The Randall-Sundrum Model

Maxime Gabella

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IPPC, EPFL

Abstract

The Randall-Sundrum model was conceived in 1999 to address the Higgs Hierarchy Problem in particle physics. It arose enormous interest from theoreticians and phenomenologists ever since and revealed a fruitful tool to explore the physics of extra dimensions. The aim of this paper is to provide an introductory exposition of this model. After a short survey of Kaluza-Klein theories, the setup of the RS model will be exposed and its metric derived. We will explain how an exponential hierarchy between the gravity scale and the weak scale can be naturally generated, and how the standard 4D gravity emerges from this model in the Newtonian limit. The Golberger-Wise mechanism will be presented as a way to stabilize the radius of the extra dimension without reintroducing a fine-tuning. Those topics will be presented in an utterly pedagogical way. Here you will find what textbooks feel free to disregard as too advanced but research papers consider as too basic to even be mentioned.

Contents

1 Basics of Kaluza-Klein theories

The existence of extra dimensions of space was first put forth in the middle of the 1920's by Theodor Kaluza and Oskar Klein [1] as a means of unifying the electromagnetic and gravitational fields as components of a single higher-dimensional field. As an illustration, consider the case of a five-dimensional theory, with the extra dimension periodically identified:

$$x^5 \sim x^5 + 2\pi R.$$

This procedure is called toroidal compactification [2]. The space obtained is the product of the traditional four-dimensional Minkowski space with a circle, noted $M^4 \otimes S^1$, which can be imagined as a 5D cylinder of radius R. In such a theory, a massless scalar field $\phi(x^\mu, x^5)$ would have a quantized momentum in the periodic dimension:

$$p^5 = \frac{n}{R},$$

with $n \in \mathbb{Z}$. We may then expand the field in Fourier series:

$$\phi(x^\mu, x^5) = \sum_{n=-\infty}^{\infty} \phi^n(x^\mu) e^{i\frac{n}{R}x^5}.$$

With this decomposition, the five-dimensional equation of motions $(\partial_\mu \partial^\mu + \partial_5 \partial^5)\phi = 0$ becomes

$$\partial_\mu \partial^\mu \phi^n(x^\mu) = \frac{n^2}{R^2} \phi^n(x^\mu).$$

In this way, an infinite tower of fields with masses $m^2 = n^2/R^2$ is generated. At energies small compared to R^{-1}, only the x^5-independent massless zero-mode remains and the physics is effectively four-dimensional. At energies above R^{-1}, the tower of Kaluza-Klein (KK) states comes into play.

An experimental bound on the size of the compactification radius R is imposed by the fact that those KK states have not been detected at colliders up to TeV energies. Their masses would thus have to be greater, $n/R >$TeV, which implies a strong constraint on R:

$$R \lesssim 10^{-21} cm.$$

It is nearly hopeless to seek experimental confirmations of such minuscule dimensions.

A way out of this restriction was suggested in 1998 by Arkani-Hamed, Dimopoulos and Dvali (ADD) [3], based on an idea formulated in 1983 by Rubakov and Shaposhnikov [4]. If the extra dimensions are accessible only

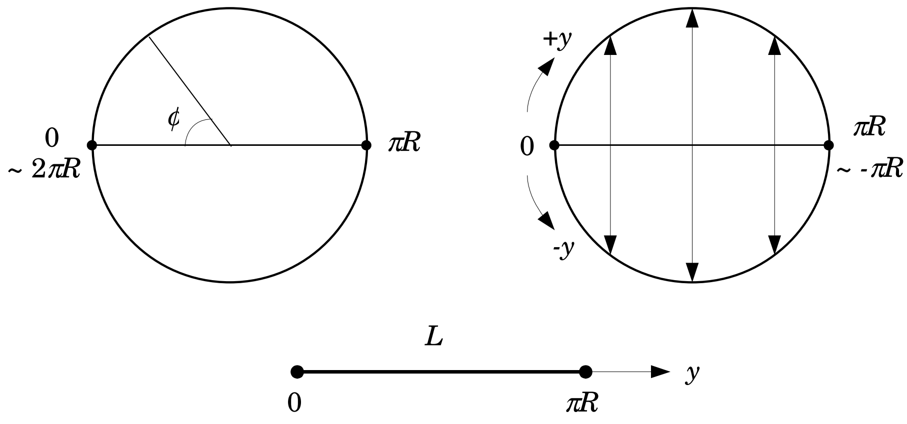

Figure 1: $S^1/\mathbb{Z}_2$ orbifold.

to gravity and not to the SM fields, the bound on their size is fixed by experimental tests of Newton's law of gravitation, which has only been led down to about a millimeter:

$$R \lesssim 1mm.$$

Such large extra dimensions could then perfectly exist and nevertheless have escaped our vigilance so far!

In addition, this scenario provides a solution to one of the central problems of particle physics: the Hierarchy Problem. This problem arises in quantum field theory because of the quadratically divergent corrections to the Higgs field mass, which require an incredible fine-tuning in order to get the expected mass of a few hundreds GeV. This problem can be equivalently formulated in terms of the unnatural discrepancy between the strength of gravity and those of the other three forces. In the ADD scenario, the weakness of gravity compared to the other forces finds an explanation in the fact that gravity gets diluted in the large volume of the extra dimensions. The hierarchy between the four-dimensional Planck scale $M_{Pl} \simeq 10^{19}$GeV and the scale of weak interactions $M_W \simeq$TeV would in reality be only apparent.

However, this solution merely translates the Hierarchy Problem into the problem of the discrepancy between the large size of the extra dimensions $R \simeq 1$mm and their natural value $R \simeq l_{Pl} \simeq 10^{-33}$cm.

The model presented in [5] and [6] by Lisa Randall and Raman Sundrum in 1999 provides a new explanation of the Hierarchy Problem.

2 Setup

The Randall-Sundrum model assumes the existence of one extra dimension compactified on a circle whose upper and lower halves are identified (see fig. 1).

Formally, this means we work in $S^1/\mathbb{Z}_2$ orbifold, where S^1 is the one-dimensional sphere (*i.e.* the circle) and $\mathbb{Z}_2$ is the multiplicative group $\{-1, 1\}$. This construction entails two fixed points, one at the origin $y = 0$ and one at the other extremity of the circle, at $y = \pi R \equiv L$. On each of these boundaries stands a four-dimensional world like the one we live in. By analogy with membranes enclosing a volume, these worlds with 3+1 dimensions enclosing the 5D bulk have been called 3-branes. The picture is then two 3-branes, at a distance L one from another, enclosing a 5D bulk (cf. fig. 2).

Taking into account the 5D cosmological constant Λ (which unlike the effective 4D cosmological constant does not need to be vanishing or even small) the fundamental action is the sum of the Hilbert-Einstein action S_H and a matter part S_M:

$$S = S_H + S_M = \int d^4x \int_{-L}^{+L} dy \sqrt{-g}(M^3 R - \Lambda), \tag{1}$$

where M is the fundamental 5D mass scale, R the 5D Ricci scalar and g the determinant of the metric, whose explicit form will be investigated in the next section.

3 Warped metric

The first step is to find the metric for such a setup. Since we are looking for solutions to the 5D Einstein equations that might fit the real world, we require that the metric should preserve Poincaré invariance: the 4D universe derived from this theory should appear flat and static. This leads to the following Ansatz:

$$ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the 4D Minkowski metric. The prefactor $e^{-2A(y)}$, called the warp factor, is written as an exponential for convenience. Its dependence on the extra dimension coordinate y causes this metric to be non-factorisable, which means that, unlike the metrics appearing in the usual Kaluza-Klein scenarios, it cannot be expressed as a product of the 4D Minkowski metric and a manifold of extra dimensions. To determine the function $A(y)$, we have to calculate the 5D Einstein equations:

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN} R = \kappa^2 T_{MN},$$

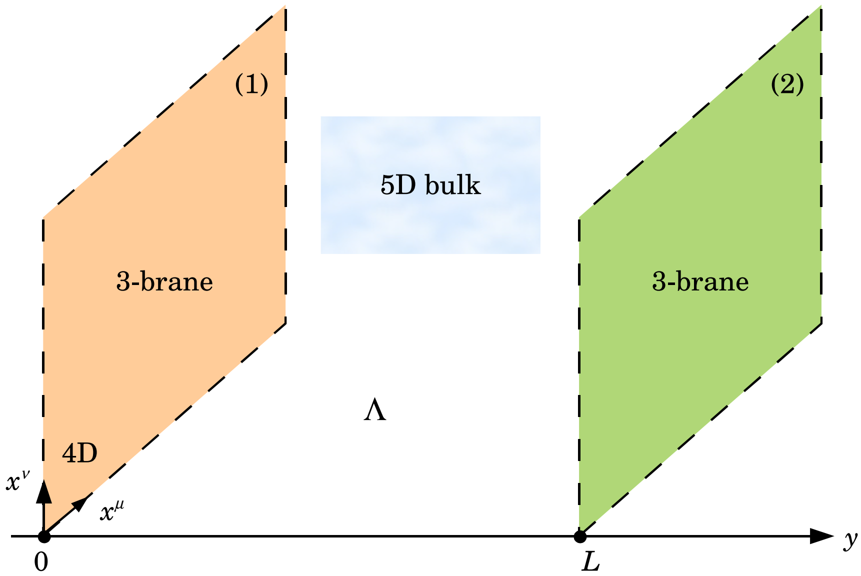

Figure 2: Randall-Sundrum setup.

where the capital indices M and N take the values $0, 1, 2, 3$ and 5, *i.e.* $M = (\mu, 5)$ with μ the usual 4D Lorentz index, and so on. The 5D Newton constant is defined as

$$\kappa^2 \equiv \frac{1}{2M^3},$$

and the energy-momentum tensor as

$$T_{MN} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{MN}}, \tag{3}$$

so that a term in the action like $\sqrt{-g}V$ with V constant corresponds to an energy-momentum tensor equal to $V g_{MN}$. The Einstein tensors for the metric parametrized by the Ansatz (2) are worked out in appendix A. The 55 component of the Einstein equation gives

$$G_{55} = 6A'^2 = \frac{-\Lambda}{2M^3}.$$

Notice that a real solution for A only exists if the 5D cosmological constant Λ is negative, which means that the space between the branes is anti-de Sitter space, noted AdS_5. The case where A is purely imaginary corresponds to an oscillating warp factor, which is not the concern of the RS model.

From that equation, we see that A'^2 is equal to a constant, which we call k^2:

$$A'^2 = \frac{-\Lambda}{12M^3} \equiv k^2. \tag{4}$$

Integrating over y gets us the expression for A:

$$A(y) = \pm ky.$$

As we want a solution that respects the orbifold symmetry, *i.e.* invariance under the transformation $y \to -y$, we choose

$$A(y) = k|y|.$$

Finally, the background metric in the Randall-Sundrum model is parametrized by

$$\boxed{ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2,} \tag{5}$$

with $-L \leq y \leq L$.

Let us look now at the $\mu\nu$ component of the 5D Einstein equations. Appendix A gives

$$G_{\mu\nu} = (6A'^2 - 3A'')g_{\mu\nu}.$$

From the solution we just found for A we see that the first derivative of A is

$$A' = \mathrm{sgn}(y)k.$$

The term $\mathrm{sgn}(y)$ may be written as a combination of Heaviside functions as

$$\mathrm{sgn}(y) = \theta(y) - \theta(-y),$$

so the second derivative is

$$A'' = 2k\delta(y).$$

This delta function arose from the kink of A at the origin $y = 0$ (cf. fig. 3). In the same way, the kink at $y = L$ gives rise to another delta function, and the complete expression for A'' is

$$A'' = 2k\big(\delta(y) - \delta(y - L)\big).$$

Plugging those results into the expression of the Einstein tensor gives

$$G_{\mu\nu} = 6k^2 g_{\mu\nu} - 6k\big(\delta(y) - \delta(y - L)\big)g_{\mu\nu}.$$

The first term is equal to the $\mu\nu$ components of the energy-momentum tensor multiplied by the 5D Newton constant:

$$\kappa^2 T_{\mu\nu} = \frac{-\Lambda}{2M^3}g_{\mu\nu} = 6k^2 g_{\mu\nu}.$$

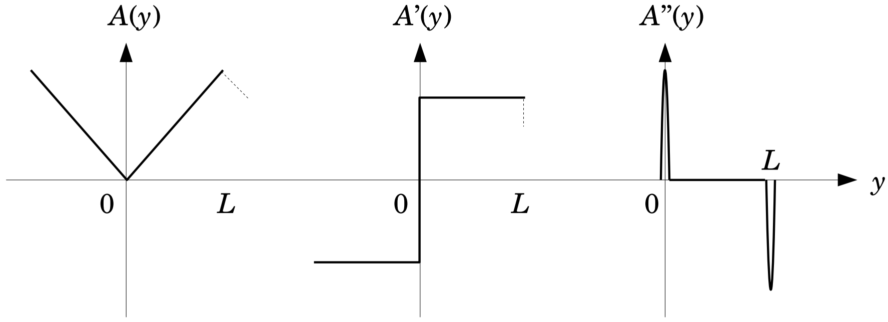

Figure 3: The function $A(y)$ and its first and second derivatives.

The second term however seems to have nothing to be matched to. The resolution of this situation is to take into account the energy densities of the branes themselves, called brane tensions. This is done by adding to the action one term for each brane, corresponding to the brane tensions λ_1 and λ_2:

$$\begin{aligned} S_1 &= -\int d^4x\sqrt{-g_1}\lambda_1 = -\int d^4xdy\sqrt{-g}\lambda_1\delta(y), \\ S_2 &= -\int d^4x\sqrt{-g_2}\lambda_2 = -\int d^4xdy\sqrt{-g}\lambda_2\delta(y-L). \end{aligned} \tag{6}$$

The terms g_1 and g_2 stand for the determinants of the metrics induced on the first brane and on the second brane respectively. The induced metrics define distances along the branes:

$$\begin{aligned} ds^2 &= g^i_{\mu\nu}dx^\mu dx^\nu \\ &= g_{\mu\nu}(x,y_i)dx^\mu dx^\nu, \end{aligned}$$

with $i = 1, 2$ and $y_1 = 0, y_2 = L$. Notice that with the metric given by (5), $g_1 = g\delta(y)$ and $g_2 = g\delta(y-L)$ because $g_{55} = 1$.

In order to satisfy the Einstein equations we need to impose the relation

$$\lambda_1 = -\lambda_2 = 12kM^3. \tag{7}$$

Moreover, by the definition of k we have

$$\Lambda = -\frac{\lambda_1^2}{12M^3}.$$

Those two relations are consequences of the requirement that the 4D universe be flat and static. The 4D brane sources are balanced by the 5D bulk cosmological constant in order to get a vanishing effective 4D cosmological constant.

4 Exponential hierarchy

Having presented the setup and found the metric of the Randall-Sundrum model, we would like to investigate what the physical scales would be if, in the spirit of [4], the matter fields were confined on the second brane at $y = L$. Consider the Higgs scalar field with the action

$$\begin{aligned} S_{Higgs} &= \int d^4x\sqrt{g_2}\left[g_2^{\mu\nu}D_\mu H^\dagger D_\nu H - \lambda\big(H^\dagger H - v^2\big)^2\right] \\ &= \int d^4x e^{-4kL}\left[e^{2kL}\eta^{\mu\nu}D_\mu H^\dagger D_\nu H - \lambda\big(H^\dagger H - v^2\big)^2\right]. \end{aligned}$$

To get a canonically normalized action we redefine the Higgs field as $H = e^{kL}\tilde{H}$. The action becomes

$$S_{Higgs} = \int d^4x\left[\eta^{\mu\nu}D_\mu\tilde{H}^\dagger D_\nu\tilde{H} - \lambda\big(\tilde{H}^\dagger\tilde{H} - (e^{-kL}v)^2\big)^2\right].$$

This is the action of a normal Higgs scalar, except for the vacuum expectation value (vev) which is exponentially suppressed:

$$\boxed{v_{eff} = e^{-kL}v.}$$

As the Higgs vev sets all the mass parameters in the Standard Model, this means that all mass parameters are submitted to an exponential suppression on the second brane. If the value of the bare Higgs mass is of the order of the Planck scale, the physical Higgs mass could be warped down to the weak scale, where we expect it to be. For this reason, the first brane at $y = 0$ is often called the "Planck" brane, whereas the second brane is called the "TeV" brane. Since $M_W \simeq 10^{-16}M_{Pl}$, the appropriate value for the size of the extra dimension is given by

$$kL \simeq \ln 10^{16} \simeq 35.$$

We will see in section 8 how such a value can be obtained for the size L of the extra dimension without reintroducing a fine-tuning.

To understand whether or not this exponential suppression is useful to address the Hierarchy Problem, we must know how the effective scale of gravity behaves with respect to the extra dimension. This information is to be obtained from the way the 5D action S contains the 4D action S^{4D}. Perturbating the action 1 around the background metric given by 5 produces a term with the schematic form

$$\begin{aligned} S &\ni M^3\int d^4x\int_{-L}^{+L} dy e^{-2k|y|}\sqrt{-g^{(0)}}R^{4D}(h^{(0)}_{\mu\nu}) \\ &= M^3\frac{1-e^{-2kL}}{k}\int d^4x\sqrt{-g^{(0)}}R^{4D}(h^{(0)}_{\mu\nu}). \end{aligned}$$

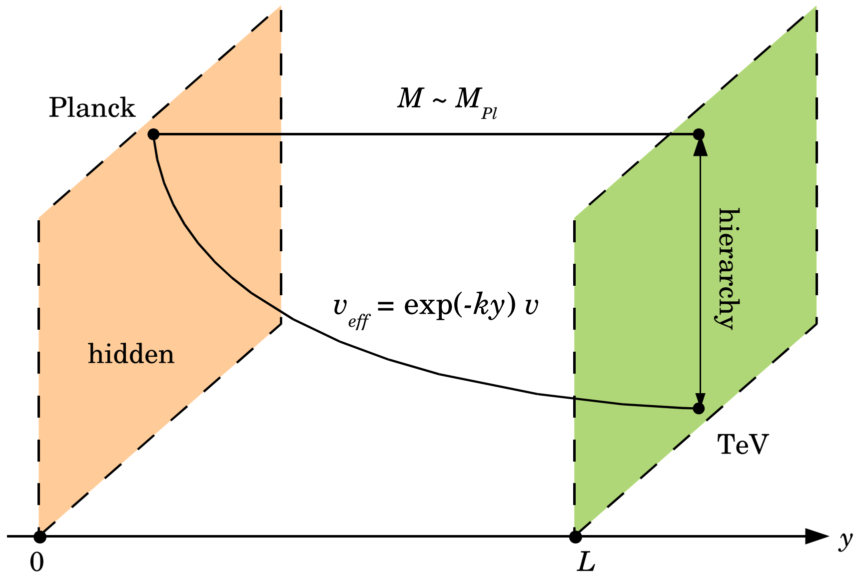

Figure 4: The generation of an exponential hierarchy.

This term corresponds to the 4D action, so that we can read off the value of the effective 4D Planck mass:

$$\boxed{M_{Pl}^2 = (1 - e^{-2kL})M^3/k.}$$

We see that it weakly depends on the size of the extra dimension L, provided kL is moderately large.

Putting our two last results together, we see that the weak scale is exponentially suppressed along the extra dimension, while the gravity scale is mostly independent of it (see fig.4).

In conclusion, in a theory where the values of all the bare parameters $(M, \Lambda, \lambda_1, v)$ are determined by the Planck scale, an exponential hierarchy can be naturally generated between the weak and the gravity scales. Thus the Randall-Sundrum model provides an original solution to the Hierarchy Problem.

Remarkably, the effective Planck mass remains finite even if we take the decompactification limit $L \to \infty$. This case where there is only one brane is known as the Randall-Sundrum II model (RS2). The fact that there could be an infinite extra dimension and still a 4D gravity as we experience it results from the localization of gravity around the brane at $y = 0$, which we now turn our attention to.

5 Graviton modes

In order to understand how gravity works in the Randall-Sundrum model, we first have to find explicit expressions for the gravitons, which correspond to small fluctuations $h_{MN}(x, y)$ around the background metric given by

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2.$$

That will be achieved by computing the solutions of the linearized Einstein equation.

Conformally flat metric It is convenient to work with a conformally flat metric, *i.e.* a metric proportional to flat space. To achieve this, we define a new extra dimension variable z related to y through

$$dy^2 \equiv e^{-2k|y|}dz^2.$$

The integration of this equation produces a constant, which we set so as to have the zero value of y corresponding to the zero value of z. The result is

$$k|z| = e^{k|y|} - 1, \tag{8}$$

and thus

$$e^{-2k|y|} = \frac{1}{(k|z|+1)^2}.$$

With this new coordinate, the metric is given by

$$ds^2 = \frac{1}{(k|z|+1)^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2).$$

To underline the fact that it is conformally flat we rewrite it in the following way:

$$\boxed{ds^2 = e^{-2A(z)}\eta_{MN}dx^M dx^N,}$$

where we use the notation $x^5 = z$. The function $A(z)$ is given by

$$e^{-2A(z)} = \frac{1}{(k|z|+1)^2},$$

and so $A(z) = \ln(k|z|+1)$. For later reference, we give its first and second derivatives:

$$A'(z) = \frac{\mathrm{sgn}(z)k}{k|z|+1}, \tag{9}$$

and

$$A''(z) = \frac{2k(\delta(z) - \delta(z - L_z))}{k|z|+1} - \frac{k^2}{(k|z|+1)^2}, \tag{10}$$

where we have used again that $\mathrm{sgn}'(z) = 2\delta(z)$.

Linearized Einstein equations To keep the calculations as concise as possible we will not compute the Einstein tensor by brute force, but rather use a formula about conformally related metrics (see [7] appendix D). Specifically, if some metric g_{MN} is a conformal transformation of another metric $\tilde{g}_{MN}$, for example

$$g_{MN} = e^{-2A}\tilde{g}_{MN},$$

then the respective Einstein tensors are related by

$$\begin{aligned} G_{MN}(g_{MN}) = \tilde{G}_{MN}(\tilde{g}_{MN}) + (n-2)\big[&\tilde{\nabla}_M A\tilde{\nabla}_N A + \tilde{\nabla}_M\tilde{\nabla}_N A \\ &-\tilde{g}_{MN}(\tilde{\nabla}_R\tilde{\nabla}^R A - \frac{n-3}{2}\tilde{\nabla}_R A\tilde{\nabla}^R A)\big], \end{aligned}$$

where n is the number of spacetime dimensions. In the present case, the perturbed metric has the form

$$g_{MN} = e^{-2A}(\eta_{MN} + h_{MN})$$

and $n = 5$, so the formula gives, taking into account the Christoffel symbols contained inside the covariant derivatives:

$$\begin{aligned} G_{MN} = \tilde{G}_{MN} + 3\big[&\partial_M A\partial_N A + \partial_M\partial_N A - \tilde{\Gamma}^R_{MN}\partial_R A \\ &-\tilde{g}_{MN}(\partial_R\partial^R A - \tilde{\Gamma}^R_{RS}\partial^S A - \partial_R A\partial^R A)\big]. \end{aligned} \tag{11}$$

To linear order, the Christoffel symbols are easily found:

$$\tilde{\Gamma}^R_{MN} = \frac{1}{2}(\partial_M h^R_N + \partial_N h^R_M - \partial^R h_{MN}),$$

where we have used η^{MN} to rise the indices. It is particularly convenient in that kind of calculations to work with a gauge in which the fluctuations do not have any extra dimension component and are transverse and traceless:

$$\begin{aligned} &h_{M5} = 0, \\ &\partial_\mu h_{\mu\nu} = 0 \quad \text{and} \quad \eta^{\mu\nu}h_{\mu\nu} = h^\mu_\mu = 0. \end{aligned}$$

We verify that those 10 conditions restrict the number of degrees of freedom of the symmetric 5×5 tensor h_{MN} from 15 to 5, as appropriate for a spin-two 5D particle (see [8] section 2.3 and [9] section 10.6). With this gauge fixing, the second Christoffel symbol in equation (11) vanishes, whereas the first one reduces to $-\partial^5 h_{MN}/2$ given that it is contracted with $\partial_R A$, whose only non-vanishing component is $\partial_5 A$. In addition, the expression of the Einstein tensor for fluctuations around the flat metric (see [7] eq. (4.4.5)) shrinks to

$$\tilde{G}_{MN} = -\frac{1}{2}\partial_R\partial^R h_{MN}.$$

The $\mu\nu$ component of the linearized Einstein tensor in this gauge is then given by

$$G_{\mu\nu} = -\frac{1}{2}\partial_R\partial^R h_{\mu\nu} + \frac{3}{2}h'_{\mu\nu}A' - 3(\eta_{\mu\nu} + h_{\mu\nu})(A'' - A'^2). \tag{12}$$

On the other hand, we have to compute the energy-momentum tensor for the perturbed metric. Coming back to the expression of the action terms for the brane tensions (6), we have to be careful about the fact that with the conformally flat metric the determinants of the metrics induced on the branes are now related to the determinant of the full metric by

$$g = g_i g_{55} = g_i e^{-2A(z_i)},$$

with $i = 1, 2$ and $z_1 = 0$, $z_2 = L_z$. The corresponding actions read

$$\begin{aligned} S_1 &= -\int d^4x\sqrt{-g_1}\lambda_1 = -\int d^4x dz\sqrt{-g}\lambda_1 e^{A(0)}\delta(z) \\ S_2 &= -\int d^4x\sqrt{-g_2}\lambda_2 = -\int d^4x dz\sqrt{-g}\lambda_2 e^{A(L_z)}\delta(z - L_z). \end{aligned}$$

The $\mu\nu$ component of the energy-momentum tensor multiplied by the 5D Newton constant is then

$$\begin{aligned} \kappa^2 T_{\mu\nu} &= \frac{1}{2M^3}\left[-\Lambda - \lambda_1 e^A\delta(z) - \lambda_2 e^A\delta(z - L_z)\right] g_{\mu\nu} \\ &= \frac{1}{2M^3}\left[-\Lambda e^{-2A} - \lambda_1 e^{-A}\delta(z) - \lambda_2 e^{-A}\delta(z - L_z)\right](\eta_{\mu\nu} + h_{\mu\nu}). \end{aligned}$$

Remembering the definition (4) of k as well as the relation (7) between the brane-tensions and referring to the expressions (9) and (10) of the first and second derivatives of A allows us to rewrite it as

$$\begin{aligned} \kappa^2 T_{\mu\nu} &= \left[6k^2 e^{-2A} - 6k\big(\delta(z) - \delta(z - L_z)\big)e^A\right](\eta_{\mu\nu} + h_{\mu\nu}) \\ &= \left[6A'^2 - 3(A'' + A'^2)\right](\eta_{\mu\nu} + h_{\mu\nu}) \\ &= (3A'^2 - 3A'')(\eta_{\mu\nu} + h_{\mu\nu}). \end{aligned} \tag{13}$$

When we put the two sides (12) and (13) of the $\mu\nu$ component of the linearized Einstein equation together, the terms proportional to $\eta_{\mu\nu}$ reproduce the unperturbed Einstein equations, and we are left with the part due to the perturbations:

$$\boxed{-\tfrac{1}{2}\partial_R\partial^R h_{\mu\nu} + \tfrac{3}{2}A'h'_{\mu\nu} = 0.}$$

Schrödinger-like equation An elegant way of solving this equation is to rewrite it in the form of a Schrödinger equation.

As a start, in order to get rid of the first derivatives $h'_{\mu\nu}$, we make the following rescaling:

$$h_{\mu\nu} \to e^{\alpha A} h_{\mu\nu},$$

with α a constant. A pencil and a small piece of scratch paper bring the Einstein equations to

$$-\frac{1}{2}\partial_R \partial^R h_{\mu\nu} + \left(\frac{3}{2} - \alpha\right) A' h'_{\mu\nu} + \left[\left(\frac{3}{2}\alpha - \frac{1}{2}\alpha^2\right) A'^2 - \frac{1}{2}\alpha A''\right] h_{\mu\nu} = 0.$$

For the choice $\alpha = 3/2$, the coefficient of $h'_{\mu\nu}$ vanishes and we are left with

$$-\frac{1}{2}\partial_R \partial^R h_{\mu\nu} + \left[\frac{9}{8}A'^2 - \frac{3}{4}A''\right] h_{\mu\nu} = 0.$$

Performing a Kaluza-Klein decomposition,

$$h_{\mu\nu}(x, z) = \sum_{n=0}^{\infty} h^n_{\mu\nu}(x)\psi_n(z),$$

with $\Box h^n_{\mu\nu} \equiv \partial_\rho \partial^\rho h^n_{\mu\nu} = m_n^2 h^n_{\mu\nu}$, we get

$$\boxed{-\psi''_n(z) + \left[\tfrac{9}{4}A'^2(z) - \tfrac{3}{2}A''(z)\right]\psi_n(z) = m_n^2 \psi_n(z).} \tag{14}$$

That looks just like a Schrödinger equation with potential

$$\begin{aligned} V(z) &= \frac{9}{4}A'^2(z) - \frac{3}{2}A''(z) \\ &= \frac{9}{4}\frac{k^2}{(k|z|+1)^2} - \frac{3}{2}\left(\frac{2k\big(\delta(z) - \delta(z - L_z)\big)}{k|z|+1} - \frac{k^2}{(k|z|+1)^2}\right) \\ &= \frac{15}{4}\frac{k^2}{(k|z|+1)^2} - \frac{3k\big(\delta(z) - \delta(z - L_z)\big)}{k|z|+1}. \end{aligned}$$

The shape of this potential looks like a volcano (see fig. 5).

Boundary conditions To get the boundary conditions that the solutions will have to obey, we integrate equation (14) over small domains around the boundaries. For the boundary at $z = 0$ we get

$$\begin{aligned} \int_{0^-}^{0^+} dz(-\psi''_n + V\psi_n) &= \int_{0^-}^{0^+} dz m^2 \psi_n \\ -\psi'_n(0^+) + \psi'_n(0^-) - 3k\psi_n(0) &= 0. \end{aligned}$$

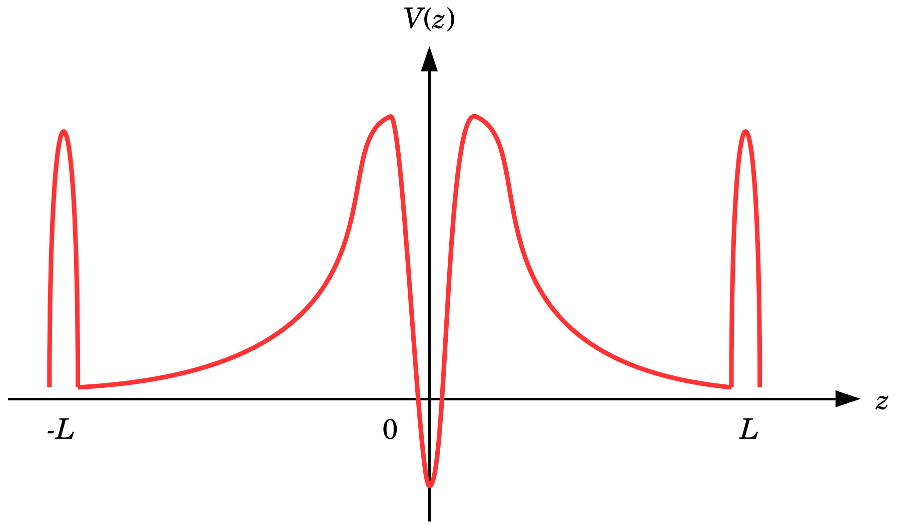

Figure 5: Volcano potential.

The wave-function has to be an even function under the transformation $z \to -z$, and so its first derivative is an odd function: $\psi_n'(0^-) = -\psi_n'(0^+)$. The boundary condition at the Planck brane is then

$$\psi_n'(0) = -\frac{3k}{2}\psi_n(0). \tag{15}$$

Similarly, we get the boundary condition at the TeV brane:

$$\psi_n'(L_z) = -\frac{3k}{2(kL_z+1)}\psi_n(L_z). \tag{16}$$

Zero-mode The zero-mode is the solution of the Schrödinger-like equation with $m_0 = 0$:

$$-\psi_0'' + \left[\frac{9}{4}A'^2 - \frac{3}{2}A''\right]\psi_0 = 0.$$

It is given by

$$\boxed{\psi_0(z) = e^{-\frac{3}{2}A} = (k|z|+1)^{-3/2},}$$

which satisfies the boundary conditions (15) and (16). We see that the graviton zero-mode has a wave function that is peaked around the origin (cf. fig. 5). As we are going to see in section 7, the gravitational interactions are predominantly mediated by the graviton zero-mode. Gravity is thus localized on the Planck brane, while on the TeV brane we feel only the tail of the graviton wave-function. So in the RS model the reason of the

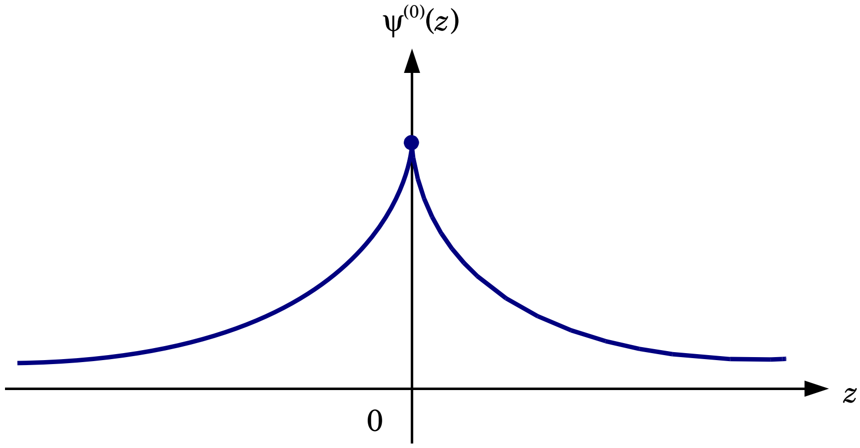

Figure 6: Localization of the graviton zero-mode around the Planck brane.

weakness of gravity is that it is localized far away from where we live — in contrast to the ADD scenario, which attributes it to the dilution of gravity in the higher-dimensional volume.

Kaluza-Klein modes Between the boundaries, the massive Kaluza-Klein modes have to satisfy the following equation:

$$\psi_n'' + \left(m_n^2 - \frac{15}{4} \frac{k^2}{(k|z|+1)^2} \right) \psi_n = 0.$$

This is a Bessel equation of order 2 (see [10] eq. 9.1.49.), and its solutions are linear combinations of Bessel functions of first and second kinds:

$$\psi_n = (|z| + 1/k)^{1/2} \left[a_n J_2\big(m_n(|z|+1/k)\big) + b_n Y_2\big(m_n(|z|+1/k)\big) \right], \quad (17)$$

with a_n and b_n some coefficients.

To get an approximation of these wave-functions, we will use asymptotic expressions of the Bessel functions (see [6]). We can rewrite the above equation as

$$\psi_n = N_n (|z| + 1/k)^{1/2} \left[Y_2\big(m_n(|z|+1/k)\big) + \frac{4k^2}{\pi m^2} J_2\big(m_n(|z|+1/k)\big) \right],$$

where N_n is a normalization constant. The coefficient in front of J_2 has been determined using the boundary condition (15) and the asymptotic expressions of the Bessel functions for small arguments ($m_n|z| \ll 1$):

$$Y_2\big(m_n(|z|+1/k)\big) \simeq -\frac{4}{\pi m_n^2 (|z|+1/k)^2} - \frac{1}{\pi}$$

and

$$J_2\big(m_n(|z|+1/k)\big) \simeq \frac{m^2(|z|+1/k)^2}{8}.$$

As $k/m_n \gg 1$, the term with J_2 dominates in the expression of the wave function. To evaluate the constant N_n, we use an approximation for large values of $m_n|z|$:

$$\sqrt{z}J_2(m_n|z|) \simeq (2/\pi m_n)^{1/2}\cos(m_n|z| - 5\pi/4),$$

which we plug in the normalization relation

$$\int_{-L}^{+L} dz|\psi|^2 = 1.$$

We get

$$\int_{-L}^{+L} dz N_n^2 \frac{32k^4}{\pi^3 m_n^5}\cos^2(m_n|z| - 5\pi/4) = N_n^2\frac{32k^4}{\pi^3 m_n^5}L = 1,$$

which implies

$$N_n = \sqrt{\frac{\pi}{2}}\frac{\pi m_n^{5/2}}{4k^2\sqrt{L}}.$$

Our approximation for the KK states wave-functions in the limit of large $m_n|z|$ is then

$$\boxed{\psi_n = \cos(m_n|z| - 5\pi/4)/\sqrt{L}.} \tag{18}$$

6 Graviton spectrum

The presence of two branes induces the quantization of the masses of the KK states. To see it, let us look at the effect of the two boundary conditions (15) and (16) on the general solutions (17). The derivative of these solutions turns out to be (cf. [10] eq. 9.1.29)

$$\begin{aligned}\psi_n' &= m_n(|z|+1/k)^{1/2}\left[a_n J_1\big(m_n(|z|+1/k)\big) + b_n Y_1\big(m_n(|z|+1/k)\big)\right] \\ &-\frac{3}{2}(|z|+1/k)^{-1/2}\left[a_n J_2\big(m_n(|z|+1/k)\big) + b_n Y_2\big(m_n(|z|+1/k)\big)\right],\end{aligned}$$

so the boundary conditions become

$$\begin{aligned} a_n J_1(m_n/k) + b_n Y_1(m_n/k) &= 0, \\ a_n J_1\big(m_n(L_z+1/k)\big) + b_n Y_1\big(m_n(L_z+1/k)\big) &= 0. \end{aligned}$$

This system has solutions only if its determinant vanishes, *i.e.* only if

$$J_1(m_n/k)Y_1\big(m_n(L_z+1/k)\big) - J_1\big(m_n(L_z+1/k)\big)Y_1(m_n/k) = 0.$$

Coming back to the coordinate y, which effectively represents to the distance along the extra dimension (see equation (8)),

$$L_z \simeq \frac{1}{k}e^{kL} \gg 1,$$

we can write

$$J_1(m_n/k)Y_1(m_n e^{kL}/k) - J_1(m_n e^{kL}/k)Y_1(m_n/k) = 0.$$

In the approximation of small masses ($m_n/k \ll 1$), the Bessel functions of first order behave like $J_1(m_n/k) \sim m_n/k$ and $Y_1(m_n/k) \sim \ln(m_n/2k)m_n/k$ (see [10] eqs. 9.4.4 and 9.4.5.), so that we can assume that $-Y_1(m_n/k) \gg J_1(m_n/k)$. The requirement that the determinant vanishes reduces to

$$J_1\left(m_n e^{kL}/k\right) = 0.$$

The masses of the KK tower are thus given by

$$\boxed{m_n = k e^{-kL} j_n,}$$

where j_n are the zeros of the Bessel function: $J_1(j_n) = 0$.

As the value of k is supposed to be of order of the Planck scale and the factor $\exp(-kL)$ at the TeV brane has been fixed to solve the Hierarchy Problem, the masses of the KK states are of order TeV. Furthermore, $j_{n+1} - j_n \simeq \pi$, so that the splitting of the masses is also of order TeV. This implies the possibility to observe individual resonances of the first KK states at colliders in the very near future [11]. Figure 6 shows an extrapolation of the cross-section for the process $e^+e^- \to \mu^+\mu^-$ at a linear collider, with different values of the ratio k/M_{Pl}. The resonances for the first and second KK modes are well-defined and can be seen individually, in dramatic opposition to the phenomenology of the ADD scenario, which predicts only collective effects, given the smallness of the splitting between the KK modes.

7 Newtonian limit

We would like to verify that the gravitational interactions mediated by the gravitons modes that we found are in agreement with Newton's law. For that purpose, we consider a minimal coupling of matter to gravity and look for the values of the coupling constants. The action is composed of a gravity part

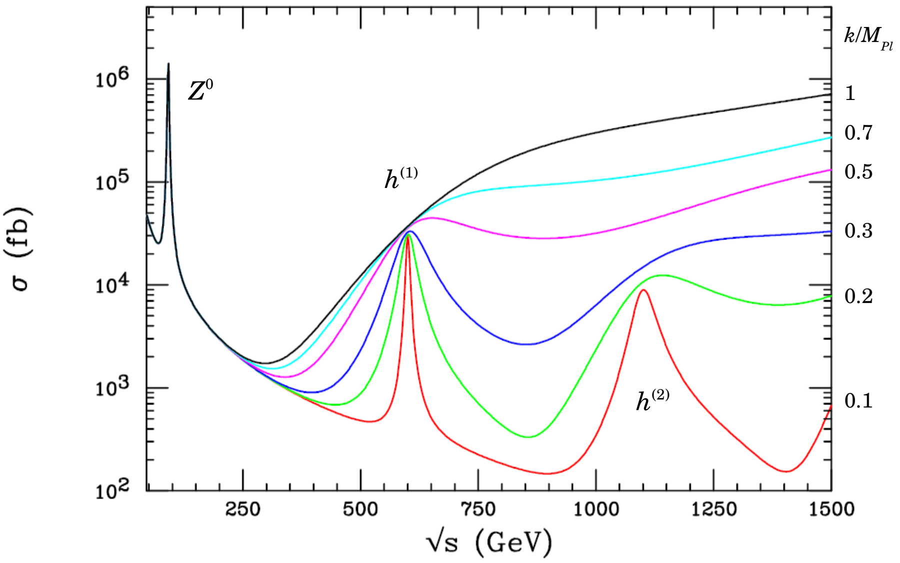

Figure 7: Individual resonances of the Randall-Sundrum gravitons.

S_G given by equations (1) and (6), and a part accounting for the interactions between matter and gravity:

$$S = S_G + \int d^4x dy \sqrt{-g} \mathcal{L}_M(\Phi, g_{MN}),$$

where Φ stands for the fields residing on the branes.

For small graviton perturbations around the background metric

$$g_{MN} = e^{-2A} \eta_{MN} \rightarrow g'_{MN} = e^{-2A}(\eta_{MN} + h_{MN}),$$

we expand the matter Lagrangian in Taylor series up to first order:

$$\mathcal{L}_M(\Phi, g'_{MN}) = \mathcal{L}_M(\Phi, g_{MN}) + h_{\mu\nu} \frac{\delta \mathcal{L}_M}{\delta g'_{\mu\nu}} \Big]_{g'_{\mu\nu} = g_{\mu\nu}} + \mathcal{O}(h^2).$$

Using the definition (3) of the energy-momentum tensor,

$$\begin{aligned} T^{\mu\nu} &= \frac{-2}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_M}{\delta g_{\mu\nu}} \Big]_{g'_{\mu\nu} = g_{\mu\nu}} \\ &= -\mathcal{L}_M g^{\mu\nu} - 2 \frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}} \Big]_{g'_{\mu\nu} = g_{\mu\nu}}, \end{aligned}$$

and the formula $\sqrt{\det(\eta_{\mu\nu} + h_{\mu\nu})} = 1 + h/2 + \mathcal{O}(h^2)$ with $h = g^{\mu\nu} h_{\mu\nu}$, we can write

$$\begin{aligned}
\sqrt{g'}\mathcal{L}_M(\phi, g'_{MN}) &= \sqrt{g}(1+h/2)\mathcal{L}_M(\phi, g'_{MN}) + \mathcal{O}(h^2) \\
&= \sqrt{g}\Big[\mathcal{L}_M(\Phi, g_{MN}) + h_{\mu\nu}\frac{\delta\mathcal{L}_M}{\delta g'_{\mu\nu}}\Big]_{g'_{\mu\nu}=g_{\mu\nu}} \\
&\quad + \frac{h}{2}\mathcal{L}_M(\Phi, g_{MN})\Big] + \mathcal{O}(h^2) \\
&= \sqrt{g}\left[\mathcal{L}_M(\Phi, g_{MN}) - \frac{1}{2}h_{\mu\nu}T^{\mu\nu}\right] + \mathcal{O}(h^2).
\end{aligned}$$

On the other side, when we expand S_G up to the second order in the perturbations, we get the following terms: one part independent of $h_{\mu\nu}$ that vanishes because of the requirement of the vanishing of the effective cosmological constant; one linear part, which is the action leading to the linear equations of motion, so that it vanishes on shell; and one quadratic part, which corresponds the usual Pauli-Fierz Lagrangian $\mathcal{L}_{PF}$. Remembering the solution we found for the KK modes after a rescaling by $\exp(3A/2)$ and a KK decomposition drives us to

$$\begin{aligned}
\mathcal{L}_M(\Phi, g'_{MN}) &= \mathcal{L}_M(\Phi, g_{MN}) + M^3\sum_n \mathcal{L}_{PF}(h^n_{\mu\nu}(x)) \\
&\quad - \sum_n \frac{e^{\frac{3}{2}A}\psi^{(n)}(z)}{2}h^n_{\mu\nu}(x)T^{\mu\nu}
\end{aligned}$$

In order to get a canonically normalized Pauli-Fierz Lagrangian, we proceed to a field redefinition:

$$h^n_{\mu\nu}(x) \to \frac{1}{\sqrt{M^3}}h^n_{\mu\nu}(x),$$

and we finally obtain

$$\mathcal{L}_M(\phi, g) = \mathcal{L}_M(\Phi, \eta) + \sum_n \mathcal{L}_{PF}(h^n_{\mu\nu}(x)) - \sum_n \frac{e^{\frac{3}{2}A}\psi_n(z)}{2\sqrt{M^3}}h^n_{\mu\nu}(x)T^{\mu\nu},$$

from which we can read the expression of the gravity-matter coupling constants:

$$a_n = \frac{e^{\frac{3}{2}A}\psi_n(z)}{2\sqrt{M^3}}.$$

We can now compute the gravitational potential between two particles with unit masses on the TeV brane at $z = L_z$, *i.e.* the static potential generated by the exchange of the zero-mode and the massive KK states. Like in the case of a Yukawa interaction (see [12] eq. (4.127)), it is given by

$$V(r) = -\sum_{n=0}^{\infty}\frac{a_n^2}{4\pi}\frac{e^{-m_n r}}{r}.$$

The contribution of the zero-mode $\psi_0(z) = \exp(-3A/2)$ to the gravitational interaction is

$$\begin{aligned} V_0(r) &= -\frac{1}{16\pi M^3}\frac{1}{r} \\ &= -\frac{G_N}{r}, \end{aligned}$$

with G_N the Newton constant. This reproduces the 4D gravity.

With the help of the approximation (18) for the KK states wave-functions, the non-relativistic gravitational potential mediated by the n^{th} massive graviton on the TeV brane reads

$$\begin{aligned} V_n(r) &= -\frac{k^3L^2}{16\pi M^3}\cos^2(m_nL_z - 5\pi/4)\frac{e^{-m_nr}}{r} \\ &= -\frac{G_N}{r}k^3L^2\cos^2(m_nL_z - 5\pi/4)e^{-m_nr}. \end{aligned}$$

These contributions to the gravitational potential are exponentially suppressed, and thus may be neglected down to distances of order of the fermi, $r \lesssim 10^{-13}$ cm. The actual experimental tests of gravity having only probed down to the millimeter scale, there is no perspective of detecting such small corrections any time soon.

In conclusion, gravity in the RS model corresponds effectively to 4D gravity as we experience it.

8 Radius stabilization

Until now we have treated the length, or equivalently the radius, of the extra dimension as a parameter, and we felt free to set it to the appropriate value to solve the Hierarchy Problem (see section 4). However, such a degree of freedom would imply the existence in the effective theory of a massless scalar field, corresponding to the fluctuations of the radius along the extra dimension: the radion. This massless radion would cause a fifth force in violation to the equivalence principle. Therefore, to preserve the viability of the Randall-Sundrum model, the radion has to obtain a mass, *i.e.* to be stabilized.

A way to do it is the Goldberger-Wise mechanism [13]. The idea is to introduce a massive scalar field ϕ in the bulk with a potential $V(\phi)$ and add some potentials $V_1(\phi)$ and $V_2(\phi)$ on the two branes at the boundaries. The corresponding action reads

$$\begin{aligned} S &= \int d^4xdy\sqrt{-g}\big[M^3R + \frac{1}{2}\partial_M\phi\partial^M\phi - V(\phi) \\ &\quad -V_1(\phi)\delta(y) - V_2(\phi)\delta(y-L)\big]. \end{aligned}$$

The requirement of Poincaré invariance imposes to choose the metric given by

$$ds^2 = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2,$$

and to restrict the dependence of the scalar field to the extra dimension:

$$\phi(x, y) = \phi(y).$$

To find $\phi(y)$, the scalar field and the Einstein equations should be solved simultaneously. The scalar equation is

$$\frac{1}{\sqrt{-g}}\partial_M \sqrt{-g} g^{MN} \partial_N \phi = -\frac{\partial V_{tot}}{\partial \phi},$$

with $V_{tot} = -V - V_1\delta(y) - V_2\delta(y - L)$. Only the 55 component gives a non-vanishing result:

$$\phi'' - 4A'\phi' = \frac{\partial V}{\partial \phi} + \frac{\partial V_1}{\partial \phi}\delta(y) + \frac{\partial V_2}{\partial \phi}\delta(y - L).$$

Referring to the expression of the Einstein tensors found in appendix A, the 55 and $\mu\nu$ components of the Einstein equations $G_{MN} = \kappa^2 T_{MN}$ are

$$A'^2 = \frac{\kappa^2}{12}\phi'^2 - \frac{\kappa^2}{6}V(\phi), \tag{19}$$

and

$$2A'^2 - A'' = \frac{\kappa^2}{6}\phi'^2 - \frac{\kappa^2}{3}(V + V_1\delta(y) + V_2\delta(y - L)).$$

We can simplify the second result by using the first one to eliminate ϕ':

$$A'' = \frac{\kappa^2}{3}(V_1\delta(y) + V_2\delta(y - L)).$$

To obtain the boundary conditions, we integrate those results on very small domains around the positions $y_1 = 0$ and $y_2 = L$. The kinks of ϕ and A at those positions will result in jumps in the derivatives:

$$\phi'\Big]_{y_i^-}^{y_i^+} = \frac{\partial V_i}{\partial \phi},$$

and

$$A'\Big]_{y_i^-}^{y_i^+} = \frac{\kappa^2}{3}V_i.$$

Together with the scalar and Einstein equations these equations form the gravity-scalar system. It is quite hard to solve generally, so we will restrain our study to a special case.

Suppose that V has the special form

$$V(\phi) = \frac{1}{8}\left(\frac{\partial W(\phi)}{\partial\phi}\right)^2 - \frac{\kappa^2}{6}W^2(\phi),$$

for some function $W(\phi)$, called "superpotential". As equation (19) can be written as

$$V(\phi) = \frac{1}{2}\phi'^2 - \frac{6}{\kappa^2}A'^2,$$

we conclude that

$$\phi' = \frac{1}{2}\frac{\partial W}{\partial\phi} \quad \text{and} \quad A' = \frac{\kappa^2}{6}W(\phi).$$

We want the bulk potential to include a cosmological constant term (independent of ϕ) and a mass term (quadratic in ϕ), so we choose for example

$$W = \frac{6k}{\kappa^2} - u\phi^2,$$

with u a parameter. From that we have

$$\phi' = \frac{1}{2}\frac{\partial W}{\partial\phi} = -u\phi,$$

whose solution is easily found:

$$\phi(y) = \phi_P e^{-uy}.$$

On the TeV brane we get

$$\phi_T = \phi_P e^{-uL},$$

which can be inverted to

$$\boxed{L = \ln(\phi_P/\phi_T)/u.}$$

The value of the radius is thus determined by the equation of motion. To solve the Hierarchy Problem, we need $kL \simeq 35$, which only implies a modest tuning of order $\mathcal{O}(50)$ on the input parameters. Thus the solution to the Hierarchy Problem provided by the Randall-Sundrum model does not arise – like in the case of large extra dimensions – at the cost of introducing another fine-tuning.

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A Einstein tensor

We want to calculate the Einstein tensor for the metric

$$\begin{aligned} ds^2 &= e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \\ &= g_{MN}(y)dx^M dx^N, \end{aligned}$$

with

$$g_{MN}(y) = e^{-2A(y)}\eta_{\mu\nu} + \delta^5_M\delta^5_N.$$

The inverse metric is

$$g^{MN}(y) = e^{2A(y)}\eta^{\mu\nu} + \delta_5^M\delta_5^N.$$

Christoffel symbols

$$\Gamma^P_{MN} = \frac{1}{2}g^{PR}(\partial_M g_{NR} + \partial_N g_{RM} - \partial_R g_{MN}).$$

As g_{MN} is a function of the extra dimension only, and this only in its $\mu\nu$ components, we have

$$\partial_L g_{MN} = \partial_5 g_{MN} = \partial_5 g_{\mu\nu}.$$

That implies that only two types of Christoffel symbols are non-vanishing:

$$\begin{aligned} \Gamma^5_{\mu\nu} &= \frac{1}{2}g^{5R}(-\partial_R g_{\mu\nu}) \\ &= \frac{1}{2}g^{55}(-\partial_5 g_{\mu\nu}) \\ &= A'e^{-2A}\eta_{\mu\nu}, \end{aligned}$$

and

$$\begin{aligned} \Gamma^\nu_{\mu 5} &= \frac{1}{2}g^{\nu R}(\partial_5 g_{R\mu}) \\ &= \frac{1}{2}e^{2A}\eta^{\nu\rho}(-2A'e^{-2A}\eta_{\rho\mu}) \\ &= -A'\delta^\nu_\mu. \end{aligned}$$

Ricci tensor

$$R_{MN} = \partial_P\Gamma^P_{MN} - \partial_N\Gamma^P_{MP} + \Gamma^P_{PQ}\Gamma^Q_{MN} - \Gamma^P_{NQ}\Gamma^Q_{MP}.$$

$$\begin{aligned} R_{\mu\nu} &= \partial_5\Gamma^5_{\mu\nu} + \Gamma^\sigma_{\sigma 5}\Gamma^5_{\mu\nu} - \Gamma^\sigma_{\nu 5}\Gamma^5_{\mu\sigma} - \Gamma^5_{\nu\sigma}\Gamma^\sigma_{\mu 5} \\ &= (A'' - 2A'^2)e^{-2A}\eta_{\mu\nu} - 4A'^2e^{-2A}\eta_{\mu\nu} \\ &\quad + A'^2e^{-2A}\eta_{\mu\nu} + A'^2e^{-2A}\eta_{\mu\nu} \\ &= (A'' - 4A'^2)g_{\mu\nu}. \end{aligned}$$

$$R_{\mu 5} = 0.$$

$$\begin{aligned} R_{55} &= -\partial_5 \Gamma^{\sigma}_{5\sigma} - \Gamma^{\sigma}_{5\rho}\Gamma^{\rho}_{5\sigma} \\ &= 4A'' - 4A'^2. \end{aligned}$$

Ricci scalar

$$\begin{aligned} R &= g^{MN} R_{MN} \\ &= g^{\mu\nu} R_{\mu\nu} + g^{55} R_{55} \\ &= 4(A'' - 4A'^2) + 4A'' - 4A'^2 \\ &= 8A'' - 20A'^2. \end{aligned}$$

Einstein tensor

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ &= (6A'^2 - 3A'') g_{\mu\nu}. \\ G_{55} &= R_{55} - \frac{1}{2} g_{55} R \\ &= 6A'^2. \end{aligned}$$