SOME ABELIAN INVARIANTS OF 3-MANIFOLDS

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Some invariants for closed orientable 3-manifolds are defined using a series of representations of the symplectic groups and the theory of Heegaard splittings. They are natural extensions of the $U(1)$ Chern-Simons-Witten invariants. These representations come from the functional equation satisfied by the theta functions of level k. We analyze the values of these invariants for lens spaces.

Keywords: Theta functions, Heegaard splitting, tensor representation, symplectic groups.
AMS Classification: 57 A 10, 14 K 25, 32 G 15.

0 . INTRODUCTION

The aim of this paper is to construct invariants of 3-manifolds using the endomorphisms of 1-homologies of surfaces determined by Heegaard splittings and representations of the symplectic group. This leads us to the study of actions of such endomorphisms on the space of theta functions on the Siegel space. The construction goes as follows. Any three manifold can be given an Heegaard decomposition, and hence can be written as the union of two handlebodies identified along a homeomorphism of the surface boundary. After a choice of a basis of the 1-homology the homeomorphism induces an element of the symplectic group. The indeterminacy in the choice of this matrix can be analyzed to give invariants of the three manifold in question. We develop a particular invariant using actions on spaces of modular forms and analyze it in the case of lens spaces.

Although the idea to consider theta functions is transparent from the notes of Oxford seminar [2] there is not an explicit treatment of Abelian Witten's theory, from this perspective, on the author's knowledge. Thus the goal of the present paper is to provide such a rigorous construction and a natural extension of it which leads us to some more general Abelian invariants.

In ([12, 9]) the U(1) invariants are introduced as complex numbers modulo U(1) (or the group of roots of unity). In [26] some invariants are constructed in terms of the linking matrices of 3-manifolds, and their absolute value is the U(1) invariant. Our first task will be to establish a family of invariants using representations of the symplectic group, and to check for the smallest group of roots of unity which have to appear as indeterminacy. Roughly speaking the usual $U(1)$ gauge theory comes with a one dimensional vacuum vector associated to a handlebody and corresponding to a theta function with trivial characteristics. We generalize it to the case where the vacuum is degenerate, and is represented by a vector subspace of the space associated to the surface. Alternatively this amounts to consider a new representation of the symplectic group which is an exterior power of the former. This way we derive nontrivial refinement $f_{p,k}$ of the usual Abelian invariants $f_{k,k}$ which depend on the level k and a divisor p of k. The interest in considering such an extension is that, starting with the standard U(1) TQFT we obtain other invariants (and furthermore also TQFT) which contain more topological information, as it can be deduced from the computations on lens spaces. It seems that this procedure could be carried on over some other TQFT.

Another construction of 3-manifold invariants via representations of the mapping class groups was obtained by Kohno [20] for the SU(2) TQFT. Our invariants are certainly less sensitive than Kohno's invariants: in particular the SU(2)-invariants can in some cases distinguish between a homology sphere and the standard sphere, but our invariants cannot do that, because they are defined on the symplectic group level rather than the mapping class group. However it is not at all clear whether all our invariants $f_{p,k}$ can be deduced from the SU(2)-invariants.

Some of the results of this paper have been announced in [9] and several related articles appeared (see [12, 26, 24, 7, 30, 31, 23]). Another semi-Abelian version was described in [11].

Aknowledgements. I am indebted to my thesis advisor V.Poénaru for the discussions we had on this subject, to V.Turaev and L.Guillou for their pertinent observations which improved the clarity of this paper. This paper is an expanded and updated version of the first chapter of the author's Ph.D. thesis at University of Paris-Sud.

1. STATEMENT OF THE MAIN RESULT

Let M^3 be a closed connected and oriented 3-manifold. Consider a Heegaard splitting of $M^3 = T_g \cup_\varphi \overline{T}_g$ into two handlebodies of genus g glued together along their common surface Σ_g using the homeomorphism $\varphi : \Sigma_g \longrightarrow \Sigma_g$. Notice that φ is not uniquely determined by the Heegaard splitting. In fact it can be composed (to the left and to the right) by any homeomorphism which extends to the whole handlebody T_g bounding the surface (i.e. an extendable homeomorphism as considered by Suzuki [28] and Kohno [20]). Set $\mathcal{M}_g$ for the mapping class group of the genus g surface and $\mathcal{M}_g^+$ for the image in $\mathcal{M}_g$ of the subgroup of extendable homeomorphisms. We have a canonical surjection $s : \mathcal{M}_g \longrightarrow \mathrm{Sp}(2g, \mathbf{Z})$ onto the symplectic group. Assume that a symplectic basis in the homology of the surface Σ_g is chosen. Therefore $s(\mathcal{M}_g^+) = \mathrm{Sp}^+(2g, \mathbf{Z})$ can be easily described as the set of symplectic matrices having the form $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ with respect to the usual splitting into $g \times g$ matrices. Remark that the tower of groups $\mathrm{Sp}(2g, \mathbf{Z})$ has an exterior multiplication law, namely the symplectic sum

$$\oplus_s : \mathrm{Sp}(2g, \mathbf{Z}) \times \mathrm{Sp}(2h, \mathbf{Z}) \longrightarrow \mathrm{Sp}(2(g+h), \mathbf{Z}),$$

given by the formula:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \oplus_s \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A \oplus A' & B \oplus B' \\ C \oplus C' & D \oplus D' \end{bmatrix}.$$

Let A denote an arbitrary set.

Definition 1.1. The set of functions $F_g : \mathrm{Sp}(2g, \mathbf{Z}) \longrightarrow A$, $g \in \mathbf{Z}_+$, is called an Abelian invariant if the following two conditions are fulfilled:

1. $F_g(axb) = F_g(x)$, for all $x \in \mathrm{Sp}(2g, \mathbf{Z})$, $a, b \in \mathrm{Sp}^+(2g, \mathbf{Z})$, $g \in \mathbf{Z}_+$,
2. $F_{g+1}(x \oplus_s \tau) = F_g(x)$, for all $x \in \mathrm{Sp}(2g, \mathbf{Z})$, $g \in \mathbf{Z}_+$, where $\tau = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}(2, \mathbf{Z})$.

Observe that an (Abelian) invariant defines a topological invariant for closed 3-manifolds by means of the formula:

$$F(M^3) = F_g(s(\varphi)).$$

The two conditions stated above for F_* and the Reidemester-Singer theorem (see [6, 27]) prove the independence on the various choices which can be made on the right hand term.

The natural way to get Abelian invariants is to use the representations of $\mathrm{Sp}(2g, \mathbf{Z})$.

Definition 1.2. A tensor representation of the symplectic group consists in the following data:

1. The hermitian vector spaces $W_g \subset V_g$ satisfying $W_g = W_1^{\otimes g}$, $V_g = V_1^{\otimes g}$.
2. A sequence of unitary representations $\rho_g : \mathrm{Sp}(2g, \mathbf{Z}) \longrightarrow U(V_g)$ which fulfills the conditions
   - $\rho_{g+h}(x \oplus_s y) = \rho_g(x) \otimes \rho_h(y)$ for all x, y, g, h appropriately chosen.
   - W_g is $\rho_g(\mathrm{Sp}^+(2g, \mathbf{Z}))$-invariant.
   - Let π_{W_g} denote the projection of V_g onto W_g. For $x \in End(V_g)$ set $\det_{W_g}(x) = \det(\pi_{W_g} \circ x)$. We will assume that
     - $\det_{W_1}\rho_1(\tau) \neq 0$,
     - 0 does not belong to $\bigcup_{g>0} \det_{W_g}(\rho_g(\mathrm{Sp}^+(2g, \mathbf{Z})))$,

     hold.

We denote by $R(\rho_*, V_*, W_*)$ the (multiplicative) group generated by $\bigcup_{g>0} \det_{W_g}(\rho_g(\mathrm{Sp}^+(2g, \mathbf{Z}))) \subset \mathbf{C}^*$.

From such data we can find an invariant by means of

LEMMA 1.3. *To each tensor representation $\rho = (\rho_*, V_*, W_*)$ of the symplectic group we can associate an Abelian invariant $F_g(\rho) : \mathrm{Sp}(2g, \mathbf{Z}) \longrightarrow \mathbf{C}/R(\rho)$, by means of the following formula:*

$$F_g(\rho; x) = [\det_{W_1}\rho_1(\tau))]^{-g} \left[\det_{W_g}(\rho_g(x))\right]^{m(g)},$$

where $m(g) = (dim(W_1))^{1-g}$.

Proof. The following equality

$$\det_{W_g}(\rho_g(cx)) = \det_{W_g}(\rho_g(c))\det_{W_g}(\rho_g(x))$$

holds whenever $c \in \mathrm{Sp}^+(2g, \mathbf{Z})$, because W_g is $\rho_g(\mathrm{Sp}^+(2g, \mathbf{Z}))$-invariant. Next we derive that

$$\det_{W_{g+1}}(\rho_{g+1}(x \oplus_s \tau)) = \det_{W_g \otimes W_1}(\rho_g(x) \otimes \rho_1(\tau)) = \left[\det_{W_g}(\rho_g(x))\right]^{dimW_1} [\det_{W_1}(\rho_1(\tau))]^{dimW_g},$$

so our claim follows. □

Our main result consists in the construction of a tensor representation of the symplectic group. We need first some notations:

$V_g(k) = \mathbf{C} < \theta_m; m \in (\mathbf{Z}/k\mathbf{Z})^g >$, for even k,

$W_g(p, k) = \mathbf{C} < \theta_{pm}; m \in (\mathbf{Z}/k\mathbf{Z})^g >$, where p divides k.

Let us denote $l = \frac{k}{p}$. Then set $R_{k,p}$ for the group of roots of unity generated by $\exp\left(p\frac{\pi\sqrt{-1}}{N(k,p)}\right)$ and $\exp\left(\frac{\pi\sqrt{-1}k}{8p}\right)$, where $N(k,p) = \text{g.c.d.}(l(\text{mod } 6), 6)$. Here $l(\text{mod } 6) \in \{1,2,3,4,5,6\}$ is the residue of l mod 6.

It is well-known ([3, 25]) that $\text{Sp}(2g, \mathbf{Z})$ is generated by the matrices having one of the following forms: $\begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}$ where $B = B^\top$ has integer entries, $\begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}$ where $A \in \text{GL}(g, \mathbf{Z})$, and $\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}$. We set:

$$\rho_g \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix} = \text{diag}(\exp(\frac{\pi\sqrt{-1}}{k} < m, Bm >)). \tag{1}$$

$$\rho_g \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix} = (\delta_{A^\top m,n})_{m,n \in (\mathbf{Z}/k\mathbf{Z})^g}. \tag{2}$$

$$\rho_g \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} = k^{-g/2} \exp(-2\pi\sqrt{-1}k^{-1} < m, l >)_{m,l \in (\mathbf{Z}/k\mathbf{Z})^g}. \tag{3}$$

We will prove that these formulas define a representation ρ_g of $\text{Sp}(2g, \mathbf{Z})$ in $U(V_g(k))/R_8$. Here R_8 the group of roots of unity of order 8 is viewed as a subgroup of scalar matrices in the unitary group. Let us denote $\mathcal{F}_{p,k} = (\rho_*, V_*(k), W_*(p,k))$.

THEOREM 1.1. *Let consider an even number k and p a divisor of k such that* $\text{g.c.d.}(\frac{k}{p}, p) = 1$. *Then the collection $\mathcal{F}_{p,k}$ is a tensor representation of the symplectic group, up to an eighth root of unity indeterminacy.*

According to the preceding lemma we have

COROLLARY 1.1. $F(\mathcal{F}_{p,k}) \in \mathbf{C}/R_{k,p}$ *is a topological invariant for closed oriented 3-manifolds.*

Let denote this invariant by $f_{p,k}$. The geometric interpretation which will be given in the last section enables us to consider $f_{p,p}$ be the exact Abelian Witten's theory, as already considered in [12, 26]. Therefore $\mid f_{k,k}(M^3) \mid \in \mathbf{R}$ can be expressed in terms of classical cohomology invariants, as follows:

$$\mid f_{k,k}(M) \mid \; = \begin{cases} \mid H^1(M, \mathbf{Z}/k\mathbf{Z}) \mid^{1/2}, & \text{if } \alpha \cup \alpha \cup \alpha = 0, \forall \alpha \in H^1(M, \mathbf{Z}/k\mathbf{Z}), \\ 0 & \text{elsewhere} \end{cases}$$

The calculations for lens spaces show that in general $f_{p,k}$ do not vanish although $\alpha \cup \alpha \cup \alpha \neq 0$, and thus they represent a non trivial extension of $f_{k,k}$.

If we had relaxed the requirement $V_g = V_1^{\otimes g}$, to the weaker inclusion condition $V_g \otimes V_h \hookrightarrow V_{g+h}$ (and accordingly for the ρ_g action), in definition 1.2., then we would find that for each tensor representation with arbitrary vacuum space W_g there exists another representation (in this broader sense) which yields the same topological invariants for 3-manifolds and has a one dimensional vacuum space in all genera. We just replace V_g, W_g and ρ_g by $\bigwedge^{\dim W_g} V_g$, $\bigwedge^{\dim W_g} W_g$ and $\bigwedge^{\dim W_g} \rho_g$.

In the third part we analyze the values of these invariants for lens spaces. Motivated by these we give a definition of the invariant in terms of some linking matrix and some additional homological structure of the manifold, likewise the case $k = p$. It seems that $f_{p,k}$ are homotopic invariants.

2. PROOF OF THE THEOREM

2.1. Preliminaries on theta functions

Let $\mathcal{S}_g$ be the Siegel space of $g \times g$ symmetric matrices Ω of complex entries having the imaginary part $\mathrm{Im}\Omega$ positive defined. There is a natural $\mathrm{Sp}(2g, \mathbf{Z})$ action on $\mathbf{C}^g \times \mathcal{S}_g$ given by

$$\gamma \cdot (z, \Omega) = ((((C\Omega + D)^\top)^{-1})z, (A\Omega + B)(C\Omega + D)^{-1}). \tag{4}$$

The dependence of the classical theta function $\theta(z, \Omega)$ on Ω is expressed by a functional equation which describes its behaviour under the action of $\mathrm{Sp}(2g, \mathbf{Z})$. Let $\Gamma(1,2)$ be the so-called theta group consisting of elements $\gamma \in \mathrm{Sp}(2g, \mathbf{Z})$ which preserve the quadratic form

$$Q(n_1, n_2, ..., n_{2g}) = \sum_{i=1}^{g} n_i n_{i+g} \in \mathbf{Z}/2\mathbf{Z},$$

which means that $Q(\gamma(x)) = Q(x)(\mathrm{mod}\ 2)$. We represent any element $\gamma \in \mathrm{Sp}(2g, \mathbf{Z})$ as $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are $g \times g$ matrices. Then $\Gamma(1,2)$ may be alternatively described as the set of those elements γ having the property that the diagonals of $A^\top C$ and $B^\top D$ are even. Let $<,>$ denote the standard hermitian product on $\mathbf{C}^{2g}$. The functional equation, as stated in [25] is:

$$\theta((C\Omega + D)^{\top^{-1}} z, (A\Omega+B)(C\Omega+D)^{-1}) = \zeta_\gamma \mathrm{det}(C\Omega+D)^{1/2} \exp(\pi\sqrt{-1} < z, (C\Omega+D)^{-1}Cz >)\theta(z, \Omega),$$

for $\gamma \in \Gamma(1,2)$, where ζ_γ is a certain 8^{th} root of unity.

If $g = 1$ we may suppose that $C > 0$ or $C = 0$ and $D > 0$ so the imaginary part $Im(C\Omega + D) \geq 0$ for Ω in the upper half plane. Then we will choose the square root $(C\Omega + D)^{1/2}$ in the first quadrant. Now we can express the dependence of ζ_γ on γ as follows:

1. for even C and odd D, $\zeta_\gamma = \sqrt{-1}^{(D-1)/2}(\frac{C}{|D|})$,
2. for odd C and even D, $\zeta_\gamma = \exp(-\pi\sqrt{-1}C/4)(\frac{D}{C})$,

where $(\frac{x}{y})$ is the usual Jacobi symbol ([14]).

For $g > 1$ it is less obvious to describe this dependence. We fix first the choice of the square root of $\det(C\Omega + D)$ in the following manner: let $\det^{\frac{1}{2}}\left(\frac{Z}{\sqrt{-1}}\right)$ be the unique holomorphic function on $\mathcal{S}_g$ satisfying

$$\left(\det^{\frac{1}{2}}\left(\frac{Z}{\sqrt{-1}}\right)\right)^2 = \det\left(\frac{Z}{\sqrt{-1}}\right),$$

and taking in $\sqrt{-1}\mathbf{1}_g$ the value 1. Next define

$$\det^{\frac{1}{2}}(C\Omega + D) = \det^{\frac{1}{2}}(D)\det^{\frac{1}{2}}\left(\frac{\Omega}{\sqrt{-1}}\right)\det^{\frac{1}{2}}\left(\frac{-\Omega^{-1} - D^{-1}C}{\sqrt{-1}}\right),$$

where the square root of $\det(D)$ is taken to lie in the first quadrant. Using this convention we may express ζ_γ as a Gauss sum for invertible D (see [8],p.26-27)

$$\zeta_\gamma = \det^{-\frac{1}{2}}(D) \sum_{l \in \mathbf{Z}^g/D\mathbf{Z}^g} \exp(\pi\sqrt{-1} < l, BD^{-1}l >), \tag{5}$$

and in particular we recover the formula from above for $g=1$. On the other hand for $\gamma = \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}$ we have $\zeta_\gamma = (\det A)^{-1/2}$. When $\gamma = \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}$ then the multiplier system is trivial, $\zeta_\gamma = 1$, and eventually for $\gamma = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}$ we have $\zeta_\gamma = \exp(\pi\sqrt{-1}g/4)$. Actually this data determines completely ζ_γ.

Denote $\det^{\frac{1}{2}}(C\Omega + D) = j(\gamma, \Omega)$. Then there exists a map

$$s : \mathrm{Sp}(2g, \mathbf{R}) \times \mathrm{Sp}(2g, \mathbf{R}) \longrightarrow \{-1, 1\}$$

satisfying

$$j(\gamma_1\gamma_2, \Omega) = s(\gamma_1, \gamma_2) j(\gamma_1, \gamma_2\Omega) j(\gamma_2, \Omega).$$

We recall that a multiplier system ([8]) for a subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbf{R})$ is a map $m : \Gamma \longrightarrow \mathbf{C}^*$ such that

$$m(\gamma_1) = s(\gamma_1, \gamma_2) m(\gamma_1) m(\gamma_2).$$

An easy remark is that, once a multiplier system m is chosen, the product $A(\gamma, \Omega) = m(\gamma)j(\gamma, \Omega)$ verifies the cocycle condition

$$A(\gamma_1\gamma_2, \Omega) = A(\gamma_1, \gamma_2\Omega) A(\gamma_2, \Omega),$$

for $\gamma_i \in \Gamma$. Then another formulation of the dependence of ζ_γ on γ is to say that it is the multiplier system defined on $\Gamma(1,2)$. Remark that using the theorem of Mennicke any two multiplier systems defined on a subgroup of the theta group are identical on some congruence subgroup.

Consider now the level k theta functions. For $m \in (\mathbf{Z}/k\mathbf{Z})^g$ these are defined by

$$\theta_m(z, \Omega) = \sum_{l \in m + k\mathbf{Z}^g} \exp\left(\frac{\pi\sqrt{-1}}{k} (< l, \Omega l > + 2 < l, z >) \right) \tag{6}$$

or, equivalently, by

$$\theta_m(z, \Omega) = \theta(m/k, 0)(kz, k\Omega).$$

where $\theta(*, *)$ are the theta functions with rational characteristics ([25]) given by

$$\theta(a, b)(z, \Omega) = \sum_{l \in \mathbf{Z}^g} \exp\left(\frac{\pi\sqrt{-1}}{k} (< l + a, \Omega(l + a) > + 2 < l + a, z + b >) \right) \tag{7}$$

for $a, b \in \mathbf{Q}^g$. Obviously $\theta(0, 0)$ is the usual theta function.

Let us denote by $R_8 \subset \mathbf{C}$ the group of 8^{th} roots of unity. Then R_8 becomes also a subgroup of the unitary group $U(n)$ acting by scalar multiplication. Consider also the theta vector of level k:

$$\Theta_k(z, \Omega) = (\theta_m(z, \Omega))_{m \in (\mathbf{Z}/k\mathbf{Z})^g}.$$

2.2. The functional equation

In order to prove the theorem we need first to show that ρ_g is indeed a representation of the symplectic group. This will be done by noticing that the level k theta vector satisfies a functional equation:

PROPOSITION 2.1. *The theta vector satisfies the following functional equation:*

$$\Theta_k(\gamma \cdot (z, \Omega)) = \zeta_\gamma \det(C\Omega + D)^{1/2} \exp(k\pi\sqrt{-1} < z, (C\Omega + D)^{-1} C z >) \rho_g(\gamma)(\Theta_k(z, \Omega)) \tag{8}$$

where

1. γ *belongs to the theta group* $\Gamma(1,2)$ *if* k *is odd and to* $\mathrm{Sp}(2g,\mathbf{Z})$ *elsewhere.*

2. $\zeta_\gamma \in R_8$ *is the (fixed) multiplier system described above.*

3. $\rho_g : \Gamma(1,2) \longrightarrow U(V_g(k))$ *is a group homomorphism. For even* k *the corresponding map* ρ_g : $\mathrm{Sp}(2g,\mathbf{Z}) \longrightarrow U(V_g(k))$ *becomes a group homomorphism (denoted also by* ρ_g *when no confusion arises) when passing to the quotient* $U(V_g(k))/R_8$.

4. ρ_g *is determined by the formulas* (1-3).

Remark 2.1. This result is stated also in [17] for some modified theta functions but in less explicit form.

Proof of the proposition. Remark first that the map

$$w(\gamma, z, \Omega) = \exp(k\pi\sqrt{-1} < z, (C\Omega + D)^{-1}Cz >),$$

verifies

$$w(\gamma_1\gamma_2, (z, \Omega)) = w(\gamma_1, \gamma_2 \cdot (z, \Omega))w(\gamma_2, (z, \Omega)),$$

for all $\gamma_1, \gamma_2 \in \mathrm{Sp}(2g,\mathbf{Z})$. We observed before that $A(\gamma, (z, \Omega)) = \zeta_\gamma j(\gamma, \Omega)$ verifies the cocycle identity:

$$A(\gamma_1\gamma_2, (z, \Omega)) = A(\gamma_1, \gamma_2(z, \Omega))A(\gamma_2, (z, \Omega)),$$

for all $\gamma_1, \gamma_2 \in \Gamma(1,2)$(see also [8] p.14). Therefore if the equation (8) holds for γ_1 and γ_2, then it will be fulfilled also by $\gamma_1\gamma_2$ with $\rho_g(\gamma_1\gamma_2)$ replaced by $\rho_g(\gamma_1)\rho_g(\gamma_2)$. But the theta functions of level k form a basis for the vector space $H^0(Ab_\Omega, \Theta^k)$, where $Ab_\Omega = \mathbf{C}^g/(\mathbf{Z}^g \oplus \Omega\mathbf{Z}^g)$ is the Abelian variety corresponding to Ω and Θ is the theta line bundle (giving the principal polarization) over Ab_Ω (see [13]). Thus we obtain in fact a representation of the theta group $\Gamma(1,2)$ and it is sufficiently to check the relation (8) for a system of generators. It is known that $\Gamma(1,2)$ is generated by the matrices of the form $\left[\begin{array}{cc} A & 0 \\ 0 & (A^\top)^{-1}\end{array}\right]$ with $A \in GL(g,\mathbf{Z})$, $\left[\begin{array}{cc} 0 & -1_g \\ 1_g & 0\end{array}\right]$, and $\left[\begin{array}{cc} 1_g & B \\ 0 & 1_g\end{array}\right]$, where B is symmetric, integral with even diagonal.

We remark that the theta functions of level k can be expressed as

$$\theta_m(z, \Omega) = \exp\left(\frac{\pi\sqrt{-1}}{k} < m, \Omega m + 2kz >\right)\theta(kz + \Omega m, k\Omega) \tag{9}$$

This relation follows immediately from [17] p.50.

We check now the relation (8) for the generators. The first case is

$$\theta_m(z, \Omega + B) = \exp\left(\frac{\pi\sqrt{-1}}{k} < m, (\Omega + B)m + 2kz >\right)\theta(kz + (\Omega + B)m, k(\Omega + B)).$$

But $\left[\begin{array}{cc} 1_g & kB \\ 0 & 1_g\end{array}\right]$ belongs to $\Gamma(1,2)$ for even k (arbitrary B) or odd k and B having an even diagonal. Therefore from the functional equation satisfied by the classical theta function we obtain:

$$\theta(kz + (\Omega + B)m, k(\Omega + B)) = \theta(kz + (\Omega + B)m, k\Omega).$$

Since θ is periodic we obtain

$$\theta(kz + (\Omega + B)m, k\Omega) = \theta(kz + \Omega m, k\Omega).$$

Using (9) it follows that

$$\theta_m(z, \Omega + B) = \exp(\frac{\pi\sqrt{-1}}{k} < m, Bm >)\theta_m(z, \Omega) \tag{10}$$

holds. This yields (1).
In the second case we have

$$\theta_m(Az, A\Omega A^\top) = \exp(\frac{\pi\sqrt{-1}}{k} < m, A\Omega A^\top m + 2kAz >)\theta(kAz + A\Omega A^\top m, kA\Omega A^\top m)).$$

We derive

$$\theta(Az, kA\Omega A^\top) = \zeta_\gamma(\det A)^{-1/2}\theta(z, k\Omega),$$

which leads to:

$$\theta_m(Az, A\Omega A^\top) = \zeta_\gamma(\det A)^{-1/2}\theta_{A^\top m}(z, \Omega) \tag{11}$$

and (2) is verified.
In order to handle the last case we recall first the well-known Poisson summation formula ([25]):

LEMMA 2.2. *Let f be a smooth function on R^g which decreases to zero faster than any rational function at infinity and*

$$f^\#(\xi) = \int_{R^g} f(x)\exp(2\pi\sqrt{-1} < x, \xi >)\mathrm{d}x$$

be its Fourier transform. Then the following identity

$$\sum_{n\in\mathbf{Z}^g} f(n) = \sum_{n\in\mathbf{Z}^g} f^\#(n) \tag{12}$$

holds.

Consider now $f(x) = \exp(\pi\sqrt{-1} < kx + m, k^{-1}\Omega(kx + m) + 2z >)$. A simple computation gives us

$$f^\#(\xi) = \zeta(\det\Omega)^{1/2}k^{-g/2}\exp(-2\pi\sqrt{-1}k^{-1} < m, \xi >)\exp(-\pi < z + k^{-1}\xi, k\Omega^{-1}(z + k^{-1}\xi >).$$

Here $\zeta = \exp(\frac{\pi\sqrt{-1}g}{4})$. Next we are interested in computing:

$$\begin{aligned}
S_l &= \sum_{\xi\in -l+k\mathbf{Z}^g} \exp(-\pi\sqrt{-1} < z + k^{-1}\xi, k\Omega^{-1}(z + k^{-1}\xi) >) \\
&= \sum_{\eta\in\mathbf{Z}^g} \exp(-\pi\sqrt{-1} < z - \eta - k^{-1}l, k\Omega^{-1}(z - \eta - k^{-1}l) >) \\
&= \exp(-\pi\sqrt{-1} < \eta, k\Omega^{-1}z >)\exp(-\pi\sqrt{-1} < l, \Omega^{-1}l >)\exp(2\pi\sqrt{-1} < l, \Omega^{-1}z >) \times \\
&\quad (\sum_{\eta\in\mathbf{Z}^g} \exp(-\pi\sqrt{-1} < \eta, k\Omega^{-1}l > +2\pi\sqrt{-1} < \eta, -\Omega^{-1}l + k\Omega^{-1}z >) \\
&= \exp(-\pi\sqrt{-1} < z, k\Omega^{-1}z >)\exp(-\pi < l, \Omega^{-1}l >) \times \\
&\quad \exp(2\pi\sqrt{-1} < l, \Omega^{-1}z)\theta(-\Omega^{-1}l + k\Omega^{-1}z, -k\Omega^{-1}) \\
&= \exp(-\pi\sqrt{-1} < z, k\Omega^{-1}z >)\theta_l(\Omega^{-1}z, -\Omega^{-1}).
\end{aligned}$$

From the Poisson formula (12) we deduce

$$\theta_m(z,\Omega) = \sum_{n\in\mathbf{Z}^g} f(n) = \sum_{n\in\mathbf{Z}^g} f^{\#}(n) = \zeta(\det\Omega)^{-1/2}\exp(-\pi\sqrt{-1} < z, k\Omega^{-1}z >)\times$$

$$\sum_{l\in(\mathbf{Z}/k\mathbf{Z})^g} k^{-g/2}\exp(2\pi\sqrt{-1}k^{-1} < m, l >)\theta_l(\Omega^{-1}z, -\Omega^{-1}).$$

Therefore we have:

$$\rho_g \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} = k^{-g/2}\exp(-2\pi\sqrt{-1}k^{-1} < m, l >)_{m,l\in(\mathbf{Z}/k\mathbf{Z})^g}. \tag{13}$$

This proves the claim for odd k. We observed above that for even k the action of $\begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}$ on the theta vector can be computed (and the value is that claimed in (1)) for arbitrary symmetric integral B. If we add these matrices to the system of generators of $\Gamma(1,2)$ then a system generating all of $\mathrm{Sp}(2g,\mathbf{Z})$ is obtained. It suffices then to see to what extent the property to be a homomorphism is preserved. Set then $\zeta_\gamma = 1$, for $\gamma = \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}$, and $j(\gamma,\Omega) = \det^{\frac{1}{2}}(C\Omega + D)$, for an arbitrary symplectic matrix γ. If $\gamma = \gamma_1\gamma_2...\gamma_n$ is written in terms of generators γ_i then we may express each $\rho(\gamma_i)$ using the previous formulas and then collecting all the terms we find that:

$$\Theta_k(\gamma(z,\Omega)) = \zeta_{\gamma_1}...\zeta_{\gamma_n}\varepsilon j(\gamma,\Omega)w(\gamma,z,\Omega)\rho(\gamma_1)...\rho(\gamma_n)\Theta_k(z,\Omega),$$

where

$$j(\gamma_1,\gamma_2...\gamma_n\Omega)j(\gamma_2,\gamma_3...\gamma_n\Omega)...j(\gamma_n,\Omega) = j(\gamma,\Omega)\varepsilon,\ \varepsilon\in\{-1,1\}.$$

We used the fact that the absolute value $\mid j(\gamma,\Omega) \mid$ is a cocycle. Therefore, if we put $\rho(\gamma) = \rho(\gamma_1)...\rho(\gamma_n)$ is well-defined (i.e. independent on the particular decomposition we choose) up to an eighth root of unity.

Notice that this indeterminacy is nontrivial. In fact the formulas (1-3) define a representation of a central extension of $\mathrm{Sp}(2g,\mathbf{Z})$. For instance when $g = 1$ we have a presentation for $SL_2(\mathbf{Z})$ with two generators S, T and relations $S^4 = 1$, $(ST)^3 = S^2$. Consider then the central extension by $\mathbf{Z}_8$ of $SL_2(\mathbf{Z})$ which has the presentation

$$< S', T', C \mid S'^4 = 1, (S'T')^3S^2 = C, C^8 = 1, [C,S] = [C,T] = 1 >,$$

obtained by introducing the central element C of order 8. It follows then that ρ_1 defines a genuine representation for this central extension. Let S' acts as $\rho_1\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and T' as $\rho_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

In another words we cannot find an extension (as a multiplier system) of the theta multiplier system from $\Gamma(1,2)$ to all of the symplectic group. If we want to extend it to a plain map, then $A(\gamma,z,\Omega)$ is no more a cocycle: it should satisfy a relation

$$A(\gamma_1\gamma_2,(z,\Omega)) = \mu(\gamma_1,\gamma_2)A(\gamma_1,\gamma_2(z,\Omega))A(\gamma_2,(z,\Omega)),$$

for some 2-cocycle $\mu : \mathrm{Sp}(2g,\mathbf{Z})\times\mathrm{Sp}(2g,\mathbf{Z}) \longrightarrow R_8$. This is the 2-cocycle expressing the central extension of $\mathrm{Sp}(2g,\mathbf{Z})$ on which ρ_g becomes a linear representation. $\square$

LEMMA 2.3. *We have*

(14) $$\rho_g(\gamma_1 \oplus_s \gamma_2) = \rho_g(\gamma_1) \otimes \rho_g(\gamma_2).$$

Proof of the lemma. We will use now the fact that ρ_g arises in fact as a monodromy representation. Specifically, we consider $\Omega = \Omega_1 \oplus \Omega_2$. From the definition of level k theta functions we find that

$$\theta_{mm'}((z_1, z_2), \Omega) = \theta_m(z_1, \Omega_1)\, \theta_{m'}(z_2, \Omega_2),$$

hence

$$\theta_{mm'}(\gamma_1 \oplus_s \gamma_2((z_1, z_2), \Omega)) = \theta_m(\gamma_1(z_1, \Omega_1))\, \theta_{m'}(\gamma_2(z_2, \Omega_2)).$$

Since the theta functions of level k give a basis for the vector space $H^0(Ab_\Omega, \Theta^k)$ and $A(\gamma, (z, \Omega))$ behaves multiplicatively we derive the claim. □

LEMMA 2.4. *$W_g(p, k)$ is a $\rho_g(\mathrm{Sp}^+(2g, \mathbf{Z}))$-invariant subspace of $V_g(k)$.*

Proof of the lemma. We have

$$W_1(p, k) = \{x \in V_1(k); x_m = 0 \text{ for } m \in \mathbf{Z}/k\mathbf{Z} \text{ which is not a multiple of } p\},$$

and we have an identification $W_g(p, k) = W_1(p, k)^{\otimes k}$. Now a system of generators for $\mathrm{Sp}^+(2g, \mathbf{Z})$ is provided by the matrices having the form $\begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}$ Matrices of the first type act diagonally hence leave $W_g(p, k)$ invariant. Next choose $\gamma = \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}$ and $x \in W_g(p, k)$. Then x has the form

$$x = \sum_{m \in (\mathbf{Z}/k\mathbf{Z})^g} x_{pm} \theta_{pm}$$

hence

$$\rho_g(\gamma)x = \sum_{m \in (\mathbf{Z}/k\mathbf{Z})^g} x_{pm} \theta_{pA^\top m}.$$

and therefore $\rho_g(\gamma)x \in W_g(p, k)$. □

This lemma ends the proof of our theorem. □

Proof of the corollary. It suffices to check the equality $R(\mathcal{F}_{k,p}) = R_{k,p}$. Let us consider first $c = \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}$. Then $\rho_g(c)$ is a permutation matrix whose restriction to $W_g(p, k)$ is again a permutation matrix, so its determinant is 1 or -1. Choose further $c = \begin{bmatrix} 1_g & B \\ 0 & 1_g \end{bmatrix}$. It suffices to check out the case when $B = E_{st}$ where E_{st} is the matrix having only a non-zero entry which equals 1 and lies on the st position. Since $\rho_g(c)$ is diagonal and leaves therefore $W_*(p, k)$ invariant we find

$$\det_{W_*(p,k)} \rho_g(c) = \exp\Big(\sum_{i=1}^{g} \sum_{m_i=1}^{k/p} \frac{\pi\sqrt{-1}}{k} p^2 m_s m_t\Big).$$

Here $m \in (\mathbf{Z}/k\mathbf{Z})^g$ is a vector whose components are m_j, and the sum is taken over all these vectors m. Then this determinant equals $u = \exp(\pi\sqrt{-1}(\frac{k}{p})^{g-2} pl(l+1)^2/4)$, if $s \neq t$ (so $g > 1$), and $v = \exp(\pi\sqrt{-1}(\frac{k}{p})^{g-1}(l+1)(2l+1)p/6)$, for $s = t$. We have to find now the smallest group of roots of unity containing these numbers for arbitrary g, which it turns to be determined only by the values for $g=2$ and respectively $g=1$. Set for $g=2$ $u=\exp(\pi\sqrt{-1}U)$, and for $g=1$, $v=\exp(\pi\sqrt{-1}V)$. We have then the following tables of values for U and V determining the group $R_{k,p}$. Remark that $k=pl$ is even, so that for even l the value of p is necessarily even. The periodicity of U has length 12 and that of V has length 8, but the group $R_{k,p}$ depends only on $l(\text{mod } 6)$. Notice that there is already the 8-th root of unity indeterminacy which yields one of the generators, namely $\exp\left(\frac{\pi\sqrt{-1}l}{4}\right)$, for $R_{k,p}$. Here are the explicit values for the first 12 terms out of the 24:

$l(\text{mod } 12)$	U	V	The other generator for $R_{k,p}$
1	p	p	1
2	$p/2$	$p/2$	$\exp\left(p\frac{\pi\sqrt{-1}}{2}\right)$
3	$4p/6$	0	$\exp\left(p\frac{\pi\sqrt{-1}}{3}\right)$
4	$3p/2$	p	$\exp\left(p\frac{\pi\sqrt{-1}}{2}\right)$
5	p	p	1
6	$7p/6$	$3p/2$	$\exp\left(p\frac{\pi\sqrt{-1}}{6}\right)$
7	0	0	1
8	$3p/2$	0	$\exp\left(p\frac{\pi\sqrt{-1}}{2}\right)$
9	$10p/6$	p	$\exp\left(p\frac{\pi\sqrt{-1}}{3}\right)$
10	$p/2$	$p/2$	$\exp\left(p\frac{\pi\sqrt{-1}}{2}\right)$
11	0	0	1
12	$p/6$	p	$\exp\left(p\frac{\pi\sqrt{-1}}{6}\right)$

This proves our claim.□

3. COMPUTATIONS FOR LENS SPACES

3.1. The normalization factor d

Let $L_{a,b}$ denote the usual lens space. We may choose for $s(\varphi)$ any element $\begin{bmatrix} b & c \\ a & d \end{bmatrix}$ with $bd-ac=1$. If $a=1, b=0$ then $L_{a,b}$ is the sphere S^3. It is obvious that

$$f_{p,k}(S^3)=1.$$

Remember that $l=k/p$. Our first task will be now to compute $\det_{W_1(p,k)}\rho_1(\tau)=d$ in order to check that d is indeed nonzero. This is the last requirement to verify in definition 1.2. Equivalently, we have to compute the invariant for $S^2\times S^1$.

PROPOSITION 3.1. *If* g.c.d.$(p,l)\neq 1$ *then* $d=0$. *If* g.c.d.$(p,l)=1$ *then*

$$f_{p,k}(S^2\times S^1)=d^{-1}=p^{\frac{l}{2}}\in \mathbf{C}/R_{k,p}$$

Proof. We obtain $S^2\times S^1$ as $L_{0,1}$. Therefore $f_{p,k}(S^2\times S^1)=d^{-1}$. Further

$$d=k^{-l/2}\det\left(\exp\left(\frac{2\pi\sqrt{-1}p^2mn}{k}\right)\right)_{m,n=1,l}$$

Now if g.c.d.$(l,p)>1$ the above considered determinant vanishes since it have two equal lines. Thus the invariants are not defined in this case. Consider now g.c.d.$(l,p)=1$.

LEMMA 3.1. *For* g.c.d.$(q,l)=1$ *the determinant*

$$\det\left(\exp\left(\frac{2\pi\sqrt{-1}qmn}{l}\right)\right)_{m,n=1,l}=l^{l/2}\sqrt{-1}^{\frac{l(l-1)}{2}}(-1)^{h(q,l)}\in \mathbf{C}/R_{ql,q}$$

where

$$h(q,l)=\frac{l(l+1)}{2}+\sum_{r=1}^{l-1}(l-r)\left[\frac{rq}{l}\right]$$

Proof of the lemma: An easy computation shows that the

$$\left(\exp\left(\frac{2\pi\sqrt{-1}qmn}{l}\right)\right)^2_{m,n=1,l}=l(\delta_{m,-n})_{m,n=1,l},$$

where $\delta_{m,-n}$ is the Kronecker delta defined to be 1 if $m+n=0(\text{mod } l)$ and 0 otherwise. Hence the absolute value of the considered determinant is $l^{l/2}$ and its phase is a 4^{th} root of unity, because its square is 1 or -1. Let compute the phase explicitly. On the other hand, this is a Vandermonde determinant hence its value is given by

$$\det\left(\exp\left(\frac{2\pi\sqrt{-1}qmn}{l}\right)\right)_{m,n=1,l}=\prod_{j>h\geq 1}^{l}\left(\exp\left(\frac{2\pi\sqrt{-1}qj}{l}\right)-\exp\left(\frac{2\pi\sqrt{-1}qh}{l}\right)\right).$$

We apply the identity

$$\exp(\sqrt{-1}x)-\exp(\sqrt{-1}y)=2\sin\left(\frac{x-y}{2}\right)\exp\left(\sqrt{-1}\frac{x+y}{2}\right),$$

to transform each member of the last product into

$$\exp\left(\frac{2\pi\sqrt{-1}qj}{l}\right)-\exp\left(\frac{2\pi\sqrt{-1}qh}{l}\right)=2\sqrt{-1}\,\sin\left(\frac{\pi(h-j)q}{l}\right)\exp(\frac{\pi\sqrt{-1}(h+j)q}{l}).$$

The phase has now two contributions: that from the sign of the product of sinuses and that from the rest. The latter gives

$$\frac{\pi l(l-1)}{2}+\sum_{l\geq j>h\geq 1}\frac{\pi(j+h)q}{l}=\frac{\pi l(l-1)}{2}+\frac{\pi q(l^2-1)}{2}.$$

The contribution of the sinuses product is

$$e=(-1)^{\frac{l(l+1)}{2}}\prod_{j>h\geq 1}^{l}\operatorname{sgn}\sin\left(\frac{\pi q(h-j)}{l}\right),$$

after we inverted the order of h and j. Now the argument $\frac{\pi q(h-j)}{l}$ is going all over the set $\{\frac{\pi q}{l},\frac{\pi 2q}{l},\dots,\frac{\pi q(l-1)}{l}\}$, each value $\frac{\pi qr}{l}$ being taken exactly $(l-r)$ times. But the sign of $\sin\frac{\pi qr}{l}$ is given by $(-1)^{\left[\frac{qr}{l}\right]}$, so that

$$e=(-1)^{h(q,l)}.$$

An elementary analysis shows that $\exp(\frac{\pi\sqrt{-1}q(l-1)}{2})\in R_{ql,l}$, and our claim follows.□
The computation of d is by now straightforward.□

3.2. The lens spaces $L_{a,1}$

We may restrict ourselves, when working with lens spaces, to the case when $0 < 2b < a$ (see [5, 15]) since $L_{a,b}$ is homeomorphic to $L^*_{a,a-b}$. It is also known that $\pi_1(L_{a,b}) = \mathbf{Z}/a\mathbf{Z}$. Then $L_{a,b}$ is homeomorphic to $L_{a,b'}$ if and only if $b' = \varepsilon b (\mathrm{mod}\ a)$ or $bb' = \varepsilon (\mathrm{mod}\ a)$ where $\varepsilon \in \{-1, 1\}$. Further the homeomorphism preserves the orientation only if $\varepsilon = 1$. Also there exists a homotopy equivalence between $L_{a,b}$ and $L_{a,b'}$ if and only if $b' = \varepsilon n^2 b (\mathrm{mod}\ a)$ for some integer n, and again, the homotopy equivalence map preserves the orientation if and only if $\varepsilon = 1$.

Let us denote by $G(u, v)$ the Gauss sum

$$G(u, v) = \sum_{x \in \mathbf{Z}/v\mathbf{Z}} \exp\left(\frac{2\pi\sqrt{-1}ux^2}{v}\right).$$

According to [21], p.85-91 (there are some errors which can be fixed easily) the value of the Gauss sum is

$$G(u, v) = dG\left(\frac{u}{d}, \frac{v}{d}\right), \text{ if g.c.d.}(u, v) = d,$$

and for g.c.d.$(u, v) = 1$ we have

$$G(u, v) = \begin{cases} \varepsilon(v) \left(\frac{u}{v}\right) \sqrt{v} & \text{for odd } v \\ 0 & \text{for } v = 2 (\mathrm{mod}\ 4) \\ \overline{\varepsilon(u)} \left(\frac{v}{u}\right) \left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right) \sqrt{2v} & \text{for } v = 0 (\mathrm{mod}\ 4). \end{cases} .$$

Here $\left(\frac{u}{v}\right)$ is the Jacobi symbol and

$$\varepsilon(a) = \begin{cases} 1 & \text{if } a = 1 (\mathrm{mod}\ 4) \\ \sqrt{-1} & \text{if } a = 3 (\mathrm{mod}\ 4). \end{cases}$$

Remember that the Jacobi (or the quadratic) symbol $\left(\frac{P}{Q}\right)$ is defined only for odd Q by the recurrent formula

$$\left(\frac{P}{Q}\right) = \prod_{i=1}^{s} \left(\frac{P}{q_i}\right),$$

where $Q = q_1 q_2 \ldots q_s$ is the prime decomposition of Q, and for prime q the quadratic symbol (also called the Legendre symbol in this setting) is

$$\left(\frac{P}{q}\right) = \begin{cases} 1 & \text{if } P = x^2 (\mathrm{mod}\ q) \\ -1 & \text{otherwise.} \end{cases}$$

The quadratic symbol verifies the following reciprocity law

$$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = (-1)^{\frac{P-1}{2}\frac{Q-1}{2}},$$

in the case when both P and Q are odd.

PROPOSITION 3.2. *If* g.c.d.$(a, k) = 1$ *then*

$$f_{p,k}(L_{a,1}) = 1 \in \mathbf{C}/R_{k,p}.$$

Proof. Denote $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ by $t(a)$. Consider now the classical expansion in continued fraction ([14]): $\frac{b}{a} = \{-a_1, a_2, ..., (-1)^m a_m\}, (-1)^m a_m \geq 2, (-1)^j a_j > 0$ for all j.

LEMMA 3.2. *There exist some natural numbers $c, d \in \mathbf{Z}$ satisfying the Diophantine equation $bd - ac = 1$, such that $\begin{bmatrix} a & d \\ b & c \end{bmatrix}$ may be decomposed as follows*

$$(-1)^{[\frac{m+1}{2}]}\tau t(a_1)\tau t(a_2)\tau ... \tau t(a_m)\tau.$$

where the right brackets state for the integer part.

Proof of the lemma: It suffices to prove that

$$\begin{bmatrix} a & d \\ b & c \end{bmatrix} = \begin{cases} s(-a_1)t(a_2)s(-a_3)...s((-1)^m a_m) & \text{for odd } m \\ s(-a_1)t(a_2)s(-a_3)...t((-1)^m a_m)\tau & \text{otherwise,} \end{cases}$$

where

$$s(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = -\tau t(-a)\tau.$$

This well-known arithmetical result can be proved by recurrence. We omit the details.□

In our proposition we considered $b = 1$. We have then:

$$-k\rho_1(s(a))_{mn} = \sum_{r=1}^{k} \exp\left(\frac{-\pi\sqrt{-1}(2(n+m)r + ar^2)}{k}\right) =$$

$$= \exp\left(\frac{\pi\sqrt{-1}(n+m)^2 a^*}{k}\right) \sum_{r=1}^{k} \exp\left(\frac{-\pi\sqrt{-1}a(r + (n+m)a^*)^2}{k}\right),$$

where $aa^* = 1 (\text{mod } k)$ and thus a^* is well-defined in $\mathbf{Z}/k\mathbf{Z}$. We remark that the value of the sum does not change when we consider the sum over any other subset S of $\mathbf{Z}/2k\mathbf{Z}$, which has k elements and maps onto $\mathbf{Z}/k\mathbf{Z}$ under the natural morphism $\mathbf{Z}/2k\mathbf{Z} \to \mathbf{Z}/k\mathbf{Z}$. This is a consequence of the equality $(r+k)^2 = r^2 (\text{mod } 2k)$, for even k. Furthermore

$$\sum_{r=1}^{k} \exp\left(\frac{-\pi\sqrt{-1}a(r + (n+m)a^*)^2}{k}\right) = \sum_{r=1}^{k} \exp\left(\frac{-\pi\sqrt{-1}ar^2}{k}\right) = \frac{1}{2}\sum_{r=1}^{2k} \exp\left(\frac{-\pi\sqrt{-1}ar^2}{k}\right).$$

We obtained this way

$$\rho_1(s(a))_{mn} = -\frac{k^{-1}}{2} \exp\left(\frac{\pi\sqrt{-1}(n+m)^2 a^*}{k}\right) G(-a, 2k).$$

We have to compute now the determinant of the submatrix corresponding to m, n divisible by p. Although $a^* \in \mathbf{Z}/k\mathbf{Z}$ an expression like $\exp(\frac{\pi r\sqrt{-1}a^*}{l})$ makes sense since we may consider a^* also as an element of $\mathbf{Z}/l\mathbf{Z}$ using the canonical reduction mod l morphism $\mathbf{Z}/k\mathbf{Z} \to \mathbf{Z}/l\mathbf{Z}$. Then the determinant to compute reads:

$$\det\left(\exp\left(\frac{\pi\sqrt{-1}p(n+m)^2 a^*}{l}\right)\right)_{m,n=1,l} = \det\left(\exp\left(\frac{\pi\sqrt{-1}p(a^*n^2 + a^*m^2 + 2a^*mn)}{l}\right)\right)_{m,n=1,l} =$$

$$\prod_{n=1}^{l} \exp\left(\frac{\pi\sqrt{-1}pa^*n^2}{l}\right) \prod_{m=1}^{l} \exp\left(\frac{\pi\sqrt{-1}pa^*m^2}{l}\right) \det\left(\exp\left(\frac{2\pi\sqrt{-1}pa^*mn}{l}\right)\right)_{m,n=1,l} =$$

$$\exp\left(\frac{2\pi\sqrt{-1}pa^*(l+1)(2l+1)}{6}\right) \det\left(\exp\left(\frac{2\pi\sqrt{-1}pa^*mn}{l}\right)\right)_{m,n=1,l}.$$

Now the matrix $(\exp(\frac{2\pi\sqrt{-1}pa^*mn}{l}))_{m,n}$ is invertible if and only if pa^* is invertible, when considered in $\mathbf{Z}/l\mathbf{Z}$. We know already that g.c.d.$(p,l)=1$ (since the invariants are defined) and that g.c.d.$(a,k)=1$. This shows that indeed pa^* is invertible, and so, from lemma 3.1 we derive that

$$f_{p,k}(L_{a,1}) = (-1)^{l+h(pa^*,l)-h(p,l)}\frac{k^{-l/2}}{2^l}G(-a,2k)^l \in \mathbf{C}/R_{k,p},$$

because $\exp(\frac{2\pi\sqrt{-1}pa^*(l+1)(2l+1)}{6}) \in R_{k,p}$. Furthermore $2k = 0 (\mathrm{mod}\ 4)$ because k is even, and thus we have to replace above the explicit value of the Gauss sum to obtain the claimed result. □

PROPOSITION 3.3. *Let assume that $a = -va_0$, $k = vk_0$, with $gcd(a_0, k_0) = 1$, $v > 1$ and even a_0k_0.*

- *Suppose now that v divides p, so that $v_0 = v/\mathrm{g.c.d.}(p,v) = 1$. We have then:*

  (15) $$f_{p,k}(L_{a,1}) = \lambda v^{l/2},$$

  where

  (16) $$\lambda = \exp\left(-\pi\sqrt{-1}\left(\frac{2p_0a_0^*(l+1)(2l+1)}{6} + \frac{p_0(l-1)}{2}\right)\right),$$

  and $p_0 = p/\mathrm{g.c.d.}(p,v)$.

- *Assume that v_0, which is a divisor of l, verifies $v_0 = l$. Then*

  (17) $$f_{p,k}(L_{a,1}) = v^{l/2}l^{-l/2}.$$

  holds.

Proof. We begin with

LEMMA 3.3. *Assume that $a = -va_0$, $k = vk_0$, with* g.c.d.$(a_0,k_0) = 1$ *and $v > 1$. Then the entry $\rho_1(s(a))_{mn}$ has the following form:*

- *For even a_0k_0,*

$$\rho_1(s(a))_{mn} = \begin{cases} -k^{-1}v\sum_{r=1}^{k_0}\exp\left(-\pi\sqrt{-1}\frac{2sr+a_0r^2}{k_0}\right) & \text{if } n+m = sv, \\ 0 & \text{if } n+m \neq 0(\mathrm{mod}\ v). \end{cases}$$

- *For odd a_0k_0,*

$$\rho_1(s(a))_{mn} = \begin{cases} -k^{-1}v\sum_{r=1}^{k_0}\exp\left(-\pi\sqrt{-1}\frac{sr+a_0r^2}{k_0}\right) & \text{if } n+m = sv_1 \text{ with odd } s, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of the lemma. We have from the definition that $\rho_1(s(a))_{mn}$ is equal to:

$$-k^{-1}\sum_{r=1}^{k}\exp\left(-\pi\sqrt{-1}\frac{2(n+m)r+ar^2}{k}\right) =$$

$$= -k^{-1}\sum_{j=0}^{v-1}\sum_{r=1}^{k_0} exp\left(-\pi\sqrt{-1}\frac{2(n+m)(r+jk_0)+a(r+jk_0)^2}{k}\right) =$$

$$= -k^{-1}\sum_{j=0}^{v-1}\exp\left(-\pi\sqrt{-1}\frac{2(n+m)jk_0+ak_0^2j^2}{k}\right)\sum_{r=1}^{k_0}\exp\left(-\pi\sqrt{-1}\frac{2(n+m)r+ar^2-2ak_0rj}{k}\right) =$$

$$= -k^{-1}\sum_{j=0}^{k-1}\exp\left(-\pi\sqrt{-1}\frac{2(n+m)jk_0}{k}+a_0k_0j^2\right)\sum_{r=1}^{k_0}\exp\left(\pi\sqrt{-1}\frac{2(n+m)r+ar^2}{k}\right).$$

For even a_0k_0 we use the identity

$$\sum_{j=0}^{v-1}\exp\left(\frac{2\pi\sqrt{-1}Nj}{v}\right) = \begin{cases} v & \text{if } N = 0(\text{mod } v),\\ 0 & \text{otherwise.}\end{cases}$$

and see that the first sum is zero unless $n+m = sv$ for some $s \in \mathbf{Z}$. When replacing it in the second sum we find

$$\rho_1(s(a))_{mn} = \begin{cases} -k^{-1}v\sum_{r=1}^{k_0}\exp\left(-\pi\sqrt{-1}\frac{2sr+a_0r^2}{k_0}\right) & \text{if } n+m = sv,\\ 0 & \text{if } n+m \neq 0(\text{mod } v).\end{cases}$$

For odd a_0k_0 the factor $\exp(\pi\sqrt{-1}a_0k_0j^2)$ is alternatively 1 and -1. Moreover since k is even then $v = 2v_1$ must be even too. We compute first

$$\sum_{j=0}^{v-1}(-1)^j\exp\left(\frac{2\pi\sqrt{-1}Nj}{v}\right) = 2\sum_{j=0}^{v_1-1}\exp\left(\frac{2\pi\sqrt{-1}2Nj}{v}\right) - \sum_{j=0}^{v-1}\exp\left(\frac{2\pi\sqrt{-1}Nj}{v}\right).$$

When $N = 0(\text{mod } v)$ the first sum is $v/2$ and the second is v so that their total contribution is 0. If $N \neq 0(\text{mod } v_1)$ then both sums are vanishing. The only possibility left is $N = sv_1$ with odd s when the first sum is $v/2$ and the second is 0. Then summing up

$$\sum_{j=0}^{v-1}(-1)^j\exp\left(\frac{2\pi\sqrt{-1}Nj}{v}\right) = \begin{cases} v & \text{if } N = sv_1 \text{ with odd } v,\\ 0 & \text{otherwise.}\end{cases}$$

It follows then that for odd a_0k_0 the entries are;

$$\rho_1(s(a))_{mn} = \begin{cases} -k^{-1}v\sum_{r=1}^{k_0}\exp\left(-\pi\sqrt{-1}\frac{sr+a_0r^2}{k_0}\right) & \text{if } n+m = sv_1 \text{ with odd } s,\\ 0 & \text{otherwise.}\end{cases}$$

This establishes the lemma.□

Let us denote by a_0^* the inverse of a_0 mod k, so that $a_0a_0^* = 1(\text{mod } k_0)$.

Assume we have an even a_0k_0 and that v divides p. The sum of quadratic exponentials can be expressed as a Gauss sum, as we did before in the proof of proposition 3.2. Then the entries of the submatrix $\rho_1(s(a))_{pm,pn}$ are all nonzero and given respectively by

$$-k^{-1}v\frac{1}{2}G(a_0,2k_0)\exp\left(-\pi\sqrt{-1}\frac{a_0^*p_0(m+n)^2}{l}\right).$$

We know that g.c.d.$(a_0,k_0) = 1$ so that either a_0 is even and k_0 odd, or else a_0 is odd and k_0 is even. In the former case

$$G(a_0,2k_0) = 2G(\frac{a_0}{2},k_0) = 2\varepsilon(k_0)\left(\frac{a_0}{2k_0}\right)\sqrt{k_0},$$

in the latter

$$G(a_0,2k_0)=\overline{\varepsilon(a_0)}\left(\frac{2k_0}{a_0}\right)(1+\sqrt{-1})\sqrt{2k_0}.$$

The determinant can be therefore computed using the same method as in lemma 3.1. and find then

$$\det(\rho_1(s(a)_{pm,pn})_{m,n=1,l}=\lambda(-1)^{l+h(-p_0a_0^*,l)}l^{-l/2}v^lk^{-l}\frac{1}{2^l}G(a_0,2k_0)^l \tag{18}$$

where

$$\lambda=\exp\left(-\pi\sqrt{-1}\left(\frac{2p_0a_0^*(l+1)(2l+1)}{6}+\frac{p_0(l-1)}{2}\right)\right). \tag{19}$$

In general we cannot get rid of this term λ which is a 12-th root of unity. Now the value of the invariant is that claimed in the first part of the proposition when mod out the $R_{k,p}$ indeterminacy.

Consider further the case when the divisor v_0 of l reaches its maximum value, namely $v_0=l$. We have $p(m+n)=0(\text{mod } v)$ iff $m+n=0(\text{mod } v_0)$, so that the only nonzero entries $\rho_1(s(a))_{pm,pn}$ are those for which $m+n=l$ or $m+n=2l$. The determinant of such a matrix is therefore $(-1)^{l(l+1)/2}$ times the product of all its nonzero entries. In our case this gives $(-1)^{l(l+1)/2}\exp(\pi\sqrt{-1}p_0(l-1)(-k^{-1}vG(a_0,2k_0))^l$. Observe that $pl=k_0\ v_0$ g.c.d.(v,p) so that $p_0=k_0$. Thus for even k_0 we have $(-1)^{p_0(l-1)}=1$. This establishes our claim.□

PROPOSITION 3.4. *Assume now that $a=-a_0v$, $k=k_0v$, with even a_0k_0 and additionally $l=2v_0$. Then k_0 is even, a_0 is odd and*

$$f_{p,k}(L_{a,1})=v^{l/2}l^{-l/2}d_l,$$

where

$$d_l=(-1)^l\frac{2}{\sqrt{5}}\left(\left(\frac{-3+\sqrt{5}}{2}\right)^{[\frac{l}{2}]-1}-\left(\frac{-3-\sqrt{5}}{2}\right)^{[\frac{l}{2}]-1}\right).$$

Proof. The associated matrix $\rho_1(s(a))_{pm,pn}$ has only four nonzero skew diagonals (i.e. parallel to the diagonal going from the top right corner to the bottom left corner) and the remaining cases are filled up with zeroes. Moreover these are equidistant, corresponding to $m+n\in\{v_0,2v_0,3v_0,4v_0\}$. On each skew diagonal all the elements are the same, and the respective common values are (up to a factor of $-k^{-1}vG(a_0,2k_0)$) $\alpha=\sqrt{-1}$, $\beta=1$, $\gamma=\sqrt{-1}$, $\delta=1$. Here we used the fact that $a_0^*p_0$ is odd. In fact we know that $k=k_0v=pl$, g.c.d.$(p,l)=1$ and g.c.d.$(a_0,k_0)=1$. Then $k_0=pl/v=p_0l/v_0=2p_0$ is even, so that a_0 must be odd, since relatively prime with k_0. Also l even implies p odd henceforth p_0 is odd. Choose a lift a_0^* of a_0^* in the first k_0 integers (we will use the same letter from now on for such lifts); then $a_0a_0^*=1+tk_0$ so a_0^* is odd.

Let us consider the determinant D_l of the l by l matrix having three equidistant skew diagonals filled up with α,β,γ respectively, and corresponding to $m+n\in\{\left[\frac{l}{2}\right],l,2l-\left[\frac{l}{2}\right]\}$. Set d_l for the determinant we want to compute, and which has four skew diagonals (the last one is just the bottom right corner element). For odd l the position of the other diagonals is like that for D_l. Then an easy induction gives us the recurrence relations:

$$d_l=\delta D_{l-1}+\gamma^2D_{l-2}.$$

$$D_l=\alpha\gamma D_{l-2}+(-1)^{l-1}\beta D_{l-1}.$$

We have then to find out d_l for $\alpha = \gamma = \sqrt{-1}$, $\beta = \delta = 1$, in a closed form. This can be done as follows. We have a recurrence relation which can be written as

$$(D_l, D_{l-1}) = A_l(D_{l-1}, D_{l-2})^\top,$$

where $A_l = \begin{pmatrix} (-1)^{l-1} & -1 \\ 1 & 0 \end{pmatrix}$. Further we can iterate this formula and get

$$(D_{2m}, D_{2m-1}) = B^m(D_2, D_1)^T.$$

where $B = A_0 A_1$. We diagonalize B, compute B^m and eventually find that

$$D_{2m+2} = -\frac{2}{\sqrt{5}}\left(\left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{-3+\sqrt{5}}{2}\right)^m + \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{-3-\sqrt{5}}{2}\right)^m\right),$$

$$D_{2m+1} = -\frac{2}{\sqrt{5}}\left(\left(\frac{-3+\sqrt{5}}{2}\right)^m - \left(\frac{-3+\sqrt{5}}{2}\right)^m\right).$$

This establishes our claim.□

PROPOSITION 3.5. *For odd a_0k_0, if v divides p or $v/2$g.c.d.$(p, v/2) = l$ then*

$$f_{p,k}(L_{a,1}) = 0$$

For odd a_0k_0 such that $v/$g.c.d.$(p, v/2) = l$ we have

$$f_{p,k}(L_{a,1}) = l^{-l/2}v^{l/2}.$$

Proof. For odd a_0k_0 we want first to compute the elements of the matrix $\rho_1(s(a))_{pm,pn}$. In order that the entry on the position pm, pn be nonzero we must have $p(m+n) = sv_1$, where $v = 2v_1$ and odd s. Then k_0 is odd and there exists some s' for which $s = 2s' (\text{mod } k_0)$ (actually $s' = 2^*s$). Notice that we don't have any control on the parity of 2^*. We know however that $2s' = s + \mu k_0$ implies μ is odd and thus:

$$\sum_{r=1}^{k_0} \exp\left(\pi\sqrt{-1}\frac{sr + a_0r^2}{k_0}\right) = \sum_{r=1}^{k_0}(-1)^r \exp\left(\pi\sqrt{-1}\frac{2s'r + a_0r^2}{k_0}\right) =$$

$$\left(\exp\left(-\frac{\pi\sqrt{-1}a_0^* s'^2}{k_0}\right)\right)\sum_{r=1}^{k_0}(-1)^r \exp\left(\frac{\pi\sqrt{-1}a_0(r + s'a_0^*)^2}{k_0}\right).$$

Let us check the value of the last sum, which up to a factor of $(-1)^{s'} = (-1)^{s'a_0^*}$ coincides with

$$\sum_{r=1}^{k_0}(-1)^r \exp\left(\frac{\pi\sqrt{-1}a_0r^2}{k_0}\right) = 2\sum_{r=1}^{k_0/2} \exp\left(\frac{4\pi\sqrt{-1}a_0r^2}{k_0}\right) - \sum_{r=1}^{k_0} \exp\left(\frac{\pi\sqrt{-1}a_0r^2}{k_0}\right).$$

Now k_0 is odd $k_0 = 2k_1 + 1$, and by switching r into $k_0 - r$ we obtain:

$$\sum_{r=1}^{k_1} \exp\left(\frac{4\pi\sqrt{-1}a_0r^2}{k_0}\right) = \sum_{r=k_1+1}^{2k_1} \exp\left(\frac{4\pi\sqrt{-1}a_0r^2}{k_0}\right),$$

from which we derive that

$$2\sum_{r=1}^{k_0/2} \exp\left(\frac{4\pi\sqrt{-1}a_0 r^2}{k_0}\right) = G(2a_0, k_0) - 1.$$

On the other hand, if we change r into $2k_0 - r$ we derive

$$\sum_{r=1}^{k_0} \exp\left(\frac{\pi a_0 r^2}{k_0}\right) = \sum_{r=k_0}^{2k_0-1} \exp\left(\frac{\pi a_0 r^2}{k_0}\right),$$

so that

$$\sum_{r=1}^{k_0} \exp\left(\frac{\pi a_0 r^2}{k_0}\right) = \frac{1}{2}(G(a_0, 2k_0) - 1 + (-1)^{k_0}).$$

Now k_0 is odd implies that $G(a_0, 2k_0) = 0$ and thus for those m, n for which the corresponding entry is non-zero, we have

$$-kv^{-1}\rho_1(s(a))_{mn} = (-1)^{s'} G(2a_0, k_0)\left(\exp\left(-\frac{\pi\sqrt{-1}a_0^* s'^2}{k_0}\right)\right).$$

Observe that $s' = 2^* p(m+n)/v_1$, so that the last term is actually a $2l$-th root of unity. Suppose now that $v = 2v_1$ divides p, or equivalently that v_1 divides p and p is even. Then $p(m+n)$ is always an even number times v_1 so that all the entries of the matrix are zero. It follows that the invariant vanishes in this case.

We discard for the moment the factor $-k^{-1}vG(2a_0, k_0)$, which is common to all entries of the matrix.

Let consider $v_2 = v_1/\text{g.c.d.}(v_1, p)$. Then $p(m+n) = 0 (\text{mod } v_1)$ is equivalent to $m+n = 0 (\text{mod } v_2)$. It is possible to have $v_2 = l$ and a necessary condition is that the maximal power of 2 which divides p is at least equal to that corresponding to v. Then we have $v_2 = v/\text{g.c.d.}(v, p)$. For instance if p is odd then $2v_2$ is a divisor of l hence $v_2 \leq l/2$. It follows that all the non-zero entries sit on the skew diagonal $m+n = l$, because $m+n = 2l$ is forbidden. Then the determinant is zero, so the invariants are zero too.

The following case is $2v_2 = l$. Then we obtain a determinant of the same shape as d_l, which was considered previously, with the new parameters $\alpha = \gamma = 1$, $\beta = \delta = 0$. The value of this determinant is then -1, using the recurrence formulas provided above. This ends the proof of the proposition.□

3.3. $L_{a,b}$ and generalized Gauss sums

We consider now the general case of lens spaces $L_{a,b}$ with $b \geq 2$, which we will treat in the same manner. Let us define the sequences $\Delta(a_1, a_2, ..., a_n)$ by the recurrence relations

$$\Delta(a_1) = a_1,$$

$$\Delta(a_1, a_2) = a_1 a_2 - 1,$$

$$\Delta(a_1, a_2, ..., a_{n+1}) = a_{n+1}\Delta(a_1, a_2, ..., a_n) - \Delta(a_1, a_2, ..., a_{n-1}).$$

Then it is easy to check out that $\Delta = \Delta(a_1, a_2, ..., a_m) = (-1)^{\left[\frac{m+1}{2}\right]}a$, $\Delta(a_1, a_2, ..., a_{m-1}) = (-1)^{\left[\frac{m}{2}\right]}d$, $\Delta(a_2, a_3, ..., a_m) = (-1)^{\left[\frac{m}{2}\right]}b$. Assume now a is invertible in $\mathbf{Z}/2k\mathbf{Z}$ and so there exists an inverse $a^* \in \mathbf{Z}/2k\mathbf{Z}$. Set then:

$$A = A(a_1, ..., a_m) = \Delta^*\Delta(a_2, ..., a_m),$$

$$B = B(a_1, ..., a_m) = \Delta^* \Delta(a_1, ..., a_{m-1}).$$

PROPOSITION 3.6. *If a is invertible in $\mathbf{Z}/k\mathbf{Z}$ then $f_{p,k}(L_{a,b}) = 1$.*

Proof. Let us denote by $G(Q, n)$ the generalized Gauss sum $G(Q, k) = \sum_{r_j=1,k} exp(\frac{2\pi i}{k} Q(r_1, r_2, ..., r_m))$, where Q states for a quadratic form in m variables. If $w = (a_1, a_2, ..., a_m)$ set also

$$Q_w(r_1, r_2, ..., r_m) = \sum_{j=1,m} a_j r_j^2 + 2 \sum_{j=1,m-1} r_j r_{j+1}.$$

We will establish first a weaker claim, namely:

LEMMA 3.4. *If a is invertible in $\mathbf{Z}/k\mathbf{Z}$, then we have*

$$f_{p,k}(L_{a,b}) = p^{\frac{l}{2}} k^{-\frac{l(m+1)}{2}} \frac{1}{2^{ml}} G(Q_w, 2k)^l,$$

where $w = (a_1, a_2, ..., a_m)$.

Proof. From lemma 3.2 together with the explicit forms for the matrices $\rho_1(\tau)$ and $\rho_1(t(a))$ we derive that:

$$(-1)^{\left[\frac{m+1}{2}\right]} k^{\frac{m+1}{2}} \rho_1(\left[\begin{array}{cc} b & c \\ a & d \end{array}\right])_{st} = \sum_{r_1,...,r_m=1}^{k} \exp\left(-\frac{\pi\sqrt{-1}}{k}\left(\sum_{i=1}^{m} a_i r_i^2 + 2\sum_{i=1}^{m-1} r_i r_{i+1} + 2sr_1 + 2r_m t\right)\right)$$

so we have to compute the right hand side sum, which is very close to be a generalized Gauss sum.

LEMMA 3.5. *If a is invertible mod k then the formula:*

$$(-1)^{\left[\frac{m+1}{2}\right]} k^{\frac{m+1}{2}} \rho_1(\left[\begin{array}{cc} b & c \\ a & d \end{array}\right])_{st} = \frac{1}{2^m} \exp\left(-\frac{\pi\sqrt{-1}}{k}\left(As^2 + Bt^2 + 2(-1)^m a^* st\right)\right) G(Q_w, 2k).$$

holds.

Notice that the exponential on the right hand side makes sense since $A, B \in \mathbf{Z}/2k\mathbf{Z}$.

Proof of the lemma. Let us consider the matrix

$$L = \left[\begin{array}{ccccccc} a_1 & 1 & 0 & ... & 0 & 0 & 0 \\ 1 & a_2 & 1 & 0 & ... & 0 & 0 \\ 0 & 1 & a_3 & 1 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & ... & 0 & 0 & 1 & a_m \end{array}\right].$$

Then a recurrence argument shows that $\Delta(a_1, ..., a_m) = \det(L)$. Since $a = -\Delta$ was supposed invertible mod k (and thereby mod $2k$ because k is even) there exist solutions $\lambda_j \in \mathbf{Z}/2k\mathbf{Z}$ of the linear equation $L\lambda = (s, 0, 0, ...0, t)^\top$. Let us check that

$$Q_w(\lambda) = As^2 + Bt^2 + 2(-1)^m a^* st.$$

In fact, we know that $Q_w(\lambda) = \lambda_1 s + \lambda_m t$, and $\lambda = L^{-1}(s, 0, ..., 0, t)$. All we need from the matrix L^{-1} are the entries sitting on the corners, which can be easily found

$$L^{-1}_{11} = A,\ L^{-1}_{mm} = B,\ L^{-1}_{1m} = L^{-1}_{m1} = (-1)^{m+1}\Delta^*.$$

We can replace the original sum over the integer r_j's with a sum over the first k classes mod $2k$, as we already observed. On the other hand we have

$$Q_w(r+\lambda) = Q_w(r) + 2sr_1 + 2tr_m + Q_w(\lambda) \in \mathbf{Z}/2k\mathbf{Z},$$

and thus

$$\sum_{r_j=1,k} \exp\left(-\frac{\pi\sqrt{-1}}{k}\left(Q_w(r) + 2sr_1 + 2tr_m\right)\right) = \sum_{r_j=1,k} \exp\left(-\frac{\pi\sqrt{-1}}{k}\left(Q_w(r+\lambda) - Q_w(\lambda)\right)\right) =$$

$$= \exp\left(-\frac{\pi\sqrt{-1}}{k}(As^2 + Bt^2 + 2(-1)^{m+1}\Delta^* st)\right) \sum_{r_j=1,k} \exp\left(-\frac{\pi\sqrt{-1}}{k} Q_w(r+\lambda)\right) =$$

$$= \exp\left(-\frac{\pi\sqrt{-1}}{k}(As^2 + Bt^2 + 2(-1)^m a^* st)\right) \frac{1}{2^m} G(Q_w, 2k).$$

This proves the lemma 3.5.□

In the last equality we used the fact that the sum over any k consecutive classes we choose out of the $2k$ elements of $\mathbf{Z}/2k\mathbf{Z}$ is independent on the choice we made.

It remains therefore to compute the determinant of

$$\left(\exp\left(\frac{\pi\sqrt{-1}}{l}\left(pAs^2 + pBt^2 + 2(-1)^m pa^* st\right)\right)\right)_{s,t=1,l}.$$

Here it is understood that A, B, a^* are reduced mod l. We already encountered such a determinant in the $m = 1$ case. Specifically we have

$$\begin{aligned} &\det\left(\exp\left(\frac{\pi\sqrt{-1}}{l}\left(pAs^2 + pBt^2 + 2pCst\right)\right)\right)_{s,t=1,l} \\ &= \exp\left(-\frac{\pi\sqrt{-1}p(A+B)(l+1)(2l+1)}{6}\right) \det\left(\exp\left(-\frac{2\pi\sqrt{-1}pCst}{l}\right)\right)_{s,t=1,l}. \end{aligned}$$

Now $pa^* \in \mathbf{Z}/2l\mathbf{Z}$ must be invertible so from the lemma 3.1 the last determinant equals

$$(-1)^{h((-1)^m pa^*, l)} \sqrt{-1}^{\frac{l(l-1)}{2}} l^{l/2} \exp(-\frac{\pi\sqrt{-1}p(l-1)}{2}).$$

Since $\exp(\frac{\pi\sqrt{-1}p(A+B)(l+1)(2l+1)}{6}) \in R_{k,p}$ the lemma 3.4. follows. □

Observe that the generalized Gauss sums $G(Q, 2k)$ may be computed as a product of usual Gauss sums if the matrix L can be diagonalized over $\mathbf{Z}/2k\mathbf{Z}$. Actually the monoid of symmetric (or skew) bilinear forms over a finite Abelian group was computed by Wall ([32]) for all p-groups with $p > 2$ and completed by Kawauchi and Kojima ([19]) for $p = 2$. Thus $G(Q, n)$ can be, in principle, calculated but this method is rather cumbersome. Another, more direct, way to calculate it was given in [26].

Consider L be an arbitrary m-by-m matrix with integer entries, N a fixed natural number and q a primitive root of unity of order N if N is odd, and of order $2N$ for even N. Then set, after [26]:

$$G_N(L,q) = \sum_{x\in(\mathbf{Z}/N\mathbf{Z})^m} q^{<Lx,x>},$$

where $<,>$ is the standard scalar product. Notice that for even N the scalar product is a map

$$<,>: (\mathbf{Z}/N\mathbf{Z})^m \times (\mathbf{Z}/N\mathbf{Z})^m \longrightarrow \mathbf{Z}/2N\mathbf{Z},$$

and thus the sum is well-defined. In our case, since k is even, we have

$$G(< Lx, x >, 2k) = 2^m G_k(L, \exp\left(\frac{\pi\sqrt{-1}}{k}\right)).$$

Furthermore the absolute value of this more general Gauss sum is easy to compute (suppose N is even from now on):

$$\mid G_N(L,q) \mid^2 \;\; = N^m \sum_{x\in \ker L} q^{<Lx,x>}.$$

Let φ be the restriction of the quadratic function $< Lx, x >$ at $\ker L : \mathbf{Z}/N\mathbf{Z} \longrightarrow \mathbf{Z}/N\mathbf{Z}$, so that $\varphi_L : \ker L \longrightarrow \{0, N\} \subset \mathbf{Z}/2N\mathbf{Z}$ (q is a $2N$-th root of unity). We have then

$$\mid G_N(L,q) \mid \;\; = \begin{cases} N^{m/2} \mid \ker L \mid^{1/2} & \text{if } \varphi_L = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If L is a symmetric matrix there exists a (framed) link $\mathcal{L}$ whose linking matrix is L and the 3-manifold M_L obtained by Dehn surgery on $\mathcal{L}$ has the linking matrix L. According to the interpretation given in [26] we have

$$\mid \ker L \mid \;\; = \;\; \mid H^1(M_L, \mathbf{Z}/N\mathbf{Z}) \mid,$$

and $\varphi_L = 0$ if and only if $\alpha \cup \alpha \cup \alpha = 0$, for any $\alpha \in H^1(M_L, \mathbf{Z}/N\mathbf{Z})$.

On the other hand it is shown that the phase of $G_N(L, q)$ is always an eighth root of unity $\phi_N(L, q)$. Moreover the value $\phi_N(M) = \phi_N(L, q)\lambda_k^{-\sigma(L)}$ is a homotopy invariant, of the 3-manifold $M = M_L$ in terms of the linking matrix L, which generalizes the Brown invariant. Its explicit computation rely on the formula

$$G_{NK}(L,q) = G_N(L, q^{K^2}) G_K(L, q^{N^2}),$$

which permits first to work out only the cases $N = p^r$, and further to use the stable decomposition of L into standard components over p-groups, in order to reduce the computation to that of the generators of the module of bilinear forms. We won't be concerned with this final evaluation of ϕ_k which is explicit in [26]. This proves the proposition.$\square$

Let us introduce the more general Gauss sums depending on a parameter $\alpha \in (\mathbf{Z}/k\mathbf{Z})^m$

$$G_k(L,q)_\alpha = \sum_{x\in(\mathbf{Z}/k\mathbf{Z})^m} q^{<Lx,x>+2<\alpha,x>},$$

for even k, a primitive $2k$-th root of unity q and a symmetric bilinear form L. These sums are the entries of the matrix $\rho_1(\tau t(a_1)\tau ... \tau)$ for the particular $\alpha_{st} = (s, 0, 0, ..., 0, t)$.

Let us begin to compute the absolute value of these:

$$\mid G_N(L,q)_\alpha \mid^2 = \sum_{x,y\in(\mathbf{Z}/k\mathbf{Z})^m} q^{<Lx,x>+2<\alpha,x>-<Ly,y>-2<\alpha,y>} = \sum_{z\in(\mathbf{Z}/k\mathbf{Z})^m} q^{<Lz,z>+2<\alpha,z>} \sum_{y\in(\mathbf{Z}/k\mathbf{Z})^m} q^{2<Lz,y>},$$

where $z = x - y$. It is known that

$$\sum_{x\in(\mathbf{Z}/k\mathbf{Z})^m} q^{2<\alpha,x>} = \begin{cases} k^m & \text{if } \alpha = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the relevant terms in the sum are those for which $z \in \ker L$, and then it transforms into

$$k^m \sum_{x\in\ker L} q^{<Lz,z>+2<\alpha,z>} = k^m \sum_{x\in\ker L} \chi_L(z) q^{2<\alpha,z>},$$

where

$$\chi_L(z) = q^{\varphi_L(z)} : \ker L \longrightarrow \{-1, 1\}.$$

Assume first that $\varphi_L = 0$.

LEMMA 3.6. *If $M \subset (\mathbf{Z}/k\mathbf{Z})^m$ is a $\mathbf{Z}/k\mathbf{Z}$-submodule then*

$$\sum_{x\in M} q^{2<\alpha,x>} = \begin{cases} \mid M \mid & \text{if } <\alpha, M>= 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof of the lemma. Let X_i be an orthogonal basis for M. Then the sum to be computed transforms into

$$\prod_i \sum_{z_i} q^{2z_i<a,X_i>},$$

which is non-zero, from the 1-dimensional statement, only if $< a, X_i >= 0$ for all i. □

Let denote for a submodule M as above by $M^{\perp}$ the orthogonal submodule formed by those of α which satisfy $< \alpha, M >= 0$. As a consequence we derive that, for $\varphi_L = 0$ we have:

$$\mid G_k(L, q)_\alpha \mid \;= \begin{cases} k^{m/2} \mid \ker L \mid^{1/2} & \text{if } < \alpha, \ker L >= 0 \\ 0 & \text{otherwise.} \end{cases}$$

Consider now the case when $\varphi_L \neq 0$. We need more information on the character χ_L. Set $W = \{x; \chi_L(x) = 1\}$ and $V = \{x; \chi_L(x) = -1\}$, and pick up some $\mu_L \in V$.

LEMMA 3.7.

1. *The function χ_L is multiplicative, namely $\chi_L(z + t) = \chi_L(z)\chi_L(t)$;*
2. *W is a $\mathbf{Z}/k\mathbf{Z}$-submodule;*
3. *$V = \mu_l + W$.*

The proof is immediate.□

The sum to compute is therefore

$$k^m \sum_{x\in\ker L} \chi_L(z) q^{2<\alpha,z>} = k^m \sum_{x\in W} q^{2<\alpha,z>} - k^m \sum_{x\in V} q^{2<\alpha,z>} = k^m \sum_{x\in W} q^{2<\alpha,z>}(1 - q^{2<\mu_L,\alpha>}).$$

The result is a real number, and this forces $q^{2<\mu_L,\alpha>} \in \{-1, 1\}$. Therefore the previous lemmas give in the case when $\varphi_L \neq 0$

$$\mid G_k(L, q)_\alpha \mid \;= \begin{cases} k^{m/2}(2 \mid W \mid)^{1/2} & \text{if } < \alpha, W >= 0,\ 2 < \alpha, V >= k (\text{mod } 2k) \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $| W | = | V |$, so that $2 | W | = | \ker L |$. Thus, when non-zero, all the absolute values $| G_k(L,q)_\alpha |$ are the same.

Let work out now the phase $\phi(L,q)_\alpha$ of $G_k(L,q)_\alpha$. Assume first that $\varphi_L = 0$. Then the only nonzero terms are those for which $\alpha \in \ker L^\perp$.

LEMMA 3.8. *We have* $\ker L^\perp = \mathrm{Im}\ L$, *where* $\mathrm{Im}\ L$ *is the image of* L.

Proof. It is a general fact that $\mathrm{Im}\ L \subset \ker L^\perp$. In fact $x = Ly$ implies $< x, \ker L >=< Ly, \ker L >=< y, L \ker L >= 0$. Let then A_{ij} be $(-1)^{i+j}$ times the complementary minor of the element sitting in the ji position. We can suppose that $\det L \neq 0$ (over $\mathbf{Z}$), because this happens in the case of lens spaces. If g.c.d.$(\det L, k) = v$ then a description of $\mathrm{Im}\ L$ is provided by:

$$\mathrm{Im}\ L = \{x; \sum_j A_{ij} x_j = 0 (\mathrm{mod}\ v),\ i = 1, m\}.$$

Furthermore if A is the matrix made of the A_{ij} then $LA = \det L \mathbf{1}_m$. Let $e_i = (A_{i1}, A_{i2}, ..., A_{in})$. Then $\frac{k}{v} e_i \in \ker L$. Consider $x \in \ker L^\perp$. Then $< x, \frac{k}{v} e_i >= 0$, and thus $\sum_j \frac{k}{v} A_{ij} x_j = 0 (\mathrm{mod}\ k)$. This implies $x \in \mathrm{Im}\ L$ according to the previous description of the image. Therefore $\ker L^\perp \subset \mathrm{Im}\ L$, and the claim is proved.□

Then the non-zero $G_N(L,q)_\alpha$ correspond to $\alpha \in \mathrm{Im}\ L$. Choose some z_α for which $Lz_\alpha = \alpha$.

LEMMA 3.9. *If* $\varphi_L = 0$ *and* $\alpha \in \ker L^\perp$ *then*

$$\phi_k(L,q)_\alpha = q^{-<\alpha, z_\alpha>} \phi_k(L,q)_0.$$

Proof. The same argument which we used in the proof of lemma 3.4. applies word-by-word now. □

LEMMA 3.10. *Assume that* $\varphi_L \neq 0$, α, β *verify* $< \alpha, W >=< \beta, W >= 0$ *and* $2 < \alpha, V >= 2 < \beta, V >= k (\mathrm{mod}\ 2k)$, *so that* $G_k(L,q)_\alpha, G_k(L,q)_\beta$ *are nonzero. Then*

$$\phi(L,q)_\alpha = q^{-<z_{\alpha-\beta}, \beta>} \phi(L,q)_\beta,$$

where $Lz_{\alpha-\beta} = \alpha - \beta$.

Proof. We have

$$G_k(L,q)_\alpha \overline{G_k(L,q)_\beta} = \sum_{x,y \in (\mathbf{Z}/k\mathbf{Z})^m} q^{<Lx,x>+2<\alpha,x>-<Ly,y>-2<\beta,y>} =$$

$$\sum_{z \in (\mathbf{Z}/k\mathbf{Z})^m} q^{<Lz,z>+2<\alpha,z>} \sum_{y \in (\mathbf{Z}/k\mathbf{Z})^m} q^{2<Lz+a-b,y>}.$$

We have $< \alpha - \beta, \ker L >= 0$, and thus there exists $z_{\beta-\alpha}$ such that $Lz_{\beta-\alpha} = \beta - \alpha$. Therefore we obtain

$$k^m \sum_{z \in z_{\beta-\alpha} + \ker L} q^{<Lz,z>+2<\alpha,z>} = k^m q^{<z_{\beta-\alpha}, \beta>} \sum_{z \in \ker L} q^{<Lz,z>+2<\beta,z>} = q^{<z_{\beta-\alpha},\beta>} k^m 2 \mid W \mid.$$

Now the lemma follows.□

Let consider now, for $\varphi_L = 0$:

$$Y_{st}(L,q) = \begin{cases} q^{-<z_{\alpha_{st}},\alpha_{st}>} & \text{if } \alpha_{st} \in \ker L^{\perp} \\ 0 & \text{otherwise.} \end{cases}$$

$$Y(L,p,q) = (Y_{ps,pt}(L,q))_{s,t=1,l}.$$

If $\varphi_L \neq 0$ then put in a similar way:

$$Y_{st}(L,q) = \begin{cases} q^{-<z_{\alpha_{st}-u},\alpha_{st}>} & \text{if } \alpha_{st} \in W^{\perp}, 2 < \alpha_{st}, V >= k (\text{mod } 2k) \\ 0 & \text{otherwise.} \end{cases}$$

$$Y(L,p,q) = (Y_{ps,pt}(L,q))_{s,t=1,l}.$$

where u is a fixed element satisfying $< u, W >= 0, 2 < u, V >= k(\text{mod } 2k)$. Then in order to compute the invariant it remains to find out the determinant of $Y(L,p,q)$. We did not succeeded in having a general close formula for this determinant, but we can give an alternative form for Y which is considerably simpler. We resume the calculations only for lens spaces.

LEMMA 3.11. *Set $q_l = q^p$ and assume that $\varphi_L = 0$. Then, for lens spaces we have*

$$Y(L,p,q) = \begin{cases} q_l^{-a_0^* p_0 (\frac{bs+(-1)^{m+1}t}{v_0} s + \frac{(-1)^{m+1}s+dt}{v_0} t)} & \textit{if } bs + (-1)^{m+1}t = 0 (\text{mod } v_0) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We have to identify first $\{(s,t); \alpha_{st} \in \text{Im } L\}$ and $\{(s,t); bs+(-1)^{m+1}t = 0(\text{mod } v)\}$. Let L' be the matrix obtained from L by deleting the last row and the last column. Then $\det(L') = d$, as we remarked before, and g.c.d.$(d,v) = 1$ since $bd - ac = 1$. If $x(r)$ is the solution in $\mathbf{Z}/v\mathbf{Z}$ of the equation $L'x(r) = (s,0,...,0,r)$, then $y = (x(r), -a_m x(r)_{m-1})$ is a solution of $Ly = (s,0,...,0,(-1)^m bs)$. This settles the case when $k = v$. If $k_0 \neq 1$ we set $x = y + zv$, where y are viewed as integers and $z \in \mathbf{Z}/k_0\mathbf{Z}$. The equation $Lx = (s,0,...,(-1)^m bs + vt')$, viewed as an equation in z has therefore solutions since $\det L = a$. This proves one inclusion between the two sets. The necessity of the condition $bs + (-1)^{m+1}t = 0(\text{mod } v)$ follows from the form of the corner entries in the inverse matrix L^{-1} (over $\mathbf{Z}$). We have furthermore

$$z_{\alpha_{st}} = (a_o^* \frac{bs + (-1)^{m+1}}{v}, *, *, ..., *, a_0^* \frac{(-1)^{m+1} + dt}{v}),$$

and now the lemma follows.□

LEMMA 3.12. *For lens spaces $\varphi_L \neq 0$ is equivalent to $a_0 k_0$ odd, and consequently v is even, $v = 2v_1$. Assume that $\varphi_L \neq 0$, $u = \alpha_{s_0 t_0}$. Then*

$$Y(L,p,q) = \begin{cases} q^{-a_0^* p^2 (\frac{bs'+(-1)^{m+1}t'}{v} s + \frac{(-1)^{m+1}s'+dt'}{v} t)} & \textit{if } bs + (-1)^{m+1}t = rv_1, \text{ odd } r \\ 0 & \textit{otherwise.} \end{cases}$$

Here $s' = s - s_0$, and $t' = t - t_0$.

Proof. It is a consequence of the previous proof.□

PROPOSITION 3.7. *We have*

$$f_{p,k}(L_{a,b}) = \text{g.c.d.}(a,k)^l \det(Y(L,p,q)).$$

Proof. It suffices to identify now (see [26]) $\ker L$ and $H^1(M_L, \mathbf{Z}/k\mathbf{Z})$. □

Notice that we computed these determinants for small values of l/v_0 in propositions 3.3-3.5.

Remark 3.1. $L_{7,2}$ is not distinguished from $L_{7,3}$ by the $f_{p,k}$'s. More generally, assume that σ is a permutation which preserves the set $\{1, m\}$, and we have $\frac{a}{b} = \{a_1, ..., a_m\}$, and $\frac{a'}{b'} = \{a_{\sigma(1)}, ..., a_{\sigma(m)}\}$. Then we have the equality $f_{p,k}(L_{a,b}) = f_{p,k}(L_{a',b'})$, for any p, k. In particular we cannot expect this set of invariants $f_{p,k}$ to lead to a classification of lens spaces.

Proof. We remark that the symmetry of the situation will imply that $f_{p,k}(L_{7,2}) = f_{p,k}(L_{7,3})$, for all k, p. This is a consequence of the relation $f_{p,k}(M^*) = \overline{f_{p,k}(M)}$, and the reality of $f_{p,k}$ for lens spaces. The most general case follows from the fact that those permutations fixing 1 and m are symmetries of the quadratic form $Q_w + 2sr_1 + 2r_m t$, and those interchanging 1 and m change the matrix into its transpose.

Remark 3.2.

1. We have $f_{p,k}(M \sharp N) = f_{p,k}(M) f_{p,k}(N)$ and $f_{p,k}(M^*) = \overline{f_{p,k}(M)}$, where the bar in the right member denotes the complex conjugation.
2. Every homology sphere could be obtained by twisting the homeomorphism corresponding to the standard Heegaard decomposition of genus g of S^3 by an homeomorphism lying in the Torelli group $\ker(\mathcal{M}_g \to \text{Sp}(2g, \mathbf{Z}))$, (see [3]). Therefore our invariants are trivial for integer homology spheres.

3.4. Other manifolds

It is known that $f_{k,k}$ can be defined in terms of the linking matrix associated to a Dehn surgery presentation of the 3-manifold.

The computations made for lens spaces involve only the linking matrix L associated to a surgery presentation of $L_{a,b}$. We would like to know if there exists such a description of $f_{k,p}$, in terms only of the linking matrix for arbitrary manifolds. The straightforward way to do that is to use the determinant $d(L) = \det(G_k(L,q)_{\alpha_{ps,pt}})_{s,t=1,l}$, with a certain normalization factor depending on $m, k, l, \sigma(L)$. The interesting point is that, in general $d(L \oplus (1)) = 0$ if g.c.d.$(\det(L), k) > 1$. In fact, we saw before that the non-zero entries are precisely those for which $\alpha_{st} \in \ker L^{\perp}$, if $\varphi_L = 0$. Then $\alpha_{st} \in \ker(L \oplus (1))^{\perp}$ (here α_{st} has $m+1$ components) is equivalent to arbitrary t and the m-component $\alpha_{s0} \in \ker L^{\perp}$. For generic L the last condition singles out values of the type $s = 0 (\text{mod } v)$. This means that the matrix associated to $L \oplus (1)$ has rows filled up with zeroes, thus its determinant vanishes. Thus the method from [26] cannot be applied in this context in this straightforward way.

However the invariants of type $f_{k,k}$ had a canonical extension to cobordisms between parametrized surfaces. Let (M, F, F') be a 3-dimensional cobordism with connected M, and parametrized (or rigid) boundaries F and F' which are supposed to be connected, of genera g and g' respectively. Let consider 3-valent framed graphs G, G' and a framed link L, all embedded in S^3, such that (G, G', L) represent a generalized Dehn surgery presentation of the cobordism (M, F, F'). This means that M is diffeomorphic to $M_L - (\mathrm{int}(N(G)) \cup \mathrm{int}(N(G')))$, where M_L is the manifold obtained by Dehn surgery on L, N means the tubular neighborhood. The boundaries of the tubular neighborhoods are naturally parametrized by the graph framings, and it is required that this and the former parametrizations coincide.

Let then A be the linking matrix of $\tilde{G} \cup L \cup \tilde{G}'$, where $\tilde{G}$ is the link obtained from the loops of the graph G slightly deformed by the framing. Then $\tilde{G}, L, \tilde{G}'$ have respectively g, n, g' components. We have also a natural induced basis on the spaces $V_g(k)$ and $V_{g'}(k)$, given by the theta functions from the introductions. Therefore the basis correspond to indices in $\mathbf{Z}^g$ and $\mathbf{Z}^{g'}$. The invariant associated to the cobordism is then a linear mapping from $V_g(k)$ to $V_{g'}(k)$. In terms of the theta basis the invariant considered in [26] (and extending $f_{k,k}$) is given by the matrix:

$$Z(M,k)_{h,h'} = c(A) \sum_{l \in \mathbf{Z}/k\mathbf{Z}} \exp\left(\frac{2\pi\sqrt{-1}}{k} < A(h',l,h),(h',l,h) > \right),$$

where c is a normalization factor

$$c(A) = G(1,k)^{-\sigma(A)} \mid G(1,k) \mid^{-n-\frac{g}{2}-\frac{g'}{2}+\sigma(A)} .$$

Thus (up to a 8-th root of unity) we can compute also the representation ρ_g using the invariant associated to the mapping cylinder, in terms of the linking matrix associated.

Let consider now a closed 3-manifold M with a Heegaard splitting which we write $M = H \cup C(\varphi) \cup \bar{H}$. This means we have two handlebodies H and $\bar{H}$ and we inserted the cylinder $C(\varphi)$ of the gluing map φ on ∂H. Consider therefore a Dehn presentation of $C(\varphi)$ having trivial framings on the boundaries. Choose for instance two copies of the graph G, the spine of H and an intermediary link L. Notice that $\tilde{G} \cup L \cup \tilde{G}'$ is a link presentation for M, and thus the matrix A is nothing but the linking matrix of this presentation. However it is not only the plain linking matrix which is needed, but also the more subtle decompositions of A into blocks. Since we have the trivial framings on the boundaries we have a linking matrix

$$A = \begin{bmatrix} 0 & U & 0 \\ U^\top & L & V \\ 0 & V^\top & 0 \end{bmatrix}$$

where the blocks U and V correspond to the linking matrices of L with $\tilde{G}$, and L with $\tilde{G}'$ respectively. Therefore the matrix $M_{h,h'} = \rho_g(\varphi_*)$ is given by

$$M_{h,h'} = c(A)\ G\left(L, \exp(\frac{2\pi\sqrt{-1}}{k})\right)_{I(h,h')},$$

where

$$I(h,h') = U^\top h' + Vh.$$

It follows that:

PROPOSITION 3.8. *For an arbitrary closed 3-manifold M the invariant $f_{p,k}$ can be expressed as follows:*

$$f_{p,k}(M) = d^{-g} c(A)^{k^{1-g}} \det\left(G\left(L, \exp(\frac{2\pi\sqrt{-1}}{k})\right)_{I(ph,ph')} \right)_{h,h' \in p(\mathbf{Z}/k\mathbf{Z})^g}.$$

The proof follows from the definition of the $f_{p,k}$. Remark that although L is a block in the linking matrix A it is not clear if it can be retrieved only from A. For instance assume that a suitable linking matrix is choosed such that A has the required form: the surgery link contains two trivial copies of g unlinked circles. Does this matrix come then from a Heegaard splitting ? In general the answer is negative. A necessary condition is that L should be unimodular, because it describes the mapping class group representation.

4. COMMENTS

Consider A an Abelian variety with a principal polarization ω. Then for a positive line bundle $L \longrightarrow A$ with first Chern class ω we have, according to [13] $h^0(A, \mathcal{O}(L)) = 1$. Moreover $H^0(A, \mathcal{O}(L))$ is generated by the classical theta function θ. Therefore the divisor θ is determined up to translation by (A, ω). Let now $\mathcal{S}_g$ be the Siegel space and $\Omega \in \mathcal{S}_g$. Then $(1, \Omega)$ determine a lattice in $\mathbf{C}^g$, hence an Abelian variety Ab_Ω which has a natural principal polarization given by the ample line bundle L_Ω. Now it is known that L_Ω's glue together i.e. they can be viewed as the fibers of a line bundle $L \longrightarrow \mathcal{S}_g$ over the Siegel space (see [16, 33, 34]). If $V_g(k)(\Omega) = H^0(Ab_\Omega, L_\Omega^{\otimes k})$ then $V_g(k)(\Omega)$ are also the fibers of a vector bundle $V_g(k)$ over $\mathcal{S}_g$. Next a local frame for $V_g(k)$ is provided by the theta functions of level k. Now a result of Welters extended to the non-Abelian case by Hitchin ([16, 33, 34]) asserts that V_k has a projectively flat connection. This follows from the fact that θ_m are global solutions of the heat equation: we identify the tangent space of $\mathcal{S}_g$ with the space of symmetric tensors (as any symmetric tensor give a deformation of the Kahler polarization of a torus). Therefore in this trivialization the heat operator takes the form

$$\partial_{\Omega_{st}} + \frac{\sqrt{-1}}{4\pi k}\partial^2_{z_s z_t}.$$

Thus θ_m are the covariant constant sections of this connection. Now the vector bundle $V_k \longrightarrow \mathcal{S}_g$ support the action of $\mathrm{Sp}(2g, \mathbf{Z})$. With respect to this action the above connection is not natural. If we modify θ_m as is done in [25] for the case when k equals one, we can obtain a natural connection on $V_g(k)$ whose covariant constant sections are the modified theta functions. This is explained by the factor

$$\det(C\Omega + D)^{1/2} \exp(\pi < \sqrt{-1}z, (C\Omega + D)^{-1}Cz >)$$

appearing in the equation (5).

However this connection is not flat but only projectively flat. Hence if we compute the holonomy of this connection (more precisely of the induced connection on the moduli space of principally polarized Abelian varieties) using the theta functions of level k, we shall obtain not a linear but a projective unitary representation of the symplectic group. This gives a geometric interpretation for the messy factor $\zeta_\gamma \in R_8$ (see also [1]). By choosing carefully the multiplier system ζ_γ we may lift the projective representation to a linear representation ρ_g of a central extension of the symplectic group. The invariants for 3-manifolds which we derived are defined up to a root of unity lying in $R_{k,p}$. It seems that this ambiguity can be removed by adding a supplementary structure on the 3-manifold M^3, for instance a spin structure.

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