * : I}\,(I_{\mathcal{R}})$$

$$\frac{\Gamma, x : A, y : B \vdash f : C}{\Gamma, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : C}\,(\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash e : A \quad \Delta \vdash f : B}{\Gamma, \Delta \vdash e \otimes f : A \otimes B}\,(\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash e : A \quad x : B, \Delta \vdash f : C}{\Gamma, g : A \multimap B, \Delta \vdash f[(ge)/x] : C}\,(\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash (\lambda x.e) : A \multimap B}\,(\multimap_{\mathcal{R}})$$

Figure 6.2: Term assignment for the sequent calculus

Moreover, the two naturality equations in Figure 6.3 induce the equations in Figure 6.4, which we shall refer to as *commuting equalities*. They assert that $(I_{\mathcal{L}})$ and $(\otimes_{\mathcal{L}})$ commute with various other structural rules at the level of terms.

We further remark that the cut elimination reductions from the previous section translate to the β and naturality equations in Figure 6.3. Most reductions produce terms which are identical (up to α-equivalence), but AR2, SR1 and SR2 produce the β equalities and the symmetries of CR1 and CR2 produce the naturality equations. Therefore our system enjoys cut elimination at the level of terms.

The fact that (6.1)–(6.8) are the necessary and sufficient equalities can be seen in the following two theorems, found in [BBPH92].

Theorem 6.2.1 *(Soundness) Every symmetric monoidal closed category* $\mathbb{C}$ *induces a term assignment system for which equations (6.1)–(6.8) all hold in the sense that interpretations of either term give rise to the same morphism in the category* $\mathbb{C}$*.*

Theorem 6.2.2 *(Completeness) Given a term assignment system satisfying equalities (6.1)–(6.8), there exists a categorical model* $\mathbb{C}$ *such that if* $\Gamma \vdash A$ *is provable, then* $\Gamma \vdash s : A$ *and* $\Gamma \vdash t : A$ *are interpreted as the same morphism* $\Gamma \to A$ *in* $\mathbb{C}$ *precisely when* $s = t : A$ *is provable via equalities (6.1)–(6.8).*

Therefore, our term assignment system on a given set of types can be viewed as an interpretation of the free symmetric monoidal category on a corresponding set of objects.

$$\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ e = e \tag{6.1}$$

$$\mathsf{let}\ u \otimes v\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f = f[u/x, v/y] \tag{6.2}$$

$$(\lambda x.f)e = f[e/x] \tag{6.3}$$

$$\mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ f[*/z] = f[x/z] \tag{6.4}$$

$$\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f[x \otimes y/z] = f[u/z] \tag{6.5}$$

$$\lambda x.fx = f \tag{6.6}$$

$$f[(\mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ e)/w] = \mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ f[e/w] \tag{6.7}$$

$$f[(\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ e)/w] = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f[e/w] \tag{6.8}$$

Figure 6.3: Categorical equalities for term assignment

$$(\mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ f)[e/x] = \mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ f[e/x] \tag{6.9}$$

$$\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ (\mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ e) = \mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ e) \tag{6.10}$$

$$(\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e) \otimes f = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e \otimes f \tag{6.11}$$

$$e \otimes (\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ f) = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e \otimes f \tag{6.12}$$

$$f[(g(\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e))/x] = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ f[(ge)/x] \tag{6.13}$$

$$(\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ f)[(ge)/x] = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ f[(ge)/x] \tag{6.14}$$

$$\lambda x.(\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ e) = \mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ (\lambda x.e) \tag{6.15}$$

$$(\mathsf{let}\ u\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ e)[f/x] = \mathsf{let}\ u\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ e[f/x] \tag{6.16}$$

$$\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ (\mathsf{let}\ v\ \mathsf{be}\ p \otimes q\ \mathsf{in}\ f) = \mathsf{let}\ v\ \mathsf{be}\ p \otimes q\ \mathsf{in}\ (\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f) \tag{6.17}$$

$$(\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ e) \otimes f = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ e \otimes f \tag{6.18}$$

$$e \otimes (\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ \otimes f) = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ e \otimes f \tag{6.19}$$

$$f[g(\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ e)/z] = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f[(ge)/z] \tag{6.20}$$

$$(\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f)[(ge)/z] = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f[(ge)/z] \tag{6.21}$$

$$\lambda z.(\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ e) = \mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ (\lambda z.e) \tag{6.22}$$

Figure 6.4: Commuting equalities for term assignment

6.3 MILL Proof Nets

We anticipate that a proof net for MILL should be a graph which illustrates a correct proof in the sequent calculus for MILL. Moreover, we would like there to be a tight correspondence between proof nets and the term assignment system provided above, so that equivalent terms correspond to proof nets which are equivalent, in some sense yet to be defined.

As with proof nets for MLL, we define a MILL proof net through the intermediary notion of a MILL proof structure. A proof structure for MILL is, roughly speaking, a finite directed graph (digraph) with signed nodes. Nodes are representative of types, with negative nodes, written A^-, corresponding to inputs of data, and positive nodes, written A^+, corresponding to outputs of data.

Definitions 6.3.1 A *digraph* $G = (v(G), e(G))$ is a finite set of nodes $v(G)$ together with a set $e(G) \subseteq v(G) \times v(G)$ of directed edges. (That is, if $(X, Y) \in e(G)$ then $X \to Y$ is an edge in G.) The *size* of G is the number of nodes in $v(G)$.

A *source* of a connected digraph G is a node X for which there is no edge $Y \to X$ in G. A *sink* of a connected digraph G is a node X for which there is no edge $X \to Y$ in G.

A *path of length* n in G is a sequence of distinct nodes $X_1, \ldots, X_{n+1}$ such that $X_i \to X_{i+1}$ is an edge in G, for all $i = 1, \ldots, n$. If $n > 0$, then the path is called *non-trivial*.

Definition 6.3.2 A *signed digraph* is a digraph G equipped with

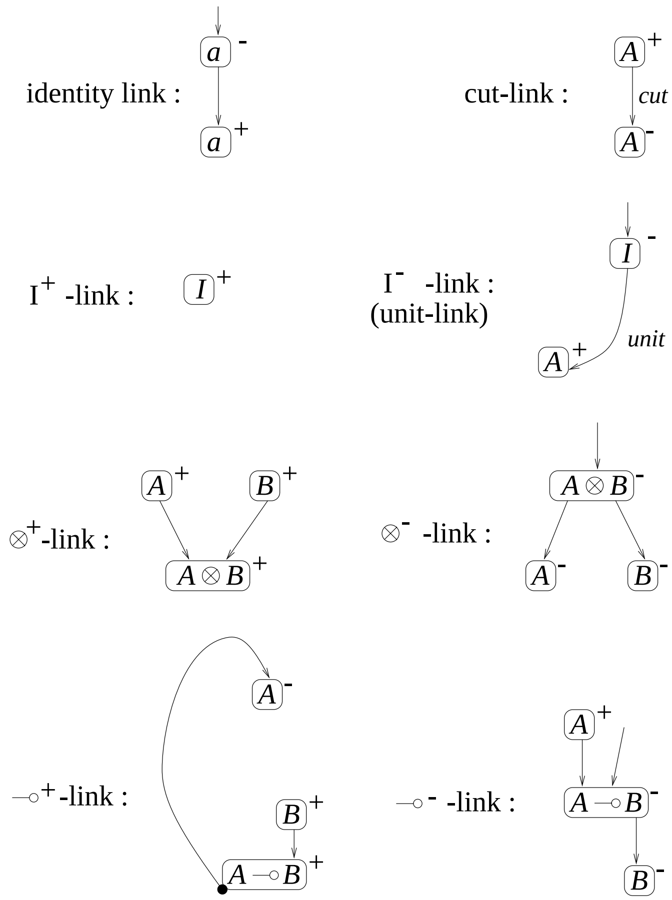
- a signing function $\lambda_G : v(G) \to \{+, -\}$. If $\lambda_G(X) = +$, then we write the node X as X^+ and call it *positive*, and if $\lambda_G(X) = -$, then we write the node X as X^- and call it *negative*.

- a subset $e^\bullet(G) \subseteq e(G)$ of "marked" edges. Graphically, we represent such an edge by placing a bullet $\bullet$ at the head of the arrow. ($\bullet\!\!\longrightarrow$)

Definitions 6.3.3 A *generalised* path in a signed digraph G is a path $X_1, \ldots, X_n$ in G such that $(X_i, X_{i+1}) \notin e^\bullet(G)$ for all $i = 1, \ldots, n-1$. We call $e(G) \setminus e^\bullet(G)$ the set of *generalised edges* of G.

A positive node X^+ is *terminal* in G if it is a sink in the subdigraph $(v(G), e(G) \setminus e^\bullet(G))$.

Definition 6.3.4 A *MILL proof structure* is a connected signed digraph with precisely one terminal node, all nodes labelled with MILL formulae, built from the following links.



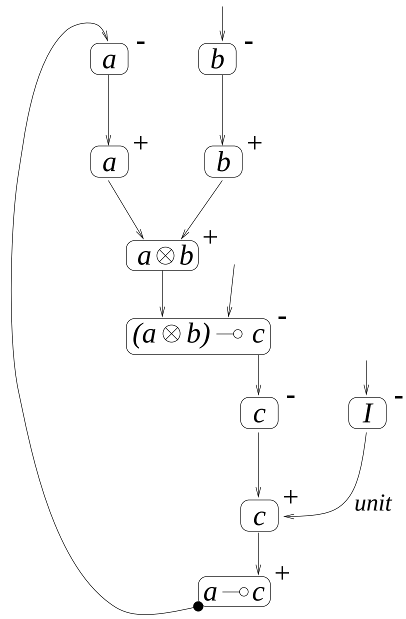
Remark A reference to a node A in a MILL proof structure will be understood to be a reference to a particular *occurrence* of a node of type A. If we wish to speak of a general node of type (or *form*) A, then we will say so explicitly. Therefore, $X \neq A$ means that X and A are distinct nodes, but it is entirely possible that X and A are of the same form.

Definitions 6.3.5 Let G be a MILL proof structure. A negative node X^- is *initial* in G if it is either a source of G or it is a node $(A \multimap B)^-$ for which there is precisely one edge $Y \to (A \multimap B)^-$ in G, namely $A^+ \to (A \multimap B)^-$.

Naturally, a MILL proof structure which contains no cut-link will be called a *cut-free* MILL proof structure.

Example 6.3.6 Even without terms, one can easily see how a MILL proof structure is formed by mimicking the structural rules.

$$\dfrac{\dfrac{\dfrac{a \vdash a \quad b \vdash b}{a, b \vdash a \otimes b}(\otimes_{\mathcal{R}}) \qquad \dfrac{c \vdash c}{c, I \vdash c}(I_{\mathcal{L}})}{a, b, (a \otimes b) \multimap c, I \vdash c}(\multimap_{\mathcal{L}})}{b, (a \otimes b) \multimap c, I \vdash a \multimap c}(\multimap_{\mathcal{R}})$$

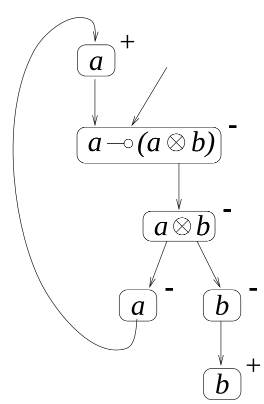


Here, the initial nodes are b^-, $((a \otimes b) \multimap c)^-$ and I^-, and the terminal node is $(a \multimap c)^+$.

We now seek a net criterion for a MILL proof structure G. Recall that we want MILL proof nets to be representative of correct proofs in MILL. There will be two conditions to check. The first will be

- G is generalised acyclic, i.e. G does not admit a cyclic generalised path.

We can see that the first condition is necessary by the following example. The following MILL proof structure has initial node $(a \multimap (a \otimes b))^-$ and terminal node b^+, but the sequent $a \multimap (a \otimes b) \vdash b$ is not provable in MILL.



Assuming that this first condition is imposed on G, we can now prove the following fact.

Proposition 6.3.7 *Let G be a MILL proof structure which is generalised acyclic. Then given any node X in G, there exists a generalised path from X to the terminal node Z^+ in G.*

Proof Let

$$\mathcal{A} = \{X \mid \text{ there is no generalised path from } X \text{ to } Z^+\}$$

and

$$\mathcal{B} = \{X \mid \text{ there exists a generalised path from } X \text{ to } Z^+\}.$$

Then clearly $\mathcal{A}$ and $\mathcal{B}$ are disjoint, and $\mathcal{B}$ is non-empty since $Z^+ \in \mathcal{B}$. Suppose that $\mathcal{A}$ is non-empty. Choose $X_1 \in \mathcal{A}$. Since G contains precisely one terminal node (and it isn't X_1), there exists a generalised edge $X_1 \to X_2$ for some $X_2 \neq X_1$. We cannot have $X_2 \in \mathcal{B}$, for then there would exist a generalised path from X_1 to Z^+, so $X_2 \in \mathcal{A}$. We can now repeat this argument with X_1 replaced by X_2. So there exists a generalised edge $X_2 \to X_3$ some node $X_3 \in \mathcal{A}$, and hence a generalised path from X_1 to X_3. At the n-th step, we obtain a node X_{n+1} such that there exists a generalised path from X_k to X_{n+1} for all $k = 1, \ldots, n$. We cannot continue this process indefinitely because G has only a finite number of nodes, so eventually we obtain a node X_{n+1} such that $X_{n+1} = X_m$ for some $m \leq n$. But then we obtain a generalised cycle containing the node X_m, which is a contradiction. Therefore $\mathcal{A}$ must be empty. ■

Corollary 6.3.8 *Let G be a MILL proof structure which is generalised acyclic. Then G is generalised connected.*

We now define a relation $\geq$ on a MILL proof structure G which is generalised acyclic. Given two nodes X and Y in G, we write $X \geq Y$ if and only if every generalised path $X, \ldots, Z^+$ in G passes through Y. We write $X > Y$ if $X \geq Y$ but $X \neq Y$.

Proposition 6.3.9 *The relation $\geq$ is reflexive and transitive.*

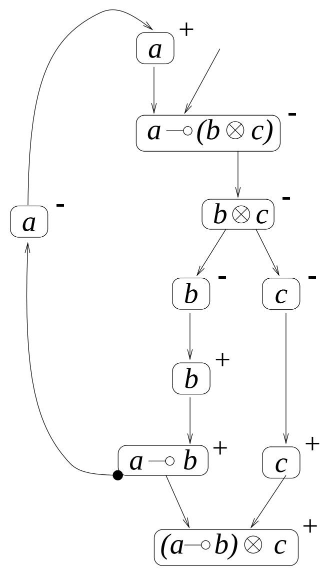
Proof Reflexivity is immediate from the definition. Suppose that $W \geq X$ and $X \geq Y$. If one of these relations is an equality, then we immediately have $W \geq Y$. So assume that $W > X$ and $X > Y$. Then every generalised path from W to Z^+ passes through X, and any generalised path from X to Z^+ passes through Y. Therefore, any generalised path from W to Z^+ passes through Y. So $W \geq Y$. ■

We are now ready to define the second correctness condition for a MILL proof net.

- Whenever there exists a $\multimap^+$-node between A^-, B^+ and $(A \multimap B)^+$, we have $A^- \geq B^+$.

We can see that this second condition is also necessary by the following example. The following MILL proof structure has initial node $(a \multimap (b \otimes c))^-$ and terminal node

$((a \multimap b) \otimes c)^+$ but the sequent $a \multimap (b \otimes c) \vdash (a \multimap b) \otimes c$ is not provable in MILL.



Aside. As far as I can tell upon a preliminary reading, this second net condition on $\multimap^+$-nodes was not mentioned in Lamarche's report [Lam94], and therefore further investigation is required to determine the connection between his net criterion in its current form, and the one provided above.

Definition 6.3.10 A *MILL proof net* G is a MILL proof structure such that

- G is generalised acyclic, i.e. G does not admit a cyclic generalised path;
- Whenever there exists a $\multimap^+$-node between A^-, B^+ and $(A \multimap B)^+$, we have $A^- \geq B^+$.

Observe that the MILL proof structure in the Example 6.3.6 above is indeed a MILL proof net (as one would hope, since the proof was correct!)

Throughout this chapter, we will often make use of the following facts.

Proposition 6.3.11 *Let $\mathcal{N}$ be a MILL proof net with terminal node Z^+. Suppose that $X \geq W_1$ and $X \geq W_2$. Then either $W_1 \geq W_2$ or $W_2 \geq W_1$.*

Proof Let γ be a generalised path from W_1 to Z^+. Then γ is a fragment of some generalised path $\delta = X, \ldots, \gamma$ from X to Z^+. Since $X \geq W_2$, δ must pass through W_2. So either

$$\delta = X, \ldots, W_1, \ldots, W_2, \ldots, Z^+$$

or

$$\delta = X, \ldots, W_2, \ldots, W_1, \ldots, Z^+.$$

In the former case, it is now impossible for there to exist a generalised path from W_2 to W_1 for then there would exist a cycle containing W_1 and W_2, so *all* generalised paths from W_1 to Z^+ must pass through W_2, i.e. $W_1 \geq W_2$.

In the latter case, we now know that it is impossible for there to exist a generalised path from W_1 to W_2, so by the same argument applied to W_2, *any* generalised path from W_2 to Z^+ must pass through W_1, i.e. $W_2 \geq W_1$. ■

Lemma 6.3.12 *Let $\mathcal{N}$ be a MILL proof net with terminal node Z^+. Then the reflexive transitive relation $\geq$ on the nodes of $\mathcal{N}$ defines a tree.*

Proof Define the graph T with vertices labelled by the nodes of $\mathcal{N}$ such that X and Y form an edge in T if $X \geq Y$ and there is no W such that $X \geq W \geq Y$.

Fix the node X. By Proposition 6.3.11, the set of nodes $\{Y \mid X \geq Y\}$ is well-ordered by $\geq$, thus forming a unique path $X \geq Y_1 \geq \cdots \geq Y_n = Z^+$ in T. Therefore T is a tree with root Z^+. ■

6.3.1 Translation from terms to proof nets

We now demonstrate the translation from terms to proof nets. This direction is straightforward. The reverse translation, however, requires further work. In particular, we will discuss the notion of an "empire", thus enabling us to form an inductive procedure for the reverse translation.

Theorem 6.3.13 *There is a translation procedure Net which takes a proof of the term $x_1 : A_1, \ldots, x_n : A_n \overset{\pi}{\vdash} f : Z$ to a MILL proof net $Net(\pi)$ with initial nodes $A_1^-, \ldots, A_n^-$ and terminal node Z^+. Moreover, if π is a cut-free proof, then $Net(\pi)$ is a cut-free MILL proof net.*

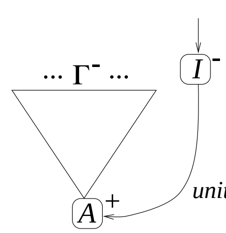
Proof The proof is by induction on the depth of π.

The identity $x : \alpha \vdash x : \alpha$ is taken to the identity link $\alpha^- \to \alpha^+$ (which is a proof net).

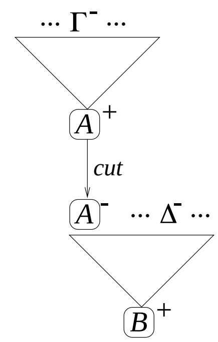
The axiom $\vdash * : I$ is taken to the node I^+ (which is a proof net).

The translation procedure on the structural rules is presented below.

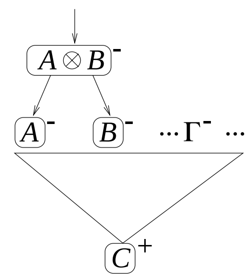
$$\frac{\Gamma \vdash e : A}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\ (I_{\mathcal{L}})$$



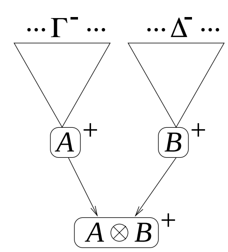
$$\frac{\Gamma \vdash e : A \quad x : A, \Delta \vdash f : B}{\Gamma, \Delta \vdash f[e/x] : B}\,(Cut)$$



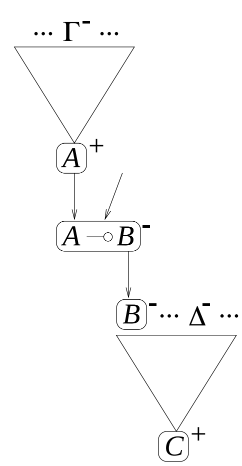
$$\frac{\Gamma, x : A, y : B \vdash f : C}{\Gamma, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : C}\,(\otimes_{\mathcal{L}})$$



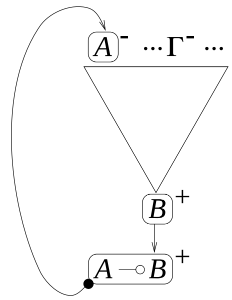
$$\frac{\Gamma \vdash e : A \quad \Delta \vdash f : B}{\Gamma, \Delta \vdash e \otimes f : A \otimes B}\,(\otimes_{\mathcal{R}})$$



$$\frac{\Gamma \vdash e : A \quad x : B, \Delta \vdash f : C}{\Gamma, g : A \multimap B, \Delta \vdash f[(ge)/x] : C}\,(\multimap_{\mathcal{L}})$$



$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.e : A \multimap B} (\multimap_{\mathcal{R}})$$



The structures constructed by these rules will be MILL proof nets if the triangular frames are themselves MILL proof nets. It is easily seen that generalised acyclicity is preserved. In the final case, where we create a $\multimap^+$-link, we are assuming that the triangular frame is a MILL proof net with an initial node A^- and terminal node B^+, so we immediately have $A^- \geq B^+$.

It is also evident that a cut-link only arises from a (Cut) rule. Thus if the proof is cut-free, so will be the MILL proof net. This completes the proof. ■

6.3.2 Empires

This subsection is purely combinatorial. We stress once again that a reference to a node A is a reference to *an occurrence of* a node of type A. If we wish to speak of a general node of type (or *form*) A, we will say so explicitly. Therefore, $X \neq A$ means that X and A are distinct nodes, but it is entirely possible that they are of the same form.

Definition 6.3.14 Define the *level value* $l(X) = l(X, \mathcal{N})$ of a node X in $\mathcal{N}$ to be the maximum length of all generalised paths from X to the terminal node of $\mathcal{N}$.

Thus $l(X) = 0$ if and only if X is the terminal node of $\mathcal{N}$.

Definition 6.3.15 Let $\mathcal{N}$ be a MILL proof net. The *empire* $\hat{e}(A^-) = \hat{e}(A^-, \mathcal{N})$ *of the negative node* A^- *in* $\mathcal{N}$ is the largest subnet of $\mathcal{N}$ with A^- as an initial node. The *empire* $\hat{e}(A^+) = \hat{e}(A^+, \mathcal{N})$ *of the positive node* A^+ *in* $\mathcal{N}$ is the largest subnet of $\mathcal{N}$ with A^+ as its terminal node.

By abuse of notation, we will refer to the nodes of a proof net as the proof net itself. In context, it should be clear to which we are referring.

Providing an eloquent description of $\hat{e}(A^+)$ is easy, but for $\hat{e}(A^-)$, the task is non-trivial. We characterise three sets of nodes related to A^-. The first two sets consist of nodes which sit "above" A^-, while the third set consists of selected nodes which sit "below" A^-.

If A^- is not a $\multimap^-$-node, we will say that a node X is *ill-above* A^- if there exists a non-trivial generalised path $X, \ldots, A^-$.

If A^- is a $\multimap^-$-node, say $A^- = (U \multimap V)^-$, we will say that a node X is *ill-above* $(U \multimap V)^-$ if there exists a non-trivial generalised path $X, \ldots, W, (U \multimap V)^-$ such that $W \neq U^+$.

Intuitively, nodes ill-above A^- are those which prevent A^- from being an initial node. Denote by $Ill(A^-)$ the set of nodes which are ill-above A^-.

We will also say that a node X is *well-above* $(U \multimap V)^-$ if $X \geq U^+$. Denote by $Well((U \multimap V)^-)$ the set of nodes which are well-above $(U \multimap V)^-$. If A^- is not a $\multimap^-$-node, we set $Well(A^-) = \varnothing$.

Finally, we define

$$Bel(A^-) = \{(X \multimap W)^+ \mid \text{there is a generalised path from } A^- \text{ to } W^+ \text{ and } X^- \in Ill(A^-)\}.$$

We will now show that when $Bel(A^-)$ is non-empty, it can be well-ordered by $\geq$.

Proposition 6.3.16 *Suppose that $(X_1 \multimap W_1)^+$ and $(X_2 \multimap W_2)^+$ are distinct nodes in $Bel(A^-)$. Then either $(X_1 \multimap W_1)^+ \geq (X_2 \multimap W_2)^+$ or $(X_2 \multimap W_2)^+ \geq (X_1 \multimap W_1)^+$. Consequently, there exists a unique $(X \multimap W)^+ \in Bel(A^-)$ of maximal level value.*

Proof We equivalently prove that either $W_1^+ \geq W_2^+$ or $W_2^+ \geq W_1^+$. Given any node $(X \multimap W)^+ \in Bel(A^-)$, we have $X^- \geq W^+$, and the existence of generalised paths from X^- to A^- and from A^- to W^+. Thus any generalised path from A^- to Z^+ is a fragment of a generalised path from X^- to Z^+, and must therefore pass through W^+, i.e. $A^- \geq W^+$. By Proposition 6.3.11, we now have either $W_1^+ \geq W_2^+$ or $W_2^+ \geq W_1^+$. ■

Theorem 6.3.17 *Let $\mathcal{N}$ be a MILL proof net with terminal node Z^+, and let A be a node in $\mathcal{N}$.*

(a) $\hat{e}(A)$ is the smallest substructure $\mathcal{A}$ closed under the following conditions.

E1. $A \in \hat{e}(A)$.

E2. If there exists an identity link between α^- and α^+, then if one of these nodes belongs to $\hat{e}(A)$, then so does the other node.

E3. If there exists a cut link between X^+ and X^-, then
if $X^+ \in \hat{e}(A)$, $X^+ \neq A$, then $X^- \in \hat{e}(A)$;
if $X^- \in \hat{e}(A)$, $X^- \neq A$, then $X^+ \in \hat{e}(A)$.

E4. If there exists a unit link between I^- and X^+, then if one of these nodes belongs to $\hat{e}(A)$, then so does the other node.

E5. If there exists a $\otimes^-$*-link between* $(X \otimes Y)^-$, X^- *and* Y^-, *then*

if $(X \otimes Y)^- \in \hat{e}(A)$ *then* $X^-, Y^- \in \hat{e}(A)$;

if $X^-, Y^- \in \hat{e}(A), X^- \neq A \neq Y^-$, *then* $(X \otimes Y)^- \in \hat{e}(A)$.

E6. If there exists a $\otimes^+$*-link between* X^+, Y^+ *and* $(X \otimes Y)^+$, *then*

if $(X \otimes Y)^+ \in \hat{e}(A)$ *then* $X^+, Y^+ \in \hat{e}(A)$;

if $X^+ \in \hat{e}(A), X^+ \neq A$, *then* $(X \otimes Y)^+ \in \hat{e}(A)$;

if $Y^+ \in \hat{e}(A), Y^+ \neq A$, *then* $(X \otimes Y)^+ \in \hat{e}(A)$.

E7. If there exists a $\multimap^-$*-link between* X^+, $(X \multimap Y)^-$ *and* Y^-, *then*

if $(X \multimap Y)^- \in \hat{e}(A)$ *then* $X^+, Y^- \in \hat{e}(A)$;

if $X^+ \in \hat{e}(A), X^+ \neq A$, *then* $(X \multimap Y)^- \in \hat{e}(A)$;

if $Y^- \in \hat{e}(A), Y^- \neq A$, *then* $(X \multimap Y)^- \in \hat{e}(A)$.

E8. If there exists a $\multimap^+$*-link between* Y^+, $(X \multimap Y)^+$ *and* X^-, *then*

if $(X \multimap Y)^+ \in \hat{e}(A)$ *then* $X^-, Y^+ \in \hat{e}(A)$;

if $X^-, Y^+ \in \hat{e}(A), X^- \neq A \neq Y^+$, *then* $(X \multimap Y)^+ \in \hat{e}(A)$.

(b) Suppose that A *is positive. Then* $\hat{e}(A^+)$ *is the largest substructure* $\mathcal{B}$ *containing all nodes* X *such that* $X \geq A^+$.

(c) Suppose that A *is negative. If* $Bel(A^-)$ *is empty, put* $(Z')^+ = Z^+$. *If* $Bel(A^-)$ *is non-empty, choose the unique* $(X \multimap W)^+ \in Bel(A^-)$ *of maximal level value, and put* $(Z')^+ = W^+$. *Then* $\hat{e}(A^-)$ *is the substructure* $\mathcal{C} = \hat{e}((Z')^+) \setminus Ill(A^-)$.

Proof Denote by Z^+ the terminal node of $\mathcal{N}$. Suppose firstly that A is positive.

1. We first prove that $\mathcal{A} \subseteq \mathcal{B}$, i.e. $\mathcal{B}$ is closed under conditions E1–E8 (with $\hat{e}(A)$ replaced by $\mathcal{B}$).

   Since $A^+ \geq A^+$, E1 holds.

   Suppose that $\alpha^- \to \alpha^+$ is an identity link. We always have $\alpha^- \geq \alpha^+$ with no distinct X such that $\alpha^- \geq X \geq \alpha^+$, so $\alpha^+ \geq A^+$ if and only if $\alpha^- \geq A^+$. Thus $\mathcal{B}$ is closed under E2.

   In much the same way, we can verify that E3–E8 hold. Therefore, $\mathcal{A} \subseteq \mathcal{B}$ when A is positive.

2. $(\mathcal{B} \subseteq \mathcal{A})$ Put $p = l(A^+)$. We prove that

$$X \geq A^+ \Rightarrow X \in \mathcal{A} \tag{6.23}$$

   by induction on the level value n of X. (Note that $n \geq p$.)

The result is true when $n = p$, since both $A^+ \geq A^+$ and $A^+ \in \mathcal{A}$ by E1. Suppose it holds true for all nodes with level value $p, \ldots, p+k$, and suppose that $Y \geq A^+$, $l(Y) = p + k + 1$. We have two cases.

If Y is the negative node $(X_1 \otimes X_2)^-$, then $l(Y) = max(l(X_1), l(X_2)) + 1$, so $l(Y) > l(X_i)$ for $i = 1, 2$. Since $Y \geq A^+$, this implies that $X_i^- \geq A^+$ for both i. By the induction hypothesis, $X_i \in \mathcal{A}$ for both i. By E5, $Y \in \mathcal{A}$.

If Y is not a $\otimes^-$ node, then there is only one generalised edge $Y \to X$. (For example, Y could form part of an identity link, a cut link, a $\multimap^+$-link, a $\otimes^+$-link etc.) Then $Y \geq A^+$ implies that $X \geq A^+$. Note that $l(Y) = l(X) + 1$, so in particular, the induction hypothesis applied to X implies that $X \in \mathcal{A}$. By conditions E1–E8, we deduce that $Y \in \mathcal{A}$.

Therefore (6.23) holds for all nodes $X \geq A^+$ by induction. Hence $\mathcal{B} \subseteq \mathcal{A}$, and combining with 1., we have equality.

3. Finally, we prove that $\mathcal{A}$ is indeed the empire of A^+. It is clear that $\mathcal{B} = \{X \mid X \geq A^+\}$ is a MILL proof net with terminal node A^+. It is generalised acyclic because $\mathcal{N}$ is generalised acyclic. If $(X \multimap Y)^+ \in \mathcal{B} = \mathcal{A}$ then $X^-, Y^+ \in \mathcal{A}$ by E7. Since $X^- \geq Y^+$ in $\mathcal{N}$, we have $X^- \geq Y^+$ in $\mathcal{B}$.

   Furthermore, if $X \notin \mathcal{B}$ then there exists a generalised path from X to Z^+ which does not pass through A^+, so it cannot belong to a subnet with terminal node A^+. (See Proposition 6.3.7.) Therefore $\mathcal{A} = \mathcal{B} = \hat{e}(A^+)$.

Now suppose that A is negative.

4. We show that $\mathcal{A} \subseteq \mathcal{C}$, i.e. $\mathcal{C}$ is closed under conditions E1–E8 (with $\hat{e}(A^-)$ replaced by $\mathcal{C}$).

   Certainly $A^- \geq (Z')^+$ and $A^- \notin Ill(A^-)$, so $A^- \in \mathcal{C}$. Therefore $\mathcal{C}$ is closed under E1.

   Suppose there exists an identity link $\alpha^- \to \alpha^+$. Then $\alpha^+ \geq (Z')^+$ if and only if $\alpha^- \geq (Z')^+$ and $\alpha^+ \notin Ill(A^-)$ if and only if $\alpha^- \notin Ill(A^-)$. So $\alpha^+ \in \mathcal{C}$ if and only if $\alpha^- \in \mathcal{C}$. Therefore $\mathcal{C}$ is closed under E2.

   In much the same way, we can verify that E3–E8 hold. Therefore $\mathcal{A} \subseteq \mathcal{C}$ when A is negative.

5. We prove that

$$Y \in \mathcal{C} \Rightarrow Y \in \mathcal{A}. \tag{6.24}$$

Certainly $A^- \in \mathcal{C}$ and $A^- \in \mathcal{A}$ by $E1$. If A^- is a $\multimap^-$-node, say $A^- = (U \multimap V)^-$, then $\mathit{Well}(A^-) = \{X \mid X \geq U^+\} = \hat{e}(U^+)$. By E7, $U^+ \in \mathcal{A}$. It is now straightforward to verify that all nodes in $\hat{e}(U^+)$ belong to $\mathcal{A}$ by induction of the level value of the nodes. Hence $\mathit{Well}(A^-) \subseteq \mathcal{A}$.

We can now prove that (6.24) holds for all nodes in $\hat{e}((Z')^+) \setminus (\mathit{Well}(A^-) \cup \mathit{Ill}(A^-))$. In most cases, if X and Y form a generalised edge in this set, and $X \in \mathcal{A}$ then E1–E8 easily imply that $Y \in \mathcal{A}$. The exception is when $Y = (E \multimap X)^+$ is a $\multimap^+$-node and $E^- \in \mathit{Well}(A^-) \cup \mathit{Ill}(A^-)$. But we cannot have $E^- \in \mathit{Ill}(A^-)$ because this would contradict our choice of $(Z')^+$. So $E^- \in \mathit{Well}(A^-) \subseteq \mathcal{A}$, and therefore $(E \multimap X)^+ \in \mathcal{A}$ by E8.

Therefore $\mathcal{C} \subseteq \mathcal{A}$ and combining with 4., we have equality.

6. Finally, we prove that $\mathcal{A}$ is indeed the empire of A^-. It is clear that $\mathcal{C} = \hat{e}((Z')^+) \setminus \mathit{Ill}(A^-)$ is a MILL proof net with an initial node A^-, since $\hat{e}((Z')^+)$ is itself a MILL proof net, and the exclusion of $\mathit{Ill}(A^-)$ frees A^- to be an initial node.

   Suppose that $X \notin \mathcal{C}$. If $X \in \mathit{Ill}(A^-)$ then X cannot belong to any subnet with initial node A^-. If $(Z')^+ \neq Z^+$, and $X \not\geq (Z')^+$ then any subnet including A^- and X must also include the node $(E \multimap Z')^+$ with $E^- \in \mathit{Ill}(A^-)$, so again X cannot belong to any subnet with initial node A^-. Therefore $\mathcal{A} = \mathcal{C} = \hat{e}(A^-)$.

■

We now make some useful observations about empires.

Lemma 6.3.18 *Suppose that $\mathcal{N}$ is a MILL proof net. If $X \notin \hat{e}(A^+)$ then either there is no generalised path from X to A^+, or there exists a generalised path from X to some node $(E \otimes F)^-$ or $(F \otimes E)^-$ such that E^- belongs to $\hat{e}(A^+)$ but F^- does not.*

Proof By Theorem 6.3.17, $\hat{e}(A^+) = \{X \mid X \geq A^+\}$. In a generalised path $Y_0, Y_1, \ldots, Y_k, A^+$ such that $Y_1, \ldots, Y_k \geq A^+$, the only way we can obtain $Y_0 \not\geq A^+$ is if $Y_0 \not\geq Y_1$. From this we deduce that $Y_0 = (Y_1 \otimes W)^-$ or $Y_0 = (W \otimes Y_1)^-$ with $W \not\geq A^+$. I.e. $Y_1^- \in \hat{e}(A^+)$ but $W^- \notin \hat{e}(A^+)$.

So if $X_0 \notin \hat{e}(A^+)$, then either there is no generalised path from X_0 to A^+, or there does exist a generalised path $X_0, X_1, \ldots, X_n, A^+$ with maximal i such that $X_i \not\geq A^+$, i.e. X_i is a $\otimes^-$-node similar to Y_0 above. ■

Lemma 6.3.19 *Suppose that $\mathcal{N}$ is a MILL proof net.*

(i) If there exists a $\otimes^+$-link between C^+, D^+ and $(C \otimes D)^+$, then $\hat{e}(C^+) \cap \hat{e}(D^+) = \varnothing$.

(ii) If there exists a $\multimap^-$-link between A^+, $(A \multimap B)^-$ and B^-, then

$\hat{e}(A^+) \cap \hat{e}(B^-) = \varnothing$.

Proof *(i)* This follows from the fact that $C^+, D^+ \geq (C \otimes D)^+$, with $C^+ \to (C \otimes D)^+$ a distinct edge from $D^+ \to (C \otimes D)^+$. Since $\geq$ defines a tree, $\hat{e}(C^+) = \{X \mid X \geq C^+\}$ and $\hat{e}(D^+) = \{X \mid X \geq D^+\}$ are therefore disjoint.

(ii) If $X \in \hat{e}(A^+)$ then every generalised path from X to Z^+ passes through A^+, and hence through $(A \multimap B)^-$ as well. So $X \notin \hat{e}(B^-)$. ■

Definition 6.3.20 Let $\mathcal{N}$ be a MILL proof net. We say that $\mathcal{N}$ *splits at a cut link* $A^+ \to A^-$ if $\mathcal{N}$ is the (disjoint) union of $\hat{e}(A^+)$ and $\hat{e}(A^-)$.

Suppose that $\mathcal{N}$ has an initial node $(A \multimap B)^-$. We say that $\mathcal{N}$ *splits at* $(A \multimap B)^-$ if $\mathcal{N}$ is the (disjoint) union of $\hat{e}(A^+)$, $\hat{e}(B^-)$ and $\{(A \multimap B)^-\}$.

Suppose that $\mathcal{N}$ has terminal node $(C \otimes D)^+$. We say that $\mathcal{N}$ *splits at* $(C \otimes D)^+$ if $\mathcal{N}$ is the (disjoint) union of $\hat{e}(C^+)$, $\hat{e}(D^+)$ and $\{(C \otimes D)^+\}$.

The following theorem will enable us to complete the translation procedure from MILL proof nets to terms in the sequent calculus.

Theorem 6.3.21 *(Splitting Theorem) Let $\mathcal{N}$ be a MILL proof net satisfying the following criteria.*

- *$\mathcal{N}$ has size greater than 2;*
- *$\mathcal{N}$ does not split at a cut link;*
- *the terminal node of $\mathcal{N}$ is not of the form $(X \multimap Y)^+$;*
- *no initial node of $\mathcal{N}$ is unit-linked to the terminal node;*
- *no initial node is of the form $(X \otimes Y)^-$.*

Then either there exists an initial node $(A \multimap B)^-$ at which $\mathcal{N}$ splits, or the terminal node is of the form $(C \otimes D)^+$, at which $\mathcal{N}$ splits.

Proof By assumption, $\mathcal{N}$ will either have at least one initial $\multimap^-$-node or the terminal node will take the form $(C \otimes D)^+$ (or both).

If $\mathcal{N}$ has at least one initial $\multimap^-$-node, then choose one such initial node $(A \multimap B)^-$ such that $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$ is maximal with respect to inclusion. If the terminal node is of the form $(C \otimes D)^+$ and $\hat{e}(C^+) \cup \hat{e}(D^+) \cup \{(C \otimes D)^+\}$ forms a larger subnet of $\mathcal{N}$ than $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$, then we will split $\mathcal{N}$ at $(C \otimes D)^+$. Otherwise we will split $\mathcal{N}$ at the chosen $(A \multimap B)^-$. Whichever node we have chosen, call this the "splitting node".

We now prove two technical propositions.

Proposition 6.3.22 *Let $\mathcal{N}$ be a MILL proof net satisfying the conditions of the Splitting Theorem and suppose that we have chosen the splitting node $(C \otimes D)^+$. If $(E \otimes F)^-$ is a $\otimes^-$-node such that $E^- \in \hat{e}(C^+)$ but $F^- \notin \hat{e}(C^+)$, then there exists an initial node $(G \multimap H)^-$ such that $(E \otimes F)^- \in \hat{e}(H^-)$.*

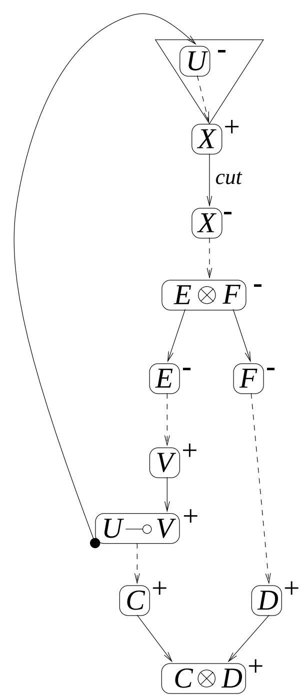
Proof By assumption, no $\otimes^-$-node can be initial, so $(E \otimes F)^-$ is not initial. We prove this result by traversing up the net, starting from the node $X^- = (E \otimes F)^-$. There are three cases to consider.

1. X^- forms part of a $\multimap^-$-link with a node $(Y \multimap X)^-$. If $(Y \multimap X)^-$ is initial, then we are done. Otherwise, repeat this process with X^- replaced by $(Y \multimap X)^-$.

2. X^- forms part of a $\otimes^-$-link with $(X \otimes Y)^-$, say. (The symmetrical case is identical.) By assumption, $(X \otimes Y)^-$ cannot be initial, so repeat this process with X^- replaced by $(X \otimes Y)^-$.

3. X^- forms part of a cut link, $X^+ \to X^-$. This, however, is impossible. By assumption, $\mathcal{N}$ does not split at a cut-link, so there exists a node $(U \multimap V)^+$ such that there exists a generalised path from X^- to V^+ and $U^- \in \hat{e}(X^+)$. Since there exists a generalised path from U^- to $(E \otimes F)^-$, we deduce that $(E \otimes F)^- \geq V^+$. Therefore $E^- \geq V^+$. But we also have $E^- \geq C^+$, so either $V^+ \geq C^+$ or $C^+ \geq V^+$ by Proposition 6.3.11. The latter, however, is impossible since $(C \otimes D)^+$ is the terminal node of $\mathcal{N}$, so there is "no room" for $(U \multimap V)^+$. Therefore $V^+ \geq C^+$ and hence $U^- \geq C^+$.

   But we know that $U^- \not\geq C^+$ because there exists a generalised path from U^- to $(C \otimes D)^+$ which passes through $F^- \notin \hat{e}(C^+)$. This contradicts the second MILL proof net condition.



Since $\mathcal{N}$ has finite size, this process must eventually stop at Case 1, with some initial node $(G \multimap H)^-$ such that there exists a generalised path from H^- to $(E \otimes F)^-$ containing negative nodes only, i.e. $(E \otimes F)^- \in \hat{e}(H^-)$. ■

Proposition 6.3.23 *Let $\mathcal{N}$ be a MILL proof net satisfying the conditions of the Splitting Theorem and suppose that we have chosen the splitting node $(A \multimap B)^-$. If $(E \otimes F)^-$*

is a $\otimes^-$*-node such that* $E^- \in \hat{e}(A^+)$ *but* $F^- \notin \hat{e}(A^+)$*, then there exists an initial node* $(G \multimap H)^-$ *such that* $(E \otimes F)^- \in \hat{e}(H^-)$.

Proof The proof is similar to Proposition 6.3.22, where we traverse up the net starting from the node $X^- = (A \multimap B)^-$. However, Case 3 requires more care.

1. X^- forms part of a $\multimap^-$-link with a node $(Y \multimap X)^-$. If $(Y \multimap X)^-$ is initial, then we are done. Otherwise, repeat this process with X^- replaced by $(Y \multimap X)^-$.

2. X^- forms part of a $\otimes^-$-link with $(X \otimes Y)^-$, say. (The symmetrical case is identical.) By assumption, $(X \otimes Y)^-$ cannot be initial, so repeat this process with X^- replaced by $(X \otimes Y)^-$.

3. X^- forms part of a cut link, $X^+ \to X^-$. By assumption, $\mathcal{N}$ does not split at a cut link, so there exists $(U \multimap V)^+$ such that there exists a generalised path from $X^-, \ldots, V^+$ and $U^- \in \hat{e}(X^+)$ (hence $U^- \in Ill(B^-)$).

   We cannot have $V^+ \in \hat{e}(A^+)$, for otherwise $U^- \geq X^+$ and $U^- \geq V^+$ imply that $X^+ \geq V^+ \geq A^+$. But we know that $X^+ \not\geq A^+$ because there exists a generalised path from X^+ to Z^+ which passes through $F^- \notin \hat{e}(A^+)$.

   From this, we deduce that $U^- \geq V^+$ and the existence of a generalised path from U^- to B^- imply that $B^- \geq V^+$. Therefore $(U \multimap V)^+ \in Bel(B^-)$, and in particular, the terminal node of $\hat{e}(B^-)$, $(Z')^+$, is not the terminal node of $\mathcal{N}$.

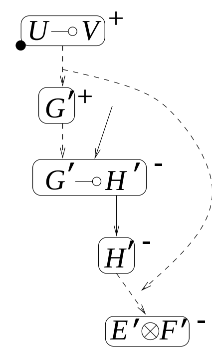
   We also remark that $(Z')^+ \geq V^+$ by definition of $(Z')^+$, and therefore

$$\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\} \subseteq \hat{e}(V^+) \subseteq \hat{e}((U \multimap V)^+).$$

   Since the terminal node of $\mathcal{N}$ is not a $\multimap^+$-node, it is either a $\otimes^+$-node, $(C \otimes D)^+$, or it is atomic, a^+.

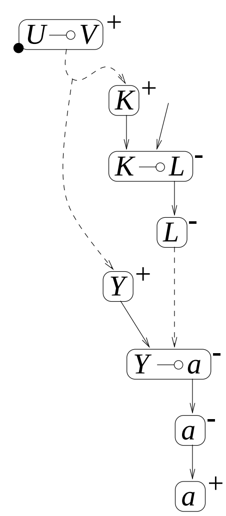
   Suppose $Z^+ = (C \otimes D)^+$. If $\hat{e}((U \multimap V)^+) \subseteq \hat{e}(C^+)$ then this contradicts choosing $(A \multimap B)^-$ as the splitting node, since $\hat{e}(C^+) \cup \hat{e}(D^+) \cup \{(C \otimes D)^+\}$ forms a larger subnet than $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$. We obtain a similar contradiction if $\hat{e}((U \multimap V)^+) \subseteq \hat{e}(D^+)$.

If $\hat{e}((U \multimap V)^+)$ is not contained in either $\hat{e}(C^+)$ or $\hat{e}(D^+)$, then by Lemma 6.3.18, $(U \multimap V)^+$ passes through some node $(E' \otimes F')^-$ such that $E^- \in \hat{e}(C^+)$ but $F^- \notin \hat{e}(C^+)$, say. By Proposition 6.3.22, there exists initial $(G' \multimap H')^-$ such that $(E' \otimes F')^- \in \hat{e}((H')^-)$. Either $(U \multimap V)^+ \geq \hat{e}((G')^+)$, so that $\hat{e}((U \multimap V)^+) \subseteq \hat{e}((G')^+)$, or $(U \multimap V)^+ \not\geq \hat{e}((G')^+)$, in which case the existence of a generalised path from $(U \multimap V)^+$ to $(E \otimes F)^-$ implies that $(U \multimap V)^+ \in \hat{e}((H')^-)$ and hence $\hat{e}((U \multimap V)^+) \subseteq \hat{e}((H')^-)$. Now, we deduce that $\hat{e}((G')^+) \cup \hat{e}((H')^-) \cup \{(G' \multimap H')^-\}$ forms a larger subnet than $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$, which contradicts choosing $(A \multimap B)^-$ as the splitting node. (See figure on the right.)



On the other hand, suppose that $Z^+ = a^+$. Then $\mathcal{N}$ must contain a $\multimap^-$-node $(Y \multimap a)^-$ attached to the identity link $a^- \to a^+$. If $(Y \multimap a)^-$ is initial, then we must have $\hat{e}((U \multimap V)^+) \in \hat{e}(Y^+)$. But then $\hat{e}(Y^+) \cup \hat{e}(a^-) \cup \{(Y \multimap a)^-\}$ forms a larger subnet than $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$, another contradiction. So $(Y \multimap a)^-$ is not initial.

We can now perform another trace up the net from the node $(Y \multimap a)^-$. The difference here is that the problem of a cut-link never arises since there is only an identity link $a^- \to a^+$ beneath the node $(Y \multimap a)^-$. Since $\mathcal{N}$ is finite, the process stops at step 1., with some initial node $(K \multimap L)^-$ such that $(Y \multimap a)^- \in \hat{e}(L^-)$. Either $(U \multimap V)^+ \geq Y^+$, in which case $\hat{e}((U \multimap V)^+) \subseteq \hat{e}(Y^+) \subseteq \hat{e}(L^-)$, or $(U \multimap V)^+ \notin \hat{e}(Y^+)$ in which case there exists a generalised path from $(U \multimap V)^+$ to a^+ which passes through $(Y \multimap a)^-$. Since the generalised path from L^- to $(Y \multimap a)^-$ consists entirely of negative nodes, we must have $(U \multimap V)^+ \geq K^+$, and hence $\hat{e}((U \multimap V)^+) \subseteq \hat{e}(K^+)$. In either case, $\hat{e}(K^+) \cup \hat{e}(L^-) \cup \{(K \multimap L)^-\}$ forms a larger subnet than $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$, another contradiction. (See figure on the right.)



So Case 3 is impossible. Since $\mathcal{N}$ is finite, we must eventually stop at Case 1, with some initial node $(G \multimap H)^-$ such that $(E \otimes F)^- \in \hat{e}(H^-)$.

■

Proof of Splitting Theorem. Suppose we have chosen the splitting node $(C \otimes D)^+$. Suppose there exists a node $X \notin \hat{e}(C^+) \cup \hat{e}(D^+) \cup \{(C \otimes D)^+\}$. Then $X \notin \hat{e}(C^+)$ implies that there exists a generalised path from X to a node $(E \otimes F)^-$ such that $E^- \in \hat{e}(C^+)$ and $F^- \notin \hat{e}(C^+)$, say, by Lemma 6.3.18. By Proposition 6.3.22, there exists an initial node $(G \multimap H)^-$ such that $(E \otimes F)^- \in \hat{e}(H^-)$. But then $(G \multimap H)^-$ is an initial node such that $\hat{e}(G^+) \cup \hat{e}(H^-) \cup \{(G \multimap H)^-\}$ forms a larger subnet than $\hat{e}(C^+) \cup \hat{e}(D^+) \cup \{(C \otimes D)^+\}$. This contradicts choosing $(C \otimes D)^+$ as the splitting node. Therefore $\hat{e}(C^+) \cup \hat{e}(D^+) \cup \{(C \otimes D)^+\}$ must cover the whole proof net.

On the other hand, suppose we have chosen the splitting node $(A \multimap B)^-$. Suppose there exists a node $X \notin \hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$. Then $X \notin \hat{e}(A^+)$ implies that there exists a generalised path from X to the terminal node Z^+ which does not pass through A^+. Also, $X \notin \hat{e}(B^-)$ implies that there exists a generalised path from X to B^- which passes through $(A \multimap B)^-$, and hence through A^+.

By Lemma 6.3.18, there exists a generalised path from X to some node $(E \otimes F)^-$ such that $E^- \in \hat{e}(A^+)$ and $F^- \notin \hat{e}(A^+)$, say. By Proposition 6.3.23, there exists an initial node $(G \multimap H)^-$ such that $(E \otimes F)^- \in \hat{e}(H^-)$. But then $\hat{e}(G^+) \cup \hat{e}(H^-) \cup \{(G \multimap H)^-\}$ forms a larger subnet than $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$. This contradicts choosing $(A \multimap B)^-$ as the splitting node. Therefore $\hat{e}(A^+) \cup \hat{e}(B^-) \cup \{(A \multimap B)^-\}$ covers the whole proof net. ■

6.3.3 Translation from proof nets to terms

Theorem 6.3.24 *Let $\mathcal{N}$ be a MILL proof net with initial nodes $A_1^-, \ldots, A_n^-$ and terminal node Z^+. Then there exists a class $Pf(\mathcal{N})$ of proofs π in the sequent calculus of the form $x_1 : A_1, \ldots, x_n : A_n \vdash f : Z$, such that $Net(\pi) = \mathcal{N}$ for every $\pi \in Pf(\mathcal{N})$. If $\mathcal{N}$ is cut-free, then every proof in $Pf(\mathcal{N})$ is cut-free. Furthermore, all proofs in $Pf(\mathcal{N})$ belong to the same equivalence class of terms.*

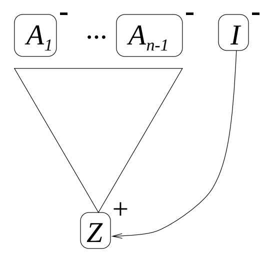
Proof The proof is by induction on the size of the proof net.

Case 0a. If $\mathcal{N}$ is the node I^+, then let π be $\vdash * : I$.

Case 0b. If $\mathcal{N}$ is the identity link $\alpha^- \to \alpha^+$, then let π be $x : \alpha \vdash x : \alpha$.

Now suppose that $\mathcal{N}$ is non-trivial and the theorem holds for all MILL proof nets smaller than $\mathcal{N}$.

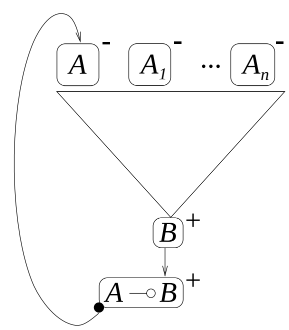
Case 1. If $\mathcal{N}$ has an initial node $A_n = I^-$, say, unit-linked to the terminal node, then remove this link and consider the subnet $\mathcal{N}_1$ with initial nodes $\Gamma^- = A_1^-, \ldots, A_{n-1}^-$ and terminal node Z^+. By the induction hypothesis, there exists a derivation $\Gamma \overset{\pi_1}{\vdash} f : Z$ such that $Net(\pi_1) = \mathcal{N}_1$. Now define π as



$$\frac{\Gamma \overset{\pi_1}{\vdash} f : Z}{\Gamma, x_n : I \vdash \mathsf{let}\ x_n\ \mathsf{be}\ *\ \mathsf{in}\ f : Z}\,(I_{\mathcal{L}}). \tag{6.25}$$

Then $Net(\pi) = \mathcal{N}$.

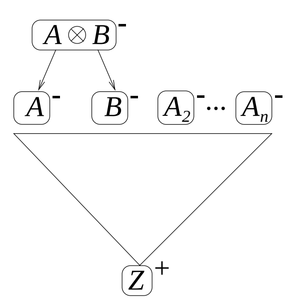
Case 2. We can now assume that no initial node is unit-linked to the terminal node. If the terminal node is a $\multimap^+$-node $(A \multimap B)^+$, then remove this node and consider the subnet $\mathcal{N}_2$ with initial nodes $A^-, A_1^-, \ldots, A_n^-$ and terminal node B^+. By induction, there exists a derivation $x : A, x_1 : A_1, \ldots, x_n : A_n \overset{\pi_2}{\vdash} e : B$ such that $Net(\pi_2) = \mathcal{N}_2$. Now define π as



$$\frac{x : A, x_1 : A_1, \ldots, x_n : A_n \overset{\pi_2}{\vdash} e : B}{x_1 : A_1, \ldots, x_n : A_n \vdash (\lambda x.e) : A \multimap B}\,(\multimap_{\mathcal{R}}). \tag{6.26}$$

Then $Net(\pi) = \mathcal{N}$.

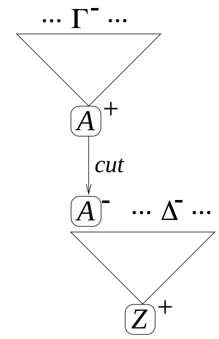
Case 3. We can now assume that no initial node is unit-linked to the terminal node, and the terminal node is not of the form $(X \multimap Y)^+$. If an initial node, A_1^- say, is a $\otimes^-$-node $(A \otimes B)^-$, then remove this node and consider the subnet $\mathcal{N}_3$ with initial nodes $A^-, B^-, \Delta^- = A_2^-, \ldots, A_n^-$ and terminal node Z^+. By induction, there exists a derivation $x : A, y : B, \Delta \overset{\pi_3}{\vdash} f : Z$ such that $Net(\pi_3) = \mathcal{N}_3$. Now define π as



$$\frac{x : A, y : B, \Delta \overset{\pi_3}{\vdash} f : Z}{z : A \otimes B, \Delta \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : Z}\,(\otimes_{\mathcal{L}}). \tag{6.27}$$

Then $Net(\pi) = \mathcal{N}$.

Case 4. We can now assume that no initial node is unit-linked to the terminal node, the terminal node is not of the form $(X \multimap Y)^+$, and there is no initial node of the form $(X \otimes Y)^-$. If $\mathcal{N}$ splits at a cut-link $A^+ \to A^-$, then remove this link and consider the resulting two subnets. By induction, there exist proofs $\Gamma \overset{\pi_4}{\vdash} e : A$ and $x : A, \Delta \overset{\pi_5}{\vdash} f : Z$ such that $Net(\pi_4) = \hat{e}(A^+)$ and $Net(\pi_5) = \hat{e}(A^-)$, with $\Gamma, \Delta = A_1, \ldots, A_n$. Now define π as

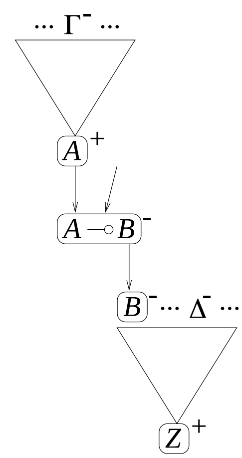


$$\frac{\Gamma \overset{\pi_4}{\vdash} e : A \quad x : A, \Delta \overset{\pi_5}{\vdash} f : Z}{\Gamma, \Delta \vdash f[e/x] : Z}\,(Cut). \tag{6.28}$$

Then $Net(\pi) = \mathcal{N}$.

Case 5. We can now assume that $\mathcal{N}$ is a MILL proof net satisfying the conditions of the Splitting Theorem. Therefore, either $\mathcal{N}$ splits at an initial node $(A \multimap B)^-$ or $\mathcal{N}$ splits at the terminal node $(C \otimes D)^+$.

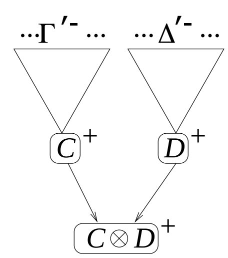
Case 5a. Suppose that $\mathcal{N}$ splits at an initial node $A_i^- = (A \multimap B)^-$. Since $\hat{e}(A^+)$ and $\hat{e}(B^-)$ are themselves MILL proof nets, by induction there exist derivations $\Gamma \overset{\pi_6}{\vdash} e : A$ and $y : B, \Delta \overset{\pi_7}{\vdash} g : Z$ such that $Net(\pi_6) = \hat{e}(A^+)$, $Net(\pi_7) = \hat{e}(B^-)$, and $\Gamma \cup \Delta = \{A_1, \ldots, A_n\} \setminus \{A_i\}$. Now define π as



$$\frac{\Gamma \overset{\pi_6}{\vdash} e : A \quad y : B, \Delta \overset{\pi_7}{\vdash} g : Z}{\Gamma, z : A \multimap B, \Delta \vdash g[ze/y] : Z}\,(\multimap_{\mathcal{L}}). \tag{6.29}$$

Then $Net(\pi) = \mathcal{N}$.

Case 5b. Suppose that $\mathcal{N}$ splits at the terminal node $(C \otimes D)^+$. Since $\hat{e}(C^+)$ and $\hat{e}(D^+)$ are themselves MILL proof nets, by induction there exist derivations $\Gamma' \overset{\pi_8}{\vdash} e : C$ and $\Delta' \overset{\pi_9}{\vdash} f : D$ such that $Net(\pi_8) = \hat{e}(C^+)$, $Net(\pi_9) = \hat{e}(D^+)$, and $\Gamma' \cup \Delta' = \{A_1, \ldots, A_n\}$. Now define π as



$$\frac{\Gamma' \overset{\pi_8}{\vdash} e : C \quad \Delta' \overset{\pi_9}{\vdash} f : D}{\Gamma', \Delta' \vdash e \otimes f : C \otimes D}\,(\otimes_{\mathcal{R}}). \tag{6.30}$$

Then $Net(\pi) = \mathcal{N}$.

The collection $Pf(\mathcal{N})$ arises from the indeterminacy of the translation procedure described. For example, Case 1 and Case 3 are interchangeable, or there may be multiple applications of Case 3. It suffices to consider permuting two cases and looking at their corresponding terms. Some will be identical (up to α-equivalence), while others will be equivalent by the naturality equations (6.8)–(6.10), (6.14), (6.16)–(6.22).

It is evident that an application of (Cut) will only arise if there was a cut-link in $\mathcal{N}$, so we have defined a translation procedure which takes cut-free MILL proof nets to cut-free proofs in the sequent calculus. ■

6.4 Normalisation of MILL proof nets

Given two distinct MILL proof nets with the same initial nodes and terminal node, it is natural to ask the question: are the terms they represent equivalent? We now present a MILL proof net normalisation procedure, for which proof nets with the same normal form represent equivalent terms. Conversely, equivalent terms will give rise to MILL proof nets with the same normal form, thus establishing a bijective correspondence between equivalence classes of terms and equivalence classes of proof nets (i.e. those with the same normal form). There are two issues to consider. The first is rewiring of the unit nodes I^-, the second is cut elimination.

6.4.1 Unit rewirings

Definition 6.4.1 Suppose that $\mathcal{N}$ is a MILL proof net with a node I^- unit-linked to B^+. Suppose that the terminal node of $\hat{e}(I^-,\mathcal{N})$ is $A^+ \neq B^+$. A (single) *unit rewiring* of I^- in $\mathcal{N}$ is the removal of the unit-linkage from I^- to B^+, replacing it with a unit-linkage from I^- to A^+.

Proposition 6.4.2 *Unit rewirings preserve the net criterion.*

Proof It is sufficient to prove the result for a single unit rewiring. Let $\mathcal{N}'$ be the MILL proof net resultant after rewiring the unit I^- in $\mathcal{N}$. Let Y^+ be the terminal node of $\hat{e}(I^-,\mathcal{N}') = \hat{e}(I^-,\mathcal{N})$. Suppose there is a generalised cyclic path in $\mathcal{N}'$. Then it must surely involve the node I^-, so there exists a generalised path from Y^+ to I^- in $\mathcal{N}'$. This is impossible since there was no such generalised path in $\mathcal{N}$. (If there was, there would have been a generalised cycle in $\mathcal{N}$.)

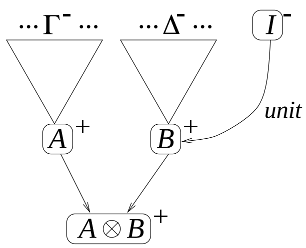
A $\multimap^+$-link from B^+ to $(A \multimap B)^+$ to A^- will only be altered by the unit rewiring if there is a generalised path from A^- to B^+ passing through I^-. In particular, $A^- \in Ill(I^-,\mathcal{N})$, and moreover, $A^- \geq B^+$ implies that $I^- \geq B^+$. So $(A \multimap B)^+ \in Bel(I^-,\mathcal{N})$. In particular, $Y^+ \geq B^+$, so in $\mathcal{N}'$ we still have $I^- \geq B^+$, and hence $A^- \geq B^+$. Therefore $\mathcal{N}'$ is a

MILL proof net. ■

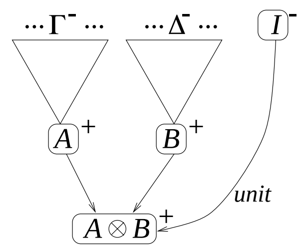
Proposition 6.4.3 *Let $\mathcal{N}$ be a MILL proof net and let $\mathcal{N}'$ be the result of rewiring a single unit. Then $Pf(\mathcal{N})$ and $Pf(\mathcal{N}')$ belong to the same equivalence class of terms.*

Proof W.l.o.g. we can assume that $\mathcal{N} = \hat{e}(I^-)$ where I^- is the unit to be rewired.

Consider the two proof nets



and



corresponding to the commutation of $(I_{\mathcal{L}})$ and $(\otimes_{\mathcal{R}})$. Corresponding terms are

$$(\mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e) \otimes f \quad \text{and} \quad \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ (e \otimes f).$$

which are equivalent by (6.11). The proof is similar for when I^- is attached to A^+, using (6.12). Similarly, one can check the commutation of $(I_{\mathcal{L}})$ with each of $(\multimap_{\mathcal{L}})$, $(\multimap_{\mathcal{R}})$, and (Cut), corresponding to (6.13), (6.15) and (6.7) respectively. ■

This means that unit rewirings in a MILL proof net do not affect the terms which the net interprets. This will be crucial in proof normalisation. But before discussing cut elimination, we will first prove that the process of rewiring all units is strongly normalising and confluent.

Consider a MILL proof net $\mathcal{N}$ with n I^--nodes, labelled 1 to n, or simply denoted $I_1^-, \ldots, I_n^-$. For each i, suppose that I_i^- is unit-linked to B_i^+ and the terminal node of $\hat{e}(I_i^-, \mathcal{N})$ is A_i^+. Define the *unit rank* $u(I_i^-, \mathcal{N})$ of I_i^- in $\mathcal{N}$ by

$$u(I_i^-, \mathcal{N}) = (l(B_i^+), l(A_i^+)).$$

Then we can define the *set* $U(\mathcal{N})$ *of unit ranks* of $\mathcal{N}$ by

$$U(\mathcal{N}) = \{u(I_i^-, \mathcal{N}) \mid i = 1, \ldots, n\}.$$

Observe that if $\mathcal{N}'$ is the MILL proof net resultant from rewiring the unit I^- in $\mathcal{N}$, then $u(I^-, \mathcal{N}') = (l(A^+), l(A^+))$.

The proof of strong normalisation will again call on the standard multiset ordering induced by the partial order $>$ on $\mathbb{N} \times \mathbb{N}$, mentioned earlier.

Proposition 6.4.4 *If $\mathcal{N}'$ is the MILL proof net resultant from rewiring I_i^- in $\mathcal{N}$, then either $u(I_j^-, \mathcal{N}) > u(I_j^-, \mathcal{N}')$ or $u(I_j^-, \mathcal{N}) = u(I_j^-, \mathcal{N}')$ for all $j = 1, \ldots, n$.*

Proof If $j = i$, then $u(I_i^-, \mathcal{N}) = (l(B_i^+), l(A_i^+)) > (l(A_i^+), l(A_i^+)) = u(I_i^-, \mathcal{N}')$. Suppose that $j \neq i$.

1) If $\hat{e}(I_i^-, \mathcal{N})$ and $\hat{e}(I_j^-, \mathcal{N})$ are disjoint, then $\hat{e}(I_j^-, \mathcal{N}) = \hat{e}(I_j^-, \mathcal{N}')$, since I_i^- cannot link to any node in $\hat{e}(I_j^-, \mathcal{N})$. So $u(I_j^-, \mathcal{N}') = u(I_j^-, \mathcal{N})$.

2) Suppose that $\hat{e}(I_i^-, \mathcal{N})$ and $\hat{e}(I_j^-, \mathcal{N})$ are not disjoint. There there exists a node X in $\mathcal{N}$ such that $X \geq A_i^+$ and $X \geq A_j^+$. By Lemma 6.3.11, either $A_i^+ \geq A_j^+$ or $A_j^+ \geq A_i^+$.

   2a) Suppose that $A_i^+ = A^+ = A_j^+$. Then either $A^+ = Z^+$, in which case trivially, $u(I_j^-, \mathcal{N}) = u(I_j^-, \mathcal{N}')$. So now suppose that $A^+ \neq Z^+$. Then A^+ forms a $\multimap^+$-link with a node $(W \multimap A)^+$ such that $W^- \in Ill(I_i^-, \mathcal{N})$, $i = 1, 2$.

   Either $I_i^- \geq I_j^-$ or $I_i^- \not\geq I_j^-$. In the former case, I_i^- is no longer ill-above I_j^- in $\mathcal{N}'$, so $\hat{e}(I_j^-, \mathcal{N}')$ includes the node $(W \multimap A)^+$. Therefore $u(I_j^-, \mathcal{N}) > u(I_j^-, \mathcal{N}')$. In the latter case, there exists a generalised path from W^- to I_j^- which does not pass through I_i^-, so W^- remains ill-above I_j^- in $\mathcal{N}'$, and therefore $u(I_j^-, \mathcal{N}) = u(I_j^-, \mathcal{N}')$.

   2b) Suppose that $A_i^+ \neq A_j^+$. Note that there is no generalised path from I_i^- to I_j^- or vice versa. For suppose that there was a generalised path from I_i^- to I_j^-. Then $Ill(I_i^-, \mathcal{N}) \subseteq Ill(I_j^-, \mathcal{N})$. The node A_i^+ must form a $\multimap^+$-link with a node $(U \multimap A_i)^+$ such that $U^- \in Ill(I_i^-, \mathcal{N})$, hence $U^- \in Ill(I_j^-, \mathcal{N})$. If there exists a generalised path from I_i^- to A_i^+ which passes through I_j^-, then we are back to Case 2a) with $A_i^+ = A_j^+$, a contradiction. Otherwise, there exists a generalised path from $(U \multimap A_i)^+$ to I_j^- in which case we are back to Case 1) with $\hat{e}(I_i^-, \mathcal{N})$ and $\hat{e}(I_j^-, \mathcal{N})$ disjoint, another contradiction.

   In particular, rewiring I_i^- does not alter the set of nodes ill-above I_j^-, and therefore A_j^+ remains as the terminal node of $\hat{e}(I_j^-)$ in $\mathcal{N}'$. Therefore $u(I_j^-, \mathcal{N}) = u(I_j^-, \mathcal{N}')$.

■

Theorem 6.4.5 *The process of unit rewirings is strongly normalising.*

Proof Whenever $\mathcal{N}'$ is the MILL proof net resultant from a single unit rewiring in $\mathcal{N}$, we have $U(\mathcal{N}) \gg U(\mathcal{N}')$ by Proposition 6.4.4. The process terminates when no further rewirings can be performed, i.e. when $U(\mathcal{N}')$ is a set of ordered pairs (x, x). ■

Theorem 6.4.6 *The process of unit rewirings is confluent.*

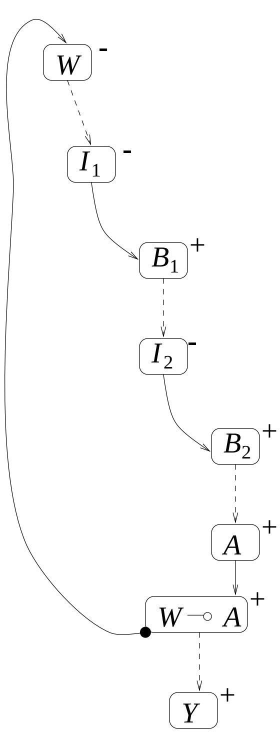
Proof Following strong normalisation, it is sufficient to prove weak confluence, i.e. if $\mathcal{N}_i$ are a MILL proof nets obtained from a single unit rewiring performed I_i^- in $\mathcal{N}$ $(i = 1, 2)$, with $\mathcal{N}_1$ distinct from $\mathcal{N}_2$, then there exists a MILL proof net $\mathcal{N}'$ such that both $\mathcal{N}_1$ and $\mathcal{N}_2$ can both be unit rewired to produce $\mathcal{N}'$.

Let A_i^+ denote the terminal node of $\hat{e}(I_i^-,\mathcal{N})$, and let I_i^- be unit-linked to B_i^+ in $\mathcal{N}$, $i = 1, 2$. Let Z^+ denote the terminal node of $\mathcal{N}$.

1) If $\hat{e}(I_1^-,\mathcal{N})$ and $\hat{e}(I_2^-,\mathcal{N})$ are disjoint, then $\hat{e}(I_2^-,\mathcal{N}_1) = \hat{e}(I_2^-,\mathcal{N})$ (see argument in Proposition 6.4.4), so $u(I_2^-,\mathcal{N}_1) = (l(B_2^+), l(A_2^+))$. Similarly $\hat{e}(I_1^-,\mathcal{N}_2) = \hat{e}(I_1^-,\mathcal{N})$, so $u(I_1^-,\mathcal{N}_2) = (l(B_1^+), l(A_1^+))$. Thus we can rewire I_1^- in $\mathcal{N}_2$ and rewire I_2^- in $\mathcal{N}_1$, and obtain the same MILL proof net $\mathcal{N}'$ with $u(I_i^-,\mathcal{N}') = (l(A_i^+), l(A_i^+))$, $i = 1, 2$.

2) Suppose that $\hat{e}(I_1^-,\mathcal{N})$ and $\hat{e}(I_2^-,\mathcal{N})$ are not disjoint. There there exists a node X in $\mathcal{N}$ such that $X \geq A_1^+$ and $X \geq A_2^+$. By Lemma 6.3.11, either $A_1^+ \geq A_2^+$ or $A_2^+ \geq A_1^+$. W.l.o.g. assume that $A_1^+ \geq A_2^+$. There are two cases to consider.

2a) Suppose that $A_1^+ = A^+ = A_2^+$. Then either $A^+ = Z^+$, in which case the proof is trivial – rewiring I_1^- and I_2^- in either order will produce the same MILL proof net. So now suppose that $A^+ \neq Z^+$. Then A^+ forms a $\multimap^+$-link with a node $(W \multimap A)^+$ such that $W \in Ill(I_i^-,\mathcal{N})$, $i = 1, 2$. It follows that there exists a generalised path from I_1^- to I_2^-, or vice versa, or neither. Suppose that there exists a generalised path from I_1^- to I_2^-. (The argument for the case when there exists a generalised path from I_2^- to I_1^- is symmetrical.) Then $I_1^- \in Ill(I_2^-,\mathcal{N})$, i.e. $I_1^- \notin \hat{e}(I_2^-,\mathcal{N})$.



Initially, we have $u(I_i^-,\mathcal{N}) = (l(B_i^+), l(A^+))$. If we rewire I_1^- to produce $\mathcal{N}_1$, then $\hat{e}(I_2^-,\mathcal{N}_1)$ will now include the node I_1^-, and hence the node $(W \multimap A)^+$. Therefore, the terminal node of $\hat{e}(I_2^-,\mathcal{N}_1)$ is some new node Y^+, i.e. $u(I_2,\mathcal{N}_1) = (l(B_2^+), l(Y^+))$. We can rewire I_2^- in $\mathcal{N}_1$ to produce a MILL proof net $\mathcal{N}'$ with $u(I_1^-,\mathcal{N}') = (l(A^+), l(A^+))$ and $u(I_2^-,\mathcal{N}') = (l(Y^+), l(Y^+))$.

On the other hand, if we rewire I_2^- to produce $\mathcal{N}_2$, then the terminal node of $\hat{e}(I_1^-,\mathcal{N}_2)$ is still A^+. Thus we can rewire I_1^- in $\mathcal{N}_2$ to produce a MILL proof net $\mathcal{N}_3$. Again, $\hat{e}(I_2^-,\mathcal{N}_3)$ will now include the node I_1^-, and hence the nodes A^+ and $(W \multimap A)^+$, so the terminal node of $\hat{e}(I_2^-,\mathcal{N}_3)$ is again Y^+, i.e. $u(I_2^-,\mathcal{N}_3) = (l(A^+), l(Y^+))$. Now rewire I_2^- in $\mathcal{N}_3$ to produce a MILL proof net $\mathcal{N}''$ with $u(I_1^-,\mathcal{N}'') = (l(A^+), l(A^+))$ and $u(I_2^-,\mathcal{N}'') = (l(Y^+), l(Y^+))$. In other words, $\mathcal{N}' = \mathcal{N}''$.

If there is no generalised path between I_1^- and I_2^-, the proof is trivial – rewiring I_1^- and I_2^- in either order will produce the same MILL proof net.

2b) Suppose that $A_1^+ \neq A_2^+$. Then there is no generalised path from I_1^- to I_2^- or vice versa. (See argument in Proposition 6.4.4.) In particular, $I_1^- \notin Ill(I_2^-,\mathcal{N})$, and $I_1^- \geq A_1^+ \geq A_2^+$, so $I_1^- \in \hat{e}(I_2^-,\mathcal{N})$.

Thus, we can rewire I_1^- in $\mathcal{N}$ to produce $\mathcal{N}_1$. Then $\hat{e}(I_2^-,\mathcal{N}_1) = \hat{e}(I_2^-,\mathcal{N})$, and so $u(I_2^-,\mathcal{N}_1) = (l(B_2^+), l(A_2^+))$. Similarly, if we rewire I_2^- in $\mathcal{N}$ to produce $\mathcal{N}_2$, then A_1^+ is still the terminal node of $\hat{e}(I_1^-,\mathcal{N}_2)$, so $u(I_1^-,\mathcal{N}_2) = (l(B_1^+), l(A_1^+))$.

Therefore, we can rewire I_2^- in $\mathcal{N}_1$ and rewire I_1^- in $\mathcal{N}_2$ to produce the same MILL proof net $\mathcal{N}'$ with $u(I_1^-,\mathcal{N}') = (l(A_1^+), l(A_1^+))$ and $u(I_2^-,\mathcal{N}') = (l(A_2^+), l(A_2^+))$. This completes the proof. ■

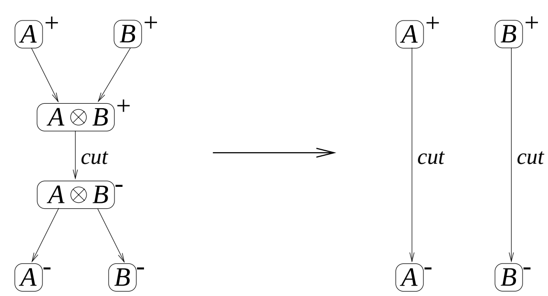
6.4.2 Cut elimination for MILL proof nets

The process of cut elimination will incorporate the process of unit rewirings. In particular, we will perform all possible unit rewirings before and after each cut reduction. As we have shown, unit rewirings correspond to commuting structural rules with $(I_\mathcal{L})$ in a way that does not affect the terms. This removes any difficulties of removing nodes with a unit linked to it. Indeed, if the net has been rewired, a node with a unit linked to it should not be removed at all.

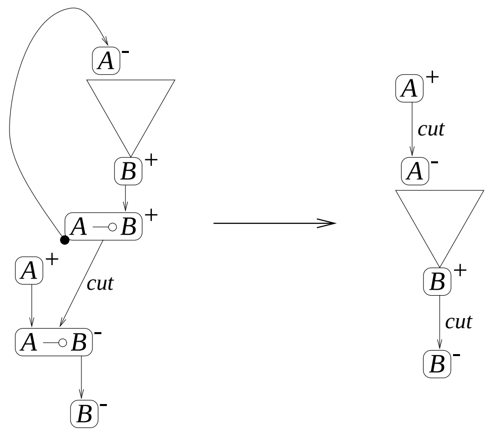
Observe that if I^- is unit-linked to A^+, where A^+ forms a cut-link $A^+ \to A^-$, then the empire of I^- includes the cut-link in question and hence the empire of A^-, so I^- will always be able to slide down the net and is in no danger of being halted at A^+.

The reductions are straightforward.

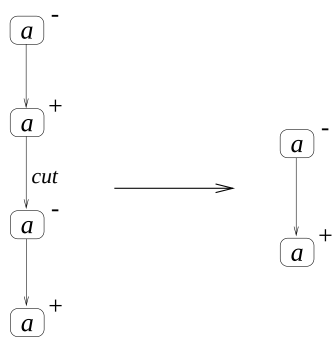
MC1.



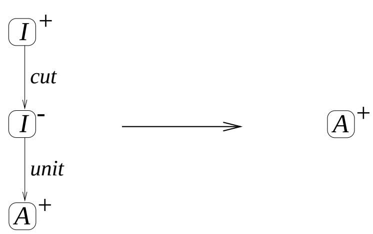
MC2.



SC1.



SC2.



In fact, the only reduction which will induce further unit rewirings is MC2, when there is a unit I^- linked to B^+, with $A^- \in Ill(I^-, \mathcal{N})$. If we apply MC2, then B^+ is no longer the terminal node of the empire of I^-, so I^- will require additional rewirings.

Lemma 6.4.7 *Cut elimination preserves the net criterion.*

Proof With reductions MC1 and MC2, any generalised cycle in the reduced net would have to involve one of the two new edges, $A^+ \to A^-$ or $B^+ \to B^-$. W.l.o.g assume that it is $A^+ \to A^-$. Since there was no generalised path from A^- to A^+ in the original graph, there can be none such in the reduced graph. Therefore the reduced graph is also a MILL proof net. We can use a similar argument for SC1. SC2 is trivial once we ensure that there is no unit linked to the I^+ node in question.

It is clear that reductions SC1 and SC2 will preserve all $\multimap^+$-links, i.e. given a $\multimap^+$-link between Y^+, $(X \multimap Y)^+$ and X^-, the property $X^- \geq Y^+$ will be preserved after performing either reduction. Consider reduction MC1. Any $\multimap^+$-link between Y^+, $(X \multimap Y)^+$ and

X^- must lie entirely above or entirely below the cut-link $(A \otimes B)^+ \to (A \otimes B)^-$, for otherwise we could not guarantee $X^- \geq Y^+$. Therefore all $\multimap^+$-links are preserved after the removing the cut-link.

Finally, consider reduction MC2. Let $\mathcal{N}$ be the original net and let $\mathcal{N}'$ be the reduced net. Note that $A^- \geq B^+$, $B^+ \geq B^-$ and $A^+ \geq B^-$ in $\mathcal{N}$ are preserved in $\mathcal{N}'$. Therefore, given any $\multimap^+$-link between Y^+, $(X \multimap Y)^+$ and X^- (with $(X \multimap Y)^+ \neq (A \multimap B)^+$), the property $X^- \geq Y^+$ in $\mathcal{N}$ will be preserved in $\mathcal{N}'$. For example, if $X^- \geq A^+ \geq B^- \geq Y^+$ in $\mathcal{N}$ then $X^- \geq A^+ \geq B^- \geq Y^+$ in $\mathcal{N}'$, or if $X^- \geq B^+ \geq B^- \geq Y^+$ in $\mathcal{N}$ then $X^- \geq B^+ \geq B^- \geq Y^+$ in $\mathcal{N}'$ etc. ∎

Theorem 6.4.8 *Cut elimination, incorporating unit rewirings, for MILL proof nets is strongly normalising.*

Proof The process of unit rewirings terminates, by Theorem 6.4.5.

Suppose that $\mathcal{N}$ has n cut links, labelled 1 to n. If the ith cut link is $A_i^+ \to A_i^-$ then put $c_i = |A_i|$. Define the *set of cut values of* $\mathcal{N}$ by

$$C(\mathcal{N}) = \{c_i \mid i = 1, \dots, n\}.$$

(Note that this set is left unaltered by unit rewirings.) Then if one applies MC1 or MC2 to the ith cut link in $\mathcal{N}$, then $c_i = |A| + |B| + 1$ is replaced by $|A|$ and $|B|$. If one applies SC1 or SC2 to the ith cut link, then c_i is deleted.

Using the usual well-founded partial order $>$ on $\mathbb{N}$, we obtain the standard well-founded multiset ordering $\gg$ on sets $C(\mathcal{N})$ whereby $C(\mathcal{N}) \underset{\neq}{\gg} C(\mathcal{N}')$ whenever $\mathcal{N}'$ is the result of performing unit rewirings followed by a single cut elimination step.

Finally, we remark that unit rewirings do not create any additional cut-links. Therefore this reduction process terminates. ∎

Theorem 6.4.9 *Cut elimination, incorporating unit rewirings, for MILL proof nets is confluent.*

Proof It is again sufficient to prove weak confluence. That is, if $\mathcal{N}_1$ and $\mathcal{N}_2$ are distinct MILL proof nets resultant from two cut eliminations, then they can both be reduced to a MILL proof net $\mathcal{N}'$. In the case when neither $\mathcal{N}_1$ nor $\mathcal{N}_2$ involve further unit rewirings, this is immediate. Two such cut links will never interfere with each other, so performing the two reductions in either order will give us the same proof net.

We need only concern ourselves with the case when one of the two reductions is MC2, with I^- linked to B^+ and $A^- \in Ill(I^-)$, for now we may have to perform unit rewirings inbetween each cut reduction. It is straightforward to show that performing either reduction followed by unit rewirings (so that we perform unit rewirings twice) will produce the same net. We provide an example in Figure 6.5. ■

Turbo cut elimination

We can speed up cut elimination by performing several reductions all in one step.

Lemma 6.4.10 *Let A be a formula built from the atomic formulae $a_1, \ldots, a_n$ $(n > 1)$. (e.g. $A = (a_1 \otimes \cdots \otimes a_{n-1}) \multimap a_n$.) Then*

- *a MILL proof net with terminal node A^+ contains identity links in bijective correspondence with all atomic subformulae of A, i.e. there exist identity links $a_i^- \to a_i^+$ for all i;*
- *a MILL proof net with initial node A^- contains identity links in bijective correspondence with all atomic subformulae of A.*

Proof We prove both results simultaneously by induction on the rank of the formula A.

The result is obvious when A is atomic. A proof net with terminal node a^+ must be the identity link $a^- \to a^+$. Any initial node a^- (in fact any node a^-) must be attached to an identity link $a^- \to a^+$.

Suppose that $A = A_1 \otimes A_2$, and suppose that $\mathcal{N}$ is a MILL proof net with terminal node $(A_1 \otimes A_2)^+$. By induction, any proof net with conclusion A_i contains identity links in bijective correspondence with all atomic subformulae of A_i, $i = 1, 2$. Since $\hat{e}(A_1^+)$ and $\hat{e}(A_2^+)$ are disjoint, we know that $\mathcal{N}$ contains identity links in bijective correspondence with all atomic subformulae of $A_1 \otimes A_2$. The proof for a MILL proof net with initial node $(A_1 \otimes A_2)^-$ is similar.

Now suppose that $A = A_1 \multimap A_2$, and suppose that $\mathcal{N}$ is a MILL proof net with terminal node $(A_1 \multimap A_2)^+$. Then we can remove this $\multimap^+$-link and recover the subnet with initial node A_1^- and terminal node A_2^+. By induction, such a net will contain both identity links in bijective correspondence with all atomic subformulae of A_1, and identity links in bijective correspondence with all atomic subformulae of A_2. Therefore, $\mathcal{N}$ contains identity links in bijective correspondence with all atomic subformulae of $A_1 \multimap A_2$. The proof is similar for a MILL proof net with initial node $(A_1 \multimap A_2)^-$. ■

Lemma 6.4.10 shows that a MILL proof net with a cut-link $A^+ \to A^-$ will contain *two* sets of identity links in bijective correspondence with the atomic subformulae of A – one set of links lying in $\hat{e}(A^+)$, the other set of links lying in $\hat{e}(A^-)$. Therefore, the sets

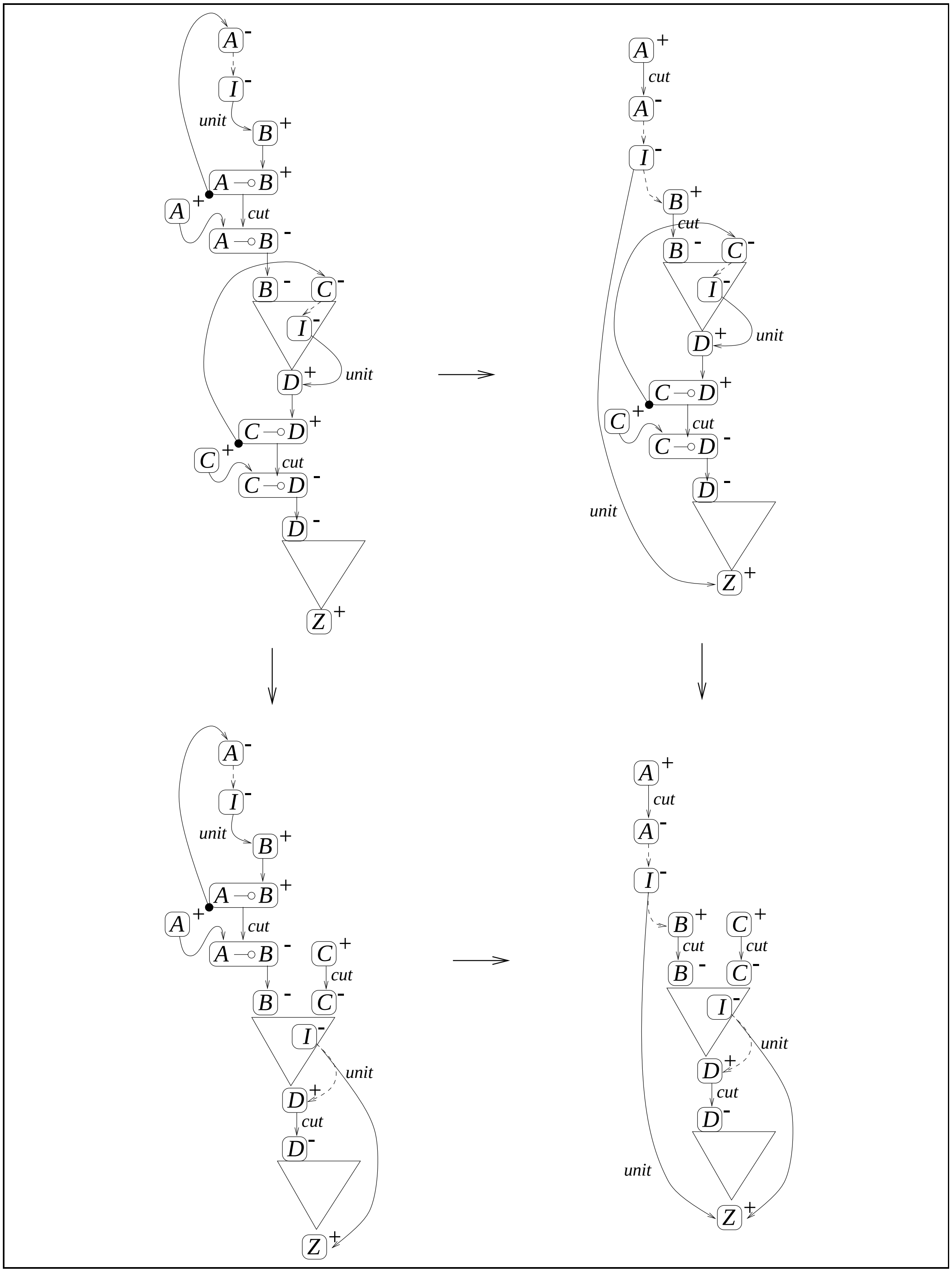
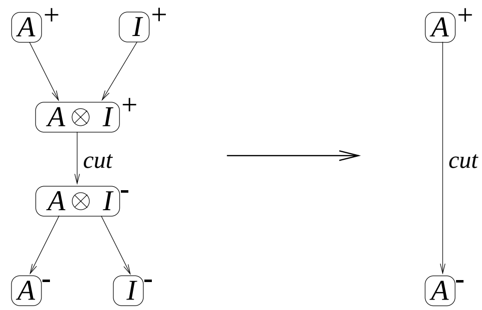


Figure 6.5: Two cut reductions incorporating unit rewirings.

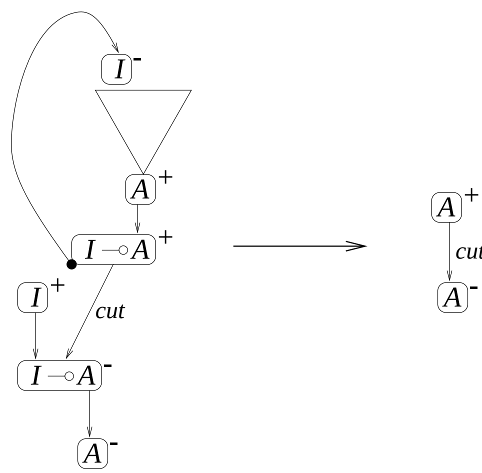
of identity links are themselves in bijective correspondence. We can simply *overlap* these links, and remove all other nodes that built A^+ and A^-.

When dealing with formulae containing occurrences of I, we can use the following reductions to eliminate the unit nodes, and then apply turbo cut elimination on the remaining proof net.

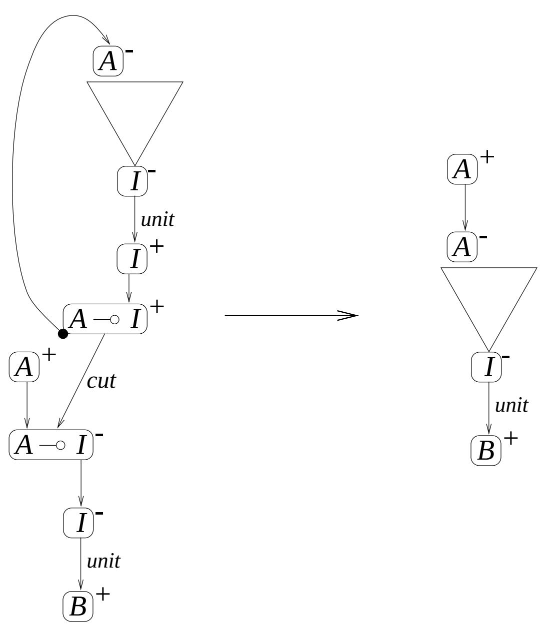
$A \otimes I$:



$I \multimap A$:



$A \multimap I$ $(A \neq I)$:



Corollary 6.4.11 *(Global rewriting) Turbo cut elimination, incorporating unit rewirings, for MILL proof nets is strongly normalising and confluent.*

Proof Immediate from Theorems 6.4.8 and 6.4.9. ■

Proposition 6.4.12 *Let $\mathcal{N}'$ be the result of a single cut link elimination applied to $\mathcal{N}$. Then $Pf(\mathcal{N})$ and $Pf(\mathcal{N}')$ belong to the same equivalence class of terms.*

Proof W.l.o.g. assume that $\mathcal{N}$ splits at the cut link in question. If we apply MC1, then the corresponding terms are equivalent by equality (6.2). If we apply MC2, then the corresponding terms are equivalent by (6.3). SC1 follows from the fact that

$$x : \alpha, \Delta \vdash f[x/z] : B$$

is α-equivalent to $z : \alpha, \Delta \vdash f : B$. SC2 follows from (6.1). ■

Definition 6.4.13 A MILL proof net is in *(rewired, cut-free)* normal form when there are no possible unit rewirings and it is cut-free.

By Theorem 6.4.9, all MILL proof nets have a *unique* normal form, and therefore we can speak of *the* normal form of a proof net.

Definition 6.4.14 Two MILL proof nets are *equivalent* if their normal forms are identical.

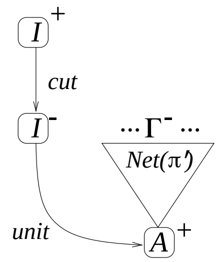
Theorem 6.4.15 *Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent proof nets. Then $Pf(\mathcal{N}_1)$ and $Pf(\mathcal{N}_2)$ belong to the same equivalence class of terms.*

Proof It suffices to prove the result when $\mathcal{N}_2$ is the normal form of $\mathcal{N}_1$. The result now follows from Propositions 6.4.3 and 6.4.12. ■

Theorem 6.4.16 *Let π_1 and π_2 be equivalent proofs of $\Gamma \vdash t : A$. Then $Net(\pi_1)$ and $Net(\pi_2)$ are equivalent MILL proof nets.*

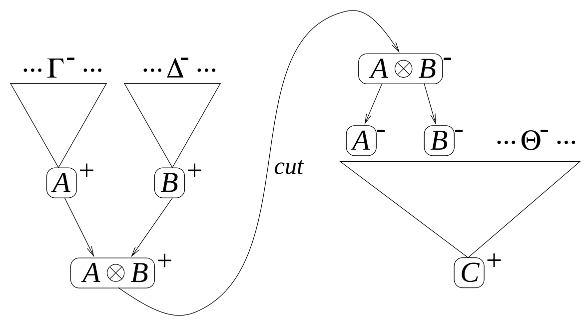
Proof It suffices to check that proof nets induced by β-equations (6.1)–(6.3) and naturality equations (6.7), (6.8) have the same normal forms.

Equation (6.1) arises from AR2. Let the left hand proof be π. Then $Net(\pi)$ looks like



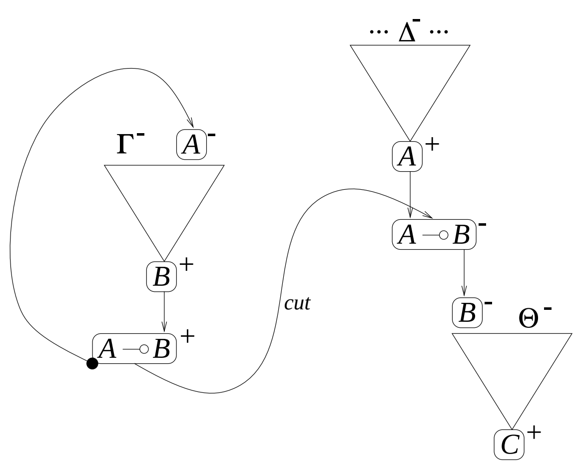
which reduces to $Net(\pi')$ by SC2. So $Net(\pi)$ and $Net(\pi')$ will reduce to the same normal form and are therefore equivalent.

Equation (6.2) arises from SR1. The proof net of the left hand proof, looks like



which after applying CR1, looks like the proof net for the right hand proof. Therefore the two nets are equivalent.

Equation (6.3) arises from SR2. The proof net corresponding to the left hand proof looks like

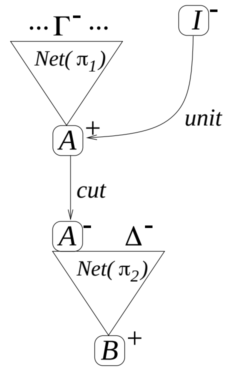


which after applying SC2 looks like the proof net corresponding to the right hand proof. Hence these nets are equivalent.

Equation (6.7) arises from the symmetry of CR1:

$$\dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A}{\Gamma, I \vdash A}(I_{\mathcal{L}}) \qquad A, \Delta \overset{\pi_2}{\vdash} B}{\Gamma, \Delta, I \vdash B}(Cut) \quad \equiv \quad \dfrac{\dfrac{\Gamma \overset{\pi_1}{\vdash} A \quad A, \Delta \overset{\pi_2}{\vdash} B}{\Gamma, \Delta \vdash A}(Cut)}{\Gamma, \Delta, I \vdash B}(I_{\mathcal{L}}).$$

The left hand proof net looks like



which after rewiring the displayed unit, looks like the right hand proof net. So these nets are equivalent.

Finally, equation (6.8) corresponds to the symmetry of CR2.

$$\dfrac{\dfrac{\Gamma, B, C \overset{\pi_1}{\vdash} A}{\Gamma, B \otimes C \vdash A}(\otimes_{\mathcal{L}}) \qquad A, \Delta \overset{\pi_2}{\vdash} D}{\Gamma, B \otimes C, \Delta \vdash D}(Cut) \quad \equiv \quad \dfrac{\dfrac{\Gamma, B, C \overset{\pi_1}{\vdash} A \quad A, \Delta \overset{\pi_2}{\vdash} D}{\Gamma, B, C, \Delta \vdash D}(Cut)}{\Gamma, B \otimes C, \Delta \vdash D}(\otimes_{\mathcal{L}}).$$

The proof nets for both derivations are identical, hence equivalent. ■

Corollary 6.4.17 *There is a bijective correspondence between the equivalence classes of terms in the sequent calculus with respect to equations (6.1)–(6.8) and the equivalence classes of proof nets with respect to unit rewirings and cut elimination.*

6.5 Comparison with Trimble

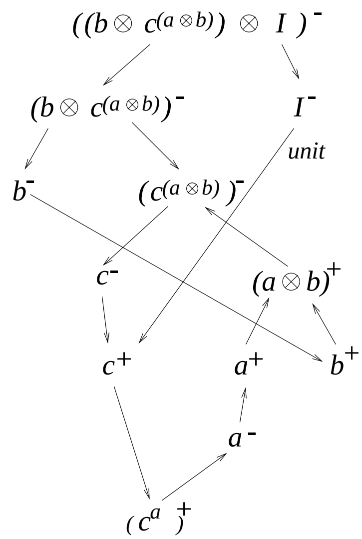
Trimble's multilinear DR-graphs [Tri95] produce a similar proof net system to the nets we have described in this chapter.

The first notable difference between MILL proof structures and DR-graphs is that MILL proof structures are representative of *multimaps*, i.e. several inputs to one output. DR-graphs, on the other hand, have precisely one input and one output. So while we can easily translate a DR-graph into a MILL proof structure, a MILL proof structure can only translate to a DR-graph in which the initial nodes of the proof structure have been "tensored" together.

The second difference between DR-graphs and MILL proof structures is that DR-graphs use an exponential notation B^A to denote $A \multimap B$. Thus $\multimap^-$ and $\multimap^+$ links are conveyed by the links

However, we will still refer to these links as $\multimap^+$ and $\multimap^-$ links, since there is no easy way to refer to them when there is no visible connective.

Example 6.5.1 The corresponding DR-graph for the MILL proof structure in Example 6.3.6 is



The soundness condition on DR-graphs requires a notion of *switching*, similar to that for Girard's original proof nets. A switching here indicates the selection of one edge (and removal of the other) in each $\otimes^-$-link and each $\multimap^+$-link. A DR-graph is *multilinear* if for every switching, the resulting graph is acyclic and connected.

Theorem 6.5.2 *The MILL proof net condition and the multilinear DR-graph condition are equivalent.*

Proof Suppose G is a multilinear DR-graph and $\mathcal{N}$ is its corresponding MILL proof structure. If we verify that G is acyclic over all switchings whereby we always choose the edges $B^+ \to (B^A)^+$ for all $\multimap^+$-links in G, then we are verifying that $\mathcal{N}$ is generalised acyclic.

Suppose that there exists a $\multimap^+$-link in $\mathcal{N}$, between B^+, $(A \multimap B)^+$ and A^-. Suppose that $A^- \not\geq B^+$. Then by Proposition 6.3.18, either there is no generalised path from A^- to B^+, or there exists a generalised path from A^- to $(E \otimes F)^-$ such that $E^- \in \hat{e}(B^+)$ but $F^- \notin \hat{e}(B^+)$, say.

In the former case, consider deleting the edge $B^+ \to (A \multimap B)^+$. Since $B^+ \geq (A \multimap B)^+$, this proves that $\hat{e}(B^+)$ is now disjoint from the rest of the proof net. Equivalently, if we choose a switching in G which removes the edge $B^+ \to (B^A)^+$, then the resulting graph is disconnected. This contradicts the multilinearity of G.

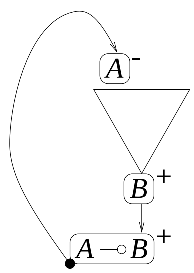
In the latter case, consider deleting the edges $(E \otimes F)^- \to E^-$ and $B^+ \to (A \multimap B)^+$. Then $\hat{e}(B^+)$ will be disjoint from the rest of the proof net. Equivalently, if we choose a switching in G which removes the edges $(E \otimes F)^- \to E^-$ and $B^+ \to (B^A)^+$, then the resulting graph is disconnected. This also contradicts the multilinearity of G.

Therefore $A^- \geq B^+$ in $\mathcal{N}$. Hence $\mathcal{N}$ is a MILL proof net.

Conversely, suppose that $\mathcal{N}$ is a MILL proof net with precisely one initial node and let G be the corresponding DR-graph. (W.l.o.g. we can assume that $\mathcal{N}$ has precisely one initial node. If $\mathcal{N}$ has more than one initial node, then attach $\otimes^-$-links so that there is precisely one initial node. The resulting structure will still be a MILL proof net.)

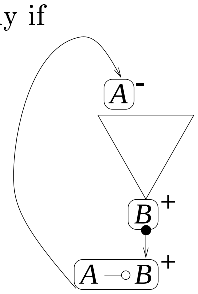
A generalised path which passes through a node $(X \otimes Y)^-$ in $\mathcal{N}$ passes through either X^- or Y^-. Thus the MILL proof net condition encompasses all possible switchings on all $\otimes^-$-links.

Consider a $\multimap^+$ link between the nodes A^-, B^+, and $(A \multimap B)^+$ in $\mathcal{N}$. Observe that



is generalised acyclic,

if and only if the inner substructure with initial node A^- and terminal node B^+ is generalised acyclic, if and only if



is generalised acyclic.

Therefore, replacing $B^+ \to (A \multimap B)^+$ in the set of generalised edges of $\mathcal{N}$ with the edge $(A \multimap B)^+ \to A^-$ does not alter the generalised acyclicity (nor the connectedness) of $\mathcal{N}$. To change such an edge is equivalent to considering an alternative switching on a $\multimap^-$-node in the corresponding DR-graph. Therefore, verifying that $\mathcal{N}$ is a MILL proof net is sufficient to prove that G is a multilinear DR-graph. ■

Thus is would appear that the notion of switching for proof nets in MILL is unnecessary when using the formulation presented here.

6.6 Comparison with Cockett and Seely

We also draw attention to a proof net system for weakly distributive categories [BCST96] (now renamed linearly distributive categories). Since a symmetric linearly distributive category with linear negation is equivalent to a $*$-autonomous category, this system models MLL.

The system has recently been expanded in [CS97b] to model other variants of Linear Logic, such as Hyland and de Paiva's Full Intuituionistic Linear Logic (FILL) [HP93]. In particular, this system encompasses MILL. We will make some brief comments about Cockett and Seely's system. For clarity of comparison, we will avoid the term "proof net" and speak only of typed "circuits".

- The system presented in [CS97b] is motivated by natural deduction.
- A circuit is built from a collection of subcircuits with typed wires. An identity wire exists for all formulae, not just propositional atoms, so it is necessary to introduce expansion rewrites as well as reduction rewrites.
- A notion of switching is required.
- A circuit is called "sequential" if it represents the proof of a sequent. In the commutative case, the sequential condition is equivalent to a Danos-Regnier condition, but in the non-commutative case, a circuit satisfying the Danos-Regnier condition may not represent a proof in the logic. Instead, a "sequentialisation" process is adopted. In effect, it is a hands-on attempt at constructing a proof from the circuit. There appears to be no immediate way to see that a circuit is not sequential without applying the sequentialisation process. As a consequence, a rewrite can only be performed if the resulting circuit remains sequential.
- There are numerous one-step unit rewirings. In the non-commutative case, this procedure cannot be globalised, since after each rewiring, one must verify that the resulting circuit remains sequential.

It would appear that the proof net system presented in this chapter is a specific simplification of the proof circuit system, designed to model MILL. We require no notion of switching, and rewiring a unit requires only one rewrite. Furthermore, the cut elimination reductions can be globalised without compromising strong normalisation or confluence.

Chapter 7

Summary

It is hoped that the reader will now be convinced that there are models of MLL for which we can observe elegant ways of proving full completeness. To summarise, we have demonstrated two ways of exploiting full completeness in a compact closed category to derive full completeness in a $*$-autonomous category.

The glueing construction in Chapter 1 provided us with a series of parallel results to demonstrate this phenomenon. There was a forgetful functor $U : \mathbf{G}\mathbb{C} \to \mathbb{C}$ which enabled us characterise the dinatural transformations in $*$-autonomous $\mathbf{G}\mathbb{C}$ using the characterisation of dinatural transformations in compact closed $\mathbb{C}$. Loader's work on **LLP/GRel** ([Loa94b]) motivated this study, but there the interaction between **Rel** and **GRel** was not emphasised. Chapters 3 and 4 applied the same construction to the category of finite dimensional vector spaces and a category of Conway games. The evident similarity between the compact closed full completeness results obtained in **Rel** and **FDVec** is itself worth noting. It would be of interest to identify more compact closed categories whose natural transformations are merely "permutations on the tensor factors". From the results observed in this thesis, proving $*$-autonomous full completeness for the glued category would almost be a certainty.

In Chapter 2, it was possible (using the category $\mathbb{S}^1$) to prove that a dinatural transformation in **GRel** was the representation of a single morphism in the free $*$-autonomous category, as opposed to at least one such morphism. However, I was unable to achieve this refinement in Chapter 3 – there is no obvious interpretation of the category $\mathbb{S}^1$ in **GFDVec**. As mentioned in that chapter, I believe that **GFDVec** does satisfy this property, having tried a few simple examples. At the time of submitting this thesis, the general proof has yet to be established.

In Chapter 4, the compact closed full completeness result obtained was not for Joyal's category of Conway games and winning strategies, but rather for a restriction to Conway games with a decomposition and history-free winning strategies. This is perhaps not the most ideal choice of category, since some properties of the strategies which we might have hoped would come from the definition of uniformity appears to be built into the category.

Chapter 5 demonstrated another usage of compact closed full completeness. This time, **FDVec** existed as a full subcategory of **ExChu**, and we applied a density argument to show that the behaviour of a dinatural transformation in **ExChu** was uniquely determined by its behaviour on objects in **FDVec**. As it stands, many existing full completeness results, including Blute and Scott's result for **ExChu/RTVec**, currently rely on a restriction to binary sequents, and it is highly desirable to do away with this. The work presented in Chapter 5 has no such restriction. I mention here another category of interest. Since **Rel** exists as a full subcategory of $\mathbb{L}$, the category of complete sup semilattices, it may be possible to obtain a full completeness result for $\mathbb{L}$. Since **FDVec** exists as a full subcategory of **ExChu**, I believe these two problems to run in parallel.

Chapter 6 presented a proof net system for MILL. Thus every normalised proof net was a unique representation of a canonical morphism in a free symmetric monoidal closed category. At the time of writing this thesis, this chapter appears independent, but the intention was to tie it in with the previous work. Soloviev [Sol95] has identified a denotational model of MILL in the form of a "test" category of vector spaces. However, his proof of "full completeness" is highly syntactical and the model remains well above the levels of intuition. It is hoped that one could prove full completeness using the MILL proof net system, i.e. a dinatural transformation in the test category induces a MILL proof net and is therefore induced by a morphism in the free symmetric monoidal closed category. Moreover, it is not unreasonable to hope that full completeness in an underlying compact closed structure, namely **FDVec**, could once again provide us with the means to characterise the identity linkages for the MILL proof net.

To conclude, I hope that this work has provided some ground in establishing standard techniques for proving full completeness, in both compact closed categories and $*$-autonomous categories.

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