WEIL'S CONJECTURE FOR FUNCTION FIELDS

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Abstract. Let X be an algebraic curve defined over a finite field $\mathbf{F}_q$ and let G be a smooth affine group scheme over X with connected fibers whose generic fiber is semisimple and simply connected. In this paper, we affirm a conjecture of Weil by establishing that the Tamagawa number of G is equal to 1.

Contents

1. Overview

Let K be a number field, let $\mathbf{A}$ denote the ring of adeles of K, and let G be a connected semisimple algebraic group over K. A conjecture of Weil (now a theorem, thanks to the work of Kottwitz, Lai, and Langlands) asserts that if G is simply connected, then the Tamagawa measure $\mu_{\mathrm{Tam}}(G(K)\backslash G(\mathbf{A}))$ is equal to 1. Our goal in this paper is to prove an analogous result in the case where K is the function field of an algebraic curve defined over a finite field. In this section, we will recall the statement of Weil's conjecture, translate the function-field analogue into a problem in algebraic geometry, and outline our approach to that problem.

We begin in §1.1 by reviewing the Smith-Minkowski-Siegel mass formula for integral quadratic forms (Theorem 1.1.15). We then reformulate the mass formula as a statement about the volumes of adelic groups (following ideas of Tamagawa and Weil) and state the general form of Weil's conjecture. In §1.2 we consider the function field analogue of Weil's conjecture. Reversing the chain of reasoning given in §1.1, we reformulate this conjecture as a problem of counting principal G-bundles on an algebraic curve X defined over a finite field $\mathbf{F}_q$ (here we take G to be a group scheme over the curve X, whose generic fiber is an algebraic group over the function field K_X).

Principal G-bundles on X can be identified with points of an algebraic stack $\mathrm{Bun}_G(X)$, called the *moduli stack of G-bundles on X*. In §1.3, we will state a version of the Grothendieck-Lefschetz trace formula for $\mathrm{Bun}_G(X)$ which reduces the problem of counting G-bundles on X to the problem of computing the trace of the (arithmetic) Frobenius endomorphism of the cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X) \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q; \mathbf{Z}_\ell)$. Our goal then is to understand the topology of the moduli stack $\mathrm{Bun}_G(X)$. In §1.4, we discuss the analogous problem in the case where X is defined over the field of complex numbers, and describe several "local-to-global" principles which can be used to compute algebro-topological invariants of $\mathrm{Bun}_G(X)$ in terms of the local structure of G at the points of X. The bulk of this paper is devoted to developing analogous ideas over an arbitrary algebraically closed ground field (such as $\overline{\mathbf{F}}_q$); we provide a brief outline in §1.5.

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1.1. The Mass Formula and Weil's Conjecture.

Let R be a commutative ring and let V be an R-module. A *quadratic form* on V is a map $q : V \to R$ satisfying the following conditions:

(a) The construction $(v, w) \mapsto q(v+w)-q(v)-q(w)$ determines an R-bilinear map $V \times V \to R$.

(b) For every element $\lambda \in R$ and every $v \in V$, we have $q(\lambda v) = \lambda^2 q(v)$.

A *quadratic space* over R is a pair (V, q), where V is a finitely generated projective R-module and q is a quadratic form on V.

One of the basic problems in the theory of quadratic forms can be formulated as follows:

Question 1.1.1. Let R be a commutative ring. Can one classify quadratic spaces over R (up to isomorphism)?

Example 1.1.2. Let V be a vector space over the field $\mathbf{R}$ of real numbers. Then any quadratic form $q : V \to \mathbf{R}$ can be diagonalized: that is, we can choose a basis $e_1, \ldots, e_n$ for V such that

$$q(\lambda_1 e_1 + \cdots + \lambda_n e_n) = \lambda_1^2 + \cdots + \lambda_a^2 - \lambda_{a+1}^2 - \cdots - \lambda_{a+b}^2$$

for some pair of nonnegative integers a, b with $a+b \leq n$. Moreover, the integers a and b depend only on the isomorphism class of the pair (V, q) (a theorem of Sylvester). In particular, if we assume that q is nondegenerate (in other words, that $a + b = n$), then the isomorphism class (V, q) is completely determined by the dimension n of the vector space V and the difference $a - b$, which is called the *signature* of the quadratic form q.

Example 1.1.3. Let $\mathbf{Q}$ denote the field of rational numbers. There is a complete classification of quadratic spaces over $\mathbf{Q}$, due to Minkowski (later generalized by Hasse to the case of an arbitrary number field). Minkowski's result is highly nontrivial, and represents one of the great triumphs in the arithmetic theory of quadratic forms: we refer the reader to [49] for a detailed and readable account.

Let us now specialize to the case $R = \mathbf{Z}$. We will refer to quadratic spaces (V, q) over $\mathbf{Z}$ as *even lattices* (since the associated bilinear form $b(x, y) = q(x+y) - q(x) - q(y)$ has the property that $b(x, x) = 2q(x)$ is always even). The classification of even lattices up to isomorphism is generally regarded as an intractable problem (see Remark 1.1.17 below). Let us therefore focus on the following variant of Question 1.1.1:

Question 1.1.4. Let (V, q) and (V', q') be even lattices. Is there an effective procedure for determining whether or not (V, q) and (V', q') are isomorphic?

Let (V, q) be a quadratic space over a commutative ring R, and suppose we are given a ring homomorphism $\phi : R \to S$. We let V_S denote the tensor product $S \otimes_R V$. An elementary calculation shows that there is a unique quadratic form $q_S : V_S \to S$ for which the diagram

$$\begin{array}{ccc} V & \xrightarrow{q} & R \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ V_S & \xrightarrow{q_S} & S \end{array}$$

is commutative. The construction $(V, q) \mapsto (V_S, q_S)$ carries quadratic spaces over R to quadratic spaces over S; we refer to (V_S, q_S) as the *extension of scalars* of (V, q). If (V, q) and (V', q') are isomorphic quadratic spaces over R, then extension of scalars yields isomorphic quadratic spaces (V_S, q_S) and (V'_S, q'_S) over S. Consequently, if we understand the classification of quadratic spaces over S and can tell that (V_S, q_S) and (V'_S, q'_S) are *not* isomorphic, it follows that (V, q) and (V', q') are not isomorphic.

Example 1.1.5. Let $q : \mathbf{Z} \to \mathbf{Z}$ be the quadratic form given by $q(n) = n^2$. Then the even lattices $(\mathbf{Z}, q)$ and $(\mathbf{Z}, -q)$ cannot be isomorphic, because they are not isomorphic after extension of scalars to $\mathbf{R}$: the quadratic space $(\mathbf{R}, q_{\mathbf{R}})$ has signature 1, while $(\mathbf{R}, -q_{\mathbf{R}})$ has signature -1.

Example 1.1.6. Let $q, q' : \mathbf{Z}^2 \to \mathbf{Z}$ be the quadratic forms given by

$$q(m, n) = m^2 + n^2 \qquad q'(m, n) = m^2 + 3n^2.$$

Then $(\mathbf{Z}^2, q)$ and $(\mathbf{Z}^2, q')$ become isomorphic after extension of scalars to $\mathbf{R}$ (since both quadratic forms are positive-definite). However, the quadratic spaces $(\mathbf{Z}^2, q)$ and $(\mathbf{Z}^2, q')$ are not isomorphic, since they are not isomorphic after extension of scalars to the field $\mathbf{F}_3 = \mathbf{Z}/3\mathbf{Z}$ (the quadratic form $q_{\mathbf{F}_3}$ is nondegenerate, but $q'_{\mathbf{F}_3}$ is degenerate).

Using variants of the arguments provided in Examples 1.1.5 and 1.1.6, one can produce many examples of even lattices (V, q) and (V', q') that cannot be isomorphic: for example, by arranging that q and q' have different signatures (after extension of scalars to $\mathbf{R}$) or have nonisomorphic reductions modulo n for some integer $n > 0$ (which can be tested by a finite calculation). This motivates the following definition:

Definition 1.1.7. Let (V, q) and (V', q') be positive-definite even lattices. We say that (V, q) and (V', q') *of the same genus* if (V, q) and (V', q') are isomorphic after extension of scalars to $\mathbf{Z}/n\mathbf{Z}$, for every positive integer n (in particular, this implies that V and V' are free abelian groups of the same rank).

Remark 1.1.8. One can also define study genera of lattices which are neither even nor positive definite, but we will restrict our attention to the situation of Definition 1.1.7 to simply the exposition.

More informally, we say that two even lattices (V, q) and (V', q') are of the same genus if we cannot distinguish between them using variations on Example 1.1.5 or 1.1.6. It is clear that isomorphic even lattices are of the same genus, but the converse is generally false. However, the problem of classifying even lattices within a genus has a great deal of structure. One can show that there are only finitely many isomorphism classes of even lattices in the same genus as (V, q). Moreover, the celebrated *Smith-Minkowski-Siegel mass formula* allows us to say exactly how many (at least when counted with multiplicity).

Notation 1.1.9. Let (V, q) be a quadratic space over a commutative ring R. We let $\mathrm{O}_q(R)$ denote the automorphism group of (V, q): that is, the group of R-module isomorphisms $\alpha : V \to V$ such that $q = q \circ \alpha$. We will refer to $\mathrm{O}_q(R)$ as the *orthogonal group* of the quadratic space (V, q). More generally, if $\phi : R \to S$ is a map of commutative rings, we let $\mathrm{O}_q(S)$ denote the automorphism group of the quadratic space (V_S, q_S) over S obtained from (V, q) by extension of scalars to S.

Example 1.1.10. Suppose q is a positive-definite quadratic form on an real vector space V of dimension n. Then $\mathrm{O}_q(\mathbf{R})$ can be identified with the usual orthogonal group $\mathrm{O}(n)$. In particular, $\mathrm{O}_q(\mathbf{R})$ is a compact Lie group of dimension $\frac{n^2-n}{2}$.

Example 1.1.11. Let (V, q) be a positive-definite even lattice. For every integer d, the group $\mathrm{O}_q(\mathbf{Z})$ acts by permutations on the set $V_{\leq d} = \{v \in V : q(v) \leq d\}$. Since q is positive-definite, each of the sets $V_{\leq d}$ is finite. Moreover, for $d \gg 0$, the action of $\mathrm{O}_q(\mathbf{Z})$ on $V_{\leq d}$ is faithful (since an automorphism of V is determined by its action on a finite generating set for V). It follows that $\mathrm{O}_q(\mathbf{Z})$ is a finite group.

Let (V, q) be a positive-definite even lattice. The mass formula gives an explicit formula for the sum $\sum_{(V',q')} \frac{1}{|\mathrm{O}_{q'}(\mathbf{Z})|}$, where the sum is taken over all isomorphism classes of even lattices (V', q') in the genus of (V, q). The explicit formula is rather complicated in general, depending on the reduction of (V, q) modulo p for various primes p. For simplicity, we will restrict our attention to the simplest possible case.

Definition 1.1.12. Let (V, q) be an even lattice. We will say that (V, q) is *unimodular* if the bilinear form $b(v, w) = q(v + w) - q(v) - q(w)$ induces an isomorphism of V with its dual $\mathrm{Hom}(V, \mathbf{Z})$.

Remark 1.1.13. Let (V,q) be a positive-definite even lattice. The condition that (V,q) be unimodular depends only on the reduction of q modulo p for all primes p. In particular, if (V,q) is unimodular and (V',q') is in the genus of (V,q), then (V',q') is also unimodular. In fact, the converse also holds: any two unimodular even lattices of the same rank are of the same genus (though this is not obvious from the definitions).

Remark 1.1.14. The condition that an even lattice (V,q) be unimodular is very strong: for example, if q is positive-definite, it implies that the rank of V is divisible by 8.

Theorem 1.1.15 (Mass Formula: Unimodular Case)**.** *Let n be an integer which is a positive multiple of 8. Then*

$$\begin{aligned}\sum_{(V,q)} \frac{1}{|\operatorname{O}_q(\mathbf{Z})|} &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(\frac{n}{2})}{2^{n-1}\pi^{n(n+1)/4}} \\ &= \frac{B_{n/4}}{n}\prod_{1\leq j<n/2}\frac{B_j}{4j}.\end{aligned}$$

Here ζ denotes the Riemann zeta function, B_j denotes the jth Bernoulli number, and the sum is taken over all isomorphism classes of positive-definite even unimodular lattices (V,q) of rank n.

Example 1.1.16. Let $n = 8$. The right hand side of the mass formula evaluates to $\frac{1}{696729600}$. The integer $696729600 = 2^{14}3^5 5^2 7$ is the order of the Weyl group of the exceptional Lie group E_8, which is also the automorphism group of the root lattice of E_8 (which is an even unimodular lattice). Consequently, the fraction $\frac{1}{696729600}$ also appears as one of the summands on the left hand side of the mass formula. It follows from Theorem 1.1.15 that no other terms appear on the left hand side: that is, the root lattice of E_8 is the *unique* positive-definite even unimodular lattice of rank 8, up to isomorphism.

Remark 1.1.17. Theorem 1.1.15 allows us to count the number of positive-definite even unimodular lattices of a given rank with multiplicity, where a lattice (V,q) is counted with multiplicity $\frac{1}{|\operatorname{O}_q(\mathbf{Z})|}$. If the rank of V is positive, then $\operatorname{O}_q(\mathbf{Z})$ has order at least 2 (since $\operatorname{O}_q(\mathbf{Z})$ contains the group $\langle \pm 1\rangle$), so that the left hand side of Theorem 1.1.15 is at most $\frac{C}{2}$, where C is the number of isomorphism classes of positive-definite even unimodular lattices. In particular, Theorem 1.1.15 gives an inequality

$$C \geq \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2}{2})\cdots\Gamma(\frac{n}{2})\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(\frac{n}{2})}{2^{n-2}\pi^{n(n+1)/4}}.$$

The right hand side of this inequality grows very quickly with n. For example, when $n = 32$, we can deduce the existence of more than eighty million pairwise nonisomorphic (positive-definite) even unimodular lattices of rank n.

We now describe a reformulation of Theorem 1.1.15, following ideas of Tamagawa and Weil. Suppose we are given a positive-definite even lattice (V,q), and that we wish to classify other even lattices of the same genus. If (V',q') is a lattice in the genus of (V,q), then for every positive integer n we can choose an isomorphism $\alpha_n : V/nV \simeq V'/nV'$ which is compatible with the quadratic forms q and q'. Using a compactness argument (or some variant of Hensel's lemma) we can assume without loss of generality that the isomorphisms $\{\alpha_n\}_{n>0}$ are compatible with one another: that is, that the diagrams

$$\begin{array}{ccc} V/nV & \xrightarrow{\alpha_n} & V'/nV' \\ \downarrow & & \downarrow \\ V/mV & \xrightarrow{\alpha_m} & V'/mV' \end{array}$$

commute whenever m divides n. In this case, the data of the family $\{\alpha_n\}$ is equivalent to the data of a single isomorphism $\alpha : \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} V \to \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} V'$, where $\widehat{\mathbf{Z}} \simeq \varprojlim_{n>0} \mathbf{Z}/n\mathbf{Z}$ denotes the profinite completion of the ring $\mathbf{Z}$.

By virtue of the Chinese remainder theorem, the ring $\widehat{\mathbf{Z}}$ can be identified with the product $\prod_p \mathbf{Z}_p$, where p ranges over all prime numbers and $\mathbf{Z}_p$ denotes the ring $\varprojlim \mathbf{Z}/p^k\mathbf{Z}$ of p-adic integers. It follows that (V, q) and (V', q') become isomorphic after extension of scalars to $\mathbf{Z}_p$, and therefore also after extension of scalars to the field $\mathbf{Q}_p = \mathbf{Z}_p[p^{-1}]$ of p-adic rational numbers. Since the lattices (V, q) and (V', q') are positive-definite and have the same rank, they also become isomorphic after extending scalars to the field of real numbers. It follows from Minkowski's classification that the quadratic spaces $(V_{\mathbf{Q}}, q_{\mathbf{Q}})$ and $(V'_{\mathbf{Q}}, q'_{\mathbf{Q}})$ are isomorphic (this is known as the *Hasse principle*: to show that quadratic spaces over $\mathbf{Q}$ are isomorphic, it suffices to show that they are isomorphic over every completion of $\mathbf{Q}$; see §3.3 of [49]). We may therefore choose an isomorphism $\beta : V_{\mathbf{Q}} \to V'_{\mathbf{Q}}$ which is compatible with the quadratic forms q and q'.

Let $\mathbf{A}_f$ denote the ring of *finite adeles*: that is, the tensor product $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$. The isomorphism $\widehat{\mathbf{Z}} \simeq \prod_p \mathbf{Z}_p$ induces an injective map

$$\mathbf{A}_f \simeq \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q} \hookrightarrow \prod_p (\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathbf{Q}) \simeq \prod_p \mathbf{Q}_p,$$

whose image is the *restricted product* $\prod_p^{\mathrm{res}} \mathbf{Q}_p \subseteq \prod_p \mathbf{Q}_p$: that is, the subset consisting of those elements $\{x_p\}$ of the product $\prod_p \mathbf{Q}_p$ such that $x_p \in \mathbf{Z}_p$ for all but finitely many prime numbers p. The quadratic spaces (V, q) and (V', q') become isomorphic after extension of scalars to $\mathbf{A}_f$ in two different ways: via the isomorphism α which is defined over $\widehat{\mathbf{Z}}$, and via the isomorphism β which is defined over $\mathbf{Q}$. Consequently, the composition $\beta^{-1} \circ \alpha$ can be regarded as an element of $\mathrm{O}_q(\mathbf{A}_f)$. This element depends not only the quadratic space (V', q'), but also on our chosen isomorphisms α and β. However, any other isomorphism between $(V_{\widehat{\mathbf{Z}}}, q_{\widehat{\mathbf{Z}}})$ and $(V'_{\widehat{\mathbf{Z}}}, q'_{\widehat{\mathbf{Z}}})$ can be written in the form $\alpha \circ \gamma$, where $\gamma \in \mathrm{O}_q(\widehat{\mathbf{Z}})$. Similarly, the isomorphism β is well-defined up to right multiplication by elements of $\mathrm{O}_q(\mathbf{Q})$. Consequently, the composition $\beta^{-1} \circ \alpha$ is really well-defined as an element of the set of double cosets

$$\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}_f) / \mathrm{O}_q(\widehat{\mathbf{Z}}).$$

Let us denote this double coset by $[V', q']$.

It is not difficult to show that the construction $(V', q') \mapsto [V', q']$ induces a bijection from the set of isomorphism classes of even lattices (V', q') in the genus of (V, q) with the set $\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}_f) / \mathrm{O}_q(\widehat{\mathbf{Z}})$ (the inverse of this construction is given by the procedure of *regluing*; see Construction 1.2.15). Moreover, if $\gamma \in \mathrm{O}_q(\mathbf{A}_f)$ is a representative of the double coset $[V', q']$, then the group $\mathrm{O}_{q'}(\mathbf{Z})$ is isomorphic to the intersection

$$\mathrm{O}_q(\widehat{\mathbf{Z}}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q}) \gamma.$$

Consequently, the left hand side of the mass formula can be written as a sum

$$\sum_{\gamma} \frac{1}{|\operatorname{O}_q(\widehat{\mathbf{Z}}) \cap \gamma^{-1} \operatorname{O}_q(\mathbf{Q})\gamma|}, \tag{1}$$

where γ ranges over a set of double coset representatives.

At this point, it will be technically convenient to introduce two modifications of the calculation we are carrying out. For every commutative ring R, let $\operatorname{SO}_q(R)$ denote the subgroup of $\operatorname{O}_q(R)$ consisting of those automorhisms of (V_R, q_R) which have determinant 1 (if R is an integral domain, this is a subgroup of index at most 2). Let us instead attempt to compute the sum

$$\sum_{\gamma} \frac{1}{|\operatorname{SO}_q(\widehat{\mathbf{Z}}) \cap \gamma^{-1} \operatorname{SO}_q(\mathbf{Q})\gamma|}, \tag{2}$$

where γ runs over a set of representatives for the collection of double cosets

$$X = \operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A}_f)/\operatorname{SO}_q(\widehat{\mathbf{Z}}).$$

If q is unimodular, expression (2) differs from the expression (1) by an overall factor of 2 (in general, the expressions differ by a power of 2).

Remark 1.1.18. Fix an orientation of the $\mathbf{Z}$-module V (that is, a generator of the top exterior power of V). Quantity (2) can be written as a sum $\sum \frac{1}{|\operatorname{SO}_{q'}(\mathbf{Z})|}$, where the sum is indexed by all isomorphism classes of *oriented* even unimodular positive-definite lattices (V', q') which are isomorphic to (V, q) as *oriented* quadratic spaces after extension of scalars to $\mathbf{Z}/n\mathbf{Z}$, for every integer $n > 0$.

Let $\mathbf{A}$ denote the ring of *adeles*: that is, the ring $\mathbf{A}_f \times \mathbf{R}$. Then we can identify X with the collection of double cosets $\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A})/\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. The virtue of this maneuver is that $\mathbf{A}$ has the structure of a locally compact commutative ring containing $\mathbf{Q}$ as a *discrete* subring. Consequently, $\operatorname{SO}_q(\mathbf{A})$ is a locally compact topological group which contains $\operatorname{SO}_q(\mathbf{Q})$ as a discrete subgroup and $\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ as a compact open subgroup.

Let μ be a Haar measure on the group $\operatorname{SO}_q(\mathbf{A})$. One can show that the group $\operatorname{SO}_q(\mathbf{A})$ is unimodular: that is, the measure μ is invariant under both right and left translations. In particular, μ determines a measure on the quotient $\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A})$, which is invariant under the right action of $\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. We will abuse notation by denoting this measure also by μ. Write $\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A})$ as a union of orbits $\bigcup_{x \in X} O_x$ for the action of the group $\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. If $x \in X$ is a double coset represented by an element $\gamma \in \operatorname{SO}_q(\mathbf{A})$, then we can identify the orbit O_x with the quotient of $\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ by the finite subgroup $\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \operatorname{SO}_q(\mathbf{Q})\gamma$. We therefore have

$$\sum_{\gamma} \frac{1}{|\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \operatorname{SO}_q(\mathbf{Q})\gamma|} = \sum_{x \in X} \frac{\mu(O_x)}{\mu(\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))} \tag{3}$$

$$= \frac{\mu(\operatorname{SO}_q(\mathbf{Q})\backslash \operatorname{SO}_q(\mathbf{A}))}{\mu(\operatorname{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}. \tag{4}$$

Of course, the Haar measure μ on $\operatorname{SO}_q(\mathbf{A})$ is only well-defined up to scalar multiplication. Rescaling the measure μ has no effect on the right hand side of the preceding equation, since μ appears in both the numerator and the denominator of the right hand side. However, it is possible to say more: it turns out that there is a canonical normalization of the Haar measure, known as *Tamagawa measure*.

Construction 1.1.19. Let G be a linear algebraic group of dimension d over the field $\mathbf{Q}$ of rational numbers. Let Ω denote the collection of all left invariant d-forms on G, so that Ω is a 1-dimensional vector space over $\mathbf{Q}$. Choose a nonzero element $\omega \in \Omega$.

The vector ω determines a left-invariant differential form of top degree on the smooth manifold $G(\mathbf{R})$, which in turn determines a Haar measure $\mu_{\mathbf{R},\omega}$ on $G(\mathbf{R})$. For every prime number p, an analogous construction yields a measure $\mu_{\mathbf{Q}_p,\omega}$ on the p-adic analytic manifold $G(\mathbf{Q}_p)$. Assuming that G is connected and and has no nontrivial characters, one can show that the product of these measures determines a measure μ_{Tam} on the restricted product

$$G(\mathbf{R}) \times \prod_p^{\mathrm{res}} G(\mathbf{Q}_p) \simeq G(\mathbf{A}).$$

Let λ be a nonzero rational number. Then an elementary calculation gives

$$\mu_{\mathbf{R},\lambda\omega} = |\lambda| \mu_{\mathbf{R},\omega} \qquad \mu_{\mathbf{Q}_p,\lambda\omega} = |\lambda|_p \mu_{\mathbf{Q}_p,\omega};$$

here $|\lambda|_p$ denotes the p-adic absolute value of λ. Combining this with the product formula $\prod_p |\lambda|_p = \frac{1}{|\lambda|}$, we deduce that μ_{Tam} is independent of the choice of nonzero element $\omega \in \Omega$. We will refer to μ_{Tam} as the *Tamagawa measure* of the algebraic group G.

If (Λ, q) is a positive-definite even lattice, then the restriction of the functor $R \mapsto \mathrm{SO}_q(R)$ to $\mathbf{Q}$-algebras can be regarded as a semisimple algebraic group over $\mathbf{Q}$. We may therefore apply Construction 1.1.19 to obtain a canonical measure μ_{Tam} on the group $\mathrm{SO}_q(\mathbf{A})$. We may therefore specialize equation (4) to obtain an equality

$$\sum_\gamma \frac{1}{|\mathrm{SO}_{q'}(\mathbf{Z})|} = \frac{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A}))}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}, \tag{5}$$

where it makes sense to evaluate the numerator and the denominator of the right hand side independently.

Remark 1.1.20. The construction $R \mapsto \mathrm{O}_q(R)$ also determines a semisimple algebraic group over $\mathbf{Q}$. However, this group is not connected, and the infinite product $\prod_p \mu_{\mathbf{Q}_p,\omega}$ does not converge to a measure on the restricted product $\prod_p^{\mathrm{res}} \mathrm{O}_q(\mathbf{Q}_p) = \mathrm{O}_q(\mathbf{A}_f)$. This is one reason for preferring to work with the group SO_q in place of O_q.

Remark 1.1.21. The numerator on the right hand side of (5) is called the *Tamagawa number* of the algebraic group SO_q. More generally, if G is a connected semisimple algebraic group over $\mathbf{Q}$, we define the *Tamagawa number* of G to be the Tamagawa measure of the quotient $G(\mathbf{Q}) \backslash G(\mathbf{A})$.

The denominator on the right hand side of (5) is computable: if we choose a differential form ω as in Construction 1.1.19, it is given by the product

$$\mu_{\mathbf{R},\omega}(\mathrm{SO}_q(\mathbf{R})) \prod_p \mu_{\mathbf{Q}_p,\omega}(\mathrm{SO}_q(\mathbf{Z}_p)).$$

The first term is the volume of a compact Lie group, and the second term is a product of local factors which are related to counting problems over finite rings. Carrying out these calculations leads to a very pretty reformulation of Theorem 1.1.15:

Theorem 1.1.22 (Mass Formula, Adelic Formulation)**.** *Let (V, q) be a nondegenerate quadratic space over $\mathbf{Q}$. Then $\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A})) = 2$.*

The appearance of the number 2 in the statement of Theorem 1.1.22 results from the fact that the algebraic group SO_q is not simply connected. Let Spin_q denote the (2-fold) universal cover of SO_q, so that Spin_q is a simply connected semisimple algebraic group over $\mathbf{Q}$. We then have the following more basic statement:

Theorem 1.1.23. *Let (V, q) be a positive-definite quadratic space over $\mathbf{Q}$. Then*

$$\mu_{\mathrm{Tam}}(\mathrm{Spin}_q(\mathbf{Q})\backslash \mathrm{Spin}_q(\mathbf{A})) = 1.$$

Remark 1.1.24. For a deduction of Theorem 1.1.22 from Theorem 1.1.23, see [43].

Theorem 1.1.23 motivates the following:

Conjecture 1.1.25 (Weil's Conjecture on Tamagawa Numbers)**.** Let G be a simply connected semisimple algebraic group over $\mathbf{Q}$. Then $\mu_{\mathrm{Tam}}(G(\mathbf{Q})\backslash G(\mathbf{A})) = 1$.

Conjecture 1.1.25 was proved by Weil in a number of special cases. The general case was proven by Langlands in the case of a split group ([31]), by Lai in the case of a quasi-split group ([29]), and by Kottwitz for arbitrary simply connected algebraic groups satisfying the Hasse principle ([28]; this is now known to be all simply connected semisimple algebraic groups over $\mathbf{Q}$, by work of Chernousov).

The goal of this paper is to address the function field analogue of Conjecture 1.1.25, which we will discuss in the next section.

1.2. **Weil's Conjecture for Function Fields.**

In this section, we will review the definition of Tamagawa measure for algebraic groups G which are defined over function fields. We will then state the function field analogue of Weil's conjecture, and explain how to reformulate it as a counting problem (using the logic of §1.1 in reverse).

Notation 1.2.1. Let $\mathbf{F}_q$ denote a finite field with q elements, and let X be an algebraic curve over $\mathbf{F}_q$ (which we assume to be smooth, proper, and geometrically connected). We let K_X denote the function field of the curve X (that is, the residue field of the generic point of X).

We will write $x \in X$ to mean that x is a *closed* point of the curve X. For each point $x \in X$, we let $\kappa(x)$ denote the residue field of X at the point x. Then $\kappa(x)$ is a finite extension of the finite field $\mathbf{F}_q$. We will denote the degree of this extension by $\deg(x)$ and refer to it as the *degree* of x. We let $\mathcal{O}_x$ denote the completion of the local ring of X at the point x: this is a complete discrete valuation ring with residue field $\kappa(x)$, noncanonically isomorphic to a power series ring $\kappa(x)[[t]]$. We let K_x denote the fraction field of $\mathcal{O}_x$.

For every finite set S of closed point of X, let $\mathbf{A}^S$ denote the product $\prod_{x\in S} K_x \times \prod_{x\notin S} \mathcal{O}_x$. We let $\mathbf{A}$ denote the direct limit

$$\varinjlim_{S\subseteq X} \mathbf{A}^S .$$

We will refer to $\mathbf{A}$ as the *ring of adeles of* K_X. It is a locally compact commutative ring, equipped with a ring homomorphism $K_X \to \mathbf{A}$ which is an isomorphism of K_X onto a discrete subset of $\mathbf{A}$. We let $\mathbf{A}_0 = \prod_{x\in X} \mathcal{O}_x$ denote the ring of *integral adeles*, so that $\mathbf{A}_0$ is a compact open subring of $\mathbf{A}$.

Let G_0 be a linear algebraic group of dimension d defined over the field K_X. It will often be convenient to assume that we are given an *integral model* of G_0: that is, that G_0 is given as the generic fiber of a smooth affine group scheme G over the curve X.

Remark 1.2.2. If G_0 is a simply connected semisimple algebraic group over K_X, then it is always possible to find a smooth affine group scheme with generic fiber G_0. See, for example, [12] or §7.1 of [11].

Remark 1.2.3. In what follows, it will sometimes be convenient to assume that the group scheme $G \to X$ has connected fibers. This can always be arranged by passing to an open subgroup $G^{\circ} \subseteq G$ (which does not injure our assumption that G is an affine group scheme over X, since the open immersion $G^{\circ} \hookrightarrow G$ is complementary to a Cartier divisor and is therefore an affine morphism).

For every commutative ring R equipped with a map $\operatorname{Spec} R \to X$, we let $G(R)$ denote the group of R-points of G. Then:

- For each closed point $x \in X$, $G(K_x)$ is a locally compact group, which contains $G(\mathcal{O}_x)$ as a compact open subgroup.
- We can identify $G(\mathbf{A})$ with the restricted product $\prod_{x \in X}^{\text{res}} G(K_x)$: that is, with the subgroup of the product $\prod_{x \in X} G(K_x)$ consisting of those elements $\{g_x\}_{x \in X}$ such that $g_x \in G(\mathcal{O}_x)$ for all but finitely many values of X.
- The group $G(\mathbf{A})$ is locally compact, containing $G(K_X)$ as a discrete subgroup and $G(\mathbf{A}_0) \simeq \prod_{x \in X} G(\mathcal{O}_x)$ as a compact open subgroup.

1.2.1. Let us now review the construction of Tamagawa measure on the locally compact group $G(\mathbf{A})$. Let $\Omega_{G/X}$ denote the relative cotangent bundle of the smooth morphism $\pi : G \to X$. Then $\Omega_{G/X}$ is a vector bundle on G of rank $d = \dim(G_0)$. We let $\Omega^d_{G/X}$ denote the top exterior power of $\Omega_{G/X}$, so that $\Omega^d_{G/X}$ is a line bundle on G. Let $\mathcal{L}$ denote the pullback of $\Omega^d_{G/X}$ along the identity section $e : X \to G$. Equivalently, we can identify $\mathcal{L}$ with the subbundle of $\pi_* \Omega^d_{G/X}$ consisting of left-invariant sections. Let $\mathcal{L}_0$ denote the generic fiber of $\mathcal{L}$, so that $\mathcal{L}_0$ is a 1-dimensional vector space over the function field K_X. Fix a nonzero element $\omega \in \mathcal{L}_0$, which we can identify with a left-invariant differential form of top degree on the algebraic group G_0.

For every point $x \in X$, ω determines a Haar measure $\mu_{x,\omega}$ on the locally compact topological group $G(K_x)$. Concretely, this measure can be defined as follows. Let t denote a uniformizing parameter for $\mathcal{O}_x$ (so that $\mathcal{O}_x \simeq \kappa(x)[[t]]$), and let $G_{\mathcal{O}_x}$ denote the fiber product $\operatorname{Spec} \mathcal{O}_x \times_X G$. Choose a local coordinates $y_1, \ldots, y_d$ for the group $G_{\mathcal{O}_x}$ near the identity: that is, coordinates which induce a map $u : G_{\mathcal{O}_x} \to \mathbf{A}^d_{\mathcal{O}_x}$ which is étale at the origin of $G_{\mathcal{O}_x}$. Let $v_x(\omega)$ denote the order of vanishing of ω at the point x. Then, in a neighborhood of the origin in $G_{\mathcal{O}_x}$, we can write $\omega = t^{v_x(\omega)} \lambda dy_1 \wedge \cdots \wedge dy_d$, where λ is an invertible regular function. Let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_x$, and let $G(\mathfrak{m}_x)$ denote the kernel of the reduction map $G(\mathcal{O}_x) \to G(\kappa(x))$. Since $y_1, \ldots, y_d$ are local coordinates near the origin, the map u induces a bijection $G(\mathfrak{m}_x) \to \mathfrak{m}_x^d$. The measure defined by the differential form $dy_1 \wedge \cdots \wedge dy_d$ on $G(\mathfrak{m}_x)$ is obtained by pulling back the "standard" measure on K_x^d along the map u, where this standard measure is normalized so that $\mathcal{O}_x^d$ has measure 1. It follows that the measure of $G(\mathfrak{m}_x)$ (with respect to the differential form $dy_1 \wedge \cdots \wedge dy_d$) is given by $\frac{1}{|\kappa(x)|^d}$. We then define

$$\mu_{x,\omega}(G(\mathfrak{m}_x)) = q^{-\deg(x) v_x(\omega)} \frac{1}{|\kappa(x)|^d}.$$

The smoothness of G implies that the map $G(\mathcal{O}_x) \to G(\kappa(x))$ is surjective, so that we have

$$\mu_{x,\omega}(G(\mathcal{O}_x)) = q^{-\deg(x) v_x(\omega)} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

Remark 1.2.4. Since $G(\mathcal{O}_x)$ is a compact open subgroup of $G(K_x)$, there is a unique left-invariant measure μ on $G(\mathcal{O}_x)$ satisfying

$$\mu(G(\mathcal{O}_x)) = q^{-\deg(x) v_x(\omega)} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

The reader can therefore take this expression as the *definition* of the measure $\mu_{x,\omega}$. However, the analytic perspective is useful for showing that this measure is independent of the choice of integral model chosen. We refer the reader to [57] for more details.

The key fact is the following:

Proposition 1.2.5. *Suppose that G_0 is connected and semisimple, and let ω be a nonzero element of $\mathcal{L}_0$. Then the measures $\mu_{x,\omega}$ on the groups $G(K_x)$ determine a well-defined product measure on $G(\mathbf{A}) = \prod_{x \in X}^{\text{res}} G(K_x)$. Moreover, this product measure is independent of ω.*

Proof. To check that the product measure is well-defined, it suffices to show that it is well-defined when evaluated on a compact open subgroup of $G(\mathbf{A})$, such as $G(\mathbf{A}_0)$. This is equivalent to the absolute convergence of the infinite product

$$\prod_{x \in X} \mu_{x,\omega}(G(\mathcal{O}_x)) = \prod_{x \in X} q^{-\deg(x) v_x(\omega)} \frac{|G(\kappa(x))|}{|\kappa(x)|^d},$$

which we will discuss in §6.5.

The fact that the product measure is independent of the choice of ω follows from the fact that the infinite sum

$$\sum_{x \in X} \deg(x) v_x(\omega) = \deg(\mathcal{L})$$

is independent of ω. □

Definition 1.2.6. Let G_0 be a connected semisimple algebraic group over K_X. Let d denote the dimension of G_0, and let g denote the genus of the curve X. The *Tamagawa measure* on $G(\mathbf{A})$ is the Haar measure given informally by the product

$$\mu_{\text{Tam}} = q^{d(1-g)} \prod_{x \in X} \mu_{x,\omega}$$

Remark 1.2.7. Equivalently, we can define Tamagawa measure μ_{Tam} to be the unique Haar measure on $G(\mathbf{A})$ which is normalized by the requirement

$$\mu_{\text{Tam}}(G(\mathbf{A}_0)) = q^{d(1-g)-\deg(\mathcal{L})} \prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

Remark 1.2.8. To ensure that the Tamagawa measure μ_{Tam} is well-defined, it is important that the quotients $\frac{|G(\kappa(x))|}{|\kappa(x)|^d}$ converge swiftly to 1, so that the infinite product $\prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}$ is absolutely convergent. This can fail dramatically if G_0 is not connected. However, it is satisfied for some algebraic groups which are not semisimple: for example, the additive group $\mathbf{G}_a$.

Remark 1.2.9. If the group G_0 is semisimple, then any left-invariant differential form ω of top degree on G_0 is also right-invariant. It follows that the group $G(\mathbf{A})$ is unimodular. In particular, the measure μ_{Tam} on $G(\mathbf{A})$ descends to a measure on the quotient $G(K_X) \backslash G(\mathbf{A})$, which is invariant under the action of $G(\mathbf{A})$ by right translation. We will denote this measure also by μ_{Tam}, and refer to it as *Tamagawa measure*. It is characterized by the following requirement: for every positive measurable function f on $G(\mathbf{A})$, we have

$$\int_{x \in G(\mathbf{A})} f(x) d\mu_{\text{Tam}} = \int_{y \in G(K_X)/G(\mathbf{A})} (\sum_{\pi(x)=y} f(x)) d\mu_{\text{Tam}}, \tag{6}$$

where $\pi : G(\mathbf{A}) \to G(K_X) \backslash G(\mathbf{A})$ denotes the projection map.

An important special case occurs when f is the characteristic function of a coset γH for some compact open subgroup $H \subseteq G(\mathbf{A})$. In this case, each element of $\pi(\gamma H)$ has exactly $o(\gamma)$

preimages in U, where $o(\gamma)$ denotes the order of the finite group $G(K_X) \cap \gamma H \gamma^{-1}$ (this group is finite, since it is the intersection of a discrete subgroup of $G(\mathbf{A})$ with a compact subgroup of $G(\mathbf{A})$). Applying formula (6), we deduce that $\mu_{\mathrm{Tam}}(\pi(\gamma H)) = \frac{\mu_{\mathrm{Tam}}(H)}{o(\gamma)}$.

Example 1.2.10. Let $G = \mathbf{G}_a$ be the additive group. Then the dimension d of G is equal to 1, and the line bundle $\mathcal{L}$ of left-invariant top forms is isomorphic to the structure sheaf $\mathcal{O}_X$ of X. Moreover, we have an equality $|G(\kappa(x))| = |\kappa(x)|$ for each $x \in X$. Consequently, the Tamagawa measure μ_{Tam} is characterized by the formula $\mu_{\mathrm{Tam}}(G(\mathbf{A}_0)) = q^{1-g}$. Note that we have an exact sequence of locally compact groups

$$0 \to \mathrm{H}^0(X; \mathcal{O}_X) \to G(\mathbf{A}_0) \to G(K_X) \backslash G(\mathbf{A}) \to \mathrm{H}^1(X; \mathcal{O}_X) \to 0,$$

so that the Tamagawa measure of the quotient $G(K_X) \backslash G(\mathbf{A})$ is given by

$$\frac{|\mathrm{H}^1(X; \mathcal{O}_X)|}{|\mathrm{H}^0(X; \mathcal{O}_X)|} \mu_{\mathrm{Tam}}(G(\mathbf{A}_0)) = \frac{q^g}{q} q^{1-g} = 1.$$

Remark 1.2.11. One might ask the motivation for the auxiliary factor $q^{d(1-g)}$ appearing in the definition of the Tamagawa measure. Remark 1.2.10 provides one answer: the auxiliary factor is exactly what we need in order to guarantee that Weil's conjecture holds for the additive group $\mathbf{G}_a$.

Another answer is that the auxiliary factor is necessary to obtain invariance under *Weil restriction*. Suppose that $f : X \to Y$ is a separable map of algebraic curves over $\mathbf{F}_q$. Let K_Y be the fraction field of Y (so that K_X is a finite separable extension of K_Y), let $\mathbf{A}_Y$ denote the ring of adeles of K_Y, and let H_0 denote the algebraic group over K_Y obtained from G_0 by Weil restriction along the field extension $K_Y \hookrightarrow K_X$. Then we have a canonical isomorphism of locally compact groups $G_0(\mathbf{A}) \simeq H_0(\mathbf{A}_Y)$. This isomorphism is compatible with the Tamagawa measures on each side, but only if we include the auxiliary factor $q^{d(1-g)}$ indicated in Definition 1.2.6. See [42] for more details.

1.2.2. Our goal in this paper is to address the following version of Weil's conjecture:

Conjecture 1.2.12 (Weil)**.** Suppose that G_0 is semisimple and simply connected. Then

$$\mu_{\mathrm{Tam}}(G(K_X) \backslash G(\mathbf{A})) = 1.$$

Let us now reformulate Conjecture 1.2.12 in more elementary terms. Note that the quotient $G(K_X) \backslash G(\mathbf{A})$ carries a right action of the compact group $G(\mathbf{A}_0)$. We may therefore write $G(K_X) \backslash G(\mathbf{A})$ as a union of orbits, indexed by the collection of double cosets

$$G(K_X) \backslash G(\mathbf{A}) / G(\mathbf{A}_0).$$

Applying Remark 1.2.9, we calculate

$$\begin{aligned} \mu_{\mathrm{Tam}}(G(K_X) \backslash G(\mathbf{A})) &= \sum_{\gamma} \frac{\mu_{\mathrm{Tam}}(G(\mathbf{A}_0))}{|G(\mathbf{A}_0) \cap \gamma^{-1} G(K_X) \gamma|} \\ &= q^{d(1-g) - \deg(\mathcal{L})} (\prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}) \sum_{\gamma} \frac{1}{|G(\mathbf{A}_0) \cap \gamma^{-1} G(K_X) \gamma|}. \end{aligned}$$

We may therefore reformulate Weil's conjecture as follows:

Conjecture 1.2.13 (Weil)**.** Suppose that G_0 is semisimple and simply connected. Then we have an equality

$$\prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|} = q^{d(1-g) - \deg(\mathcal{L})} \sum_{\gamma} \frac{1}{|G(\mathbf{A}_0) \cap \gamma^{-1} G(K_X) \gamma|},$$

where the sum on the right hand side is taken over a set of representatives for the double quotient $G(K_X)\backslash G(\mathbf{A})/G(\mathbf{A}_0)$.

Remark 1.2.14. In the statement of Conjecture 1.2.13, the product on the left hand side and the sum on the right hand side are generally both infinite. The convergence of the left hand side is equivalent to the well-definedness of Tamagawa measure μ_{Tam}, and the convergence of the right hand side is equivalent to the statement that $\mu_{\mathrm{Tam}}(G(K_X)\backslash G(\mathbf{A}))$ is finite.

1.2.3. We now give an algebro-geometric interpretation of the sum appearing on the right hand side of Conjecture 1.2.13. In what follows, we will assume that the reader is familiar with the theory of principal G-bundles; we will give a brief review in §A.1.

Construction 1.2.15 (Regluing)**.** Let γ be an element of the group $G(\mathbf{A})$. We can think of γ as given by a collection of elements $\gamma_x \in G(K_x)$, having the property that there exists a finite set S such that $\gamma_x \in G(\mathcal{O}_x)$ whenever $x \notin S$.

We define a G-bundle $\mathcal{P}_\gamma$ on X as follows:

- (a) The bundle $\mathcal{P}_\gamma$ is equipped with a trivialization ϕ on the open set $U = X - S$.
- (b) The bundle $\mathcal{P}_\gamma$ is equipped with a trivialization ψ_x over the scheme $\operatorname{Spec} \mathcal{O}_x$ of each point $x \in S$.
- (c) For each $x \in S$, the trivializations of $\mathcal{P}_\gamma|_{\operatorname{Spec} K_x}$ determined by ϕ and ψ_x differ by multiplication by the the element $\gamma_x \in G(K_x)$.

Note that the G-bundle $\mathcal{P}_\gamma$ is canonically independent of the choice of S, so long as S contains all points x such that $\gamma_x \notin G(\mathcal{O}_x)$.

Remark 1.2.16. Let $\gamma, \gamma' \in G(\mathbf{A})$. The G-bundles $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma'}$ come equipped with trivializations at the generic point of X. Consequently, giving an isomorphism between the restrictions $\mathcal{P}_\gamma|_{\operatorname{Spec} K_X}$ and $\mathcal{P}_{\gamma'}|_{\operatorname{Spec} K_X}$ is equivalent to giving an element $\beta \in G(K_X)$. Unwinding the definitions, we see that this isomorphism admits an (automatically unique) extension to an isomorphism of $\mathcal{P}_\gamma$ with $\mathcal{P}_{\gamma'}$ if and only if $\gamma'^{-1}\beta\gamma$ belongs to $G(\mathbf{A}_0)$. This has two consequences:

- (a) The G-bundles $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma'}$ are isomorphic if and only if γ and γ' determine the same element of $G(K_X)\backslash G(\mathbf{A})/G(\mathbf{A}_0)$.
- (b) The automorphism group of the G-torsor $\mathcal{P}_\gamma$ is the intersection $G(\mathbf{A}_0) \cap \gamma^{-1} G(K_X)\gamma$.

Remark 1.2.17. Let $\mathcal{P}$ be a G-bundle on X. Then $\mathcal{P}$ can be obtained from Construction 1.2.15 if and only if the following two conditions are satisfied:

- (i) There exists an open set $U \subseteq X$ such that $\mathcal{P}|_U$ is trivial.
- (ii) For each point $x \in X - U$, the restriction of $\mathcal{P}$ to $\operatorname{Spec} \mathcal{O}_x$ is trivial.

By a direct limit argument, condition (i) is equivalent to the requirement that $\mathcal{P}|_{\operatorname{Spec} K_X}$ be trivial: that is, that $\mathcal{P}$ is classified by a trivial element of $\mathrm{H}^1(\operatorname{Spec} K_X; G_0)$. If G_0 is semisimple and simply connected, then $\mathrm{H}^1(\operatorname{Spec} K_X; G_0)$ vanishes (see [24]).

If the map $G \to X$ is smooth and has connected fibers, then condition (ii) is automatic (the restriction $\mathcal{P}|_{\operatorname{Spec} \kappa(x)}$ can be trivialized by Lang's theorem (see [30]), and any trivialization of $\mathcal{P}|_{\operatorname{Spec} \kappa(x)}$ can be extended to a trivialization of $\mathcal{P}|_{\operatorname{Spec} \mathcal{O}_x}$ by virtue of the assumption that G is smooth.

Suppose now that G has connected fibers. Combining Remarks 1.2.16 and 1.2.17, we obtain the formula

$$\mu_{\mathrm{Tam}}(G(K_X)\backslash G(\mathbf{A})) \simeq q^{d(1-g)-\deg(\mathcal{L})} \Big(\prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}\Big) \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}.$$

Here the sum is taken over all isomorphism classes of generically trivial G-bundles on X. We may therefore reformulate Conjecture 1.2.12 as follows:

Conjecture 1.2.18 (Weil)**.** Let $G \to X$ be a smooth affine group scheme with connected fibers whose generic fiber is semisimple and simply connected. Then

$$\prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|} = q^{d(1-g)-\deg(\mathcal{L})} \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}.$$

The assertion of Conjecture 1.2.18 can be regarded as a function field version of Theorem 1.1.15. More precisely, we have the following table of analogies:

Classical Mass Formula	Conjecture 1.2.18
Number field $\mathbf{Q}$	Function field K_X
Quadratic space $(V_{\mathbf{Q}}, q_{\mathbf{Q}})$ over $\mathbf{Q}$	Algebraic Group G_0
Even lattice (V, q)	Integral model G
Even lattice (V', q') of the same genus	Principal G-bundle $\mathcal{P}$
$\sum_{q'} \frac{1}{|\operatorname{O}_{q'}(\mathbf{Z})|}$	$\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}$.

1.2.4. There are a number of tools that are available for attacking Conjecture 1.2.18 that have no analogue in the case of a number field. More specifically, we would like to take advantage of the fact that the collection of all G-bundles on X admits an algebro-geometric parametrization.

Notation 1.2.19. For every $\mathbf{F}_q$-algebra R, let $\operatorname{Bun}_G(X)(R)$ denote the category of principal G-bundles on the relative curve $X_R = \operatorname{Spec} R \times_{\operatorname{Spec} \mathbf{F}_q} X$ (where morphisms are given by isomorphisms of G-bundles). The construction $R \mapsto \operatorname{Bun}_G(X)(R)$ determines an algebraic stack, which we will denote by $\operatorname{Bun}_G(X)$ and refer to as the *moduli stack of G-bundles on X*.

By definition, we can identify $\mathbf{F}_q$-valued points of $\operatorname{Bun}_G(X)$ with principal G-bundles on X. We will denote the sum $\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}$ by $|\operatorname{Bun}_G(X)(\mathbf{F}_q)|$: we can think of it as a (weighted) count of the objects of $\operatorname{Bun}_G(X)(\mathbf{F}_q)$, which properly takes into account the fact that $\operatorname{Bun}_G(X)(\mathbf{F}_q)$ is a groupoid rather than a set.

Remark 1.2.20. One can show that $\operatorname{Bun}_G(X)$ is a smooth algebraic stack over $\mathbf{F}_q$. Moreover, for every G-bundle $\mathcal{P}$ on X, the dimension of $\operatorname{Bun}_G(X)$ at the point determined by $\mathcal{P}$ is given by the Euler characteristic

$$-\chi(\mathfrak{g}_{\mathcal{P}}) = \mathrm{H}^1(X; \mathfrak{g}_{\mathcal{P}}) - \mathrm{H}^0(X; \mathfrak{g}_{\mathcal{P}}),$$

where $\mathfrak{g}_{\mathcal{P}}$ denotes the vector bundle on X obtained by twisting the Lie algebra $\mathfrak{g}$ of G using the torsor $\mathcal{P}$. Since the generic fiber G_0 is semisimple, the group G acts trivially on the top exterior power $\bigwedge^d \mathfrak{g}$, so that

$$\bigwedge^d \mathfrak{g}_{\mathcal{P}} \simeq \bigwedge^d \mathfrak{g} \simeq \mathcal{L}^{\vee}.$$

It follows that the vector bundle $\mathfrak{g}_{\mathcal{P}}$ has degree $-\deg(\mathcal{L})$, so that so that the Riemann-Roch theorem gives $\chi(\mathfrak{g}_{\mathcal{P}}) = d(1-g) - \deg(\mathcal{L})$ is independent of $\mathcal{P}$. Applying the same analysis to any R-valued point of $\operatorname{Bun}_G(X)$, we conclude that $\operatorname{Bun}_G(X)$ is equidimensional of dimension $d(g-1) + \deg(\mathcal{L})$. We may therefore rewrite the right hand side of Conjecture 1.2.18 as a fraction

$$\frac{|\operatorname{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\operatorname{Bun}_G(X))}}.$$

Heuristically, this is a normalized count of the number G-bundles on X, where the normalization factor $q^{\dim(\mathrm{Bun}_G(X))}$ can be regarded as a naive estimate determined by the dimension of $\mathrm{Bun}_G(X)$.

1.2.5. For every closed point $x \in X$, let G_x denote the fiber product $\operatorname{Spec} \kappa(x) \times_X G$, so that G_x is a connected algebraic group over $\kappa(x)$. Let BG_x denote the classifying stack of G_x: this is a smooth algebraic stack of dimension $-d$ over $\operatorname{Spec} \kappa(x)$. Then $\mathrm{BG}_x(\mathbf{F}_q)$ is the category of G_x-bundles on $\operatorname{Spec} \kappa(x)$. If G_x is connected, then Lang's theorem implies that every G_x-bundle on $\operatorname{Spec} \kappa(x)$ is trivial. Moreover, the automorphism group of the trivial G_x-bundle is given by $G_x(\kappa(x)) = G(\kappa(x))$. Consequently, we have an identity

$$\frac{|\kappa(x)|^d}{|G(\kappa(x))|} = \frac{|\,\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim(\mathrm{BG}_x)}}.$$

We may therefore rewrite Weil's conjecture in the suggestive form

$$\frac{|\,\mathrm{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\mathrm{Bun}_G)}} = \prod_{x \in X} \frac{|\,\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim(\mathrm{BG}_x)}}. \tag{7}$$

Roughly speaking, formula (7) reflects the idea that $\mathrm{Bun}_G(X)$ can be viewed as a "continuous product" of the classifying stacks BG_x, where x ranges over the closed points of X. Most of this paper will be devoted to making this heuristic idea more precise.

1.3. **Cohomological Formulation.** Throughout this section, we let X denote an algebraic curve defined over a finite field $\mathbf{F}_q$ and G a smooth affine group scheme over X. The analysis given in §1.2 shows that Weil's conjecture can be reduced to the problem of computing the sum $\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}$, where $\mathcal{P}$ ranges over all isomorphism classes of G-bundles on X. Roughly speaking, we can think of this quantity as counting the number of $\mathbf{F}_q$-points of the moduli stack $\mathrm{Bun}_G(X)$.

1.3.1. Let us begin by discussing the analogous counting problem where we replace $\mathrm{Bun}_G(X)$ by an algebraic variety Y defined over $\mathbf{F}_q$. Let $\overline{\mathbf{F}}_q$ be an algebraic closure of $\mathbf{F}_q$, and let $\overline{Y} = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} Y$ denote the associated algebraic variety over $\overline{\mathbf{F}}_q$. We let $\mathrm{Frob} : \overline{Y} \to \overline{Y}$ denote the product of the identity map from $\operatorname{Spec} \overline{\mathbf{F}}_q$ to itself with the absolute Frobenius map from Y to itself. We refer to Frob as the *geometric Frobenius map* on Y. If Y is a quasi-projective variety equipped with an embedding $j : Y \hookrightarrow \mathbf{P}^n$, then the map Frob is given in homogeneous coordinates by the construction

$$[x_0 : \cdots : x_n] \mapsto [x_0^q : \cdots x_n^q]$$

(this map carries $\overline{Y}$ to itself, since $\overline{Y}$ can be described using homogeneous polynomials with coefficients in $\mathbf{F}_q$).

Let $Y(\mathbf{F}_q)$ denote the finite set of $\mathbf{F}_q$-points of Y. Then $Y(\mathbf{F}_q)$ can be identified with the fixed point locus of the map $\mathrm{Frob} : \overline{Y} \to \overline{Y}$. Weil had the beautiful insight that one should be able to compute the integers $|Y(\mathbf{F}_q)|$ using the Lefschetz fixed-point formula, provided that one had a sufficiently robust cohomology theory for algebraic varieties. Motivated by this heuristic, he made a series of famous conjectures about the behavior of the integers $|Y(\mathbf{F}_q)|$.

Weil's conjectures were eventually proven by the Grothendieck school using the theory of *ℓ-adic cohomology*. We will give a brief summary here, and a more detailed discussion in §2. Fix a prime number ℓ which is invertible in $\mathbf{F}_q$. To every algebraic variety V over $\overline{\mathbf{F}}_q$, the theory of ℓ-adic cohomology assigns ℓ-adic cohomology groups $\{\mathrm{H}^n(V; \mathbf{Q}_\ell)\}_{n \geq 0}$ and compactly supported ℓ-adic cohomology groups $\{\mathrm{H}^n_c(V; \mathbf{Q}_\ell)\}_{n \geq 0}$, which are finite dimensional vector spaces over $\mathbf{Q}_\ell$.

If Y is an algebraic variety over $\mathbf{F}_q$, then the geometric Frobenius map $\mathrm{Frob} : \overline{Y} \to \overline{Y}$ is proper and therefore determines a pullback map from $\mathrm{H}^*_c(\overline{Y}; \mathbf{Q}_\ell)$ to itself. We will abuse notation by denoting this map also by Frob. We then have the following:

Theorem 1.3.1 (Grothendieck-Lefschetz Trace Formula)**.** *Let Y be an algebraic variety over $\mathbf{F}_q$. Then the number of $\mathbf{F}_q$-points of Y is given by the formula*

$$|Y(\mathbf{F}_q)| = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\mathrm{Frob} \,|\, \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell)).$$

1.3.2. For our purposes, it will be convenient to write the Grothendieck-Lefschetz trace formula in a slightly different form. Suppose now that Y is a smooth variety of dimension n over $\mathbf{F}_q$. Then, from the perspective of ℓ-adic cohomology, Y behaves as if it were a smooth manifold of dimension $2n$. In particular, it satisfies Poincare duality: that is, there is a perfect pairing

$$\mu : \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) \to \mathbf{Q}_\ell \,.$$

This pairing is not quite Frob-equivariant: instead, it fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell \\ \downarrow {\scriptstyle \mathrm{Frob} \otimes \mathrm{Frob}} & & \downarrow {\scriptstyle q^n} \\ \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell, \end{array}$$

reflecting the idea that the geometric Frobenius map $\mathrm{Frob} : \overline{Y} \to \overline{Y}$ has degree q^n. In particular, pullback along the geometric Frobenius map Frob induces an isomorphism from $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ to itself, and we have the identity

$$q^{-n} \operatorname{Tr}(\mathrm{Frob} \,|\, \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell)) \simeq \operatorname{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell)).$$

We may therefore rewrite Theorem 1.3.1 as follows:

Theorem 1.3.2 (Grothendieck-Lefschetz Trace Formula, Dual Version)**.** *Let Y be an algebraic variety over $\mathbf{F}_q$ which is smooth of dimension n. Then the number of $\mathbf{F}_q$-points of Y is given by the formula*

$$\frac{|Y(\mathbf{F}_q)|}{q^n} = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^i(\overline{Y}; \mathbf{Q}_\ell)).$$

1.3.3. We would like to apply an analogue of Theorem 1.3.2 to the problem of counting G-bundles on an algebraic curve X.

Notation 1.3.3. Let $\mathbf{C}$ denote the field of complex numbers, and fix an embedding $\iota : \mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. Let M be a $\mathbf{Z}_\ell$-module for which $\mathbf{C} \otimes_{\mathbf{Z}_\ell} M$ is a finite-dimensional vector space over $\mathbf{C}$. If ψ is any endomorphism of M as a $\mathbf{Z}_\ell$-module, we let $\operatorname{Tr}(\psi | M) \in \mathbf{C}$ denote the trace of $\mathbf{C}$-linear map $\mathbf{C} \otimes_{\mathbf{Z}_\ell} M \to \mathbf{C} \otimes_{\mathbf{Z}_\ell} M$ determined by ψ. More generally, if ψ is an endomorphism of a graded $\mathbf{Z}_\ell$-module M^*, we let $\operatorname{Tr}(\psi | M^*)$ denote the alternating sum $\sum_{i \geq 0} (-1)^i \operatorname{Tr}(\psi | M^i)$ (provided that this sum is convergent).

Let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles on X. We let $\overline{\mathrm{Bun}}_G(X)$ denote the fiber product $\operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} \mathrm{Bun}_G(X)$, which we regard as a smooth algebraic stack over $\overline{\mathbf{F}}_q$. For every $\overline{\mathbf{F}}_q$-algebra R, we can identify the category of R-valued points of $\overline{\mathrm{Bun}}_G(X)$ with the category of principal G-bundles on the relative curve $X_R = \operatorname{Spec} R \times_{\operatorname{Spec} \mathbf{F}_q} X$.

Note that if R is an $\overline{\mathbf{F}}_q$-algebra, then the construction $a \mapsto a^q$ determines an $\mathbf{F}_q$-algebra homomorphism from R to itself, and therefore induces a map $\mathrm{Frob}_R : X_R \to X_R$ (which is the identity on X). If $\mathcal{P}$ is a principal G-bundle on X_R, then $\mathrm{Frob}_R^* \mathcal{P}$ is another principal

G-bundle on X_R. The construction $\mathcal{P} \mapsto \mathrm{Frob}_R^* \mathcal{P}$ determines a morphism of algebraic stacks $\mathrm{Frob} : \overline{\mathrm{Bun}}_G(X) \to \overline{\mathrm{Bun}}_G(X)$, which we will refer to as the *geometric Frobenius morphism* on $\overline{\mathrm{Bun}}_G(X)$.

We let $\mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)$ denote the ℓ-adic cohomology ring of the algebraic stack $\overline{\mathrm{Bun}}_G(X)$ (for a definition, see §2.3). The geometric Frobenius map $\mathrm{Frob} : \overline{\mathrm{Bun}}_G(X) \to \overline{\mathrm{Bun}}_G(X)$ determines an endomorphism of $\mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)$, which we will denote also by Frob. One can show that this map is an automorphism of $\mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)$ (it is inverse to the map given by pullback along the Frobenius automorphism of $\mathrm{Spec}\, \overline{\mathbf{F}}_q$).

Weil's conjecture is an immediate consequence of the following pair of results:

Theorem 1.3.4. *[Grothendieck-Lefschetz Trace Formula for $\mathrm{Bun}_G(X)$] Assume that the fibers of G are connected and that the generic fiber of G is semisimple. Then we have an equality*

$$\frac{|\mathrm{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\mathrm{Bun}_G(X))}} = \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)).$$

Theorem 1.3.5 (Weil's Conjecture, Cohomological Form)**.** *Suppose that G has connected fibers and that the generic fiber of G is semisimple and simply connected. Then there is an equality*

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)) = \prod_{x \in X} \frac{|\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim(\mathrm{BG}_x)}}$$

In particular, the sum on the left hand side and the product on the right hand side are both absolutely convergent.

Warning 1.3.6. Neither the left or right hand side of the identity asserted by Theorem 1.3.4 is *a priori* well-defined. We should therefore state it more carefully as follows:

(a) For each integer i, the tensor product $\mathbf{C} \otimes_{\mathbf{Z}_\ell} \mathrm{H}^i(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)$ is a finite-dimensional vector space over $\mathbf{C}$, so that the trace $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^i(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell))$ is well-defined.

(b) The sum

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)) = \sum_{i \geq 0} (-1)^i \, \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^i(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell))$$

is absolutely convergent (note that, in contrast with the situation of Theorem 1.3.2, this sum is generally infinite).

(c) The sum $|\mathrm{Bun}_G(X)(\mathbf{F}_q)| = \sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}$ is convergent.

(d) We have an equality

$$\frac{|\mathrm{Bun}_G(X)(\mathbf{F}_q)|}{q^{\dim(\mathrm{Bun}_G)}} = \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)).$$

Remark 1.3.7. Assertion (b) of Warning 1.3.6 relies crucially on the fact that we are summing eigenvalues of the arithmetic Frobenius map Frob^{-1} (which are small), rather than the eigenvalues of the geometric Frobenius Frob (which are large).

Remark 1.3.8. For each closed point $x \in X$, the stack BG_x satisfies the Grothendieck-Lefschetz trace formula. In particular, if we set

$$\overline{\mathrm{BG}}_x = \mathrm{Spec}\, \overline{\mathbf{F}}_q \times_{\mathrm{Spec}\, \kappa(x)} \mathrm{BG}_x$$

and let Frob_x denote the geometric Frobenius morphism of $\overline{\mathrm{BG}}_x$, then we have equalities

$$\frac{|\mathrm{BG}_x(\kappa(x))|}{|\kappa(x)|^{\dim(\mathrm{BG}_x)}} = \mathrm{Tr}(\mathrm{Frob}_x^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Z}_\ell))$$

(see Remark 1.2.20). Theorem 1.3.5 can therefore be reformulated as an identity

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)) = \prod_{x \in X} \mathrm{Tr}(\mathrm{Frob}_x^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Z}_\ell)). \tag{8}$$

We can regard Theorem 1.3.4 as an analogue of Theorem 1.3.1, where the smooth $\mathbf{F}_q$-scheme Y is replaced by the algebraic stack $\mathrm{Bun}_G(X)$. The principal difficulty in verifying Theorem 1.3.4 comes not from the fact that $\mathrm{Bun}_G(X)$ is a stack, but from the fact that it fails to be quasi-compact. For every quasi-compact open substack $U \subseteq \mathrm{Bun}_G(X)$, one can write U as the stack-theoretic quotient of an algebraic space $\widetilde{U}$ by the action of an algebraic group H over $\mathbf{F}_q$ (for example, we can take $\widetilde{U}$ to be a fiber product $U \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G(X, D)$, where $\mathrm{Bun}_G(X, D)$ denotes the moduli stack of G-bundles on X which are equipped with a trivialization on some sufficiently large effective divisor $D \subseteq X$). One can then show that U satisfies the Grothendieck-Lefschetz trace formula by applying Theorem 1.3.2 to $\widetilde{U}$ and H (see §10.1). One might hope to prove Theorem 1.3.4 by writing $\mathrm{Bun}_G(X)$ as the union of a sequence of well-chosen quasi-compact open substacks

$$U_0 \hookrightarrow U_1 \hookrightarrow U_2 \hookrightarrow \cdots,$$

and making some sort of convergence argument. Using this method, Behrend has proven Theorem 1.3.4 in a number of special cases (see [5]). In §10, we will use the same technique to prove the general case of Theorem 1.3.4.

The bulk of this paper is devoted to the proof of Theorem 1.3.5. Roughly speaking, the idea of the proof is to show that $\mathrm{H}^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)$ is the cohomology of a chain complex $C^*(\overline{\mathrm{Bun}}_G(X); \mathbf{Z}_\ell)$, which can be identified (in a Galois-equivaraint way) with a continuous tensor product of chain complexes $C^*(\overline{\mathrm{BG}}_x; \mathbf{Z}_\ell)$, where x ranges over the points of X. In §1.4, we will formulate this "local-to-global" principle in more detail, using ideas which are inspired by homotopy theory and the theory of chiral algebras.

1.4. **Analyzing the Homotopy Type of** $\mathrm{Bun}_G(X)$**.** Let X be an algebraic curve over an algebraically closed field k, let G be a smooth affine group scheme over X, and let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles on X. Our objective in this paper is to describe the cohomology of $\mathrm{Bun}_G(X)$. In the special case where X and G are actually defined over a finite field $\mathbf{F}_q \subseteq k$, understanding the structure of the ℓ-adic cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ (along with the action of Frobenius) is the key to proving Theorem 1.3.5. In this section, we summarize (without proofs) the "classical" situation where k is the field $\mathbf{C}$ of complex numbers, where we can identify X with a compact Riemann surface (in particular, it is a real manifold of dimension 2). For a more detailed discussion, we refer the reader to [4].

1.4.1. To simplify the discussion, let us assume that all fibers of the group scheme G are semisimple and simply connected. Fix a G-bundle $\mathcal{P}_{\mathrm{sm}}$ in the category of smooth manifolds. The tangent bundle of $\mathcal{P}_{\mathrm{sm}}$ is a G-equivariant vector bundle on $\mathcal{P}_{\mathrm{sm}}$, and can therefore be written the pullback of a smooth vector bundle $\mathcal{E}$ on X. This vector bundle fits into an exact sequence

$$0 \to \mathcal{E}_0 \to \mathcal{E} \to T_X \to 0, \tag{9}$$

where $\mathcal{E}_0$ denotes the vector bundle associated by $\mathcal{P}_{\mathrm{sm}}$ to the adjoint representation of G. In particular, we can regard $\mathcal{E}_0$ as a complex vector bundle on X. A $\overline{\partial}$-*connection* on $\mathcal{P}_{\mathrm{sm}}$ is a choice of complex structure on the vector bundle $\mathcal{E}$ for which (9) is an exact sequence of *complex* vector bundles on X. Let Ω denote the collection of all $\overline{\partial}$-connections on X. Then Ω can be regarded as a torsor for the infinite-dimensional vector space of $\mathbf{C}$-antilinear bundle maps from T_X into $\mathcal{E}_0$: in particular, it is an infinite-dimensional affine space, and therefore contractible.

Proposition 1.4.1. *Let $\mathcal{G} = \mathrm{Aut}(\mathcal{P}_{\mathrm{sm}})$ denote the group of all automorphisms of the smooth G-bundle $\mathcal{P}_{\mathrm{sm}}$. Then the moduli stack $\mathrm{Bun}_G(X)$ can be identified, as a differentiable stack, with the quotient of the contractible space Ω by the action of the $\mathcal{G}$. In particular, $\mathrm{Bun}_G(X)$ has the homotopy type of the classifying space $\mathrm{B}\mathcal{G}$.*

Sketch. Every G-bundle $\rho : \mathcal{P} \to X$ is a fiber bundle with simply connected fibers, and is therefore trivial in the category of smooth G-bundles (since X is real manifold of dimension 2). In particular, for every complex-analytic G-bundle $\mathcal{P}$ on X, we can choose an isomorphism of smooth G-bundles $\alpha : \mathcal{P}_{\mathrm{sm}} \to \mathcal{P}$. We can identify isomorphism classes of pairs $(\mathcal{P}, \alpha)$ with complex-analytic structures on the bundle $\mathcal{P}_{\mathrm{sm}}$; since X has dimension ≤ 1, these are in bijection with points of Ω. Then $\mathcal{G}$ acts on the space Ω, and the homotopy quotient of Ω by $\mathcal{G}$ classifies complex-analytic G-bundles on X. Since X is a projective algebraic variety, the category of complex-analytic G-bundles on X is equivalent to the category of algebraic vector bundles on X. □

Remark 1.4.2. The argument we sketched above really proves that the groupoid $\mathrm{Bun}_G(X)(\mathbf{C})$ of $\mathbf{C}$-valued points of $\mathrm{Bun}_G(X)$ can be identified with the groupoid quotient of Ω (regarded as a set) by $\mathcal{G}$ (regarded as a discrete group). To formulate a stronger claim, we would need to be more precise about the procedure which associates a homotopy type to an algebraic stack over $\mathbf{C}$. A reader who is concerned with this technical point should feel free to take Principle 1.4.1 as a *definition* of the homotopy type of $\mathrm{Bun}_G(X)$.

Warning 1.4.3. The validity of Principle 1.4.1 relies crucially on the fact that X is an algebraic curve. If X is a smooth projective variety of higher dimension, then smooth G-bundles on X need not be trivial, and $\overline{\partial}$-connections on a smooth G-bundle $\mathcal{P}_{\mathrm{sm}}$ need not be integrable. Consequently, the homotopy type of $\mathrm{Bun}_G(X)$ is not so easy to describe.

Note that since the G-bundle $\mathcal{P}_{\mathrm{sm}}$ is trivial, we can identify the gauge group $\mathcal{G}$ with the space of all smooth sections of the projection map $G \to X$. We would like to use this information to describe the homotopy type of the classifying space of $\mathrm{B}\mathcal{G}$ in terms of the individual classifying spaces $\{\mathrm{BG}_x\}_{x \in X}$. We next outline three approaches to this problem: the first allows us to express $\mathrm{H}^*(\mathrm{B}\mathcal{G}; \mathbf{Q})$ as the cohomology of a certain differential graded Lie algebra (Theorem 1.4.4), while the remaining two express $\mathrm{H}^*(\mathrm{B}\mathcal{G}; \mathbf{Q})$ and $\mathrm{H}_*(\mathrm{B}\mathcal{G}; \mathbf{Q})$ as the homology of certain factorization algebras on X (Theorems 1.4.9 and 1.4.13).

1.4.2. *First Approach: Rational Homotopy Theory.* Let H be a path-connected topological group. Then the homology $\mathrm{H}_*(H; \mathbf{Q})$ has the structure of a cocommutative Hopf algebra: the multiplication on $\mathrm{H}_*(H; \mathbf{Q})$ is given by pushforward along the product map $H \times H \to H$, and the comultiplication on $\mathrm{H}_*(H; \mathbf{Q})$ is given by pushforward along the diagonal map $\delta : H \to H \times H$. With more effort, one can construct an analogue of this Hopf algebra structure at the level of chains, rather than homology. More precisely, Quillen's work on rational homotopy theory gives a functorial procedure for associating to each topological group H a differential graded Lie algebra $\mathfrak{g}(H)$ (defined over the field of rational numbers) with the following properties:

(*a*) Let $\mathrm{H}_*(\mathfrak{g}(H))$ denote the homology groups of the underlying chain complex of $\mathfrak{g}(H)$. Then we have a canonical isomorphism

$$\mathbf{Q} \otimes \pi_* H \simeq \mathrm{H}_*(\mathfrak{g}(H)).$$

Under this isomorphism, the Whitehead product on $\pi_{*+1} \mathrm{BH} \simeq \pi_* H$ corresponds to the Lie bracket on $\mathrm{H}_*(\mathfrak{g}(H))$.

(*b*) The singular chain complex $C_*(H;\mathbf{Q})$ is canonically quasi-isomorphic to the universal enveloping algebra $U(\mathfrak{g}(H))$. This quasi-isomorphism induces a Hopf algebra isomorphism

$$\mathrm{H}_*(U(\mathfrak{g}(H))) \simeq \mathrm{H}_*(H;\mathbf{Q}).$$

(*c*) The differential graded Lie algebra $\mathfrak{g}(H)$ is a complete invariant of the rational homotopy type of the classifying space BH. More precisely, from $\mathfrak{g}(H)$ one can functorially construct a pointed topological space Z for which there exists a pointed map $\mathrm{BH} \to Z$ which induces an isomorphism on rational cohomology. In particular, the cohomology ring $\mathrm{H}^*(\mathrm{BH};\mathbf{Q})$ can be functorially recovered as the *Lie algebra cohomology* of $\mathfrak{g}(H)$.

Let us now apply the above reasoning to our situation. For every open subset $U \subseteq X$, let $\mathcal{G}_U$ denote the (topological) group of all smooth sections of the projection map $G \times_X U \to U$, and let $\mathfrak{g}(\mathcal{G}_U)$ be the associated differential graded Lie algebras. The construction $U \mapsto \mathfrak{g}(\mathcal{G}_U)$ is contravariantly functorial in U. For each integer n, let $\mathcal{F}_n$ denote the presheaf of rational vector spaces on X given by $\mathcal{F}_n(U) = \mathfrak{g}(\mathcal{G}_U)_n$, and let $\overline{\mathcal{F}}_n$ be the associated sheaf. Ignoring the Lie algebra structures on the differential graded Lie algebras $\mathfrak{g}(\mathcal{G}_U)$ and remember only the underlying chain complexes, we obtain a chain complex of presheaves

$$\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F}_{-1} \to \mathcal{F}_{-2} \to \cdots,$$

hence a chain complex of sheaves

$$\cdots \to \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_0 \to \overline{\mathcal{F}}_{-1} \to \overline{\mathcal{F}}_{-2} \to \cdots.$$

In this language, we can formulate a local-to-global principle as follows:

Theorem 1.4.4. *The canonical map*

$$\mathfrak{g}(\mathcal{G}) = \Gamma(X;\mathcal{F}_*) \to \Gamma(X;\overline{\mathcal{F}}_*) \to R\Gamma(X;\overline{\mathcal{F}}_*)$$

is a quasi-isomorphism of differential graded Lie algebras. In other words, the cohomology groups of the differential graded Lie algebra $\mathfrak{g}(\mathcal{G})$ can be identified with the hypercohomology groups of the chain complex $\overline{\mathcal{F}}_*$ of sheaves on X.

Proof. This follows from the compatibility of the construction $H \mapsto \mathfrak{g}(H)$ with (suitable) homotopy inverse limits. □

Remark 1.4.5. Fix a point $x \in X$. If $U \subseteq X$ is an open disk containing x, then evaluation at x induces a homotopy equivalence of topological groups $\mathcal{G}_U \to G_x$. Passing to the direct limit, we obtain a quasi-isomorphism of chain complexes $\mathcal{F}_{*,x} \simeq \overline{\mathcal{F}}_{*,x} \to \mathfrak{g}(G_x)$. In particular, the nth homology of the complex $\overline{\mathcal{F}}_*$ is a locally constant sheaf on X. Theorem 1.4.4 then supplies a convergent spectral sequence

$$\mathrm{H}^s(X;\mathbf{Q}\otimes\pi_t(G_\bullet)) \Rightarrow \mathbf{Q}\otimes\pi_{t-s}\,\mathcal{G},$$

where $\mathbf{Q}\otimes\pi_t(G_\bullet)$ denotes the local system of rational vector spaces on X given by $x \mapsto \mathbf{Q}\otimes\pi_t(G_x)$.

Example 1.4.6 (Atiyah-Bott)**.** Suppose that G is constant: that is, it is the product of X with a simply connected semisimple algebraic group G_0 over $\mathbf{C}$. In this case, the chain complex $\overline{\mathcal{F}}_*$ is quasi-isomorphic to the chain complex of constant sheaves with value $\mathfrak{g}(G_0)$. In this case, Theorem 1.4.4 supplies a quasi-isomorphism

$$\mathfrak{g}(\mathcal{G}) \simeq C^*(X;\mathbf{Q}) \otimes_{\mathbf{Q}} \mathfrak{g}(G_0)$$

.

The rational cohomology of the classifying space BG_0 is isomorphic to a polynomial ring $\mathbf{Q}[t_1, \ldots, t_r]$, where r is the rank of the semisimple algebraic group G_0 and each t_i is a homogeneous element of $\mathrm{H}^*(\mathrm{BG}_0; \mathbf{Q})$ of some even degree d_i. From this, one can deduce that the differential graded Lie algebra $\mathfrak{g}(G_0)$ is *formal*: that is, it is quasi-isomorphic to a graded vector space V on generators $t_i^\vee$ of (homological) degree $d_i - 1$, where the differential and the Lie bracket vanish. It follows that $\mathfrak{g}(\mathcal{G})$ is quasi-isomorphic to the tensor product $\mathrm{H}^*(X; \mathbf{Q}) \otimes_{\mathbf{Q}} V$, where the differential and Lie bracket vanish. From this, one can deduce that $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q})$ is isomorphic to a (graded) symmetric algebra on the graded vector space $\mathrm{H}_*(X; \mathbf{Q}) \otimes_{\mathbf{Q}} V^\vee[-1]$. In other words, $\mathrm{H}^*(\mathcal{M}; \mathbf{Q})$ is a tensor product of a polynomial ring on $2r$ generators in even degrees with an exterior algebra on $2gr$ generators in odd degrees.

1.4.3. *Second Approach: Factorization Homology.* Theorem 1.4.4 asserts that that the differential graded Lie algebra $\mathfrak{g}(\mathcal{G})$ can be recovered as the hypercohomology of a "local system" of differential graded Lie algebras given by $x \mapsto \mathfrak{g}(G_x)$. Roughly speaking, this reflects the idea that the gauge group $\mathcal{G}$ can be identified with a "continuous product" of the groups G_x, and that the construction $H \mapsto \mathfrak{g}(H)$ is compatible with "continuous products" (at least in good cases).

Our ultimate goal is to formulate a local-to-global principle which will allow us to compute the rational cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}) \simeq \mathrm{H}^*(\mathrm{B}\mathcal{G}; \mathbf{Q})$. It is possible to formulate such a principle directly, without making a detour through the theory of differential graded Lie algebras. However, the basic mechanism of the local-to-global principle takes a more complicated form.

Definition 1.4.7. For each open set $U \subseteq X$, let $\mathcal{B}(U)$ denote the rational cochain complex $C^*(\mathrm{B}\,\mathcal{G}_U; \mathbf{Q})$. Then the construction $U \mapsto \mathcal{B}(U)$ determines a covariant functor from the partially ordered set of open subsets of X to the category of chain complexes of rational vector spaces.

Let $\mathcal{U}$ denote the collection of all open subsets of X which can be written as a disjoint union of disks. We let $\int \mathcal{B}$ denote a homotopy colimit of the diagram $\{\mathcal{B}(U)\}_{U \in \mathcal{U}}$ (in the category of chain complexes of rational vector spaces). We refer to the homology of the chain complex $\int \mathcal{B}$ as the *factorization homology* of $\mathcal{B}$.

Example 1.4.8. Suppose that $U \subseteq X$ is an open set which can be written as a disjoint union $U_1 \cup \cdots \cup U_n$, where each U_i is an open disk. Choose a point $x_i \in U_i$ for $1 \leq i \leq n$. Then $\mathcal{G}_U$ is homeomorphic to a product $\prod_{1 \leq i \leq n} \mathcal{G}_{U_i}$, and evaluation at the points x_i determine homotopy equivalences $\mathcal{G}_{U_i} \to G_{x_i}$. Consequently, there is a canonical quasi-isomorphism of chain complexes

$$\begin{aligned} \bigotimes_{1 \leq i \leq n} C^*(\mathrm{BG}_{x_i}; \mathbf{Q}) &\xrightarrow{\sim} \bigotimes_{1 \leq i \leq n} C^*(\mathrm{B}\,\mathcal{G}_{U_i}; \mathbf{Q}) \\ &\xrightarrow{\sim} C^*(\mathrm{B}\mathcal{G}_U; \mathbf{Q}) \\ &= \mathcal{B}(U). \end{aligned}$$

In other words, each term in the diagram $\{\mathcal{B}(U)\}_{U \in \mathcal{U}}$ can be identified with a tensor product $\bigotimes_{x \in S} C^*(\mathrm{BG}_x; \mathbf{Q})$, where S is some finite subset of X. We can therefore think of the factorization homology $\int \mathcal{B}$ as a kind of continuous tensor product $\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q})$. We refer the reader to [35] for more details.

We can now formulate a second local-to-global principle for describing the cohomology of $\mathrm{Bun}_G(X)$:

Theorem 1.4.9. *If the fibers of G are semisimple and simply connected, then the canonical map*

$$\int \mathcal{B} = \mathrm{hocolim}_{U \in \mathcal{U}} \mathcal{B}(U) \to \mathcal{B}(X) = C^*(\mathrm{B}\mathcal{G}; \mathbf{Q}) = C^*(\mathrm{Bun}_G(X); \mathbf{Q})$$

is a quasi-isomorphism. In other words, we can identify the cohomology of the moduli stack $\mathrm{Bun}_G(X)$ with the factorization homology of $\mathcal{B}$.

1.4.4. *Third Approach: Nonabelian Poincare Duality.* The local-to-global principle expressed by Theorem 1.4.9 is based on the idea of approximating the moduli stack $\mathrm{Bun}_G(X) \simeq \mathrm{B}\mathcal{G}$ "from the right". For any finite set $S \subseteq X$, evaluation at the points of S defines a map of classifying spaces

$$\mathrm{B}\mathcal{G} \to \prod_{x \in S} \mathrm{BG}_x,$$

hence a map of cochain complexes

$$\mu_S : \bigotimes_{x \in S} C^*(\mathrm{BG}_x; \mathbf{Q}) \to C^*(\mathrm{B}\mathcal{G}; \mathbf{Q}).$$

Roughly speaking, Theorem 1.4.9 asserts that if we allow S to vary continuously over all finite subsets of X, then we can use these maps to recover the chain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Q})$ up to quasi-isomorphism. We now explore an parallel approach, which is based on the idea of realizing $\mathrm{B}\mathcal{G}$ as direct limit, rather than an inverse limit.

Notation 1.4.10. For each open set $U \subseteq X$, let $\mathcal{G}_U^c$ denote the subgroup of $\mathcal{G}$ consisting of those automorphisms of $\mathcal{P}_{\mathrm{sm}}$ which are the identity outside of a compact subset of U, and let $\mathcal{A}(U)$ denote the chain complex $C_*(\mathrm{B}\mathcal{G}_U^c; \mathbf{Q})$. Note that $\mathcal{G}_U^c \subseteq \mathcal{G}_V^c$ whenever $U \subseteq V$, so that we can regard the construction $U \mapsto \mathcal{A}(U)$ is a covariant functor from the partially ordered set of open subsets of X to the category of chain complexes.

Let $\mathcal{U}$ denote the collection of all open subsets of X which can be written as a disjoint union of disks. We let $\int \mathcal{A}$ denote a homotopy colimit of the diagram $\{\mathcal{A}(U)\}_{U \in \mathcal{U}}$ (in the category of chain complexes of rational vector spaces). We refer to the homology of the chain complex $\int \mathcal{A}$ as the *factorization homology* of $\mathcal{A}$.

Example 1.4.11. Let $U \subseteq X$ be an open disk containing a point $x \in X$. Then $U \times_X G$ is diffeomorphic to a product $U \times G_x$, so that $\mathcal{G}_U^c$ can be identified with the space of compactly supported maps from U into G_x. A choice of homeomorphism $U \simeq \mathbf{R}^2$ then determines a homotopy equivalence of $\mathcal{G}_U^c$ with the two-fold loop space $\Omega^2(G_x)$, so that $\mathrm{B}\mathcal{G}_U^c$ can be identified with $\Omega^2(\mathrm{BG}_x) \simeq \Omega(G_x)$.

More generally, if U can be written as a disjoint union of disks $U_1 \cup \cdots \cup U_n$ containing points $x_i \in U_i$, then $\mathcal{G}_U^c$ is homeomorphic to a product $\prod_{1 \leq i \leq n} \mathcal{G}_{U_i}^c$, so we obtain a quasi-isomorphism of chain complexes

$$\begin{aligned}
\bigotimes_{1 \leq i \leq n} C_*(\Omega^2 \mathrm{BG}_{x_i}; \mathbf{Q}) &\simeq \bigotimes_{1 \leq i \leq n} C_*(\mathrm{B}\mathcal{G}_{U_i}^c; \mathbf{Q}) \\
&\simeq C_*(\mathrm{B}\mathcal{G}_U^c; \mathbf{Q}) \\
&= \mathcal{A}(U).
\end{aligned}$$

In other words, each term in the diagram $\{\mathcal{A}(U)\}_{U \in \mathcal{U}}$ can be identified with a tensor product

$$\bigotimes_{x \in S} C_*(\Omega^2(\mathrm{BG}_x); \mathbf{Q}),$$

where S is some finite subset of X. We can therefore think of the factorization homology $\int \mathcal{A}$ as a kind of continuous tensor product $\bigotimes_{x \in X} C_*(\Omega^2(\mathrm{BG}_x); \mathbf{Q})$.

Remark 1.4.12. The double loop space $\Omega^2(\mathrm{BG}_x)$ is homotopy equivalent to the quotient space $G(K_x)/G(\mathcal{O}_x)$, where $\mathcal{O}_x$ denotes the completed local ring of X at x, and K_x denotes its fraction field. We will denote this quotient by $\mathrm{Gr}_{G,x}$ and refer to it as the *affine Grassmannian* of the group G at the point x. This paper depends crucially on the fact that $\mathrm{Gr}_{G,x}$ admits an algebro-geometric incarnation (as the direct limit of a sequence of algebraic varieties) and can be defined over ground fields different from $\mathbf{C}$.

We have the following analogue of Theorem 1.4.9:

Theorem 1.4.13 (Nonabelian Poincare Duality)**.** *The canonical map*

$$\int \mathcal{A} = \mathrm{hocolim}_{U \in \mathcal{U}} \mathcal{A}(U) \to \mathcal{A}(X) = C_*(\mathrm{B}\mathcal{G}; \mathbf{Q}) = C_*(\mathrm{Bun}_G(X); \mathbf{Q})$$

is a quasi-isomorphism. In other words, we can identify the homology of the moduli stack $\mathrm{Bun}_G(X)$ *with the factorization homology of* $\mathcal{A}$*.*

Remark 1.4.14. Theorem 1.4.13 can be regarded as version of Poincare duality for the manifold X with coefficients in the nonabelian group G. We will explain this idea in more detail in §3.

1.4.5. Let us now outline the relationship between Theorems 1.4.4, 1.4.9, and 1.4.13.

- Theorem 1.4.4 is the weakest of the three results. It only gives information about the rational homotopy type of the moduli stack $\mathrm{Bun}_G(X)$, while Theorems 1.4.9 and 1.4.13 remain valid with integral coefficients. In fact, Theorem 1.4.13 is even true at the "unstable" level: that is, it gives a procedure for reconstructing the space $\mathrm{B}\mathcal{G}$ itself, rather than just the singular chain complex of $\mathrm{B}\mathcal{G}$). However, Theorem 1.4.4 gives information in a form which is most amenable to further calculation, since it articulates a local-to-global principle using the familiar language of sheaf cohomology, rather than the comparatively exotic language of factorization homology.
- Theorem 1.4.13 can also be regarded as the strongest of the three results because it requires the weakest hypotheses: if it is formulated correctly, we only need to assume that the fibers of the map $G \to X$ are connected, rather than simply connected.
- Theorems 1.4.13 and 1.4.9 can be regarded as duals of one another. More precisely, the construction $x \mapsto C^*(\mathrm{BG}_x; \mathbf{Q})$ determines a factorization algebra on X which is *Koszul dual* to the factorization algebra $x \mapsto C_*(\Omega^2 \mathrm{BG}_x; \mathbf{Q})$. Using this duality, one can construct a duality pairing between the chain complexes $\int \mathcal{A}$ and $\int \mathcal{B}$, which identifies each with the $\mathbf{Q}$-linear dual of the other (under the assumption that the fibers of G are simply connected).
- Theorems 1.4.4 and 1.4.9 can also be regarded as duals of one another, but in a different sense. Namely, each of the differential graded Lie algebras $\mathfrak{g}(G_x)$ can be regarded as the Koszul dual of $C^*(\mathrm{BG}_x; \mathbf{Q})$, which we regard as an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$. One can exploit this to deduce Theorem 1.4.4 from Theorem 1.4.9 and vice versa.

1.5. **Summary of this Paper.** Fix an algebraically closed field k, an algebraic curve X over k, and a smooth affine group scheme G over X. Let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles on X and let ℓ denote the a prime number which is invertible in k. Our main goal in this paper is to formulate and prove various "local-to-global" principles which can be used to compute the ℓ-adic cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$, which are analogous to Theorems 1.4.4, 1.4.9, and 1.4.13 in the case $k = \mathbf{C}$.

We begin in §3 by proving an analogue of nonabelian Poincare duality (Theorem 1.4.13) in the algebro-geometric setting. Let $\mathrm{Ran}_G(X)$ denote the *Beilinson-Drinfeld* Grassmannian of G, which classifies pairs $(\mathcal{P}, \gamma)$, where $\mathcal{P}$ is a G-bundle on X and γ is a trivialization of $\mathcal{P}$ outside of a finite subset of X (see Definition 3.2.3). Our main result (Theorem 3.2.9) asserts that if the generic fiber of G is semisimple and simply connected, then the evident map $\mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$ induces an isomorphism on ℓ-adic cohomology (and homology). Roughly speaking, the idea of the proof is to show that for each G-bundle $\mathcal{P}$, the fiber product $\mathrm{Ran}_G(X) \times_{\mathrm{Bun}_G(X)} \{\mathcal{P}\}$ (which parametrizes "rational sections" of $\mathcal{P}$) is contractible.

Let BG denote the classifying stack of G (which we regard as an algebraic stack over X), let $\pi : \mathrm{BG} \to X$ denote the projection map, and let ω_X denote the ℓ-adic dualizing complex of X. Then the ℓ-adic complex $\pi_* \pi^* \omega_X$ can be regarded as a *factorizable* ℓ-adic complex on X. In particular, it extends naturally to a sheaf $\mathcal{B}$ on the space $\mathrm{Ran}(X)$ of all nonempty finite subsets of X. In §4, we introduce an analogue of factorization homology in the setting of ℓ-adic sheaves, and construct a map of chain complexes

$$\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell).$$

The second main result of this paper asserts that the map ρ induces an isomorphism from the factorization homology of $\mathcal{B}$ to the ℓ-adic cohomology of $\mathrm{Bun}_G(X)$ (Theorem 5.4.5). The proof of this result will be given in §9. Roughly speaking, the idea is to reduce Theorem 5.4.5 to Theorem 3.2.9 using Verdier duality on the space $\mathrm{Ran}(X)$. Since $\mathrm{Ran}(X)$ is infinite-dimensional, the theory of Verdier duality is somewhat subtle: to guarantee that it is well-behaved, we will need to work with sheaves on $\mathrm{Ran}(X)$ which have (degreewise) finite-dimensional support. The sheaf $\mathcal{B}$ does *not* satisfy this condition. To address this point (and others of the same nature), it will be convenient to introduce "reduced" version of $\mathcal{B}$, which we will denote by $\mathcal{B}_{\mathrm{red}}$. In §8, we show that the process of replacing $\mathcal{B}$ by $\mathcal{B}_{\mathrm{red}}$ has a very mild effect on factorization homology: in particular, Theorem 5.4.5 implies that the factorization homology of $\mathcal{B}_{\mathrm{red}}$ can be identified with the *reduced* ℓ-adic cohomology of $\mathrm{Bun}_G(X)$ (Theorem 8.2.18).

Specializing to the case where $k = \overline{\mathbf{F}}_q$ is the algebraic closure of a finite field (and where X and G are defined over $\mathbf{F}_q$), the above suggests that we should be able to use the Grothendieck-Lefschetz trace formula to compute the trace of Frobenius on $\mathrm{H}^*_{\mathrm{red}}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ as a sum of "local" contributions coming from $\mathbf{F}_q$-valued points of $\mathrm{Ran}(X)$. Ignoring issues of convergence, this leads to a heuristic proof of Theorem 1.3.5. In §6, we address the convergence problem by using Theorem 5.4.5 to deduce an ℓ-adic analogue of Theorem 1.4.4. More precisely, we show that the cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}]$ admits an exhaustive filtration whose successive quotients can be identified with the symmetric powers of a particular chain complex of $\mathbf{Q}_\ell$-modules M, which can be computed as the hypercohomology of an ℓ-adic complex $\mathcal{M}(G)$ on X (here M can be described as the "cotangent fiber" of $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}]$ as an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}_\ell$, and $\mathcal{M}(G)$ bears a similar relationship to the sheaf $\mathcal{B}$). By applying the (usual) Grothendieck-Lefschetz trace formula to the sheaf $\mathcal{M}(G)$, we will compute the trace of (arithmetic) Frobenius on the symmetric powers of M and therefore also on the cohomology $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$. Combining this calculation with the Grothendieck-Lefschetz trace formula for $\mathrm{Bun}_G(X)$ (which we prove in §10), we will complete the proof of Weil's conjecture.

Throughout this paper, we will make extensive use of the theory of ℓ-adic cohomology, both of algebraic varieties and of more exotic algebro-geometric objects (such as $\mathrm{Ran}(X)$). In §2 and §4, we supply a quick introduction to the formalism of ℓ-adic sheaves, using the language of ∞-categories developed in [34] and [35]. For convenience, we will adopt the following reference conventions:

(*HTT*) We will indicate references to [34] using the letters HTT.
(*HA*) We will indicate references to [35] using the letters HA.
(*SAG*) We will indicate references to [36] using the letters SAG.

For example, Theorem HTT.6.1.0.6 refers to Theorem 6.1.0.6 of [34].

2. Generalities on ℓ-adic Homology and Cohomology

The ultimate goal of this paper is to describe the ℓ-adic cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$, where X is an algebraic curve defined over an algebraically closed field k, and G is a (sufficiently nice) group scheme over X. However, we will first need to address a more basic question: how is the ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ defined? Theories of of ℓ-adic sheaves on Artin stacks have been developed by a number of authors (see, for example, [7] and [32]). However, we will also need to work with more exotic algebro-geometric objects (like the "space" $\mathrm{Ran}(X)$ of nonempty finite subsets of X), which are infinite-dimensional in nature and cannot be realized as algebraic stacks. Moreover, at several points we will need to make "homotopy coherent" constructions which are not easily described using the traditional formalism of derived categories. Consequently, we devote this section to giving an exposition of ℓ-adic cohomology from a perspective which is adequate for our needs.

We begin in §2.1 by giving a brief overview of the theory of ∞-categories (also known as *quasi-categories* and *weak Kan complexes* in the literature). In particular, we introduce the ∞-category Mod_Λ whose objects are chain complexes of Λ-modules, where Λ is an arbitrary commutative ring. This is a mathematical object which can be regarded as intermediate between the ordinary category $\mathrm{Chain}(\Lambda)$ of chain complexes of Λ-modules and the derived category $\mathcal{D}(\Lambda)$ obtained from $\mathrm{Chain}(\Lambda)$ by inverting quasi-isomorphisms, and furnishes a convenient language for various constructions in homological algebra.

Let k be an algebraically closed field, and let ℓ be a prime number which is invertible in k. In §2.2, we define the *ℓ-adic cohomology groups*

$$\mathrm{H}^*(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \qquad \mathrm{H}^*(Y; \mathbf{Z}_\ell) \qquad \mathrm{H}^*(Y; \mathbf{Q}_\ell)$$

as well as the (formally dual) *ℓ-adic homology groups*

$$\mathrm{H}_*(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \qquad \mathrm{H}_*(Y; \mathbf{Z}_\ell) \qquad \mathrm{H}_*(Y; \mathbf{Q}_\ell)$$

associated to a quasi-projective k-scheme Y, and review some of their basic properties (referring to the literature for proofs).

In §2.3, we introduce the notion of a *prestack* over the field k. Roughly speaking, a prestack is a rule which associates to every finitely generated k-algebra R a category $\mathcal{C}_R$, which depends functorially on R. The collection of prestacks can be organized into a 2-category which contains the category of finite-type k-schemes as a full subcategory. It also contains several other objects which will be relevant to us in this paper, such as the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles on X. After reviewing the basic definitions, we will explain how to generalize the theory of ℓ-adic homology and cohomology to the setting of prestacks, and establish some of their basic formal properties.

In §2.4, we introduce a prestack $\mathrm{Ran}(X)$, called the *Ran space* of X, which parametrizes nonempty finite subsets of X. We then prove show that if X is connected, then $\mathrm{Ran}(X)$ is acyclic with respect to $\mathbf{Z}_\ell$-homology (an ℓ-adic version of a basic foundational result of Beilinson and Drinfeld).

The formation of ℓ-adic cohomology is functorial: every map of prestacks $f : Y \to Z$ induces a pushforward map $f^* : \mathrm{H}^*(Z; \mathbf{Z}_\ell) \to \mathrm{H}^*(Y; \mathbf{Z}_\ell)$ on ℓ-adic cohomology groups. In the course of this paper, we will encounter many situations in which we want to prove that the map f^* is an isomorphism. In §2.5, we will show that this condition holds whenever f is a *universal homology equivalence*: roughly speaking, this means that the "fibers" of f are acyclic with respect to $\mathbf{Z}_\ell$-homology (the caveat is that one must consider the "fiber" of f over any R-valued point of

Z, where R is a finitely generated k-algebra). We then use the theory of universal homology equivalences to compare several different version of the prestack $\mathrm{Ran}(X)$ (Theorem 2.4.5).

Throughout this section, we will confine our attention to the case of ℓ-adic cohomology with *constant* coefficients. This is all that we will need in §3 to formulate and prove the first main result of this paper (Theorem 3.2.9), an ℓ-adic version of nonabelian Poincare duality). In the later portions of this paper, we will need the more robust formalism of ℓ-adic sheaves, which we will review in §4.1.

2.1. **Higher Category Theory.**

2.1.1. *Homological Algebra.* Let Λ be a commutative ring. Throughout this section, we let $\mathrm{Chain}(\Lambda)$ denote the abelian category whose objects are chain complexes

$$\cdots \to V_2 \to V_1 \to V_0 \to V_{-1} \to V_{-2} \to \cdots$$

of Λ-modules. We will always employ *homological* conventions when discussing chain complexes (so that differential on a chain complex lowers degree). If V_* is a chain complex, then its *homology* $\mathrm{H}_*(V_*)$ is given by

$$\mathrm{H}_n(V_*) = \{x \in V_n : dx = 0\}/\{x \in V_n : (\exists y \in V_{n-1})[x = dy]\}.$$

Any map of chain complexes $\alpha : V_* \to W_*$ induces a map $\mathrm{H}_*(V_*) \to \mathrm{H}_*(W_*)$. We say that α is a *quasi-isomorphism* if it induces an isomorphism on homology.

For many purposes, it is convenient to treat quasi-isomorphisms as if they are isomorphisms (emphasizing the idea that a chain complex is just a vessel for carrying information about its homology). One can make this idea explicit using Verdier's theory of *derived categories*. The derived category $\mathcal{D}(\Lambda)$ can be described as the category obtained from $\mathrm{Chain}(\Lambda)$ by formally inverting all quasi-isomorphisms.

The theory of derived categories is a very useful tool in homological algebra, but has a number of limitations. Many of these stem from the fact that $\mathcal{D}(\Lambda)$ is not very well-behaved from a categorical point of view. The category $\mathcal{D}(\Lambda)$ does not generally have limits or colimits, even of very simple types. For example, a morphism $f : X \to Y$ in $\mathcal{D}(\Lambda)$ generally does not have a cokernel in $\mathcal{D}(\Lambda)$. However, there is a substitute: every morphism f in $\mathcal{D}(\Lambda)$ fits into a "distinguished triangle"

$$X \xrightarrow{f} Y \to \mathrm{Cn}(f) \to \Sigma X.$$

Here we refer to $\mathrm{Cn}(f)$ is called the *cone* of f, and it behaves in some respects like a cokernel: every map $g : Y \to Z$ such that $g \circ f = 0$ factors through $\mathrm{Cn}(f)$, though the factorization is generally not unique. The object $\mathrm{Cn}(f) \in \mathcal{D}(\Lambda)$ (and, in fact, the entire diagram above) is well-defined up to isomorphism, but *not* up to canonical isomorphism: there is no functorial procedure for constructing the cone $\mathrm{Cn}(f)$ from the data of a morphism f in the category $\mathcal{D}$. And this is only a very simple example: for other types of limits and colimits (such as taking invariants or coinvariants with respect to the action of a group), the situation is even worse.

Let $f, g : V_* \to W_*$ be maps of chain complexes. Recall that a *chain homotopy* from f_* to g_* is a collection of maps $h_n : V_n \to W_{n+1}$ such that $f_n - g_n = d \circ h_n + h_{n-1} \circ d$. In this case, we say that f and g are *chain-homotopic*. Chain-homotopic maps induce the same map from $\mathrm{H}_*(V_*)$ to $\mathrm{H}_*(W_*)$, and have the same image in the derived category $\mathcal{D}(\Lambda)$. In fact, there is an alternative description of the derived category $\mathcal{D}(\Lambda)$, which places an emphasis on the notion of chain-homotopy rather than quasi-isomorphism. More precisely, one can define a category $\mathcal{D}'(\Lambda)$ equivalent to $\mathcal{D}(\Lambda)$ as follows:

Definition 2.1.1.

- The objects of $\mathcal{D}'(\Lambda)$ are the *K-injective* chain complexes of Λ-modules, in the sense of [51]. A chain complex V_* is K-injective if, for every chain

complex $W_* \in \mathrm{Chain}(\Lambda)$ and every subcomplex $W'_* \subseteq W_*$ which is quasi-isomorphic to W_*, every chain map $f : W'_* \to V_*$ can be extended to a chain map $\overline{f} : W_* \to V_*$.
- A morphism from V_* to W_* in $\mathcal{D}'(\Lambda)$ is a chain-homotopy equivalence class of chain maps from V_* to W_*.

Remark 2.1.2. If $V_* \in \mathrm{Chain}(\Lambda)$ is K-injective, then each V_n is an injective Λ-module. The converse holds if $V_n \simeq 0$ for $n \gg 0$ or if the commutative ring Λ has finite injective dimension (for example, if $\Lambda = \mathbf{Z}$), but not in general. For example, the chain complex of $\mathbf{Z}/4\mathbf{Z}$-modules

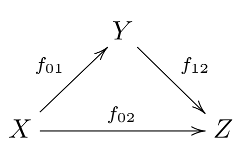
$$\cdots \to \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \to \cdots$$

is not K-injective.

From the perspective of Definition 2.1.1, categorical issues with the derived category stem from the fact that we are identifying chain-homotopic morphisms in $\mathcal{D}'(\Lambda)$ without remembering *how* they are chain-homotopic. For example, suppose that we wish to construct the cone of a morphism $[f] : V_* \to W_*$ in $\mathcal{D}'(\Lambda)$. By definition, $[f]$ is an equivalence class of chain maps from V_* to W_*. If we choose a representative f for the equivalence class $[f]$, then we can construct the mapping cone $\mathrm{Cn}(f)$ by equipping the direct sum $W_* \oplus V_{*-1}$ with a differential which depends on f. If h is a chain-homotopy from f to g, we can use h to construct an isomorphism of chain complexes $\alpha_h : \mathrm{Cn}(f) \simeq \mathrm{Cn}(g)$. However, the isomorphism α_h *depends on* h: different choices of chain homotopy can lead to different isomorphisms, even up to chain-homotopy.

2.1.2. *The Differential Graded Nerve.* It is possible to correct many of the deficiencies of the derived category by keeping track of more information. To do so, it is useful to work with mathematical structures which are a bit more elaborate than categories, where the primitive notions include not only "object" and "morphism" but also a notion of "homotopy between morphisms." Before giving a general definition, let us spell out the structure that is visible in the example of chain complexes over Λ.

Construction 2.1.3. We define a sequence of sets $S_0, S_1, S_2, \ldots$ as follows:
- Let S_0 denote the set of *objects* under consideration: in our case, these are chain complexes X of K-injective chain complexes of Λ-modules (strictly speaking this is not a set but a proper class, because we are trying to describe a "large" category).
- Let S_1 denote the set of *morphisms* under consideration. That is, S_1 is the collection of all chain maps $f : X \to Y$, where X and Y are chain complexes of injective abelian groups.
- Let S_2 denote the set of all pairs consisting of a non-necessarily commutative diagram



together with a chain homotopy f_{012} from f_{02} to $f_{12} \circ f_{01}$. Here X, Y, and Z are chain complexes of injective abelian groups.
- More generally, we let S_n denote the collection of all n-tuples $\{X(0), X(1), \ldots, X(n)\}$ of chain complexes of injective abelian groups, together with chain maps $f_{i,j} : X(i) \to X(j)$ which are compatible with composition *up to coherent homotopy*. More precisely, this means that for every subset $I = \{i_- < i_1 < \ldots < i_m < i_+\} \subseteq \{0, \ldots, n\}$, we supply a collection of maps $f_I : X(i_-)_k \to X(i_+)_{k+m}$ satisfying the identities

$$d(f_I(x)) = (-1)^m f_I(dx) + \sum_{1 \leq j \leq m} (-1)^j (f_{I-\{i_j\}}(x) - (f_{\{i_j,\ldots,i_+\}} \circ f_{\{i_-,\ldots i_j\}})(x)).$$

Suppose we are given an element $(\{X(i)\}_{0 \leq i \leq n}, \{f_I\})$ of S_n. Then for $0 \leq i \leq n$, we can regard $X(i)$ as an element of S_0. If we are given a pair of integers $0 \leq i < j \leq n$, then $f_{\{i,j\}}$ is a chain map from $X(i)$ to $X(j)$, which we can regard as an element of S_1. More generally, given any nondecreasing map $\alpha : \{0, \ldots, m\} \to \{0, \ldots, n\}$, we can define a map $\alpha^* : S_n \to S_m$ by the formula

$$\alpha^*(\{X(j)\}_{0 \leq j \leq n}, \{f_I\}) = (\{X(\alpha(j))\}_{0 \leq j \leq m}, \{g_J\}),$$

where

$$g_J(x) = \begin{cases} f_{\alpha(J)}(x) & \text{if } \alpha|_J \text{ is injective} \\ x & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') \\ 0 & \text{otherwise.} \end{cases}$$

This motivates the following:

Definition 2.1.4. A *simplicial set* $X_\bullet$ consists of the following data:

- For every integer $n \geq 0$, a set X_n (called the *set of n-simplices of* $X_\bullet$).
- For every nondecreasing map of finite sets $\alpha : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$, a map of sets $\alpha^* : X_n \to X_m$.

This data is required to be be compatible with composition: that is, we have

$$\mathrm{id}^*(x) = x \qquad (\alpha \circ \beta)^*(x) = \beta^*(\alpha^*(x))$$

whenever α and β are composable nondecreasing maps.

If $X_\bullet$ is a simplicial set, we will refer to X_n as the *set of n-simplices* of $X_\bullet$.

Example 2.1.5 (The Nerve of a Category)**.** Let $\mathcal{C}$ be a category. We can associate to $\mathcal{C}$ a simplicial set $\mathrm{N}(\mathcal{C})_\bullet$, whose n-simplices are given by chains of composable morphisms

$$C_0 \to C_1 \to \cdots \to C_n$$

in $\mathcal{C}$. We refer to $\mathrm{N}(\mathcal{C})_\bullet$ as the *nerve* of the category $\mathcal{C}$.

Example 2.1.6. Let Λ be a commutative ring and let $\mathrm{Chain}'(\Lambda)$ denotes the full subcategory of $\mathrm{Chain}(\Lambda)$ spanned by the K-injective chain complexes of Λ-modules. Construction 2.1.3 yields a simplicial set $\{S_n\}_{n \geq 0}$ which we will denote by Mod_Λ. The simplicial set Mod_Λ can be regarded as an enlargement of the nerve $\mathrm{N}(\mathrm{Chain}'(\Lambda))_\bullet$ (more precisely, we can identify $\mathrm{N}(\mathrm{Chain}'(\Lambda))_\bullet$ with the simplicial subset of Mod_Λ whose n-simplices are pairs $(\{X(i)\}_{0 \leq i \leq n}, \{f_I\})$ for which $f_I = 0$ whenever I has cardinality > 2.

The construction $\mathrm{Chain}'(\Lambda) \mapsto \mathrm{Mod}_\Lambda$ can be regarded as a variant of Example 2.1.5 which takes into account the structure of $\mathrm{Chain}'(\Lambda)$ as a *differential graded* category. We refer to §HA.1.3.1 for more details.

From the nerve of a category $\mathcal{C}$, we can recover $\mathcal{C}$ up to isomorphism. For example, the objects of $\mathcal{C}$ are just the 0-simplices of $\mathrm{N}(\mathcal{C})_\bullet$ and the morphisms of $\mathcal{C}$ are just the 1-simplices of $\mathrm{N}(\mathcal{C})_\bullet$. Moreover, given a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, the composition $h = g \circ f$ is the unique 1-morphism in $\mathcal{C}$ for which there exists a 2-simplex $\sigma \in \mathrm{N}(\mathcal{C})_2$ satisfying

$$\alpha_0^*(\sigma) = g \qquad \alpha_1^*(\sigma) = h \qquad \alpha_2^*(\sigma) = f,$$

where $\alpha_i : \{0, 1\} \to \{0, 1, 2\}$ denotes the unique injective map whose image does not contain i.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, then there is a bijective correspondence between functors $F : \mathcal{C} \to \mathcal{D}$ and maps of simplicial sets $\mathrm{N}(\mathcal{C})_\bullet \to \mathrm{N}(\mathcal{D})_\bullet$. We can summarize the situation as follows: the construction $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})_\bullet$ furnishes a fully faithful embedding from the category of (small) categories to the category of simplicial sets. It is therefore natural to ask about the essential image of this construction: which simplicial sets arise as the nerves of categories? To answer this question, we need a bit of terminology:

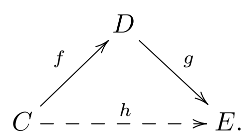
Notation 2.1.7. Let $X_\bullet$ be a simplicial set. For $0 \leq i \leq n$, we define a set $\Lambda^n_i(X_\bullet)$ as follows:

- To give an element of $\Lambda^n_i(X_\bullet)$, one must give an element $\sigma_J \in X_m$ for every subset $J = \{j_0 < \cdots < j_m\} \subseteq \{0, \ldots, n\}$ which does not contain $\{0, 1, \ldots, i-1, i+1, \ldots, n\}$. These elements are subject to the compatibility condition $\sigma_I = \alpha^* \sigma_J$ whenever $I = \{i_0 < \cdots < i_\ell\} \subseteq \{j_0 < \cdots < j_m\}$ and α satisfies $i_k = j_{\alpha(k)}$.

More informally, $\Lambda^n_i(X_\bullet)$ is the set of "partially defined" n-simplices of $X_\bullet$, which are missing their interior and a single face. There is an evident restriction map $X_n \to \Lambda^n_i(X_\bullet)$.

Proposition 2.1.8. *Let $X_\bullet$ be a simplicial set. Then $X_\bullet$ is isomorphic to the nerve of a category if and only if, for each $0 < i < n$, the restriction map $X_n \to \Lambda^n_i(X_\bullet)$ is bijective.*

For example, the bijectivity of the map $X_2 \to \Lambda^2_1(X_\bullet)$ encodes the existence and uniqueness of composition: it says that every pair of composable morphisms $f : C \to D$ and $g : D \to E$ can be completed uniquely to a commutative diagram



Example 2.1.9. Let Z be a topological space. We can associate to Z a simplicial set $\mathrm{Sing}(Z)_\bullet$, whose n-simplices are continuous maps $\Delta^n \to Z$ (here Δ^n denotes the standard n-simplex: that is, the convex hull of the standard basis for $\mathbf{R}^{n+1}$). The simplicial set $\mathrm{Sing}(Z)_\bullet$ is called the *singular simplicial set* of Z.

From the perspective of homotopy theory, the singular simplicial set $\mathrm{Sing}(Z)_\bullet$ is a complete invariant of X. More precisely, from $\mathrm{Sing}(Z)_\bullet$ one can functorially construct a topological space which is (weakly) homotopy equivalent to Z. Consequently, the simplicial set $\mathrm{Sing}(Z)_\bullet$ can often serve as a surrogate for Z. For example, there is a combinatorial recipe for extracting the homotopy groups of Z directly from $\mathrm{Sing}(Z)_\bullet$. However, this recipe works only for a special class of simplicial sets:

Definition 2.1.10. Let $X_\bullet$ be a simplicial set. We say that X is a *Kan complex* if, for $0 \leq i \leq n$, the map $X_n \to \Lambda^n_i(X_\bullet)$ is surjective.

Example 2.1.11. For any topological space Z, the singular simplicial set $\mathrm{Sing}(Z)_\bullet$ is a Kan complex. To see this, let H denote the topological space obtained from the standard n-simplex Δ^n by removing the interior and the ith face. Then $\Lambda^n_i(\mathrm{Sing}(Z)_\bullet)$ can be identified with the set of continuous maps from H into Z. Any continuous map from H into Z can be extended to a map from Δ^n into Z, since H is a retract of Δ^n.

The converse of Example 2.1.11 fails: not every Kan complex is isomorphic to the singular simplicial set of a topological space. However, every Kan complex $X_\bullet$ is *homotopy equivalent* to the singular simplicial set of a topological space, which can be constructed explicitly from $X_\bullet$. In fact, something stronger is true: the construction $Z \mapsto \mathrm{Sing}(Z)_\bullet$ induces an equivalence from the homotopy category of nice spaces (say, CW complexes) to the homotopy category of Kan complexes (which can be defined in a purely combinatorial way).

Example 2.1.12. A *simplicial Λ-module* is a simplicial set $X_\bullet$ for which each of the sets X_n is equipped with the structure of a Λ-module, and each of the maps $\alpha^* : X_n \to X_m$ is a Λ-module homomorphism homomorphism. One can show that every simplicial Λ-module is a Kan complex, so that one has homotopy groups $\{\pi_n X_\bullet\}_{n \geq 0}$. According to the classical *Dold-Kan correspondence*, the category of simplicial Λ-modules is equivalent to the category $\mathrm{Chain}_{\geq 0}(\Lambda) \subseteq$

Chain(Λ) of nonnegatively graded chain complexes of Λ-modules. Under this equivalence, the homotopy groups of a simplicial Λ-module $X_\bullet$ can be identified with the homology groups of the corresponding chain complex.

The hypothesis of Proposition 2.1.8 resembles the definition of a Kan complex, but is different in two important respects. Definition 2.1.10 requires that every element of $\Lambda_i^n(X_\bullet)$ can be extended to an n-simplex of X. Proposition 2.1.8 requires this condition only in the case $0 < i < n$, but demands that the extension be unique. Neither condition implies the other, but they admit a common generalization:

Definition 2.1.13. A simplicial set $X_\bullet$ is an ∞-*category* if, for each $0 < i < n$, the map $X_n \to \Lambda_i^n(X_\bullet)$ is surjective.

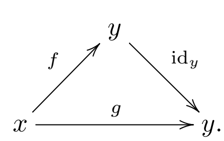
Remark 2.1.14. A simplicial set $X_\bullet$ satisfying the requirement of Definition 2.1.13 is also referred to as a *quasi-category* or a *weak Kan complex* in the literature.

Example 2.1.15. Any Kan complex is an ∞-category. In particular, for any topological space Z, the singular simplicial set $\mathrm{Sing}(Z)_\bullet$ is an ∞-category.

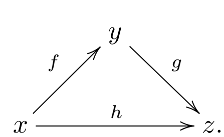
Example 2.1.16. For any category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})_\bullet$ is an ∞-category.

By virtue of the discussion following Example 2.1.5, no information is lost by identifying a category $\mathcal{C}$ with the simplicial set $\mathrm{N}(\mathcal{C})_\bullet$. It is often convenient to abuse notation by identifying $\mathcal{C}$ with its nerve, thereby viewing a category as a special type of ∞-category. We will generally use category-theoretic notation and terminology when discussing ∞-categories. Here is a brief sampler; for a more detailed discussion of how the basic notions of category theory can be generalized to this setting, we refer the reader to the first chapter of [34].

- Let $\mathcal{C} = \mathcal{C}_\bullet$ be an ∞-category. An *object* of $\mathcal{C}$ is an element of the set $\mathcal{C}_0$ of 0-simplices of $\mathcal{C}$. We will indicate that x is an object of $\mathcal{C}$ by writing $x \in \mathcal{C}$.
- A *morphism* of $\mathcal{C}$ is an element f of the set $\mathcal{C}_1$ of 1-simplices of $\mathcal{C}$. More precisely, we will say that f is a *morphism from x to y* if $\alpha_0^*(f) = x$ and $\alpha_1^*(f) = y$, where $\alpha_i : \{0\} \hookrightarrow \{0, 1\}$ denote the map given by $\alpha_i(0) = i$. We will often indicate that f is a morphism from x to y by writing $f : x \to y$.
- For any object $x \in \mathcal{C}$, there is an *identity morphism* id_x, given by $\beta^*(x)$ where $\beta : \{0, 1\} \to \{0\}$ is the unique map.
- Given a pair of morphisms $f, g : x \to y$ in $\mathcal{C}$, we say that f and g are *homotopic* if there exists a 2-simplex $\sigma \in \mathcal{C}_2$ whose faces are as indicated in the diagram



  In this case, we will write $f \simeq g$, and we will say that σ *is a homotopy from f to g*. One can show that homotopy is an equivalence relation on the collection of morphisms from x to y.
- Given a pair of morphisms $f : x \to y$ and $g : y \to z$, it follows from Definition 2.1.13 that there exists a 2-simplex with boundary as indicated in the diagram



Definition 2.1.13 does *not* guarantee that the morphism h is unique. However, one can show that h is unique up to homotopy. We will generally abuse terminology and refer to h as the composition of f and g, and write $h = g \circ f$.

- Composition of morphisms in $\mathcal{C}$ is associative up to homotopy. Consequently, we can define an ordinary category $\mathrm{h}\mathcal{C}$ as follows:
  - The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
  - Given objects $x, y \in \mathcal{C}$, the set of morphisms from x to y in $\mathrm{h}\mathcal{C}$ is the set of equivalence classes (under the relation of homotopy) of morphisms from x to y in $\mathcal{C}$.
  - Given morphisms $[f] : x \to y$ and $[g] : y \to z$ in $\mathrm{h}\mathcal{C}$ represented by morphisms f and g in $\mathcal{C}$, we define $[g] \circ [f]$ to be the morphism from x to z in $\mathrm{h}\mathcal{C}$ given by the homotopy class of $g \circ f$.

  We refer to $\mathrm{h}\mathcal{C}$ as the *homotopy category* of $\mathcal{C}$.
- We will say that a morphism f in $\mathcal{C}$ is an *equivalence* if its image $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$ (in other words, f is an equivalence if it admits an inverse up to homotopy). We say that two objects $x, y \in \mathcal{C}$ are *equivalent* if there exists an equivalence $f : x \to y$.

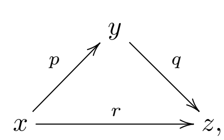
The theory of ∞-categories allows us to treat topological spaces (via their singular simplicial sets) and ordinary categories (via the nerves) as examples of the same type of object. This is often very convenient.

Definition 2.1.17. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. A *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a map of simplicial sets from $\mathcal{C}$ to $\mathcal{D}$.

Remark 2.1.18. Let $\mathcal{C}$ be an ∞-category. The homotopy category of $\mathcal{C}$ admits another characterization: it is universal among ordinary categories for which there exists a functor from $\mathcal{C}$ to (the nerve of) $\mathrm{h}\mathcal{C}$.

Example 2.1.19. Let Z be a topological space and let $\mathcal{C}$ be a category. Unwinding the definitions, we see that a functor from $\mathrm{Sing}(Z)_\bullet$ to $\mathrm{N}(\mathcal{C})_\bullet$ consists of the following data:

(1) For each point $z \in Z$, an object $C_z \in \mathcal{C}$.
(2) For every path $p : [0,1] \to Z$, a morphism $\alpha_p : C_{p(0)} \to C_{p(1)}$, which is an identity morphism if the map p is constant.
(3) For every continuous map $\Delta^2 \to Z$, which we write informally as



we have $\alpha_r = \alpha_q \circ \alpha_p$ (an equality of morphisms from C_x to C_z).

Here condition (3) encodes simultaneously the assumption that the map α_p depends only on the homotopy class of p, and that the construction $p \mapsto \alpha_p$ is compatible with concatenation of paths. Moreover, it follows from condition (3) that each of the maps α_p is an isomorphism (since every path is invertible up to homotopy). Consequently, we see that the data of a functor from $\mathrm{Sing}(Z)_\bullet$ into $\mathrm{N}(\mathcal{C})_\bullet$ recovers the classical notion of a *local system on Z with values in* $\mathcal{C}$.

One of the main advantages of working in the setting of ∞-categories is that the collection of functors from one ∞-category to another can easily be organized into a third ∞-category.

Notation 2.1.20. For every integer $n \geq 0$, we let Δ^n denote the simplicial set given by the nerve of the linearly ordered set $\{0 < 1 < \cdots < n\}$. We refer to Δ^n as the *standard n-simplex*. By definition, an m-simplex of Δ^n is given by a nondecreasing map $\{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$.

Let X and Y be simplicial sets. We let $\mathrm{Fun}(X, Y)$ denote the simplicial of maps from X to Y. More precisely, $\mathrm{Fun}(X, Y)$ is the simplicial set whose n-simplices are maps $\Delta^n \times X \to Y$ (more generally, giving a map of simplicial sets $Z \to \mathrm{Fun}(X, Y)$ is equivalent to giving a map $Z \times X \to Y$).

One can show that if the simplicial set Y is an ∞-category, then $\mathrm{Fun}(X, Y)$ is also an ∞-category (for any simplicial set X). Note that the objects of $\mathrm{Fun}(X, Y)$ are functors from X to Y, in the sense of Definition 2.1.17. We will refer to $\mathrm{Fun}(X, Y)$ as the *∞-category of functors from X to Y*.

Example 2.1.21. Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories. Then the simplicial set

$$\mathrm{Fun}(\mathrm{N}(\mathcal{C})_\bullet, \mathrm{N}(\mathcal{D})_\bullet)$$

is isomorphic to the nerve of the category of functors from $\mathcal{C}$ to $\mathcal{D}$. In particular, there is a bijection between the set of functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of classical category theory) to the set of functors from $\mathrm{N}(\mathcal{C})_\bullet$ to $\mathrm{N}(\mathcal{D})_\bullet$ (in the sense of Definition 2.1.17).

Remark 2.1.22. It follows from Example 2.1.21 that no information is lost by passing from a category $\mathcal{C}$ to the associated ∞-category $\mathrm{N}(\mathcal{C})$. For the remainder of this paper, we will generally abuse notation by identifying each category $\mathcal{C}$ with its nerve.

Example 2.1.23. Let Λ be a commutative ring and let $\mathrm{Mod}_\Lambda = \{S_n\}_{n \geq 0}$ denote the simplicial set introduced in Construction 2.1.3. Then Mod_Λ is an ∞-category, which we will refer to as the *derived ∞-category of Λ-modules*. It can be regarded as an enhancement of the usual derived category $\mathcal{D}(\Lambda)$ of Λ-modules, in the sense that the homotopy category of Mod_Λ is equivalent to $\mathcal{D}(\Lambda)$ (in fact, the homotopy category of Mod_Λ is *isomorphic* to the category $\mathcal{D}'(\Lambda)$ defined above).

Notation 2.1.24. Let Λ be a commutative ring. For every integer n, the construction $M_* \mapsto \mathrm{H}_n(M_*)$ determines a functor from the ∞-category Mod_Λ to the ordinary abelian category of Λ-modules. We will say that an object $M_* \in \mathrm{Mod}_\Lambda$ is *discrete* if $\mathrm{H}_n(M_*) \simeq 0$ for $n \neq 0$. One can show that the construction $M_* \mapsto \mathrm{H}_0(M_*)$ induces an equivalence from the ∞-category of discrete objects of Mod_Λ to the ordinary category of Λ-modules. We will generally abuse notation by identifying the abelian category of Λ-modules with its inverse image under this equivalence. We will sometimes refer to Λ-modules as as *discrete Λ-modules* or *ordinary Λ-modules*, to distinguish them from more general objects of Mod_Λ.

Remark 2.1.25. The ∞-category Mod_Λ is, in many respects, easier to work with than the usual derived category $\mathcal{D}(\Lambda)$. For example, we have already mentioned that there is no functorial way to construct the cone of a morphism in $\mathcal{D}(\Lambda)$. However, Mod_Λ does not suffer from the same problem: the formation of cones is given by a functor $\mathrm{Fun}(\Delta^1, \mathrm{Mod}_\Lambda) \to \mathrm{Mod}_\Lambda$.

The theory of ∞-categories is a robust generalization of ordinary category theory. In particular, many important notions of ordinary category theory (adjoint functors, Kan extensions, Pro-objects and Ind-objects, ...) can be generalized to the setting of ∞-categories in a natural way. We will make use of these notions throughout this paper. For a detailed introduction (including complete definitions and proofs of the basic categorical facts we will need), we refer the reader to [34].

Notation 2.1.26. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Throughout this paper, we will often need to consider a *limit* or *colimit* of a functor $F : \mathcal{C} \to \mathcal{D}$. Roughly speaking, a *limit* of F is an object $D \in \mathcal{D}$ which is universal among those objects which are equipped with a family of morphisms $\{D \to F(C)\}_{C \in \mathcal{C}}$ (together with appropriate higher coherence data), and a *colimit* of F is an object $D' \in \mathcal{D}$ which is universal among those objects equipped with a compatible family of morphisms $\{F(C) \to D\}_{C \in \mathcal{C}}$ (together with higher coherence data). We refer the reader to §HTT.1.2.13 for a more detailed discussion.

The limit and colimit of a functor $F : \mathcal{C} \to \mathcal{D}$ are determined uniquely up to equivalence if they exist. We will generally abuse terminology by referring to *the* limit or colimit of a functor F, which we will denote by $\varprojlim_{C \in \mathcal{C}} F(C)$ and $\varinjlim_{C \in \mathcal{C}} F(C)$, respectively.

If F is given instead as a functor from the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$, we will generally denote a limit and colimit of F also by the notation

$$\varprojlim_{C \in \mathcal{C}} F(C) \qquad \varinjlim_{C \in \mathcal{C}} F(C).$$

There is little danger of conflict between these notations, provided that it is clear from context whether the domain of the functor F is the ∞-category $\mathcal{C}$ or its opposite $\mathcal{C}^{\mathrm{op}}$.

2.2. **ℓ-adic Cohomology of Algebraic Varieties.** Let k denote an algebraically closed field and ℓ a prime number which is invertible in k. In this section, we will review the theory of ℓ-adic cohomology in the setting of quasi-projective k-schemes.

Remark 2.2.1. The restriction to quasi-projective k-schemes is not essential in what follows; we could just as well work in the category of k-schemes of finite type, or even some larger category. However, such generalizations will be subsumed by the setting of prestacks which we discuss in §2.3.

Definition 2.2.2. Let Sch_k denote the category of quasi-projective k-schemes, and let $\mathcal{C}$ be an ∞-category which admits limits. A *$\mathcal{C}$-valued presheaf* on Sch_k is a functor $\mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}$.

Let $\mathcal{F} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}$ be a $\mathcal{C}$-valued presheaf on Sch_k. We will say that $\mathcal{F}$ is a *$\mathcal{C}$-valued sheaf on* Sch_k if the following condition is satisfied:

- $(*)$ Let X be a quasi-projective k-scheme and suppose we are given a jointly surjective collection of étale morphisms $u_\alpha : U_\alpha \to X$. Let $\mathcal{U}$ denote the category of quasi-projective k-schemes Y equipped with a map $Y \to X$ which factors through some u_α (the factorization need not be specified). Then $\mathcal{F}$ induces an equivalence

$$\mathcal{F}(X) \to \varprojlim_{Y \in \mathcal{U}} \mathcal{F}(Y)$$

in the ∞-category $\mathcal{C}$.

Example 2.2.3. Let $\mathcal{C}$ be an ordinary category which admits limits and let $\mathrm{N}(\mathcal{C})_\bullet$ be its nerve. Then the data of a $\mathrm{N}(\mathcal{C})_\bullet$-valued sheaf on Sch_k (in the sense of Definition 2.2.2) is equivalent to the data of a $\mathcal{C}$-valued sheaf on Sch_k (in the sense of classical category theory).

Example 2.2.4. Let $\mathcal{C}$ be an arbitrary ∞-category. For each object $M \in \mathcal{C}$, the constant functor $\mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}$ taking the value of M is a $\mathcal{C}$-valued presheaf on Sch_k, which we will denote by c_M.

Definition 2.2.5. Let $\mathrm{Mod}_{\mathbf{Z}}$ denote the derived ∞-category of abelian groups (see Example 2.1.23). We let $\mathrm{Sh}(\mathrm{Sch}_k; \mathbf{Z})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ spanned by those functors $\mathcal{F}$ which are $\mathrm{Mod}_{\mathbf{Z}}$-valued sheaves.

One can show that the inclusion

$$\mathrm{Sh}(\mathrm{Sch}_k^{\mathrm{op}}; \mathbf{Z}) \hookrightarrow \mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$$

admits a left adjoint, which assigns to each presheaf $\mathcal{F} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ its *sheafification* $\mathcal{F}^\dagger$ with respect to the étale topology.

Definition 2.2.6. Let M be a finite abelian group, which we regard as an object of $\mathrm{Mod}_{\mathbf{Z}}$, and let $c_M : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ denote the constant presheaf taking the value M. We will denote the sheafification of c_M by $X \mapsto C^*(X; M)$. For each quasi-projective k-scheme X, we will refer to $C^*(X; M)$ as the *complex of M-valued cochains on X*.

Remark 2.2.7. Let **Ab** denote the *ordinary* category of abelian groups. Then each finite abelian group M, we can regard c_M as an **Ab**-valued presheaf on Sch_k. This presheaf admits a sheafification (in the ordinary category of **Ab**-valued presheaves on Sch_k) which we will denote by $\underline{M}$. Concretely, the functor $\underline{M}$ assigns to each quasi-projective k-scheme X the group of all continuous M-valued functions on X. Note that we can regard $\underline{M}$ as a $\mathrm{Mod}_{\mathbf{Z}}$-valued presheaf on Sch_k (via the inclusion $\mathbf{Ab} \hookrightarrow \mathrm{Mod}_{\mathbf{Z}}$), but that it is *not* a $\mathrm{Mod}_{\mathbf{Z}}$-valued sheaf (since the inclusion $\mathbf{Ab} \hookrightarrow \mathrm{Mod}_{\mathbf{Z}}$ does not preserve inverse limits).

Let $\mathcal{A}$ denote the abelian category of **Ab**-valued sheaves on Sch_k. The abelian category $\mathcal{A}$ has enough injectives, so that we can choose an injective resolution

$$0 \to \underline{M} \to I^0 \to I^1 \to \cdots$$

in the category $\mathcal{A}$. The construction $X \mapsto I^*(X)$ determines a functor from $\mathrm{Sch}_k^{\mathrm{op}}$ to the category $\mathrm{Chain}(\mathbf{Z})$ of chain complexes of abelian groups, hence also a functor $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$. One can show that the evident maps $M \to I^*(X)$ exhibit the functor $X \mapsto I^*(X)$ as a sheafification of c_M (see Proposition SAG.2.1.1.8).

In other words, the object $C^*(X; M) \in \mathrm{Mod}_{\mathbf{Z}}$ can be explicitly described described as the chain complex

$$\cdots \to 0 \to I^0(X) \to I^1(X) \to \cdots .$$

In particular, the cohomology groups of the chain complex $C^*(X; M)$ can be identified with the usual *étale cohomology groups* of X with values in M, which we will denote simply by $\mathrm{H}^*(X; M)$.

Definition 2.2.6 makes sense also in the case where the abelian group M is not finite. However, it is generally badly behaved if $M = \mathbf{Z}$ or $M = \mathbf{Q}$. Consequently, we will use the notation $C^*(X; M)$ for a slightly different chain complex in general.

Definition 2.2.8. Let $\mathbf{Z}_\ell$ denote the ring of ℓ-adic integers. For every quasi-projective k-scheme X, we let $C^*(X; \mathbf{Z}_\ell)$ denote the limit $\varprojlim C^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$, formed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. We will refer to $C^*(X; \mathbf{Z}_\ell)$ as the *complex of $\mathbf{Z}_\ell$-valued cochains on X*.

Warning 2.2.9. The construction $X \mapsto C^*(X; \mathbf{Z}_\ell)$ is a sheaf for the étale topology, in the sense of Definition 2.2.2. However, it is *not* the sheafification of the constant functor $c_{\mathbf{Z}_\ell}$. It can be described instead as the ℓ-adic completion of the sheafification $c^{\dagger}_{\mathbf{Z}_\ell}$.

Definition 2.2.10. The inclusion $\mathbf{Z} \hookrightarrow \mathbf{Z}[\ell^{-1}]$ determines a base change functor $\mathrm{Mod}_{\mathbf{Z}} \to \mathrm{Mod}_{\mathbf{Z}[\ell^{-1}]}$, which we will denote by $M \mapsto M[\ell^{-1}]$. For every quasi-projective k-scheme X, we define

$$C^*(X; \mathbf{Q}_\ell) = C^*(X; \mathbf{Z}_\ell)[\ell^{-1}].$$

We will denote the cohomology groups of the chain complexes $C^*(X; \mathbf{Z}_\ell)$ and $C^*(X; \mathbf{Q}_\ell)$ by $\mathrm{H}^*(X; \mathbf{Z}_\ell)$ and $\mathrm{H}^*(X; \mathbf{Q}_\ell)$, respectively. We refer to either of these as the *ℓ-adic cohomology of X*.

2.2.1. *The Cup Product.* If X is a quasi-projective k-scheme, then one can define a cup-product map

$$\mathrm{H}^p(X; \mathbf{Z}_\ell) \times \mathrm{H}^q(X; \mathbf{Z}_\ell) \to \mathrm{H}^{p+q}(X; \mathbf{Z}_\ell)$$

which endows $\mathrm{H}^*(X;\mathbf{Z}_\ell)$ with the structure of a graded commutative ring. In fact, one can be more precise: the cup product arises from a multiplication on the ℓ-adic cochain complex $C^*(X;\mathbf{Z}_\ell)$ itself. In this section, we will describe how this structure arises from an ∞-categorical point of view.

We begin by observing that the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$ admits a *symmetric monoidal* structure: that is, it is equipped with a tensor product functor

$$\otimes_{\mathbf{Z}} : \mathrm{Mod}_{\mathbf{Z}} \times \mathrm{Mod}_{\mathbf{Z}} \to \mathrm{Mod}_{\mathbf{Z}}$$

which is commutative and associative up to *coherent* homotopy (see Definition HA.2.0.0.7). Concretely, this can be described as the left derived functor of the usual tensor product (to compute with it, it is convenient to work with an alternative definition of $\mathrm{Mod}_{\mathbf{Z}}$ using chain complexes of free modules rather than chain complexes of injective modules).

Since $\mathrm{Mod}_{\mathbf{Z}}$ is a symmetric monoidal ∞-category, it makes sense to consider associative or commutative algebra objects of $\mathrm{Mod}_{\mathbf{Z}}$ (see §HA.2.1.3). These can be thought of as chain complexes of abelian groups which are equipped with an algebra structure which is required to be associative (respectively commutative and associative) up to coherent homotopy. In the case of associative algebras, it is always possible to *rectify* the multiplication by choosing a quasi-isomorphic chain complex which is equipped with a multiplication which is strictly associative: that is, a differential graded algebra (see Proposition HA.7.1.4.6). In the commutative case this is not always possible: in concrete terms, a commutative algebra structure on an object of $\mathrm{Mod}_{\mathbf{Z}}$ is equivalent to the data of an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}$.

The symmetric monoidal structure on $\mathrm{Mod}_{\mathbf{Z}}$ induces a symmetric monoidal structure on the functor ∞-category $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$, given by pointwise tensor product (see Remark HA.2.1.3.4):

$$(\mathcal{F} \otimes \mathcal{F}')(X) = \mathcal{F}(X) \otimes_{\mathbf{Z}} \mathcal{F}'(X).$$

One can show that this symmetric monoidal structure determines a symmetric monoidal structure on the subcategory $\mathrm{Sh}(\mathrm{Sch}_k;\mathbf{Z}) \subseteq \mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ whose underlying tensor product $\otimes : \mathrm{Sh}(\mathrm{Sch}_k;\mathbf{Z}) \times \mathrm{Sh}(\mathrm{Sch}_k;\mathbf{Z}) \to \mathrm{Sh}(\mathrm{Sch}_k;\mathbf{Z})$ fits into a diagram

$$\begin{array}{ccc}
\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}}) \times \mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}}) & \xrightarrow{\otimes} & \mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}}) \\
\downarrow & & \downarrow \\
\mathrm{Sh}(\mathrm{Sch}_k;\mathbf{Z}) \times \mathrm{Sh}(\mathrm{Sch}_k;\mathbf{Z}) & \xrightarrow{\otimes} & \mathrm{Sh}(\mathrm{Sch}_k;\mathbf{Z})
\end{array}$$

which commutes up to equivalence, where the vertical maps are given by the sheafification functors. It follows that the sheafification functor carries commutative algebra objects of the ∞-category $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ to commutative algebra objects of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$. In particular, since each of the constant functors $c_{\mathbf{Z}/\ell^d\mathbf{Z}}$ is a commutative algebra object of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$, we can regard the construction $X \mapsto C^*(X;\mathbf{Z}/\ell^d\mathbf{Z})$ as a functor which takes values in commutative algebra objects of $\mathrm{Mod}_{\mathbf{Z}}$. This structure passes to the limit in d (see §HA.3.2.2), and determines commutative algebra structures on $C^*(X;\mathbf{Z}_\ell)$ and $C^*(X;\mathbf{Q}_\ell)$. At the level of cohomology, this endows the groups

$$\mathrm{H}^*(X;\mathbf{Z}/\ell^d\mathbf{Z}) \qquad \mathrm{H}^*(X;\mathbf{Z}_\ell) \qquad \mathrm{H}^*(X;\mathbf{Q}_\ell)$$

with the structure of a graded-commutative ring (which agrees with the usual *cup product* of cohomology classes).

Remark 2.2.11. Any commutative ring object Λ can be identified with a commutative algebra object of the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. Moreover, one can identify the ∞-category of Λ-modules in

$\mathrm{Mod}_{\mathbf{Z}}$ with the ∞-category Mod_Λ of Example 2.1.23 (Proposition HA.7.1.1.15). In particular, Mod_Λ inherits the structure of a symmetric monoidal ∞-category (see §HA.4.5.2).

For $\Lambda \in \{\mathbf{Z}/\ell\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$, our assumption that k is algebraically closed implies that we have a canonical equivalence

$$C^*(\operatorname{Spec} k; \Lambda) \simeq \Lambda.$$

For every quasi-projective k-scheme X, we have morphism

$$\Lambda \simeq C^*(\operatorname{Spec} k; \Lambda) \to C^*(X; \Lambda)$$

of commutative algebra objects of $\mathrm{Mod}_{\mathbf{Z}}$. This implies that $C^*(X;\Lambda)$ can be promoted to a (commutative algebra) object of the ∞-category Mod_Λ.

Warning 2.2.12. The tensor product functor $\otimes_\Lambda$ on Mod_Λ does not agree with the usual tensor product on discrete Λ-modules. If M and N are discrete Λ-modules, then the tensor product $M \otimes_\Lambda N$ (formed in Mod_Λ) is obtained by tensoring M with some projective resolution of N, or vice versa. In particular, we have canonical isomorphisms

$$\mathrm{H}_i(M \otimes_\Lambda N) = \mathrm{Tor}_i^\Lambda(M, N).$$

In particular $M \otimes_\Lambda N$ is discrete if and only if the groups $\mathrm{Tor}_i^\Lambda(M,N) \simeq 0$ for $i > 0$ (this is automatic, for example, if M or N is flat over Λ).

Unless otherwise specified, we will always use the notation $\otimes_\Lambda$ to indicate the tensor product in the ∞-category Mod_Λ, rather than the ordinary category of discrete Λ-modules.

Let Λ be a commutative ring. An object $M \in \mathrm{Mod}_\Lambda$ is said to be *perfect* if it is a compact object of Mod_Λ: that is, if the functor $N \mapsto \mathrm{Map}_{\mathrm{Mod}_\Lambda}(M, N)$ preserves filtered colimits. Equivalently, M is perfect if it is quasi-isomorphic to a finite complex of finitely generated projective Λ-modules.

Let us now study the behavior of the chain complexes $C^*(X;\Lambda)$ as the ring Λ varies.

Proposition 2.2.13. *Let X be a quasi-projective k-scheme. For every $d \geq 0$, the canonical map*

$$\theta : (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} C^*(X; \mathbf{Z}_\ell) \to C^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$$

is an equivalence in the ∞-category $\mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$.

Proof. For every object $M \in \mathrm{Mod}_{\mathbf{Z}_\ell}$, let $c_M^\dagger : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ denote the sheafification of the constant functor $c_M : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ taking the value M. Then θ factors as a composition

$$\begin{aligned}
(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} \varprojlim \{c^\dagger_{\mathbf{Z}/\ell^e\mathbf{Z}}\}_{e \geq 0} &\xrightarrow{\theta_0} \varprojlim \{\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}_\ell} c^\dagger_{\mathbf{Z}/\ell^e\mathbf{Z}}(X)\}_{e\geq 0} \\
&\xrightarrow{\theta_1} \varprojlim \{c^\dagger_{M_{d,e}}(X)\}_{e \geq 0} \\
&\xrightarrow{\theta_2} c^\dagger_{\mathbf{Z}/\ell^d\mathbf{Z}}(X),
\end{aligned}$$

where $M_{d,e}$ denotes the (left-derived) tensor product $(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} (\mathbf{Z}/\ell^e\mathbf{Z})$. The maps θ_0 and θ_1 are equivalences by virtue of the fact that $\mathbf{Z}/\ell^d\mathbf{Z}$ is perfect as a $\mathbf{Z}_\ell$-module, and the map θ_2 is determined by the identification of $\mathbf{Z}/\ell^d\mathbf{Z}$ with the group $\mathrm{Tor}_0^{\mathbf{Z}_\ell}(\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}/\ell^e\mathbf{Z})$ for $e \geq d$. It follows that the fiber of θ_2 can be identified (up to suspension) with the limit

$$\varprojlim_{e \geq d} c^\dagger_{\mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}/\ell^e\mathbf{Z})}(X).$$

This limit vanishes, because the tower $\{\mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}/\ell^e\mathbf{Z})\}_{e \geq d}$ is trivial as a pro-object in the abelian category of $\mathbf{Z}_\ell$-modules. □

Corollary 2.2.14. *Let $0 \leq d \leq e$ be integers and let X be a quasi-projective k-scheme. Then the canonical map*

$$\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}/\ell^e\mathbf{Z}} C^*(X; \mathbf{Z}/\ell^e\mathbf{Z}) \to C^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$.

Proposition 2.2.15. *Let X be a quasi-projective k-scheme and let $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$. Then $C^*(X;\Lambda)$ is a perfect object of* Mod_Λ.

Proof. Using either Proposition 2.2.13 or the definition, we obtain an equivalence

$$C^*(X;\Lambda) \simeq \Lambda \otimes_{\mathbf{Z}_\ell} C^*(X; \mathbf{Z}_\ell).$$

It therefore suffice to treat the case where $\Lambda = \mathbf{Z}_\ell$. Since $C^*(X; \mathbf{Z}_\ell)$ is ℓ-complete (see Definition 4.3.1), it is a perfect object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$ if and only if $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}_\ell} C^*(X;\mathbf{Z}_\ell) \simeq C^*(X; \mathbf{Z}/\ell\mathbf{Z})$ is a perfect object of $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$: that is, if and only if the total cohomology $\mathrm{H}^*(X; \mathbf{Z}/\ell\mathbf{Z})$ is a finite-dimensional vector space over $\mathbf{Z}/\ell\mathbf{Z}$. This is proven in [13] (Theorem 1.1 of the seventh part). □

Definition 2.2.16. Let $\mathcal{C}$ be a symmetric monoidal category with unit object $\mathbf{1}$. Recall that an object $M \in \mathcal{C}$ is said to be *dualizable* if there exists another object $M^\vee \in \mathcal{C}$ together with maps

$$e : M^\vee \otimes M \to \mathbf{1} \qquad c : \mathbf{1} \to M \otimes M^\vee$$

for which the composite maps

$$M \xrightarrow{c \times \mathrm{id}} M \otimes M^\vee \otimes M \xrightarrow{\mathrm{id} \times e} M$$
$$M^\vee \xrightarrow{\mathrm{id} \times c} M^\vee \otimes M \otimes M^\vee \xrightarrow{e \times \mathrm{id}} M^\vee$$

are equal to id_M and $\mathrm{id}_{M^\vee}$, respectively.

If $\mathcal{C}$ is a symmetric monoidal ∞-category, we say that an object $M \in \mathcal{C}$ is *dualizable* if it is dualizable as an object of the homotopy category of $\mathcal{C}$.

Example 2.2.17. Let Λ be a commutative ring and let $M \in \mathrm{Mod}_\Lambda$. Then M is dualizable if and only if it is perfect (see Proposition HA.7.2.5.4). In this case, the dual $M^\vee$ is canonically determined. More precisely, the construction $M \mapsto M^\vee$ determines a contravariant functor from the full subcategory $\mathrm{Mod}_\Lambda^{\mathrm{pf}} \subseteq \mathrm{Mod}_\Lambda$ of perfect Λ-modules to itself. We will refer to $M^\vee$ as the *dual* of M, or as the Λ-*linear dual* of M if we wish to emphasize its dependence on the ring Λ.

Definition 2.2.18. If X is a quasi-projective k-scheme, then Proposition 2.2.15 asserts that the cochain complex $C^*(X;\Lambda)$ is perfect for $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$. We will denote their respective duals by

$$C_*(X; \mathbf{Z}/\ell^d\mathbf{Z}) \in \mathrm{Mod}_{\mathbf{Z}/l^d\mathbf{Z}} \qquad C_*(X;\mathbf{Z}_\ell) \in \mathrm{Mod}_{\mathbf{Z}_\ell} \qquad C_*(X;\mathbf{Q}_\ell) \in \mathrm{Mod}_{\mathbf{Q}_\ell}.$$

We will refer to $C_*(X;\Lambda)$ as the *complex of* Λ-*valued chains on* X. For each integer n, we will denote the nth homology groups of these chain complexes by

$$\mathrm{H}_n(X; \mathbf{Z}/l^d\mathbf{Z}) \qquad \mathrm{H}_n(X;\mathbf{Z}_\ell) \qquad \mathrm{H}_n(X;\mathbf{Q}_\ell).$$

We refer to these groups as the *ℓ-adic homology groups of X*.

Remark 2.2.19. Let X be a quasi-projective k-scheme. Then we have canonical isomorphisms

$$\mathrm{H}^*(X;\mathbf{Q}_\ell) \simeq \mathrm{H}^*(X;\mathbf{Z}_\ell)[\ell^{-1}] \qquad \mathrm{H}_*(X;\mathbf{Q}_\ell) \simeq \mathrm{H}_*(X;\mathbf{Z}_\ell)[\ell^{-1}],$$

and (noncanonically split) "universal coefficient" exact sequences

$$0 \to \mathrm{Ext}_{\mathbf{Z}_\ell}(\mathrm{H}^{n+1}(X;\mathbf{Z}_\ell), \Lambda) \to \mathrm{H}_n(X;\Lambda) \to \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathrm{H}^n(X;\mathbf{Z}_\ell);\Lambda) \to 0$$

for $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$. The homology groups $\mathrm{H}_n(X; \mathbf{Z}/\ell\mathbf{Z})$ and $\mathrm{H}_n(X; \mathbf{Q}_\ell)$ can be described more concretely as the duals of the (finite-dimensional) vector spaces $\mathrm{H}^n_{\mathrm{et}}(X; \mathbf{Z}/\ell\mathbf{Z})$ and $\mathrm{H}^n_{\mathrm{et}}(X; \mathbf{Q}_\ell)$.

Let X and Y be quasi-projective k-schemes and let $X \times Y = X \times_{\operatorname{Spec} k} Y$ denote their product. Then the multiplication on $C^*(X \times Y; \Lambda)$ induces a map

$$\begin{aligned} C^*(X;\Lambda) \otimes_\Lambda C^*(Y;\Lambda) &\to C^*(X \times Y;\Lambda) \otimes_\Lambda C^*(X \times Y;\Lambda) \\ &\to C^*(X \times Y;\Lambda). \end{aligned}$$

Passing to Λ-linear duals, we also have a map

$$C_*(X \times Y;\Lambda) \to C_*(X;\Lambda) \otimes_\Lambda C_*(Y;\Lambda).$$

Theorem 2.2.20 (Künneth Formula)**.** *For every pair of quasi-projective k-schemes X and Y, if $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$, then the canonical maps*

$$C^*(X;\Lambda) \otimes_\Lambda C^*(Y;\Lambda) \to C^*(X \times Y;\Lambda)$$
$$C_*(X \times Y;\Lambda) \to C_*(X;\Lambda) \otimes_\Lambda C_*(Y;\Lambda)$$

are equivalences in Mod_Λ.

See Remark 4.6.6 for a proof (assuming some of the fundamental base change properties for étale sheaves).

Remark 2.2.21. When $\Lambda \in \{\mathbf{Z}/\ell\mathbf{Z}, \mathbf{Q}_\ell\}$ is a field, Theorem 2.2.20 asserts that we have canonical isomorphisms

$$\mathrm{H}^*(X \times Y;\Lambda) \simeq \mathrm{H}^*(X;\Lambda) \otimes_\Lambda \mathrm{H}^*(Y;\Lambda)$$
$$\mathrm{H}_*(X \times Y;\Lambda) \simeq \mathrm{H}_*(X;\Lambda) \otimes_\Lambda \mathrm{H}_*(Y;\Lambda).$$

2.2.2. *Cohomological Descent.* The category Sch_k admits many Grothendieck topologies different from the étale topology. In particular, we have the following variation on Definition 2.2.2:

Variant 2.2.22. Let $\mathcal{F} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ be a $\mathrm{Mod}_{\mathbf{Z}}$-valued presheaf on $\mathrm{Sch}_k^{\mathrm{op}}$. We will say that $\mathcal{F}$ is an *fppf sheaf* if the following condition is satisfied

(∗) Let X be a quasi-projective k-scheme and suppose we are given a jointly surjective collection of flat morphisms $u_\alpha : U_\alpha \to X$. Let $\mathcal{U}$ denote the category of quasi-projective k-schemes Y equipped with a map $Y \to X$ which factors through some u_α (the factorization need not be specified). Then $\mathcal{F}$ induces an equivalence

$$\mathcal{F}(X) \to \varprojlim_{Y \in \mathcal{U}} \mathcal{F}(Y)$$

in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$.

Let $\mathrm{Sh}_{\mathrm{fppf}}(\mathrm{Sch}_k; \mathbf{Z})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ spanned by those functors which are fppf sheaves. Then the inclusion $\mathrm{Sh}_{\mathrm{fppf}}(\mathrm{Sch}_k; \mathbf{Z}) \hookrightarrow \mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ admits a left adjoint. Applying this left adjoint to the constant functor $c_M : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$, when M is a finite abelian group, we obtain a new functor $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ which we will denote by $X \mapsto C^*_{\mathrm{fppf}}(X; M)$.

Every sheaf for the fppf topology on Sch_k is also a sheaf for the étale topology on Sch_k. In particular, the construction $X \mapsto C^*_{\mathrm{fppf}}(X; M)$ is an étale sheaf. We therefore obtain a map $\alpha_X : C^*(X; M) \to C^*_{\mathrm{fppf}}(X; M)$ which depends functorially on X. At the level of cohomology, this induces the usual map from the étale cohomology of X to the fppf cohomology of X (with coefficients in the finite abelian group M), which is an isomorphism (Theorem 11.7 of [22]). It follows that each of the maps α_X is an equivalence, so that the construction $X \mapsto C^*(X; M)$ is a sheaf for the fppf topology.

Remark 2.2.23. The collection of those functors $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ which are sheaves with respect to the fppf topology is closed under inverse limits. It follows that the construction $X \mapsto C^*(X; \mathbf{Z}_\ell)$ is also a sheaf for the fppf topology.

Notation 2.2.24. Let $\mathbf{\Delta}$ denote the category whose objects are the linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and whose morphisms are nondecreasing maps of linearly ordered sets. If $\mathcal{C}$ is an ∞-category, we will refer to functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ as *simplicial objects* of $\mathcal{C}$. If $C_\bullet$ is a simplicial object of $\mathcal{C}$, then we define its *geometric realization* to be the colimit $\varinjlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} C_n$, provided that the colimit exists in $\mathcal{C}$. We will denote the geometric realization of $C_\bullet$ by $|C_\bullet|$.

Proposition 2.2.25. *Let $u : U_0 \to X$ be a faithfully flat map between quasi-projective k-schemes and let $U_\bullet$ be the simplicial scheme given by the nerve of f (so that U_n is the $(n+1)$-fold fiber power of U_0 over X). Then:*

(1) *The canonical map $|C_*(U_\bullet; \mathbf{Z}_\ell)| \to C_*(X; \mathbf{Z}_\ell)$ is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.*
(2) *For every integer $d \geq 0$, the canonical map $|C_*(U_\bullet; \mathbf{Z}/\ell^d\mathbf{Z})| \to C_*(X; \mathbf{Z}/\ell^d\mathbf{Z})$ is an equivalence in $\mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$.*
(3) *The canonical map $|C_*(U_\bullet; \mathbf{Q}_\ell)| \to C_*(X; \mathbf{Q}_\ell)$ is an equivalence in $\mathrm{Mod}_{\mathbf{Q}_\ell}$.*

Proof. We will prove (1); assertions (2) and (3) are then immediate consequences. Let K denote the fiber of the map $\varinjlim C_*(U_\bullet; \mathbf{Z}_\ell) \to C_*(X; \mathbf{Z}_\ell)$ in the ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$, and let $K^\vee$ denote its $\mathbf{Z}_\ell$-linear dual. Then $K^\vee$ is the cofiber of the canonical map $C^*(X; \mathbf{Z}_\ell) \to \varprojlim C^*(U_\bullet; \mathbf{Z}_\ell)$, which is an equivalence by virtue of Remark 2.2.23. It follows that $K^\vee \simeq 0$, and we wish to prove that $K \simeq 0$. Note that the fiber of the map $K^\vee \xrightarrow{\ell} K^\vee$ can be identified with the $\mathbf{Z}/\ell\mathbf{Z}$-linear dual of $K_0 = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} K$. Since $\mathbf{Z}/\ell\mathbf{Z}$ is a field, it follows that $K_0 \simeq 0$, so that the map $K \xrightarrow{\ell} K$ is an equivalence and therefore the homology groups of K are vector spaces over $\mathbf{Q}_\ell$. We will complete the proof by showing that the homology groups of K are finitely generated as $\mathbf{Z}_\ell$-modules. Since $C_*(X; \mathbf{Z}_\ell)$ is a perfect object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$, its homology groups are finitely generated modules over $\mathbf{Z}_\ell$. It will therefore suffice to show that the homology groups of the limit $\varinjlim C_*(U_\bullet; \mathbf{Z}_\ell)$ are finitely generated over $\mathbf{Z}_\ell$. For each integer $m \geq 0$, the skeleton

$$\varinjlim_{[p] \in \mathbf{\Delta}, p \leq m} C_*(U_\bullet; \mathbf{Z}_\ell)$$

is a finite colimit of perfect objects of $\mathrm{Mod}_{\mathbf{Z}_\ell}$, hence perfect, and therefore has finite-dimensional homology in each degree. The desired result now follows from the observation that the map

$$\varinjlim_{[p] \in \mathbf{\Delta}, p \leq m} C_*(U_\bullet; \mathbf{Z}_\ell) \to \varinjlim_{[p] \in \mathbf{\Delta}} C_*(U_\bullet; \mathbf{Z}_\ell)$$

induces an isomorphism on homology in degree n provided that $m \gg n$. □

2.3. **ℓ-adic Cohomology of Prestacks.** Let k be an algebraically closed field and let ℓ be a prime number which is invertible in k. In §2.2, we reviewed the theory of ℓ-adic cohomology for quasi-projective k-schemes. Unfortunately, this is not sufficiently general for our purposes: in this paper, we will need to study the ℓ-adic cohomology of more general algebro-geometric objects, such as the moduli stack $\mathrm{Bun}_G(X)$. In this section, we will extend the theory of ℓ-adic cohomology to arbitrary (pre)stacks over k. We begin with a brief review of the language of prestacks.

Notation 2.3.1. For every commutative ring R, we let Ring_R denote the category of finitely generated commutative R-algebras (we will use this notation only when R is Noetherian, in which case Ring_R can also be described as the category of finitely *presented* R-algebras).

If X is a k-scheme and $R \in \mathrm{Ring}_k$, then an *R-valued point* of X is a map $\operatorname{Spec} R \to X$ in the category of k-schemes. The collection of all R-valued points of X forms a set $X(R)$. The construction $R \mapsto X(R)$ determines a functor from Ring_k to the category of sets. We refer to this functor as the *functor of points* of X. If X is of finite type over k (or if we were to enlarge Ring_k to include k-algebras which are not finitely generated), then X is determined by its functor of points up to canonical isomorphism. In this case, we will generally abuse notation by identifying X with its functor of points.

Suppose that G is a smooth affine group scheme over an algebraic curve X. We would like to introduce an algebro-geometric object $\mathrm{Bun}_G(X)$ which parametrizes G-bundles on X. In other words, we would like the R-points of $\mathrm{Bun}_G(X)$ to be G-bundles on the relative curve $X_R = \operatorname{Spec} R \times_{\operatorname{Spec} k} X$. Here some caution is in order. The collection of all G-bundles on X_R naturally forms a category, rather than a set. Let us denote this category by $\mathrm{Bun}_G(X)(R)$. If $\phi : R \to R'$ is a k-algebra homomorphism, then ϕ determines a map of categories $\phi^* : \mathrm{Bun}_G(X)(R) \to \mathrm{Bun}_G(X)(R')$, given on objects by the formula

$$\phi^* \mathcal{P} = X_{R'} \times_{X_R} \mathcal{P}.$$

However, this construction is not strictly functorial: given another ring homomorphism $\psi : R' \to R''$, the iterated pullback

$$\psi^*(\phi^* \mathcal{P}) = X_{R''} \times_{X_{R'}} (X_{R'} \times_{X_R} \mathcal{P})$$

is canonically isomorphic to $X_{R''} \times_{X_R} \mathcal{P}$, but might not be literally identical.

It is possible to axiomatize the functorial behavior exhibited by the construction $R \mapsto \mathrm{Bun}_G(X)(R)$ using the language of 2-categories (or ∞-categories). However, it is often more convenient to encode the same data in a different package, where the functoriality is "implicit" rather than "explicit".

Definition 2.3.2. Let X be an algebraic curve over k and let G be a smooth group scheme over X. We define a category $\mathrm{Bun}_G(X)$ as follows:

(1) The objects of $\mathrm{Bun}_G(X)$ are pairs $(R, \mathcal{P})$, where R is a finitely presented k-algebra and $\mathcal{P}$ is a G-bundle on the relative curve $X_R = \operatorname{Spec} R \times_{\operatorname{Spec} k} X$.
(2) A morphism from $(R, \mathcal{P})$ to $(R', \mathcal{P}')$ consists of a k-algebra homomorphism $\phi : R \to R'$ together with a G-bundle isomorphism α between $\mathcal{P}'$ and $X_{R'} \times_{X_R} \mathcal{P}$.

We will refer to $\mathrm{Bun}_G(X)$ as the *moduli stack of G-bundles*.

By construction, the assignment $(R, \mathcal{P}) \mapsto R$ determines a forgetful functor $\pi : \mathrm{Bun}_G(X) \to \mathrm{Ring}_k$. For every finitely generated k-algebra R, we can recover the category $\mathrm{Bun}_G(X)(R)$ as the fiber product $\mathrm{Bun}_G(X) \times_{\mathrm{Ring}_k} \{R\}$. Moreover, the map π also encodes the functoriality of the construction $R \mapsto \mathrm{Bun}_G(X)(R)$: given an object $(R, \mathcal{P}) \in \mathrm{Bun}_G(X)(R)$ and a ring homomorphism $\phi : R \to R'$, we can choose any lift of ϕ to a morphism $(\phi, \alpha) : (R, \mathcal{P}) \to (R', \mathcal{P}')$ in $\mathrm{Bun}_G(X)$. Such a lift then exhibits $\mathcal{P}'$ as a fiber product $X_{R'} \times_{X_R} \mathcal{P}$.

More generally, for any functor $\pi : \mathcal{C} \to \mathcal{D}$ and any object $D \in \mathcal{D}$, let $\mathcal{C}_D$ denote the fiber product $\mathcal{C} \times_{\mathcal{D}} \{D\}$. We might then ask if $\mathcal{C}_D$ depends functorially on D, in some sense. This requires an assumption on the functor π.

Definition 2.3.3. Let $\pi : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We say that a morphism $\alpha : C \to C'$ in $\mathcal{C}$ is *π-coCartesian* if, for every object $C'' \in \mathcal{C}$, composition with α induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(C', C'') \to \mathrm{Hom}_{\mathcal{C}}(C, C'') \times_{\mathrm{Hom}_{\mathcal{D}}(\pi C, \pi C'')} \mathrm{Hom}_{\mathcal{D}}(\pi C', \pi C'').$$

We will say that π is a *coCartesian fibration* if, for every object $C \in \mathcal{C}$ and every morphism $\alpha_0 : \pi C \to D$ in the category $\mathcal{D}$, there exists an π-coCartesian morphism $\alpha : C \to \overline{D}$ with $\alpha_0 = \pi(\alpha)$.

Remark 2.3.4. The definition of a coCartesian fibration can be generalized to the setting of ∞-categories. We refer the reader to §HTT.2.4 for more details.

Remark 2.3.5. A functor $\pi : \mathcal{C} \to \mathcal{D}$ between categories is said to be a *Cartesian fibration* if the induced map $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a coCartesian fibration.

Remark 2.3.6. A functor $\pi : \mathcal{C} \to \mathcal{D}$ satisfying the requirements of Definition 2.3.3 is more often referred to as an *op-fibration* or *cofibration* of categories. We use the term *coCartesian fibration* to remain consistent with [34] and to avoid conflict with other uses of the word "cofibration" in homotopy theory.

Example 2.3.7 (Grothendieck Construction)**.** Let $\mathcal{D}$ be a category, and let U be a functor from $\mathcal{D}$ to the category $\mathcal{C}\mathrm{at}$ of categories. We can define a new category $\mathcal{D}_U$ as follows:

(1) The objects of $\mathcal{D}_U$ are pairs (D, u) where D is an object of $\mathcal{D}$ and u is an object of the category $U(D)$.
(2) A morphism from (D, u) to (D', u') consists of a pair (ϕ, α), where $\phi : D \to D'$ is a morphism in $\mathcal{D}$ and $\alpha : U(\phi)(u) \to u'$ is a morphism in $U(D')$.

The construction $(D, u) \mapsto D$ determines a forgetful functor $\mathcal{D}_U \to \mathcal{D}$ which is a coCartesian fibration. The passage from U to $\mathcal{D}_U$ is often called the *Grothendieck construction.*

For any coCartesian fibration $F : \mathcal{C} \to \mathcal{D}$, the category $\mathcal{C}$ is equivalent to $\mathcal{D}_U$, for some functor $U : \mathcal{D} \to \mathcal{C}\mathrm{at}$. Moreover, the data of F and the data of the functor U are essentially equivalent to one another (in a suitable 2-categorical sense).

Definition 2.3.8. A *prestack* is a category $\mathcal{C}$ equipped with a coCartesian fibration $\pi : \mathcal{C} \to \mathrm{Ring}_k$.

Warning 2.3.9. Definition 2.3.8 is not standard: many authors use the term *prestack* to refer to a coCartesian fibration $\pi : \mathcal{C} \to \mathrm{Ring}_k$ which satisfies some weak form of descent with respect to a Grothendieck topology on Ring_k (see Remark 2.3.23).

Remark 2.3.10. We will generally abuse notation by identifying a prestack $\pi : \mathcal{C} \to \mathrm{Ring}_k$ with its underlying category $\mathcal{C}$ and simply say that $\mathcal{C}$ is a prestack, or that π *exhibits* $\mathcal{C}$ *as a prestack.*

Example 2.3.11. The forgetful functor $\mathrm{Bun}_G(X) \to \mathrm{Ring}_k$ is a coCartesian fibration, and therefore exhibits $\mathrm{Bun}_G(X)$ as a prestack.

Remark 2.3.12. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack. For every finitely generated k-algebra R, we let $\mathcal{C}(R)$ denote the fiber product $\mathcal{C} \times_{\mathrm{Ring}_k} \{R\}$. According to Example 2.3.7, the data of the prestack π is essentially equivalent to the data of the construction $R \mapsto \mathcal{C}(R)$. We will often describe prestacks informally by specifying the categories $\mathcal{C}(R)$, rather than the "total category" $\mathcal{C}$.

Example 2.3.13. Let X be a k-scheme. We can associate to X a category $\mathcal{C}_X$, which we call the *category of points* of X. By definition, an object of $\mathcal{C}_X$ is a pair (R, ϕ), where R is a finitely presented k-algebra and $\phi : \operatorname{Spec} R \to X$ is a map of k-schemes. A morphism from (R, ϕ) to

(R', ϕ') is a k-algebra homomorphism $\psi : R \to R'$ for which the diagram

$$\begin{array}{ccc} \operatorname{Spec} R' & \xrightarrow{\operatorname{Spec}(\psi)} & \operatorname{Spec} R \\ & {\scriptstyle\phi'}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & X & \end{array}$$

commutes. The construction $(R, \phi) \mapsto R$ defines a coCartesian fibration $\mathcal{C}_X \to \mathrm{Ring}_k$, so that we can view $\mathcal{C}_X$ as a prestack. For any commutative ring R, we have a canonical equivalence $\mathcal{C}_X(R) \simeq X(R)$, where we view the set $X(R)$ as a category with only identity morphisms. In other words, the prestack $\mathcal{C}_X \to \mathrm{Ring}_k$ encodes (via the Grothendieck construction) the functor of points of X.

Definition 2.3.14. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a functor. We say that π is a *prestack in groupoids* if it is prestack with the property that each of the categories $\mathcal{C}(R)$ is a groupoid. We will say that $\mathcal{C}$ is a *prestack in sets* if each of the categories $\mathcal{C}(R)$ is discrete (that it, if it has only identity morphisms).

Remark 2.3.15. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack. Then $\mathcal{C}$ is a prestack in groupoids if and only if every morphism in $\mathcal{C}$ is π-coCartesian.

Example 2.3.16. For every k-scheme X, the category of points $\mathcal{C}_X$ is a prestack in sets. The moduli stack $\mathrm{Bun}_G(X)$ of Definition 2.3.2 is a prestack in groupoids.

Remark 2.3.17. Though prestacks in groupoids are often technically easier to work with, many of the prestacks which we study in this paper (such as the Ran space $\mathrm{Ran}(X)$) are more conveniently described as prestacks which do *not* satisfy the requirement of Definition 2.3.14.

Definition 2.3.18. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ and $\pi' : \mathcal{C}' \to \mathrm{Ring}_k$ be prestacks. A *weak morphism of prestacks* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ for which the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ & \searrow \quad \swarrow & \\ & \mathrm{Ring}_k & \end{array}$$

commutes. A *morphism of prestacks* from $\mathcal{C}$ to $\mathcal{C}'$ is a weak morphism of prestacks which carries π-coCartesian morphisms to π'-coCartesian morphisms.

The collection of all morphisms of prestacks from $\mathcal{C}$ to $\mathcal{C}'$ forms a category $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$, where a morphism from $F : \mathcal{C} \to \mathcal{C}'$ to $G : \mathcal{C} \to \mathcal{C}'$ is a natural transformation of functors $\alpha : F \to G$ such that, for each object $C \in \mathcal{C}$, the map $\pi'(\alpha_C)$ is an identity morphism in Ring_k. If $\pi'' : \mathcal{C}'' \to \mathrm{Ring}_k$ is another prestack, we have evident composition functors

$$\mathrm{Hom}(\mathcal{C}, \mathcal{C}') \times \mathrm{Hom}(\mathcal{C}', \mathcal{C}'') \to \mathrm{Hom}(\mathcal{C}, \mathcal{C}'').$$

We can summarize the situation by saying that the collection of all prestacks forms a (strict) 2-category.

Remark 2.3.19. Let $\mathcal{C}$ and $\mathcal{D}$ be prestacks. If $\mathcal{D}$ is a prestack in groupoids, then every weak morphism of prestacks from $\mathcal{C}$ to $\mathcal{D}$ is automatically a morphism of prestacks from $\mathcal{C}$ to $\mathcal{D}$.

Remark 2.3.20. Let $\pi : \mathcal{C} \to \mathcal{D}$ be a morphism of prestacks, where $\mathcal{D}$ is a prestack in groupoids. Then π is a coCartesian fibration if and only if it satisfies the following condition:

- $(*)$ For each object $C \in \mathcal{C}$ and each isomorphism $\alpha : \pi(C) \to D$ in $\mathcal{D}$, there exists an isomorphism $\overline{\alpha} : C \to \overline{D}$ in $\mathcal{C}$ with $\pi(\overline{\alpha}) = \alpha$.

We can always arrange that condition (2) is satisfied by replacing $\mathcal{C}$ by an equivalent prestack.

Remark 2.3.21. Let X and Y be k-schemes, and let $\mathrm{Hom}_k(X,Y)$ be the *set* of k-scheme maps from X to Y. We regard $\mathrm{Hom}_k(X,Y)$ as a category, having no morphisms other than the identities. We have an evident functor $\mathrm{Hom}_k(X,Y) \to \mathrm{Hom}(\mathcal{C}_X, \mathcal{C}_Y)$. If X is locally of finite type over k, then this map is an isomorphism of categories. In particular, the construction $X \mapsto \mathcal{C}_X$ determines a fully faithful embedding from the ordinary category of schemes which are locally of finite type over k to the 2-category of prestacks. In other words, if X is a scheme which is locally of finite type over k, then X can be functorially reconstructed from the associated prestack $\mathcal{C}_X$. Because of this, we will generally abuse notation by identifying X with the prestack $\mathcal{C}_X$.

Example 2.3.22. Let R be a finitely generated k-algebra, and let $\mathrm{Spec}\, R$ be the associated affine scheme. Then the prestack associated to $\mathrm{Spec}\, R$ is equivalent to the category Ring_R of finitely generated R-algebras (viewed as a prestack via the functor $\mathrm{Ring}_R \to \mathrm{Ring}_k$ which "forgets" the R-algebra structure).

Remark 2.3.23. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack. We say that π is a *stack for the étale topology* if the following condition is satisfied:

($*$) Let R be a finitely generated k-algebra, let $\{R_\alpha\}$ be a collection of étale R-algebras for which the map $\coprod \mathrm{Spec}\, R_\alpha \to \mathrm{Spec}\, R$ is surjective, and let $\mathcal{D}$ be the full subcategory of Ring_R spanned by those finitely generated R-algebras A which admit the structure of an R_α algebra for some α. Then the forgetful functor

$$\mathcal{C}(R) \simeq \mathrm{Hom}(\mathrm{Ring}_R, \mathcal{C}) \to \mathrm{Hom}(\mathcal{D}, \mathcal{C})$$

is an equivalence of categories. Here $\mathrm{Hom}(\mathcal{D}, \mathcal{C})$ denotes the category of prestack morphisms from $\mathcal{D}$ to $\mathcal{C}$, in the sense of Definition 2.3.18.

We say that a morphism of prestacks $f : \mathcal{C} \to \mathcal{C}'$ *exhibits* $\mathcal{C}'$ *as an étale stackification of* $\mathcal{C}$ if $\mathcal{C}'$ is a stack for the étale topology and, for every prestack $\mathcal{C}''$ which is a stack for the étale topology, composition with f induces an equivalence of categories $\mathrm{Hom}(\mathcal{C}', \mathcal{C}'') \to \mathrm{Hom}(\mathcal{C}, \mathcal{C}'')$. One can show that an étale stackification of $\mathcal{C}$ always exists and is uniquely determined up to equivalence (in the 2-category of prestacks).

Many of the prestacks we are interested in (such as the moduli stack $\mathrm{Bun}_G(X)$) are stacks for the étale topology. However, it will be technically convenient to work with prestacks which do not satisfy this condition.

Variant 2.3.24. In Remark 2.3.23, we can replace the étale topology on Ring_k by any other Grothendieck topology, such as the Zariski topology on the fppf topology.

Definition 2.3.25. Let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$, where ℓ is a prime number which is invertible in k. For any prestack $\pi : \mathcal{C} \to \mathrm{Ring}_k$, we define chain complexes $C^*(\mathcal{C}; \Lambda)$ and $C_*(\mathcal{C}; \Lambda)$ by the formulae

$$C^*(\mathcal{C}; \Lambda) = \varprojlim_{C \in \mathcal{C}} C^*(\mathrm{Spec}\, \pi(C); \Lambda) \qquad C_*(\mathcal{C}; \Lambda) = \varinjlim_{C \in \mathcal{C}} C_*(\mathrm{Spec}\, \pi(C); \Lambda).$$

Here the limit and colimit are computed in the ∞-category Mod_Λ.

We let $\mathrm{H}^*(\mathcal{C}; \Lambda)$ denote the cohomology groups of $C^*(\mathcal{C}; \Lambda)$, and $\mathrm{H}_*(\mathcal{C}; \Lambda)$ the homology groups of $C_*(\mathcal{C}; \Lambda)$. We refer the groups $\mathrm{H}_*(\mathcal{C}; \Lambda)$ ($\mathrm{H}_*(\mathcal{C}; \Lambda)$) as the *$\ell$-adic (co)homology groups of* $\mathcal{C}$ *with coefficients in* Λ.

Remark 2.3.26. The notation of Definition 2.3.25 is slightly abusive: the chain complexes $C^*(\mathcal{C}; \Lambda)$ and $C_*(\mathcal{C}; \Lambda)$ depend not only on the category $\mathcal{C}$, but also the coCartesian fibration $\pi : \mathcal{C} \to \mathrm{Ring}_k$.

Example 2.3.27. Let $X \in \mathrm{Sch}_k$ be a quasi-projective k-scheme, and let $\mathcal{C}_X$ be the associated prestack. If $X = \operatorname{Spec} R$ is affine, then the category $\mathcal{C}_X$ has a final object (given by the pair (R, id)), so we have canonical equivalences

$$C^*(\mathcal{C}_X; \Lambda) \simeq C^*(\operatorname{Spec} R; \Lambda) \qquad C_*(\mathcal{C}_X; \Lambda) \simeq C_*(\operatorname{Spec} R; \Lambda).$$

Using the fact that the functor $X \mapsto C^*(X; \Lambda)$ is a sheaf for the étale topology, we deduce the existence of equivalences

$$C^*(\mathcal{C}_X; \Lambda) \simeq C^*(X; \Lambda) \qquad C_*(\mathcal{C}_X; \Lambda) \simeq C_*(X; \Lambda)$$

for any quasi-projective k-scheme X.

Warning 2.3.28. Let $\mathcal{C}$ be a prestack. Then $C^*(\mathcal{C}; \Lambda)$ can be identified with the Λ-linear dual of $C_*(\mathcal{C}; \Lambda)$. In particular, if $\Lambda \in \{\mathbf{Q}_\ell, \mathbf{Z}/\ell\mathbf{Z}\}$ is a field, then we have canonical isomorphisms

$$\mathrm{H}^i(\mathcal{C}; \Lambda) \simeq \mathrm{H}_i(\mathcal{C}; \Lambda)^\vee.$$

However, $C_*(\mathcal{C}; \Lambda)$ need not be the Λ-linear dual of $C^*(\mathcal{C}; \Lambda)$ (if Λ is a field, this is true if and only if each $\mathrm{H}_i(\mathcal{C}; \Lambda)$ is a finite-dimensional vector space).

Warning 2.3.29. Let $\mathcal{C}$ be a prestack. Then $C_*(\mathcal{C}; \mathbf{Q}_\ell)$ is equivalent to $C_*(\mathcal{C}; \mathbf{Z}_\ell)[\ell^{-1}]$, since the process of "inverting ℓ" commutes with colimits. However, it generally does not commute with limits, so that the canonical map

$$C^*(\mathcal{C}; \mathbf{Z}_\ell)[\ell^{-1}] \to C^*(\mathcal{C}; \mathbf{Q}_\ell)$$

is not an equivalence in general.

Remark 2.3.30. Suppose that $\mathcal{C}$ is an algebraic stack which is of finite type over k Then we can present $\mathcal{C}$ by a simplicial scheme $X_\bullet$ where each X_n is an affine scheme of finite type over k. In this case, the canonical map

$$\theta : C^*(\mathcal{C}; \mathbf{Z}_\ell)[\ell^{-1}] \to C^*(\mathcal{C}; \mathbf{Q}_\ell)$$

is an equivalence: it can be identified with the natural map

$$\mathrm{Tot}(C^*(X_\bullet; \mathbf{Z}_\ell))[\ell^{-1}] \to \varinjlim \mathrm{Tot}(C^*(X_\bullet; \mathbf{Z}_\ell)[\ell^{-1}]),$$

and the formation of totalizations commutes with filtered colimits in the ∞-category $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$.

Warning 2.3.31. For every prestack $\mathcal{C}$, we have a canonical equivalence

$$C^*(\mathcal{C}; \mathbf{Z}_\ell) \simeq \varprojlim C^*(\mathcal{C}; \mathbf{Z}/\ell^d\mathbf{Z}).$$

However, the canonical map $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to \varprojlim C_*(\mathcal{C}; \mathbf{Z}/\ell^d\mathbf{Z})$ need not be an equivalence in general, since the formation of colimits generally does not commute with the formation of limits.

Remark 2.3.32. Let $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$. The constructions $\mathcal{C} \mapsto C_*(\mathcal{C}; \Lambda)$ and $\mathcal{C} \mapsto C^*(\mathcal{C}; \Lambda)$ depend functorially on $\mathcal{C}$. More precisely, every weak morphism (see Definition 2.3.18) of prestacks $f : \mathcal{C} \to \mathcal{C}'$ induces pushforward and pullback maps

$$f_\star : C_*(\mathcal{C}; \Lambda) \to C_*(\mathcal{D}; \Lambda) \qquad f^\star : C^*(\mathcal{D}; \Lambda) \to C^*(\mathcal{C}; \Lambda),$$

and every natural transformation $\alpha : f \to g$ (which projects to the identity in Ring_k) determines homotopies $f_\star \simeq g_\star$ and $f^\star \simeq g^\star$. Note that this holds regardless of whether or not α is an isomorphism: the existence of a noninvertible 2-morphism from f to g is enough to guarantee that f and g induce the same map at the level of homology and cohomology.

Remark 2.3.33. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of prestacks which exhibits $\mathcal{D}$ as the stackification of $\mathcal{C}$ with respect to the étale topology (or the Zariski topology, or the fppf topology). Then the induced maps

$$f_\star : C_*(\mathcal{C}; \Lambda) \to C_*(\mathcal{D}; \Lambda) \qquad f^\star : C^*(\mathcal{D}; \Lambda) \to C^*(\mathcal{C}; \Lambda)$$

are equivalences (for $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell\mathbf{Z}\}$). This is a formal consequence of the fact that the construction $X \mapsto C^*(X; \Lambda)$ introduced in §2.2 is a sheaf for the étale topology (Zariski topology, fppf topology; see Variant 2.2.22).

Proposition 2.3.34. *Let R be a finitely generated k-algebra and suppose we are given a map of prestacks $\nu : \mathcal{C} \to \operatorname{Spec} R$. Suppose we are given a faithfully flat morphism of finitely generated k-algebras $\phi : R \to R^0$, and let $R^\bullet$ be the associated cosimplicial ring (so that R^n is given by the $(n+1)$st tensor power of R^0 over R). Then the canonical map*

$$\theta : |C_*(\operatorname{Spec} R^\bullet \times_{\operatorname{Spec} R} \mathcal{C}; \Lambda)| \to C_*(\mathcal{C}; \Lambda)$$

is an equivalence for $\Lambda \in \{\mathbf{Z}/\ell\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$.

Before giving the proof of Proposition 2.3.34, it will be convenient to review of a bit of category theory.

Notation 2.3.35. Let $\mathcal{C}$ be any category, and let $C \in \mathcal{C}$ be an object. We let $\mathcal{C}_{C/}$ denote the category whose objects are morphisms $f : C \to D$ in $\mathcal{C}$, and whose morphisms are given by commutative diagrams

$$\begin{array}{ccc} & C & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ D & \longrightarrow & D'. \end{array}$$

The construction $(f : C \to D) \mapsto D$ determines a forgetful functor $\mathcal{C}_{C/} \to \mathcal{C}$. We will generally abuse notation by not distinguishing between an object of $\mathcal{C}_{C/}$ and its image in $\mathcal{C}$ (in other words, we will simply refer to D as an object of $\mathcal{C}_{C/}$ if the map f is understood).

There is an evident dual construction, which produces a category $\mathcal{C}_{/C}$ whose objects are morphisms $f : D \to C$ in the original category $\mathcal{C}$.

Example 2.3.36. Let $\pi : \mathcal{C} \to \operatorname{Ring}_k$ be a prestack in groupoids. Then for any object $C \in \mathcal{C}$ with $\pi(C) = R \in \operatorname{Ring}_k$, the functor π induces an equivalence of categories

$$\mathcal{C}_{C/} \to (\operatorname{Ring}_k)_{R/} \simeq \operatorname{Ring}_R .$$

In other words, a choice of object $C \in \mathcal{C}(R)$ determines a morphism of prestacks $\operatorname{Spec} R \to \mathcal{C}$ which restricts to an equivalence $\operatorname{Spec} R \to \mathcal{C}_{C/}$.

Remark 2.3.37. We say that a category $\mathcal{C}$ is *weakly contractible* if its nerve $\mathrm{N}(\mathcal{C})_\bullet$ is a weakly contractible simplicial set (in other words, if $\mathcal{C}$ is nonempty and every map from $\mathrm{N}(\mathcal{C})_\bullet$ to a Kan complex X is homotopic to a constant map).

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We will say that F is *left cofinal* if, for every object $D \in \mathcal{D}$, the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible. We say that F is *right cofinal* if, for every object $D \in \mathcal{D}$, the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible. For every ∞-category $\mathcal{E}$ and every functor $G : \mathcal{D} \to \mathcal{E}$, we have canonical maps

$$\phi : \varinjlim_{C \in \mathcal{C}} (G \circ F)(C) \to \varinjlim_{D \in \mathcal{D}} G(D) \qquad \psi : \varprojlim_{D \in \mathcal{D}} G(D) \to \varprojlim_{C \in \mathcal{C}} (G \circ F)(C)$$

(provided that the relevant limits and colimits exist in $\mathcal{E}$). The map ϕ is an equivalence whenever F is left cofinal, and the map ψ is an equivalence whenever F is right cofinal. We refer the reader to §HTT.4.1 for more details.

Note that F is left cofinal if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is right cofinal, and vice-versa. Consequently, for any functor $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$, the canonical map $\varinjlim_{C\in\mathcal{C}}(G \circ F)(C) \to \varinjlim_{D\in\mathcal{D}} G(D)$ is an equivalence when F is right cofinal, and the canonical map $\varprojlim_{D\in\mathcal{D}} G(D) \to \varprojlim_{C\in\mathcal{C}}(G \circ F)(C)$ is an equivalence when F is left cofinal (see Notation 2.1.26).

Remark 2.3.38. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then F admits a right adjoint G if and only if, for each object $D \in \mathcal{D}$, the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ has a final object. In particular, if F admits a right adjoint, then F is right cofinal. Similarly, if F admits a left adjoint, then it is left cofinal.

Remark 2.3.39. Let R be a finitely generated k-algebra and let $\nu : \mathcal{C} \to \operatorname{Spec} R$ be a coCartesian fibration between prestacks. For every finitely generated R-algebra R', the projection map

$$\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{C} \to \mathcal{C}$$

admits a left adjoint F, which carries an object $C \in \mathcal{C}$ to the codomain of a ν-coCartesian morphism $C \to F(C)$ covering the ring homomorphism $\nu(C) \to \nu(C) \otimes_R R'$. Since the functor F admits a right adjoint, it is right cofinal (Remark 2.3.38). We therefore have a canonical equivalence

$$C_*(\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{C}; \Lambda) \simeq \varinjlim_{C\in\mathcal{C}} C_*(\operatorname{Spec}(\nu(C) \otimes_R R'); \Lambda)$$

for $\Lambda \in \{\mathbf{Z}/\ell\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$.

Proof of Proposition 2.3.34. Replacing $\mathcal{C}$ by an equivalent prestack if necessary, we may assume that ν is a coCartesian fibration (see Remark 2.3.20). Using Remark 2.3.39, we can identify θ with a colimit of maps of the form

$$|C_*(\operatorname{Spec}(R^\bullet \otimes_{\operatorname{Spec} R} A); \Lambda)| \to C_*(\operatorname{Spec} A; \Lambda),$$

each of which is a quasi-isomorphism by Proposition 2.2.25. □

Let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell\mathbf{Z}\}$ be a commutative ring. Then Λ-linear duality determines a contravariant equivalence of the symmetric monoidal ∞-category of perfect objects of $\operatorname{Mod}_\Lambda$ with itself. If X is a quasi-projective k-scheme, then we can view $C^*(X; \Lambda)$ as a commutative algebra object of $\operatorname{Mod}_\Lambda$, so that $C_*(X; \Lambda)$ inherits the structure of a commutative coalgebra object of $\operatorname{Mod}_\Lambda$ (see Remark 2.2.11), depending functorially on X. It follows that if $\pi : \mathcal{C} \to \operatorname{Ring}_k$ is a prestack, then the chain complexes

$$C^*(\mathcal{C}; \Lambda) \simeq \varprojlim_{C\in\mathcal{C}} C^*(\operatorname{Spec} \pi(C); \Lambda) \qquad C_*(\mathcal{C}; \Lambda) = \varinjlim_{C\in\mathcal{C}} C_*(\operatorname{Spec} \pi(C); \Lambda)$$

inherit the structure of commutative algebra and coalgebra objects of $\operatorname{Mod}_\Lambda$, respectively. In particular, for every pair of k-prestacks $\mathcal{C}$ and $\mathcal{C}'$, we have a canonical map

$$\begin{aligned} C_*(\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}'; \Lambda) &\to C_*(\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}'; \Lambda) \otimes_\Lambda C_*(\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}'; \Lambda) \\ &\to C_*(\mathcal{C}; \Lambda) \otimes_\Lambda C_*(\mathcal{C}'; \Lambda). \end{aligned}$$

Proposition 2.3.40 (Künneth Formula for Prestacks)**.** *Let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$. Then, for every pair of prestacks $\pi : \mathcal{C} \to \operatorname{Ring}_k$ and $\pi' : \mathcal{C}' \to \operatorname{Ring}_k$, the canonical map*

$$C_*(\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}'; \Lambda) \to C_*(\mathcal{C}; \Lambda) \otimes_\Lambda \mathcal{C}_*(\mathcal{C}'; \Lambda)$$

is an equivalence in $\operatorname{Mod}_\Lambda$.

Proof. We have an evident functor $g : \mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}' \to \mathcal{C} \times \mathcal{C}'$. This functor admits a G admits a left adjoint $f : \mathcal{C} \times \mathcal{C}' \to \mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}'$, given by $(C, C') \mapsto (C_A, C'_A)$, where $A = \pi(C) \otimes_k \pi'(C')$, C_A denotes the image of C under the functor $\mathcal{C}(\pi(C)) \to \mathcal{C}(A)$, and C'_A is defined similarly. Since the functor f admits a right adjoint, it is right cofinal. Combining this observation with Proposition 2.2.20, we obtain equivalences

$$\begin{aligned} C_*(\mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}'; \Lambda) &\simeq \varinjlim_{(C,C') \in \mathcal{C} \times \mathcal{C}'} C_*(\mathrm{Spec}(\pi(C) \otimes_k \pi'(C'); \Lambda) \\ &\simeq \varinjlim_{(C,C') \in \mathcal{C} \times \mathcal{C}'} C_*(\mathrm{Spec}\,\pi(C); \Lambda) \otimes_\Lambda C_*(\mathrm{Spec}\,\pi'(C'); \Lambda) \\ &\simeq (\varinjlim_{C \in \mathcal{C}} C_*(\mathrm{Spec}\,\pi(C); \Lambda)) \otimes_\Lambda (\varinjlim_{C' \in \mathcal{C}'} C_*(\mathrm{Spec}\,\pi'(C'); \Lambda)) \\ &\simeq C_*(\mathcal{C}; \Lambda) \otimes_\Lambda C_*(\mathcal{D}; \Lambda). \end{aligned}$$

□

Remark 2.3.41. One can use Proposition 2.3.40 to show that the construction $\mathcal{C} \mapsto C_*(\mathcal{C}; \Lambda)$ determines a symmetric monoidal functor from the 2-category of prestacks (with symmetric monoidal structure given by the Cartesian product) to the ∞-category Mod_Λ (with symmetric monoidal structure given by tensor product over $\mathbf{Z}_\Lambda$). This observation has several useful consequences:

(1) Suppose that G is a *group-valued* prestack: that is, a prestack in sets equipped with a multiplication map $G \times_{\mathrm{Spec}\,k} G \to G$ which determines a group structure on each of the sets $G(R)$. Then the ℓ-adic chain complex $C_*(G; \Lambda)$ inherits the structure of an associative algebra object of Mod_Λ: that is, it can be viewed as an $\mathbb{E}_1$-algebra over Λ (and is therefore quasi-isomorphic to a differential graded algebra over Λ; see Proposition HA.7.1.4.6).
(2) Let G be a group-valued prestack and let X be a prestack in sets equipped with a (left) action of G. Then the ℓ-adic chain complex $C_*(X; \Lambda)$ inherits the structure of a (left) modue over the algebra $C_*(G; \Lambda)$.
(3) Let G be a group-valued prestacks and suppose we are given prestacks in sets Y and X equipped with right and left actions of G, respectively. Let Z denote the prestack in sets obtained from $Y \times_{\mathrm{Spec}\,k} X$ by dividing out by the diagonal action of G. Then we have a canonical map
$$C_*(Y; \Lambda) \otimes_{C_*(G;\Lambda)} C_*(X; \Lambda) \to C_*(Z; \Lambda).$$
(4) Taking $Y = \mathrm{Spec}\,k$ in (3), we obtain a canonical map
$$\Lambda \otimes_{C_*(G;\Lambda)} C_*(X; \Lambda) \to C_*(G \backslash X; \Lambda).$$
If G acts freely on X, then this map is an equivalence.

For later use, we record the following consequence of Remark 2.3.41:

Corollary 2.3.42. *Let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$, let $\iota : G_0 \hookrightarrow G$ be a monomorphism between group-valued prestacks, and suppose that that the canonical map $G/G_0 \to \mathrm{Spec}\,k$ induces an isomorphism*
$$\mathrm{H}_*(G/G_0; \Lambda) \to \mathrm{H}_*(\mathrm{Spec}\,k; \Lambda) \simeq \Lambda.$$
Then:

(a) *If ι induces an isomorphism $\mathrm{H}_0(G_0; \Lambda) \to \mathrm{H}_0(G; \Lambda)$, then it induces an isomorphism $\mathrm{H}_*(G_0; \Lambda) \simeq \mathrm{H}_*(G; \Lambda)$.*

(*b*) *If the induced map* $\mathrm{H}_0(G_0;\Lambda) \to \mathrm{H}_0(G;\Lambda)$ *factors through the augmentation*

$$\mathrm{H}_0(G_0;\Lambda) \to \mathrm{H}_0(\operatorname{Spec} k;\Lambda) \simeq \Lambda,$$

then $\mathrm{H}_0(G;\Lambda) \simeq \Lambda$ *(that is, G is connected).*

Proof. We first prove (*a*). Remark 2.3.41 supplies an equivalence

$$\Lambda \otimes_{C_*(G_0;\Lambda)} C_*(G;\Lambda) \simeq \Lambda.$$

Consequently, if K denotes the cofiber of the canonical map $C_*(G_0;\Lambda) \to C_*(G;\Lambda)$, then we have $K \otimes_{C_*(G_0;\Lambda)} C_*(G;\Lambda) \simeq 0$. If $K \neq 0$, then there exists some smallest integer m such that $\mathrm{H}_m(K) \neq 0$. In this case we obtain an isomorphism

$$\mathrm{Tor}_0^{\mathrm{H}_0(G_0;\Lambda)}(\mathrm{H}_m(K), \mathrm{H}_0(G;\Lambda)) \simeq \mathrm{H}_m(K \otimes_{C_*(G_0;\Lambda)} C_*(G;\Lambda)) \simeq 0$$

(see Corollary HA.7.2.1.23). If $\mathrm{H}_0(G_0;\Lambda) \simeq \mathrm{H}_0(G;\Lambda)$, we conclude that $\mathrm{H}_m(K) \simeq 0$, contrary to our assumption on m.

We now prove (*b*). Note that Corollary HA.7.2.1.23 also supplies an isomorphism

$$\mathrm{Tor}_0^{\mathrm{H}_0(G_0;\Lambda)}(\Lambda, \mathrm{H}_0(G;\Lambda)) \simeq \mathrm{H}_0(G/G_0;\Lambda) \simeq \Lambda.$$

In other words, the augmentation ideal of $\mathrm{H}_0(G;\Lambda)$ is generated by the image of the augmentation ideal of $\mathrm{H}_0(G_0;\Lambda)$. In particular, if the map $\mathrm{H}_0(G_0;\Lambda) \to \mathrm{H}_0(G;\Lambda)$ annihilates the augmentation ideal of $\mathrm{H}_0(G_0;\Lambda)$, then the augmentation $\mathrm{H}_0(G;\Lambda) \to \Lambda$ is an isomorphism. □

For every pair of objects $M, N \in \mathrm{Mod}_\Lambda$, we have an evident map $M^\vee \otimes_\Lambda N^\vee \to (M \otimes_\Lambda N)^\vee$. This map is an equivalence whenever either M or N is perfect, but not in general. Combining this observation with Proposition 2.3.40, we obtain:

Corollary 2.3.43. *Let* $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$. *Let* $\mathcal{C}$ *be a prestack for which* $C_*(\mathcal{C};\Lambda) \in \mathrm{Mod}_\Lambda$ *is perfect. Then, for every prestack* $\mathcal{C}'$, *the canonical map*

$$C^*(\mathcal{C};\Lambda) \otimes_\Lambda C^*(\mathcal{C}';\Lambda) \to C^*(\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}';\Lambda)$$

is an equivalence.

Warning 2.3.44. The analogous Künneth formula does not necessarily hold for cochain complexes $C^*(\mathcal{C};\Lambda)$, because in general the formation of tensor products does not distribute over inverse limits. This is one reason that it will be convenient for us to work with the homology of prestacks.

2.4. **Acyclicity of the Ran Space.**

2.4.1. *The Ran Space in Topology.* Let M be a topological space, and let $\mathrm{Ran}(M)$ denote the collection of all nonempty finite subsets of M. For every collection of disjoint open sets $U_1, \ldots, U_M \subseteq M$, let $\mathrm{Ran}(U_1, \ldots, U_m)$ denote the subset of $\mathrm{Ran}(M)$ consisting of those nonempty finite sets $S \subseteq M$ satisfying

$$S \subseteq U_1 \cup \cdots \cup U_m \qquad S \cap U_1 \neq \emptyset \qquad \cdots \qquad S \cap U_m \neq \emptyset.$$

The collection of sets $\mathrm{Ran}(U_1, \ldots, U_m)$ form a basis for a topology on $\mathrm{Ran}(M)$. Following [9], we will refer to $\mathrm{Ran}(M)$ as the *Ran space* of M.

Remark 2.4.1. Suppose that the topology on M is defined by a metric d. Then the topology on $\mathrm{Ran}(M)$ is also defined by a metric, where the distance from a nonempty finite set $S \subseteq M$ to another nonempty finite set $T \subseteq M$ is given by

$$\max\{\max_{s \in S} \min_{t \in T} d(s,t), \max_{t \in T} \min_{s \in S} d(s,t)\}.$$

2.4.2. *The Ran Space in Algebraic Geometry.* In this section, we will study an analogue of the construction $M \mapsto \mathrm{Ran}(M)$ in the setting of algebraic geometry. Fix an algebraically closed field k and a prime number ℓ which is invertible in k. For every quasi-projective k-scheme X, we will define an algebro-geometric object $\mathrm{Ran}^u(X)$ which parametrizes nonempty finite subsets of X. Since the collection of such points is "infinite-dimensional", it is unreasonable to expect to realize $\mathrm{Ran}^u(X)$ as a k-scheme: instead, we will describe it as a prestack over k, whose R-valued points are nonempty finite subsets of $X(R)$:

Definition 2.4.2. Let X be a quasi-projective k-scheme. We define a category $\mathrm{Ran}^u(X)$ as follows:

(1) The objects of $\mathrm{Ran}^u(X)$ are pairs (R, S) where R is a finitely generated k-algebra and S is a nonempty finite subset of $X(R)$.
(2) A morphism from (R, S) to (R', S') in $\mathrm{Ran}^u(X)$ is a k-algebra homomorphism $\phi : R \to R'$ having the property that S' is the image of the induced map $S \subseteq X(R) \stackrel{X(\phi)}{\to} X(R')$.

The construction $(R, S) \mapsto R$ determines a forgetful functor $\mathrm{Ran}^u(X) \to \mathrm{Ring}_k$. It is easy to see that this functor is a coCartesain fibration, so that we can regard $\mathrm{Ran}^u(X)$ as a prestack. We will refer to $\mathrm{Ran}^u(X)$ as the *unlabelled Ran space of* X.

Remark 2.4.3. We can regard $\mathrm{Ran}^u(X)$ as obtained by performing the Grothendieck construction (Example 2.3.7) using the set-valued functor which assigns to each $R \in \mathrm{Ring}_k$ the set of all nonempty finite subsets of $X(R)$. In particular, it is a prestack in groupoids.

Warning 2.4.4. The prestack $\mathrm{Ran}^u(X)$ usually not a stack for the étale topology. For example, suppose that $X = \mathrm{Spec}\, R$ is an affine k-scheme equipped with a free action of a finite group Γ. Every element $\gamma \in \Gamma$ determines an automorphism of X, which we can regard as an R-valued point of X. Then $(R, \{\gamma\}_{\gamma \in \Gamma})$ is an R-valued point of the prestack $\mathrm{Ran}^u(X)$ which is invariant under the action of Γ. However, this point does not *descend* to an R^Γ-valued point of $\mathrm{Ran}^u(X)$ unless we can choose a section of the quotient map $X \to X/\Gamma$.

We can now state the main result of this section:

Theorem 2.4.5 (Beilinson-Drinfeld). *Suppose that $X \in \mathrm{Sch}_k$ is connected, and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$. Then the canonical map $\mathrm{Ran}^u(X) \to \mathrm{Spec}\, k$ induces a quasi-isomorphism*

$$C_*(\mathrm{Ran}^u(X); \Lambda) \to C_*(\mathrm{Spec}\, k; \Lambda).$$

In other words, we have canonical isomorphisms

$$\mathrm{H}_*(\mathrm{Ran}^u(X); \Lambda) \simeq \begin{cases} \Lambda & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.4.6. For any prestack $\mathcal{C}$, we have a fiber sequence

$$C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell) \stackrel{\ell}{\to} C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell) \to C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}/\ell\mathbf{Z}).$$

If $C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}/\ell\mathbf{Z})$ is acyclic, then multiplication by ℓ induces a quasi-isomorphism from the chain complex $C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell)$ to itself. In this case, we have $C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell) \simeq C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Q}_\ell)$. Consequently, to prove Theorem 2.4.5, it will suffice to treat the special cases where $\Lambda \in \{\mathbf{Q}_\ell, \mathbf{Z}/\ell\mathbf{Z}\}$ is a field.

Notation 2.4.7. For any prestack $\mathcal{C}$, we let $C_*^{\mathrm{red}}(\mathcal{C}; \Lambda)$ denote the fiber of the canonical map $C_*(\mathcal{C}; \Lambda) \to C_*(\mathrm{Spec}\, k; \Lambda)$. We will refer to $C_*^{\mathrm{red}}(\mathcal{C}; \Lambda)$ as the *reduced chain complex of* $\mathcal{C}$ with coefficients in Λ. Theorem 2.4.5 is equivalent to the assertion that the chain complex $C_*^{\mathrm{red}}(\mathrm{Ran}^u(X); \Lambda)$ is acyclic.

We will prove Theorem 2.4.5 by reducing to the following special case, which we will establish at the end of this section:

Proposition 2.4.8. *Suppose that $X \in \mathrm{Sch}_k$ is connected. Then the map $\mathrm{Ran}^u(X) \to \operatorname{Spec} k$ induces an isomorphism of abelian groups $\mathrm{H}_0(\mathrm{Ran}^u(X); \Lambda) \to \mathrm{H}_0(\operatorname{Spec} k; \Lambda) \simeq \Lambda$.*

Proof of Theorem 2.4.5. Using Remark 2.4.6, we may assume that Λ is a field (though this reduction is not really essential). By virtue of Proposition 2.4.8, it will suffice to show that $\mathrm{H}_n(\mathrm{Ran}^u(X); \Lambda) \simeq 0$ for $n > 0$. Proceeding by induction on n, we may assume without loss of generality that $\mathrm{H}_i(\mathrm{Ran}^u(X); \Lambda) \simeq 0$ for $0 < i < n$. Set $V = \mathrm{H}_n(\mathrm{Ran}^u(X); \Lambda)$. Using Propositions 2.4.8 and 2.3.40, we obtain an isomorphism

$$\mathrm{H}_n(\mathrm{Ran}^u(X) \times_{\operatorname{Spec} k} \mathrm{Ran}^u(X); \Lambda) \simeq V \oplus V.$$

We have an evident "multiplication" map $m : \mathrm{Ran}^u(X) \times_{\operatorname{Spec} k} \mathrm{Ran}^u(X) \to \mathrm{Ran}^u(X)$, given on objects by the formula

$$((R, S), (R, S')) \mapsto (R, S \cup S').$$

Passing to homology, we obtain a map $\lambda : V \oplus V \to V$, which we can identify with a pair of maps $\lambda_1, \lambda_2 : V \to V$. By symmetry, we have $\lambda_1 = \lambda_2$. Note that the composite map

$$\mathrm{Ran}^u(X) \xrightarrow{\delta} \mathrm{Ran}^u(X) \times_{\operatorname{Spec} k} \mathrm{Ran}^u(X) \xrightarrow{m} \mathrm{Ran}^u(X)$$

is the identity. From this, we deduce that $v = \lambda(v, v) = \lambda_1(v) + \lambda_2(v) = 2\lambda_2(v)$ for $v \in V$.

Choose a k-rational point $x \in X$. Then $\{x\}$ can be identified with a k-rational point of $\mathrm{Ran}^u(X)$: that is, with a map of prestacks $\iota : \operatorname{Spec} k \to \mathrm{Ran}^u(X)$. Let F denote the composite map

$$\mathrm{Ran}^u(X) \simeq \operatorname{Spec} k \times_{\operatorname{Spec} k} \mathrm{Ran}^u(X) \xrightarrow{(\iota, \mathrm{id})} \mathrm{Ran}^u(X) \times_{\operatorname{Spec} k} \mathrm{Ran}^u(X) \xrightarrow{m} \mathrm{Ran}^u(X).$$

Note that the point x determines a map $\operatorname{Spec} k \to \mathrm{Ran}^u(X)$ which induces an isomorphism $\Lambda \simeq \mathrm{H}^0(\operatorname{Spec} k; \Lambda) \simeq \mathrm{H}^0(\mathrm{Ran}^u(X); \Lambda)$. It follows that on homology, F induces the map from V to V given by $v \mapsto \lambda_2(v)$. Since $F^2 = F$, we have

$$2\lambda_2(v) = 2\lambda_2(\lambda_2(v)) = \lambda_2(v),$$

so that $\lambda_2(v) = 0$ and therefore $v = 2\lambda_2(v) = 0$. Since this is true for all $v \in V$, we conclude that $V \simeq 0$. ☐

To execute the proof of Proposition 2.4.8 (and many other arguments throughout this paper), it will be convenient to work with a slight variant of $\mathrm{Ran}^u(X)$, whose points are *parametrized* finite subsets of X.

Definition 2.4.9. Let X be a quasi-projective k-scheme. We define a category $\mathrm{Ran}(X)$ as follows:

- An object of $\mathrm{Ran}(X)$ is a triple (R, S, μ) where R is a finitely generated k-algebra, S is a nonempty finite set, and $\mu : S \to X(R)$ is a map of sets.
- A morphism from (R, S, μ) to (R', S', μ') in $\mathrm{Ran}(X)$ consists of a k-algebra homomorphism $\phi : R \to R'$ together with a surjection of finite sets $S \to S'$ for which the diagram

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R') \end{array}$$

commutes.

It is easy to see that the forgetful functor $(R, S, \mu) \mapsto R$ determines an coCartesain fibration $\mathrm{Ran}(X) \to \mathrm{Ring}_k$, so that we can regard $\mathrm{Ran}(X)$ as a prestack.

Remark 2.4.10. More informally, we can describe $\mathrm{Ran}(X)$ as the prestack which assigns to each finitely generated k-algebra R the category whose objects are pairs (S, μ), where S is a nonempty finite set and $\mu : S \to X(R)$ is a map of sets.

Warning 2.4.11. The prestack $\pi : \mathrm{Ran}(X) \to \mathrm{Ring}_k$ is not a stack in groupoids. A morphism $(R, S, \mu) \to (R', S', \mu')$ in $\mathrm{Ran}(X)$ is π-coCartesian if and only if the map of finite sets $S \to S'$ is bijective.

Let us now describe the relationship between $\mathrm{Ran}(X)$ and $\mathrm{Ran}^u(X)$. We have an evident functor $F : \mathrm{Ran}(X) \to \mathrm{Ran}^u(X)$, which carries a map of $\mu : S \to X(R)$ to its image $\mu(S) \subseteq X(R)$. We can regard F as a map of prestacks from $\mathrm{Ran}(X)$ to $\mathrm{Ran}^u(X)$. The functor F admits a right adjoint G, which carries a subset $T \subseteq X(R)$ to the inclusion map $\iota : T \to X(R)$. The functor G is a *weak* morphism of prestacks (in the sense of Definition 2.3.18), but does not preserve coCartesian morphisms (since a map of k-algebras $R \to R'$ need not induce an injective map $X(R) \to X(R')$). Nevertheless, Remark 2.3.32 yields the following:

Lemma 2.4.12. *Let $X \in \mathrm{Sch}_k$ and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$. Then the canonical map $\mathrm{Ran}(X) \to \mathrm{Ran}^u(X)$ induces an isomorphism*

$$\mathrm{H}_*(\mathrm{Ran}(X); \Lambda) \to \mathrm{H}_*(\mathrm{Ran}^u(X); \Lambda).$$

Proof of Proposition 2.4.8. By virtue of Lemma 2.4.12, it will suffice to show that the canonical map $\mathrm{H}_0(\mathrm{Ran}(X); \Lambda) \to \mathrm{H}_0(\mathrm{Spec}\, k; \Lambda)$ is an isomorphism.

Let $\mathrm{Fin}^{\mathrm{s}}$ denote the category whose objects are nonempty finite sets and whose morphisms are surjections. The construction $(R, S, \mu) \mapsto S$ determines a Cartesian fibration of categories $\phi : \mathrm{Ran}(X) \to \mathrm{Fin}^{\mathrm{s}}$ (see Remark 2.3.5), whose fiber over an object $S \in \mathrm{Fin}^{\mathrm{s}}$ can be identified with X^S (which we regard as a prestack). We therefore obtain

$$C_*(\mathrm{Ran}(X); \Lambda) \simeq \varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}\,\mathrm{op}}} C_*(X^S; \Lambda)$$

in the ∞-category Mod_Λ.

Each of the chain complexes $C_*(X^S; \Lambda)$ is *connective*: that is, its homologies are concentrated in non-negative degrees. It follows that we can identify $\mathrm{H}_0(\mathrm{Ran}(X); \Lambda)$ with the direct limit $\varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}\,\mathrm{op}}} \mathrm{H}_0(X^S; \Lambda)$, computed in the ordinary category of abelian groups. Since X is connected, the construction $S \mapsto \mathrm{H}_0(X^S; \Lambda)$ is equivalent to the constant functor taking the value Λ. The category $\mathrm{Fin}^{\mathrm{s}}$ has weakly contractible nerve (since it has a final object), so that the colimit $\varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathrm{H}_0(X^S; \Lambda)$ is also isomorphic to Λ. □

Corollary 2.4.13 (Acyclicity of the Ran Space)**.** *If $X \in \mathrm{Sch}_k$ is connected, then the reduced chain complex $C_*^{red}(\mathrm{Ran}(X); \Lambda)$ is acyclic for $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$.*

Proof. Combine Lemma 2.4.12 with Theorem 2.4.5. □

2.5. **Universal Homology Equivalences.** Fix an algebraically closed field k and a prime number ℓ which is invertible in k. At many points in this paper, we will need to argue that a (weak) morphism of prestacks $f : \mathcal{C} \to \mathcal{D}$ induces an isomorphism of ℓ-adic homology groups $f_\star : \mathrm{H}_*(\mathcal{C}; \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathcal{D}; \mathbf{Z}_\ell)$ (and therefore also an isomorphism $f^\star : \mathrm{H}^*(\mathcal{D}; \mathbf{Z}_\ell) \to \mathrm{H}^*(\mathcal{C}; \mathbf{Z}_\ell)$, by duality). We have already encountered several formal conditions on f which guarantee this:

- The map $f_\star$ is an isomorphism if f is right cofinal (see Remark 2.3.37).
- The map $f_\star$ is an isomorphism if f admits an adjoint (on either side) which is compatible with the projection to Ring_k (this is an immediate consequence of Remark 2.3.32).

- The map $f_\star$ is an isomorphism if f induces an equivalence after stackification with respect to the fppf topology (Remark 2.3.33).

In this section, we will study another class of morphisms f which induce isomorphisms on ℓ-adic cohomology, which we call *universal homology equivalences.*

Definition 2.5.1. Suppose we are given a morphism of prestacks

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & \mathrm{Ring}_k . & \end{array}$$

We say that F is a *universal homology equivalence* if, for each object $D \in \mathcal{D}$, the canonical map

$$\varinjlim_{C \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}} C_*(\operatorname{Spec} \pi(C); \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} \pi'(D); \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 2.5.2. In the situation of Definition 2.5.1, suppose that the functor $F : \mathcal{C} \to \mathcal{D}$ is a Cartesian fibration. Then, for each object $D \in \mathcal{D}$, the inclusion

$$\mathcal{C} \times_{\mathcal{D}} \{D\} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$$

is right cofinal. Consequently, π is a universal homological equivalence if and only if, for each object $D \in \mathcal{D}$, the canonical map

$$\varinjlim_{\pi(C)=D} C_*(\operatorname{Spec} \pi(C); \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} \pi'(D); \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 2.5.3. In the situation of Definition 2.5.1, suppose that F is a universal homology equivalence. It follows immediately that for each object $D \in \mathcal{D}$, the canonical maps

$$\varinjlim_{C \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}} C_*(\operatorname{Spec} \pi(C); \Lambda) \to C_*(\operatorname{Spec} \pi'(D); \Lambda)$$

$$\varprojlim_{C \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}} C^*(\operatorname{Spec} \pi(C); \Lambda) \to C^*(\operatorname{Spec} \pi'(D); \Lambda)$$

are equivalences in Mod_Λ, for $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$.

Example 2.5.4. In the situation of Definition 2.5.1, we can identify the direct limit

$$\varinjlim_{C \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}} C_*(\operatorname{Spec} \pi(C); \mathbf{Z}_\ell)$$

with the complex of $\mathbf{Z}_\ell$-chains on the prestack $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$.

Suppose that $\mathcal{D}$ is a prestack in groupoids, and let $D \in \mathcal{D}$ be an object with $\pi'(D) = R$. Then π' induces an equivalence $\mathcal{D}_{D/} \simeq \mathrm{Ring}_R$ (Example 2.3.36). We may therefore identify the forgetful functor $\mathcal{D}_{D/} \to \mathcal{D}$ with a map $\operatorname{Spec} R \to \mathcal{D}$. In this case, Definition 2.5.1 requires that the canonical map

$$C_*(\mathcal{C} \times_{\mathcal{D}} \operatorname{Spec} R; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$$

be an equivalence.

Remark 2.5.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (small) ∞-categories. Then composition with F induces a functor $F^* : \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell})$. Under some mild set-theoretic hypotheses on $\mathcal{C}$ and $\mathcal{D}$, one can show that the functor F^* admits a left adjoint $F_! : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell}) \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell})$. Concretely, the functor $F_!$ is given by the formula

$$F_!(\mathcal{F})(D) = \varinjlim_{C \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}} \mathcal{F}(C).$$

We refer to $F_!(\mathcal{F})$ as a *left Kan extension of F along D*.

Suppose now that F is a map of prestacks

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {}_{\pi}\searrow \quad \swarrow_{\pi'} & \\ & \mathrm{Ring}_k . & \end{array}$$

Define functors $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ring}_k$, $\mathcal{F}' : \mathcal{D}^{\mathrm{op}} \to \mathrm{Ring}_k$ by the formulae

$$\mathcal{F}(C) = C_*(\operatorname{Spec} \pi(C); \mathbf{Z}_\ell) \qquad \mathcal{F}'(D) = C_*(\operatorname{Spec} \pi'(D); \mathbf{Z}_\ell).$$

Then $\mathcal{F} = F^* \mathcal{F}'$, so we obtain a canonical map $\alpha : F_! \mathcal{F} \to \mathcal{F}'$. The map F is a universal homology equivalence if and only if the natural transformation α is an equivalence.

Remark 2.5.6. Suppose that $F : \mathcal{C} \to \mathcal{E}$ is a Cartesian fibration between categories. Then, for every object $E \in \mathcal{E}$, the inclusion functor

$$\mathcal{C}_E = \mathcal{C} \times_{\mathcal{E}} \{E\} \hookrightarrow \mathcal{C} \times_{\mathcal{E}} \mathcal{E}_{E/}$$

is right cofinal. It follows that for each object $\mathcal{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell})$, the left Kan extension $F_!(\mathcal{F})$ is given by the formula $F_!(\mathcal{F})(E) = \varinjlim_{C \in \mathcal{C}_E} \mathcal{F}(C)$.

Remark 2.5.7. Suppose we are given a map of prestacks $F : \mathcal{C} \to \mathcal{D}$, and auxiliary category $\mathcal{E}$, and a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {}_{G}\searrow \quad \swarrow_{H} & \\ & \mathcal{E} & \end{array}$$

where the vertical maps are Cartesian fibrations. Let $\alpha : F_! \mathcal{F} \to \mathcal{F}'$ be as in Remark 2.5.5. Then F induces a map on $\mathbf{Z}_\ell$-valued chains $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathcal{D}; \mathbf{Z}_\ell)$ which is given by the composition

$$\begin{aligned} C^*(\mathcal{C}; \mathbf{Z}_\ell) &\simeq \varinjlim_{E \in \mathcal{E}} (G_! \mathcal{F})(E) \\ &\simeq \varinjlim_{E \in \mathcal{E}} H_!(F_! \mathcal{F}) \\ &\overset{F_!(\alpha)}{\to} \varinjlim_{E \in \mathcal{E}} (H_! \mathcal{F}')(E) \\ &\simeq C^*(\mathcal{D}; \mathbf{Z}_\ell) \end{aligned}$$

By virtue of Remark 2.5.6, to show that F induces an isomorphism on $\mathbf{Z}_\ell$-homology, it will suffice to show that the induced map

$$\varinjlim_{C \in \mathcal{C}_E} \mathcal{F}(C) \to \varinjlim_{D \in \mathcal{D}_E} \mathcal{F}'(D)$$

is an equivalence for each object $E \in \mathcal{E}$.

The construction which carries a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ to its colimit $\varinjlim_{C \in \mathcal{C}} \mathcal{F}(C) \in \mathrm{Mod}_{\mathbf{Z}_\ell}$ can be regarded as a special kind of left Kan extension: namely, left Kan extension along the projection map $\mathcal{C} \to \Delta^0$. Invoking the transitivity of the formation of left Kan extensions, we obtain the following:

Proposition 2.5.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be a universal homology equivalence of prestacks. Then F induces an equivalence $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathcal{D}; \mathbf{Z}_\ell)$.*

Corollary 2.5.9. *Let $u : \mathcal{C} \to \mathcal{D}$ be a universal homology equivalence of prestacks. Then u induces equivalences*

$$\begin{array}{ccc} C_*(\mathcal{C}; \mathbf{Z}/\ell^d\mathbf{Z}) \to C_*(\mathcal{D}; \mathbf{Z}/\ell^d\mathbf{Z}) & C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathcal{D}; \mathbf{Z}_\ell) & C_*(\mathcal{C}; \mathbf{Q}_\ell) \to C_*(\mathcal{D}; \mathbf{Q}_\ell) \\ C^*(\mathcal{D}; \mathbf{Z}/\ell^d\mathbf{Z}) \to C^*(\mathcal{C}; \mathbf{Z}/\ell^d\mathbf{Z}) & C^*(\mathcal{D}; \mathbf{Z}_\ell) \to C^*(\mathcal{C}; \mathbf{Z}_\ell) & C^*(\mathcal{D}; \mathbf{Q}_\ell) \to C^*(\mathcal{C}; \mathbf{Q}_\ell). \end{array}$$

Remark 2.5.10. The collection of universal homology equivalences is closed under composition. This follows immediately from the characterization of universal homology equivalences supplied by Remark 2.5.5.

Proposition 2.5.11. *Suppose we are given a pullback diagram of prestacks*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & \mathcal{D}' \\ \downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

where u is a coCartesian fibration. If f is a universal homology equivalence, then f' is a universal homology equivalence.

Proof. To prove this, it suffices to show that for every object $D' \in \mathcal{D}'$ having image $D \in \mathcal{D}$, the canonical map $\mathcal{C}' \times_{\mathcal{D}'} \mathcal{D}'_{D'/} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is right cofinal. In fact, this functor has a right adjoint, by virtue of our assumption that $\mathcal{D}' \to \mathcal{D}$ is a coCartesian fibration. □

Corollary 2.5.12. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of prestacks, and suppose that $\mathcal{D}$ is a prestack in groupoids. The following conditions are equivalent:*

(1) *The morphism f is a universal homology equivalence.*
(2) *For every homotopy pullback diagram*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{D}' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle u} \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

in the 2-category of prestacks, the morphism f' is a universal homology equivalence.

(3) *For every homotopy pullback diagram*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{D}' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle u} \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

in the 2-category of prestacks, the morphism f' induces an isomorphism of ℓ-adic homology groups

$$f'_\star : \mathrm{H}_*(\mathcal{C}'; \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathcal{D}'; \mathbf{Z}_\ell)$$

.

(4) *For every map* $\eta : \operatorname{Spec} R \to \mathcal{D}$*, the induced map*
$$\mathrm{H}_*(\operatorname{Spec} R \times_{\mathcal{D}} \mathcal{C}; \mathbf{Z}_\ell) \to \mathrm{H}_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$$
is an equivalence.

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 2.5.11 and Remark 2.3.20. The implication (2) $\Rightarrow$ (3) follows from Proposition 2.5.8. The implication (3) $\Rightarrow$ (4) is immediate, and the equivalence (1) $\Leftrightarrow$ (4) follows from Example 2.3.36. $\square$

Example 2.5.13. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack. Then π is a universal homology equivalence (when regarded as a morphism from $\mathcal{C}$ to $\operatorname{Spec} k$ in the 2-category of prestacks) if and only if it induces an isomorphism
$$\mathrm{H}_*(\mathcal{C}; \mathbf{Z}_\ell) \to \mathrm{H}_*(\operatorname{Spec} k; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell.$$
The "only if" direction is clear, and the converse follows from Proposition 2.3.40.

Example 2.5.14. Let X be a k-scheme of finite type, and let $\mathcal{E}$ be a vector bundle over X. Then the projection map $\mathcal{E} \to X$ is a universal homology equivalence. To prove this, we must show that for every map $\operatorname{Spec} R \to X$, the induced map
$$C_*(\operatorname{Spec} R \times_X \mathcal{E}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$$
is an equivalence. By virtue of Proposition 2.3.34, this assertion can be tested locally with respect to the Zariski topology on $\operatorname{Spec} R$. We may therefore reduce to the case where $\operatorname{Spec} R \times_X \mathcal{E} \simeq \operatorname{Spec} R \times_{\operatorname{Spec} k} \mathbf{A}^n$. Using Proposition 2.3.40, we are reduced to proving that $C_*(\mathbf{A}^1; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$, which follows from our assumption that ℓ is invertible in the field k.

Proposition 2.5.15. *Let R be a finitely generated k-algebra, and suppose we are given a diagram of prestacks σ :*
$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} R. & \end{array}$$
Let R' be a finitely generated R-algebra for which the map $\operatorname{Spec} R' \to \operatorname{Spec} R$ is faithfully flat. If the map $\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{C} \to \operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{D}$ is a universal homology equivalence, then the map $\mathcal{C} \to \mathcal{D}$ is a universal homology equivalence.

Proof. Fix an object $D \in \mathcal{D}$ whose image in $\operatorname{Spec} R$ is a finitely generated R-algebra A, and let $\mathcal{E}$ denote the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$. We wish to prove that the canonical map $\mathcal{E} \to \operatorname{Spec} A$ induces an equivalence $C_*(\mathcal{E}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$. Let $R^\bullet$ denote the cosimplicial R-algebra determined by R' (so that R^n is the $(n+1)$st tensor power of R' over R) and set $A^\bullet = R^\bullet \otimes_R A$. We have a commutative diagram
$$\begin{array}{ccc} |C_*(\operatorname{Spec} R^\bullet \times_{\operatorname{Spec} R} \mathcal{E}; \mathbf{Z}_\ell)| & \longrightarrow & |C_*(\operatorname{Spec} A^\bullet; \mathbf{Z}_\ell)| \\ \downarrow & & \downarrow \\ C_*(\mathcal{E}; \mathbf{Z}_\ell) & \longrightarrow & C_*(\operatorname{Spec} R; \mathbf{Z}_\ell), \end{array}$$
where the vertical maps are equivalences by virtue of Proposition 2.3.34. We are therefore reduced to proving that each of the maps $C_*(\operatorname{Spec} R^n \times_{\operatorname{Spec} R} \mathcal{E}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A^n; \mathbf{Z}_\ell)$ is an equivalence, which follows from our assumption that the map $\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{C} \to \operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{D}$ is a universal homology equivalence. $\square$

We now describe some examples of universal homology equivalences which are related to the constructions of §2.4.

Construction 2.5.16. Let X be a quasi-projective k-scheme. We define a category $\mathrm{Ran}^+(X)$ as follows:

- The objects of $\mathrm{Ran}^+(X)$ are pairs (R, S) where R is a finitely generated k-algebra, and S is a (possibly empty) finite subset of $X(R)$.
- A morphism from (R, S) to (R', S') is a k-algebra homomorphism $\phi : R \to R'$ for which the induced map $X(R) \to X(R')$ carries S to a subset of S'.

Put more informally: the category $\mathrm{Ran}^+(X)$ is defined in the same way as the category $\mathrm{Ran}^u(X)$, except that we do not require our subsets of X to be nonempty or the maps between them to be surjective. Note that we can regard $\mathrm{Ran}^u(X)$ as a (non-full) subcategory of $\mathrm{Ran}^+(X)$.

Remark 2.5.17. Heuristically, we can think of the prestack $\mathrm{Ran}^+(X)$ as a geometric object obtained from $\mathrm{Ran}(X)$ by adding a new point (corresponding to the empty set) and drastically modifying its topology, so that a finite subset $S \subseteq X$ lies in the closure of a finite subset $S' \subseteq X$ whenever $S' \subseteq S$. In particular, the empty subset $\emptyset \subseteq X$ can be regarded as a "generic point" of $\mathrm{Ran}^+(X)$.

Remark 2.5.18. Let X be a quasi-projective k-scheme. The empty set $\emptyset \subseteq X(k)$ determines a k-valued point $\operatorname{Spec} k \to \mathrm{Ran}^+(X)$, which is a section of the projection map $\mathrm{Ran}^+(X) \to \operatorname{Spec} k$. It follows immediately from the definitions that these morphisms are adjoint (in the 2-category of prestacks), and therefore induce mutually inverse isomorphisms on the level of ℓ-adic homology and cohomology (Remark 2.3.32). In particular, we obtain isomorphisms

$$\mathrm{H}_*(\mathrm{Ran}^+(X); \mathbf{Z}_\ell) \simeq \begin{cases} \mathbf{Z}_\ell & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.5.19. *Let X be a quasi-projective k-scheme. If X is connected, then the maps*

$$\mathrm{Ran}(X) \to \mathrm{Ran}^u(X) \to \mathrm{Ran}^+(X)$$

are universal homology equivalences.

Remark 2.5.20. If $X \in \mathrm{Sch}_k$ is connected, then Theorem 2.5.19 and Remark 2.5.18 supply isomorphisms

$$\mathrm{H}_*(\mathrm{Ran}(X); \mathbf{Z}_\ell) \simeq \mathrm{H}_*(\mathrm{Ran}^u(X); \mathbf{Z}_\ell) \simeq \begin{cases} \mathbf{Z}_\ell & \text{if } * = 0 \\ 0 & \text{otherwise,} \end{cases}$$

which is the content of Theorem 2.4.5 (and Corollary 2.4.13). However, this result does not come for free: we will use Theorem 2.4.5 in our proof of Theorem 2.5.19.

Example 2.5.21. Let us outline an application of Theorem 2.5.19 which we will need in §7.7. Let X be a connected quasi-projective k-scheme and suppose we are given nonempty closed subscheme $Y \subseteq X$. Let $\mathrm{Ran}^+(X)'$ denote the full subcategory of $\mathrm{Ran}^+(X)$ consisting of those pairs $(R, S \subseteq X(R))$ having the property that for every k-valued point $\eta \in \operatorname{Spec} R$, the image of S in $X(k)$ has nonempty intersection with $Y(k)$. Let $\mathrm{Ran}(X)'$ denote the inverse image of $\mathrm{Ran}^+(X)'$ in $\mathrm{Ran}(X)$. We claim that the inclusion maps

$$u : \mathrm{Ran}(X)' \hookrightarrow \mathrm{Ran}(X) \qquad u^+ : \mathrm{Ran}^+(X)' \hookrightarrow \mathrm{Ran}^+(X)$$

induce isomorphisms on $\mathbf{Z}_\ell$-homology. Note first that the inclusion $\mathrm{Ran}^+(X)' \hookrightarrow \mathrm{Ran}^+(X)$ is a coCartesian fibration (since the condition that the image of a map $S \to X(k)$ intersects $Y(k)$ is stable under enlarging S). Applying Proposition 2.5.11 to the pullback diagram

$$\begin{array}{ccc} \mathrm{Ran}(X)' & & \mathrm{Ran}^+(X)' \\ \downarrow u & & \downarrow u^+ \\ \mathrm{Ran}(X) & \longrightarrow & \mathrm{Ran}^+(X), \end{array}$$

we deduce from Theorem 2.5.19 that horizontal arrows are universal homology equivalences. Consequently, to prove that u induces an isomorphism on $\mathbf{Z}_\ell$-homology, it will suffice to show that u^+ induces an isomorphism on $\mathbf{Z}_\ell$-homology. To prove this, choose a point $y \in Y(k)$, and for each $R \in \mathrm{Ring}_k$ let y_R denote the image of y in $Y(R)$. The construction $(R, S) \mapsto (R, S \cup \{y_R\})$ determines a morphism of prestacks $v : \mathrm{Ran}^+(X) \to \mathrm{Ran}^+(X)'$. Applying the two-out-of-six property to the diagram

$$\mathrm{Ran}^+(X)' \stackrel{u^+}{\to} \mathrm{Ran}^+(X) \stackrel{v}{\to} \mathrm{Ran}^+(X)' \stackrel{u^+}{\to} \mathrm{Ran}^+(X),$$

we are reduced to proving that the composite maps

$$v \circ u^+ : \mathrm{Ran}^+(X)' \to \mathrm{Ran}^+(X)'$$
$$u^+ \circ v : \mathrm{Ran}^+(X) \to \mathrm{Ran}^+(X)$$

induce isomorphisms on homology. Both of these maps are the identity on homology, since they are related to the identity map by a 2-morphism (in the 2-category of prestacks); see Remark 2.3.32.

We can break the statement of Theorem 2.5.19 into two parts:

Proposition 2.5.22. *Let X be a quasi-projective k-scheme. Then the prestack morphism $\mathrm{Ran}(X) \to \mathrm{Ran}^u(X)$ is a universal homology equivalence.*

Proposition 2.5.23. *Let $X \in \mathrm{Sch}_k$ be a connected quasi-projective k-scheme. Then the inclusion $\mathrm{Ran}^u(X) \to \mathrm{Ran}^+(X)$ is a universal homology equivalence.*

Proof of Proposition 2.5.22. The proof is a slight elaboration Lemma 2.4.12 (which can be regarded as a special case). Fix an R-valued point of $\mathrm{Ran}^u(X)$ given by a nonempty finite subset $S \subseteq X(R)$, and let $\mathcal{C} = \mathrm{Ran}(X) \times_{\mathrm{Ran}^u(X)} \mathrm{Ran}^u(X)_{(R,S)/}$. We wish to prove that the projection map $f : \mathcal{C} \to \mathrm{Spec}\, R$ induces an isomorphism on $\mathbf{Z}_\ell$-homology. For every finitely generated R-algebra R', let $S_{R'}$ denote the image of S under the induced map $X(R) \to X(R')$. Unwinding the definitions, we can identify the objects of $\mathcal{C}$ with triples (R', S', μ), where R' is a finitely generated R-algebra, S' is a nonempty finite set, and $\mu : S' \to X(R')$ is a map of sets with $\mu(S') = S_{R'}$. The functor f admits a right adjoint, given on objects by $R' \mapsto (R', S_{R'}, \mathrm{id})$. It follows that the maps f and g induce mutually inverse isomorphisms on the level of homology (Remark 2.3.32).

□

Proof of Proposition 2.5.23. Fix an object of $\mathrm{Ran}^+(X)$ given by a pair (R, S) where R is a finitely generated k-algebra and S is a finite subset of $X(R)$. Let $\mathcal{C}$ denote the fiber product $\mathrm{Ran}^u(X) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}^+(X)_{(R,S)/}$. We wish to show that the canonical map $\theta : C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell)$ is a quasi-isomorphism.

For every finitely generated R-algebra R', let $S_{R'}$ denote the image of S in $X(R')$. Let us identify the fiber product $\mathrm{Spec}\, R \times_{\mathrm{Spec}\, k} \mathrm{Ran}^u(X)$ with the prestack whose objects are pairs

(R', S'), where R' is a finitely generated R'-algebra and S' is a nonempty finite subset of $X(R')$. Unwinding the definitions, we see that there is a fully faithful embedding

$$f : \mathcal{C} \hookrightarrow \operatorname{Spec} R \times_{\operatorname{Spec} k} \operatorname{Ran}^u(X),$$

whose essential image consist of those pairs (R', S') for which S' contains $S_{R'}$. The functor f admits a left inverse g, given on objects by the formula $g(R', S') = (R', S' \cup S_{R'})$. We therefore have a commutative diagram

$$\begin{array}{ccccc} C_*(\mathcal{C}; \mathbf{Z}_\ell) & \xrightarrow{f} & C_*(\operatorname{Spec} R \times_{\operatorname{Spec} k} \operatorname{Ran}^u(X); \mathbf{Z}_\ell) & \longrightarrow & C_*(\mathcal{C}; \mathbf{Z}_\ell) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} \\ C_*(\operatorname{Spec} R; \mathbf{Z}_\ell) & \longrightarrow & C_*(\operatorname{Spec} R; \mathbf{Z}_\ell) & \longrightarrow & C_*(\operatorname{Spec} R; \mathbf{Z}_\ell), \end{array}$$

where the upper horizontal composition is the identity map. By a diagram chase, we are reduced to proving that the map θ' is a quasi-isomorphism. This follows immediately from Proposition 2.3.40 and Theorem 2.4.5. □

3. Nonabelian Poincare Duality

Let k be an algebraically closed field, let X be an algebraic curve over k, and let ℓ be a prime number which is invertible in k. To every smooth affine group scheme over X, we can associate a moduli stack $\operatorname{Bun}_G(X)$ of principal G-bundles on X. Our goal in this section is to prove an ℓ-adic version of Theorem 1.4.13: that is, to articulate a "local-to-global" principle which controls the structure of the ℓ-adic chain complex $C_*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$ (and, by extension, the structure of the ℓ-adic homology and cohomology of $\operatorname{Bun}_G(X)$). We begin in §3.1 by giving a more leisurely exposition of the theory of nonabelian Poincare duality in the setting of classical topology. In §3.2, we adapt this discussion to the algebraic setting by introducing a prestack $\operatorname{Ran}_G(X)$ which classifies principal G-bundles trivialized away from a finite subset of X (Definition 3.2.3). We then formulate our main result: the forgetful map $\operatorname{Ran}_G(X) \to \operatorname{Bun}_G(X)$ induces an isomorphism on ℓ-adic homology (Theorem 3.2.9) whenever the generic fiber of G is semisimple and simply connected.

The remainder of this section is devoted to the proof of Theorem 3.2.9. Roughly speaking, the idea is to show that the map $\operatorname{Ran}_G(X) \to \operatorname{Bun}_G(X)$ has acyclic fibers. For a more detailed outline of our strategy, we refer the reader to the end of §3.3.

Remark 3.0.1. The material of §3.1 is presented purely for motivation, and is not logically necessary for our proof of Theorem 3.2.9. Readers suffering from a surfeit of motivation can safely skip directly to §3.2.

3.1. Motivation: Poincare Duality in Topology.

Recall the statement of Poincare duality for (possibly noncompact) oriented manifolds:

Theorem 3.1.1 (Poincare Duality)**.** *Let M be an oriented topological manifold of dimension n and let A be an abelian group. Then there is a canonical isomorphism*

$$\mathrm{H}^*_c(M; A) \simeq \mathrm{H}_{n-*}(M; A);$$

here $\mathrm{H}^*_c(M; A)$ denotes the compactly supported cohomology of M with coefficients in A.

In this section, we will discuss a generalization of Theorem 3.1.1 to the case of "nonabelian" coefficients. This result can be regarded as a prototype (in the setting of classical topology) for the main result of §3. The material here is presented purely for motivation, and may be safely skipped by the reader who prefers to work entirely in the setting of algebraic geometry. Since the rest of the paper is logically independent of these ideas, we will not give any proofs; for a more detailed discussion, we refer the reader to §HA.5.5.6 (see also [48], [39], and [46] for some related results.).

3.1.1. *Poincare Duality for Abelian Coefficients.* Let M be an oriented n-manifold. We let $\mathcal{U}(M)$ denote the partially ordered set of all open subsets of M and $\mathcal{U}_0(M)$ the subset of $\mathcal{U}(M)$ consisting of open subsets U which are homeomorphic to Euclidean space $\mathbf{R}^n$. For each open set $U \in \mathcal{U}(M)$, let $C_*(U; A)$ denote the singular chain complex of U, and let $C^*_c(U; A)$ denote the compactly supported cochain complex of U. Then the constructions

$$U \mapsto C_*(U; A) \qquad U \mapsto C^*_c(U; A)$$

can be regarded as functors from $\mathcal{U}(M)$ to the category Chain of chain complexes of abelian groups, or to the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$ introduced in §2.1. Theorem 3.1.1 is an immediate consequence of the following three facts:

Proposition 3.1.2. *The constructions $U \mapsto C^*_c(U; A)$ and $U \mapsto C_{*-n}(U; A)$ determine equivalent functors from $\mathcal{U}_0(M)$ into $\mathrm{Mod}_{\mathbf{Z}}$.*

Proposition 3.1.3. *The canonical map*

$$\varinjlim_{U \in \mathcal{U}_0(M)} C_*(U; A) \to C_*(M; A)$$

is an equivalence (here the colimit is computed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$).

Proposition 3.1.4. *The canonical map*

$$\varinjlim_{U \in \mathcal{U}_0(M)} C^*_c(U; A) \to C^*_c(M; A)$$

is an equivalence (here the colimit is computed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$).

Remark 3.1.5. Proposition 3.1.2 amounts to a local calculation: the compactly supported cohomology of Euclidean space is given by

$$\mathrm{H}^*_c(\mathbf{R}^n; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{H}^*_c(\mathbf{R}^n; A) \simeq \mathrm{H}^*_c(\mathbf{R}^n; \mathbf{Z}) \otimes_{\mathbf{Z}} A.$$

The isomorphism $\mathrm{H}^n_c(\mathbf{R}^n; \mathbf{Z}) \simeq \mathbf{Z}$ is unique up to a sign, and a choice of isomorphism is equivalent to the choice of an orientation on the manifold $\mathbf{R}^n$.

It follows that if $U \subseteq M$ is homeomorphic to $\mathbf{R}^n$, then the homologies of the chain complexes $C^*_c(U; A)$ and $C_*(U; A)$ are concentrated in a single degree. A choice of orientation of U determines an isomorphism of abelian groups $\mathrm{H}^n_c(U; A) \simeq \mathrm{H}_0(U; A)$ which lifts to an equivalence between $C^*_c(U; A)$ and $C_*(U; A)$ (after applying a suitable shift). An orientation of the manifold M allows us to choose these equivalences functorially in U.

Remark 3.1.6. Let $\mathcal{C}$ be an ∞-category which admits colimits and let $\mathcal{F} : \mathcal{U}(M) \to \mathcal{C}$ be a functor. We say that $\mathcal{F}$ is a *$\mathcal{C}$-valued cosheaf on M* if the following condition is satisfied:

- $(*)$ For every open set $U \subseteq M$ and every open cover $\{U_\alpha\}$ of U, the canonical map

  $$\varinjlim_{V} \mathcal{F}(V) \to \mathcal{F}(U)$$

  is an equivalence in $\mathcal{C}$, where the colimit is taken over all open sets $V \subseteq M$ which are contained in some U_α.

Using the fact that M has finite covering dimension and that $\mathcal{U}_0(M)$ forms a basis for the topology of M, one can show that any $\mathcal{C}$-valued cosheaf $\mathcal{F}$ determines an equivalence

$$\varinjlim_{U \in \mathcal{U}_0(M)} \mathcal{F}(U) \to \mathcal{F}(M).$$

Propositions 3.1.3 and 3.1.4 can be deduced from the following more basic facts:

- (a) For any topological space M, the construction $U \mapsto C_*(U; A)$ determines a $\mathrm{Mod}_{\mathbf{Z}}$-valued cosheaf on M.
- (b) For any locally compact topological space M, the construction $U \mapsto C^*_c(U; A)$ determines a $\mathrm{Mod}_{\mathbf{Z}}$-valued cosheaf on M.

Assertions (a) and (b) articulate the idea that homology and compactly supported cohomology satisfy excision. For example, if U and V are open subsets of M, then condition (a) implies that the diagram

$$\begin{array}{ccc} C_*(U \cap V; A) & \longrightarrow & C_*(U; A) \\ \downarrow & & \downarrow \\ C_*(V; A) & \longrightarrow & C_*(U \cup V; A) \end{array}$$

is a pushout square in $\mathrm{Mod}_{\mathbf{Z}}$, which in turn implies (and is morally equivalent to) the existence of a long exact Mayer-Vietoris sequence

$$\cdots \to \mathrm{H}_*(U \cap V; A) \to \mathrm{H}_*(U; A) \oplus \mathrm{H}_*(V; A) \to \mathrm{H}_*(U \cup V; A) \to \mathrm{H}_{*-1}(U \cap V; A) \to \cdots$$

3.1.2. *Poincare Duality for Nonabelian Coefficients.* Recall that cohomology is a *representable* functor on the homotopy category of spaces. More precisely, for every abelian group A and every integer $n \geq 0$, one can construct a topological space $K(A, n)$ and a cohomology class $\eta \in \mathrm{H}^n(K(A, n); A)$ with the following universal property: for any sufficiently nice space M, the pullback of η induces a bijection

$$[M, K(A, n)] \simeq \mathrm{H}^n(M; A),$$

where $[M, K(A, n)]$ denotes the set of homotopy classes of maps from M into $K(A, n)$. The space $K(A, n)$ is called an *Eilenberg-MacLane space*. It is characterized (up to weak homotopy equivalence) by the existence of isomorphisms

$$\pi_i K(A, n) \simeq \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

When $n = 1$, one can define an Eilenberg-MacLane space $K(G, n)$ even when the group G is nonabelian. In this case, $K(G, 1)$ is called a *classifying space* of G, and denoted by BG. It can be constructed as the quotient of a contractible space by a free action of G. This motivates one possible definition of nonabelian cohomology:

Definition 3.1.7. Let G be a discrete group and let M be a manifold (or any other reasonably nice topological space). We let $\mathrm{H}^1(M; G)$ denote the set of homotopy classes of maps from M into $K(G, 1) = \mathrm{BG}$.

Definition 3.1.7 has many other formulations: the set $\mathrm{H}^1(M; G)$ can be identified with the set of isomorphism classes of G-torsors on M, or (in the case where M is connected) with the set of conjugacy classes of group homomorphisms $\pi_1 M \to G$. However, the formulation given above suggests a much broader notion of nonabelian cohomology:

Definition 3.1.8. Let Y be a topological space, and let M be a manifold (or any other sufficiently nice space). Then the *cohomology of M with coefficients in Y* is the set of homotopy classes of maps from M into Y, which we will denote by $[M, Y]$.

We have the following table of analogies:

Abelian Cohomology	Nonabelian Cohomology
Abelian group A	Pointed topological space (Y, y)
$\mathrm{H}^n(M; A)$	$[M, Y] = \pi_0 \operatorname{Map}(M, Y)$
$C^*(M; A)$	$\operatorname{Map}(M, Y)$
$C^*_c(M; A)$	$\operatorname{Map}_c(M, Y)$
$C_*(M; A)$	???

Here $\operatorname{Map}(M, Y)$ denotes the space of continuous maps from M into Y, and $\operatorname{Map}_c(M, Y)$ denotes the subspace consisting of maps which are *compactly supported*: that is, maps $f : M \to Y$ such that the set $\{x \in M : f(x) \neq y\}$ has compact closure (to avoid technicalities, it is sometimes convenient to view $\operatorname{Map}(M, Y)$ and $\operatorname{Map}_c(M, Y)$ as simplicial sets, rather than topological spaces; we will ignore the distinction in what follows).

We can now ask if there is any analogue of Poincare duality in the above setting. That is, if M is a manifold, does the space $\operatorname{Map}_c(M, Y)$ of compactly supported maps from M into Y admit some sort of "homological" description? By analogy with classical Poincare duality, we can break this question into two parts:

(a) What does the mapping space $\operatorname{Map}_c(M, Y)$ look like when $M \simeq \mathbf{R}^n$?
(b) Can we recover the mapping space $\operatorname{Map}_c(M, Y)$ from the mapping spaces $\operatorname{Map}_c(U, Y)$, where U ranges over the open disks in M?

Question (a) is easy to address. The space of compactly supported maps from $\mathbf{R}^n$ into a pointed space (Y, y) is homotopy equivalent to the space of maps which are supported in the unit ball of $\mathbf{R}^n$: that is, the n-fold based loop space $\Omega^n(Y)$.

To address question (b), we note that the construction $U \mapsto \operatorname{Map}_c(U, Y)$ can be regarded as a *covariant* functor of U: any compactly supported map from U into Y can be extended to a compactly supported map on any open set containing U (by carrying the complement of U to the base point of Y). We can regard this construction as a functor from the partially ordered set $\mathcal{U}(M)$ to the ∞-category $\mathcal{S}$ of spaces. We might then ask the following:

Question 3.1.9. Let (Y, y) be a topological space. Is the construction $U \mapsto \operatorname{Map}_c(U, Y)$ a $\mathcal{S}$-valued cosheaf on M?

For example, Question 3.1.9 asks if, for any pair of open sets $U, V \subseteq M$, the diagram of spaces

$$\begin{array}{ccc} \operatorname{Map}_c(U \cap V, Y) & \longrightarrow & \operatorname{Map}_c(U, Y) \\ \downarrow & & \downarrow \\ \operatorname{Map}_c(V, Y) & \longrightarrow & \operatorname{Map}_c(U \cup V, Y) \end{array}$$

is a pushout square in the ∞-category $\mathcal{S}$ (such a diagram of spaces is commonly referred to as a *homotopy pushout* square). This is an unreasonable demand: if it were true, then the diagram

$$\begin{array}{ccc} \pi_0 \operatorname{Map}_c(U \cap V, Y) & \longrightarrow & \pi_0 \operatorname{Map}_c(U, Y) \\ \downarrow & & \downarrow \\ \pi_0 \operatorname{Map}_c(V, Y) & \longrightarrow & \pi_0 \operatorname{Map}_c(U \cup V, Y) \end{array}$$

would be a pushout square in the ordinary category of sets. In other words, any compactly supported map from $U \cup V$ into Y would need to be homotopic (through compactly supported maps) to a map which is supported either in U or in V. This is generally not true.

To understand why Question 3.1.9 has a negative answer, we should emphasize that the ∞-categories $\mathcal{S}$ and $\operatorname{Mod}_{\mathbf{Z}}$ have very different behavior. The cosheaf property for the functor $U \mapsto C^*_c(U; A)$ implies (and is essentially equivalent to) the existence of Mayer-Vietoris sequences

$$\cdots \to \mathrm{H}^*_c(U \cap V; A) \to \mathrm{H}^*_c(U; A) \oplus \mathrm{H}^*_c(V; A) \to \mathrm{H}^*_c(U \cup V; A) \xrightarrow{\delta} \mathrm{H}^{*+1}_c(U \cap V; A) \to \cdots$$

The existence of such a sequence says that any compactly supported cohomology class $u \in \mathrm{H}^m_c(U \cup V; A)$ satisfying the condition $\delta(u) = 0$ can be written as a sum $u = u' + u''$, where u' is supported on U and u'' is supported on V. Here it is crucial that we can add cohomology classes (and the cocycles that represent them): there is no reason to expect that we can arrange that u' or u'' is equal to zero.

In the setting of nonabelian cohomology, there is generally no way to "add" a compactly supported map $u' : U \to Y$ to a compactly supported map $u'' : V \to Y$ to obtain a compactly supported map from $u : U \cup V \to Y$. However, there is an obvious exception: if U and V are disjoint, then there is a canonical homeomorphism $\operatorname{Map}_c(U, Y) \times \operatorname{Map}_c(V, Y) \simeq \operatorname{Map}_c(U \cup V, Y)$, which we can think of as a type of "addition". It turns out that if we take this structure into account, then we can salvage Proposition 3.1.4.

Theorem 3.1.10 (Nonabelian Poincare Duality for Manifolds)**.** *Let M be a manifold of dimension n, let $\mathcal{U}_1(M)$ denote the collection of all open subsets of M which are homeomorphic to a disjoint union of finitely many open disks, and let Y be a pointed topological space which is $(n-1)$-connected. Then the canonical map*

$$\varinjlim_{U \in \mathcal{U}_1(M)} \operatorname{Map}_c(U, Y) \to \operatorname{Map}_c(M, Y)$$

is an equivalence in the ∞-category $\mathcal{S}$. In other words, $\operatorname{Map}_c(M, Y)$ *can be realized as the homotopy colimit of the diagram* $\varinjlim_{U \in \mathcal{U}_1(M)} \operatorname{Map}_c(U, Y)$.

For a proof, we refer the reader to Theorem HA.5.5.6.6.

Remark 3.1.11. The formation of singular chain complexes $T \mapsto C_*(T; \mathbf{Z})$ determines a functor of ∞-categories $\mathcal{S} \to \operatorname{Mod}_{\mathbf{Z}}$ which preserves colimits and carries products of spaces to tensor products in $\operatorname{Mod}_{\mathbf{Z}}$. Consequently, Theorem 3.1.10 implies that the chain complex $C_*(\operatorname{Map}_c(M, Y); \mathbf{Z})$ can be realized as a colimit

$$\varinjlim_{U_1, \ldots, U_n} C_*(\operatorname{Map}_c(U_1 \cup \cdots \cup U_n, Y); \mathbf{Z}) \simeq \varinjlim_{U_1, \ldots, U_n} \bigotimes C_*(\operatorname{Map}_c(U_i, Y); \mathbf{Z}),$$

where the U_i range over all collections of disjoint open disks in M. This expresses the informal idea that $C_*(\operatorname{Map}_c(M, Y); \mathbf{Z})$ can be obtained as a continuous tensor product of copies of cochain complex $C_*(\Omega^n Y; \mathbf{Z})$, indexed by the points of M (or open disks in M).

Warning 3.1.12. When $Y = K(A, m)$ is an Eilenberg-MacLane space, the homotopy groups of the mapping space $\mathrm{Map}_c(M, Y)$ can be identified with the compactly supported cohomology groups $\mathrm{H}_c^*(M; A)$. However, Theorem 3.1.10 is perhaps better understood as supplying information about the *homology* groups of the mapping space $\mathrm{Map}_c(M, Y)$ (see Remark 3.1.11). Nevertheless, Theorem 3.1.10 can be regarded as a generalization of classical Poincare duality. More precisely, it can be used to deduce the local-to-global principle articulated by Proposition 3.1.4. To prove this, we first note that the singular chain complex construction $X \mapsto C_*(X; \mathbf{Z})$ determines a functor from the ∞-category of spaces to the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. This functor admits a right adjoint, which we will denote by $N_* \mapsto K(N_*)$. More concretely, this right adjoint carries a chain complex of abelian groups N_* to the *generalized Eilenberg-MacLane space* $K(N_*)$, whose homotopy groups are given by

$$\pi_n K(N_*) \simeq \mathrm{H}_n(N_*)$$

In particular, if N_* is quasi-isomorphic to the chain complex consisting of a single abelian group A concentrated in homological degree n, then $K(N_*)$ can be identified with the usual Eilenberg-MacLane space $K(A, n)$). More generally, for every manifold M and every integer $m \geq 0$, there is a canonical homotopy equivalence

$$K(C_c^{*+m}(M; A)) \simeq \mathrm{Map}_c(M, K(A, m)).$$

To prove Proposition 3.1.4, we wish to show that the composition

$$\varinjlim_{U \in \mathcal{U}_0(M)} C_c^*(U; A) \xrightarrow{\phi} \varinjlim_{U \in \mathcal{U}_1(M)} C_c^*(U; A) \xrightarrow{\psi} C_c^*(M; A)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}}$. Using the additivity of the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$, one shows that the construction $(U \in \mathcal{U}_1(M)) \mapsto C_c^*(U; A)$ is left Kan extension of its restriction to $\mathcal{U}_0(M)$, so that the map ϕ is an equivalence. To prove that ψ is an equivalence, it will suffice to show that the induced map of spaces

$$K(\psi) : K(\varinjlim_{U \in \mathcal{U}_1(M)} C_c^{*+m}(U; A)) \to K(C_c^{*+m}(M; A)) \simeq \mathrm{Map}_c(M, K(A, m))$$

is a homotopy equivalence for $m \gg 0$. Unwinding the definitions, this map fits into a commutative diagram

$$\begin{array}{ccc} \varinjlim_{U \in \mathcal{U}_1(M)} \mathrm{Map}_c(U, K(A, m)) & \xrightarrow{\theta} & K(\varinjlim_{U \in \mathcal{U}_1(M)} C_c^{*+m}(U; A)) \\ & \searrow^{\rho} \quad \swarrow_{K(\psi)} & \\ & \mathrm{Map}_c(M, K(A, m)). & \end{array}$$

If $m \geq \dim(M)$, then one can show that the map θ is a homotopy equivalence (in other words, that passage to the colimit commutes with the functor K), and the map ρ is a homotopy equivalence by virtue of Theorem 3.1.10. For more details, we refer the reader to §HA.5.5.6.

Remark 3.1.13. Theorem 3.1.10 provides a convenient mechanism for analyzing the homotopy type of the mapping space $\mathrm{Map}_c(M, Y)$: the partially ordered set $\mathcal{U}_1(M)$ indexing the colimit depends only on manifold M, and the individual terms $\mathrm{Map}_c(U, Y)$ are (noncanonically) homotopy equivalent to products of finitely many copies of $\Omega^n Y$, which depends only on Y.

Remark 3.1.14. The hypothesis that Y be $(n-1)$-connected is necessary for Theorem 3.1.10. For example, if $n > 0$ and Y is disconnected, then the constant map from a compact manifold

M to a point $y' \in Y$ belonging to a different connected component than the base point $y \in Y$ cannot be homotopic to a map which is supported in a union of open disks of M.

On the other hand, suppose that Y is $(n-1)$-connected and that M is a compact manifold which admits a triangulation. Then any continuous map $f : M \to Y$ is nullhomotopic on the $(n-1)$-skeleton of M, and therefore homotopic to a map which is support on the interiors of the n-simplices of M. This implies that the map $\varinjlim_{U \in \mathcal{U}_1(M)} \mathrm{Map}_c(U, Y) \to \mathrm{Map}_c(M, Y)$ is surjective on connected components.

Theorem 3.1.10 asserts the existence of a homotopy equivalence

$$\varinjlim_{U \in \mathcal{U}_1(M)} \mathrm{Map}_c(U, Y) \to \mathrm{Map}_c(M, Y),$$

whose codomain can be viewed as a kind measuring the (compactly supported) nonabelian cohomology of the manifold M with coefficients in Y. As in the case of classical Poincare duality, the left hand side can be viewed as a kind of homology. However, it is not the homology of M itself, but of the Ran space $\mathrm{Ran}(M)$.

Theorem 3.1.15 (Nonabelian Poincare Duality)**.** *Let M be a topological manifold of dimension n, let Y be a pointed space which is $(n-1)$-connected, and let $\mathrm{Ran}(M)$ be defined as in §2.4. Then there exists an $\mathcal{S}$-valued cosheaf $\mathcal{F}$ on the topological space $\mathrm{Ran}(M)$ with the following property: for every collection of disjoint connected open sets $U_1, \ldots, U_k \subseteq M$, we have*

$$\mathcal{F}(\mathrm{Ran}(U_1, \ldots, U_k)) \simeq \mathrm{Map}_c(U_1, Y) \times \cdots \times \mathrm{Map}_c(U_k, Y).$$

Theorem 3.1.15 is essentially a reformulation of Theorem 3.1.10. If M is connected, it implies that we can recover $\mathrm{Map}_c(M, Y) \simeq \mathcal{F}(M)$ as a homotopy colimit

$$\varinjlim_{U_1, \ldots, U_k} \mathrm{Map}_c(U_1, Y) \times \cdots \times \mathrm{Map}_c(U_k, Y),$$

where the colimit is taken over all collections of disjoint open disks in M (this follows from the fact that sets of the form $\mathrm{Ran}(U_1, \ldots, U_k)$ form a basis for the topology of $\mathrm{Ran}(M)$). This is essentially the same as the colimit which appears in the statement of Theorem 3.1.10 (though there are a few subtleties; see §HA.5.5.6 for a more detailed discussion).

Remark 3.1.16. The cosheaf $\mathcal{F}$ appearing in the statement of Theorem 3.1.15 is not locally constant. Unwinding the definitions, one can identify the costalk of $\mathcal{F}$ at a point $S = \{x_1, \ldots, x_m\} \in \mathrm{Ran}(M)$ with the product $\prod_i \mathrm{Map}_c(U_i, M)$, where $\{U_i\}_{1 \leq i \leq m}$ is a collection of disjoint open disks around the points $\{x_i\}_{1 \leq i \leq m}$. In particular, the costalk of $\mathcal{F}$ at S is noncanonically equivalent to $\Omega^n(Y)^m$: a homotopy type which depends only on the target space Y, and not on the manifold M.

3.1.3. *Heuristic Reformulation.* Let $f : E \to B$ be any map of topological spaces (not necessarily a fibration). For each open set $V \subseteq B$, let $\mathcal{F}(V) = f^{-1}(V)$. Then $\mathcal{F}$ can be regarded as a covariant functor from the partially ordered set of open subsets of B to the ∞-category $\mathcal{S}$. One can show that this construction determines a $\mathcal{S}$-valued cosheaf on B. Conversely, any sufficiently nice $\mathcal{S}$-valued cosheaf $\mathcal{F}$ on B arises via this construction (this is true, for example, if $\mathcal{F}$ is constructible with respect to some triangulation of B). We therefore obtain the following heuristic version of Theorem 3.1.15:

- $(*)$ Let M be an n-manifold and let Y be an $(n-1)$-connected pointed space. Then there should exist a map $\pi : E \to \mathrm{Ran}(M)$ with the following property: for every finite collection of disjoint connected open sets $U_1, \ldots, U_k \subseteq M$, the inverse image $f^{-1} \mathrm{Ran}(U_1, \ldots, U_k)$ is homotopy equivalent to $\mathrm{Map}_c(U_1 \cup \cdots \cup U_k, Y)$. In particular, if M is connected, then $E \simeq \mathrm{Map}_c(M, Y)$.

Remark 3.1.17. The singular cochain functor $T \mapsto C^*(T; \mathbf{Z})$ determines a contravariant functor which carries colimits in $\mathcal{S}$ to limits in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. It therefore follows from Theorem 3.1.15 that if M is an n-manifold and Y is an $(n-1)$-connected space, then there exists a $\mathrm{Mod}_{\mathbf{Z}}$-valued sheaf $\mathcal{A}$ on $\mathrm{Ran}(M)$ with the property that $\mathcal{A}(\mathrm{Ran}(U_1, \ldots, U_m)) \simeq C^*(\prod_i \mathrm{Map}_c(U_i, Y); \mathbf{Z})$ when the U_i are disjoint connected open subsets of M (take $\mathcal{A}(V) = C^*(\mathcal{F}(V); \mathbf{Z})$). If $\pi : E \to \mathrm{Ran}(M)$ is as in $(*)$, then the sheaf $\mathcal{A}$ can be described by the formula $\mathcal{A}(U) = C^*(\pi^{-1}(U); \mathbf{Z})$.

If M is connected, then the cochain complex $C^*(\mathrm{Map}_c(M, Y); \mathbf{Z})$ is given by the global sections of the sheaf $\mathcal{A}$. One of the main goals of this paper will be to construct an analogue of the sheaf $\mathcal{A}$ in the setting of algebraic geometry (where we replace M by an algebraic curve and Y by the moduli stack of principal bundles).

3.2. **Statement of the Theorem.** Fix an algebraically closed field k, a prime number ℓ which is invertible in k, and an algebraic curve X over k. Let G_0 be an algebraic group over k and let BG_0 denote its classifying stack, so that the moduli stack of G_0-bundles on X can be identified with the moduli stack of maps from X into BG_0. Our goal in this section is to formulate an analogue of Theorem 3.1.10, which asserts that the ℓ-adic cohomology of this moduli stack does not change if we restrict our attention to maps which are supported on a finite subset of X (Theorem 3.2.9).

For our applications, it will be convenient to consider a more general situation:

- We replace the algebraic group G_0 over k with a smooth affine group scheme G over X, which we do not assume to be constant (so our result can more properly be regarded as version of Poincare duality for a *non-constant* coefficient system).
- We formulate our result not only for G-bundles on X, but also "compactly supported" G-bundles on nonempty open subsets $U \subseteq X$ (which we take to mean G-bundles on X which are trivialized on some divisor with support $X - U$).

We begin by introducing some notation.

Definition 3.2.1. Let G be a smooth affine group scheme over X, and let $D \subseteq X$ be an effective divisor. For every finitely generated k-algebra R, we let D_R denote the fiber product $D \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$, which we regard as an effective divisor in the relative curve X_R. We define a category $\mathrm{Bun}_G(X, D)$ as follows:

- The objects of $\mathrm{Bun}_G(X, D)$ are triples $(R, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}|_{D_R}$.
- A morphism from $(R, \mathcal{P}, \gamma)$ to $(R', \mathcal{P}', \gamma')$ in $\mathrm{Bun}_G(X, D)$ consists of a k-algebra homomorphism $R \to R'$ together with an isomorphism of G-bundles $\mathcal{P} \times_{X_R} X_{R'} \simeq \mathcal{P}'$ which carries γ to γ'.

The construction $(R, \mathcal{P}, \gamma) \mapsto R$ determines a coCartesian fibration of categories $\mathrm{Bun}_G(X, D) \to \mathrm{Ring}_k$, which exhibits $\mathrm{Bun}_G(X, D)$ as a prestack. We will refer to $\mathrm{Bun}_G(X, D)$ as the *moduli stack of G-bundles on X with a trivialization along D*.

Remark 3.2.2. In the situation of Definition 3.2.1, the prestack $\mathrm{Bun}_G(X, D)$ is a smooth algebraic stack over k (in particular, it satisfies descent for the fppf topology). In the special case where D is empty, $\mathrm{Bun}_G(X, D)$ is isomorphic to the moduli stack $\mathrm{Bun}_G(X)$ introduced in Definition 2.3.2.

We next introduce the *Beilinson-Drinfeld Grassmannian* $\mathrm{Ran}_G(X)$ associated to a group scheme G over X. Roughly speaking, $\mathrm{Ran}_G(X)$ parametrizes G-bundles on X which are equipped with a trivialization outside of a (specified) finite subset of X. We begin by formulating a more precise definition:

Definition 3.2.3. Let G be a smooth affine group scheme over X. We define a category $\mathrm{Ran}^+_G(X)$ as follows:

- The objects of $\mathrm{Ran}^+_G(X)$ are quadruples $(R, \mathcal{P}, S, \gamma)$ where R is a finitely generated k-algebra, $\mathcal{P}$ is a G-bundle on the relative curve $X_R = \operatorname{Spec} R \times_{\operatorname{Spec} k} X$, S is a finite subset of $X(R)$, and γ is a trivialization of $\mathcal{P}$ on the open set $X_R - |S|$ determined by S.
- A morphism from $(R, \mathcal{P}, S, \gamma)$ to $(R', \mathcal{P}', S', \gamma')$ in $\mathrm{Ran}^+_G(X)$ consists of a k-algebra homomorphism $\phi : R \to R'$ which carries $S \subseteq X(R)$ into $S' \subseteq X(R)$, together with an isomorphism of G-bundles $X_R \times_{X_R} \mathcal{P} \simeq \mathcal{P}'$ which carries γ to γ'.

The construction $(R, \mathcal{P}, S, \gamma) \mapsto (R, S)$ determines a forgetful functor $\theta : \mathrm{Ran}^+_G(X) \to \mathrm{Ran}^+(X)$. It follows immediately from the definitions that θ is a coCartesian fibration, so that $\mathrm{Ran}^+_G(X)$ can be regarded as a prestack and θ as a morphism of prestacks.

We define two more prestacks by the formulae

$$\mathrm{Ran}_G(X) = \mathrm{Ran}(X) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}^+_G(X) \qquad \mathrm{Ran}^u_G(X) = \mathrm{Ran}^u(X) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}^+_G(X).$$

Remark 3.2.4. Each of the prestacks introduced in Definition 3.2.3 can be described informally as a "moduli space for G-bundles on X which are trivialized away from a finite set $S \subseteq U$", where U is a nonempty open subset of X. They differ slightly in details of implementation: whether the set S is given as an abstract set or as a subset of U, and whether we require maps between our finite sets to be surjections.

Remark 3.2.5. In the situation of Definition 3.2.3, we need not require the algebraic curve X to be complete. In particular, for every nonempty open subset $U \subseteq X$, we can consider the prestack $\mathrm{Ran}^+_G(U)$, which is equivalent to the fiber product

$$\mathrm{Ran}^+_G(X) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}^+(U).$$

That is, we can identify $\mathrm{Ran}^+_G(U)$ with the full subcategory of $\mathrm{Ran}^+_G(X)$ spanned by those quadruples $(R, \mathcal{P}, S, \gamma)$ where S is contained in the subset $U(R) \subseteq X(R)$ (similar remarks apply to the variants $\mathrm{Ran}^u_G(U)$ and $\mathrm{Ran}_G(U)$).

Example 3.2.6. Suppose that the group scheme G is trivial. Then the projection maps

$$\mathrm{Ran}_G(U) \to \mathrm{Ran}(U) \qquad \mathrm{Ran}^u_G(U) \to \mathrm{Ran}^u(U) \qquad \mathrm{Ran}^+_G(U) \to \mathrm{Ran}(U)$$

are equivalences of prestacks.

Remark 3.2.7. The definition of the Beilinson-Drinfeld Grassmannian $\mathrm{Ran}_G(X)$ is local with respect to the étale topology. More precisely, suppose we are given an étale morphism $f : U \to V$ between smooth (not necessarily complete) algebraic curves over k. Let G_V be a smooth affine group scheme over V and let $G_U = U \times_V G_V$. Suppose that R is a finitely generated k-algebra and S is a finite subset of $U(R)$ having image $f(S)$ in $V(R)$ for which the map of divisors $|S| \to |f(S)|$ is bijective. Then the commutative diagram

$$\begin{array}{ccc} U_R - |S| & \longrightarrow & U_R \\ \downarrow & & \downarrow \\ V_R - |f(S)| & \longrightarrow & V_R \end{array}$$

determines a pullback square of categories of G-torsors

$$\begin{array}{ccc} \mathrm{Tors}_{G_U}(U_R - |S|) & \longleftarrow & \mathrm{Tors}_{G_U}(U_R) \\ \uparrow & & \uparrow \\ \mathrm{Tors}_{G_V}(X_R - |f(S)|) & \longleftarrow & \mathrm{Tors}_G(X_R) \end{array}$$

(see Notation A.1.3). It follows that we have a canonical equivalence of prestacks

$$\mathrm{Ran}_{G_V}(V) \times_{\mathrm{Ran}(V)} \mathrm{Ran}'(V) \simeq \mathrm{Ran}_{G_U}(V) \times_{\mathrm{Ran}(U)} \mathrm{Ran}'(V),$$

where $\mathrm{Ran}'(V)$ denotes the full subcategory of $\mathrm{Ran}(V)$ spanned by those triples (R, S, μ) for which the map of divisors $|\mu(S)| \to |f(\mu(S))|$ is bijective. In particular, we have a canonical equivalence

$$\mathrm{Ran}_{G_V}(V) \times_{\mathrm{Ran}(V)} V \simeq \mathrm{Ran}_{G_U}(U) \times_{\mathrm{Ran}(U)} V.$$

Since the forgetful functor $\theta : \mathrm{Ran}^+_G(U) \to \mathrm{Ran}^+(U)$ is a coCartesian fibration, Proposition 2.5.11 and Theorem 2.5.19 immediately yield the following result:

Proposition 3.2.8. *Let G be a smooth affine group scheme over X and let $U \subseteq X$ be a nonempty open subset. Then the maps*

$$\mathrm{Ran}_G(U) \to \mathrm{Ran}^u_G(U) \to \mathrm{Ran}^+_G(U)$$

are universal homology equivalences.

We can think of the prestack $\mathrm{Ran}_G(U)$ as parametrizing G-bundles on X which are trivialized away from a finite subset $S \subseteq U$, and are therefore "supported" on a union of small (formal) disks around the points of S. Heuristically, this is an algebro-geometric incarnation of the homotopy colimit

$$\varinjlim_{V_1, \ldots, V_k \subseteq U} \mathrm{Map}_c(V_1, \mathrm{BG}) \times \cdots \times \mathrm{Map}_c(V_k, \mathrm{BG})$$

appearing in Proposition 3.1.10 (at least in the case where G is a constant group). We have the following table of analogies:

Abelian Cohomology	Nonabelian Cohomology	Algebraic Geometry
Manifold M	Manifold M	Open Curve $U = X - D$
Abelian group A	Pointed space (Y, y)	G (or BG)
$C^*_c(M; A)$	$\mathrm{Map}_c(M, Y)$	$\mathrm{Bun}_G(X, D)$
Open disk $V \subseteq M$	Open disk $V \subseteq M$	Completion of U at $x \in U$
$C^*_c(V; A)$	$\mathrm{Map}_c(V, Y)$	G-bundles trivialized on $X - \{x\}$
$C_*(M; A)$	$\varinjlim_{V_1, \ldots, V_k} \prod \mathrm{Map}_c(V_i, Y)$	$\mathrm{Ran}_G(U)$

Note that if $D \subseteq X$ is an effective divisor, we have an evident forgetful functor

$$\mathrm{Ran}^u_G(X - D)^+ \to \mathrm{Bun}_G(X, D),$$

given on objects by the formula $(R, \mathcal{P}, S, \gamma) \mapsto (R, \mathcal{P}, \gamma|_{D_R})$. Composing with the forgetful functor $\mathrm{Ran}_G(X - D) \to \mathrm{Ran}^u_G(X - D)^+$, we obtain a map $\mathrm{Ran}_G(X - D) \to \mathrm{Bun}_G(X, D)$. We can now formulate the first main result of this paper:

Theorem 3.2.9 (Nonabelian Poincare Duality)**.** *Let G be a smooth affine group scheme over X, let $D \subseteq X$ be an effective divisor, and suppose that the generic fiber of G is semisimple and simply connected. Then the forgetful functor $\rho : \mathrm{Ran}_G(X - D) \to \mathrm{Bun}_G(X, D)$ induces an isomorphism*

$$\mathrm{H}_*(\mathrm{Ran}_G(X - D); \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathrm{Bun}_G(X, D); \mathbf{Z}_\ell).$$

Remark 3.2.10. The hypothesis that the generic fiber of G be simply connected can be considerably weakened. Our proof depends only on the fact that the character lattice of the geometric generic fiber of G is a permutation representation of the Galois group of the fraction field of X. This is also true if the generic fiber of G is split reductive or has trivial center.

Remark 3.2.11. In the special case where the group scheme G is trivial, Theorem 3.2.9 reduces to Theorem 2.4.5.

Taking the divisor D to be empty in Theorem 3.2.9, we obtain the following:

Corollary 3.2.12. *Let G be a smooth affine group scheme over X suppose that the generic fiber of G is semisimple and simply connected. Then the forgetful functor $\rho : \mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$ induces an isomorphism*

$$\mathrm{H}_*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell).$$

We will deduce Theorem 3.2.9 from the following slightly stronger result:

Theorem 3.2.13. *Let G be a smooth affine group scheme over X whose generic fiber is semisimple and simply connected and let $D \subseteq X$ be an effective divisor. Then the forgetful functor*

$$\mathrm{Ran}^+_G(X - D) \to \mathrm{Bun}_G(X, D)$$

is a universal homology equivalence.

Remark 3.2.14. The projection map $\theta : \mathrm{Ran}_G(X-D) \to \mathrm{Bun}_G(X, D)$ factors as a composition

$$\mathrm{Ran}_G(X - D) \to \mathrm{Ran}^+_G(X - D) \to \mathrm{Bun}_G(X, D).$$

Consequently, it follows from Theorem 3.2.13 (together with Proposition 3.2.8) that θ is a universal homology equivalence. Similarly, the map $\theta^u : \mathrm{Ran}^u_G(X) \to \mathrm{Bun}_G(X, D)$ is a universal homology equivalence.

3.3. **Outline of Proof.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, and an effective divisor $D \subseteq X$.

Because $\mathrm{Bun}_G(X, D)$ is a prestack in groupoids, Theorem 3.2.13 is equivalent to the assertion that for every map $\eta : \mathrm{Spec}\, R \to \mathrm{Bun}_G(X, D)$, the projection map

$$\pi : \mathrm{Spec}\, R \times_{\mathrm{Bun}_G(X,D)} \mathrm{Ran}^+_G(X - D) \to \mathrm{Spec}\, R$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology (Corollary 2.5.12). We can identify the map η with a pair $(\mathcal{P}, \gamma_0)$, where $\mathcal{P}$ is a G-bundle on the curve X_R, and γ_0 is a trivialization of $\mathcal{P}$ on the divisor D_R. Let us denote the domain of the projection map π by $\mathrm{Sect}_D(\mathcal{P})$. Roughly speaking, we can think of $\mathrm{Sect}_D(\mathcal{P})$ as a moduli space for *rational* trivializations of $\mathcal{P}$, which are defined and equal to γ_0 on the divisor D_R. Unwinding the definitions, we can describe the prestack $\mathrm{Sect}_D(\mathcal{P})$ as follows:

- The objects of $\mathrm{Sect}_D(\mathcal{P})$ are triples (A, S, γ), where A is a finitely generated R-algebra, S is a finite subset of $X(A)$ such that $|S| \cap D_A = \emptyset$, and γ is a map of schemes which fits into a commutative diagram

$$\begin{array}{ccc} X_A - |S| & \xrightarrow{\gamma} & \mathcal{P} \\ & \searrow & \downarrow \\ & & X_R \end{array}$$

whose restriction to D_A is given by γ_0.
- A morphism from (A, S, γ) to (A', S', γ') is an R-algebra homomorphism $\phi : A \to A'$ having the property that S' contains the image of S under the induced map $X(A) \to X(A')$, and the diagram of schemes

$$\begin{array}{ccc} X_{A'} - |S'| & \longrightarrow & X_A - |S| \\ & {\scriptstyle\gamma'}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & \mathcal{P} & \end{array}$$

commutes.

Using Corollary 2.5.12, we can reformulate Theorem 3.2.13 as follows:

Theorem 3.3.1 (Acyclicity of Spaces of Rational Sections)**.** *Let G be a smooth affine group scheme over X whose generic fiber is semisimple and simply connected, let R be a finitely generated k-algebra, let $\mathcal{P}$ be a G-bundle on X_R, and let γ_0 be a trivialization of $\mathcal{P}$ along the divisor $D_R \subseteq X_R$. Then the canonical map*

$$\mathrm{H}_*(\mathrm{Sect}_D(\mathcal{P}); \mathbf{Z}_\ell) \to \mathrm{H}_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$$

is an isomorphism.

The remainder of §3 is devoted to the proof of Theorem 3.3.1. Our strategy is to first treat the following special case, which we will discuss in §3.4:

Theorem 3.3.2. *Let G be a smooth affine group scheme over X whose generic fiber is semisimple and simply connected and let $\gamma_0 : D \to G$ be the restriction to D of the identity section of G. Then the canonical map*

$$\mathrm{H}_*(\mathrm{Sect}_D(G); \mathbf{Z}_\ell) \to \mathrm{H}_*(\operatorname{Spec} k; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$$

is an isomorphism.

Example 3.3.3. Consider the case where $D = \emptyset$ and G is the constant group scheme associated to the multiplicative group $\mathbf{G}_m$ (strictly speaking, this is not a special case of Theorem 3.3.2, because the multiplicative group $\mathbf{G}_m$ is not semisimple). In this case, we can think of $\mathrm{Sect}_D(G)$ as a parameter space for *rational* maps from the algebraic curve X into $\mathbf{G}_m$. We can therefore embed $\mathrm{Sect}_D(G)$ into a larger prestack $\overline{\mathrm{Sect}}_D(G)$, which parametrizes rational maps from X into the affine line $\mathbf{A}^1$. The prestack $\overline{\mathrm{Sect}}_D(G)$ behaves like an affine space of infinite dimension (it is an algebro-geometric incarnation of the function field K_X of the curve X), and the prestack $\mathrm{Sect}_D(G)$ behaves as if it were obtained from $\overline{\mathrm{Sect}}_D(G)$ by removing the origin. From this heuristic description, it is natural to expect that $\mathrm{Sect}_D(G)$ and $\overline{\mathrm{Sect}}_D(G)$ are both acyclic with respect to $\mathbf{Z}_\ell$-homology.

In order to reduce Theorem 3.3.1 to Theorem 3.3.2, the main obstacle we need to overcome is that a G-bundle $\mathcal{P}$ on a relative curve X_R need not be trivial. However, since we are only interested in studying *rational* sections, the following weaker condition can serve as a replacement:

Definition 3.3.4. Let R be a finitely generated k-algebra and let U be an open subset of X_R. We will say that U is *full* if the composite map $U \hookrightarrow X_R \to \operatorname{Spec} R$ is surjective. In other words, U is full if it contains the generic point of each fiber of the map $X_R \to \operatorname{Spec} R$.

Let $D \subseteq X$ be an effective divisor, let $\mathcal{P}$ be a G-bundle on X_R, and let γ_0 be a trivialization of $\mathcal{P}$ along the divisor D_R. We say that γ_0 *extends to a rational trivialization of* $\mathcal{P}$ if there exists a full open subset $U \subseteq X_R$ which contains D_R, and a trivialization of $\mathcal{P}|_U$ which extends γ_0.

Remark 3.3.5. If the divisor $D \subseteq X$ has positive degree, then any open set $U \subseteq X_R$ which contains D_R is automatically full.

The second main ingredient in our proof of Theorem 3.3.1 is the following result, which we will prove in §3.7

Theorem 3.3.6 (Existence of Rational Trivializations)**.** *Let R be a finitely generated k-algebra, and let $\mathcal{P}$ be a G-bundle on X_R equipped with a trivialization γ_0 on D_R. Then, after passing to an étale covering of* $\operatorname{Spec} R$*, we can arrange that γ_0 extends to a rational trivialization of $\mathcal{P}$.*

We devote the remainder of this section to the deduction of Theorem 3.3.1 from Theorems 3.3.2 and 3.3.6. The main idea is that the classification of rational maps depends only on the generic behavior of the G-bundle $\mathcal{P}$.

Definition 3.3.7. Let R be a finitely generated k-algebra, let $\mathcal{P}$ be a G-bundle on X_R equipped with a trivialization γ_0 over the relative divisor $D_R \subseteq X_R$. Suppose we are given a finite subset $S_0 \subseteq X(R)$ such that $|S_0| \cap D_R = \emptyset$.

We let $\operatorname{Sect}_D^{\supseteq S_0}(\mathcal{P})$ denote the full subcategory of $\operatorname{Sect}_D(\mathcal{P})$ spanned by those triples (A, S, γ) for which S contains the image of the map $S_0 \subseteq X(R) \to X(A)$.

In the situation of Definition 3.3.7, the inclusion functor $\operatorname{Sect}_D^{\supseteq S_0}(\mathcal{P}) \hookrightarrow \operatorname{Sect}_D(\mathcal{P})$ admits a left adjoint (in the 2-category of prestacks), given on objects by $(A, S, \gamma) \mapsto (A, S', \gamma')$, where S' is the union of S with the image of S_0, and γ' is the restriction of γ. Applying Remark 2.3.32, we obtain the following:

Lemma 3.3.8. *In the situation of Definition 3.3.7, the inclusion* $\operatorname{Sect}^{\supseteq S}(\mathcal{P}) \to \operatorname{Sect}(\mathcal{P}_R)$ *induces an equivalence*

$$C_*(\operatorname{Sect}_D^{\supseteq S}(\mathcal{P}); \mathbf{Z}_\ell) \to C_*(\operatorname{Sect}_D(\mathcal{P}); \mathbf{Z}_\ell)$$

in $\operatorname{Mod}_{\mathbf{Z}_\ell}$.

Proof of Theorem 3.3.1. Let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R equipped with a trivialization γ_0 on the relative divisor D_R. We will show that the map $\operatorname{Sect}_D(\mathcal{P}) \to \operatorname{Spec} R$ is a universal homology equivalence. By virtue of Proposition 2.5.15, this assertion can be tested locally with respect to the fppf topology on $\operatorname{Spec} R$. We may therefore use Theorem 3.3.6 and Corollary A.2.10 to reduce to the case where there exists a finite subset $S \subseteq X(R)$ such that $|S| \cap D_R = \emptyset$ and γ_0 extends to a trivialization γ of $\mathcal{P}|_{X_R - |S|}$. Then γ determines an equivalence of prestacks $\operatorname{Sect}_D^{\supseteq S}(\mathcal{P}) \simeq \operatorname{Sect}_D^{\supseteq S}(G \times_X X_R)$ (where the projection map $G \times_X X_R \to X_R$ is equipped with the unit section over D). Using Lemma 3.3.8, we can replace $\mathcal{P}$ by the trivial G-bundle $G \times_X X_R$ (and γ_0 by its tautological section). In this case, $\mathcal{P}$ and γ_0 are defined over k. We may therefore assume without loss of generality that $R = k$, in which case the desired result follows from Theorem 3.3.2 (together with Example 2.5.13). □

3.4. **Proof of Theorem 3.3.2.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, and an algebraic curve X over k.

Suppose that $D \subseteq X$ is an effective divisor, and that G is a smooth affine group scheme over X whose generic fiber is semisimple and simply connected. Our goal is to show that that the projection map $\mathrm{Sect}_D(G) \to \mathrm{Spec}\, k$ induces an isomorphism on ℓ-adic homology (where the prestack $\mathrm{Sect}_D(G)$ is defined using the map $D \to G$ given by the identity section). To prove this, we will exploit the fact that the generic fiber of G is automatically quasi-split, so that there is a close relationship between sections of the map $G \to X$ and equivariant maps from some ramified cover $\widetilde{X}$ of X into an algebraic group which is split over k. To formulate this relationship more precisely, it will be convenient to introduce a bit of notation.

Notation 3.4.1. Let $\widetilde{X}$ be an algebraic curve over k and let Γ be a finite group with a faithful (but not necessarily free) action on $\widetilde{X}$ for which the quotient $\widetilde{X}/\Gamma$ is isomorphic to X (here the quotient is formed in the category of k-schemes). Let $D \subseteq X$ be an effective divisor, and let $\widetilde{D}$ denote the fiber product $D \times_X \widetilde{X}$ (which we regard as an effective divisor in $\widetilde{X}$).

For every finitely generated k-algebra R, we let $\widetilde{X}_R$ denote the fiber product $\mathrm{Spec}\, R \times_{\mathrm{Spec}\, k} \widetilde{X}$, and X_R the quotient $\widetilde{X}_R/\Gamma \simeq \mathrm{Spec}\, R \times_{\mathrm{Spec}\, k} X$. Let $D_R = D \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$ and $\widetilde{D}_R = \widetilde{D} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$.

Definition 3.4.2. Suppose we are given a finitely generated k-algebra R, a k-scheme Y equipped with an action of Γ, and a Γ-equivariant map $\pi : Y \to \widetilde{X}_R$ equipped with a Γ-equivariant section $\alpha_0 : \widetilde{D}_R \to Y$ over the relative divisor $\widetilde{D}_R \subseteq \widetilde{X}_R$.

We define a category $\mathrm{Sect}_{\Gamma,D}(Y)$ as follows:

- The objects of $\mathrm{Sect}_{\Gamma,D}(Y)$ are triples (A, S, α) where A is a finitely generated R-algebra, S is a finite subset of $X(A)$ such that $|S| \cap D_A = \emptyset$, and $\alpha : \widetilde{X} \times_X (X_A - |S|) \to Y$ is a Γ-equivariant map of $\widetilde{X}_R$-schemes whose restriction to $\widetilde{D}_R$ coincides with α_0.
- A morphism from (A, S, α) to (A', S', α') in $\mathrm{Sect}_{\Gamma,D}(Y)$ is an R-algebra homomorphism $A \to A'$ which carries $S \subseteq X(A)$ to a subset of $S' \subseteq X(A')$, and for which the diagram

$$\begin{array}{ccc} \widetilde{X} \times_X (X_{A'} - |S'|) & \longrightarrow & \widetilde{X} \times_X (X_A - |S|) \\ & {\scriptstyle \alpha'} \searrow \quad \swarrow {\scriptstyle \alpha} & \\ & Y & \end{array}$$

commutes.

In the special case where $D = \emptyset$, we will denote $\mathrm{Sect}_{\Gamma,D}(Y)$ by $\mathrm{Sect}_{\Gamma}(Y)$. If, in addition, the group Γ is trivial, we will denote $\mathrm{Sect}_{\Gamma}(Y)$ by $\mathrm{Sect}(Y)$.

Remark 3.4.3. In the special case where the group Γ is trivial and $\mathcal{P} \to X_R$ is a bundle for some group scheme G on X, the prestack $\mathrm{Sect}_{\Gamma,D}(\mathcal{P})$ agrees with the prestack $\mathrm{Sect}_D(\mathcal{P})$ introduced in §3.2.

Remark 3.4.4. In the situation of Notation 3.4.2, the construction $(A, S, \alpha) \mapsto A$ determines a coCartesian fibration $\mathrm{Sect}_{\Gamma,D}(Y) \to \mathrm{Ring}_k$, so that we can regard $\mathrm{Sect}_{\Gamma,D}(Y)$ as a prestack. We will refer to $\mathrm{Sect}_{\Gamma,D}(Y)$ as the *prestack of* Γ*-equivariant rational sections of* π *extending* α_0.

The following result is the main technical ingredient in our proof of Theorem 3.3.2:

Proposition 3.4.5. *Let* $\widetilde{X}$, D, *and* Γ *be as in Notation 3.4.1. Let* G *be a simply connected semisimple algebraic group over the field* k, *and suppose that we are given an action of* Γ *on* G *which preserves a pinning (see §A.4), and consider the constant* Γ*-equivariant map* $\widetilde{D} \to G$

given by the identity element of G. Then the canonical map $\mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\mathrm{Spec}\, k} G) \to \mathrm{Spec}\, k$ *is a universal homology equivalence.*

The proof of Proposition 3.4.5 will be given in §3.5. The remainder of this section is devoted to the deduction of Theorem 3.3.2 from Proposition 3.4.5. We begin by introducing some notation.

Notation 3.4.6. Let H be a smooth affine group scheme over X. For each integer $n \geq 0$, let $H(-nD)$ denote the X-scheme obtained from H by dilitation of H at the divisor nD along its identity section (that is, the scheme obtained by iteratively taking the $(n\lambda_i)$th order dilitation at the points x_i, if $D = \sum \lambda_i x_i$); see §A.3. Then each $H(-nD)$ is a smooth affine group scheme over X, equipped with a map $H(-nD) \to H$ which is an isomorphism over the open subset $X - D \subseteq X$. Moreover, if $n > 0$, then the fiber of $H(-nD)$ at a point $x \in D$ is a vector group (that is, it is isomorphic to a product of finitely many copies of the additive group).

Our proof of Theorem 3.3.2 depends on the following:

Lemma 3.4.7. *Let n be a nonnegative integer. Then:*

(*a*) *If the canonical map* $\iota : \mathrm{Sect}_D(G(-nD)) \to \mathrm{Sect}_D(G)$ *induces an isomorphism*

$$\mathrm{H}_0(\mathrm{Sect}_D(G(-nD)); \mathbf{Z}_\ell) \to \mathrm{H}_0(\mathrm{Sect}_D(G); \mathbf{Z}_\ell),$$

then it induces isomorphisms $\mathrm{H}_i(\mathrm{Sect}_D(G(-nD)); \mathbf{Z}_\ell) \to \mathrm{H}_i(\mathrm{Sect}_D(G); \mathbf{Z}_\ell)$ *for all* $i \geq 0$.

(*b*) *If the map* $\mathrm{H}_0(\mathrm{Sect}_D(G(-nD)); \mathbf{Z}_\ell) \to \mathrm{H}_0(\mathrm{Sect}_D(G); \mathbf{Z}_\ell)$ *factors through the augmentation* $\epsilon : \mathrm{H}_0(\mathrm{Sect}_D(G(-nD)); \mathbf{Z}_\ell) \to \mathrm{H}_0(\mathrm{Spec}\, k; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$*, then* $\mathrm{H}_0(\mathrm{Sect}_D(G); \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$.

Proof of Theorem 3.3.2. Let G be a smooth affine group scheme over X whose generic fiber is semisimple and simply connected. Since the ground field k is algebraically closed, Tsen's theorem implies that the function field K_X has dimension 1 (that is, every finite extension of K_X has trivial Brauer group). It follows that the generic fiber $G_0 = \mathrm{Spec}\, K_X \times_X G$ is quasi-split (see [10]). Let G' denote the split form of G_0, regarded as a semisimple algebraic group over k. Since G_0 is quasi-split, we can choose a finite Galois extension $\widetilde{K}_X$ of K_X with Galois group $\Gamma = \mathrm{Gal}(\widetilde{K}_X / K_X)$, an action of Γ on G' which preserves a pinning, and a Γ-equivariant isomorphism

$$u_0 : \mathrm{Spec}\, \widetilde{K}_X \times_{\mathrm{Spec}\, k} G' \simeq \mathrm{Spec}\, \widetilde{K}_X \times_X G.$$

The field $\widetilde{K}_X$ is the function field of an algebraic curve $\widetilde{X}$ over k, which comes equipped with a faithful action of Γ and an isomorphism $\widetilde{X}/\Gamma \simeq X$. Let $H = \widetilde{X} \times_{\mathrm{Spec}\, k} G'$, which we regard as Γ-equivariant group scheme over $\widetilde{X}$. For each open subset $U \subseteq X$, let $\widetilde{U}$ denote the inverse image of U in $\widetilde{X}$. Choosing U sufficiently small, we may assume that Γ acts freely on $\widetilde{U}$ and that u_0 extends to a Γ-equivariant isomorphism

$$u : \widetilde{U} \times_{\widetilde{X}} H \simeq \widetilde{U} \times_X G.$$

If the divisor D were contained in U (this is automatic, for example, if the divisor D is empty), then the desired result would easily from Proposition 3.4.5. To handle the general case, we will need to work a bit harder. Shrinking U if necessary, we may assume that $U = X - (D \cup D')$, where $D' \subseteq X$ is an effective divisor which does not intersect D. Let $\widetilde{D}$ and $\widetilde{D}'$ denote the (scheme-theoretic) inverse images of D and D' in $\widetilde{X}$, respectively. Using Proposition A.3.11, we can choose an integer $n \gg 0$ with the property that the map

$$\widetilde{U} \times_{\widetilde{X}} H(-n\widetilde{D}) \to \widetilde{U} \times_{\widetilde{X}} H \xrightarrow{u} \widetilde{U} \times_X G$$

extends (uniquely) to a map of $(\widetilde{X} - \widetilde{D}')$-schemes

$$\overline{u} : (\widetilde{X} - \widetilde{D}') \times_{\widetilde{X}} H(-n\widetilde{D}) \to (\widetilde{X} - \widetilde{D}') \times_X G.$$

Similarly, we can choose an integer $m \gg n$ such that the inverse u^{-1} extends to a map

$$\overline{v} : (\widetilde{X} - \widetilde{D}') \times_X G(-mD) \to (\widetilde{X} - \widetilde{D}') \times_{\widetilde{X}} H(-n\widetilde{D}).$$

Note that $\overline{u}$ and $\overline{v}$ are Γ-equivariant homomorphisms between group schemes over $\widetilde{X} - \widetilde{D}'$ (since this can be checked over the dense open subscheme $\widetilde{U} \subseteq \widetilde{X} - \widetilde{D}'$).

Note that the natural map $\mathrm{Sect}_D(G(-mD)) \to \mathrm{Sect}_D(G)$ factors through

$$\mathrm{Sect}_{\Gamma,D}(H(-n\widetilde{D})) \simeq \mathrm{Sect}_{\Gamma,(n+1)D}(H)$$

(see Remark A.3.10). Proposition 3.4.5 implies that the projection map

$$\mathrm{Sect}_{\Gamma,(n+1)D}(H) \to \mathrm{Spec}\, k$$

induces an isomorphism on homology, so that the map

$$\mathrm{H}_0(\mathrm{Sect}_D(G(-mD); \mathbf{Z}_\ell) \to \mathrm{H}_0(\mathrm{Sect}_D(G); \mathbf{Z}_\ell)$$

factors through $\mathrm{H}_0(\mathrm{Spec}\, k; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$. Applying (b) of Lemma 3.4.7, we deduce that that $\mathrm{Sect}_D(G)$ is connected: that is, we have $\mathrm{H}_0(\mathrm{Sect}_D(G); \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$. Applying the same argument to $G(-mD)$, we obtain $\mathrm{H}_0(\mathrm{Sect}_D(G(-mD))) \simeq \mathbf{Z}_\ell$, so that the map $\mathrm{Sect}_D(G(-mD)) \to \mathrm{Sect}_D(G)$ induces an isomorphism on degree zero homology. Combining this with part (a) of Lemma 3.4.7, we conclude that the natural map

$$\mathrm{H}_*(\mathrm{Sect}_D(G(-mD)); \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathrm{Sect}_D(G; \mathbf{Z}_\ell)$$

is an isomorphism. It follows that the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_*(\mathrm{Sect}_D(G(-mD)) & \longrightarrow & \mathrm{H}_*(\mathrm{Sect}_{\Gamma,D}(H(-n\widetilde{D})); \mathbf{Z}_\ell) & \longrightarrow & \mathrm{H}_*(\mathrm{Sect}_D(G); \mathbf{Z}_\ell) \\ \downarrow \phi & & \downarrow \phi' & & \downarrow \phi'' \\ \mathrm{H}_*(\mathrm{Spec}\, k; \mathbf{Z}_\ell) & \xrightarrow{\mathrm{id}} & \mathrm{H}_*(\mathrm{Spec}\, k; \mathbf{Z}_\ell) & \xrightarrow{\mathrm{id}} & \mathrm{H}_*(\mathrm{Spec}\, k; \mathbf{Z}_\ell). \end{array}$$

exhibits ϕ'' as a retract of ϕ'. Since ϕ' is an isomorphism(Proposition 3.4.5), it follows that ϕ is an isomorphism, so that the projection map $\mathrm{Sect}_D(G) \to \mathrm{Spec}\, k$ induces an isomorphism on ℓ-adic homology and is therefore a universal homology equivalence (Example 2.5.13). □

The proof of Lemma 3.4.7 will require a bit more preparation.

Notation 3.4.8. Let H be a smooth affine group scheme over X and let $D \subseteq X$ be a divisor. For every finitely generated k-algebra A and each finite subset $S \subseteq X(A)$ with $|S| \cap D_A = \emptyset$, let $F(S, A)$ denote the set of all X-scheme morphisms $\gamma : X_A - |S| \to H$ which vanish on the divisor D. Let $F(A)$ denote the direct limit $\varinjlim_S F(S, A)$, where S ranges over all finite subsets of $X(A)$ such that $|S| \cap D_A = \emptyset$. We let $\mathrm{Sect}^g_D(H)$ denote the prestack in sets determined by the functor F (so that the objects of $\mathrm{Sect}^g_D(H)$ are given by pairs (A, η) where $A \in \mathrm{Ring}_k$ and $\eta \in F(A)$). Note that:

- We can regard F as a functor from Ring_k to the category of groups, so that $\mathrm{Sect}^g_D(H)$ is a group-valued prestack in the sense of Remark 2.3.41.
- There is an evident forgetful functor $\mathrm{Sect}_D(H) \to \mathrm{Sect}^g_D(H)$, given on objects by $(A, S, \gamma) \mapsto (A, \gamma)$. This functor is right cofinal, and therefore induces an equivalence $C_*(\mathrm{Sect}_D(H); \mathbf{Z}_\ell) \simeq C_*(\mathrm{Sect}^g_D(H); \mathbf{Z}_\ell)$.

- Let $n \geq 0$ be an integer, and let H_{nD} denote the group scheme given by the Weil restriction of $nD \times_X H$ along the finite flat map $nD \to \operatorname{Spec} k$. Then restriction to nD yields a map of group-valued prestacks $\operatorname{Sect}^g_D(H) \to H_{nD}$, and the composite map $\operatorname{Sect}^g_D(H) \to H_{nD} \to H_D$ vanishes.

Remark 3.4.9. The prestack $\operatorname{Sect}^g_D(H)$ can be regarded as an alternate version of $\operatorname{Sect}_D(H)$ which is slightly more convenient for some purposes (for example, it is a prestack in sets).

Lemma 3.4.10. *For each integer $n > 0$, the restriction map*

$$\operatorname{Sect}^g_D(G) \to \ker(G_{nD} \to G_D)$$

becomes a surjection after sheafification with respect to the fppf topology.

Remark 3.4.11. Lemma 3.4.10 implies in particular that the sequence

$$\operatorname{Sect}^g_D(G) \to G_{nD} \to G_D$$

is exact at the level of k-valued points. This can be proven by a more elementary argument: it suffices to show that any section of G over nD can be extended to a section of G over an open subset of X, which is a special case of Lemma 3.9.7.

Proof of Lemma 3.4.7, assuming Lemma 3.4.10. Note that we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Sect}_D(G(-nD)) & \longrightarrow & \operatorname{Sect}_D(G) \\ \downarrow & & \downarrow \\ \operatorname{Sect}^g_D(G(-nD)) & \xrightarrow{\theta} & \operatorname{Sect}^g_D(G), \end{array}$$

where the vertical maps induce isomorphisms on homology (see Notation 3.4.8). Note that we can regard θ as an inclusion between group-valued prestacks, let $\mathcal{C}$ denote the quotient of $\operatorname{Sect}^g_D(G)$ by the action of $\operatorname{Sect}^g_D(G(-nD))$. Then restriction of germs defines fully faithful embedding of prestacks $\theta : \mathcal{C} \hookrightarrow \ker(G_{nD} \to G_D)$, and Lemma 3.4.10 (together with Remark A.3.10) shows that θ becomes an equivalence after sheafification with respect to the fppf topology. Since G is smooth, the kernel $\ker(G_{nD} \to G_D)$ is isomorphic to a finite extension of vector groups and is therefore isomorphic (in the category of k-schemes) to an affine space $\mathbf{A}^d$. It follows that

$$\mathrm{H}_*(\mathcal{C}; \mathbf{Z}_\ell) \simeq \begin{cases} \mathbf{Z}_\ell & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Assertions (a) and (b) now follows immediately from the corresponding assertions of Corollary 2.3.42. □

Proof of Lemma 3.4.10. Let $H = G(-D)$ and let H_0 be the generic fiber of H. Since K_X has dimension 1, the group H_0 is quasi-split. We may therefore choose a pair of Borel subgroups $B_-, B_+ \subseteq G_0$ which are in general position (so that the intersection $B_- \cap B_+$ is a maximal torus in H_0). Let U_- and U_+ denote the unipotent radicals of B_- and B_+, and let $C_0 = U_- T U_+$ denote the associated big cell (which we regard as an open subset of G_0).

Since H_0 is simply connected, the maximal torus T is a product of induced tori and is therefore isomorphic (as a K_X-scheme) to an open subset of an affine space. Similarly, the unipotent groups U_- and U_+ admit finite filtrations whose successive quotients are vector groups, and are therefore isomorphic (as K_X-schemes) to affine spaces. We may therefore choose an open immersion $j : C_0 \hookrightarrow \mathbf{A}^d \times_{\operatorname{Spec} k} \operatorname{Spec} K_X$, where d is the dimension of G_0. By a direct limit argument, we can choose a dense open subset $V \subseteq X$, an affine open subset $C \subseteq H \times_X V$ with generic fiber C_0, and an open immersion $\overline{j} : C \hookrightarrow \mathbf{A}^d \times_{\operatorname{Spec} k} V$ extending j.

Shrinking V if necessary, we may assume that C contains the identity section of $H \times_X V$ and that $V = X - (D \cup D')$, where D' is a nonempty effective divisor in X which does not intersect D.

Enlarging D' if necessary, we may further assume that D' can be written as the vanishing locus of a regular function ϕ on $X - D$. For each integer q, let $\overline{j}_q$ denote the composition of $\overline{j}$ with the open immersion

$$\mathbf{A}^d \times_{\operatorname{Spec} k}(X - (D \cup D')) \rightarrow \mathbf{A}^d \times_{\operatorname{Spec} k}(X - D)$$

$$(y_1, \ldots, y_d, x) \mapsto (\phi(x)^q y_1, \ldots, \phi(x)^q y_d, x).$$

Form a pushout diagram

$$\begin{array}{ccc} C & \longrightarrow & H \times_X (X - D') \\ \downarrow{\scriptstyle \overline{j}_q} & & \downarrow \\ \mathbf{A}^d \times_{\operatorname{Spec} k}(X - D) & \longrightarrow & C(q). \end{array}$$

Then $C(q)$ is an X-scheme which is not necessarily separated. However, the scheme $C(q)$ admits a covering by two open affine subsets (the images of $H \times_X (X - D')$ and $\mathbf{A}^d \times_{\operatorname{Spec} k}(X - D)$) with affine intersection (by virtue of our assumption that C is affine), so that the diagonal map $C(q) \rightarrow C(q) \times_{\operatorname{Spec} k} C(q)$ is affine. It follows that the Weil restriction of $C(q)$ along the projection map $X \rightarrow \operatorname{Spec} k$ is representable by an algebraic space $Y(q)$ which is locally of finite type over k (see, for example, Theorem SAG.5.4.3.1). Moreover, the unit section of $H \times_X (X - D')$ extends to a section s_q of the projection map $C(q) \rightarrow X$, and therefore determines a k-valued point y_q of $Y(q)$.

Note that each $C(q)$ is a smooth X-scheme. Let $T_{C(q)/X}$ denote the relative tangent bundle of $C(q)$ over X and let $\mathcal{E}_q$ denote the vector bundle on X obtained by pullback of $T_{C(q)/X}$ along s_q. Every section of the map $C(q) \rightarrow X$ determines a map from a formal neighborhood of D into the group scheme H; this observation determines a map of algebraic spaces $\rho_q : Y(q) \rightarrow H_{(n-1)D}$ for each integer $n > 0$. Using obstruction theory, one sees that s_q belongs to the smooth locus of ρ_q provided that the cohomology group $\mathrm{H}^1(X; \mathcal{E}_q(-(n-1)D))$ vanishes. Note that the vector bundles $\mathcal{E}_q$ are related to one another by the formula $\mathcal{E}_{q+1} = \mathcal{E}_q(D')$, so that we have $\mathrm{H}^1(X; \mathcal{E}_q(-(n-1)D))$ provided that q is sufficiently large (compared with n). We may therefore choose $q \gg 0$ so that s_q belongs to the smooth locus of the map ρ_q. It follows that there exists an étale map $u : \operatorname{Spec} R \rightarrow H_{(n-1)D}$ whose image contains the identity element of $H_{(n-1)D}$ such that u factors through ρ_q. By definition, u classifies a map $u' : (n-1)D \times_X \operatorname{Spec} R \rightarrow H$. The assumption that u factors through ρ_q guarantees that u' extends to a map $\overline{u}' : X_R \rightarrow C(q)$. The inverse image $\overline{u}'^{-1}(H \times_X (X - D')) \subseteq X_R$ is an open subset containing the divisor D_R. Using Corollary A.2.10, we can choose an fppf covering $\operatorname{Spec} R' \rightarrow \operatorname{Spec} R$ and a finite subset $S \subseteq X(R')$ such that $|S| \cap D_{R'} = \emptyset$ and $\overline{u}'|_{X_{R'} - |S|}$ factors through $H \times_X (X - D')$. It follows that the induced map $\gamma : \operatorname{Spec} R' \rightarrow H_{(n-1)D}$ factors through the restriction map

$$r : \operatorname{Sect}^g_D(G) \rightarrow \ker(G_{nD} \rightarrow G_D) \simeq H_{(n-1)D}.$$

Let $W \subseteq H_{(n-1)D}$ be the image of the map γ. Then W is a nonempty open subset of $H_{(n-1)D}$, and γ determines a faithfully flat surjection $\operatorname{Spec} R' \rightarrow W$. Note that the fibers H_x are vector groups for $x \in D$, so that $H_{(n-1)D}$ is a finite extension of vector groups and therefore connected. It follows that $H_{(n-1)D}$ is equal to the union of all translates of W by the elements of $W(k)$.

Let $Z \subseteq H_{(n-1)D}$ be the sheafification (with respect to the fppf topology) of the essential image of the map r. Then Z contains W. Since Since r is a group homomorphism, Z is a subgroup of $H_{(n-1)D}$, and therefore contains every translate of W by a point of $W(k)$. It follows that $Z = H_{(-1)D}$, as desired. □

3.5. **Equivariant Sections.** Throughout this section, we let k be an algebraically closed field, ℓ a prime number which is invertible in k, $\widetilde{X}$ an algebraic curve over k which is equipped with a faithful (but not necessarily free) action of a finite group Γ. Let $X = \widetilde{X}/\Gamma$ denote the quotient of $\widetilde{X}$ by the action of Γ (formed in the category of k-schemes), let $D \subseteq X$ be an effective divisor, and let $\widetilde{D}$ denote its inverse image in X.

Our ultimate goal in this section is to give the proof of Proposition 3.4.5, which asserts the acyclicity of the prestack of Γ-equivariant rational maps from $\widetilde{X}$ into a simply connected semisimple algebraic group G over k (on which Γ acts by pinned automorphisms). The basic strategy is to compare the space of rational maps into G with the space of rational maps into the open subset of G given by the "big cell" of the Bruhat decomposition.

Notation 3.5.1. Let R be a finitely generated k-algebra, let Y be a quasi-projective k-scheme equipped with an action of Γ and a Γ-equivariant map $Y \to \widetilde{X}_R$, and let $U \subseteq Y$ be a Γ-invariant open subscheme. We let $\mathrm{Sect}_{\Gamma,D}(U \subseteq Y)$ denote the full subcategory of $\mathrm{Sect}_{\Gamma,D}(Y)$ spanned by those triples (A, S, α) for which the map $\alpha^{-1}(U) \subseteq \widetilde{X} \times_X X_A \to \operatorname{Spec} A$ is surjective. Then $\mathrm{Sect}_{\Gamma,D}(U \subseteq Y)$ inherits the structure of a prestack.

Remark 3.5.2. In the situation of Notation 3.4.2, suppose that $U \subseteq Y$ is a Γ-invariant open set which contains the image of α_0. If the divisor D is nonempty, then we have $\mathrm{Sect}_{\Gamma,D}(U \subseteq Y) = \mathrm{Sect}_{\Gamma,D}(Y)$.

Proposition 3.5.3. *Let R be a finitely generated k-algebra, let Y be a scheme with an action of Γ equipped with a Γ-equivariant map $\pi : Y \to \widetilde{X}_R$, and let $\alpha_0 : \widetilde{D}_R \to Y$ be a Γ-equivariant section of π over the relative divisor $\widetilde{D}_R \subseteq \widetilde{X}_R$. Let $U \subseteq Y$ be a Γ-invariant open subscheme which contains the image of α_0. Then the canonical map $\mathrm{Sect}_{\Gamma,D}(U) \hookrightarrow \mathrm{Sect}_{\Gamma,D}(U \subseteq Y)$ is a universal homology equivalence.*

Remark 3.5.4. In the situation of Proposition 3.5.3, suppose that we are given a Γ-equivariant open embedding $U \hookrightarrow Y'$, where Y' is another X_R-scheme equipped with an action of Γ. Then the inclusions

$$\mathrm{Sect}_{\Gamma,D}(U \subseteq Y) \hookleftarrow \mathrm{Sect}_{\Gamma,D}(U) \hookrightarrow \mathrm{Sect}_{\Gamma,D}(U \subseteq Y')$$

are universal homology equivalences. We will invoke this principle repeatedly to "simplify" the codomains of our rational maps.

Proof of Proposition 3.5.3. Fix an object $C = (A, S, \alpha)$ of the category $\mathrm{Sect}_{\Gamma,D}(U \subseteq Y)$ and let $\mathcal{C}$ denote the fiber product

$$\mathrm{Sect}_{\Gamma,D}(U) \times_{\mathrm{Sect}_{\Gamma,D}(U \subseteq Y)} \mathrm{Sect}_{\Gamma,D}(U \subseteq Y)_{C/}.$$

Let K be the image of $\alpha^{-1}(Y - U)$ under the projection map $\widetilde{X}_A \to X_A$. Since the map $\alpha^{-1}(U) \to \operatorname{Spec} A$ is surjective, K has finite intersection with each fiber of the map $X_A \to \operatorname{Spec} A$. Since α_0 factors through U, the intersection $K \cap D_A$ is empty.

We wish to prove that the canonical map $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$ is an equivalence. This assertion can be tested locally with respect to the fppf topology on $\operatorname{Spec} A$ (Proposition 2.5.15). We may therefore apply Corollary A.2.10 to reduce to the case where there exists a finite subset $T_0 \subseteq X(A)$ such that $K \subseteq |T_0|$ and $D_A \cap |T_0| = \emptyset$.

Unwinding the definitions, we see that $\mathcal{C}$ can be identified with the category whose objects are pairs (B, T), where B is a finitely generated A-algebra and T is a finite subset of $X(B)$ satisfying the following condition:

(∗) The set T contains the image of $S \subseteq X(A)$, the divisor $|T|$ contains the inverse image of K, and $|T| \cap D_A = \emptyset$.

Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those pairs (B, T) where T contains the image of $T_0 \subseteq X(A)$. Note that the inclusion $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint (in the 2-category of prestacks) given on objects by $(B, T) \mapsto (B, T')$, where T' is the union of T with the image of T_0. It follows that ι induces an isomorphism $\mathrm{H}_*(\mathcal{C}_0; \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathcal{C}; \mathbf{Z}_\ell)$. We are therefore reduced to proving that the canonical map $\mathrm{H}_*(\mathcal{C}_0; \mathbf{Z}_\ell) \to \mathrm{H}_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$ is isomorphism. This is clear, since the category $\mathcal{C}_0$ contains $(A, S \cup T_0)$ as an initial object. □

Corollary 3.5.5. *Let R be a finitely generated k-algebra, let Y be a scheme with an action of Γ equipped with a Γ-equivariant map $\pi : Y \to \widetilde{X}_R$, and let $\alpha_0 : \widetilde{D}_R \to Y$ be a Γ-equivariant section of π over the relative divisor $\widetilde{D}_R \subseteq \widetilde{X}_R$. Suppose that α_0 factors through a Γ-invariant open subset $U \subseteq Y$. If D is nonempty, then the inclusion $\mathrm{Sect}_{\Gamma,D}(U) \to \mathrm{Sect}_{\Gamma,D}(Y)$ is a universal homology equivalence.*

Proof. Combine Remark 3.5.2 with Proposition 3.5.3. □

Proof of Proposition 3.4.5 when $D \neq \emptyset$. Let G be an algebraic group over k which is semisimple, simply connected, and equipped with an action of Γ which preserves a pinning $(B, T, \{\phi_\alpha\})$. Let B' be the unique Borel subgroup of G which contains T and is in general position with respect to B. Let U and U' denote the unipotent radicals of B and B', respectively, and set $V = UTU' \subseteq G$. Then V is a Γ-invariant open subset of G which contains the image of the map α_0. By virtue of Corollary 3.5.5, the inclusion

$$\mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V) \hookrightarrow \mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} G)$$

is a universal homology equivalence. It will therefore suffice to show that the projection map $\mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V) \to \operatorname{Spec} k$ induces an isomorphism on $\mathbf{Z}_\ell$-homology.

Using the Bruhat decomposition, we can identify V with the product $U \times_{\operatorname{Spec} k} T \times_{\operatorname{Spec} k} U'$ as a k-scheme. Let $\pi : V \to T$ be the projection onto the middle factor, and let $\iota : T \to V$ be the inclusion map. We will show that the maps π and ι induce mutually inverse isomorphisms between the ℓ-adic homology of $\mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V)$ and $\mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} T)$.

Let $\alpha_1, \ldots, \alpha_r : T \to \mathbf{G}_m$ be the system of simple roots of G determined by the choice of Borel subgroup G. Then composition with $\{\alpha_i\}_{1 \leq i \leq r}$ determines a group homomorphism

$$\mathrm{Hom}(\mathbf{G}_m, T) \to \prod_{1 \leq i \leq r} \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) \simeq \mathbf{Z}^r.$$

This is an injective map between free abelian groups of the same rank, and is therefore a rational isomorphism. In particular, we can choose an integer $n > 0$ and a cocharacter $\beta : \mathbf{G}_m \to T$ such that $\alpha_i \circ \beta$ is given by the nth power map $\mathbf{G}_m \xrightarrow{n} \mathbf{G}_m$ for $1 \leq i \leq r$. Note that β is invariant under the action of Γ. Consider the map $h : \mathbf{G}_m \times_{\operatorname{Spec} k} V \to V$ given on k-points by the formula

$$h(\lambda, utu') = \beta(\lambda) u \beta(\lambda)^{-1} t \beta(\lambda)^{-1} u' \beta(\lambda).$$

Then h extends uniquely to a map $\overline{h} : \mathbf{A}^1 \times_{\operatorname{Spec} k} V \to V$, whose restriction to $\{0\} \times_{\operatorname{Spec} k} V$ coincides with the composition $\iota \circ \pi : V \to V$. The map $\overline{h}$ induces a map of prestacks

$$\theta : \mathbf{A}^1 \times_{\operatorname{Spec} k} \mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V) \to \mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V).$$

Note that the composition of θ with the inclusion

$$e_1 : \{1\} \times_{\operatorname{Spec} k} \mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V) \hookrightarrow \mathbf{A}^1 \times_{\operatorname{Spec} k} \mathrm{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V)$$

is the identity map, and therefore induces an isomorphism on $\mathbf{Z}_\ell$-homology. Since e_1 also induces an isomorphism on $\mathbf{Z}_\ell$-homology, it follows from the Künneth formula (Proposition 2.3.40) that θ induces an isomorphism of $\mathbf{Z}_\ell$-homology. It follows that the composition of θ with the inclusion

$$e_0 : \{0\} \times_{\operatorname{Spec} k} \operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V) \hookrightarrow \mathbf{A}^1 \times_{\operatorname{Spec} k} \operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V)$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology, so that the composite map

$$\operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V) \xrightarrow{\pi} \operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} T) \xrightarrow{\iota} \operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} V)$$

also induces an isomorphism on $\mathbf{Z}_\ell$-homology. We are therefore reduced to proving that the projection map $\operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} T) \to \operatorname{Spec} k$ induces an isomorphism on $\mathbf{Z}_\ell$-homology.

Since the group G is simply connected, the simple coroots $\{\alpha_i^\vee : \mathbf{G}_m \to T\}$ determine a Γ-equivariant isomorphism $\mathbf{G}_m^r \simeq T$, where Γ acts on $\mathbf{G}_m^r$ by permuting the factors. In particular, there is a Γ-equivariant open immersion $j : T \hookrightarrow \mathbf{A}^r$, where Γ acts linearly on $\mathbf{A}^r$. Modifying this open immersion by a translation if necessary, we may suppose that it carries the identity of T to the origin in $\mathbf{A}^r$. Corollary 3.5.5 implies that j induces a universal homology equivalence

$$\operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} T) \to \operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} \mathbf{A}^r).$$

We are therefore reduced to proving that the projection map

$$\operatorname{Sect}_{\Gamma,D}(\widetilde{X} \times_{\operatorname{Spec} k} \mathbf{A}^r) \to \operatorname{Spec} k$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology.

Note that the map $\mathbf{A}^1 \times \mathbf{A}^r \to \mathbf{A}^r$ given by $(\lambda, v) \mapsto v$ determines a $\mathbf{A}^1$-homotopy from the identity map $\mathrm{id} : \mathbf{A}^r \to \mathbf{A}^r$ to the zero map $\mathbf{A}^r \to \operatorname{Spec} k \xrightarrow{e} \mathbf{A}^r$. Arguing as above, we reduce to proving that the map $g : \operatorname{Sect}_{\Gamma,D}(\widetilde{X}) \to \operatorname{Spec} k$ induces an isomorphism on $\mathbf{Z}_\ell$-homology. This follows from Remark 2.3.32, since g admits a left adjoint f (in the 2-category of prestacks). □

The remainder of this section is devoted to the proof of Proposition 3.4.5 in the case $D = \emptyset$. This will require some preliminaries.

Proposition 3.5.6. *Let R be a finitely generated k-algebra, let Y be a k-scheme equipped with an action of Γ, and let $\pi : Y \to \widetilde{X}_R$ be a Γ-equivariant map. Suppose we are given a finite collection of Γ-invariant open subsets $U_1, \ldots, U_n \subseteq Y$. For every subset $I \subseteq \{1, \ldots, n\}$, let $U_I = \bigcap_{i \in I} U_i$. If the forgetful functor $\operatorname{Sect}_\Gamma(U_I \subseteq Y) \to \operatorname{Spec} R$ is a universal homology equivalence for every nonempty subset $I \subseteq \{1, \ldots, n\}$, then the forgetful functor $\operatorname{Sect}_\Gamma(\bigcup U_i \subseteq Y) \to \operatorname{Spec} R$ is a universal homology equivalence.*

We begin by proving Proposition 3.5.6 in the special case $n = 2$.

Lemma 3.5.7. *In the situation of Definition 3.4.2, suppose we are given a pair of Γ-invariant open subsets $U, V \subseteq Y$. Let $\mathcal{C}$ denote the full subcategory of $\operatorname{Sect}_\Gamma(Y)$ given by the union of $\operatorname{Sect}_\Gamma(U \subseteq Y)$ and $\operatorname{Sect}_\Gamma(V \subseteq Y)$. Then the inclusion $\mathcal{C} \hookrightarrow \operatorname{Sect}_\Gamma(U \cup V \subseteq Y)$ is a universal homology equivalence.*

Proof. Suppose we are given an object $C = (A, S, \alpha) \in \operatorname{Sect}_\Gamma(U \cup V \subseteq Y)$, and let $\mathcal{C}_{C/}$ denote the category $\mathcal{C} \times_{\operatorname{Sect}_\Gamma(U \cup V \subseteq Y)} \operatorname{Sect}_\Gamma(U \cup V \subseteq Y)_{C/}$. We wish to prove that the canonical map

$$C_*(\mathcal{C}_{C/}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$$

is an equivalence in $\operatorname{Mod}_{\mathbf{Z}_\ell}$. In fact, we claim that the map $\theta : \mathcal{C}_{C/} \to \operatorname{Spec} A$ is a universal homology equivalence. Let $W_0 \subseteq \operatorname{Spec} A$ denote the image of $\alpha^{-1} U$, and let $W_1 \subseteq \operatorname{Spec} A$ denote the image of $\alpha^{-1} V$. By assumption we have $W_0 \cup W_1 = \operatorname{Spec} A$.

By virtue of Proposition 2.5.15, the assertion that θ is a universal homology equivalence can be tested locally with respect to the fppf topology. We may therefore assume that either $W_0 = \operatorname{Spec} A$ or $W_1 = \operatorname{Spec} A$. In this case, the map θ is an equivalence and there is nothing to prove. □

Remark 3.5.8. Let $\mathcal{C}$ be a category containing full subcategories $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ which satisfy the following conditions:

- The subcategories $\mathcal{C}_0$ and $\mathcal{C}_1$ are *cosieves* in $\mathcal{C}$: that is, for any morphism $f : C \to D$ in $\mathcal{C}$, if C belongs to $\mathcal{C}_i$, then so does D.
- Every object of $\mathcal{C}$ belongs to either $\mathcal{C}_0$ or $\mathcal{C}_1$.

Then the diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{N}(\mathcal{C}_0 \cap \mathcal{C}_1) & \longrightarrow & \mathrm{N}(\mathcal{C}_0) \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathcal{C}_1) & \longrightarrow & \mathrm{N}(\mathcal{C}) \end{array}$$

is a pushout square. It follows that for any ∞-category $\mathcal{D}$ which admits colimits and any functor $F : \mathcal{C} \to \mathcal{D}$, the diagram

$$\begin{array}{ccc} \varinjlim_{C \in \mathcal{C}_0 \cap \mathcal{C}_1} F(C) & \longrightarrow & \varinjlim_{C \in \mathcal{C}_0} F(C) \\ \downarrow & & \downarrow \\ \varinjlim_{C \in \mathcal{C}_1} F(C) & \longrightarrow & \varinjlim_{C \in \mathcal{C}} F(C) \end{array}$$

is a pushout square in $\mathcal{D}$ (see §HTT.4.2.3).

Proof of Proposition 3.5.6. Using induction on n, we can reduce to the case where $n = 2$. Let $\pi : Y \to \widetilde{X}_R$ be as in Definition 3.4.2. For each Γ-invariant open set $U \subseteq Y$, the full subcategory $\operatorname{Sect}_\Gamma(U \subseteq Y) \subseteq \operatorname{Sect}_\Gamma(Y)$ is a cosieve. If V is another Γ-invariant open subset of Y, then the irreducibility of $\widetilde{X}$ gives an equality

$$\operatorname{Sect}_\Gamma(U \cap V \subseteq Y) = \operatorname{Sect}_\Gamma(U \subseteq Y) \cap \operatorname{Sect}_\Gamma(V \subseteq Y).$$

Combining Remark 3.5.8 with Lemma 3.5.7, we obtain a pushout diagram

$$\begin{array}{ccc} C_*(\operatorname{Sect}_\Gamma(U \cap V \subseteq Y); \mathbf{Z}_\ell) & \longrightarrow & C_*(\operatorname{Sect}_\Gamma(U \subseteq Y); \mathbf{Z}_\ell) \\ \downarrow & & \downarrow \\ C_*(\operatorname{Sect}_\Gamma(V \subseteq Y); \mathbf{Z}_\ell) & \longrightarrow & C_*(\operatorname{Sect}_\Gamma(U \cup V \subseteq Y); \mathbf{Z}_\ell) \end{array}$$

in the ∞-category $\operatorname{Mod}_{\mathbf{Z}_\ell}$. □

The main ingredient we will need is for the proof of Proposition 3.4.5 is the following result, which we will prove in §3.6:

Proposition 3.5.9. *Let $\mathcal{E}$ be a Γ-equivariant vector bundle on $\widetilde{X}$, let R be a finitely generated k-algebra, let $V \subseteq \widetilde{X}_R$ be a Γ-invariant open subset, let Y be a Γ-equivariant $\mathcal{E}$-torsor over V, and let $U \subseteq Y$ be a Γ-invariant open subset such that the projection map $U \to \operatorname{Spec} R$ is surjective. Then the map $C_*(\operatorname{Sect}_\Gamma(U \subseteq Y); \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$ is an equivalence in $\operatorname{Mod}_{\mathbf{Z}_\ell}$.*

Our proofs of Proposition 3.4.5 and Proposition 3.5.9 both depend on the following "transitivity" result:

Lemma 3.5.10. *Let R be a finitely generated k-algebra, let Y and Z be schemes equipped with an action of Γ, and suppose we are given a commutative diagram of Γ-equivariant maps*

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Z \\ & \searrow \quad \swarrow & \\ & \widetilde{X}_R. & \end{array}$$

Let $U \subseteq Y$ be a Γ-invariant open set, let $V = \phi(U)$, and assume that V is open. Suppose that the following condition is satisfied:

- $(*)$ *For every object $(A, S, \alpha) \in \mathrm{Sect}_\Gamma(V \subseteq Z)$, set $Y_0 = Y \times_Z (\widetilde{X} \times_X (X_A - |S|))$ and set $U_0 = Y_0 \times_Y U$. Then the canonical map $C_*(\mathrm{Sect}_\Gamma(U_0 \subseteq Y_0); \mathbf{Z}_\ell) \to C_*(\mathrm{Spec}\, A; \mathbf{Z}_\ell)$ is an equivalence.*

Then the map $\mathrm{Sect}_\Gamma(U \subseteq Y) \to \mathrm{Sect}_\Gamma(V \subseteq Z)$ is a universal homology equivalence. In particular, if the map $\mathrm{Sect}_\Gamma(V \subseteq Z) \to \mathrm{Spec}\, R$ is a universal homology equivalence, then the map $\mathrm{Sect}_\Gamma(U \subseteq Y) \to \mathrm{Spec}\, R$ has the same property.

Proof. Fix an object $C = (A, S, \alpha) \in \mathrm{Sect}_\Gamma(V \subseteq Z)$, and set $\mathcal{C} = \mathrm{Sect}_\Gamma(U \subseteq Y) \times_{\mathrm{Sect}_\Gamma(V \subseteq Z)} \mathrm{Sect}_\Gamma(V \subseteq Z)_{C/}$. We wish to prove that the map $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathrm{Spec}\, A; \mathbf{Z}_\ell)$ is a quasi-isomorphism. Define $U_0 \subseteq Y_0$ as in $(*)$. Unwinding the definitions, we can identify $\mathcal{C}$ with the full subcategory of $\mathrm{Sect}_\Gamma(U_0 \subseteq Y_0)$ spanned by those objects (B, T, β) where T contains the image of S in $X(B)$. The inclusion $\mathcal{C} \hookrightarrow \mathrm{Sect}_\Gamma(U_0 \subseteq Y_0)$ admits a left adjoint (in the 2-category of prestacks), given on objects by $(B, T, \beta) \mapsto (B, T \cup S_B, \beta')$ where S_B denotes the image of S in $X(B)$ and β' is the restriction of β. It follows that $C_*(\mathcal{C}; \mathbf{Z}_\ell) \simeq C_*(\mathrm{Sect}_\Gamma(U_0 \subseteq Y_0); \mathbf{Z}_\ell)$, so that the desired result follows from assumption $(*)$. □

Proof of Proposition 3.4.5 when $D = \emptyset$. Write G as a product of simple factors $\prod_{i \in I} G_i$, so that Γ permutes the set I. For each $i \in I$, let Γ_i denote its stabilizer in Γ. Then the prestack $\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} G)$ is equivalent to $\prod \mathrm{Sect}_{\Gamma_i}(\widetilde{X} \times_{\mathrm{Spec}\, k} G_i)$, where the product is taken over a set of representatives for the orbits of Γ on I. Using the Künneth formula (Proposition 2.3.40), we can reduce to the case where the group G is simple.

We now argue that it suffices to prove the analogue of Proposition 3.4.5 where the group G has been replaced by an open subset of the big Bruhat cell of G (assertion $(*)$ below). Let $(B, T, \{\phi_\alpha\})$ be a pinning of G which is invariant under the action of Γ, and let B' be the unique Borel subgroup of G which contains T and is in general position with respect to B. Let U and U' denote the unipotent radicals of B and B', respectively, and set $V = UTU' \subseteq G$. Then V is a dense open subset of G. Note that an element $g \in G(k)$ belongs to $V(k)$ if and only if $gB'g^{-1}$ is in general position with respect to B.

Let G_0 denote the identity component of the fixed point set G^Γ, and let V^+ denote the open subset of G given by $\bigcup_{g \in G_0(k)} gV$. Using Propositions A.4.2 and A.4.4, we see that either $V^+ = G$ or (in the special case where $G = \mathrm{SL}_{2n+1}$, the characteristic of k is equal to 2, and Γ is nontrivial) the difference $G - V^+$ can be written as a disjoint union of components $K_- \amalg K_+$ which are permuted by Γ. Since $\widetilde{X}$ is connected, no Γ-invariant map from $\widetilde{X}$ to G can factor through $K_- \amalg K_+$, so in any case we have

$$\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} V^+ \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} G) = \mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} G).$$

Since V^+ is quasi-compact, we can choose finitely many points $g_1, \ldots, g_n \in G^\Gamma(k)$ such $V^+ = \bigcup_{1 \leq j \leq n} g_j V$. For every subset $J \subseteq \{1, \ldots, n\}$, let $V_J = \bigcap_{j \in J} g_j^{-1} V$. By virtue of

Proposition 3.5.6, it will suffice to show that the map

$$\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} V_J \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} G) \to \mathrm{Spec}\, k$$

is a universal homology equivalence for every nonempty subset $J \subseteq \{1, \ldots, n\}$. Choose an element $j \in J$, so that multiplication by g_j determines an isomorphism of V_I with a Γ-invariant open subset of V. We are therefore reduced to proving the following:

- $(*)$ Let W be a nonempty Γ-invariant open subset of the big cell $V \subseteq G$. Then the forgetful functor $\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} W \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} G) \to \mathrm{Spec}\, k$ is a universal homology equivalence.

Using Remark 3.5.4, we see that $(*)$ can be reformulated as follows:

- $(*')$ Let W be a nonempty Γ-invariant open subset of the big cell $V \subseteq G$. Then the forgetful functor $\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} W \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} V) \to \mathrm{Spec}\, k$ is a universal homology equivalence.

Note that U admits a Γ-equivariant filtration

$$* = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d = U,$$

where each quotient U_i/U_{i-1} is a vector group equipped with a linear action of Γ; choose a similar filtration

$$* = U'_0 \subseteq U'_1 \subseteq \cdots \subseteq U'_d = U'.$$

For $0 \leq i \leq d$, let V_i denote the double quotient $U_i \backslash V / U'_i$, and let W_i denote the image of W in V_i. Applying Lemma 3.5.10 and Proposition 3.5.9, we deduce that each of the maps

$$\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} W_i \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} V_i) \to \mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} W_{i+1} \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} V_{i+1})$$

is a universal homology equivalence.

Note that V_d is isomorphic to a maximal torus $T \subseteq G$. Consequently, we are reduced to proving the following:

- $(*'')$ Let W be a nonempty Γ-invariant open subset of T. Then the map

  $$\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} W \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} T) \to \mathrm{Spec}\, k$$

  is a universal homology equivalence.

Since G is simply connected, the character lattice of the torus T is freely generated by the fundamental weights of G, which are permuted among themselves by the group Γ. Consequently, there exists a Γ-equivariant open immersion $T \hookrightarrow \mathbf{A}^r$, where r is the rank of G and the group Γ acts linearly on $\mathbf{A}^r$. Invoking Remark 3.5.4 again, we are reduced to proving the following assertion:

- $(*''')$ Let $\mathbf{A}^r$ be an affine space equipped with a linear action of Γ, and let W be a nonempty Γ-invariant open subset of $\mathbf{A}^r$. Then the map $\mathrm{Sect}_\Gamma(\widetilde{X} \times_{\mathrm{Spec}\, k} W \subseteq \widetilde{X} \times_{\mathrm{Spec}\, k} \mathbf{A}^r) \to \mathrm{Spec}\, k$ is a universal homology equivalence.

This is a special case of Proposition 3.5.9. □

3.6. **Sections of Vector Bundles.**

Throughout this section, we let k denote an algebraically closed field, ℓ a prime number which is invertible in k, and X an algebraic curve over k. Our goal is to give a proof of Proposition 3.5.9. The main step is to establish the following special case:

Lemma 3.6.1. *Let R be a finitely generated k-algebra and let $U \subseteq X_R \times_{\mathrm{Spec}\, k} \mathbf{A}^1$ be a full open subset. Then the map*

$$C_*(\mathrm{Sect}(U \subseteq X_R \times_{\mathrm{Spec}\, k} \mathbf{A}^1); \mathbf{Z}_\ell) \to C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Let us first show that Lemma 3.6.1 implies Proposition 3.5.9.

Proof of Proposition 3.5.9. Let $\widetilde{X}$ be an algebraic curve equipped with a faithful action of a finite group Γ with $X = \widetilde{X}/\Gamma$, let $\mathcal{E}$ be a Γ-equivariant vector bundle on $\widetilde{X}$, let R be a finitely generated k-algebra, let $V \subseteq \widetilde{X}_R$ be a Γ-invariant open subset, let Y be a Γ-equivariant $\mathcal{E}$-torsor over V, and let $U \subseteq Y$ be a Γ-invariant open subset such that the projection map $U \to \operatorname{Spec} R$ is surjective. We wish to show that the canonical map

$$C_*(\operatorname{Sect}_\Gamma(U \subseteq Y); \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$$

is an equivalence in $\operatorname{Mod}_{\mathbf{Z}_\ell}$.

Let $W \subseteq X$ be a dense open subset with the property that the map $\nu : \widetilde{X} \times_X W \to W$ is étale. Over the open subset $\widetilde{X} \times_X W \subseteq \widetilde{X}$, the action of Γ on $\mathcal{E}$ furnishes descent data: that is, we have a Γ-equivariant isomorphism $\mathcal{E}|_{\widetilde{X} \times_X W} \simeq \nu^* \mathcal{E}_0$, for some vector bundle $\mathcal{E}_0$ on W. Shrinking the open set W if necessary, we may suppose that the vector bundle $\mathcal{E}_0$ is trivial. Note that replacing U by $U \times_X W$ does not change the category $\operatorname{Sect}_\Gamma(U \subseteq Y)$. We may therefore assume without loss of generality that the set U is contained in the open subset $Y \times_X W \subseteq Y$. Using Remark 3.5.4, we can replace V by $V \times_X W$. Then the action of Γ on V is free, so that we can write Y as the pullback of a $\mathcal{E}_0$-torsor Y/Γ over the open subscheme $V/\Gamma \subseteq X_R$.

Working locally on $\operatorname{Spec} R$, we can use Proposition A.2.8 to reduce to the case where $X_R - V/\Gamma$ is contained in a closed subscheme $E \subseteq X_R$ which is finite and flat (of positive degree) over R. Note that replacing U with $U \times_{X_R} (X_R - E)$ does not change the category $\operatorname{Sect}_\Gamma(U \subseteq Y)$. We may therefore assume without loss of generality that $U \subseteq Y \times_{X_R} (X_R - E)$. Using Remark 3.5.4 again, we can replace V by the inverse image of $X_R - E$, and thereby reduce to the case where V/Γ is affine. It follows that every $\mathcal{E}_0$-torsor on V/Γ is trivial, so that $Y/\Gamma \simeq \mathcal{E}_0 \times_W V/\Gamma$ and we therefore have a Γ-equivariant isomorphism $Y \simeq V \times_{\operatorname{Spec} k} \mathbf{A}^n$, where Γ acts trivially on $\mathbf{A}^n$. Using induction on n and Lemma 3.5.10, we can reduce to the case where $n = 1$. Using Remark 3.5.4 again, we can replace Y by $\widetilde{X}_R \times_{\operatorname{Spec} k} \mathbf{A}^1$. Note that the data of a Γ-equivariant map $\widetilde{X} \times_X (X_A - |S|) \to Y$ of $\widetilde{X}$-schemes is equivalent to the data of a map of X-schemes from $X_A - |S|$ to $X \times_{\operatorname{Spec} k} \mathbf{A}^1$. We may therefore reduce to the case where the group Γ is trivial, in which case the desired result follows from Lemma 3.6.1. □

We now turn to the proof of Lemma 3.6.1. Here we invoke the same basic idea as in Example 3.3.3: the prestack $\operatorname{Sect}(X_R \times_{\operatorname{Spec} k} \mathbf{A}^1)$ behaves like an infinite-dimensional affine space (and is therefore acyclic), and the prestack

$$\operatorname{Sect}(U \subseteq X_R \times_{\operatorname{Spec} k} \mathbf{A}^1)$$

is obtained from $\operatorname{Sect}(X_R \times_{\operatorname{Spec} k} \mathbf{A}^1)$ by removing a closed subset of infinite codimension (which has no effect on $\mathbf{Z}_\ell$-homology).

Proof of Lemma 3.6.1. The proof can be broken into three steps:

(*a*) Every A-valued point of $\operatorname{Sect}(U \subseteq X_R \times_{\operatorname{Spec} k} \mathbf{A}^1)$ determines an effective divisor $K \subseteq X_A$ (namely, the divisor on which the relevant section is not defined). Using the acyclicity of the Ran space $\operatorname{Ran}(X)$, we reduce the proof of Lemma 3.6.1 to establishing an analogous result for a variant of the prestack $\operatorname{Sect}(U \subseteq X_R \times_{\operatorname{Spec} k} \mathbf{A}^1)$ where the divisor K has been fixed (and is nonempty); we will denote this variant by $\mathcal{C}'$.

(*b*) The collection of regular functions from $X_A - K$ into $\mathbf{A}^1$ admits a filtration, whose nth stage consists of those rational functions on X_A having poles of order at most n along the divisor K. We will define a corresponding filtration

$$\mathcal{C}'(0) \subseteq \mathcal{C}'(1) \subseteq \mathcal{C}'(2) \subseteq \cdots$$

of the prestack $\mathcal{C}'$. We are then reduced to the problem of showing that the map $\mathrm{H}_*(\mathcal{C}'(m); \mathbf{Z}_\ell) \simeq \mathrm{H}_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$ is an isomorphism for $m \gg *$.

(*c*) We show that the prestack $\mathcal{C}'(m)$ is highly connected for $m \gg 0$ by showing that it is equivalent to an open subset of an affine space having large codimension.

We begin with step (*a*). Note that the forgetful functor $\operatorname{Sect}(U \subseteq X_R \times_{\operatorname{Spec} k} \mathbf{A}^1) \to \operatorname{Ran}^+(X)$ is a coCartesian fibration of categories. Set $\mathcal{C} = \operatorname{Sect}(U \subseteq X_R \times_{\operatorname{Spec} k} \mathbf{A}^1) \times_{\operatorname{Ran}^+(X)} \operatorname{Ran}(X)$. Using Theorem 2.5.19 (and Proposition 2.5.11), we see that the canonical map

$$C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\operatorname{Sect}(U \subseteq X_R \times_{\operatorname{Spec} k} \mathbf{A}^1); \mathbf{Z}_\ell)$$

is an equivalence. Unwinding the definitions, we can identify $\mathcal{C}$ with the category whose objects are triples $(A, \mu : S \to X(A), \alpha)$, where A is a finitely generated R-algebra, S is a nonempty finite set, μ is a map of sets, and $\alpha : X_A - |\mu| \to X_R \times_{\operatorname{Spec} k} \mathbf{A}^1$ is a map of X_R-schemes for which $\alpha^{-1} U \subseteq X_A$ is full.

We wish to prove that the canonical map $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$ is an equivalence. Note that this map factors as a composition

$$C_*(\mathcal{C}; \mathbf{Z}_\ell) \xrightarrow{\theta} C_*(\operatorname{Ran}(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R; \mathbf{Z}_\ell) \xrightarrow{\theta'} C_*(\operatorname{Spec} R; \mathbf{Z}_\ell).$$

It follows from Corollary 2.4.13 (together with Proposition 2.3.40) that θ' is an equivalence. It will therefore suffice to show that θ is an equivalence. We have a commutative diagram of categories

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \operatorname{Ran}(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R \\ & \searrow \quad \swarrow & \\ & \operatorname{Fin}^{\mathrm{s}} & \end{array}$$

where the vertical maps are Cartesian fibrations. It will therefore suffice to show that for each nonempty finite set S, the induced map

$$\mathcal{C}_S \to (\operatorname{Ran}(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R) \times_{\operatorname{Fin}^{\mathrm{s}}} \{S\} \simeq X_R^S$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology, where $\mathcal{C}_S$ denotes the fiber product $\mathcal{C} \times_{\operatorname{Fin}^{\mathrm{s}}} \{S\}$.

Because the forgetful functors

$$\operatorname{Ran}(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R \to \operatorname{Fin}^{\mathrm{s}} \leftarrow \mathcal{C}$$

are Cartesian fibrations of categories, the canonical maps

$$C_*(\mathcal{C}_S; \mathbf{Z}_\ell) \to ((vu)_! u^* \mathcal{F})(S) \qquad C_*(X_R^S; \mathbf{Z}_\ell) \to (v_! \mathcal{F})(S)$$

are equivalences. We are therefore reduced to proving that the forgetful functor $\psi : \mathcal{C}_S \to X_R^S$ induces an equivalence $C_*(\mathcal{C}_S; \mathbf{Z}_\ell) \to C_*(X_R^S; \mathbf{Z}_\ell)$. In fact, we claim that ψ is a universal homology equivalence. To prove this, fix a finitely generated R-algebra A and a map of R-schemes $\operatorname{Spec} A \to X_R^S$, classifying a map of sets $\mu : S \to X(A)$. Set $\mathcal{C}' = \mathcal{C}_S \times_{X_R^S} \operatorname{Spec} A$. Unwinding the definitions, we can identify $\mathcal{C}'$ with the category whose objects are pairs (B, α), where B is a finitely generated A-algebra and $\alpha : \operatorname{Spec} B \times_{\operatorname{Spec} A} (X_A - |\mu|) \to X_R \times_{\operatorname{Spec} k} \mathbf{A}^1$ is a map of X_R-schemes satisfying the following condition:

(∗) The projection map $\alpha^{-1}(U) \to \operatorname{Spec} B$ is surjective.

We now proceed with step (b). We wish to prove that the canonical map $C_*(\mathcal{C}'; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$ is an equivalence. Let $\overline{\mathcal{C}}'$ denote the category whose objects are pairs (B, α), where B is a finitely generated A-algebra and $\alpha : \operatorname{Spec} B \times_{\operatorname{Spec} A} (X_A - |\mu|) \to X_R \times_{\operatorname{Spec} k} \mathbf{A}^1$ is a map of X_R-schemes (so that $\mathcal{C}'$ is the full subcategory of $\overline{\mathcal{C}}'$ spanned by those objects which satisfy condition $(*)$).

Let $K = |\mu|$ denote the effective divisor in the relative curve X_A determined by μ. For each integer m, we let $\mathcal{O}(mK)$ denote the sheaf on X_A determined by the divisor mK. If B is a finitely generated A-algebra, we let $\mathcal{O}(mK)_B$ denote the pullback of $\mathcal{O}(mK)$ to the relative curve X_B, and $\mathcal{O}(\infty K)_B$ the direct limit $\varinjlim \mathcal{O}(mK)_B$. Unwinding the definitions, we see that $\overline{\mathcal{C}}'$ can be identified with the category whose objects are pairs (B, α), where B is a finitely generated A-algebra and α is a global section of the quasi-coherent sheaf $\mathcal{O}(\infty K)_B$. For each integer $m \geq 0$, we let $\overline{\mathcal{C}}'_m$ denote the full subcategory of $\overline{\mathcal{C}}'$ spanned by those pairs (B, α) where α is a section of $\mathcal{O}(mK)_B$, and set $\mathcal{C}'_m = \overline{\mathcal{C}}'_m \cap \mathcal{C}'$. Let s denote the cardinality of S (so that s is the degree of the finite flat map $K \to \operatorname{Spec} A$). Using the Riemann-Roch theorem, we deduce that there exists constants m_0 and C such that for $m \geq m_0$, $\overline{\mathcal{C}}'_m$ is representable by a vector bundle E_m over $\operatorname{Spec} A$ of rank $C + ms$. Applying Example 2.5.14, we deduce that the map

$$C_*(\overline{\mathcal{C}}'_m; \mathbf{Z}_\ell) = C_*(E_m; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$$

is an equivalence for $m \geq m_0$. Consequently, to prove that the projection map $C_*(\mathcal{C}'; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$ is an equivalence, it will suffice to show that the natural map

$$\mathrm{H}_d(\mathcal{C}'(m); \mathbf{Z}_\ell) \to \mathrm{H}_d(\overline{\mathcal{C}}'(m); \mathbf{Z}_\ell)$$

is an isomorphism for $m \gg d$.

We now carry out step (c). Note that for $m \geq m_0$, the prestack $\mathcal{C}'_m$ is representable by an open subscheme $E^\circ_m \subseteq E_m$. It will now be convenient to use Verdier duality for ℓ-adic sheaves (see §4.5 for a more detailed discussion). Let ω_{E_m} denote the dualizing sheaf of E_m, so that we can identify θ with the canonical map

$$\mathrm{H}^{-d}_c(E^\circ_m; j^* \omega_{E_m}) \to \mathrm{H}^{-d}_c(E_m; \omega_{E_m}).$$

Let Y_m denote the complement of E°_m in E_m and let i denote the inclusion of Y_m into E_m. We then have an exact sequence

$$\mathrm{H}^{-d-1}_c(Y_m; i^* \omega_{E_m}) \to \mathrm{H}^{-d}_c(E^\circ_m; j^* \omega_{E_m}) \to \mathrm{H}^{-d}_c(E_m; \omega_{E_m}) \to \mathrm{H}^{-d}_c(Y_m; i^* \omega_{E_m}).$$

It will therefore suffice to show that the groups $\mathrm{H}^{-d-1}_c(Y_m; i^* \omega_{E_m})$ and $\mathrm{H}^{-d}_c(Y_m; i^* \omega_{E_m})$ vanish for large values of m.

For $m \geq m_0$, the map $E_m \to \operatorname{Spec} A$ is smooth of relative dimension $ms + C$. Since the dualizing sheaf of $\operatorname{Spec} A$ is concentrated in cohomological degrees ≤ 0, we conclude that ω_{E_m} is concentrated in cohomological degrees $\leq -2(ms + C)$. Note also that the map $Y_m \to \operatorname{Spec} A$ has finite fibers, so that $\dim(Y_m) \leq \dim(\operatorname{Spec} A)$. It follows that the desired vanishing holds whenever $-2(ms + C) + 2 \dim(\operatorname{Spec} A) < -d - 1$. □

3.7. **Existence of Rational Trivializations.** Throughout this section we let k be an algebraically closed field, X an algebraic curve over k, $D \subseteq X$ an effective divisor, and G a smooth affine group scheme over X whose generic fiber G_0 is semisimple and simply connected.

Let R be a finitely generated k-algebra, let $D_R = D \times_{\operatorname{Spec} k} \operatorname{Spec} R$, and let $\mathcal{P}$ be a G-bundle on X_R which is equipped with a trivialization γ_0 on the divisor $D_R \subseteq X_R$. To prove Theorem 3.3.6, we would like to show that, after passing to an étale cover of $\operatorname{Spec} R$, we can extend γ_0 to a rational trivializaton of $\mathcal{P}$. When $R = k$, the existence of a rational trivialization of $\mathcal{P}$ follows from the vanishing of the cohomology set $\mathrm{H}^1(K_X; G_0)$ (where G_0 denotes the generic fiber of G). Consequently, for each k-valued point y of $\operatorname{Spec} R$, we can choose a trivialization γ_y of the fiber $\mathcal{P}_y$ over a dense open subset $U \subseteq X$. Moreover, there are no *infinitesimal* obstructions to extending this trivialization to a neighborhood of y: if U is affine, then the smoothness of $\mathcal{P}$ implies that γ_y can be extended to a trivialization of $\mathcal{P}$ over $U \times_{\operatorname{Spec} k} \operatorname{Spec} R/\mathfrak{m}_y^n$, where $\mathfrak{m}_y$ denotes the maximal ideal of R determined by the point y. In other words, we can choose a generic trivialization of $\mathcal{P}$ over a *formal* neighborhood of y in $\operatorname{Spec} R$. However, there is no obvious way to extend this trivialization from a formal neighborhood to an étale neighborhood, because the collection of generic trivializations of $\mathcal{P}$ is not parametrized by any reasonable finite-dimensional algebro-geometric object. We might attempt to remedy the situation by studying trivializations which are defined over the *entire* curve X: these are parametrized by an affine R-scheme of finite type (given by the Weil restriction of $\mathcal{P}$ along the map $X_R \to \operatorname{Spec} R$). However, this Weil restriction could be empty (since G-bundles on X can be globally nontrivial).

Following Drinfeld and Simpson ([15]), we will circumvent these difficulties by first looking for a weaker structure on the G-bundle $\mathcal{P}$: namely, a reduction of structure group from G to a Borel subgroup. Since the fraction field K_X has dimension 1, the group G_0 is automatically quasi-split ([10]); we may therefore choose a Borel subgroup $B_0 \subseteq G_0$. Let B denote the scheme-theoretic closure of B_0 in G. Then B is an affine group scheme which is flat (but not necessarily smooth) over X. We will deduce Theorem 3.3.6 from the following result, which we will prove in §3.9:

Theorem 3.7.1. *Let G be a smooth affine group scheme over X such that the generic fiber of G is either semisimple and simply connected or semisimple and adjoint. Let R be a finitely generated k-algebra, let $\mathcal{P}$ be a G-bundle on X_R, and let γ_0 be a trivialization of $\mathcal{P}|_{D_R}$. Then, étale locally on* $\operatorname{Spec} R$*, the G-bundle $\mathcal{P}$ admits a B-reduction which is compatible with γ_0.*

Using Theorem 3.7.1, we can reduce the problem of finding rational trivializations of G-bundles to the problem of finding rational reductions of B-bundles, which we will discuss in §3.9.

Remark 3.7.2. In the special case where the group scheme G is constant and $D = \emptyset$, Theorem 3.7.1 is proven in [15].

The remainder of this section is devoted to the deduction of Theorem 3.3.6 from Theorem 3.7.1. Note that the generic fiber of B is a smooth algebraic group over the fraction field K_X. We may therefore choose a dense open subset $U \subseteq X$ containing D such that $B \times_X (U - D)$ is a smooth affine group scheme over $U - D$. Shrinking U if necessary, we may further assume that U is affine and that D is the vanishing locus of a regular function $t \in \mathcal{O}_X(U)$.

Let $\mathcal{P}$ be as in the statement of Theorem 3.7.1. Note that both $\mathcal{P}$ and $B\backslash\mathcal{P}$ are smooth X-schemes (Lemma 3.9.6). Let us denote their relative tangent bundles by $T = T_{\mathcal{P}/X}$ and $T' = T_{(B\backslash\mathcal{P})/X}$, respectively. Then the projection map $\pi : \mathcal{P} \to B\backslash\mathcal{P}$ induces a map of vector bundles $e : T \to \pi^* T'$ on $\mathcal{P}$. Note that e is surjective over the open set $\mathcal{P} \times_X (U - D)$. Since $\mathcal{P} \times_X (U - D)$ is affine, the map e admits a section s over the open set $\mathcal{P} \times_X (U - D) \subseteq \mathcal{P}$. For $n \gg 0$, the product $t^n s$ extends (uniquely) to a regular map $e' : (\pi^* T')|_{\mathcal{P} \times_X U} \to T|_{\mathcal{P} \times_X U}$. Replacing D by nD (and arbitrarily lifting γ_0 to a trivialization of $\mathcal{P}$ over nD), we may assume that $n = 1$. In this case, we have the following result:

Lemma 3.7.3. *Suppose we are given a B-reduction of $\mathcal{P}$, given by a map $f : X_R \to B\backslash\mathcal{P}$. Suppose that $m \geq 2$, and that we are given a map $\beta_{m+1} : (m+1)D_R \to \mathcal{P}$ in the category of*

X_R-schemes such that the diagram

$$\begin{array}{ccc} (m+1)D_R & \xrightarrow{\beta_{m+1}} & \mathcal{P} \\ \downarrow & & \downarrow \\ X_R & \xrightarrow{f} & B\backslash \mathcal{P} \end{array}$$

commutes. Then there exists a map of X_R-schemes $\beta_{m+2} : (m+2)D_R \to \mathcal{P}$ such that $\beta_{m+1}|_{mD_R} = \beta_{m+2}|_{mD_R}$ and the diagram

$$\begin{array}{ccc} (m+2)D_R & \xrightarrow{\beta_{m+2}} & \mathcal{P} \\ \downarrow & & \downarrow \\ X_R & \xrightarrow{f} & B\backslash \mathcal{P} \end{array}$$

commutes.

Proof. Since $\mathcal{P}$ is smooth, we can choose a map $\overline{\beta}_{m+2} : (m+2)D_R \to \mathcal{P}$ extending β_{m+1}. Since $m \geq 2$, we can view the structure sheaf of $(m+2)D_R$ as a square-zero extension of the structure sheaf of mD_R by an ideal $\mathcal{I}$, so that the collection of all maps $(m+2)D_R \to \mathcal{P}$ compatible with $\overline{\beta}_{m+2}|_{mD_R}$ can be identified with $\mathrm{H}^0(D_R; T|_{mD_R} \otimes \mathcal{I})$ (so that the identity element corresponds to $\overline{\beta}_{m+2}$). Similarly, we can identify the set of all maps from $(m+2)D_R \to B\backslash \mathcal{P}$ compatible with $(\pi\overline{\beta}_{m+2})|_{mD_R}$ with the set of global sections $\mathrm{H}^0(D_R; (\pi^* T')|_{mD_R} \otimes \mathcal{I})$. In particular, the restriction $f|_{(m+2)D_R}$ determines an element $[f] \in \mathrm{H}^0(D_R; (\pi^* T')|_{mD_R} \otimes \mathcal{I})$. Unwinding the definitions, we wish to prove that f belongs to the image of the map

$$\mathrm{H}^0(D_R; T|_{mD_R} \otimes \mathcal{I}) \to \mathrm{H}^0(D_R; (\pi^* T')|_{mD_R} \otimes \mathcal{I}).$$

In fact, we claim that $[f]$ belongs to the image of the composite map

$$\mathrm{H}^0(D_R; (\pi^* T')|_{mD_R} \otimes \mathcal{I}) \to \mathrm{H}^0(D_R; T|_{mD_R} \otimes \mathcal{I}) \to \mathrm{H}^0(D_R; (\pi^* T')|_{mD_R} \otimes \mathcal{I}),$$

where the first map is induced by $ts : (\pi^* T')|_{\mathcal{P} \times_X U} \to T|_{\mathcal{P} \times_X U}$. This is equivlent to the assertion that $[f]$ is divisible by t, which follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} (m+1)D_R & \xrightarrow{\beta_{m+1}} & \mathcal{P} \\ \downarrow & & \downarrow \\ X_R & \xrightarrow{f} & B\backslash \mathcal{P}. \end{array}$$

□

Lemma 3.7.4. *In the situation of Theorem 3.3.6, there exists étale covering of* $\operatorname{Spec} R$ *over which there is a reduction of $\mathcal{P}$ to a B-bundle $\mathcal{Q} \subseteq \mathcal{P}$ such that γ_0 extends to a trivialization of $\mathcal{Q}$ over the formal completion of X_R along D_R.*

Proof. Since G is smooth, we can extend γ_0 to a map of X_R-schemes $\beta_3 : 3D_R \to \mathcal{P}$. Using Theorem 3.7.1, we can choose a B-bundle $\mathcal{Q} \subseteq \mathcal{P}$ such that β_3 factors through $\mathcal{Q}$. Repeatedly applying Lemma 3.7.3, we can choose maps $\{\beta_m : mD_R \to \mathcal{Q}\}_{m \geq 3}$ such that $\beta_{m+1}|_{mD_R} = \beta_{m+2}|_{mD_R}$. Then the sequence of restrictions $\{\beta_{m+1}|_{mD_R}\}_{m \geq 2}$ determines a trivialization of $\mathcal{Q}$ on the formal completion of X_R along D_R. □

Note that to prove Theorem 3.3.6, we are free to enlarge the divisor D; in particular, we may always assume that D is nonempty. By virtue of Lemma 3.7.4, we are reduced to proving the following:

Proposition 3.7.5. *Assume that the divisor D is nonempty. Let R be a finitely generated k-algebra, let $\mathcal{Q}$ be a B-bundle on X_R, and let $\widehat{\gamma}$ be a trivialization of $\mathcal{Q}$ on the formal completion $D_R^\wedge$ of D_R in X_R. Then there exists a full open subset $U \subseteq X_R$ containing D_R and a trivialization of $\mathcal{Q}|_U$ which agrees with $\widehat{\gamma}$ on the divisor D_R.*

Our proof of Proposition 3.7.5 will rely on the following elementary algebraic fact, whose proof we defer until the end of this section.

Lemma 3.7.6. *Let R be a reduced Noetherian ring such that $\operatorname{Spec} R$ is connected. Let f be an invertible element in the Laurent series ring $R((u)) = R[[u]][u^{-1}]$. Then $f = u^n g$ for some invertible element g in $R[[u]]$.*

Proof of Proposition 3.7.5. Our argument will proceed in three steps:

- (a) We reduce the proof of Proposition 3.7.5 to a density statement concerning the set of A-valued points of the torsor $\mathcal{Q}$ (assertion $(*)$ below).
- (b) We show that $(*)$ follows from the analogous assertion for the maximal torus $T \subseteq B$.
- (c) Using the simple connectivity of G, we show that the requisite density statement for the torus T follows from elementary ring-theoretic considerations.

We begin with step (a). Let G_0 be the generic fiber of G. Since K_X has dimension 1, the group G_0 is quasi-split (see [10]). It follows that there exists a dense open subset $V \subseteq X$ such that $G_V = V \times_X G$ is a quasi-split semisimple group scheme over V. Shrinking V if necessary, we may assume that $D \cap V = \emptyset$, that $V' = V \cup D$ is affine, and that D is the vanishing locus of a regular function t on V'.

Let S denote the set of all relative divisors $D' \subseteq X_R$ such that $D' \cap D_R = \emptyset$ and $X_R - D' \subseteq V'_R$. For each $D' \in S$, the difference $X_R - D'$ is an affine open subset of X_R, which we can write as $\operatorname{Spec} A_{D'}$ for some finitely generated R-algebra $A_{D'}$. We will abuse notation by identifying t with its image on $A_{D'}$, so that $X_R - (D' \cup D_R) \simeq \operatorname{Spec} A_{D'}[t^{-1}]$. Let $\widehat{A}$ denote the t-adic completion of $A_{D'}$: note that this completion depends only on the divisor D, and not on D'. Let us regard $\widehat{A}$ and $\widehat{A}[t^{-1}]$ as equipped with the t-adic topology. Note that for any affine A-scheme $Y = \operatorname{Spec} B$, the t-adic topologies on $\widehat{A}$ and $\widehat{A}[t^{-1}]$ determine topologies on the sets $Y(\widehat{A})$ and $Y(\widehat{A}[t^{-1}])$; we will apply this observation in the case $Y = \mathcal{Q}$.

For every commutative ring C equipped with a map $\operatorname{Spec} C \to X_R$, we let $\mathcal{Q}(C)$ denote the set of trivializations of $\mathcal{Q} \times_{X_R} \operatorname{Spec} C$: that is, the set of X_R-scheme morphisms from $\operatorname{Spec} C$ into $\mathcal{Q}$.Then $\mathcal{Q}(\widehat{A})$ can be identified with the set of trivializations of $\mathcal{Q}$ over the formal completion $D_R^\wedge$. Under this identification, the collection of those trivializations which coincide with γ on the divisor D corresponds to an open subset of $\mathcal{Q}(\widehat{A})$, which we can identify with a nonempty open subset of $W \subseteq \mathcal{Q}(\widehat{A}[t^{-1}])$.

For each $D' \in S$, we have a pullback square

$$\begin{array}{ccc} \mathcal{Q}(A_{D'}) & \longrightarrow & \mathcal{Q}(A_{D'}[t^{-1}]) \\ \downarrow & & \downarrow \\ \mathcal{Q}(\widehat{A}) & \longrightarrow & \mathcal{Q}(\widehat{A}[t^{-1}]). \end{array}$$

Consequently, to show that there exists a trivialization of $\mathcal{Q}$ over $X_R - D'$ which agrees with $\widehat{\gamma}$ on the divisor D, it will suffice to show that the image of the map $\mathcal{Q}(A_{D'}[t^{-1}]) \to \mathcal{Q}(\widehat{A}[t^{-1}])$ has nonempty intersection with W. To complete the proof, it will suffice to show the following:

$(*)$ The union of the images of the maps $\mathcal{Q}(A_{D'}[t^{-1}]) \to \mathcal{Q}(\widehat{A}[t^{-1}])$ is dense in $\mathcal{Q}(\widehat{A}[t^{-1}])$ (as D' ranges over S).

We now proceed with step (b). Note that $B_V = V \times_X B$ is the scheme-theoretic closure of B_0 in G_V, which is a Borel subgroup of G_V. It therefore fits into an exact sequence

$$0 \to \operatorname{rad}_u B_V \to B_V \to T \to 0,$$

where T is a diagonalizable group scheme over V. Since G_0 is simply connected, T is isomorphic to a finite product of induced tori: that is, there exists a collection of finite étale maps $\{V_i \to V\}_{1 \leq i \leq m}$ such that T is isomorphic to the product of Weil restrictions

$$\prod_{1 \leq i \leq m} \operatorname{Res}^{V_i/V}(\mathbf{G}_m \times_{\operatorname{Spec} k} V_i).$$

Without loss of generality, we may assume that each V_i is connected. Let X_i denote the smooth projective curve having the same fraction field as V_i, so that we have finite maps $X_i \to X$ with $V_i \simeq X_i \times_X V$. Each of the projection maps $B_V \to T \to \operatorname{Res}_{V_i/V}(\mathbf{G}_m \times_{\operatorname{Spec} k} V_i)$ classifies a map $B_V \times_V V_i \to \mathbf{G}_m \times_{\operatorname{Spec} k} V_i$ of group schemes over V_i. Using Proposition A.1.8, we see that each of these maps can be extended to a map $B \times_X X_i \to \mathbf{G}_m \times_{\operatorname{Spec} k} X_i$ of group schemes over X_i. In particular, the B-bundle $\mathcal{Q}$ on X_R determines a line bundle $\mathcal{L}_i$ on X_{iR} for $1 \leq i \leq m$.

For each index i, let D_i denote the (scheme-theoretic) inverse image of D in X_i, and choose a k-rational point x_i of $X_i - D_i$. Choose an integer $N \gg 0$, so that each of the line bundles $\mathcal{L}_i(Nx_i)$ has degree $\geq 2g_i + \deg(D_i)$ along each fiber of the map $\nu_i : X_{iR} \to \operatorname{Spec} R$. It follows that $\mathrm{H}^1(X_{iR}; \mathcal{L}_i(Nx_i - D_{iR}))$ vanishes, and therefore the restriction map

$$\mathrm{H}^0(X_{iR}; \mathcal{L}_i(Nx_i)) \to \mathrm{H}^0(D_{iR}, \mathcal{L}_i\,|_{D_{iR}})$$

is surjective. Since $\mathcal{Q}$ is trivial along the divisor D, the line bundles $\mathcal{L}_i$ admit trivializations along the relative divisors D_{iR}. We may therefore lift these trivializations to sections s_i of $\mathcal{L}_i(Nx_i)$. Let $E_i \subseteq X_{iR}$ denote the union of $\{x_i\} \times \operatorname{Spec} R$ with the vanishing locus of s_i (since $D \neq \emptyset$ and s_i does not vanish on D_i, this vanishing locus is a relative divisor).

Let $S' \subseteq S$ denote the collection of those divisors $D' \in S$ which contain the image of each E_i. If $D' \in S'$, then each of the line bundles $\mathcal{L}_i$ is trivial on $X_{iR} \times_{X_R} (X_R - D')$. It follows that the T-bundle on $X_R - (D_R \cup D')$ induced from $\mathcal{Q}$ is trivial: that is, $\mathcal{Q}$ is induced from a $\operatorname{rad}_u B_V$-bundle over the open set $X_R - (D_R \cup D')$. Since $X_R - (D_R \cup D')$ is affine and $\operatorname{rad}_u B_V$ admits a finite filtration by vector groups, every $\operatorname{rad}_u B_V$-bundle on $X_R - (D_R \cup D')$ is automatically trivial. It follows that $\mathcal{Q}\,|_{X_R - (D_R \cup D')}$ is trivial, so that we have identifications

$$\mathcal{Q}(A_{D'}[t^{-1}]) \simeq B_V(A_{D'}[t^{-1}]) \qquad \mathcal{Q}(\widehat{A}[t^{-1}]) \simeq B_V(\widehat{A}[t^{-1}]).$$

Since S' is cofinal in S, assertion $(*)$ can be reformulated as follows:

$(*')$ The map

$$\varinjlim_{D' \in S'} B_V(A_{D'}[t^{-1}]) \to B_V(\widehat{A}[t^{-1}])$$

has dense image.

Note that assertion $(*')$ depends only on the structure of B_V as a V-scheme, and not on the group structure of B_V. Since T is affine and $\operatorname{rad}_u(B_V)$ is a successive extension of vector groups, we have an isomorphism (in the category of V-schemes) $B_V \simeq T \times_V \mathcal{E}$, where $\mathcal{E}$ is the total space of a vector bundle over V. It follows easily that the map $\mathcal{E}(A_{D'}[t^{-1}]) \to \mathcal{E}(\widehat{A}[t^{-1}])$ has dense image for *any* $D' \in S'$. We are therefore reduced to proving that the map

$$\varinjlim_{D' \in S'} T(A_{D'}[t^{-1}]) \to T(\widehat{A}[t^{-1}])$$

has dense image.

We now carry out step (c). Writing T as a product of induced tori $T_i = \mathrm{Res}^{V_i/V}(\mathbf{G}_m \times_{\operatorname{Spec} k} V_i)$, we are reduced to proving that each of the maps $\varinjlim_{D' \in S'} T_i(A_{D'}[t^{-1}]) \to T_i(\widehat{A}_i[t^{-1}])$ has dense image. Write $X_i \times_X (X_R - D') \simeq \operatorname{Spec} A_{iD'}$, where $A_{iD'}$ is a finite flat $A_{D'}$-module, and let $\widehat{A}_i$ be the t-adic completion of $A_{iD'}$ (which is independent of D'). We are then reduced to showing that the map

$$\varinjlim_{D' \in S'} A_{iD'}[t^{-1}]^{\times} \to \varinjlim \widehat{A}_i[t^{-1}]^{\times}$$

has dense image.

Let $\{x_1, \ldots, x_m\}$ be the closed points of X_i which belong to the divisor D_i. For $1 \leq j \leq m$, choose a rational function u_i on X_i which vanishes at the point x_j, and has neither zeroes nor poles on the set $\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m\}$. Shrinking V if necessary, we may assume that each u_i is a regular function on $X_i \times_X V'$, which vanishes only at the point x_j. Let us abuse notation by identifying each u_j with its image in $A_{iD'}$ (for each $D' \in S'$) and $\widehat{A}_i$. Then $\widehat{A}_i$ is isomorphic to the product $\prod_{1 \leq j \leq m} R[[u_j]]$, and $\widehat{A}_i[t^{-1}]$ is isomorphic to the product $\prod_{1 \leq j \leq m} R((u_j))$.

Factoring R as a product if necessary, we may assume without loss of generality that $\operatorname{Spec} R$ is connected. Let $\mathrm{rad}(R)$ denote the nilradical of R. For each $D' \in S'$, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\mathrm{rad}(R) \otimes_R A_{iD'} & \longrightarrow & A_{iD'}^{\times} & \longrightarrow & (A_{iD'}/\mathrm{rad}(R)A_{iD'})^{\times} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\
\prod_{1 \leq j \leq m} \mathrm{rad}(R)((u_j)) & \longrightarrow & \prod_{1 \leq j \leq m} R((u_j)) & \longrightarrow & \prod_{1 \leq j \leq m} (R/\mathrm{rad}(R))((u_j)) & \longrightarrow & 0
\end{array}$$

where the map ϕ' has dense image. Consequently, we may replace R by $R/\mathrm{rad}(R)$ and thereby reduce to the case where R is reduced. In this case, it follows from Lemma 3.7.6 that the units in the ring $\prod_{1 \leq j \leq m} R((u_j))$ are given by

$$u_1^{\mathbf{Z}} u_2^{\mathbf{Z}} \cdots u_m^{\mathbf{Z}} \widehat{A}_i^{\times}.$$

It will therefore suffice to proving the following:

$(*'')$ The map

$$\varinjlim_{D' \in S'} A_{iD'}^{\times} \to \widehat{A}_i^{\times}$$

has dense image.

To prove $(*'')$, choose any $D' \in S'$. Let x be an invertible element in $\widehat{A}_i^{\times}$. For each $n \geq 0$, we can choose an element $y \in A_{iD'}$ whose image in $\widehat{A}_i$ is congruent to x modulo t^n. Then y is a regular function on $X_{iR} \times_{X_R} (X_R - D')$ which does not vanish along the divisor D_{iR}. Let $E \subseteq X_{iR}$ denote the union of $D' \times_X X_i$ with the vanishing locus of y, and let D'' denote the image of E in X_R. Then D'' is an effective divisor which contains D' and does not intersect D'_R. It follows that D'' belongs to S', and the image of y in $A_{iD''}$ is invertible by construction. Since n was chosen arbitrarily, it follows that x admits arbitrarily close approximations by elements of the direct limit $\varinjlim_{D' \in S'} A_{iD'}^{\times}$. □

Proof of Lemma 3.7.6. Suppose first that R is an integral domain. For every nonzero f in $R((u))$, let $\lambda(f)u^{n(f)}$ denote the monomial of lowest degree which occurs in f. If g is another nonzero element, we have

$$\lambda(fg) = \lambda(f)\lambda(g) \qquad n(fg) = n(f) + n(g).$$

In particular, if $fg = 1$, then we have $\lambda(f)\lambda(g) = 1$, so that $\lambda(f)$ is an invertible element of R. It follows that $u^{-n(f)} f \in \lambda(f) + uR[[u]]$ is an invertible element of $R[[u]]$.

We now treat the general case. Suppose that f is an invertible element of $R((u))$. Then for every prime ideal $\mathfrak{p} \subseteq R$, the image of f in $(R/\mathfrak{p})((u))$ is invertible. We may therefore write the image of f as a product $u^{n(\mathfrak{p})} g_{\mathfrak{p}}$, where $g_{\mathfrak{p}}$ is an invertible element of $(R/\mathfrak{p})[[u]]$. Note that $n(\mathfrak{p})$ is uniquely determined by $\mathfrak{p}$. Moreover, if $\mathfrak{p} \subseteq \mathfrak{q}$, then the image of $g_{\mathfrak{p}}$ in $(R/\mathfrak{q})[[u]]$ is invertible, so that $g_{\mathfrak{p}} = g_{\mathfrak{q}}$ and $n(\mathfrak{p}) = n(\mathfrak{q})$. It follows that the function $\mathfrak{p} \mapsto n(\mathfrak{p})$ is constant on each connected component of $\operatorname{Spec} R$. Since $\operatorname{Spec} R$ is connected, the function $\mathfrak{p} \mapsto n(\mathfrak{p})$ is constant with value n, for some integer n. Replacing f by $u^{-n} f$, we may assume that $n = 0$.

Write $f = \sum c_i u^i$. The above argument shows that for each prime ideal $\mathfrak{p} \subseteq R$, we have $c_i \in \mathfrak{p}$ for $i < 0$. Consequently, the elements $c_{-1}, c_{-2}, \ldots \in R$ belong the nilradical of R. Since R is reduced, we deduce that $c_i = 0$ for $i < 0$. Moreover, $c_0 \notin \mathfrak{p}$ for each prime ideal $\mathfrak{p} \in R$, so that c_0 is invertible in R. It follows that f is an invertible element of $R[[u]]$. $\square$

3.8. **Digression: Maps of Large Degree.** Throughout this section, we let k denote an algebraically closed field, G a semisimple algebraic group defined over k, and Γ a finite group which acts on G by automorphisms that preserve a pinning $(B, T, \{u_\alpha\})$ of G (see §A.4). Let $\widetilde{X}$ be an algebraic curve over k equipped with a faithful (but not necessarily free) action of Γ and let X denote the quotient $\widetilde{X}/\Gamma$ (formed in the category of k-schemes). Our goal in this section is to establish the following technical result which will be needed for the proof of Theorem 3.7.1:

Proposition 3.8.1. *Let $D \subseteq \widetilde{X}$ be an effective divisor and let $\mathcal{L}$ be a line bundle on $\widetilde{X}$. Then there exists a map $s : \widetilde{X} \to G/B$ with the following properties:*

- (a) *The map s is Γ-equivariant.*
- (b) *The restriction $s|_D$ is equal to the constant map from D to the base point of G/B.*
- (c) *The cohomology group $\mathrm{H}^1(\widetilde{X}; \mathcal{L} \otimes s^* T_{G/B})$ vanishes. Here $T_{G/B}$ denotes the tangent bundle to the flag variety G/B.*

Every character $\lambda : B \to \mathbf{G}_m$ of the group B determines a G-equivariant line bundle on the flag variety G/B, which we will denote by $\mathcal{L}_\lambda$. If $s : \widetilde{X} \to G/B$ is a map, we let $\deg_\lambda(s)$ denote the degree of the line bundle $s^* \mathcal{L}_\lambda$. The function $\lambda \mapsto \deg_\lambda(s)$ is linear in λ, and therefore given by pairing λ with a coweight $\deg(s) \in \operatorname{Hom}(\mathbf{G}_m, T)$.

Let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{b} \subseteq \mathfrak{g}$ the Lie algebra of the Borel subgroup B. It follows from the structure theory of reductive groups that the quotient $\mathfrak{g}/\mathfrak{b}$ admits a B-equivariant filtration, whose successive quotients are one-dimensional representations of B associated to the characters $\alpha : T \to \mathbf{G}_m$ where α is a negative root. It follows that the tangent bundle $T_{G/B}$ admits a finite filtration whose successive quotients are the line bundles $\mathcal{L}_\alpha$ where α is a negative root of G. Consequently, for each line bundle $\mathcal{L}$ on $\widetilde{X}$ and every map $s : \widetilde{X} \to G/B$, the cohomology group $\mathrm{H}^1(\widetilde{X}; \mathcal{L} \otimes s^* T_{G/B})$ admits a finite filtration, whose succesive quotients are subquotients of the groups $\mathrm{H}^1(\widetilde{X}; \mathcal{L} \otimes s^* \mathcal{L}_\alpha)$ where α is a negative root of G. These groups vanish provided that $\deg_\alpha(s) > 2g - 2 + \deg(\mathcal{L})$. Proposition 3.8.1 is therefore an immediate consequence of the following:

Proposition 3.8.2. *Let $D \subseteq \widetilde{X}$ be an effective divisor and let C be an integer. Then there exists a map $s : \widetilde{X} \to G/B$ with the following properties:*

- (a) *The map s is Γ-equivariant.*
- (b) *The restriction $s|_D$ is equal to the constant map from D to the base point of G/B.*
- (c) *For every negative root α of G, we have $\deg_\alpha(s) \geq C$.*

The proof of Proposition 3.8.2 depends on the following:

Lemma 3.8.3. *Let G be a reductive group and B a Borel subgroup of G. For every dominant weight μ of G, there exists a map $f : \mathbf{P}^1 \to G/B$ of degree $-\mu$.*

Proof. Let $\mathrm{Bun}_B^{\mu}(\mathbf{P}^1)$ denote the moduli stack of B-bundles on $\mathbf{P}^1$ having degree $-\mu$. For every such bundle $\mathcal{P}$, let $\mathcal{E}_{\mathcal{P}}$ denote the vector bundle on $\mathbf{P}^1$ associated to the represntation $\mathfrak{g}/\mathfrak{b}$ of B. Note that $\mathcal{E}_{\mathcal{P}}$ admits a filtration whose successive quotients are line bundles of degree $\langle -\mu, \alpha \rangle$, where α ranges over the negative roots of G. Since μ is dominant, the cohomology group $\mathrm{H}^1(\mathbf{P}^1; \mathcal{E}_{\mathcal{P}})$ vanishes. It follows that the inclusion of B into G induces a smooth morphism of algebraic stacks $u : \mathrm{Bun}_B^{\mu}(\mathbf{P}^1) \to \mathrm{Bun}_G(\mathbf{P}^1)$. Since $\mathrm{Bun}_B^{\mu}(\mathbf{P}^1)$ is nonempty, the image of u is a nonempty open substack of $\mathrm{Bun}_G(\mathbf{P}^1)$. According to [15], the diagonal map $\mathrm{BG} \to \mathrm{Bun}_G(\mathbf{P}^1)$ is an open immersion with dense image, so that the fiber product $\mathrm{BG} \times_{\mathrm{Bun}_G(\mathbf{P}^1)} \mathrm{Bun}_B^{\mu}(\mathbf{P}^1)$ is nonempty. In particular, we can choose a B-bundle $\mathcal{P}$ on $\mathbf{P}^1$ of degree $-\mu$ for which the associated G-bundle is trivial. Then $\mathcal{P}$ is classified by a map $\mathbf{P}^1 \to G/B$ having degree $-\mu$. □

Proof of Proposition 3.8.2. We may assume without loss of generality that D is nonempty, that G is semisimple and simply connected, and that $C > 0$. Choose a rational function on $X = \widetilde{X}/\Gamma$ which vanishes on the image of the divisor D. This choice determines a Γ-equivariant map $g : \widetilde{X} \to \mathbf{P}^1$ (where Γ acts trivially on $\mathbf{P}^1$). Composing g with a map from $\mathbf{P}^1$ to itself if necessary, we may suppose that g has degree $\geq C$. Consequently, to construct a map $s : \widetilde{X} \to G/B$ satisfying conditions (a), (b), and (c), it will suffice to construct a map $s_0 : \mathbf{P}^1 \to G/B$ satisfying the following analogous conditions:

- (a') The map s_0 is Γ-equivariant (in other words, s_0 factors through the subgroup $G^{\Gamma} \subseteq G$).
- (b') The map s_0 carries the point $0 \in \mathbf{P}^1$ to the base point of G/B.
- (c') For every negative root α of G, we have $\deg_{\alpha}(s_0) > 0$.

Let G_0 denote the identity component of G^{Γ}, and let $B_0 = B \cap G_0$. Using Remark A.4.8 and Corollary A.4.7, we see that G_0 is a reductive group and that B_0 is a Borel subgroup of G_0. According to Lemma 3.8.3, there exists a map $u : \mathbf{P}^1 \to G_0/B_0$ such that $-\deg(u)$ is a strictly dominant weight of G_0. Without loss of generality, we may assume that this map carries the origin $0 \in \mathbf{P}^1$ to the base point of G_0/B_0. Then the composite map

$$\mathbf{P}^1 \to G_0/B_0 \hookrightarrow G/B$$

evidently satisfies conditions (a') and (b'), and satisfies (c') by virtue of Proposition A.4.10. □

3.9. **Existence of Borel Reductions.** Throughout this section, we let k denote an algebraically closed field, X an algebraic curve over k, G a smooth affine group scheme over X, and assume that the generic fiber of G is either semisimple and simply connected or semisimple and adjoint. Let B_0 a Borel subgroup of the generic fiber of G, and B the scheme-theoretic closure of B_0 in G. Our goal is to give a proof of Theorem 3.7.1, which asserts that any G-bundle on X_R admits a B-reduction étale locally on $\mathrm{Spec}\, R$.

Notation 3.9.1. Let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R. Then B acts on $\mathcal{P}$ via a map $a : B \times_X \mathcal{P} \to \mathcal{P}$. Let $\pi_2 : B \times_X \mathcal{P} \to \mathcal{P}$ denote the projection onto the second factor. The diagram

$$\mathcal{P} \xleftarrow{a} B \times_X \mathcal{P} \xrightarrow{\pi_2} \mathcal{P}$$

exhibits $B \times_X \mathcal{P}$ as an fppf equivalence relation on $\mathcal{P}$. We let $B \backslash \mathcal{P}$ denote the quotient of $\mathcal{P}$ by this equivalence relation (in the category of fppf sheaves on Ring_k). It follows from a general theorem of Artin that $B \backslash \mathcal{P}$ is representable by an algebraic space ([1]).

Remark 3.9.2. In the situation of Notation 3.9.1, we have maps

$$\mathcal{P} \xrightarrow{\alpha} B \backslash \mathcal{P} \xrightarrow{\beta} X_R,$$

where $\beta \circ \alpha$ is smooth and α is faithfully flat. It follows that $B\backslash \mathcal{P}$ is smooth over X_R.

Notation 3.9.3. Let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R. Let $\mathrm{Fl}(\mathcal{P})$ denote the algebraic space obtained by Weil restriction of $B\backslash \mathcal{P}$ along the map $X_R \to \operatorname{Spec} R$ (for a general discussion of Weil restriction, we refer the reader to [41]). In other words, $\mathrm{Fl}(\mathcal{P})$ is the R-scheme whose A-valued points can be identified with commutative diagrams

$$\begin{array}{ccc} X_A & \longrightarrow & B\backslash \mathcal{P} \\ & \searrow & \downarrow \\ & & X_R. \end{array}$$

Similarly, we let $\mathrm{Fl}(\mathcal{P}_{D_R})$ denote the Weil restriction of $(B\backslash \mathcal{P}) \times_{X_R} D_R$ along the projection map $D_R \to \operatorname{Spec} R$. We have an evident restriction map $\mathrm{Fl}(\mathcal{P}) \to \mathrm{Fl}(\mathcal{P}_{D_R})$. If γ_0 is a trivialization of $\mathcal{P}|_{D_R}$, then γ_0 determines a map $\operatorname{Spec} R \to \mathrm{Fl}(\mathcal{P}_{D_R})$; in this case, we let $\mathrm{Fl}_D(\mathcal{P})$ denote the fiber product

$$\operatorname{Spec} R \times_{\mathrm{Fl}(\mathcal{P}_{D_R})} \mathrm{Fl}(\mathcal{P}).$$

Let R, $\mathcal{P}$, and γ_0 be as in Theorem 3.7.1. Unwinding the definitions, we see that there is a bijective correspondence between A-valued points of $\mathrm{Fl}_D(\mathcal{P})$ and isomorphism classes of B-reductions of the G-bundle $X_A \times_{X_R} \mathcal{P}$ which are compatible with γ_0. Consequently, Theorem 3.7.1 is equivalent to the assertion that the map $\mathrm{Fl}_D(\mathcal{P}) \to \operatorname{Spec} R$ admits étale local sections. Let $\mathrm{Fl}_D(\mathcal{P})^\circ$ denote the smooth locus of the projection map $\mathrm{Fl}_D(\mathcal{P})^\circ \to \operatorname{Spec} R$. Since a smooth surjection admits étale local sections, Theorem 3.7.1 will follow if we can show that the map $\mathrm{Fl}_D(\mathcal{P})^\circ \to \operatorname{Spec} R$ is surjective.

Let y be a k-valued point of $\operatorname{Spec} R$, and let $\overline{y}$ be a k-valued point of $\mathrm{Fl}_D(\mathcal{P})$ lying over y. Let $\mathcal{P}_y$ denote the G-bundle on X determined by y, so that $\overline{y}$ can be identified with a section s of the projection map $\pi : B\backslash \mathcal{P}_y \to X$ which is compatible with η_0. The map π is smooth; let T_π denote its relative tangent bundle (a vector bundle on $B\backslash \mathcal{P}_y$). A standard deformation-theoretic argument shows that the cohomology group $\mathrm{H}^1(X; (s^* T_\pi)(-D))$ controls obstructions to deforming the section s (where the deformation is fixed along the divisor D). In particular, if the group $\mathrm{H}^1(X; (s^* T_\pi)(-D))$ vanishes, then $\overline{y}$ belongs to the smooth locus $\mathrm{Fl}(\mathcal{P})^\circ$. It will therefore suffice to show that for each k-valued point y of $\operatorname{Spec} R$, we can choose a section s (which is compatible with γ_0) such that $\mathrm{H}^1(X; (s^* T_\pi(-D))$ vanishes. In this case, we might as well replace R by k. It will therefore suffice to prove the following:

Proposition 3.9.4. *Let $\mathcal{P}$ be a G-bundle on X, let $\pi : B\backslash \mathcal{P} \to X$ be the projection map, and let $s_0 : D \to B\backslash \mathcal{P}$ be a map of X-schemes which can be lifted to a map $D \to \mathcal{P}$. Then s_0 can be extended to a section s of π with the property that $\mathrm{H}^1(X; (s^* T_\pi)(-D)) \simeq 0$.*

Remark 3.9.5. If the group scheme B is smooth, then the projection map $\mathcal{P} \to B\backslash \mathcal{P}$ is a smooth surjection. In this case, the existence of a map $D \to \mathcal{P}$ lifting s_0 is automatic.

The proof of Proposition 3.9.4 will require some preliminaries.

Lemma 3.9.6. *Let $\mathcal{P}$ be a G-bundle on X. Then the quotient $B\backslash \mathcal{P}$ is (representable by) a scheme (automatically separated, since B is closed in G).*

Lemma 3.9.7. *Let $\mathcal{P}$ be a G-bundle on X and let γ_0 be a trivialization of $\mathcal{P}$ over D. Then there exists a dense open subset $U \subseteq X$ which contains D and a trivialization of $\mathcal{P}|_U$ which extends γ_0.*

The proofs of Lemmas 3.9.6 and 3.9.7 will be given at the end of this section.

Lemma 3.9.8. *Let $\mathcal{P}$ be a G-bundle on X, and let $s_0 : D \to B\backslash \mathcal{P}$ be a map of X-schemes which can be lifted to a map $\overline{s}_0 : D \to \mathcal{P}$. Then s_0 can be extended to a section of the projection map $\pi : B\backslash \mathcal{P} \to X$.*

Proof. Since k is an algebraically closed field, the function field K_X has dimension 1. The generic fiber of G is a connected reductive algebraic group over K_X, so that every G-bundle on $\operatorname{Spec} K_X$ is trivial (see [10]). Choose a rational trivialization of $\mathcal{P}$, determining bijections giving isomorphisms $\mathcal{P}(L) \simeq G(L)$ for every field extension L of K_X.

Since B_0 is a parabolic subgroup of G_0, the fiber product $B\backslash \mathcal{P} \times_X \operatorname{Spec} K_X$ is proper over $\operatorname{Spec} K_X$. It follows that there exists a dense open subset $V \subseteq X$ for which the quotient $B\backslash \mathcal{P} \times_X V$ is proper over V. Shrinking V if necessary, we may suppose that $V \cap D = \emptyset$. Enlarging D, we may assume that $V = X - D$.

Applying Lemma 3.9.7, we deduce that there exists an open set $U \subseteq X$ containing D and a trivialization of $\mathcal{P}|_U$ which is compatible with $\overline{s}$. This trivialization determines a map f fitting into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & B\backslash \mathcal{P} \\ \downarrow & \nearrow & \downarrow \\ X & \xrightarrow{\mathrm{id}} & X. \end{array}$$

We now complete the proof by observing that the map f admits an extension as indicated in the diagram, by virtue of the valuative criterion of properness for the map $(B\backslash \mathcal{P}) \times_X V \to V$. □

Lemma 3.9.9. *Let $\phi : \mathcal{E}' \to \mathcal{E}$ be a map of vector bundles on X which is an epimorphism at the generic point of X. If $\mathrm{H}^1(X; \mathcal{E}') \simeq 0$, then $\mathrm{H}^1(X; \mathcal{E}) \simeq 0$.*

Proof. The exact sequence of quasi-coherent sheaves $0 \to \ker(\phi) \to \mathcal{E}' \to \mathcal{E}' / \ker(\phi) \to 0$ determines an exact sequence of cohomology groups $\mathrm{H}^1(X; \mathcal{E}') \to \mathrm{H}^1(X; \mathcal{E}' / \ker(\phi)) \to \mathrm{H}^2(X; \ker(\phi))$. Since X is a curve, $\mathrm{H}^2(X; \ker(\phi)) \simeq 0$. It follows that $\mathrm{H}^1(X; \mathcal{E}' / \ker(\phi)) \simeq 0$. The short exact sequence of sheaves

$$0 \to \mathcal{E}' / \ker(\phi) \to \mathcal{E} \to \operatorname{coker}(\phi) \to 0$$

determines a short exact sequence

$$\mathrm{H}^1(X; \mathcal{E}' / \ker(\phi)) \to \mathrm{H}^1(X; \mathcal{E}) \to \mathrm{H}^1(X; \operatorname{coker}(\phi)).$$

Since ϕ is generically surjective, the sheaf $\operatorname{coker}(\phi)$ has finite support so that $\mathrm{H}^1(X; \operatorname{coker}(\phi)) \simeq 0$. It follows that $\mathrm{H}^1(X; \mathcal{E}) \simeq 0$ as desired. □

Lemma 3.9.10. *Let $\mathcal{E}$ be a vector bundle on X and let $f : \widetilde{X} \to X$ be a finite flat map of curves which is generically étale. If $\mathrm{H}^1(\widetilde{X}; f^* \mathcal{E}) \simeq 0$, then $\mathrm{H}^1(X; \mathcal{E}) \simeq 0$.*

Proof. Since f is generically étale, the trace map $f_* \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$ induces a generically surjective map of vector bundles $f_* f^* \mathcal{E} \simeq f_* \mathcal{O}_{\widetilde{X}} \otimes \mathcal{E} \to \mathcal{E}$. Because $\mathrm{H}^1(X; f_* f^* \mathcal{E}) \simeq \mathrm{H}^1(\widetilde{X}; f^* \mathcal{E}) \simeq 0$, Lemma 3.9.9 implies that $\mathrm{H}^1(X; \mathcal{E}) \simeq 0$. □

Proof of Proposition 3.9.4. Let G_0 denote the generic fiber of G, let G' denote a split semisimple algebraic group over k of the same type as G_0, let $B' \subseteq G'$ be a Borel subgroup, and let $Y = B' \backslash G'$ be the associated flag variety. Since G_0 is quasi-split, we can choose a Galois extension L of K_X such that G_0 splits over L, an action of the Galois group $\Gamma = \mathrm{Gal}(L/K_X)$ on G' (via pinned automorphisms), and a Γ-invariant isomorphism

$$\operatorname{Spec} L \times_X G \simeq \operatorname{Spec} L \times_{\operatorname{Spec} k} G'.$$

Then L is the fraction field of a smooth curve $\widetilde{X}$ equipped with a finite generically étale map $f : \widetilde{X} \to X$.

Choose a map $\overline{s}_0 : D \to \mathcal{P}$ lifting s_0. Using Lemma 3.9.7, we can extend $\overline{s}_0$ to a trivialization $\overline{s}$ of $\mathcal{P}$ over some open set $V \subseteq X$ containing D and all points $x \in X$ for which the fiber G_x is not semisimple. The proof of Lemma 3.9.8 shows that the composite map $V \xrightarrow{\overline{s}} \mathcal{P} \to B \backslash \mathcal{P}$ extends uniquely to a map $s : X \to B\backslash \mathcal{P}$ which extends s_0.

The trivialization $\overline{s}$ determines an isomorphism

$$(B\backslash \mathcal{P}) \times_X \operatorname{Spec} K_X \simeq B_0\backslash G_0,$$

where we can identify $B_0\backslash G_0$ with the quotient of $Y \times_{\operatorname{Spec} k} \operatorname{Spec} L$ by the diagonal action of Γ. It follows that there exists a dense open subset $U \subseteq V$ and a Γ-equivariant isomorphism

$$\rho : Y \times_{\operatorname{Spec} k} \widetilde{U} \to (B\backslash \mathcal{P}) \times_X \widetilde{U},$$

where $\widetilde{U} = U \times_X \widetilde{X}$ denotes the inverse image of U in $\widetilde{X}$. Shrinking U if necessary, we may suppose that $D \cap U = \emptyset$. Let $r : \widetilde{X} \to Y \times_{\operatorname{Spec} k} \widetilde{X}$ be the map whose projection onto the first factor is the constant map determined by the base point of Y, and note that the diagram

$$\begin{array}{ccc} \widetilde{U} & \longrightarrow & X \\ \downarrow{\scriptstyle r|_{\widetilde{U}}} & & \downarrow{\scriptstyle s} \\ Y \times_{\operatorname{Spec} k} \widetilde{U} & \xrightarrow{\rho} & B\backslash \mathcal{P} \end{array}$$

commutes. Write $\widetilde{X} - \widetilde{U} = \{x_1, \ldots, x_m\}$. Using Lemma 3.9.6 and Proposition A.3.11, we deduce that there exist integers $n_1, \ldots, n_m \geq 0$ and a commutative diagram

$$\begin{array}{ccc} Y \times_{\operatorname{Spec} k} \widetilde{U} & \xrightarrow{\rho} & (B\backslash \mathcal{P}) \times_X \widetilde{U} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\overline{\rho}} & (B\backslash \mathcal{P}) \times_X \widetilde{X}, \end{array}$$

where Z is obtained from $Y \times_{\operatorname{Spec} k} \widetilde{X}$ by performing an n_ith order dilitation along r at the point x_i for each $1 \leq i \leq m$. Let $D' = \sum n_i x_i$. Enlarging the integers n_i if necessary, we may suppose that D' is the inverse image of a divisor in X which contains D.

Let $p : Z \to Y$ and $q : Z \to \widetilde{X}$ denote the projection maps. Applying Remark A.3.2 repeatedly, we obtain a canonical isomorphism $T_{Z/\widetilde{X}} \simeq p^* T_Y \otimes q^* \mathcal{O}_{\widetilde{X}}(-D')$. By construction, r lifts to a map $\overline{r} : \widetilde{X} \to Z$. Invoking Proposition 3.8.1, we can choose a map $r' : \widetilde{X} \to Y \times_{\operatorname{Spec} k} \widetilde{X}$ with the following properties:

- (a) The map r' is Γ-equivariant.
- (b) Let $\widetilde{D}$ denote the inverse image of D in $\widetilde{X}$. Then the restriction of $r'|_{\widetilde{D}+D'}$ is given by the constant map from $\widetilde{D} + D'$ to the base point of Y.
- (c) The cohomology group $\mathrm{H}^1(\widetilde{X}; (r')^*(T_Y \boxtimes \mathcal{O}_{\widetilde{X}}(-\widetilde{D} - D'))$ vanishes.

Using (b) and Proposition A.3.7 (applied to the maps $r, r' : \widetilde{X} \to Y$), we deduce that $r' = p \circ \overline{r}'$ for some map $\overline{r}' : \widetilde{X} \to Z$ such that $\overline{\rho} \circ \overline{r}'|_{\widetilde{D}}$ is compatible with s_0. Using (c), we deduce that $\mathrm{H}^1(\widetilde{X}; (\overline{r}'^* T_{Z/\widetilde{X}})(-\widetilde{D})) \simeq 0$. Let $\widetilde{s}' = \overline{\rho} \circ \overline{r}'$, which we identify with a map from $\widetilde{X}$ into $B\backslash \mathcal{P}$. Then $\overline{\rho}$ induces a map of vector bundles

$$\overline{r}'^* T_{Z/\widetilde{X}} \to \widetilde{s}'^* T_\pi$$

which is a generic isomorphism. Applying Lemma 3.9.9, we conclude that the cohomology group $\mathrm{H}^1(\widetilde{X}; (\widetilde{s}'^* T_\pi)(-\widetilde{D}))$ vanishes. Since $\widetilde{s}'$ is Γ-invariant, it descends to a map $s' : X \simeq \widetilde{X}/\Gamma \to B \backslash \mathcal{P}$. Then s' is a section of the projection $\pi : B\backslash \mathcal{P} \to X$, and the cohomology group $\mathrm{H}^1(X; (s'^* T_\pi)(-D))$ vanishes by virtue of Lemma 3.9.10. □

We now turn to the proofs of Lemmas 3.9.6 and 3.9.7.

Proof of Lemma 3.9.6. Let $U \subseteq X$ be an affine open subset; we will show that $U \times_X (B\backslash \mathcal{P})$ is a scheme. Write $U = \operatorname{Spec} A$ and let $U \times_X G = \operatorname{Spec} H$, where H is a Hopf algebra over A. Then we can write $U \times_X B = \operatorname{Spec} H'$, where H' is the quotient of H by a Hopf ideal $I \subseteq H$. Let us view H as a left comodule over itself, and note that H is A-flat. Since A is a Dedekind ring, we can write H as a filtered colimit $\varinjlim H_\alpha$, where each H_α is a submodule of H which is projective of finite rank over A and invariant under the right coaction of H on itself. Let $I_\alpha = H_\alpha \cap I$. Then we have an injective map $H_\alpha / I_\alpha \hookrightarrow H'$. Since B is flat over X, H' is a torsion-free A-module so that H_α/I_α is also torsion-free. Note that $I = \bigcup_\alpha I_\alpha$. We may therefore choose an index α such that I is generated by I_α (as an ideal in H). Let $\mathcal{E}$ denote the vector bundle on U determined by H_α, let d denote the rank of I_α as an A-module, and let $\mathrm{Gr}_d(\mathcal{E})$ denote the U-scheme which parametrizes subbundles of $\mathcal{E}$ having rank d. The choice of submodule $I_\alpha \subseteq H_\alpha$ determines a map of U-schemes $s : U \to \mathrm{Gr}_d(\mathcal{E})$.

The H-comodule structure on H_α determines a right action of $G_U = U \times_X G$ on $\mathcal{E}$, and therefore also on the scheme $\mathrm{Gr}_d(\mathcal{E})$. Since I is a Hopf ideal, the comultiplication map $\Delta : H \to H \otimes_A H$ carries I into $I \otimes_A H + H \otimes_A I$, so that the composite map

$$I \hookrightarrow H \xrightarrow{\Delta} H \otimes_A H \to H \otimes_A H'$$

carries I into $I \otimes_A H'$. It follows that this map also carries I_α into $I_\alpha \otimes_A H'$, so that s is invariant under the action of $B_U = U \times_X B$ on $\mathrm{Gr}_d(\mathcal{E})$. We claim that $B_U = U \times_X B$ is precisely the stabilizer of the section s. To prove this, we let A' denote an arbitrary A-algebra, and suppose we are given a point $g \in G(A')$, which is classified by an A-algebra homomorphism $\phi : H \to A'$. If g fixes the section s, then the composite map

$$I_\alpha \hookrightarrow H \xrightarrow{\Delta} H \otimes_A H \xrightarrow{\phi} H \otimes_A A'$$

carries I_α into $I_\alpha \otimes_A A'$. Let $\epsilon : H \to A$ denote the augmentation on A, so that ϵ annihilates I_α. Then the composite map

$$I_\alpha \hookrightarrow H \xrightarrow{\Delta} H \otimes_A H \xrightarrow{\epsilon \otimes \phi} A \otimes_A A' \simeq A'$$

vanishes. It follows that ϕ annihilates I_α and therefore (since I_α generates the ideal I) the element g belongs to the subgroup $B(A') \subseteq G(A')$, as desired.

Let $B\backslash G$ denote the quotient of G by the left action of B, so that evaluation on the section s determines a monomorphism of algebraic spaces

$$U \times_X (B\backslash G) \to \mathrm{Gr}_d(\mathcal{E}).$$

This map is G-invariant, and therefore determines a map

$$\mu : U \times_X (B\backslash \mathcal{P}) \to \mathrm{Gr}_d(\mathcal{E}) \times^G \mathcal{P},$$

where $\mathrm{Gr}_d(\mathcal{E}) \times^G \mathcal{P}$ denotes the quotient of $\mathrm{Gr}_d(\mathcal{E}) \times_X \mathcal{P}$ by the diagonal action of G. Then μ is a quasi-compact monomorphism of algebraic spaces, hence quasi-affine ([27]). Consequently, to prove that $U \times_X (B/\mathcal{P})$ is a scheme, it will suffice to show that $\mathrm{Gr}_d(\mathcal{E}) \times^G \mathcal{P}$ is a scheme. This is clear: the product $\mathrm{Gr}_d(\mathcal{E}) \times^G \mathcal{P}$ can be identified with $\mathrm{Gr}_d(\mathcal{E}')$, where $\mathcal{E}'$ is the vector bundle on $U \times_X X_R$ determined by $\mathcal{E}$ and $\mathcal{P}$ (so that $\mathrm{Gr}_d(\mathcal{E}) \times^G \mathcal{P}$ is projective over $U \times_X X_R$). □

Our proof of Lemma 3.9.7 will use the following fact, which was communicated to us by Brian Conrad:

Lemma 3.9.11. *Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, let K be the fraction field of R, let Y be a smooth affine K-scheme, and let $U \subseteq Y$ be a dense open set. Then $U(K)$ is dense in $Y(K)$ (where we equip $Y(K)$ with the $\mathfrak{m}$-adic topology).*

Proof. The assertion is local with respect to the Zariski topology on Y. We may therefore assume without loss of generality that there exists an étale morphism of k-schemes $\phi : Y \to \mathbf{A}^d$, where d is the dimension of Y. Let Z denote the complement of U In Y. Since U is dense, we have $\dim(Z) < d$, so that the image under ϕ of Z is contained in a proper closed subscheme of $\mathbf{A}^d$. We may therefore choose a nonzero polynomial $f(x_1, \ldots, x_d)$ which vanishes on the points of $\phi(Z(K))$, so that $\phi(Z(K))$ cannot contain any nonempty open subset of K^n. It follows from Hensel's lemma that ϕ induces an open map $Y(K) \to K^d$, so that $Z(K)$ cannot contain any open subset of $Y(K)$ and therefore $U(K)$ is dense in $Y(K)$, as desired. □

Proof of Lemma 3.9.7. By Tsen's theorem, the fraction field K_X is a field of dimension 1. It follows that the G-bundle $\mathcal{P}$ is trivial at the generic point of X (see [10]). Let us view this trivialization as a map $\eta : \operatorname{Spec} K_X \to \mathcal{P}$ fitting into a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K_X & \xrightarrow{\eta} & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & X & \end{array}.$$

Using a direct limit argument, we see that η can be extended to a map of X-schemes $V \to \mathcal{P}$, where V is a dense open subset of X. Shrinking V if necessary, we may assume that $V \cap D = \emptyset$. Let U be the open subset of X given by the union of V and D. We wish to show that after modifying the set V and the trivialization η, we can arrange that η and s extend to a trivialization of $\mathcal{P}|_U$.

Write $D = \{x_1, \ldots, x_n\}$. Since G is smooth, we can extend γ_0 to a trivialization γ of $\mathcal{P}$ over $\amalg \operatorname{Spec} \mathcal{O}_{x_i}$, where $\mathcal{O}_{x_i}$ denotes the complete local ring of X at the point x_i (so that $\mathcal{O}_{x_i}$ is noncanonically isomorphic to a power series ring $k[[t]]$). For $1 \leq i \leq n$, let K_{x_i} denote the fraction field of $\mathcal{O}_{x_i}$, so that η and γ determine two different trivializations of $\mathcal{P}|_{\operatorname{Spec} K_{x_i}}$. These trivializations differ by some elements $g_i \in G(K_{x_i})$. Let d_i denote the multiplicity of x_i in the divisor D, let $\mathfrak{m}_i$ denote the maximal ideal of $\mathcal{O}_{x_i}$, and let S_i denote the kernel of the reduction map $G(\mathcal{O}_{x_i}) \to G(\mathcal{O}_{x_i}/\mathfrak{m}_i^{d_i})$. Unwinding the definitions, we see that γ is compatible with η if and only if each g_i belongs to the set S_i.

To complete the proof, we wish to show that we can change the trivialization η to arrange that each g_i belongs to S_i. In other words, we wish to prove that we can choose $g \in G(K_X)$ so that each of the products $gg_i \in G(K_{x_i})$ belongs to S_i.

Let us regard each $G(K_{x_i})$ as a topological space as in Lemma 3.9.11. By construction, the product $\prod_{1 \leq i \leq n} S_i g_i^{-1}$ is a nonempty open subset of $\prod_{1 \leq i \leq n} G(K_{x_i})$. It will therefore suffice to prove the following:

($*$) The map $G(K_X) \to \prod_{1 \leq i \leq n} G(K_{x_i})$ has dense image.

Let G_0 be the generic fiber of G. Since the field K_X has dimension 1, the group G_0 is quasi-split. Let B_0 be a Borel subgroup of G_0, let $T_0 \subseteq B_0$ be a maximal torus, let B_0' be the unique Borel subgroup of G_0 which contains T_0 and is in general position with respect to B_0, and let U_0 and U_0' be the unipotent radicals of B_0 and B_0', respectively. Then $W = U_0' T_0 U_0$ is a Zariski-dense open subset of G_0, so that Lemma 3.9.11 implies that $\prod_{1 \leq i \leq n} W(K_{x_i})$ is dense in $\prod_{1 \leq i \leq n} G(K_{x_i})$. It will therefore suffice to show that the map $W(K_X) \to \prod_{1 \leq i \leq n} W(K_{x_i})$ has dense image.

Using the lower central series of U_0, we obtain a sequence of surjective algebraic group homomorphisms

$$U_0 \to U_1 \to U_2 \to \cdots \to U_m = \{0\}.$$

Each of the kernels of these maps is a vector group over K_X, and therefore isomorphic (as a scheme) to an affine space over K_X. Since each U_i is an affine scheme, the maps $U_i \to U_{i+1}$ admit sections (in the category of schemes), so that U_0 is isomorphic (as a scheme) to a product of finitely many copies of the affine line $\mathbf{A}^1$ over K_X. Similarly, U_0' is isomorphic to a product of finitely many copies of $\mathbf{A}^1$. Using the decomposition $W \simeq U_0 \times T \times U_0'$ (and our assumption that the generic fiber of G is either semisimple or adjoint), we see that the K_X-scheme W is isomorphic to product of finitely many factors W_β, where each W_β is isomorphic either to the affine line $\mathbf{A^1}$ over K_X or to a restriction of scalars of the multiplicative group $\mathbf{G}_m$ from a finite extension $L_0 \subseteq L$ of K_X. The desired result now follows from the observations that the maps

$$K_X \to \prod_{1 \leq i \leq n} K_{x_i} \qquad L_0^\times \to \prod_{1 \leq i \leq n} (L_0 \otimes_{K_X} K_{x_i})^\times$$

have dense image. □

4. The Formalism of ℓ-adic Sheaves

Let k be an algebraically closed field, let ℓ be a prime number which is invertible in k, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. In §3, we proved that if the generic fiber of G is semisimple and simply connected, then the forgetful functor $\mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$ is a universal homology equivalence (see Theorem 3.2.13). In particular, the pullback map

$$\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to \mathrm{H}^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell)$$

is an isomorphism. The formulation and proof of this statement use the language of ℓ-adic cohomology, but only in its most elementary incarnation: all cohomology (or homology) is taken with constant coefficients.

Unfortunately, the calculation of $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ supplied by Theorem 3.2.13 is not adequate for our needs in this paper. In order to prove Theorem 1.3.5, we will need to establish ℓ-adic analogues of the other topological formulae for the cohomology of $\mathrm{Bun}_G(X)$ outlined in §1.4. For this purpose, the language of §2 is not sufficient: we need not only the theory of ℓ-adic cohomology, but also the more elaborate theory of ℓ-adic sheaves. Consequently, we devote this section to giving a review of the formalism of ℓ-adic sheaves in a form which is convenient for our purposes.

We begin in §4.1 by reviewing the theory of étale sheaves. To every scheme Y and every commutative ring Λ, one can associate a stable ∞-category $\mathrm{Shv}(Y; \Lambda)$ of Mod_Λ-valued étale sheaves on Y (Notation 4.1.2). This can be regarded as an "enhancement" of the derived category of the abelian category of sheaves of Λ-modules on Y, whose objects are cochain complexes

$$\cdots \to \mathcal{F}^{-2} \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots.$$

The ∞-category $\mathrm{Shv}(Y; \Lambda)$ contains a full subcategory $\mathrm{Shv}^c(Y; \Lambda)$ of constructible perfect complexes, which we will discuss in §4.2. However, the ∞-category $\mathrm{Shv}^c(Y; \Lambda)$ is too small for many of our purposes: it fails to contain many of the objects we are interested in (for example, the cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Z}/\ell\mathbf{Z})$ typically has cohomology in infinitely many degrees), and does not have good closure properties under various categorical constructions we will need to use (such as the formation of infinite direct limits). On the other hand, allowing arbitrary chain complexes (in particular, chain complexes which are not bounded below) raises some technical convergence issues. We will avoid these issues by restricting our attention to the case where the scheme Y has finite type over an algebraically closed field. In this case, the étale site of Y has finite cohomological dimension, which implies that $\mathrm{Shv}(Y;\Lambda)$ is compactly generated by the subcategory $\mathrm{Shv}^c(Y;\Lambda)$ (Proposition 3.5.4).

The construction $Y \mapsto \mathrm{Shv}(Y;\Lambda)$ depends functorially on Λ: every map of commutative rings $\Lambda \to \Lambda'$ induces base change functors

$$\mathrm{Shv}(Y;\Lambda) \to \mathrm{Shv}(Y;\Lambda') \qquad \mathrm{Shv}^c(Y;\Lambda) \to \mathrm{Shv}^c(Y;\Lambda').$$

In particular, we have a tower of ∞-categories

$$\cdots \to \mathrm{Shv}^c(Y;\mathbf{Z}/\ell^3) \to \mathrm{Shv}^c(Y;\mathbf{Z}/\ell^2) \to \mathrm{Shv}^c(Y;\mathbf{Z}/\ell\mathbf{Z}).$$

We will denote the (homotopy) inverse limit of this tower by $\mathrm{Shv}^c_\ell(Y)$, and refer to it as the *∞-category of constructible ℓ-adic sheaves on Y*. In §4.3, we will study the Ind-completion $\mathrm{Shv}_\ell(Y) = \mathrm{Ind}(\mathrm{Shv}^c_\ell(Y))$ of $\mathrm{Shv}^c_\ell(Y)$, which we refer to as the *∞-category of ℓ-adic sheaves on Y*. These ∞-categories provide a convenient formal setting for formulating most of the constructions of this paper: the ∞-category $\mathrm{Shv}_\ell(Y)$ contains all constructible ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}^c_\ell(Y)$ as well as other objects obtained by limiting procedures (such as localizations of the form $\mathcal{F}[\ell^{-1}]$). Many important foundational results in the theory of étale cohomology (such as the smooth and proper base change theorems) can be extended to the setting of ℓ-adic sheaves in a purely formal way; we will review the situation in §4.5. However, it is sometimes necessary to make convergence arguments which require us to restrict our attention to ℓ-adic sheaves satisfying boundedness conditions; we therefore devote §4.4 to a review of the construction of a t-structure on $\mathrm{Shv}^c_\ell(Y)$ (which formally determines a t-structure on the ∞-category $\mathrm{Shv}_\ell(Y)$ as well).

For any quasi-projective k-scheme Y, the usual (left derived) tensor product of sheaves determines a symmetric monoidal structure on the ∞-category $\mathrm{Shv}_\ell(Y)$, whose underlying tensor product functor we denote by

$$\otimes : \mathrm{Shv}_\ell(Y) \times \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(Y).$$

In §4.6, we will study the Verdier dual of this operation: this determines a second symmetric monoidal structure on $\mathrm{Shv}_\ell(Y)$, whose underlying tensor product we denote by

$$\otimes^! : \mathrm{Shv}_\ell(Y) \times \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(Y).$$

The unit object of $\mathrm{Shv}_\ell(Y)$ with respect to this second tensor product is not the constant sheaf, but instead the *dualizing complex* of Y which we denote by $\omega_Y \in \mathrm{Shv}_\ell(Y)$.

4.1. **Étale Sheaves.** Let X be a scheme and let Λ be a commutative ring. One can associate to X an abelian category $\mathcal{A}$ of *étale sheaves of Λ-modules* on X. The derived category $\mathcal{D}(\mathcal{A})$ provides a useful setting for performing a wide variety of sheaf-theoretic constructions. However, there are other basic constructions (such as the formation of mapping cones) which cannot be carried out functorially at the level of derived categories. One way to remedy the situation is to introduce an ∞-category $\mathrm{Shv}(X;\Lambda)$ whose homotopy category is equivalent to the derived category $\mathcal{D}(\mathcal{A})$. It is possible to produce such an ∞-category by applying a purely formal procedure to the abelian category $\mathcal{A}$ (see §HA.1.3.2 and §HA.1.3.5). However, it will be more convenient for us to define $\mathrm{Shv}(X;\Lambda)$ directly as the ∞-category of (hypercomplete) Mod_Λ-*valued* sheaves on X. Our goal in this section is to give a brief introduction to this point of view, and to review some of the basic properties of étale sheaves which will be needed in the later sections of this paper. We will confine our attention here to the most formal aspects of the theory, where the coefficient ring Λ can be taken to be arbitrary; for the essential base change and finiteness results for étale cohomology, which require additional hypotheses on Λ, will be discussed in §4.3 and 4.5.

Remark 4.1.1. Since the apparatus of étale cohomology is treated exhaustively in other sources (such as [2] and [13]; see also [17] for an expository account), we will be content to summarize the relevant definitions and give brief indications of proofs.

Let k be an algebraically closed field, which we regard as fixed throughout this section. To simplify the exposition, we will restrict our discussion to the setting of étale sheaves on quasi-projective k-schemes. This is largely unnecessary: much of the theory that we describe can be carried out for more general schemes. However, some restrictions on cohomological dimension are needed in the proof of Lemma 4.1.13 (and the many other statements which depend on it).

Notation 4.1.2. Let Sch_k denote the category of quasi-projective k-schemes. For each $X \in \mathrm{Sch}_k$, we let $\mathrm{Sch}_X^{\mathrm{et}}$ denote the category whose objects are étale maps $U \to X$ between quasi-projective k-schemes. Morphisms in $\mathrm{Sch}_X^{\mathrm{et}}$ are given by commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

We will say that a collection of morphisms $\{f_\alpha : U_\alpha \to V\}$ in $\mathrm{Sch}_X^{\mathrm{et}}$ is a *covering* if the induced map $\amalg U_\alpha \to V$ is surjective. The collection of coverings determines a Grothendieck topology on the category $\mathrm{Sch}_X^{\mathrm{et}}$, which we refer to as the *étale topology*.

Let Λ be a commutative ring, and let Mod_Λ be the ∞-category of chain complexes over Λ (see Example 2.1.23). A Mod_Λ*-valued presheaf on* X is a functor of ∞-categories

$$\mathcal{F} : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda .$$

If $\mathcal{F}$ is a Mod_Λ-valued presheaf on X and $U \in \mathrm{Sch}_X^{\mathrm{et}}$, then we can regard $\mathcal{F}(U)$ as a chain complex of Λ-modules. For each integer n, the construction $U \mapsto \mathrm{H}_n(\mathcal{F}(U))$ determines a presheaf of abelian groups on X. We let $\pi_n \mathcal{F}$ denote the étale sheaf of abelian groups on X obtained by sheafifying the presheaf $U \mapsto \mathrm{H}_n(\mathcal{F}(U))$. We will say that $\mathcal{F}$ is *locally acyclic* if, for every integer n, the sheaf $\pi_n \mathcal{F}$ vanishes.

We let $\mathrm{Shv}(X; \Lambda)$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ spanned by those Mod_Λ-valued presheaves $\mathcal{F}$ which have the following property: for every locally acyclic object $\mathcal{F}' \in \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$, every morphism $\alpha : \mathcal{F}' \to \mathcal{F}$ is nullhomotopic.

Remark 4.1.3. Let $\mathcal{F} : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda$ be a Mod_Λ-valued presheaf on a quasi-projective k-scheme X. Then $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ if and only if the following conditions are satisfied:

(1) The presheaf $\mathcal{F}$ is a sheaf with respect to the étale topology on $\mathrm{Sch}_X^{\mathrm{et}}$. That is, for every covering $\{f_\alpha : U_\alpha \to V\}$, the canonical map $\mathcal{F}(V) \to \varprojlim \mathcal{F}(U)$ is an equivalence in Mod_Λ, where the limit is taken over all objects $U \in \mathrm{Sch}_V^{\mathrm{et}}$ for which the map $U \to V$ factors through some f_α.
(2) The Mod_Λ-valued sheaf $\mathcal{F}$ is hypercomplete, in the sense of Definition SAG.1.2.1.15 (this is a technical hypothesis which is necessary only because we consider potentially unbounded complexes, where descent for Čech coverings does not necessarily imply descent for arbitrary hypercoverings).

Example 4.1.4. Let M be a finite abelian group equipped which is a module over some commutative ring Λ. For every quasi-projective k-scheme X, the construction $U \mapsto C^*(U; M)$ (see Definition 2.2.6) satisfies conditions (1) and (2) of Remark 4.1.3, and can therefore be regarded as an object of $\mathrm{Shv}(X;\Lambda)$. Condition (1) follows immediately from the definition, and condition (2) is automatic since the cochain complexes $C^*(U;M)$ are concentrated in nonnegative cohomological degrees.

Example 4.1.5. If $X = \operatorname{Spec} k$, then the ∞-category $\mathrm{Shv}(X;\Lambda)$ is equivalent to Mod_Λ. Concretely, this equivalence is implemented by the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X) \in \mathrm{Mod}_\Lambda$.

Remark 4.1.6. Let X and Λ be as in Notation 4.1.2. For each integer $n \in \mathbf{Z}$, we let $\mathrm{Shv}(X;\Lambda)_{\leq n}$ denote the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by those objects $\mathcal{F}$ for which $\pi_m \mathcal{F} \simeq 0$ for $m > 0$, and we let $\mathrm{Shv}(X;\Lambda)_{\geq n}$ denote the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by those objects $\mathcal{F}$ for which $\pi_m \mathcal{F} \simeq 0$ for $m < 0$. Then the full subcategories $(\mathrm{Shv}(X;\Lambda)_{\geq 0}, \mathrm{Shv}(X;\Lambda)_{\leq 0})$ determine a t-structure on $\mathrm{Shv}(X;\Lambda)$. Moreover, the construction $\mathcal{F} \mapsto \pi_0 \mathcal{F}$ determines an equivalence of categories from the heart

$$\mathrm{Shv}(X;\Lambda)^{\heartsuit} = \mathrm{Shv}(X;\Lambda)_{\geq 0} \cap \mathrm{Shv}(X;\Lambda)_{\leq 0}$$

of $\mathrm{Shv}(X;\Lambda)$ to the abelian category of étale sheaves of Λ-modules on X (see Proposition SAG.2.1.1.3). In what follows, we will use this equivalence to identify the abelian category of sheaves of Λ-modules on X with a full subcategory of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$. In particular, if $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$, we will generally identify the sheaves $\pi_n \mathcal{F}$ with the corresponding objects of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$.

Warning 4.1.7. In this paper, we will use homological indexing conventions when working with t-structures on triangulated categories, rather than the cohomological conventions which can be found (for example) in [8]. One can translate between conventions using the formulae

$$\mathcal{C}^{\leq n} = \mathcal{C}_{\geq -n} \qquad \mathcal{C}^{\geq n} = \mathcal{C}_{\leq -n}\,.$$

Warning 4.1.8. Let X and Λ be as in Notation 4.1.2, and let $\mathcal{F}$ be an object of the abelian category $\mathcal{A}$ of étale sheaves of Λ-modules on X. Then there are two *different* ways in which $\mathcal{F}$ can be interpreted as a Mod_Λ-valued presheaf on X:

(*a*) One can view $\mathcal{F}$ as a presheaf with values in the abelian category $\mathrm{Mod}_\Lambda^{\heartsuit}$ of (discrete) Λ-modules, which determines a functor

$$\mathcal{F}_0 : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda^{\heartsuit} \subseteq \mathrm{Mod}_\Lambda\,.$$

(*b*) Using the equivalence of abelian categories $\mathcal{A} \simeq \mathrm{Shv}(X;\Lambda)^{\heartsuit}$, one can identify $\mathcal{F}$ with an object

$$\mathcal{F}_1 \in \mathcal{A} \simeq \mathrm{Shv}(X;\Lambda)^{\heartsuit} \subseteq \mathrm{Shv}(X;\Lambda) \subseteq \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda).$$

The functors $\mathcal{F}_0$ and $\mathcal{F}_1$ are generally *not* the same. By construction, the functor $\mathcal{F}_0$ has the property that for every étale X-scheme U, the chain complex $\mathcal{F}_0(U) \in \mathrm{Mod}_\Lambda$ has homology concentrated in degree zero, but the homologies of $\mathcal{F}_1(U)$ are given by the formula

$$\mathrm{H}_n(\mathcal{F}_1(U)) \simeq \mathrm{H}_{\mathrm{et}}^{-n}(U; \mathcal{F}|_U).$$

Note also that $\mathcal{F}_1$ is a Mod_Λ-valued sheaf with respect to the étale topology on $\mathrm{Sch}_X^{\mathrm{et}}$ but $\mathcal{F}_0$ is not (in fact, $\mathcal{F}_1$ can be identified with the sheafification of $\mathcal{F}_0$ with respect to the étale topology).

Remark 4.1.9 (Functoriality)**.** Let $f : X \to Y$ be morphism of quasi-projective k-schemes and let Λ be a commutative ring. Then f determines a base-change functor $\mathrm{Sch}_Y^{\mathrm{et}} \to \mathrm{Sch}_X^{\mathrm{et}}$, given by $U \mapsto U \times_Y X$. Composition with this base-change functor induces a map $\mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(Y;\Lambda)$, which we will denote by f_* and refer to as *pushforward along* f. The functor f_* admits a left adjoint, which we will denote by f^* and refer to as *pullback along* f. If $\mathcal{F} \in \mathrm{Shv}(Y;\Lambda)$, we will sometimes denote the pullback $f^* \mathcal{F}$ by $\mathcal{F}|_X$, particularly in those cases when f exhibits X as a subscheme of Y.

Example 4.1.10. Let $f : X \to Y$ be an étale morphism between quasi-projective k-schemes. Then composition with f induces a forgetful functor $u : \mathrm{Sch}_X^{\mathrm{et}} \to \mathrm{Sch}_Y^{\mathrm{et}}$. The pullback functor $f^* : \mathrm{Shv}(Y; \Lambda) \to \mathrm{Shv}(X; \Lambda)$ is then given by composition with u. From this description, we immediately deduce that f^* preserves limits and colimits. Using Corollary HTT.5.5.2.9, we deduce that f^* admits a left adjoint which we will denote by $f_!$. In the special case where f is an open immersion, we will refer to $f_!$ as the functor of *extension by zero along* f.

Proposition 4.1.11. *Let X be a quasi-projective k-scheme, and let $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ for some commutative ring Λ. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ vanishes.*
(2) *For every k-valued point $\eta : \mathrm{Spec}\, k \to X$, the stalk $\eta^* \mathcal{F} \in \mathrm{Shv}(\mathrm{Spec}\, k; \Lambda) \simeq \mathrm{Mod}_\Lambda$ vanishes.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Suppose that $\mathcal{F}$ satisfies (2); we will show that $\mathcal{F} \simeq 0$ by proving that the identity map $\mathrm{id} : \mathcal{F} \to \mathcal{F}$ is nullhomotopic. For this, it will suffice to show that $\mathcal{F}$ is locally acyclic: that is, each of the sheaves of abelian groups $\pi_n \mathcal{F}$ vanishes. We may therefore assume without loss of generality that $\mathcal{F}$ belongs to the heart of $\mathrm{Shv}(X; \Lambda)$. We will abuse notation by identifying $\mathcal{F}$ with the corresponding sheaf of abelian groups on $\mathrm{Sch}_X^{\mathrm{et}}$. Choose an object $U \in \mathrm{Sch}_X^{\mathrm{et}}$ and a section $s \in \mathcal{F}(U)$; we wish to show that $s = 0$. Let $V \subseteq U$ be the largest open subset for which $s|_V = 0$. Suppose for a contradiction that $V \neq U$. Then we can choose a point $\eta_U : \mathrm{Spec}\, k \to U$ which does not factor through V. Let η denote the composition of η_U with the map $U \to X$, so that $\eta^* \mathcal{F} \simeq 0$ by virtue of (2). It follows that the map η_U factors as a composition

$$\mathrm{Spec}\, k \to \widetilde{U} \to U,$$

where $s|_{\widetilde{U}} = 0$. We conclude that s vanishes on the open subset of U given by the union of V with the image of $\widetilde{U}$, contradicting the maximality of V. □

Proposition 4.1.12. *Suppose we are given a diagram of quasi-projective k-schemes σ :*

$$\begin{array}{ccc} U_X & \xrightarrow{f'} & U_Y \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y, \end{array}$$

where j' and j are étale. If σ is a pullback diagram, then the associated diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}(U_X; \Lambda) & \longleftarrow & \mathrm{Shv}(U_Y; \Lambda) \\ \uparrow & & \uparrow \\ \mathrm{Shv}(X; \Lambda) & \longleftarrow & \mathrm{Shv}(Y; \Lambda) \end{array}$$

satisfies the Beck-Chevalley property: that is, the induced natural transformation $j'_! f'^* \to f^* j_!$ is an equivalence of functors from $\mathrm{Shv}(U_Y; \Lambda)$ to $\mathrm{Shv}(X; \Lambda)$ (see §4.5 for a more detailed discussion).

Proof. Passing to right adjoints, we are reduced to proving that the canonical map $j^* f_* \to f'_* j'^*$ is an equivalence. Let $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$. Using the descriptions of the pullback and pushforward functors supplied by Remark 4.1.9 and Example 4.1.10, we must show that for every object $V \in \mathrm{Sch}_{U_Y}^{\mathrm{et}}$, the restriction map $\mathcal{F}(U_X \times_{U_Y} V) \to \mathcal{F}(X \times_Y V)$ is an equivalence. This is evidently satisfied whenever σ is a pullback square. □

The theory of étale sheaves makes sense for arbitrary schemes, not just those which are quasi-projective over an algebraically closed field k. However, technical difficulties can arise when dealing with unbounded chain complexes. In the setting of quasi-projective k-schemes, these difficulties can be circumvented using the finiteness of cohomological dimension:

Lemma 4.1.13. *Let X be a quasi-projective k-scheme of Krull dimension d, and let $\mathcal{F}$ be an étale sheaf of abelian groups on X. Then the cohomology groups $\mathrm{H}^n(X;\mathcal{F})$ vanish for $n > 2d+1$.*

Proof. Let $\mathrm{Shv}_{\mathrm{Nis}}(X;\mathbf{Z})$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ spanned by those functors which are sheaves with respect to the Nisnevich topology, and let $\iota : \mathrm{Shv}(X;\mathbf{Z}) \hookrightarrow \mathrm{Shv}_{\mathrm{Nis}}(X;\mathbf{Z})$ denote the inclusion map. Let $\mathcal{F}'$ denote the object of the heart $\mathrm{Shv}(X;\mathbf{Z})^{\heartsuit}$ corresponding to $\mathcal{F}$, so that we have a canonical isomorphism $\mathrm{H}^n(X;\mathcal{F}) \simeq \mathrm{H}_{-n}\,\mathcal{F}'(X)$. Since the ∞-topos of Nisnevich sheaves on X has homotopy dimension $\leq d$ (see SAG.1.1.5), it will suffice to show that $\mathcal{F}'$ belongs to $\mathrm{Shv}_{\mathrm{Nis}}(X;\Lambda)_{\geq -d-1}$. To prove this, it will suffice to show that for every map $\eta : \operatorname{Spec} R \to X$ which exhibits R as the Henselization of X with respect to some finite extension of some residue field of X, the cohomology groups $\mathrm{H}^m(\operatorname{Spec} R; \eta^*\mathcal{F})$ vanish for $m > d+1$. Let κ' denote the residue field of R, and let $\eta_0 : \operatorname{Spec} \kappa' \to X$ be the restriction of η. Then κ' is an extension of κ of trancendence degree $\leq d$, and is therefore a field of cohomological dimension $\leq d$ (see [50]). Since the ring R is Henselian, the canonical map $\mathrm{H}^m(\operatorname{Spec} R; \eta^*\mathcal{F}) \to \mathrm{H}^m(\operatorname{Spec} \kappa'; \eta_0^*\mathcal{F})$ is an isomorphism so that that $\mathrm{H}^m(\operatorname{Spec} R; \eta^*\mathcal{F})$ vanishes for $m > d+1$ as desired. □

To any Grothendieck abelian category $\mathcal{A}$, one can associate a stable ∞-category $\mathcal{D}(\mathcal{A})$ called the *(unbounded) derived ∞-category of* $\mathcal{A}$, whose homotopy category is the classical derived category of $\mathcal{A}$; see §HA.1.3.5 for details.

Proposition 4.1.14. *Let X be a quasi-projective k-scheme and let Λ be a commutative ring. Then the inclusion $\mathrm{Shv}(X;\Lambda)^{\heartsuit} \hookrightarrow \mathrm{Shv}(X;\Lambda)$ extends to an equivalence of ∞-categories $\theta : \mathcal{D}(\mathrm{Shv}(X;\Lambda)^{\heartsuit}) \simeq \mathrm{Shv}(X;\Lambda)$. In particular, the homotopy category of $\mathrm{Shv}(X;\Lambda)$ is equivalent to the unbounded derived category of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$.*

Proof. When restricted to chain complexes which are (cohomologically) bounded below, this follows from the fact that the Grothendieck site $\mathrm{Sch}_X^{\mathrm{et}}$ is an ordinary category (rather than an ∞-category) and that Λ is an ordinary ring (rather than a ring spectrum); see Proposition SAG.2.1.1.8. It follows from Lemma 4.1.13 that the equivalence extends to unbounded complexes; see Proposition SAG.2.1.1.11 for more details. □

Remark 4.1.15. Let X be a quasi-projective k-scheme and let Λ be a commutative ring. Then an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ belongs to $\mathrm{Shv}(X;\Lambda)_{\geq 0}$ if and only if, for every point $\eta : \operatorname{Spec} k \to X$, the stalk $\eta^*\mathcal{F} \in \mathrm{Shv}(\operatorname{Spec} k;\Lambda) \simeq \mathrm{Mod}_{\Lambda}$ belongs to $(\mathrm{Mod}_{\Lambda})_{\geq 0}$. Similarly, $\mathcal{F}$ belongs to $\mathrm{Shv}(\operatorname{Spec} k;\Lambda)_{\leq 0}$ if and only if each stalk $\eta^*\mathcal{F}$ belongs to $(\mathrm{Mod}_{\Lambda})_{\leq 0}$.

Proposition 4.1.16. *Let X be a quasi-projective k-scheme. Then, for every commutative ring Λ, the full subcategory $\mathrm{Shv}(X;\Lambda) \subseteq \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_{\Lambda})$ is closed under colimits.*

Proof. The inclusion $\mathrm{Shv}(X;\Lambda) \hookrightarrow \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_{\Lambda})$ is a left exact functor between stable ∞-categories and therefore preserves finite colimits. It will therefore suffice to show that it preserves filtered colimits. Let $\{\mathcal{F}_\alpha\}$ be a filtered diagram of objects of $\mathrm{Shv}(X;\Lambda)$ having colimit $\mathcal{F}$. We wish to prove that for each $U \in \mathrm{Sch}_X^{\mathrm{et}}$, the canonical map $\varinjlim \mathcal{F}_\alpha(U) \to \mathcal{F}(U)$ is an equivalence. In other words, we want to show that for each integer n, the induced map $\varinjlim \pi_n\,\mathcal{F}_\alpha(U) \to \pi_n\,\mathcal{F}(U)$ is an isomorphism of abelian groups. Shifting if necessary, we may suppose that $n = 0$. Replacing each $\mathcal{F}_\alpha$ by a truncation if necessary, we may suppose that each $\mathcal{F}_\alpha$ belongs to $\mathrm{Shv}(X;\Lambda)_{\geq 0}$. Using Lemma 4.1.13, one can show that there exists an integer $N \gg 0$ such that the canonical map $\pi_0 \mathcal{G}(U) \to \pi_0(\tau_{\leq N} \mathcal{G})(U)$ is an isomorphism, for each $\mathcal{G} \in \mathrm{Shv}(X; \Lambda)$. Replacing each $\mathcal{F}_\alpha$ by $\tau_{\leq N} \mathcal{F}_\alpha$, we may assume that $\{\mathcal{F}_\alpha\}$ is a diagram in $\mathrm{Shv}(X; \Lambda)_{\leq N}$ for some integer N. The desired result now follows formally from the fact that the Grothendieck topology on $\mathrm{Sch}_X^{\mathrm{et}}$ is *finitary* (that is, every covering admits a finite refinement); see Corollary SAG.1.3.2.20 for more details. □

Corollary 4.1.17. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then, for every commutative ring Λ, the pushforward functor $f_* : \mathrm{Shv}(X; \Lambda) \to \mathrm{Shv}(Y; \Lambda)$ preserves colimits.*

Remark 4.1.18. Let $f : X \to Y$ be as in Corollary 4.1.17. Applying Corollary HTT.5.5.2.9, we deduce that the functor f_* admits a right adjoint. *In the special case where f is proper*, we will denote this right adjoint by $f^!$. We will refer to $f^!$ as the *exceptional inverse image functor*. We will primarily be interested in the functor $f^!$ in the special case where f is a closed immersion.

Warning 4.1.19. If the coefficient ring Λ is finite, there is a good definition of the exceptional inverse image functor $f^!$ for an arbitrary morphism $f : X \to Y$. However, the functor $f^!$ is right adjoint to the compactly supported direct image functor $f_!$, rather than the usual direct image functor f_*. Since we do not wish to address the homotopy coherence issues which arise in setting up an "enhanced" six-functor formalism, we will not consider this additional generality: that is, we consider the functor $f^!$ as defined only when f is proper, and the functor $f_!$ as defined only when f is étale (a special case of the relationship between $f^!$ and $f_!$ is articulated in Example 4.5.15).

Example 4.1.20. Let n be a positive integer which is invertible in k and let $\Lambda = \mathbf{Z}/n\mathbf{Z}$. For every k-scheme X, let μ_{nX} denote the invertible object of the abelian category $\mathrm{Shv}(X; \Lambda)^{\heartsuit}$ corresponding to the sheaf of abelian groups $U \mapsto \{f \in \mathcal{O}_X(U) : f^n = 1\}$. If $f : X \to Y$ is a proper smooth morphism of relative dimension d, then the main result of [56] supplies an equivalence

$$f^! \mathcal{F} \simeq \Sigma^{2d} \mu_{nX}^{\otimes d} \otimes_\Lambda f^* \mathcal{F}.$$

Remark 4.1.21. Let X be a quasi-projective k-scheme which is smooth of dimension d, let n be a positive integer which is invertible in k, and let $\eta : \operatorname{Spec} k \to X$ be a point of X. Then there is an equivalence

$$\eta^! \underline{\mathbf{Z}/n\mathbf{Z}} \simeq \Sigma^{-2d} \mu_n^{\otimes -d},$$

where $\mu_n = \mu_{n \operatorname{Spec} k} \in \mathrm{Shv}(\operatorname{Spec} k; \mathbf{Z}/n\mathbf{Z})$. To prove this, we can work locally with respect to the étale topology on X, and thereby reduce to the case where $X = \mathbf{P}^n$ so that there exists a proper morphism $\pi : X \to \operatorname{Spec} k$. In this case, Example 4.1.20 supplies an equivalence

$$\begin{aligned} \eta^! \underline{\mathbf{Z}/n\mathbf{Z}} &\simeq \eta^! (\pi^! \Sigma^{-2d} \mu_n^{\otimes -d}) \\ &\simeq (\pi \circ \eta)^! \Sigma^{-2d} \mu_n^{\otimes -d} \\ &\simeq \Sigma^{-2d} \mu_n^{\otimes -d} \end{aligned}$$

Proposition 4.1.22. *Let $i : Y \hookrightarrow X$ be a closed immersion of quasi-projective k-schemes. Then:*

(1) *The functor $i^!$ preserves filtered colimits.*
(2) *The functor i_* preserves compact objects.*

Proof. The implication (1) ⇒ (2) follows from Proposition HTT.5.5.7.2. We will prove (1). Since the functor i_* is a fully faithful embedding which preserves colimits (Corollary 4.1.18), it will suffice to show that the composite functor $\mathcal{F} \mapsto i_* i^! \mathcal{F}$ preserves filtered colimits. Using the existence of a fiber sequence $i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$, we are reduced to proving that the functor $\mathcal{F} \mapsto j_* j^* \mathcal{F}$ preserves filtered colimits, which follows from Corollary 4.1.17. □

Proposition 4.1.23. *Let X be a quasi-projective k-scheme. Then there exists an integer n with the following property: for every closed immersion $i : Y \hookrightarrow X$ and every commutative ring Λ, the functor $i^!$ carries $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$ into $\mathrm{Shv}(Y;\Lambda)_{\geq n}$.*

Proof. Let d be the Krull dimension of X. We will prove that $n = -2d$ has the desired property. Let $i : Y \hookrightarrow X$ be a closed immersion, and let $j : U \to X$ be the complementary open immersion. To prove that the functor $i^!$ carries $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$ into $\mathrm{Shv}(Y;\Lambda)_{\geq n}$, it will suffice to show that the composite functor $i_* i^!$ carries $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$ to $\mathrm{Shv}(X;\Lambda)_{\geq n}$. Using the fiber sequence of functors

$$\Sigma j_* j^* \to i_* i^! \to \mathrm{id},$$

we are reduced to proving that the functor j_* carries $\mathrm{Shv}(U;\Lambda)^{\heartsuit}$ into $\mathrm{Shv}(X;\Lambda)_{\geq n-1}$. Let $\mathcal{F} \in \mathrm{Shv}(U;\Lambda)^{\heartsuit}$. We will prove that $j_* \mathcal{F} \in \mathrm{Shv}(X;\Lambda)_{\geq n-1}$ by proving that $(j_* \mathcal{F})(V) \in (\mathrm{Mod}_{\Lambda})_{\geq n-1}$ for every étale map $V \to X$. Equivalently, we must show that the cohomology groups $\mathrm{H}^i(U \times_X V; \mathcal{F}|_{U \times_X V})$ vanish for $i > 2d+1$, which follows from Lemma 4.1.13. □

Remark 4.1.24. Let $i : Y \to X$ be a closed immersion of quasi-projective k-schemes schemes, let $U = X - Y$, and let $j : U \to X$ be the complementary open immersion. Then the pushforward functor $i_* : \mathrm{Shv}(Y;\Lambda) \to \mathrm{Shv}(X;\Lambda)$ is a fully faithful embedding, whose essential image is the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by those objects $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ such that $j^* \mathcal{F} \simeq 0$ (see Proposition SAG.2.2.1.14)

Let $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$. Then the fiber $\mathcal{K}$ of the canonical map $\mathcal{F} \to j_* j^* \mathcal{F}$ satisfies $j^* \mathcal{K} \simeq 0$, so we can write $\mathcal{K} \simeq i_* \mathcal{K}_0$ for some $\mathcal{K}_0 \in \mathrm{Shv}(Y;\Lambda)$. For each $\mathcal{G} \in \mathrm{Shv}(Y;\Lambda)$, we have canonical homotopy equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Shv}(Y;\Lambda)}(\mathcal{G}, \mathcal{K}_0) &\simeq \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, i_* \mathcal{K}_0) \\
&\simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, j_* j^* \mathcal{F})) \\
&\simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{Shv}(U;\Lambda)}(j^* i_* \mathcal{G}, j^* \mathcal{F})) \\
&\simeq \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, \mathcal{F}).
\end{aligned}$$

so that $\mathcal{K}_0$ can be identified with the sheaf $i^! \mathcal{F}$. In other words, we have a canonical fiber sequence

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}.$$

Using similar reasoning, we obtain a canonical fiber sequence

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}.$$

Remark 4.1.25. If $i : X \to Y$ is a closed immersion of quasi-projective k-schemes, then Remark 4.1.24 gives an explicit construction of the functor $i^!$ (which does not depend on Corollary 4.1.17): namely, for each object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$, we can identify $i^!$ with a preimage (under the functor i_*) of the fiber of the unit map $\mathcal{F} \to j_* j^* \mathcal{F}$.

4.2. **Constructible Sheaves.** Let k be an algebraically closed field, which we regard as fixed throughout this section. Let X be a quasi-projective k-scheme and let Λ be a commutative ring. In §4.1, we introduced the stable ∞-category $\mathrm{Shv}(X;\Lambda)$ of étale sheaves of Λ-modules on X. In this section, we will show that the ∞-category $\mathrm{Shv}(X;\Lambda)$ is compactly generated, and that the compact objects of $\mathrm{Shv}(X;\Lambda)$ can be identified with the (perfect) constructible sheaves on X (Proposition 4.2.5).

Notation 4.2.1. Let X be a quasi-projective k-scheme, so that there is a unique morphism of k-schemes $f : X \to \operatorname{Spec} k$. Pullback along f determines a functor

$$\mathrm{Mod}_\Lambda \simeq \mathrm{Shv}(\operatorname{Spec} k; \Lambda) \xrightarrow{f^*} \mathrm{Shv}(X; \Lambda),$$

which we will denote by $M \mapsto \underline{M}_X$. For each $M \in \mathrm{Mod}_\Lambda$, we will refer to $\underline{M}_X$ as the *constant sheaf on X with value M*. By construction, the functor $M \mapsto \underline{M}_X$ is left adjoint to the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X) \in \mathrm{Mod}_\Lambda$. In the special case where $M = \Lambda = \mathbf{Z}/\ell^d\mathbf{Z}$, the constant sheaf $\underline{M}_X$ is given by the formula $\underline{M}_X(U) = C^*(U; \mathbf{Z}/\ell^d\mathbf{Z})$ (see Definition 2.2.6).

Proposition 4.2.2. *Let X be a quasi-projective k-scheme and let Λ be a commutative ring. Then the ∞-category $\mathrm{Shv}(X; \Lambda)$ is compactly generated. Moreover, the full subcategory*

$$\mathrm{Shv}^c(X; \Lambda) \subseteq \mathrm{Shv}(X; \Lambda)$$

spanned by the compact objects is the smallest stable subcategory of $\mathrm{Shv}(X; \Lambda)$ which is closed under retracts and contains every object of the form $j_! \underline{\Lambda}_U$, where $j : U \to X$ is an object of the category Sch^{et}_X.

Proof. We first show that for each $j : U \to X$ in $\mathrm{Sch}^{\mathrm{et}}_X$, the sheaf $j_! \underline{\Lambda}_U$ is a compact object of $\mathrm{Shv}(X; \Lambda)$. To prove this, it suffices to show that the functor

$$\mathcal{F} \mapsto \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(j_! \underline{\Lambda}_U, \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Mod}_\Lambda}(\Lambda, \mathcal{F}(U))$$

commutes with filtered colimits, which follows immediately from Proposition 4.1.16.

Let $\mathcal{C} \subseteq \mathrm{Shv}(X; \Lambda)$ be the smallest full subcategory which contains every object of the form $j_! \underline{\Lambda}_U$ and is closed under retracts. Since $\mathcal{C}$ consists of compact objects of $\mathrm{Shv}(X; \Lambda)$, the inclusion $\mathcal{C} \hookrightarrow \mathrm{Shv}(X; \Lambda)$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Shv}(X; \Lambda)$ which commutes with filtered colimits (Proposition HTT.5.3.5.10). Moreover, since $\mathcal{C}$ is closed under retracts, we can identify $\mathcal{C}$ with the full subcategory of $\mathrm{Ind}(\mathcal{C})$ spanned by the compact objects. To complete the proof that $\mathrm{Shv}(X; \Lambda)$ is a compactly generated ∞-category and that $\mathcal{C}$ is the ∞-category of compact objects of $\mathrm{Shv}(X; \Lambda)$, it will suffice to show that F is an equivalence of ∞-categories. Using Corollary HTT.5.5.2.9, we deduce that F has a right adjoint G. We wish to show that F and G are mutually inverse equivalences. Since F is fully faithful, it will suffice to show that G is conservative. Since G is an exact functor between stable ∞-categories, it will suffice to show that if $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ satisfies $G(\mathcal{F}) \simeq 0$, then $\mathcal{F} \simeq 0$. This is clear, since $G(\mathcal{F}) \simeq 0$ implies that

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^n j_! \underline{\Lambda}_U, G(\mathcal{F})) \simeq \pi_0 \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\Sigma^n j_! \underline{\Lambda}, \mathcal{F}) \simeq \mathrm{H}_n(\mathcal{F}(U))$$

vanishes for each $U \in \mathrm{Sch}^{\mathrm{et}}_X$. $\square$

Remark 4.2.3. For every quasi-projective k-scheme X, we can regard $\mathrm{Shv}(X; \Lambda)$ as a symmetric monoidal ∞-category (see §SAG.2.1.1), whose unit object is the constant sheaf $\underline{\Lambda}_X$. Suppose that $\mathcal{F}$ is a dualizable object of $\mathrm{Shv}(X; \Lambda)$ with dual $\mathcal{F}^\vee$. For any diagram of objects $\{\mathcal{G}_\alpha\}$ of $\mathrm{Shv}(X; \Lambda)$, we have a commutative diagram

$$\begin{array}{ccc} \varinjlim \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F}, \mathcal{G}_\alpha) & \longrightarrow & \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F}, \varinjlim \mathcal{G}_\alpha) \\ \downarrow & & \downarrow \\ \varinjlim \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\underline{\Lambda}_X, \mathcal{F}^\vee \otimes \mathcal{G}_\alpha) & \longrightarrow & \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\underline{\Lambda}_X, \mathcal{F}^\vee \otimes \varinjlim \mathcal{G}_\alpha). \end{array}$$

Consequently, since $\underline{\Lambda}_X$ is a compact object of $\mathrm{Shv}(X; \Lambda)$ every dualizable object of $\mathrm{Shv}(X; \Lambda)$ is compact.

We now give another characterization of the compact objects of $\mathrm{Shv}(X; \Lambda)$.

Definition 4.2.4. Let X be a quasi-projective k-scheme. We will say that an object $\mathcal{F} \in \operatorname{Shv}(X;\Lambda)$ is *constant* if it is equivalent to $\underline{M}_X$, for some $M \in \operatorname{Mod}_\Lambda$. We will say that $\mathcal{F}$ is *locally constant* if there is an étale covering $\{f_\alpha : U_\alpha \to X\}$ for which each pullback $f_\alpha^* \mathcal{F} \in \operatorname{Shv}(U_\alpha;\Lambda)$ is constant.

Proposition 4.2.5. *Let X be a quasi-projective k-scheme. Then an object $\mathcal{F} \in \operatorname{Shv}(X;\Lambda)$ is compact if and only if the following conditions are satisfied:*

(1) *There exists a finite sequence of quasi-compact open subsets*
$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$$
such that, for $1 \leq i \leq n$, if Y_i denotes the locally closed reduced subscheme of X with support $U_i - U_{i-1}$, then each restriction $\mathcal{F}|_{Y_i}$ is locally constant.

(2) *For every k-valued point $\eta : \operatorname{Spec} k \to X$, the stalk $\eta^* \mathcal{F} \in \operatorname{Shv}(\operatorname{Spec} k;\Lambda) \simeq \operatorname{Mod}_\Lambda$ is perfect (that is, it is a compact object of $\operatorname{Mod}_\Lambda$).*

Definition 4.2.6. We will say that an object $\mathcal{F} \in \operatorname{Shv}(X;\Lambda)$ is *constructible* if it satisfies conditions (1) and (2) of Proposition 4.2.5 (equivalently, if it is a compact object of $\operatorname{Shv}(X;\Lambda)$). We let $\operatorname{Shv}^c(X;\Lambda)$ denote the full subcategory of $\operatorname{Shv}(X;\Lambda)$ spanned by the constructible objects.

Warning 4.2.7. When the commutative ring Λ is finite, some authors use the term *constructible* to refer to sheaves which are required to satisfy some weaker version of condition (2), such as the finiteness of the graded abelian group $\mathrm{H}_*(\eta^* \mathcal{F})$ for each point $\eta : \operatorname{Spec} k \to X$.

Proof of Proposition 4.2.5. We begin by showing that every compact object $\mathcal{F} \in \operatorname{Shv}(X;\Lambda)$ satisfies conditions (1) and (2). Using Proposition 4.2.2, we may reduce to the case where $\mathcal{F} = j_! \underline{\Lambda}_U$ for some étale map $j : U \to X$. We first show that $\mathcal{F}$ satisfies (1). We may assume that $X \neq \emptyset$, otherwise the result is vacuous. Using Noetherian induction on X (and Proposition 4.1.12), we may suppose that the restriction $\mathcal{F}|_Y$ is satisfies (1) for every nonempty closed subscheme $Y \subseteq X$. It will therefore suffice to show that $\mathcal{F}|_V$ satisfies (1) for some nonempty open subscheme $V \subseteq X$. Passing to an open subscheme, we may suppose that $j : U \to X$ is finite étale of some fixed rank r. In this case, we claim that $j_! \underline{\Lambda}$ is locally constant. Choose a finite étale surjection $\widetilde{X} \to X$ such that the fiber product $U \times_X \widetilde{X}$ is isomorphic to a disjoint union of r copies of $\widetilde{X}$. Using Proposition 4.1.12, we may replace X by $\widetilde{X}$. In this case, the sheaf $j_! \underline{\Lambda}_U \simeq \underline{\Lambda}_X^r$ is constant.

We now show that for every étale map $j : U \to X$, the sheaf $j_! \underline{\Lambda}_U$ satisfies condition (2). Using Proposition 4.1.12, we may replace X by $\operatorname{Spec} k$ and thereby reduce to the case where X is the spectrum of an algebraically closed field. In this case, U is a disjoint union of finitely many copies of X, so that $j_! \underline{\Lambda}_U$ can be identified with a free module Λ^r as an object of $\operatorname{Shv}(X;\Lambda) \simeq \operatorname{Mod}_\Lambda$.

Now suppose that $\mathcal{F}$ is a sheaf satisfying conditions (1) and (2); we wish to show that $\mathcal{F}$ is a compact object of $\operatorname{Shv}(X;\Lambda)$. Without loss of generality we may suppose that X is nonempty. Using Noetherian induction on X, we may assume that for every closed immersion $i : Y \to X$ whose image is a proper closed subset of X, the pullback $i^* \mathcal{F}$ is a compact object of $\operatorname{Shv}(Y;\Lambda)$. Using Proposition 4.1.22 we deduce that $i_* i^* \mathcal{F}$ is a compact object of $\operatorname{Shv}(Y;\Lambda)$. Let $j : U \to X$ denote the complementary open immersion, so that we have a fiber sequence
$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}.$$
It will therefore suffice to show that there exists a nonempty open subset $U \subseteq X$ such that $j_! j^* \mathcal{F}$ is a compact object of $\operatorname{Shv}(X;\Lambda)$. Since the functor j^* preserves colimits, $j_!$ preserves compact objects; it will therefore suffice to show that we can choose U such that $j^* \mathcal{F}$ is a

compact object of $\mathrm{Shv}(U;\Lambda)$. Since $\mathcal{F}$ satisfies (1), we may pass to a nonempty open subscheme of X and thereby reduce to the case where $\mathcal{F}$ is locally constant.

Choose a collection of jointly surjective étale maps $j_\alpha : U_\alpha \to X$ such that each pullback $j_\alpha^* \mathcal{F}$ is constant, hence of the form $\underline{M(\alpha)}_{U_\alpha}$ for some $M(\alpha) \in \mathrm{Mod}_\Lambda$. Using condition (2), we deduce that each $M(\alpha)$ is perfect, hence a dualizable object of Mod_Λ. It follows that each pullback $j_\alpha^* \mathcal{F}$ is a dualizable object of $\mathrm{Shv}(U_\alpha;\Lambda)$, so that $\mathcal{F}$ is a dualizable object of $\mathrm{Shv}(X;\Lambda)$ and therefore compact by virtue of Remark 4.2.3. □

Remark 4.2.8 (Extension by Zero)**.** Let $i : X \to Y$ be a locally closed immersion between quasi-projective k-schemes, so that i factors as a composition

$$X \xrightarrow{i'} \overline{X} \xrightarrow{i''} Y$$

where $\overline{X}$ denotes the scheme-theoretic closure of X in Y, i'' is a closed immersion, and i' is an open immersion. We let $i_!$ denote the composite functor

$$\mathrm{Shv}(X;\Lambda) \xrightarrow{i'_!} \mathrm{Shv}(\overline{X};\Lambda) \xrightarrow{i''_*} \mathrm{Shv}(Y;\Lambda),$$

which we will refer to as the functor of *extension by zero* from X to Y.

Remark 4.2.9. It follows from Proposition 4.2.5 that for every compact object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$, there exists a finite stratification of X by locally closed subschemes Y_α and a finite filtration of $\mathcal{F}$ whose successive quotients have the form $i_{\alpha!}\mathcal{F}_\alpha$, where $\mathcal{F}_\alpha \in \mathrm{Shv}(Y_\alpha;\Lambda)$ is a locally constant sheaf with perfect stalks, and $i_\alpha : Y_\alpha \to X$ denotes the inclusion map.

Corollary 4.2.10. *Let X be a quasi-projective k-scheme and let Λ be a field. If $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ is compact, then each truncation $\tau_{\geq n}\mathcal{F}$ and $\tau_{\leq n}\mathcal{F}$ is also a compact object of* $\mathrm{Shv}(X;\Lambda)$.

Remark 4.2.11. The conclusion of Corollary 4.2.10 holds more generally under the assumption that Λ is a ring of finite projective dimension; for example, it also holds when $\Lambda = \mathbf{Z}$.

Proposition 4.2.12. *Let X be a quasi-projective k-scheme, let Λ be a field, and let $\mathcal{F}$ be an object of* $\mathrm{Shv}(X;\Lambda)^\heartsuit$*. If $\mathcal{F}$ is constructible, then $\mathcal{F}$ is a Noetherian object of the abelian category* $\mathrm{Shv}(X;\Lambda)^\heartsuit$*.*

Proof. Proceeding by Noetherian induction, we may suppose that for each proper closed subscheme $Y \subsetneq X$ that each constructible object $\mathcal{G} \in \mathrm{Shv}(Y;\Lambda)^\heartsuit$ is Noetherian.

By virtue of Proposition 4.2.5, it will suffice to prove the following:

- $(*_n)$ Let $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)^\heartsuit$ be constructible. Suppose there exists a nonempty connected open subset $U \subseteq X$ containing a point x such that $\mathcal{F}|_U$ is locally constant and the stalk $\mathcal{F}_x$ has dimension $\leq n$ (when regarded as a vector space over Λ). Then $\mathcal{F}$ is a Noetherian object of $\mathrm{Shv}(X;\Lambda)^\heartsuit$.

The proof proceeds by induction on n. Let U and $\mathcal{F}$ satisfy the hypotheses of $(*_n)$. We will abuse notation by identifying $\mathcal{F}$ with a sheaf of Λ-vector spaces on X. Suppose we are given an ascending chain of subobjects

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$$

of $\mathcal{F}$; we wish to show that it is eventually constant. If each restriction $\mathcal{F}_m|_U$ vanishes, then we have $\mathcal{F}_m \simeq i_* i^* \mathcal{F}_m$. We are therefore reduced to proving that the sequence of inclusions

$$i^*\mathcal{F}_0 \subseteq i^*\mathcal{F}_1 \subseteq \cdots$$

stabilizes, which follows from our inductive hypothesis. We may therefore assume that some $\mathcal{F}_m|_U \neq 0$ for some integer m. Using Proposition 4.2.2 we can write $\mathcal{F}_m$ as the colimit of a filtered diagram $\{\mathcal{F}_\alpha\}$ of constructible objects of $\mathrm{Shv}(X;\Lambda)$. Using Corollary 4.2.10, we can assume that each $\mathcal{F}_\alpha$ belongs to $\mathrm{Shv}(X;\Lambda)^\heartsuit$. Choose an index α for which the map $\mathcal{F}_\alpha|_U \to \mathcal{F}_m|_U$ is nonzero. Using Proposition 4.2.5, we can choose a nonempty open subset $U' \subseteq U$ such that $\mathcal{F}_\alpha|_{U'}$ is locally constant. Choose an étale U-scheme V such that the map $\mathcal{F}_\alpha(V) \to \mathcal{F}_m(V) \subseteq \mathcal{F}(V)$ is nonzero (as a map of vector spaces over Λ), and let $V' = U' \times_U V$. Then V' is dense in V, so that the map $\mathcal{F}(V) \to \mathcal{F}(V')$ is injective. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{F}_\alpha(V) & \longrightarrow & \mathcal{F}_\alpha(V') \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V'), \end{array}$$

we deduce that the map of vector spaces $\mathcal{F}_\alpha(V') \to \mathcal{F}(V')$ is nonzero, so that the map of sheaves $\mathcal{F}_\alpha|_{U'} \to \mathcal{F}|_{U'}$ is nonzero. Replacing U by U', we may reduce to the case where $\mathcal{F}_\alpha|_U$ is locally constant. Note that the cofiber of the map $\mathcal{F}_\alpha \to \mathcal{F}$ is constructible, so that (by virtue of Corollary 4.2.10) the cokernel $\mathcal{G} = \mathrm{coker}(\mathcal{F}_\alpha \to \mathcal{F})$ is a constructible object of $\mathrm{Shv}(X;\Lambda)^\heartsuit$. For any point $x \in U$, we have $\dim \mathcal{G}_x < \dim \mathcal{F}_x$, so that our inductive hypothesis implies that $\mathcal{G}$ is a Noetherian object of $\mathrm{Shv}(X;\Lambda)^\heartsuit$. The sheaf $\mathcal{F}/\mathcal{F}_m$ is a quotient of $\mathcal{G}$, and therefore also Noetherian. It follows that the sequence of subobjects $\{\mathcal{F}_{m'}/\mathcal{F}_m \subseteq \mathcal{F}/\mathcal{F}_m\}_{m' \geq m}$ is eventually constant, so that the sequence $\{\mathcal{F}_{m'} \subseteq \mathcal{F}\}_{m' \geq m}$ is eventually constant. □

Proposition 4.2.13. *Let X be a quasi-projective k-scheme, and let Λ be a field. Then there exists an integer n with the following property: for every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X;\Lambda)^\heartsuit$, if $\mathcal{F}$ is compact object of $\mathrm{Shv}(X;\Lambda)$, then $\mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{G}) \simeq 0$ for $m > n$.*

Proof. Using Proposition 4.1.23, we can choose an integer n' such that, for every closed immersion $i : Y \hookrightarrow X$, the sheaf $i^! \mathcal{G}$ belongs to $\mathrm{Shv}(Y;\Lambda)_{\geq n'}$. Let d be the Krull dimension of X. We will prove that $n = n' + 2d + 1$ has the desired property. For this, it will suffice to prove the following:

- $(*)$ Let $\mathcal{H} \in \mathrm{Shv}(X;\Lambda)$ have the property that $i^! \mathcal{H} \in \mathrm{Shv}(Y;\Lambda)_{\geq n'}$ for every closed immersion $i : Y \to X$. Then $\mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{H}) \simeq 0$ for $m > n$.

We prove $(*)$ using Noetherian induction on X. Using Proposition 4.2.5, we can choose an open immersion $j : U \to X$ such that $j^* \mathcal{F}$ is locally constant, hence a dualizable object of $\mathrm{Shv}(X;\Lambda)^\heartsuit$. Let $i : Y \hookrightarrow X$ be a complementary closed immersion, so that we have a fiber sequence

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}.$$

We therefore obtain an exact sequence

$$\mathrm{Ext}^m_{\mathrm{Shv}(U;\Lambda)}(j^* \mathcal{F}, j^* \mathcal{H}) \to \mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{H}) \to \mathrm{Ext}^m_{\mathrm{Shv}(Y;\Lambda)}(i^* \mathcal{F}, i^! \mathcal{H}).$$

The first group can be identified with $\mathrm{H}^m(U; (j^* \mathcal{F})^\vee \otimes_\Lambda j^* \mathcal{H}))$, which vanishes for $m > n$ by virtue of Lemma 4.1.13. The third group vanishes for $m > n$ by the inductive hypothesis, so that $\mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{H})$ also vanishes for $m > n$. □

Proposition 4.2.14. *Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for $d \geq 1$. Then $\mathcal{F}$ is constructible if and only if the object $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})$ is constructible.*

Proof. It follows from Proposition 4.1.16 that the forgetful functor

$$\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})$$

preserves colimits and therefore the left adjoint $\mathcal{F} \mapsto (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{F}$ preserves compact objects; this proves the "only if" direction. For the converse, suppose that $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ has the property that $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ is constructible. Let $\mathcal{C} \subseteq \mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$ denote the full subcategory spanned by those objects M for which the functor

$$\mathcal{G} \mapsto \mathrm{Map}_{\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}, M \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{G})$$

preserves filtered colimits. Then $\mathcal{C}$ contains $\mathbf{Z}/\ell\mathbf{Z}$ and is closed under the formation of extensions. It follows that $\mathcal{C}$ contains $\mathbf{Z}/\ell^d\mathbf{Z}$, so that $\mathcal{F}$ is constructible. □

The theory of étale sheaves is particularly well-behaved when the coefficient ring Λ has the form $\mathbf{Z}/\ell^d\mathbf{Z}$, where ℓ is a prime number which is invertible in k. We close this section by recalling some of the special features of this situation, which will play an important role in our discussion of ℓ-adic sheaves in §4.3.

Proposition 4.2.15 (Persistence of Constructibility)**.** *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes, let ℓ be a prime number which is invertible in k, and let $d \geq 0$. Then:*

(1) *The pushforward functor* $f_* : \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ *carries* $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ *into* $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$.
(2) *The pullback functor* $f^* : \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ *carries* $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ *into* $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$.
(3) *If f is proper, then the exceptional inverse image functor*

$$f^! : \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$$

carries $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ *into* $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$.
(4) *If f is étale, then the functor*

$$f_! : \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$$

carries $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ *into* $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$.

Remark 4.2.16. Assertions (2) and (4) of Proposition 4.2.15 follow immediately from the fact that the functors f_* and f^* preserve filtered colimits (and remain valid when $\mathbf{Z}/\ell^d\mathbf{Z}$ is replaced by an arbitrary commutative ring).

Proof of Proposition 4.2.15. By virtue of Proposition 4.2.14, we can assume without loss of generality that $d = 1$. In this case, the desired result is proven as Corollaire 1.5 ("Théorème de finitude") on page 234 of [13] (note that our definition of constructibility is different from the notion of constructibility considered in [13], but the two notions agree in the case $d = 1$; see Warning 4.2.7). □

Remark 4.2.17. Let X be a quasi-projective k-scheme, let ℓ be a prime number which is invertible in k, and let $\mathcal{F}$ be a compact object of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$. Using Propositions 4.2.15 and 4.2.5, we see that that there exists a finite collection of locally closed immersions $i_\alpha : Y_\alpha \hookrightarrow X$ (having disjoint images) such that $\mathcal{F}$ admits a filtration with successive quotients of the form $i_{\alpha *} \mathcal{F}_\alpha$, where each $\mathcal{F}_\alpha$ is a locally constant sheaf on Y_α with perfect stalks (compare with Remark 4.2.9).

Corollary 4.2.18. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes, let ℓ be a prime number which is invertible in k, and let $d \geq 0$. Then the functor $f^! : \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ preserves filtered colimits.*

Proof. This is a reformulation of assertion (1) of Proposition 4.2.15. □

Corollary 4.2.19. *Let X be a quasi-projective k-scheme, let ℓ be a prime number which is invertible in k, let and let $\mathcal{F}, \mathcal{G} \in \operatorname{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for some integer $d \geq 0$. Then the groups $\operatorname{Ext}^i_{\operatorname{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}, \mathcal{G})$ are finite.*

Proof. Using Proposition 4.2.2, we may reduce to the case where $\mathcal{F} = j_! \underline{\mathbf{Z}/\ell^d\mathbf{Z}}_U$ for some étale morphism $j : U \to X$. In this case, we have

$$\begin{aligned}\operatorname{Ext}^i_{\operatorname{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}, \mathcal{G}) &\simeq \operatorname{Ext}^i_{\operatorname{Shv}(U;\mathbf{Z}/\ell^d\mathbf{Z})}(\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_U, j^* \mathcal{G}) \\ &\simeq \mathrm{H}^i(\pi_* j^* \mathcal{G})\end{aligned}$$

where $\pi : U \to \operatorname{Spec} k$ denotes the projection map. The desired result now follows from Proposition 4.2.15. □

Proposition 4.2.20. *Let X be a quasi-projective k-scheme, let ℓ be a prime number which is invertible in k, and let $\mathcal{F} \in \operatorname{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for some $d \geq 0$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ vanishes.*
(2) *For every point $\eta : \operatorname{Spec} k \to X$, the stalk $\mathcal{F}_\eta = \eta^* \mathcal{F}$ vanishes.*
(3) *For every point $\eta : \operatorname{Spec} k \to X$, the costalk $\eta^! \mathcal{F} \in \operatorname{Shv}(\operatorname{Spec} k; \mathbf{Z}/\ell^d\mathbf{Z})$ vanishes.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) are obvious, and the implication (2) ⇒ (1) is Proposition 4.1.11 (and is valid for any coefficient ring Λ). Assume that $\mathcal{F}$ satisfies (3); we will prove that $\mathcal{F} \simeq 0$ using Noetherian induction on X. Using Proposition 4.2.5, we can choose a nonempty open subset U such that $\mathcal{F}|_U$ is locally constant. Shrinking U if necessary, we may suppose that U is smooth of dimension $n \geq 0$. Let $i : Y \to X$ be a closed immersion complementary to U. Then $i^! \mathcal{F} \in \operatorname{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ satisfies condition (3), so that $i^! \mathcal{F} \simeq 0$ by the inductive hypothesis. We may therefore replace X by U, and thereby reduce to the case where $\mathcal{F}$ is locally constant. The assertion that $\mathcal{F}$ vanishes is local on X; we may therefore suppose further that X is smooth and $\mathcal{F} \in \operatorname{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ has the form $\underline{M}_X$ for some perfect object $M \in \operatorname{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$. Arguing as in Remark 4.1.21 (and choosing a primitive ℓ^dth root of unity in k), we see that for any point $\eta : \operatorname{Spec} k \to X$, the pullback $\eta^! \mathcal{F}$ is equivalent to $\Sigma^{-2n} M$. It then follows from (3) that $M \simeq 0$, so that $\mathcal{F} \simeq 0$ as desired. □

4.3. **ℓ-adic Sheaves.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let X be a quasi-projective k-scheme. For every commutative ring Λ, the theory outlined in §4.1 associates an ∞-category $\operatorname{Shv}(X; \Lambda)$ of (hypercomplete) $\operatorname{Mod}_\Lambda$-valued étale sheaves on X. This theory is very well-behaved when the commutative ring Λ has the form $\mathbf{Z}/\ell^d\mathbf{Z}$ for some $d \geq 0$, but badly behaved when $\Lambda = \mathbf{Z}$ or $\Lambda = \mathbf{Q}$. To remedy the situation, it is convenient to introduce the formalism of *ℓ-adic sheaves*: roughly speaking, a (constructible) ℓ-adic sheaf on X is a compatible system $\{\mathcal{F}_d\}_{d \geq 0}$, where each $\mathcal{F}_d$ is a (constructible) object of $\operatorname{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$. The collection of ℓ-adic sheaves on X can be organized into an ∞-category which we will denote by $\operatorname{Shv}_\ell(X)$. Our goal in this section is to review the definition of the ∞-categories $\operatorname{Shv}_\ell(X)$ and summarize some of the properties which we will need later in this paper.

Definition 4.3.1. Let Λ be a commutative ring, and let M be an object of $\operatorname{Mod}_\Lambda$. We will say that M is *ℓ-complete* if the limit of the diagram

$$\cdots \to M \xrightarrow{\ell} M \xrightarrow{\ell} M$$

vanishes in $\operatorname{Mod}_\Lambda$.

Remark 4.3.2. In the situation of Definition 4.3.1, we have a tower of fiber sequences

$$\{M \xrightarrow{\ell^d} M \to \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} M\}_{d \geq 0}.$$

Passing to the limit, we see that M is ℓ-complete if and only if the canonical map

$$M \to \varprojlim((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} M)$$

is an equivalence in the ∞-category Mod_Λ.

Remark 4.3.3. Let Λ be a commutative ring. Then an object $M \in \mathrm{Mod}_\Lambda$ is ℓ-complete if and only if each homology group $\mathrm{H}_n(M)$ is ℓ-complete, when regarded as a discrete object of Mod_Λ. To prove this, we may assume without loss of generality that $\Lambda = \mathbf{Z}$, in which case M is noncanonically equivalent to the product $\prod_{n \in \mathbf{Z}} \Sigma^n \mathrm{H}_n(M)$.

If Λ is Noetherian and each homology group $\mathrm{H}_n(M)$ is finitely generated as a Λ-module, then M is ℓ-complete if and only if each of the homology groups $\mathrm{H}_n(M)$ is isomorphic to its ℓ-adic completion $\varprojlim \mathrm{H}_n(M)/\ell^d \mathrm{H}_n(M)$, where the limit is taken in the abelian category of Λ-modules.

Remark 4.3.4. Let Λ be a commutative ring, let $M_\bullet$ be a simplicial object of Mod_Λ, and let $|M_\bullet| \in \mathrm{Mod}_\Lambda$ denote its geometric realization. Suppose that there exists an integer $n \in \mathbf{Z}$ such that the simplicial abelian groups $\mathrm{H}_m(M_\bullet)$ vanish for $m < n$. Then if each M_q is ℓ-complete, the geometric realization $|M_\bullet|$ is ℓ-complete. To prove this, it will suffice to show that each homology group $\mathrm{H}_i(|M_\bullet|)$ is ℓ-complete (Remark 4.3.3). We may therefore replace $M_\bullet$ by a sufficiently large skeleton, in which case $|M_\bullet|$ is a finite colimit of ℓ-complete objects of Mod_Λ.

Definition 4.3.5. Let X be a quasi-projective k-scheme and let Λ be a commutative ring. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ is *ℓ-complete* if, for every object $U \in \mathrm{Sch}_X^{\mathrm{et}}$, the object $\mathcal{F}(U) \in \mathrm{Mod}_\Lambda$ is ℓ-complete.

Remark 4.3.6. Let $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$. The following conditions are equivalent:

(1) The sheaf $\mathcal{F}$ is ℓ-complete.
(2) The limit of the tower

$$\cdots \to \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F}$$

vanishes.
(3) The canonical map $\mathcal{F} \to \varprojlim(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$ is an equivalence in $\mathrm{Shv}(X;\Lambda)$.

Remark 4.3.7. Let X be a quasi-projective k-scheme, let Λ a commutative ring, and let $\mathcal{C} \subseteq \mathrm{Shv}(X;\Lambda)$ be the full subcategory spanned by the ℓ-complete objects. Then the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Shv}(X;\Lambda)$ admits a left adjoint L, given by the formula

$$L\,\mathcal{F} = \varprojlim(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}).$$

We will refer to L as the *ℓ-adic completion functor*. Note that an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ is annihilated by the functor L if and only if the map $\ell : \mathcal{F} \to \mathcal{F}$ is an equivalence.

Remark 4.3.8. Let X be a quasi-projective k-scheme, Λ a commutative ring. We will regard $\mathrm{Shv}(X;\Lambda)$ as a symmetric monoidal ∞-category, with tensor product we will denote by $\otimes_\Lambda$. Suppose we are given objects $\mathcal{F}, \mathcal{F}' \in \mathrm{Shv}(X;\Lambda)$. If multiplication by ℓ induces an equivalence from $\mathcal{F}$ to itself, then multiplication by ℓ also induces an equivalence from $\mathcal{F} \otimes_\Lambda \mathcal{F}'$ to itself. It follows that the full subcategory $\mathcal{C} \subseteq \mathrm{Shv}(X;\Lambda)$ spanned by the ℓ-complete objects inherits the structure of a symmetric monoidal ∞-category, with tensor product $\widehat{\otimes}_\Lambda$ given by the formula

$$\mathcal{F} \,\widehat{\otimes}_\Lambda\, \mathcal{F}' = L(\mathcal{F} \otimes_\Lambda \mathcal{F}')$$

where L denotes the ℓ-adic completion functor of Remark 4.3.7.

If $\phi : \Lambda \to \Lambda'$ is a morphism of commutative rings, then ϕ induces a base-change functor $\mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(X;\Lambda')$ (which is left adjoint to the forgetful functor from $\mathrm{Shv}(X;\Lambda')$ to $\mathrm{Shv}(X;\Lambda)$). In particular, we always obtain a map of ∞-categories

$$\theta : \mathrm{Shv}(X;\Lambda) \to \varprojlim_{d\geq 0} \mathrm{Shv}(X;\Lambda/\ell^d\Lambda).$$

Proposition 4.3.9. *Let Λ be a commutative ring and suppose that ℓ is not a zero-divisor in Λ. Let X be a quasi-projective k-scheme and let $\mathcal{C} \subseteq \mathrm{Shv}(X;\Lambda)$ be the full subcategory spanned by the ℓ-complete objects. Then the composite functor*

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(X;\Lambda) \xrightarrow{\theta} \varprojlim_{d\geq 0} \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$$

is an equivalence of ∞-categories.

Proof. We first prove that θ is fully faithful when restricted to $\mathcal{C}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be objects of $\mathrm{Shv}(X;\Lambda)$. We compute

$$\begin{aligned} \mathrm{Map}(\theta(\mathcal{F}), \theta(\mathcal{F}')) &\simeq \varprojlim_{d\geq 0} \mathrm{Map}_{\mathrm{Shv}(X;\Lambda/\ell^d\Lambda)}((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}, (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}') \\ &\simeq \varprojlim_{d\geq 0} \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F}, (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}') \\ &\simeq \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F}, L\,\mathcal{F}'). \end{aligned}$$

where L is defined as in Remark 4.3.7. If $\mathcal{F}'$ is ℓ-complete, then the canonical map

$$\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F}, \mathcal{F}') \to \mathrm{Map}(\theta(\mathcal{F}), \theta(\mathcal{F}'))$$

is a homotopy equivalence.

It remains to prove essential surjectivity. Suppose we are given an object of the inverse limit $\varprojlim_{d\geq 0} \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$, which we can identify with a compatible sequence of objects

$$\{\mathcal{F}_d \in \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)\}_{d\geq 0}.$$

Let us abuse notation by identifying each $\mathcal{F}_d$ with its image in $\mathrm{Shv}(X;\Lambda)$, and set $\mathcal{F} = \varprojlim \mathcal{F}_d \in \mathrm{Shv}(X;\Lambda)$. Since each $\mathcal{F}_d$ is ℓ-complete, it follows that $\mathcal{F}$ is also ℓ-complete. Moreover, we have a canonical map $\theta(\mathcal{F}) \to \{\mathcal{F}_d\}_{d\geq 0}$ in the ∞-category $\varprojlim_{d\geq 0} \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$. To prove that this map is an equivalence, it will suffice to show that for each integer $d \geq 0$, the canonical map

$$(\Lambda/\ell^d\Lambda) \otimes_{\Lambda} \varprojlim_{e\geq d} \mathcal{F}_e \to \mathcal{F}_d$$

is an equivalence in $\mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$. Since $\Lambda/\ell^d\Lambda$ is a perfect Λ-module, we can identify this with the natural map

$$\varprojlim_{e\geq d} (\Lambda/\ell^d\Lambda) \otimes_{\Lambda} \mathcal{F}_e \simeq \varprojlim_{e\geq d} ((\Lambda/\ell^d\Lambda) \otimes_{\Lambda} (\Lambda/\ell^e\Lambda)) \otimes_{\Lambda/\ell^e\Lambda} \mathcal{F}_e \to (\Lambda/\ell^d\Lambda) \otimes_{\Lambda/\ell^e\Lambda} \mathcal{F}_e\,.$$

This map is an equivalence, since the inverse system $\{(\Lambda/\ell^d\Lambda) \otimes_{\Lambda} (\Lambda/\ell^e\Lambda)\}_{e\geq d}$ is equivalent to $\Lambda/\ell^d\Lambda$ as a Pro-object of the ∞-category Mod_{Λ}. □

Definition 4.3.10. Let X be a quasi-projective k-scheme. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X;\mathbf{Z})$ is a *constructible ℓ-adic sheaf* if it satisfies the following conditions:

(1) The sheaf $\mathcal{F}$ is ℓ-complete.

(2) For each integer $d \geq 0$, the tensor product
$$(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$$
is constructible.

We let $\mathrm{Shv}^c_\ell(X)$ denote the full subcategory of $\mathrm{Shv}(X; \mathbf{Z})$ spanned by the constructible ℓ-adic sheaves.

Remark 4.3.11. In the situation of Definition 4.3.10, it suffices to verify condition (2) in the case $d = 1$, by virtue of Proposition 4.2.14.

Remark 4.3.12. Let $\mathcal{F}$ be as in Definition 4.3.10. Then the tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ is constructible as an object of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ if and only if it is constructible as an object of $\mathrm{Shv}(X; \mathbf{Z})$. Consequently, condition (2) can be rephrased as follows:

(2′) The cofiber of the map $\ell : \mathcal{F} \to \mathcal{F}$ is a constructible object of $\mathrm{Shv}(X; \mathbf{Z})$.

Remark 4.3.13. It follows from Proposition 4.3.9 that the forgetful functor $\mathrm{Shv}(X; \mathbf{Z}_\ell) \to \mathrm{Shv}(X; \mathbf{Z})$ is an equivalence when restricted to ℓ-complete objects. Consequently, we can replace $\mathrm{Shv}(X; \mathbf{Z})$ by $\mathrm{Shv}(X; \mathbf{Z}_\ell)$ in Definition 4.3.10 without changing the notion of constructible ℓ-adic sheaf.

Warning 4.3.14. Let X be a quasi-projective k-scheme. Neither of the full subcategories $\mathrm{Shv}^c_\ell(X), \mathrm{Shv}^c(X; \mathbf{Z}) \subseteq \mathrm{Shv}(X; \mathbf{Z})$ contains the other. Objects of $\mathrm{Shv}^c(X; \mathbf{Z})$ are generally not ℓ-complete (this is true even if we replace $\mathbf{Z}$ by $\mathbf{Z}_\ell$), and objects of $\mathrm{Shv}^c_\ell(X)$ need not be locally constant when restricted to any nonempty open subset of X.

Proposition 4.3.15. *Let X be a quasi-projective k-scheme.*

(1) *For each integer $d \geq 0$, the full subcategory $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z}) \subseteq \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ contains the unit object and is stable under tensor products.*

(2) *Let $\mathcal{C} \subseteq \mathrm{Shv}(X; \mathbf{Z})$ be the full subcategory spanned by the ℓ-complete objects, and regard $\mathcal{C}$ as a symmetric monoidal ∞-category with respect to the completed tensor product $\widehat{\otimes}_{\mathbf{Z}}$ of Remark 4.3.8. Then the full subcategory $\mathrm{Shv}^c_\ell(X) \subseteq \mathcal{C}$ contains the unit object of $\mathcal{C}$ and is closed under tensor products.*

Proof. Assertion (1) follows from the characterization of constructible sheaves supplied by Proposition 4.2.5, and assertion (2) follows from (1). □

Remark 4.3.16. Let X be a quasi-projective k-scheme. Then we can identify $\mathrm{Shv}^c_\ell(X)$ (as a symmetric monoidal ∞-category) with a homotopy inverse limit of the tower of symmetric monoidal ∞-categories
$$\cdots \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell^3\mathbf{Z}) \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell^2\mathbf{Z}) \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z}).$$

Proposition 4.3.17. *Let X be a quasi-projective k-scheme. Then the equivalence of ∞-categories $\mathrm{Shv}^c_\ell(X) \simeq \varprojlim \{\mathrm{Shv}^c_\ell(X; \mathbf{Z}/\ell^d\mathbf{Z})\}_{d \geq 0}$ induces an equivalence of homotopy categories*
$$\theta : \mathrm{hShv}^c_\ell(\mathrm{X}) \to \varprojlim \{\mathrm{hShv}^c_\ell(\mathrm{X}; \mathbf{Z}/\ell^{\mathrm{d}}\mathbf{Z})\}_{\mathrm{d} \geq 0}.$$

Remark 4.3.18. Proposition 4.3.17 implies that the homotopy category of $\mathrm{Shv}^c_\ell(X)$ can be identified with the constructible derived category of $\mathbf{Z}_\ell$-sheaves considered elsewhere in the literature (see, for example [8]).

Proof of Proposition 4.3.17. It follows immediately from the definitions that θ is essentially surjective; we will show that θ is fully faithful. For every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}^c_\ell(X)$ having images $\mathcal{F}(d), \mathcal{G}(d) \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$, we have a Milnor exact sequence
$$0 \to \lim{}^1 \{\mathrm{Ext}^{n-1}(\mathcal{F}(d), \mathcal{G}(d))\} \to \mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) \to \lim{}^0 \{\mathrm{Ext}^n(\mathcal{F}(d), \mathcal{G}(d))\} \to 0.$$

Since each of the groups $\mathrm{Ext}^{n-1}(\mathcal{F}(d), \mathcal{G}(d))$ is finite (Corollary 4.2.19), the first term of this sequence vanishes. It follows that the canonical map

$$\mathrm{Map}_{\mathrm{hShv}^c_\ell(X)}(\mathcal{F}, \mathcal{G}) \to \varprojlim \mathrm{Map}_{\mathrm{hShv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}(d), \mathcal{G}(d))$$

is bijective. □

We now discuss the functorial behavior of some of the preceding constructions.

Proposition 4.3.19. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then:*

(1) *The pushforward functor $f_* : \mathrm{Shv}(X; \mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z})$ carries constructible ℓ-adic sheaves to constructible ℓ-adic sheaves sheaves.*
(2) *The resulting map from $\mathrm{Shv}^c_\ell(X)$ to $\mathrm{Shv}^c_\ell(Y)$ admits a left adjoint $f^*_\wedge$, which carries an object $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$ to the ℓ-completion of $f^* \mathcal{F}$.*

Proof. The functor f_* preserves limits, and therefore carries ℓ-complete objects to ℓ-complete objects. Assertion (1) is now a consequence of Proposition 4.2.15. To prove (2), let $L : \mathrm{Shv}(X; \mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z})$ denote the ℓ-completion functor. If $\mathcal{F} \in \mathrm{Shv}^c_\ell(Y)$, then we have a natural homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}(Y;\mathbf{Z})}(\mathcal{F}, f_* \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(Lf^* \mathcal{F}, \mathcal{G})$$

whenever $\mathcal{G} \in \mathrm{Shv}(X; \mathbf{Z}_\ell)$ is ℓ-complete. It will therefore suffice to show that $Lf^* \mathcal{F}$ is constructible. By construction, $Lf^* \mathcal{F}$ is ℓ-complete. It will therefore suffice to show that each tensor product

$$(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} Lf^* \mathcal{F} \simeq (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} f^* \mathcal{F} \simeq f^*(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$$

is a constructible object of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for each $d \geq 0$, which follows immediately from Proposition 4.2.15. □

Warning 4.3.20. In the situation of Proposition 4.3.19, the pullback functor $f^* : \mathrm{Shv}(Y; \mathbf{Z}_\ell) \to \mathrm{Shv}(X; \mathbf{Z}_\ell)$ does not preserve constructibility. For example, if $Y = \mathrm{Spec}\, k$ and $\mathcal{F} \in \mathrm{Shv}(Y; \mathbf{Z}_\ell)$ is the constant sheaf with value $\mathbf{Z}_\ell$, then the chain complex $(f^* \mathcal{F})(X)$ computes the étale cohomology of X with coefficients in the constant sheaf associated $\mathbf{Z}_\ell$, while the chain complex $(f^*_\wedge \mathcal{F})(X)$ computes its ℓ-adic completion $C^*(X; \mathbf{Z}_\ell)$.

Proposition 4.3.21. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Then the functor $f^! : \mathrm{Shv}(Y; \mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z})$ carries $\mathrm{Shv}^c_\ell(Y)$ into $\mathrm{Shv}^c_\ell(X)$.*

Proof. The functor $f^!$ preserves limits and therefore carries ℓ-complete objects to ℓ-complete objects. It will therefore suffice to show that if $\mathcal{F} \in \mathrm{Shv}^c_\ell(Y)$, then

$$(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} f^! \mathcal{F} \simeq f^!((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F})$$

is constructible for each $d \geq 0$, which follows from Proposition 4.2.15. □

Remark 4.3.22. In the situation of Proposition 4.3.21, the functor $f^! : \mathrm{Shv}^c_\ell(Y) \to \mathrm{Shv}^c_\ell(X)$ can be identified with the inverse limit of the tower of exceptional inverse image functors

$$f^! : \mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z}).$$

Example 4.3.23. Let $\mathbf{Z}_\ell(1)$ denote the inverse limit of the sequence

$$\cdots \to \mu_{\ell^3}(k) \to \mu_{\ell^2}(k) \to \mu_\ell(k),$$

where $\mu_{\ell^d}(k)$ denotes the group of ℓ^dth roots of unity in k. For each integer n, we let $\mathbf{Z}_\ell(n)$ denote the nth tensor power of $\mathbf{Z}_\ell(1)$. If $f : X \to Y$ is a smooth morphism of relative dimension n, then Example 4.1.20 supplies an equivalence

$$f^! \mathcal{F} \simeq \Sigma^{2n} \underline{\mathbf{Z}_\ell(n)}_X \otimes_{\mathbf{Z}_\ell} f^* \mathcal{F},$$

which depends functorially on $\mathcal{F} \in \mathrm{Shv}_\ell^c(Y)$. Since both sides commute with filtered colimits in $\mathcal{F}$, we obtain an equivalence of functors $f^! \simeq \Sigma^{2n} \underline{\mathbf{Z}_\ell(n)}_X \otimes_{\mathbf{Z}_\ell} f^*$.

Proposition 4.3.24. *Let $f : X \to Y$ be an étale morphism between quasi-projective k-schemes. Then:*

(1) *The pullback functor*

$$f^* : \mathrm{Shv}(X; \mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z})$$

carries $\mathrm{Shv}_\ell^c(X)$ into $\mathrm{Shv}_\ell^c(Y)$.

(2) *When regarded as a functor form $\mathrm{Shv}_\ell^c(X)$ to $\mathrm{Shv}_\ell^c(Y)$, the functor f^* admits a left adjoint $f_!^\wedge$, which carries an object $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$ to the ℓ-completion of $f_! \mathcal{F}$.*

Proof. We first prove (1). If $\mathcal{F} \in \mathrm{Shv}(Y; \mathbf{Z})$ is a constructible ℓ-adic sheaf, then $\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}$ belongs to $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$, so that Proposition 4.2.15 shows that

$$\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} f^* \mathcal{F} \simeq f^*(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$$

for each $d \geq 0$. Since f is étale, the pullback functor f^* preserves limits, and therefore carries ℓ-complete objects to ℓ-complete objects.

We now prove (2). Let $f_!$ denote the left adjoint to the pullback functor $f^* : \mathrm{Shv}(Y; \mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z})$ (see Example 4.1.10), and let $f_!^\wedge$ denote the composition of $f_!$ with the ℓ-completion functor. It follows immediately from the definitions that for every object $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z})$ and every ℓ-complete object $\mathcal{G} \in \mathrm{Shv}(Y; \mathbf{Z})$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}(Y;\mathbf{Z})}(f_!^\wedge \mathcal{F}, \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, f^* \mathcal{G}).$$

It will therefore suffice to show that if $\mathcal{F}$ is an ℓ-adic constructible sheaf, then $f_!^\wedge \mathcal{F}$ is an ℓ-adic constructible sheaf. Since $f_!^\wedge \mathcal{F}$ is ℓ-complete by construction, we are reduced to proving that each tensor product

$$\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} f_!^\wedge \mathcal{F} \simeq f_!(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$$

is a compact object of $\mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$, which follows from Proposition 4.2.15. □

Proposition 4.3.25. *Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ vanishes.*
(2) *For every point $\eta : \mathrm{Spec}\, k \to X$, the stalk $\mathcal{F}_\eta = \eta^* \mathcal{F}$ vanishes.*
(3) *For every point $\eta : \mathrm{Spec}\, k \to X$, the costalk $\eta^! \mathcal{F} \in \mathrm{Shv}_\ell(\mathrm{Spec}\, k) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$ vanishes.*

Proof. Note that since $\mathcal{F}$ is ℓ-complete, it vanishes if and only if $\mathcal{F}_1 = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ vanishes. Similarly, the stalk (costalk) of $\mathcal{F}$ at a point $\eta \in X(k)$ vanishes if and only if the stalk (costalk) of $\mathcal{F}_1$ vanishes at η. The desired result now follows from the corresponding assertion for $\mathcal{F}_1 \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ (Proposition 4.2.20). □

For our purposes in this paper, the setting of constructible ℓ-adic sheaves will be too restrictive: we will meet many examples of sheaves which are not constructible. To accommodate these examples, we introduce the following enlargement of $\mathrm{Shv}_\ell^c(X)$:

Definition 4.3.26. Let X be a quasi-projective k-scheme. We let $\mathrm{Shv}_\ell(X)$ denote the ∞-category $\mathrm{Ind}(\mathrm{Shv}^c_\ell(X))$ of Ind-objects of $\mathrm{Shv}^c_\ell(X)$ (see §HTT.5.3.5). We will refer to $\mathrm{Shv}_\ell(X)$ as the *∞-category of ℓ-adic sheaves on X*.

Remark 4.3.27. Let X be a quasi-projective k-scheme. By abstract nonsense, the fully faithful embedding $\mathrm{Shv}^c_\ell(X) \hookrightarrow \mathrm{Shv}(X; \mathbf{Z}_\ell)$ extends to a colimit-preserving functor $\theta : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}(X; \mathbf{Z}_\ell)$. However, this functor need not be an equivalence of ∞-categories, since the objects of $\mathrm{Shv}^c_\ell(X)$ need not be compact in $\mathrm{Shv}(X; \mathbf{Z}_\ell)$.

Example 4.3.28. If $X = \mathrm{Spec}\, k$, then the essential image of the inclusion $\mathrm{Shv}^c_\ell(X) \hookrightarrow \mathrm{Shv}(X; \mathbf{Z}_\ell) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$ consists precisely of the compact objects of $\mathrm{Mod}_{\mathbf{Z}_\ell}$. It follows that the forgetful functor of Remark 4.3.27 induces an equivalence $\mathrm{Shv}_\ell(X) \to \mathrm{Shv}(X; \mathbf{Z}_\ell) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 4.3.29. Let X be a quasi-projective k-scheme. Then there is a fully faithful exact functor $\mathrm{Shv}^c_\ell(X) \to \mathrm{Shv}_\ell(X)$. We will generally abuse notation by identifying $\mathrm{Shv}^c_\ell(X)$ with its essential image under this embedding.

Notation 4.3.30. Let $f : X \to Y$ be a morphism between quasi-projective k-schemes. Then the adjoint functors

$$f_* : \mathrm{Shv}^c_\ell(X) \to \mathrm{Shv}^c_\ell(Y) \qquad f^*_\wedge : \mathrm{Shv}^c_\ell(Y) \to \mathrm{Shv}^c_\ell(X)$$

extend (in an essentially unique way) to a pair of adjoint functors relating the ∞-categories $\mathrm{Shv}_\ell(X)$ and $\mathrm{Shv}_\ell(Y)$, which we will denote by

$$f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y) \qquad f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X).$$

If f is proper, then the functor $f^! : \mathrm{Shv}^c_\ell(Y) \to \mathrm{Shv}^c_\ell(X)$ admits an essentially unique extension to a functor $\mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ which commutes with filtrered colimits. This extension is a right adjoint to the pushforward functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$, and will be denoted by $f^!$.

If f is étale, then the functor $f^\wedge_! : \mathrm{Shv}^c_\ell(X) \to \mathrm{Shv}^c_\ell(Y)$ admits an essentially unique extension to a functor $\mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ which commutes with filtered colimits. This extension is left adjoint to the pullback functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$, and will be denoted by $f_!$.

Remark 4.3.31. Let X be a quasi-projective k-scheme and let $f : X \to \mathrm{Spec}\, k$ be the projection map. For $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we will often denote the direct image $f_* \mathcal{F}$ by $C^*(X; \mathcal{F})$.

Warning 4.3.32. There is some potential for confusion, because the operations introduced in Notation 4.3.30 need not be compatible with the corresponding operations on étale sheaves studied in §4.1. That is, the diagrams of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(Y) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(X) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(Y; \mathbf{Z}_\ell) & \xrightarrow{f^*} & \mathrm{Shv}(X; \mathbf{Z}_\ell) \end{array} \qquad \begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f_!} & \mathrm{Shv}_\ell(Y) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(X; \mathbf{Z}_\ell) & \xrightarrow{f_!} & \mathrm{Shv}(Y; \mathbf{Z}_\ell) \end{array}$$

$$\begin{array}{ccc} \mathrm{Shv}_\ell(Y) & \xrightarrow{f^!} & \mathrm{Shv}_\ell(X) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(Y; \mathbf{Z}_\ell) & \xrightarrow{f^!} & \mathrm{Shv}(X; \mathbf{Z}_\ell) \end{array}$$

need not commute, where the vertical maps are given by the forgetful functors of Remark 4.3.27. In the first two cases, this is because the definition of f^* and $f_!$ on ℓ-adic sheaves involves the process of ℓ-completion; in the third, it is because the functor $f^! : \mathrm{Shv}(Y; \mathbf{Z}_\ell) \to \mathrm{Shv}(X; \mathbf{Z}_\ell)$ need not preserve colimits. However, the analogous diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f_*} & \mathrm{Shv}_\ell(Y) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(X; \mathbf{Z}_\ell) & \xrightarrow{f_*} & \mathrm{Shv}(Y; \mathbf{Z}_\ell) \end{array}$$

does commute (up to canonical homotopy).

Remark 4.3.33. Let X be a quasi-projective k-scheme and let $\eta : \mathrm{Spec}\, k \to X$ be a k-valued point of X. Then the pullback functor $\eta^* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(\mathrm{Spec}\, k)$ carries each ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ to an object of $\mathrm{Shv}_\ell(\mathrm{Spec}\, k) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$. We will denote this object by $\mathcal{F}_\eta$ and refer to it as the *stalk* of $\mathcal{F}$ at the point η.

Warning 4.3.34. Proposition 4.3.25 does not extend to non-constructible ℓ-adic sheaves. It is possible to have a nonzero object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ whose stalk $\mathcal{F}_\eta$ vanishes for every k-valued point $\eta \in X(k)$.

We say that an object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ is *ℓ-complete* if the inverse limit of the tower

$$\cdots \to \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F}$$

vanishes.

Remark 4.3.35. Let X be a quasi-projective k-scheme. An object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ is ℓ-complete if and only if, for every object $\mathcal{F}' \in \mathrm{Shv}_\ell(X)$, the inverse limit of mapping spaces

$$\cdots \to \mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}', \mathcal{F}) \xrightarrow{\ell} \mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}', \mathcal{F}) \xrightarrow{\ell} \mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}', \mathcal{F})$$

is contractible. Moreover, it suffices to verify this condition when $\mathcal{F}' \in \mathrm{Shv}_\ell^c(X)$ is constructible.

Remark 4.3.36. Let X be a quasi-projective k-scheme. Then every constructible ℓ-adic sheaf $\mathcal{F}$ on X is ℓ-complete (this follows from Remark 4.3.35, since $\mathcal{F}$ is ℓ-complete when viewed as an object of $\mathrm{Shv}(X; \mathbf{Z})$).

Let X be a quasi-projective k-scheme. It is generally not true that the vanishing of an ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ can be tested stalkwise: there can exist nonzero objects of $\mathrm{Shv}_\ell(X)$ whose stalks vanish at every point $x \in X$. However, this phenomenon does not arise for ℓ-complete objects:

Proposition 4.3.37. *Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ be ℓ-complete. The followign conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ vanishes.*
(2) *For every étale morphism $f : U \to X$, the object $C^*(U; f^* \mathcal{F}) \in \mathrm{Mod}_{\mathbf{Z}_\ell}$ vanishes.*
(3) *For every k-valued point $x \in X(k)$, the stalk $x^* \mathcal{F}$ vanishes.*

Proof. The implication (1) ⇒ (2) is trivial. Conversely, suppose that (2) is satisfied. Write $\mathcal{F}$ as the colimit of a filtered diagram $\{\mathcal{F}_\alpha\}$ in $\mathrm{Shv}_\ell^c(X)$. For each integer $d \geq 0$, let $\mathcal{F}_d$ denote the cofiber of the canonical map $\ell^d : \mathcal{F} \to \mathcal{F}$, so that $\mathcal{F}_d$ can be written as a colimit $\varinjlim_\alpha \mathcal{F}_{\alpha,d}$ where $\mathcal{F}_{\alpha,d} = \mathrm{cofib}(\ell^d : \mathcal{F}_\alpha \to \mathcal{F}_\alpha)$. Note that we can identify the diagram $\{\mathcal{F}_{\alpha,d}\}$ with an object of the Ind-category $\mathrm{Ind}(\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})) \simeq \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$. Using condition (2) we see that this Ind-object vanishes, so that $\mathcal{F}_d \simeq 0$. Since $\mathcal{F}$ is ℓ-complete, it follows that $\mathcal{F} \simeq \varprojlim \mathcal{F}_d \simeq 0$. This proves that (2) $\Rightarrow$ (1). The proof that (1) and (3) are equivalent is similar (using Proposition 4.1.11). $\square$

The existence of the adjunction $(f_!, f^*)$ when $f : X \to Y$ is an étale morphism has the following consequence:

Proposition 4.3.38. *Let $f : X \to Y$ be an étale morphism between quasi-projective k-schemes. Then the pullback functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ preserves limits.*

In fact, we have the following stronger assertion:

Proposition 4.3.39. *Let $f : X \to Y$ be a smooth morphism between quasi-projective k-schemes. Then the functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ preserves limits.*

Proof. Using Corollary 4.3.42 and Proposition 4.3.38, we see that the result is local with respect to the étale topology on X. We may therefore assume without loss of generality that f factors as a composition

$$X \xrightarrow{f'} \mathbf{P}^n \times Y \xrightarrow{f''} Y,$$

where the map f' is étale. Since f'^* preserves limits (Proposition 4.3.38), we may replace f by f'' and thereby reduce to the case where f is smooth and proper. In this case, the functor f^* is equivalent to a shift of the functor $f^!$ (Example 4.3.23) and therefore admits a left adjoint (given by a shift of f_*). $\square$

Remark 4.3.40. Let X be a quasi-projective k-scheme. Then the symmetric monoidal structure on $\mathrm{Shv}^c_\ell(X)$ described in Proposition 4.3.15 determines a symmetric monoidal structure on $\mathrm{Shv}_\ell(X) = \mathrm{Ind}(\mathrm{Shv}^c_\ell(X))$, which is determined uniquely by the requirement that the inclusion $\mathrm{Shv}^c_\ell(X) \hookrightarrow \mathrm{Shv}_\ell(X)$ be a symmetric monoidal functor and that the associated tensor product functor

$$\otimes : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$$

preserves colimits separately in each variable. We will denote the unit object of $\mathrm{Shv}_\ell(X)$ by $\underline{\mathbf{Z}_\ell}_X$. Beware that this notation conflicts with that of Remark 4.3.27: the forgetful functor $\mathrm{Shv}_\ell(X) \to \mathrm{Shv}(X; \mathbf{Z}_\ell)$ of Remark 4.3.27 is not symmetric monoidal in general; in particular, it does not carry $\underline{\mathbf{Z}_\ell}_X$ to the constant sheaf given in Definition 4.2.4.

Let X be a quasi-projective k-scheme. For every commutative ring Λ, the theory of Mod_Λ-valued sheaves on X satisfies effective descent for the étale topology: that is, the construction

$$(U \in \mathrm{Sch}^{\mathrm{et}}_X) \mapsto \mathrm{Shv}(U; \Lambda)$$

is a sheaf of ∞-categories with respect to the étale topology. We now prove the analogous statement for ℓ-adic sheaves.

Proposition 4.3.41 (Effective Cohomological Descent)**.** *Let $f : U \to X$ be a surjective étale morphism between quasi-projective k-schemes, and let $U_\bullet$ denote the simplicial scheme given by the nerve of the map f (so that U_m is the $(m+1)$st fiber power of U over X). Then the canonical map*

$$\psi : \mathrm{Shv}_\ell(X) \to \varprojlim \mathrm{Shv}_\ell(U_\bullet)$$

is an equivalence of ∞-categories.

Corollary 4.3.42. *Let $f : X \to Y$ be a smooth surjection between quasi-projective k-schemes. Then the functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ is conservative.*

Proof. Since f is a smooth surjection, there exists a map $g : X' \to X$ such that the composite map $f \circ g$ is an étale surjection. Replacing X by X', we may suppose that f is étale. In this case, the desired result follows immediately from Proposition 4.3.41. $\square$

The proof of Proposition 4.3.41 depends on the following result:

Lemma 4.3.43. *Let X be a quasi-projective k-scheme. Suppose that $\mathcal{F}_\bullet$ is an augmented simplicial object of $\mathrm{Shv}^c_\ell(X)$ satisfying the following conditions:*

(a) *There exists an integer n such that $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}_\bullet$ is an augmented simplicial object of $\mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq n}$.*
(b) *The image of $\mathcal{F}_\bullet$ in $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ is a colimit diagram (that is, it exhibits $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{-1}$ as a geometric realization of the simplicial object $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_\bullet$).*

Then $\mathcal{F}_\bullet$ is a colimit diagram in both $\mathrm{Shv}(X; \mathbf{Z})$ and $\mathrm{Shv}_\ell(X)$.

Proof of Proposition 4.3.41. It follows from Corollary HA.4.7.6.3 (and the Beck-Chevalley property given in Variant 4.5.5) that the functor ψ admits a fully faithful left adjoint

$$\phi : \varprojlim \mathrm{Shv}_\ell(U_\bullet) \to \mathrm{Shv}_\ell(X).$$

To complete the proof, it will suffice to show that for each object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the counit map $v : (\phi \circ \psi)(\mathcal{F}) \to \mathcal{F}$ is an equivalence in $\mathrm{Shv}_\ell(X)$. For each $n \geq 0$, let $f_n : U_n \to X$ denote the projection map. Unwinding the definitions, we can identify v with the natural map $|f_{\bullet !} f^*_\bullet \mathcal{F}| \to \mathcal{F}$. Writing $\mathcal{F}$ as a colimit of constructible ℓ-adic sheaves, we may assume without loss of generality that $\mathcal{F}$ is constructible. By virtue of Lemma 4.3.43, it will suffice to prove this after tensoring with $\mathbf{Z}/\ell\mathbf{Z}$, in which case the desired result follows from the fact that the construction $U \mapsto \mathrm{Shv}(U; \mathbf{Z}/\ell\mathbf{Z})$ satisfies étale descent. □

Proof of Lemma 4.3.43. We first prove that $\mathcal{F}_\bullet$ is a colimit diagram in $\mathrm{Shv}(X; \mathbf{Z})$: that is, that the canonical map

$$\alpha : |\mathcal{F}_\bullet| \to \mathcal{F}_{-1}$$

is an equivalence in $\mathrm{Shv}(X; \mathbf{Z})$. Condition (b) implies that α is an equivalence after tensoring with $\mathbf{Z}/\ell\mathbf{Z}$. Since the codomain of α is ℓ-complete, it will suffice to show that the domain of α is also ℓ-complete. For each integer m, let $\mathcal{F}(m)$ denote the colimit of the m-skeleton of $\mathcal{F}_\bullet$ (formed in the ∞-category $\mathrm{Shv}(X; \mathbf{Z})$. Then each $\mathcal{F}(m)$ belongs to $\mathrm{Shv}^c_\ell(X)$ and is therefore ℓ-complete, and we have an equivalence $|\mathcal{F}_\bullet| \simeq \varinjlim \mathcal{F}(m)$. Fix an étale map $V \to X$; we wish to prove that

$$|\mathcal{F}_\bullet|(V) \simeq \varinjlim \mathcal{F}(m)(V) \in \mathrm{Mod}_{\mathbf{Z}_\ell}$$

is ℓ-complete. According to Remark 4.3.3, this is equivalent to the assertion that for every integer i, the abelian group $\varinjlim \mathrm{H}_i(\mathcal{F}(m)(V))$ is ℓ-complete (in the derived sense). To prove this, it will suffice to show that the direct system of abelian groups $\{\mathrm{H}_i(\mathcal{F}(m)(V))\}$ is eventually constant. Let $\mathcal{K}(m)$ denote the fiber of the map $\mathcal{F}(m) \to \mathcal{F}(m+1)$, so that we have an exact sequence

$$\mathrm{H}_i(\mathcal{K}(m)(V)) \to \mathrm{H}_i(\mathcal{F}(m)) \to \mathrm{H}_i(\mathcal{F}(m+1)(V)) \to \mathrm{H}_{i-1}(\mathcal{K}(m)(V)).$$

It will therefore suffice to show that the groups $\mathrm{H}_i(\mathcal{K}(m)(V))$ vanish for $m \gg i$. Since $\mathcal{K}(m)$ is ℓ-complete, we have a Milnor exact sequence

$$\lim{}^1\{\mathrm{H}_{i+1}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m)(V)) \to \mathrm{H}_i(\mathcal{K}(m)(V)) \to \varprojlim\{\mathrm{H}_i(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m)(V))\}.$$

Corollary 4.2.19 implies that the left hand side vanishes. We are therefore reduced to proving that $\mathrm{H}_i(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m)(V)) \simeq 0$ for $m \gg i$. Using induction on d, we can reduce to the case $d = 1$. Using Lemma 4.1.13, we are reduced to the problem of showing that $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m) \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq i}$ for $m \gg i$. This follows easily from assumption (a). This completes the proof that α is an equivalence in $\mathrm{Shv}(X; \mathbf{Z})$.

Note that each $\mathcal{F}(m)$ is a constructible ℓ-adic sheaf, and is a colimit of the m-skeleton of $\mathcal{F}_\bullet$ in both $\mathrm{Shv}(X;\mathbf{Z})$ and $\mathrm{Shv}_\ell(X)$. The sheaf $\mathcal{F}_{-1}$ can be identified with the colimit of the sequence

$$\mathcal{F}(0) \to \mathcal{F}(1) \to \cdots$$

in the ∞-category $\mathrm{Shv}(X;\mathbf{Z})$; we wish to show that $\mathcal{F}$ is also a colimit of this sequence in $\mathrm{Shv}_\ell(X)$. Equivalently, we wish to show that for every object $\mathcal{G} \in \mathrm{Shv}^c_\ell(X)$, the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}(m)) \to \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F})$$

is a homotopy equivalence. For each $m \geq 0$, let $\mathcal{F}'(m)$ denote the cofiber of the canonical map $\mathcal{F}(m) \to \mathcal{F}$, so that we have a fiber sequence

$$\varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}(m)) \to \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}) \to \varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}'(m)).$$

It will therefore suffice to show that the space

$$\varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}'(m))$$

is contractible. We will prove the following more precise statement: for every integer q, the mapping space $\mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}'(m))$ is q-connective for m sufficiently large (depending on q). Since $\mathcal{F}'(m)$ is ℓ-complete, we can identify $\mathrm{Map}_{\mathrm{Shv}^c(X;\mathbf{Z}_\ell)}(\mathcal{G}, \mathcal{F}'(m))$ with the limit of a tower of spaces $\mathrm{Map}_{\mathrm{Shv}^c(X;\mathbf{Z}_\ell)}(\mathcal{G}, \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m))$. It will therefore suffice to show that each of these spaces is $(q+1)$-connective. Using the existence of a fiber sequence

$$\begin{array}{c} \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathbf{Z}/\ell^{d+1}\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)), \end{array}$$

we can reduce to the case $d = 1$. That is, we are reduced to proving that the mapping spaces $\mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})}(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}, \mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m))$ are $(q+1)$-connective for $q \gg m$. This follows from Proposition 4.2.13, since condition (a) guarantees that the sheaves $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)$ are highly connected for $m \gg 0$. □

4.4. **The t-Structure on ℓ-Adic Sheaves.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. In §4.3, we associated to every quasi-projective k-scheme X an ∞-category $\mathrm{Shv}_\ell(X)$ of ℓ-adic sheaves on X. In this section, we will describe the relationship of $\mathrm{Shv}_\ell(X)$ with the abelian category of ℓ-adic sheaves introduced in [23]. Our starting point is the following:

Proposition 4.4.1. *Let X be a quasi-projective k-scheme. Then there exists a t-structure $(\mathrm{Shv}^c_\ell(X)_{\geq 0}, \mathrm{Shv}^c_\ell(X)_{\geq 0})$ on the ∞-category $\mathrm{Shv}^c_\ell(X)$ of constructible ℓ-adic sheaves on X, which is uniquely characterized by the following property:*

- *A constructible ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$ belongs to $\mathrm{Shv}^c_\ell(X)_{\geq 0}$ if and only if $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}$ belongs to $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})_{\geq 0}$.*

Warning 4.4.2. In the situation of Proposition 4.4.1, we can regard $\mathrm{Shv}^c_\ell(X)$ as a full subcategory of $\mathrm{Shv}(X;\mathbf{Z})$, which is equipped with a t-structure by virtue of Remark 4.1.6. However, the inclusion $\mathrm{Shv}^c_\ell(X) \hookrightarrow \mathrm{Shv}(X; \mathbf{Z})$ is *not* t-exact. However, it is left t-exact: see Remark 4.4.5 below.

Example 4.4.3. Let $X = \operatorname{Spec} k$, so that $\mathrm{Shv}^c_\ell(X)$ can be identified with the ∞-category $\mathrm{Mod}^{\mathrm{pf}}_{\mathbf{Z}_\ell}$ of perfect $\mathbf{Z}_\ell$-modules. Under this identification, the t-structure of Proposition 4.4.1 agrees with the usual t-structure of $\mathrm{Mod}^{\mathrm{pf}}_{\mathbf{Z}_\ell}$.

Lemma 4.4.4. *Let $\mathcal{A}$ be an abelian category. For each object $M \in \mathcal{A}$, let $M/\ell^d M$ and $M[\ell^d]$ denote the cokernel and kernel of the map $\ell^d : M \to M$. Suppose we are given a tower of objects*

$$\cdots \to M_3 \to M_2 \to M_1 \to M_0$$

satisfying the following conditions:

(*a*) *Each of the maps $M_{d+1} \to M_d$ induces an equivalence $M_{d+1}/\ell^d \simeq M_d$.*

(*b*) *The object M_1 is Noetherian.*

Then, for each integer $m \geq 0$, the tower $\{M_d[\ell^m]\}_{d \geq 0}$ is equivalent to a constant Pro-*object of $\mathcal{A}$.*

Proof. For each $d \geq 0$, let N_d denote the image of the natural map $M_{d+m}[\ell^m] \to M_d[\ell^m]$. If $d \geq m$, multiplication by ℓ^{d-m} induces a map $\theta_d : M_m \to M_d[\ell^m]$. Let N'_d denote the fiber product $M_m \times_{M_d} N_d$, which we regard as a subobject of M_m. Assumption (*a*) implies that M_m admits a finite filtration by quotients of M_1, so that M_m is Noetherian by virtue of (*b*). Note that $N'_d = \ker(\theta_{d+m}) \subseteq N'_{d+m}$, so that the subobjects $N'_d \subseteq M_m$ form an ascending chain

$$N'_m \subseteq N'_{2m} \subseteq N'_{3m} \subseteq \cdots .$$

Since M_m is Noetherian, this chain must eventually stabilize. We may therefore choose an integer a_0 such that $N'_{am} = N'_{(a-1)m} = \ker(\theta_{am})$ for $a \geq a_0$. Using the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N'_{(a-1)m} & \longrightarrow & M_m & \xrightarrow{\theta_{am}} & \mathrm{Im}(\theta_{am}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & N'_{am} & \longrightarrow & M_m & \xrightarrow{\theta_{(a+1)m}} & \mathrm{Im}(\theta_{(a+1)m}) & \longrightarrow & 0,
\end{array}$$

we see that multiplication by ℓ^m induces an isomorphism from $\mathrm{Im}(\theta_{am})$ to $\mathrm{Im}(\theta_{(a+1)m})$ for $a \geq a_0$. This isomorphism factors as a composition

$$\mathrm{Im}(\theta_{am}) \hookrightarrow M_{am}[\ell^m] \xrightarrow{\ell^m} \mathrm{Im}(\theta_{(a+1)m}),$$

so that for $a \geq a_0$ the object $M_{am}[\ell^m]$ splits as a direct sum $\mathrm{Im}(\theta_{am}) \oplus N_{am}$. Note that the restriction map $M_{(a+1)m}[\ell^m] \to M_{am}[\ell^m]$ has image N_{am} and kernel $\mathrm{Im}(\theta_{(a+1)m})$, and therefore restricts to an isomorphism $N_{(a+1)m} \to N_{am}$ for $a \geq a_0$. It follows that the tower $\{M_{am}[\ell^m]\}_{a \geq a_0}$ is isomorphic to the direct sum of a constant tower $\{N_{am}\}_{a \geq a_0}$ and a tower $\{\mathrm{Im}(\theta_{am})\}_{a \geq a_0}$ with vanishing transition maps, and is therefore equivalent to a constant Pro-object of $\mathcal{A}$. □

Proof of Proposition 4.4.1. For each integer n, let $\mathrm{Shv}^c_\ell(X)_{\leq n}$ denote the full subcategory of $\mathrm{Shv}^c_\ell(X)$ spanned by those objects $\mathcal{F}$ such that, for each object $\mathcal{G} \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$, the mapping space $\mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \Sigma^{-m} \mathcal{F})$ is contractible for $m > n$. To prove Proposition 4.4.1, it will suffice to show that for each object $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$, there exists a fiber sequence

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

where $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$ and $\mathcal{F}'' \in \mathrm{Shv}^c_\ell(X)_{\leq -1}$.

For each integer $d \geq 0$, let $\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ denote the image of $\mathcal{F}$ in $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$, so that $\mathcal{F} \simeq \varprojlim\{\mathcal{F}_d\}_{d\geq 0}$. Set $\mathcal{F}' = \varprojlim\{\tau_{\geq 0}\,\mathcal{F}_d\}_{d\geq 0}$ and $\mathcal{F}'' = \varprojlim\{\tau_{\leq -1}\,\mathcal{F}_d\}_{d\geq 0}$, where the limits are formed in $\mathrm{Shv}(X;\mathbf{Z})$. We will prove that $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$. Assuming this, it follows that $\mathcal{F}'' \in \mathrm{Shv}^c_\ell(X)$. Note that for $\mathcal{G} \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$, the mapping space

$$\begin{aligned} \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}) &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \tau_{\leq -1}\,\mathcal{F}_d) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}_\ell(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}, \tau_{\leq -1}\,\mathcal{F}_d) \end{aligned}$$

is contractible, since each tensor product $\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}$ belongs to $\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})_{\geq 0}$. It follows that $\mathcal{F}''$ belongs to $\mathrm{Shv}^c_\ell(X)_{\leq -1}$, as desired.

It remains to prove that $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$. For this, we must establish three things:

(a) The object $\mathcal{F}' \in \mathrm{Shv}(X;\mathbf{Z})$ is ℓ-complete.
(b) The tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}'$ is a compact object of $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})$.
(c) The tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}'$ belongs to $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})_{\geq 0}$.

Assertion (a) is obvious (since the collection of ℓ-complete objects of $\mathrm{Shv}(X;\mathbf{Z})$ is closed under limits). We will deduce (b) and (c) from the following:

($*$) The tower
$$\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \tau_{\geq 0}\,\mathcal{F}_d\}_{d\geq 0}$$
is constant when regarded as a Pro-object of $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})$.

Note that if a tower $\{C_d\}_{d\geq 0}$ in some ∞-category $\mathcal{C}$ is Pro-equivalent to an object $C \in \mathcal{C}$, then C can be identified with a retract of C_d for $d \gg 0$. In particular, using assertion ($*$) (and the fact that the construction $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \bullet$ preserves limits), we can identify $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}'$ with a retract of some $\mathcal{G} = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \tau_{\geq 0}\,\mathcal{F}_d$ for some $d \geq 0$. From this, assertion (c) is obvious and assertion (b) follows from Proposition 4.2.5.

Note that the tower $\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/\ell^d\mathbf{Z})\}$ determines a constant Pro-object of $\mathrm{Mod}_{\mathbf{Z}}$, so that the Pro-objects $\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_d\}_{d\geq 0}$ and $\{\tau_{\geq 0}(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}_d)\}_{d\geq 0}$ are likewise constant. For each $d \geq 0$, form a fiber sequence

$$(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \tau_{\geq 0}\,\mathcal{F}_d \to \tau_{\geq 0}((\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_d) \to \mathcal{G}_d\,.$$

To prove ($*$), it will suffice to show that the tower $\{\mathcal{G}_d\}_{d\geq 0}$ is constant. Unwinding the definitions, we see that each $\mathcal{G}_d$ belongs to the heart $\mathrm{Shv}(X;\mathbf{Z})^\heartsuit$, where it can be identified with the kernel of the map $\pi_{-1}\,\mathcal{F}_d \to \pi_{-1}\,\mathcal{F}_d$ given by multiplication by ℓ. For each integer m, let us regard $\pi_m\,\mathcal{F}$ as an object of $\mathrm{Shv}(X;\mathbf{Z})^\heartsuit$, and let $(\pi_m\,\mathcal{F})/\ell^d$ and $(\pi_m\,\mathcal{F})[\ell^d]$ denote the cokernel and kernel of the multiplication map $\ell^d : \pi_m\,\mathcal{F} \to \pi_m\,\mathcal{F}$, so that we have exact sequences

$$0 \to (\pi_{-1}\,\mathcal{F})/\ell^d \to \pi_{-1}\,\mathcal{F}_d \to (\pi_{-2}\,\mathcal{F})[\ell^d] \to 0$$

which determine an exact sequence of Pro-objects

$$0 \to \{(\pi_{-1}\,\mathcal{F})/\ell^d)[\ell]\}_{d\geq 0} \to \{\mathcal{G}_d\}_{d\geq 0} \to \{(\pi_{-2}\,\mathcal{F})[\ell]\}_{d\geq 0}.$$

The last of these Pro-objects is trivial (it has vanishing transition maps), so we are reduced to proving that the Pro-object $\{(\pi_{-1}\,\mathcal{F})/\ell^d)[\ell]\}_{d\geq 0}$ is constant. Note that $(\pi_{-1}\,\mathcal{F})/\ell$ is a subobject of $\pi_{-1}\,\mathcal{F}_1$, and is therefore a Noetherian object of the abelian category $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})^\heartsuit$ by virtue of Proposition 4.2.12. The desired result now follows from Lemma 4.4.4. □

Remark 4.4.5. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$. The proof of Proposition 4.4.1 shows that $\mathcal{F}$ belongs to $\mathrm{Shv}^c_\ell(X)_{\leq 0}$ if and only if the canonical map $\mathcal{F} \to \varprojlim \tau_{\leq 0}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$ is an equivalence in $\mathrm{Shv}(X;\mathbf{Z})$. In particular, every object of $\mathrm{Shv}^c_\ell(X)_{\leq 0}$ belongs to $\mathrm{Shv}(X;\mathbf{Z})_{\leq 0}$. In other words, the inclusion $\mathrm{Shv}^c_\ell(X) \hookrightarrow \mathrm{Shv}(X;\mathbf{Z})$ is left t-exact.

Remark 4.4.6. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$. For each integer $d \geq 0$, let $\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$. If $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)_{\leq n}$, then Remark 4.4.5 implies that $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z})_{\leq n}$ so that each $\mathcal{F}_d$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\leq n+1}$. Conversely, if $\mathcal{F}_1$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq n+1}$, then it follows by induction on d that each $\mathcal{F}_d \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\leq n+1}$, so that the proof of Proposition 4.4.1 shows that $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)_{\leq n+1}$.

Proposition 4.4.7. *Let X be a quasi-projective k-scheme. Then the t-structure on $\mathrm{Shv}_\ell^c(X)$ is right and left bounded: that is, we have*

$$\mathrm{Shv}_\ell^c(X) = \bigcup_n \mathrm{Shv}_\ell^c(X)_{\leq n} = \bigcup_n \mathrm{Shv}_\ell^c(X)_{\geq -n}.$$

Proof. Let $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$. For each integer $d \geq 0$, let $\mathcal{F}_d = \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$. The characterization of constructibility given by Proposition 4.2.5, we see that there exists an integer $n \geq 0$ such that $\mathcal{F}_1 \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq -n} \cap \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq n}$. It follows by induction on d that each $\mathcal{F}_d$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\geq -n} \cap \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\geq n}$, so that $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)_{\geq -n} \cap \mathrm{Shv}_\ell^c(X)_{\leq n}$. □

We now discuss the functorial behavior of the t-structure introduced in Proposition 4.4.1.

Proposition 4.4.8. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then the pullback functor $f^* : \mathrm{Shv}_\ell^c(Y) \to \mathrm{Shv}_\ell^c(X)$ is t-exact.*

Proof. If $\mathcal{F} \in \mathrm{Shv}_\ell^c(Y)_{\geq 0}$, then

$$\begin{aligned} (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} f^* \mathcal{F} &\simeq f^*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \\ &\in f^* \mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0} \\ &\subseteq \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0}. \end{aligned}$$

This proves that the functor f^* is right t-exact.

To prove left exactness, we must work a little bit harder. Assume that $\mathcal{F} \in \mathrm{Shv}_\ell^c(Y)_{\leq 0}$, and for $d \geq 0$ set $\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$. We have

$$\begin{aligned} \tau_{\geq 1} f^* \mathcal{F} &\simeq \varprojlim \tau_{\geq 1}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} f^* \mathcal{F}) \\ &\simeq \varprojlim \tau_{\geq 1} f^*(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \\ &= \varprojlim \tau_{\geq 1} f^* \mathcal{F}_d \\ &\simeq \varprojlim f^* \tau_{\geq 1} \mathcal{F}_d\,. \end{aligned}$$

It will therefore suffice to show that $\varprojlim f^* \tau_{\geq 1} \mathcal{F}_d$ vanishes in $\mathrm{Shv}(X; \mathbf{Z})$. Since the limit is ℓ-complete, we are reduced to proving that the limit

$$\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \varprojlim f^* \tau_{\geq 1} \mathcal{F}_d \simeq \varprojlim f^*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d)$$

vanishes. Using the characterization of $\mathrm{Shv}_\ell^c(Y)_{\leq 0}$ obtained in the proof of Proposition 4.4.1, we see that the limit $\varprojlim(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d)$ vanishes in $\mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$. It will therefore suffice to show that the natural map

$$f^* \varprojlim(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d) \to \varprojlim f^*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d)$$

is an equivalence in $\mathrm{Shv}(X; \mathbf{Z})$. This follows from assertion $(*)$ from the proof of Proposition 4.4.1. □

Corollary 4.4.9. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then the pushforward functor $f_* : \mathrm{Shv}_\ell^c(X) \to \mathrm{Shv}_\ell^c(Y)$ is left t-exact. If f is a finite morphism, then f_* is t-exact.*

Proof. The left t-exactness of f_* follows immediately from the right t-exactness of the adjoint functor f^* (Proposition 4.4.8). If f is a finite morphism, then for $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$ we have

$$\begin{aligned} \mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} f_* \mathcal{F} &\simeq f_*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \\ &\in f_* \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0} \\ &\subseteq \mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0}, \end{aligned}$$

so that f_* is left t-exact. □

Corollary 4.4.10. *Let $f : X \to Y$ be a finite morphism of quasi-projective k-schemes. Then the functor $f^! : \mathrm{Shv}_\ell^c(Y) \to \mathrm{Shv}_\ell^c(X)$ is left t-exact.*

Corollary 4.4.11. *Let X be a quasi-projective k-scheme. Then:*

- (a) *An object $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$ belongs to $\mathrm{Shv}_\ell^c(X)_{\geq 0}$ if and only if, for each point $\eta : \mathrm{Spec}\, k \to X$, the stalk $\eta^* \mathcal{F} \in \mathrm{Shv}_\ell^c(\mathrm{Spec}\, k) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}^{\mathrm{pf}}$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 0}$.*
- (b) *An object $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$ belongs to $\mathrm{Shv}_\ell^c(X)_{\leq 0}$ if and only if, for each point $\eta : \mathrm{Spec}\, k \to X$, the stalk $\eta^* \mathcal{F} \in \mathrm{Shv}_\ell^c(\mathrm{Spec}\, k) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}^{\mathrm{pf}}$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$.*

Proof. We will prove (b); the proof of (a) is similar. The "only if" direction follows immediately from Proposition 4.4.8 and Example 4.4.3. Conversely, suppose that $\eta^* \mathcal{F}$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$ for each point $\eta : \mathrm{Spec}\, k \to X$. Since the functor η^* is t-exact (Proposition 4.4.8), it follows that the canonical map $\alpha : \mathcal{F} \to \tau_{\leq 0} \mathcal{F}$ induces an equivalence after passing to the stalk at each point, so that α is an equivalence by virtue of Proposition 4.3.25. □

Our next goal is to describe the heart of the t-structure of Proposition 4.4.1. First, we need to introduce a bit of terminology:

Definition 4.4.12. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^\heartsuit$. We will say that the sheaf $\mathcal{F}$ is *imperfect constructible* if it satisfies the following conditions:

(1) There exists a finite sequence of quasi-compact open subsets

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$$

such that, for $1 \leq i \leq n$, if Y_i denotes the locally closed reduced subscheme of X with support $U_i - U_{i-1}$, then each restriction $\mathcal{F}|_{Y_i}$ is locally constant.

(2) For every k-valued point $\eta : \mathrm{Spec}\, k \to X$, the pullback $\eta^* \mathcal{F} \in \mathrm{Shv}(\mathrm{Spec}\, k; \mathbf{Z}/\ell^d\mathbf{Z}) \simeq \mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}^\heartsuit$ is finite (when regarded as an abelian group).

We let $\mathrm{Shv}^\circ(X; \mathbf{Z}/\ell^d\mathbf{Z})$ denote the full subcategory of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^\heartsuit$ spanned by the imperfect constructible objects.

Example 4.4.13. Let $\mathcal{F} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$. Then each of the cohomology sheaves $\pi_i \mathcal{F}$ is imperfect constructible.

Remark 4.4.14. If X is a quasi-projective k-scheme, then the full subcategory

$$\mathrm{Shv}^\circ(X; \mathbf{Z}/\ell^d\mathbf{Z}) \subseteq \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^\heartsuit$$

is closed under the formation of kernels, cokernels, and extensions. Consequently, it forms an abelian category.

Remark 4.4.15. For every pair of integers $d' \geq d \geq 0$, the construction

$$\mathcal{F} \mapsto \tau_{\leq 0}((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^{d'}\mathbf{Z}} \mathcal{F})$$

carries $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^{d'}\mathbf{Z})$ into $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^{d}\mathbf{Z})$. We therefore have a tower of (abelian) categories and right-exact functors

$$\cdots \to \mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^3\mathbf{Z}) \to \mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^2\mathbf{Z}) \to \mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell\mathbf{Z}).$$

We will denote the homotopy inverse limit of this tower by $\mathrm{Shv}^{\circ}(X)$.

Proposition 4.4.16. *Let X be a quasi-projective k-scheme, and let*

$$\phi : \mathrm{Shv}_{\ell}^{c}(X)_{\geq 0} \to \mathrm{Shv}^{\circ}(X)$$

be the functor given on objects by the formula

$$\phi(\mathcal{F}) = \{\tau_{\leq 0}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})\}_{d \geq 0}.$$

Then θ induces an equivalence of categories $\mathrm{Shv}_{\ell}^{c}(X)^{\heartsuit} \simeq \mathrm{Shv}^{\circ}(X)$. In particular, $\mathrm{Shv}^{\circ}(X)$ is an abelian category.

Proof. Let $\psi : \mathrm{Shv}^{\circ}(X) \to \mathrm{Shv}(X; \mathbf{Z})$ be the functor given by $\psi\{\mathcal{F}_d\}_{d \geq 0} = \varprojlim \mathcal{F}_d$ (where the limit is formed in the ∞-category $\mathrm{Shv}(X; \mathbf{Z})$). The proof of Proposition 4.4.1 shows that the composite functor $\psi \circ \phi : \mathrm{Shv}_{\ell}^{c}(X)_{\geq 0} \to \mathrm{Shv}(X; \mathbf{Z})$ is given by $\mathcal{F} \mapsto \tau_{\leq 0}\mathcal{F}$ (where the truncation is formed with respect to the t-structure of Proposition 4.4.1). Consequently, ψ is a left homotopy inverse of the restriction $\phi|_{\mathrm{Shv}(X;\mathbf{Z})}$. To complete the proof, it will suffice to show that ψ factors through the full subcategory $\mathrm{Shv}_{\ell}^{c}(X)^{\heartsuit} \subseteq \mathrm{Shv}(X; \mathbf{Z})$ and that it is a right homotopy inverse to $\phi|_{\mathrm{Shv}_{\ell}^{c}(X)^{\heartsuit}}$. To prove this, we must prove that for every object $\{\mathcal{F}_d\}_{d \geq 0}$ of $\mathrm{Shv}^{\circ}(X)$ has the following properties:

- (a) The inverse limit $\mathcal{F} = \varprojlim \mathcal{F}_d$ (formed in the ∞-category $\mathrm{Shv}(X; \mathbf{Z})$) is ℓ-complete.
- (b) The tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ is a constructible object of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$.
- (c) The limit $\mathcal{F} = \varprojlim \mathcal{F}_d$ belongs to $\mathrm{Shv}_{\ell}^{c}(X)_{\geq 0}$: in other words, the tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0}$.
- (d) The limit $\mathcal{F} = \varprojlim \mathcal{F}_d$ belongs to $\mathrm{Shv}_{\ell}^{c}(X)_{\leq 0}$.
- (e) For each integer $d \geq 0$, the canonical map $\tau_{\leq 0}(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}) \to \mathcal{F}_d$ is an equivalence in $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^{\heartsuit}$.

Assertion (a) is clear. Note that the tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ can be identified with the limit of the diagram $\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_d\}_{d \geq 0}$. For each $d \geq 1$, we have

$$\pi_i(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}_d) \simeq \begin{cases} \mathcal{F}_1 & \text{if } i = 0 \\ \ker(\ell : \mathcal{F}_d \to \mathcal{F}_d) & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 4.4.4 and Proposition 4.2.12, we see that the tower $\{\ker(\ell : \mathcal{F}_d \to \mathcal{F}_d)\}_{d \geq 1}$ is constant as a Pro-object of $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell\mathbf{Z})$. It follows that $\pi_i(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$ is $\mathcal{F}_1$ when $i = 0$, a retract of some $\ker(\ell : \mathcal{F}_d \to \mathcal{F}_d)$ if $i = 1$, and vanishes otherwise. This proves (b) and (c). To prove (d), we note that for $\mathcal{G} \in \mathrm{Shv}_{\ell}^{c}(X)_{\geq 0}$, the mapping space

$$\begin{aligned} \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{G}, \Sigma^{-1}\mathcal{F}) &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{G}, \Sigma^{-1}\mathcal{F}_d) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}, \Sigma^{-1}\mathcal{F}_d) \end{aligned}$$

is contractible because each $(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{G}$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\geq 0}$. To prove ($e$), we first observe that for $d' \geq d$, we have

$$\pi_i((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}) \simeq \begin{cases} \mathcal{F}_d & \text{if } i = 0 \\ \ker(\ell^d : \mathcal{F}_{d'} \to \mathcal{F}_{d'}) & \text{if } i = 1 \\ 0. & \text{otherwise} \end{cases}$$

Using Lemma 4.4.4 and Proposition 4.2.12, we see that the tower $\{\ker(\ell^d : \mathcal{F}_{d'} \to \mathcal{F}_{d'})\}_{d' \geq d}$ is equivalent to a constant Pro-object of $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^d\mathbf{Z})$, so that the tower $\{((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}\}_{d' \geq d}$ is constant and we obtain an equivalence

$$\tau_{\leq 0} \varprojlim_{d'} ((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}) \to \varprojlim_{d'} \tau_{\leq 0}((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}) \simeq \mathcal{F}_d$$

is an equivalence. □

Notation 4.4.17. Let X be a quasi-projective k-scheme. We let $\mathrm{Shv}_\ell(X)_{\geq 0}$ and $\mathrm{Shv}_\ell(X)_{\leq 0}$ denote the essential images of the fully faithful embeddings

$$\mathrm{Ind}(\mathrm{Shv}^c_\ell(X)_{\geq 0}) \hookrightarrow \mathrm{Ind}(\mathrm{Shv}^c_\ell(X)) = \mathrm{Shv}_\ell(X)$$
$$\mathrm{Ind}(\mathrm{Shv}^c_\ell(X)_{\leq 0}) \hookrightarrow \mathrm{Ind}(\mathrm{Shv}^c_\ell(X)) = \mathrm{Shv}_\ell(X).$$

It follows from Proposition 4.4.1 that the full subcategories $(\mathrm{Shv}_\ell(X)_{\geq 0}, \mathrm{Shv}_\ell(X)_{\leq 0})$ determine a t-structure on the ∞-category $\mathrm{Shv}_\ell(X)$.

Remark 4.4.18. Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then the pullback functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ is t-exact, and the pushforward functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ is left t-exact. If f is finite, then $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ is t-exact and $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ is left t-exact. These assertions follow immediately from Proposition 4.4.8, Corollary 4.4.9, and Corollary 4.4.10. Beware that the analogue of Corollary 4.4.11 for non-constructible ℓ-adic sheaves is generally false: for example, one can find nonzero objects of $\mathrm{Shv}_\ell(X)$ with vanishing stalks (or costalks) at every point.

Proposition 4.4.19. *Let X be a quasi-projective k-scheme. Then the t-structure on $\mathrm{Shv}^c_\ell(X)$ is right and left bounded: that is, we have*

$$\mathrm{Shv}^c_\ell(X) = \bigcup_n \mathrm{Shv}^c_\ell(X)_{\leq n} = \bigcup_n \mathrm{Shv}^c_\ell(X)_{\geq -n}.$$

Let X be a quasi-projective k-scheme. Then the t-structure on $\mathrm{Shv}_\ell(X)$ is right complete (that is, the canonical map $\mathrm{Shv}_\ell(X) \to \varprojlim_n \mathrm{Shv}_\ell(X)_{\geq -n}$ is an equivalence of ∞-categories). Moreover, the canonical map

$$\mathrm{Shv}_\ell(X) \to \varprojlim_n \mathrm{Shv}_\ell(X)_{\leq n}$$

is fully faithful.

Remark 4.4.20. We do not know if the t-structure on $\mathrm{Shv}_\ell(X)$ is left complete.

Lemma 4.4.21. *Let X be a quasi-projective k-scheme. Then there exists an integer q with the following property: if $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)_{\leq 0}$ and $\mathcal{G} \in \mathrm{Shv}_\ell(X)_{\geq q}$, then every morphism $\mathcal{F} \to \mathcal{G}$ is nullhomotopic.*

Proof. By virtue of Proposition 4.2.13, we can choose an integer n for which the groups $\mathrm{Ext}^m_{\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})}(\mathcal{F}', \mathcal{G}')$ vanish whenever $\mathcal{F}' \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z})^{\heartsuit}$, $\mathcal{G}' \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})^{\heartsuit}$, and $m > n$. We will show that $q = n+3$ has the desired property. Let $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)_{\leq 0}$ and $\mathcal{G} \in \mathrm{Shv}_\ell(X)_{\geq n+3}$; we wish to prove that $\mathrm{Ext}^0_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \mathcal{G}) \simeq 0$. Writing $\mathcal{G}$ is a filtered colimit of objects of $\mathrm{Shv}^c_\ell(X)_{\geq n+3}$, we may assume that $\mathcal{G}$ is constructible. For each $d \geq 0$, set

$$\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \qquad \mathcal{G}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{G}.$$

Note that $\mathcal{F}$ can be regarded as an object of $\mathrm{Shv}(X; \mathbf{Z})_{\leq 0}$ (Remark 4.4.5), so that each $\mathcal{F}_d$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\leq -1}$. We have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \mathcal{G}) \simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)$$

which gives rise to Milnor exact sequences

$$0 \to \lim^1 \{\mathrm{Ext}^{-1}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)\} \to \mathrm{Ext}^0_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \mathcal{G}) \to \lim^0 \{\mathrm{Ext}^0_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)\}.$$

It will therefore suffice to show that the groups $\mathrm{Ext}^i_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)$ vanish for $i \in \{0, -1\}$. Writing $\mathcal{G}_d$ as successive extension of finitely many copies of $\mathcal{G}_1$, we may reduce to the case $d = 1$. We are therefore reduced to showing that the groups $\mathrm{Ext}^i_{\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})}(\mathcal{F}_1, \mathcal{G}_1)$ vanish for $i \in \{0, -1\}$. The desired result now follows by writing $\mathcal{F}_1$ and $\mathcal{G}_1$ as successive extensions of objects belonging to the heart $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})^{\heartsuit}$. □

Proof of Proposition 4.4.19. The right completeness of the $\mathrm{Shv}_\ell(X)$ follows formally from the right boundedness of $\mathrm{Shv}^c_\ell(X)$ (Proposition 4.4.19). To see this, we first observe that the full subcategory $\mathrm{Shv}_\ell(X)_{\leq 0}$ is closed under infinite direct sums. To show that $\mathrm{Shv}_\ell(X)$ is right complete, it will suffice to show that the intersection $\bigcap \mathrm{Shv}_\ell(X)_{\leq -n}$ consists only of zero objects (Proposition HA.1.2.1.19). To prove this, let $\mathcal{F} \in \bigcap \mathrm{Shv}_\ell(X)_{\leq -n}$. If $\mathcal{F} \neq 0$, then there exists an object $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)$ and a nonzero map $\mathcal{F}' \to \mathcal{F}$. This is impossible, since $\mathcal{F}'$ belongs to $\mathrm{Shv}_\ell(X)_{\geq m}$ for some integer m (by virtue of the right boundedeness of the t-structure on $\mathrm{Shv}^c_\ell(X)$).

To complete the proof, it will suffice to show that for every object $\mathcal{G} \in \mathrm{Shv}_\ell(X)$, the canonical map $\mathcal{G} \to \varprojlim \tau_{\leq n} \mathcal{G}$ is an equivalence. Equivalently, we must show that the object $\varprojlim \tau_{\geq n} \mathcal{G}$ vanishes. To prove this, we argue that for each constructible object $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$, the mapping space $\mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \varprojlim \tau_{\geq n} \mathcal{G})$ is contractible. We have Milnor exact sequences

$$\lim^1 \{\mathrm{Ext}^{m-1}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \tau_{\geq n} \mathcal{G})\}_{n \geq 0} \to \mathrm{Ext}^m_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \varprojlim \tau_{\geq n} \mathcal{G}) \to \lim^0 \{\mathrm{Ext}^m_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \tau_{\geq n} \mathcal{G})\}_{n \geq 0}.$$

The desired result now follows from Lemma 4.4.21 (and the left boundedness of $\mathrm{Shv}^c_\ell(X)$), which guarantees that the groups $\mathrm{Ext}^{m-1}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \tau_{\geq n} \mathcal{G})$ and $\mathrm{Ext}^m_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \tau_{\geq n} \mathcal{G})$ are trivial for $n \gg 0$. □

4.5. **Base Change Theorems and Dualizing Sheaves.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. In this section, we recall some nontrivial results in the theory of ℓ-adic sheaves which will be needed in the later sections of this paper. We begin with a few general categorical remarks.

Notation 4.5.1. Suppose we are given a diagram of ∞-categories and functors σ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow g & & \downarrow g' \\ \mathcal{C}' & \xrightarrow{f'} & \mathcal{D}' \end{array}$$

which commutes up to specified homotopy: that is, we are given an equivalence of functors $\alpha : g' \circ f \simeq f' \circ g$. Suppose that f and f' admit left adjoints f^L and f'^L, respectively. Then σ determines a map $\beta : f'^L \circ g' \to g \circ f^L$, given by the composition

$$f'^L \circ g' \to f'^L \circ g' \circ f \circ f^L \xrightarrow{\alpha} f'^L \circ f' \circ g \circ f^L \to g \circ f^L$$

where the first and third maps are given by composition with the unit and counit for the adjunctions between the pairs (f^L, f) and (f'^L, f'), respectively. We will refer to β as the *left Beck-Chevalley transformation* determined by α. We will say that the diagram σ is *left adjointable* if the functors f and f' admit left adjoints and the natural transformation β is an equivalence. If f and f' admit right adjoints f^R and f'^R, then a dual construction yields a natural transformation

$$\gamma : g \circ f^R \to f'^R \circ g',$$

which we will refer to as the *right Beck-Chevalley transformation* determined by σ. We will say that σ is *right adjointable* if the functors f and f' admit right adjoints ant the natural transformation γ is an equivalence.

Remark 4.5.2. In the situation of Notation 4.5.1, suppose that the functors f, f', g and g' all admit left adjoints. We then obtain a diagram σ^L:

$$\begin{array}{ccc} \mathcal{D}' & \xrightarrow{g'^L} & \mathcal{D} \\ \downarrow{\scriptstyle f'^L} & & \downarrow{\scriptstyle f^L} \\ \mathcal{C}' & \xrightarrow{g^L} & \mathcal{C} \end{array},$$

which commutes up to (preferred) homotopy, and the vertical maps admit right adjoints g' and g. We therefore obtain a right Beck-Chevalley transformation $f'^L \circ g' \to g \circ f^L$ for σ^L, which agrees (up to canonical homotopy) with the left Beck-Chevalley transformation for σ.

Remark 4.5.3. Suppose we are given a diagram of ∞-categories σ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathcal{C}' & \xrightarrow{f'} & \mathcal{D}' \end{array}$$

where the functors f and f' admit left adjoints f^L and f'^L, and the functors g and g' admit right adjoints g^R and g'^R. Applying the Construction of Notation 4.5.1 to σ and to the transposed diagram σ^t :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{g} & \mathcal{C}' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathcal{D} & \xrightarrow{g'} & \mathcal{D}', \end{array}$$

we obtain left and right Beck-Chevalley transformations

$$\beta : f'^L \circ g' \to g \circ f^L \qquad \gamma : f \circ g^R \to g'^R \circ f'.$$

Unwinding the definitions, we see that γ is the natural transformation obtained from β by passing to right adjoints. In particular, under the assumption that the relevant adjoints exist, the diagram σ is left adjointable if and only if the diagram σ^t is right adjointable.

We now specialize to the setting of algebraic geometry. Suppose we are given a commutative diagram σ :

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

of quasi-projective k-schemes. Then σ determines a diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S) & \xrightarrow{p^*} & \mathrm{Shv}_\ell(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f'^*} \\ \mathrm{Shv}_\ell(S') & \xrightarrow{p'^*} & \mathrm{Shv}_\ell(X'). \end{array}$$

Each functor in this diagram admits a right adjoint, so we obtain a right Beck-Chevalley transformation $\beta : f^* p_* \to p'_* f'^*$. The following is summarizes some of the main foundational results in the theory of étale cohomology:

Theorem 4.5.4 (Smooth and Proper Base Change)**.** *Suppose we are given a pullback diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S. \end{array}$$

If either p is proper or f is smooth, then the Beck-Chevalley morphism $\beta : f^* p_* \to p'_* f'^*$ is an equivalence of functors from $\mathrm{Shv}_\ell(X)$ to $\mathrm{Shv}_\ell(S')$.

Proof. Let $\mathcal{F} \in \mathrm{Shv}_\ell(X')$; we wish to prove that the canonical map $\beta_{\mathcal{F}} : f^* p_* \mathcal{F} \to p'_* f'^* \mathcal{F}$ is an equivalence in $\mathrm{Shv}_\ell(S')$. Writing $\mathcal{F}$ as a filtered colimit of constructible ℓ-adic sheaves (and using the fact that the functors f^*, p_*, p'_*, and f'^* commute with filtered colimits), we can reduce to the case where $\mathcal{F}$ is constructible. In this case, the domain and codomain of $\beta_{\mathcal{F}}$ are constructible ℓ-adic sheaves (see Notation 4.3.30). We may therefore identify $\beta_{\mathcal{F}}$ with a morphism in the ∞-category $\mathrm{Shv}^c_\ell(S') \subseteq \mathrm{Shv}(S'; \mathbf{Z}_\ell)$. Since the domain and codomain of $\beta_{\mathcal{F}}$ are ℓ-complete, it will suffice to show that the induced map

$$(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} f^* p_* \mathcal{F} \to (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} p'_* f'^* \mathcal{F}$$

is an equivalence in $\mathrm{Shv}(S'; \mathbf{Z}/\ell\mathbf{Z})$. For this, it suffices to show that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}^c(S; \mathbf{Z}/\ell\mathbf{Z}) & \xrightarrow{p^*} & \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f'^*} \\ \mathrm{Shv}^c(S'; \mathbf{Z}/\ell\mathbf{Z}) & \xrightarrow{p'^*} & \mathrm{Shv}^c(X'; \mathbf{Z}/\ell\mathbf{Z}). \end{array}$$

is right adjointable: that is, that the canonical map $f^* p_* \mathcal{G} \to p'_* f'^* \mathcal{G}$ is an equivalence for each constructible object $\mathcal{G} \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$. The constructibility of $\mathcal{G}$ implies that it can be written as a finite extension of suspensions of objects belonging to the heart $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ (which we can identify with the abelian category étale sheaves of $\mathbf{Z}/\ell\mathbf{Z}$ on X). The desired result now follows from the usual smooth and proper base change theorems for étale cohomology (see [13]). □

In the situation of Theorem 4.5.4, suppose that the map f is étale. Then the pullback functors f^* and f'^* admit left adjoints $f_!$ and $f'_!$. Invoking the dual of Remark 4.5.3, we obtain the following version of Proposition 4.1.12:

Variant 4.5.5. Suppose we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S, \end{array}$$

where f is étale. Then the diagram of ∞-categories

$$\begin{CD} \mathrm{Shv}_\ell(S) @>{f^*}>> \mathrm{Shv}_\ell(S') \\ @V{p^*}VV @VV{p'^*}V \\ \mathrm{Shv}_\ell(X) @>{f'^*}>> \mathrm{Shv}_\ell(X'). \end{CD}$$

is left adjointable. In other words, the associated Beck-Chevalley transformation $\beta' : f'_! p'^* \to p^* f_!$ is an equivalence of functors from $\mathrm{Shv}_\ell(S')$ to $\mathrm{Shv}_\ell(X)$.

Remark 4.5.6. It is easy to deduce Variant 4.5.5 directly from Proposition 4.1.12; the full force of the smooth base change theorem is not required.

In the situation of Theorem 4.5.4, the right adjointability of the diagram

$$\begin{CD} \mathrm{Shv}_\ell(S) @>{p^*}>> \mathrm{Shv}_\ell(X) \\ @V{f^*}VV @VV{f'^*}V \\ \mathrm{Shv}_\ell(S') @>{p'^*}>> \mathrm{Shv}_\ell(X'). \end{CD}$$

is equivalent to the left adjointability of the diagram

$$\begin{CD} \mathrm{Shv}_\ell(X') @>{f'_*}>> \mathrm{Shv}_\ell(X) \\ @V{p'_*}VV @VV{p_*}V \\ \mathrm{Shv}_\ell(S') @>{f_*}>> \mathrm{Shv}_\ell(S) \end{CD}$$

(Remark 4.5.2). If p is proper, then the vertical maps admit right adjoints given by $p^!$ and $p'^!$, respectively. Invoking Remark 4.5.3, we obtain:

Variant 4.5.7. Suppose we are given a pullback diagram of quasi-projective k-schemes

$$\begin{CD} X' @>{f'}>> X \\ @V{p'}VV @VV{p}V \\ S' @>{f}>> S, \end{CD}$$

where p is proper. Then the diagram of ∞-categories

$$\begin{CD} \mathrm{Shv}_\ell(X') @>{p'_*}>> \mathrm{Shv}_\ell(S') \\ @V{f'_*}VV @VV{f_*}V \\ \mathrm{Shv}_\ell(X) @>{p_*}>> \mathrm{Shv}_\ell(S) \end{CD}$$

is right adjointable. In other words, the right Beck-Chevalley transformation

$$\beta'' : f'_* p'^! \to p^! f_*$$

is an equivalence of functors from $\mathrm{Shv}_\ell(S')$ to $\mathrm{Shv}_\ell(X)$.

Construction 4.5.8. Suppose that we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where f is étale, so that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p'_*} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(S') \end{array}$$

commutes up to canonical homotopy (Theorem 4.5.4). Note that the horizontal maps admit left adjoints $f'_!$ and $f_!$, respectively, so that there is an associated left Beck-Chevalley transformation $\gamma : f_! p'_* \to p_* f'_!$. By virtue of Remark 4.5.2, we can also identify γ with the right Beck-Chevalley transformation associated to the diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{p'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f'_!} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{p^*} & \mathrm{Shv}_\ell(X) \end{array}$$

of Variant 4.5.5.

Proposition 4.5.9. *Suppose that we are given a pullback diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S, \end{array}$$

where f is étale. If p is proper, then the natural transformation $\gamma : f_! p'_* \to p_* f'_!$ of Construction 4.5.8 is an equivalence. In other words, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p'_*} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(S') \end{array}$$

is left adjointable, and the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{p'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f'_!} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{p^*} & \mathrm{Shv}_\ell(X) \end{array}$$

is right adjointable.

Proof. Let $\mathcal{F} \in \mathrm{Shv}_\ell(X')$; we wish to show that the map $\gamma_{\mathcal{F}} : f_! p'_* \mathcal{F} \to p_* f'_! \mathcal{F}$ is an equivalence in $\mathrm{Shv}_\ell(S)$. Writing $\mathcal{F}$ as a filtered colimit of constructible ℓ-adic sheaves (and using the fact that the functors $f_!$, p_*, $f'_!$, and p'_* commute with filtered colimits), we can reduce to the case where $\mathcal{F}$ is constructible. In this case, the domain and codomain of $\gamma_{\mathcal{F}}$ are also constructible ℓ-adic sheaves. Using Proposition 4.3.25, we are reduced to showing that $\beta_{\mathcal{F}}$ induces an equivalence $\eta^* f_! p'_* \mathcal{F} \to \eta^* p_* f'_! \mathcal{F}$ for every point $\eta : \operatorname{Spec} k \to S$. Using Theorem 4.5.4 and Variant 4.5.5, we can replace S by $\operatorname{Spec} k$. In this case, S' is isomorphic to a disjoint union of finitely many copies of $\operatorname{Spec} k$ and the result is easy. □

Corollary 4.5.10 (Projection Formula)**.** *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Then for every pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Y)$, the canonical map*

$$\beta_{\mathcal{F},\mathcal{G}} : (f_* \mathcal{F}) \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^* \mathcal{G})$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$.

Proof of Corollary 4.5.10. The construction $(\mathcal{F}, \mathcal{G}) \mapsto \beta_{\mathcal{F},\mathcal{G}}$ commutes with filtered colimits separately in each variable. We may therefore assume without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are constructible ℓ-adic sheaves. In this case, $\beta_{\mathcal{F},\mathcal{G}}$ is a morphism between constructible ℓ-adic sheaves. Consequently, to prove that $\beta_{\mathcal{F},\mathcal{G}}$ is an equivalence, it will suffice to show that the image of $\beta_{\mathcal{F},\mathcal{G}}$ in $\mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$ is an equivalence. In other words, it suffices to prove the analogue of Corollary 4.5.10 when $\mathcal{F}$ and $\mathcal{G}$ are constructible objects of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ and $\mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$, respectively.

Let us regard $\mathcal{F}$ as fixed. Using Remark 4.1.18, we see that the construction $\mathcal{G} \mapsto \beta_{\mathcal{F},\mathcal{G}}$ preserves colimits. It follows that the collection of those objects $\mathcal{G} \in \mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$ for which $\beta_{\mathcal{F},\mathcal{G}}$ is an equivalence is closed under colimits. Using Proposition 4.2.2, we may suppose that $\mathcal{G} = j_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_U$ for some étale map $j : U \to Y$. Form a pullback diagram

$$\begin{array}{ccc} U_X & \xrightarrow{j'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & Y. \end{array}$$

Unwinding the definitions, we can identify $(f_* \mathcal{F}) \otimes \mathcal{G}$ with the object $j_! j^* f_* \mathcal{F}$, and $\mathcal{F} \otimes f^* \mathcal{G}$ with $j'_! j'^* \mathcal{F}$. Under these identifications, the map $\beta_{\mathcal{F},\mathcal{G}}$ factors as a composition

$$j_! j^* f_* \mathcal{F} \xrightarrow{\beta'} j_! f'_* j'^* \xrightarrow{\beta''} f_* j'_! j'^* \mathcal{F},$$

where β' is an equivalence by Theorem 4.5.4 (since j is étale) and β'' is an equivalence by Proposition 4.5.9 (since f is proper). □

We also have the following dual version of Construction 4.5.8:

Construction 4.5.11. Suppose that we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where p is proper, so that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{p_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle f'^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{p'_*} & \mathrm{Shv}_\ell(S') \end{array}$$

commutes up to canonical homotopy (Theorem 4.5.4). Note that the horizontal maps admit right adjoints $p^!$ and $p'^!$, so that there is an associated right Beck-Chevalley transformation $\gamma' : f'^* p^! \to p'^! f^*$ of functors from $\mathrm{Shv}_\ell(S)$ to $\mathrm{Shv}_\ell(X')$. Using Remark 4.5.2, we can also identify γ' with the left Beck-Chevalley transformation associated to the diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{f_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle p'^!} & & \downarrow{\scriptstyle p^!} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{f'_*} & \mathrm{Shv}_\ell(X) \end{array}$$

of Variant 4.5.7.

Proposition 4.5.12. *Suppose we are given a commutative diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where p is proper. If f is smooth, then the natural transformation $\gamma' : f'^* p^! \to p'^! f^*$ of Construction 4.5.11 is an equivalence. In other words, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{p_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle f'^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{p'_*} & \mathrm{Shv}_\ell(S') \end{array}$$

is right adjointable and the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{f_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle p'^!} & & \downarrow{\scriptstyle p^!} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{f'_*} & \mathrm{Shv}_\ell(X) \end{array}$$

is left adjointable.

Remark 4.5.13. In the situation of Proposition 4.5.12, suppose that p is proper and f is étale. In this case, the natural transformation $\gamma' : f'^* p^! \to p'^! f^*$ is obtained from the natural transformation $\gamma : f_! p'_* \to p_* f'_!$ of Construction 4.5.8 by passing to right adjoints. In this case, Proposition 4.5.12 reduces to Proposition 4.5.9.

Proof of Proposition 4.5.12. Fix an object $\mathcal{F} \in \mathrm{Shv}_\ell(S)$; we wish to show that the map $\gamma'_{\mathcal{F}} : f'^* p^! \mathcal{F} \to p'^! f^* \mathcal{F}$ is an equivalence. Since the construction $\mathcal{F} \mapsto \gamma'_{\mathcal{F}}$ preserves filtered colimits, we may assume without loss of generality that $\mathcal{F}$ is a constructible ℓ-adic sheaf. For every point $\eta : \operatorname{Spec} k \to X$, let i_η denote the inclusion of the fiber product $X' \times_X \operatorname{Spec} k$ into X'. By virtue of Proposition 4.3.25, it will suffice to show that $i_\eta^! \gamma_{\mathcal{F}}$ is an equivalence for each η. Let $f'' : X' \times_X \operatorname{Spec} k \to \operatorname{Spec} k$ denote the projection map, so that $i_\eta^! \gamma_{\mathcal{F}}$ fits into a commutative diagram

$$\begin{array}{ccccc} & & i_\eta^! f'^* p^! & & \\ & \overset{\beta'}{\nearrow} & & \overset{\eta^! \gamma'_{\mathcal{F}}}{\searrow} & \\ f''^* \eta^! p^! & & \xrightarrow{\beta''} & & i_\eta^! p'^! f^*. \end{array}$$

It will therefore suffice to show that β' and β'' are equivalences. We may therefore replace the map p by either η or $p \circ \eta$, and thereby reduce to the case where p is a closed immersion.

Let $j : U \to S$ be an open immersion complementary to p, let U' denote the fiber product $U \times_S S'$, and let $j' : U' \to S'$ denote the projection onto the second factor. If p is a closed immersion, then the pushforward functor p'_* is fully faithful. It will therefore suffice to show that the $p'_* \gamma'_{\mathcal{F}}$ is an equivalence. Identifying $p'_* f'^* p^! \mathcal{F}$ with $f^* p_* p^! \mathcal{F}$, we see that $p'_* \gamma'_{\mathcal{F}}$ fits into a commutative diagram of fiber sequences

$$\begin{array}{ccccc} f^* p_* p^! \mathcal{F} & \longrightarrow & f^* \mathcal{F} & \longrightarrow & f^* j_* j^* \mathcal{F} \\ \downarrow{\scriptstyle p'_* \gamma'_{\mathcal{F}}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \rho} \\ p'_* p'^! f^* \mathcal{F} & \longrightarrow & f^* \mathcal{F} & \longrightarrow & j'_* j'^* f^* \mathcal{F}. \end{array}$$

It will therefore suffice to show that ρ is an equivalence. This follows from Theorem 4.5.4, since f is smooth. □

Example 4.5.14. Let X be a quasi-projective k-scheme, and let $j : U \to X$ be an open immersion whose image is also closed in X. Then j is a proper map, and the diagram

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{id}} & U \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} \\ U & \xrightarrow{j} & X \end{array}$$

is a pullback square. Then Proposition 4.5.12 supplies a canonical equivalence

$$j^! \simeq \mathrm{id}^* j^! \simeq \mathrm{id}^! j^* \simeq j^*.$$

Example 4.5.15. Let $f : X \to Y$ be a morphism between quasi-projective k-schemes. Let $U \subseteq X$ be the locus over which f is étale, let f_0 be the restriction of f to U, let $j : U \hookrightarrow X$ be the inclusion map, and let $\delta : U \to U \times_Y X$ be the diagonal map. Then δ exhibits U as a direct summand of $U \times_Y X$, so that Example 4.5.14 supplies an equivalence $\delta^! \simeq \delta^*$. Applying Proposition 4.5.12 to the pullback square

$$\begin{array}{ccc} U \times_Y X & \xrightarrow{\pi_2} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{f_0} & Y, \end{array}$$

we obtain a natural equivalence

$$j^* f^! \simeq \delta^* \pi_2^* f^! \simeq \delta^* \pi_1^! f_0^* \simeq \delta^! \pi_1^! f_0^* \simeq f_0^*.$$

In particular, if f is both étale and proper, then the functors $f^!$ and f^* are canonically equivalent to one another (one can show that this equivalence agrees with the one supplied by Example 4.3.23).

Variant 4.5.16. Suppose we are given a commutative diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where p is proper. Let $U \subseteq X$ be an open subset for which the restriction $p|_U$ is smooth, and let $U' \subseteq X'$ denote the inverse image of U. Then the natural transformation $\gamma' : f'^* p^! \to p'^! f^*$ of Construction 4.5.11 induces an equivalence

$$(f'^* p^! \mathcal{F})|_{U'} \to (p'^! f^* \mathcal{F})|_{U'}.$$

for each object $\mathcal{F} \in \mathrm{Shv}_\ell(S)$. In particular, if p is smooth, then γ' is an equivalence.

Proof. The assertion is local on U. We may therefore assume without loss of generality that there exists an étale map of S-schemes $g : U \to \mathbf{P}^n \times_{\operatorname{Spec} k} S$. Let $\Gamma \subseteq U \times \mathbf{P}^n$ denote the graph of g, let $\overline{\Gamma} \subseteq X \times_{\operatorname{Spec} k} \mathbf{P}^n$ be the closure of Γ and let $q : \overline{\Gamma} \to X$ be the projection onto the first factor. Then q is a proper morphism which restricts to an isomorphism over the open set U. Using Example 4.5.15, we can replace X by $\overline{\Gamma}$ and thereby reduce to the case where g extends to a map $\overline{g} : \mathbf{P}^n \times_{\operatorname{Spec} k} S$. Using Example 4.5.15 again, we can replace X by $\mathbf{P}^n \times_{\operatorname{Spec} k} S$, and thereby reduce to the case where p is smooth. In this case, the desired result follows from the description of the functors $p^!$ and $p'^!$ supplied by Example 4.3.23 (and the fact that this description is compatible with base change). □

Construction 4.5.17. Suppose that we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where p is proper and f is étale. Then Proposition 4.5.12 supplies a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(S') \\ \downarrow{\scriptstyle p^!} & & \downarrow{\scriptstyle p'^!} \\ \mathrm{Shv}_\ell(X) & \xrightarrow{f'^*} & \mathrm{Shv}_\ell(X'). \end{array}$$

Note that the horizontal maps admit left adjoints $f_!$ and $f'_!$, so that there is an associated left Beck-Chevalley transformation $\delta : f'_! p'^! \to p^! f_!$ of functors from $\mathrm{Shv}_\ell(S')$ to $\mathrm{Shv}_\ell(X)$. Using Remarks 4.5.2 and 4.5.13, we see that δ can also be identified with the right Beck-Chevalley transformation associated to the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X') & \xrightarrow{p'_*} & \mathrm{Shv}_\ell(S') \\ \downarrow{\scriptstyle f'_!} & & \downarrow{\scriptstyle f_!} \\ \mathrm{Shv}_\ell(X) & \xrightarrow{p_*} & \mathrm{Shv}_\ell(S) \end{array}$$

given by Proposition 4.5.9.

Proposition 4.5.18. *Suppose we are given a commutative diagram of quasi-projective k-schemes*

$$\begin{CD} X' @>{f'}>> X \\ @V{p'}VV @VV{p}V \\ S' @>{f}>> S \end{CD}$$

where p is proper and f is étale, and let $\delta : f'_! p'^! \to p^! f_!$ be the natural transformation of Construction 4.5.17. If U is an open subset of X such that $p|_U$ is smooth, then δ induces an equivalence

$$(f'_! p'^! \mathcal{F})|_U \to (p^! f_! \mathcal{F})|_U$$

for each object $\mathcal{F} \in \mathrm{Shv}_\ell(S')$. In particular, if p is smooth, then δ is an equivalence, so that the diagrams of ∞-categories

$$\begin{CD} \mathrm{Shv}_\ell(S) @>{f^*}>> \mathrm{Shv}_\ell(S') \\ @V{p^!}VV @VV{p'^!}V \\ \mathrm{Shv}_\ell(X) @>{f'^*}>> \mathrm{Shv}_\ell(X') \end{CD} \qquad \begin{CD} \mathrm{Shv}_\ell(X') @>{p'_*}>> \mathrm{Shv}_\ell(S') \\ @V{f'_!}VV @VV{f_!}V \\ \mathrm{Shv}_\ell(X) @>{p_*}>> \mathrm{Shv}_\ell(S) \end{CD}$$

are left and right adjointable, respectively.

Proof. Arguing as in the proof of Variant 4.5.16, we may reduce to the case where p is smooth, in which case the desired result follows from the desciption of the functors $p^!$ and $p'^!$ supplied by Example 4.3.23. □

Construction 4.5.19. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes and suppose we are given objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(Y)$. Tensoring the counit map $f_* f^! \mathcal{G} \to \mathcal{G}$ with $\mathcal{F}$ and applying Corollary 4.5.10, we obtain a map

$$f_*(f^* \mathcal{F} \otimes f^! \mathcal{G}) \simeq \mathcal{F} \otimes f_* f^! \mathcal{G} \to \mathcal{F} \otimes \mathcal{G},$$

which in turn classifies a map

$$\rho_{\mathcal{F},\mathcal{G}} : f^* \mathcal{F} \otimes f^! \mathcal{G} \to f^!(\mathcal{F} \otimes \mathcal{G})$$

in $\mathrm{Shv}_\ell(X)$.

Proposition 4.5.20. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Let $U \subseteq X$ be an open subset for which $f|_U$ is smooth. Then the natural map*

$$\rho_{\mathcal{F},\mathcal{G}} : f^* \mathcal{F} \otimes f^! \mathcal{G} \to f^!(\mathcal{F} \otimes \mathcal{G})$$

induces an equivalence

$$(f^* \mathcal{F} \otimes f^! \mathcal{G})|_U \to f^!(\mathcal{F} \otimes \mathcal{G})|_U$$

for every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(Y)$.

Proof. Let $\underline{\mathbf{Z}_\ell}_Y$ denote the unit object of $\mathrm{Shv}_\ell(Y)$. Note that we have a commutative diagram

$$\begin{array}{ccc} & f^* \mathcal{F} \otimes f^! \mathcal{G} & \\ {}^{\rho_{\mathcal{G}, \underline{\mathbf{Z}_\ell}_Y}} \nearrow & & \searrow \\ f^* \mathcal{F} \otimes f^* \mathcal{G} \otimes f^! \underline{\mathbf{Z}_\ell}_Y & \xrightarrow{\rho_{\mathcal{F} \otimes \mathcal{G}, \underline{\mathbf{Z}_\ell}_Y}} & f^!(\mathcal{F} \otimes \mathcal{G}). \end{array}$$

It will therefore suffice to show that the maps $\rho_{\mathcal{G},\underline{\mathbf{Z}_\ell}_Y}$ and $\rho_{\mathcal{F}\otimes\mathcal{G},\underline{\mathbf{Z}_\ell}_Y}$ are equivalences over the open set U. We may therefore reduce to the case where $\mathcal{G} = \underline{\mathbf{Z}_\ell}_Y$. Since the construction $\mathcal{F} \mapsto \rho_{\mathcal{F},\underline{\mathbf{Z}_\ell}_Y}$ preserves filtered colimits, we may assume without loss of generality that $\mathcal{F}$ is a constructible ℓ-adic sheaf. In this case, $\rho_{\mathcal{F},\underline{\mathbf{Z}_\ell}_Y}$ is a morphism of constructible ℓ-adic sheaves. To prove that it is an equivalence over U, it will suffice to show that its image in $\mathrm{Shv}(U; \mathbf{Z}/\ell\mathbf{Z})$ is an equivalence. In other words, we are reduced to proving that for each constructible object $\mathcal{F}_1 \in \mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$, the canonical map

$$(f^* \mathcal{F}_1 \otimes f^! \underline{\mathbf{Z}/\ell\mathbf{Z}}_Y)|_U \to (f^! \mathcal{F}_1)|_U$$

is an equivalence. Using Proposition 4.2.2, we may assume without loss of generality that $\mathcal{F}_1 = g_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_V$ where $g : V \to Y$ is an étale map. In this case, the desired result follows from Proposition 4.5.18. □

Notation 4.5.21. If $f : X \to Y$ is a proper morphism of quasi-projective k-schemes, we let $\omega_{X/Y} \in \mathrm{Shv}_\ell(X)$ denote the ℓ-adic sheaf given by $f^! \underline{\mathbf{Z}_\ell}_Y$.

In the special case where $Y = \operatorname{Spec} k$, we will denote $\omega_{X/Y}$ by ω_X, and refer to it as the *dualizing sheaf* of X.

Remark 4.5.22. In §4.6, we will extend the definition of ω_X to the case where X is an arbitrary quasi-projective k-scheme (Notation 4.6.15).

Example 4.5.23. If $f : X \to Y$ is a proper smooth morphism of relative dimension d, then Example 4.3.23 supplies an equivalence $\omega_{X/Y} \simeq \Sigma^{2d} \underline{\mathbf{Z}_\ell(d)}_X$. More generally, one can show that if $U \subseteq X$ is an open subset for which $f|_U$ is a smooth morphism of relative dimension d, then $\omega_{X/Y}|_U$ is equivalent to $\Sigma^{2d} \underline{\mathbf{Z}_\ell(d)}_X$.

Remark 4.5.24. Suppose we are given a commutative diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow p \\ Y' & \longrightarrow & Y, \end{array}$$

where the vertical maps are proper. Then Construction 4.5.11 supplies a canonical map $f^* \omega_{X/Y} \to \omega_{X'/Y'}$, which is an equivalence over the inverse image of the smooth locus of p (Variant 4.5.16).

Remark 4.5.25. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. For each object $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, Construction 4.5.19 supplies a canonical map

$$f^* \mathcal{F} \otimes \omega_{X/Y} \to f^! \mathcal{F},$$

which induces an equivalence over the smooth locus of f (Proposition 4.5.20).

4.6. **Künneth Formulae and the !-Tensor Product.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k.

Notation 4.6.1. Let Sch_k denote the category of quasi-projective k-schemes, and let $\mathrm{Sch}_k^{\mathrm{pr}}$ denote the subcategory of Sch_k whose morphisms are proper maps of quasi-projective k-schemes.

If X and Y are quasi-projective k-schemes, we let $X \times Y$ denote the Cartesian product of X and Y in the category Sch_k: that is, the fiber product $X \times_{\operatorname{Spec} k} Y$ in the category of schemes. This fiber product is equipped with projection maps

$$X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y.$$

Given a pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, $\mathcal{G} \in \mathrm{Shv}_\ell(Y)$, we let $\mathcal{F} \boxtimes \mathcal{G}$ denote the tensor product $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$, formed in the ∞-category $\mathrm{Shv}_\ell(X \times Y)$. We will refer to $\mathcal{F} \boxtimes \mathcal{G}$ as the *external tensor product* of $\mathcal{F}$ and $\mathcal{G}$.

Note that if $f : X \to Y$ is a morphism of quasi-projective k-schemes and Z is another quasi-projective k-scheme, then we have a canonical equivalence

$$(f \times \mathrm{id}_Z)^*(\mathcal{H} \boxtimes \mathcal{G}) \simeq f^* \mathcal{H} \boxtimes \mathcal{G}$$

for $\mathcal{H} \in \mathrm{Shv}_\ell(Y)$, $\mathcal{G} \in \mathrm{Shv}_\ell(Z)$. Taking $\mathcal{H} = f_* \mathcal{F}$ for $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ (and composing with the counit map $f^* \mathcal{H} \to \mathcal{F}$, we obtain a map

$$\theta_{\mathcal{F},\mathcal{G}} : f_* \mathcal{F} \boxtimes \mathcal{G} \to (f \times \mathrm{id}_Z)_*(\mathcal{F} \boxtimes \mathcal{G}).$$

Proposition 4.6.2. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes and let Z be a quasi-projective k-scheme. Then for every pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Z)$, the canonical map*

$$\theta_{\mathcal{F},\mathcal{G}} : (f_* \mathcal{F}) \boxtimes \mathcal{G} \to (f \times \mathrm{id}_Z)_*(\mathcal{F} \boxtimes \mathcal{G})$$

is an equivalence in $\mathrm{Shv}_\ell(Y \times Z)$.

Proof. The construction $(\mathcal{F}, \mathcal{G}) \mapsto \theta_{\mathcal{F},\mathcal{G}}$ preserves filtered colimits in $\mathcal{F}$ and $\mathcal{G}$. We may therefore assume without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are constructible ℓ-adic sheaves. In this case, $\theta_{\mathcal{F},\mathcal{G}}$ is a morphism of constructible ℓ-adic sheaves on $Y \times Z$. Consequently, to prove that $\theta_{\mathcal{F},\mathcal{G}}$ is an equivalence, it will suffice to show that the image of $\theta_{\mathcal{F},\mathcal{G}}$ in $\mathrm{Shv}(Y \times Z; \mathbf{Z}/\ell\mathbf{Z})$ is an equivalence. It will therefore suffice to prove the analogue of Proposition 4.6.2 where $\mathcal{F}$ and $\mathcal{G}$ are compact objects of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ and $\mathrm{Shv}(Z; \mathbf{Z}/\ell\mathbf{Z})$, respectively.

We first consider two special cases:

- (a) If the map f is proper, then the desired result follows immediately from the projection formula (Corollary 4.5.10).
- (b) Suppose that Z is smooth and that $\mathcal{G}$ is locally constant. In this case, we can assume that $\mathcal{G}$ is the constant sheaf $\underline{M}_Z$, where $M \in \mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$ is perfect (since the assertion is local with respect to the étale topology on Z). The collection of those M for which $\theta_{\mathcal{F},\mathcal{G}}$ is an equivalence is closed under shifts, retracts, and finite colimits; we may therefore assume that $M = \mathbf{Z}/\ell\mathbf{Z}$. In this case, the desired result follows from the smooth base change theorem (Theorem 4.5.4).

We now treat the general case. For the remainder of the proof, we will regard $f : X \to Y$ and $\mathcal{F} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z})$ as fixed. Let d denote the dimension of Z; we will proceed by induction on d. It follows from case (a) that if we are given a proper map $g : Z \to Z'$, then we can identify $\theta_{\mathcal{F},g_* \mathcal{G}}$ with the image of $\theta_{\mathcal{F},\mathcal{G}}$ under the pushforward functor $(\mathrm{id} \times g)_* : \mathrm{Shv}(Y \times Z; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Shv}(Y \times Z'; \mathbf{Z}/\ell\mathbf{Z})$.

Since the desired conclusion can be tested locally on Z, we may assume without loss of generality that Z is affine. In this case, we can use Noether normalization to choose a finite map $g : Z \to \mathbf{A}^d$. Then $\mathrm{cofib}(\theta_{\mathcal{F},\mathcal{G}})$ vanishes if and only if $g_* \mathrm{cofib}(\theta_{\mathcal{F},\mathcal{G}}) \simeq \mathrm{cofib}(\theta_{\mathcal{F},g_* \mathcal{G}})$ vanishes. We may therefore replace $\mathcal{G}$ by $g_* \mathcal{G}$, and thereby reduce to the case where $Z = \mathbf{A}^d$ is an affine space.

Using Proposition 4.2.5, we can choose a nonempty open subset $U \subseteq Z$ such that $\mathcal{G}|_U$ is locally constant. Applying a translation if necessary, we may suppose that U contains the origin $0 \in \mathbf{A}^d = Z$. Set $\mathcal{H} = \mathrm{cofib}(\theta_{\mathcal{F},\mathcal{G}})$, so that $\mathcal{H} \in \mathrm{Shv}^c(Y \times Z; \mathbf{Z}/\ell\mathbf{Z})$. Using (b), we see that $\mathcal{H}$ vanishes on the open set $Y \times U$. We wish to prove that $\mathcal{H} \simeq 0$. Suppose otherwise: then $\mathcal{H}$ has nonvanishing stalk at some closed point (y, z) of $Y \times Z$. Since $\mathcal{H}$ vanishes on $Y \times U$, z is not the origin of $Z \simeq \mathbf{A}^d$. Applying a linear change of coordinates, we may assume without loss of generality that $z = (1, 0, \ldots, 0)$. Let $\overline{Z} = \mathbf{P}^1 \times \mathbf{A}^{d-1}$, let $j : Z \to \overline{Z}$ denote the inclusion map, let $\overline{\mathcal{G}} = j_! \, \mathcal{G}$ and $\overline{\mathcal{H}} = \theta_{\mathcal{F}, \overline{\mathcal{G}}} \in \mathrm{Shv}^c(Y \times \overline{Z}; \Lambda)$. Let $g : \overline{Z} \to \mathbf{A}^{d-1}$ denote the projection map onto the second fiber. Since $\overline{\mathcal{H}}$ vanishes on $Y \times_{\mathrm{Spec}\, k} U$, the support of $\overline{\mathcal{H}}$ has finite intersection with the fiber $(\mathrm{id} \times g)^{-1}\{(y, 0)\}$. Using the proper base change theorem (Theorem 4.5.4), we see that the stalk of $\mathcal{H}$ at (y, z) can be identified with a direct summand of the stalk of $(\mathrm{id} \times g)_* \overline{\mathcal{H}}$ at the point $(y, 0)$. In particular, we have $0 \neq (\mathrm{id} \times g)_* \overline{\mathcal{H}} \simeq \mathrm{cofib}(\theta_{\mathcal{F}, g_* \overline{\mathcal{G}}})$, contradicting our inductive hypothesis. □

Corollary 4.6.3. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes and let Z be another quasi-projective k-scheme, so that we have a pullback square*

$$\begin{array}{ccc} X \times Z & \xrightarrow{f'} & Y \times Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{f} & Y. \end{array}$$

For each sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the canonical map $g'^* f_* \mathcal{F} \to f'_* g^* \mathcal{F}$ is an equivalence.

Proof. Apply Proposition 4.6.2 in the special case $\mathcal{G} = \underline{\mathbf{Z}_\ell}_Z \in \mathrm{Shv}_\ell(Z)$. □

Corollary 4.6.4. *Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of quasi-projective k-schemes. For every pair of ℓ-adic sheaves*

$$\mathcal{F} \in \mathrm{Shv}_\ell(X) \qquad \mathcal{F}' \in \mathrm{Shv}_\ell(X')$$

the canonical map

$$(f_* \mathcal{F}) \boxtimes (f'_* \mathcal{F}') \to (f \times f')_* (\mathcal{F} \boxtimes \mathcal{F}')$$

is an equivalence in $\mathrm{Shv}_\ell(Y \times Y')$.

Corollary 4.6.5. *Let X and X' be quasi-projective k-schemes. For every pair of ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, $\mathcal{F}' \in \mathrm{Shv}_\ell(X')$, the canonical map*

$$C^*(X; \mathcal{F}) \otimes_{\mathbf{Z}_\ell} C^*(X'; \mathcal{F}') \to C^*(X \times X'; \mathcal{F} \boxtimes \mathcal{F}')$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Example 4.6.6 (Künneth Formula)**.** Let X and X' be quasi-projective k-schemes. It follows from Corollary 4.6.5 the the canonical map

$$C^*(X; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C^*(X'; \mathbf{Z}_\ell) \to C^*(X \times X'; \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

In the situation of Proposition 4.6.2, suppose that the morphism $f : X \to Y$ is proper. For every pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Z)$, we obtain a canonical map

$$(f^! \mathcal{F} \boxtimes \mathcal{G}) \to f'^! f'_* (f^! \mathcal{F} \boxtimes \mathcal{G}) \xrightarrow{\theta^{-1}} f'^! (f_* f^! \mathcal{F} \boxtimes \mathcal{G}) \to f'^! (\mathcal{F} \boxtimes \mathcal{G}).$$

where $f' = (f \times \mathrm{id}_Z) : X \times Z \to Y \times Z$ and $\theta = \theta_{f^! \mathcal{F}, \mathcal{G}}$.

Proposition 4.6.7. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes, let Z be a quasi-projective k-scheme, and let $f' = f \times \mathrm{id}_Z : X \times Z \to Y \times Z$. Then, for every pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Z)$, the map*

$$\mu_{\mathcal{F}, \mathcal{G}} : f^! \mathcal{F} \boxtimes \mathcal{G} \to f'^! (\mathcal{F} \boxtimes \mathcal{G})$$

constructed above is an equivalence.

Proof. We first treat the case where f is a closed immersion. In this case, $f_*(\mu_{\mathcal{F},\mathcal{G}})$ can be identified with a homotopy inverse to the equivalence $\theta_{f^! \mathcal{F},\mathcal{G}}$ of Proposition 4.6.2, and the desired result follows from the fact that the functor f_* is fully faithful (and, in particular, conservative).

To treat the general case, we first choose an immersion $i : X \hookrightarrow \mathbf{P}^n$. Then f factors as a composition

$$X \xrightarrow{(i,f)} \mathbf{P}^n \times Y \xrightarrow{\pi} Y,$$

where π denotes the projection onto the second factor. Since f is proper, the map $(i,f) : X \to \mathbf{P}^n \times Y$ is a closed immersion. Using the first part of the proof, we can replace f by the projection map $\pi : \mathbf{P}^n \times Y \to Y$. In this case, the desired result follows from Proposition 4.5.20 and Remark 4.5.24. □

Corollary 4.6.8. *Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of quasi-projective k-schemes. For every pair of ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$ and $\mathcal{F}' \in \mathrm{Shv}_\ell(Y')$, there is a canonical equivalence*

$$(f^! \mathcal{F}) \boxtimes (f'^! \mathcal{F}') \simeq (f \times f')^! (\mathcal{F} \boxtimes \mathcal{F}')$$

of ℓ-adic sheaves on $X \times X'$.

Remark 4.6.9. Proposition 4.6.7 (and Corollary 4.6.8) are also valid for non-proper morphisms of k-schemes provided that the exceptional inverse image functor has been appropriately defined.

We now discuss some consequences of Proposition 4.6.7.

Construction 4.6.10. Let X be a quasi-projective k-scheme and let $\delta : X \to X \times X$ denote the diagonal map. For every pair of ℓ-adic sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(X)$, we define

$$\mathcal{F} \otimes^! \mathcal{G} = \delta^! (\mathcal{F} \boxtimes \mathcal{G}) \in \mathrm{Shv}_\ell(X).$$

The construction $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes^! \mathcal{G}$ determines a functor $\otimes^! : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$, which we will refer to as the !-*tensor product* functor.

Remark 4.6.11. Let X be a quasi-projective k-scheme. We then have a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\delta}\swarrow & & \searrow^{\delta} \\ X \times X & \xrightarrow{s} & X \times X \end{array}$$

where s denotes the automorphism of $X \times X$ which interchanges the factors. For $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(X)$, we obtain equivalences

$$\begin{aligned} \mathcal{F} \otimes^! \mathcal{G} &\simeq \delta^! (\mathcal{F} \boxtimes \mathcal{G}) \\ &\simeq (s \circ \delta)^! (\mathcal{F} \boxtimes \mathcal{G}) \\ &\simeq \delta^! s^! (\mathcal{F} \boxtimes \mathcal{G}) \\ &\simeq \delta^! (\mathcal{G} \boxtimes \mathcal{F}) \\ &\simeq \mathcal{G} \otimes^! \mathcal{F}. \end{aligned}$$

In other words, the !-tensor product $\otimes^!$ is commutative up to canonical equivalence. It is also associative up to equivalence: given a triple of ℓ-adic sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Shv}_\ell(X)$, it follows from Proposition 4.6.7 that the iterated tensor products

$$(\mathcal{F} \otimes^! \mathcal{G}) \otimes^! \mathcal{H} \qquad \mathcal{F} \otimes^! (\mathcal{G} \otimes^! \mathcal{H})$$

can both be identified with $\delta^{(3)!}(\mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{H})$, where $\delta^{(3)} : X \to X \times X \times X$ denotes the ternary diagonal.

Combining the preceding arguments with appropriate organizational principles, one can prove a much stronger assertion: the !-tensor product endows the ∞-category $\mathrm{Shv}_\ell(X)$ with the structure of a symmetric monoidal ∞-category. We will prove this in §5.5.

Remark 4.6.12. Let $f : X \hookrightarrow Y$ be a closed immersion of quasi-projective k-schemes. Then the diagram

$$\begin{array}{ccc} X & \longrightarrow & X \times X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times f} \\ Y & \longrightarrow & Y \times Y \end{array}$$

is a pullback square. Using Variant 4.5.7, we deduce that for every pair of ℓ-adic sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(X)$, the canonical map

$$f_*(\mathcal{F} \otimes^! \mathcal{G}) \to f_* \mathcal{F} \otimes^! f_* \mathcal{G}$$

is an equivalence.

Proposition 4.6.13. *Let X be a quasi-projective k-scheme. Then the !-tensor product on X is unital. In other words, there exists an object $\mathcal{E} \in \mathrm{Shv}_\ell(X)$ for which the functor*

$$\mathcal{F} \mapsto \mathcal{E} \otimes^! \mathcal{F}$$

is equivalent to the identity map from $\mathrm{Shv}_\ell(X)$ to itself.

Proof. Choose an open embedding $j : X \to \overline{X}$, where $\overline{X}$ is a projective k-scheme. Let $\pi : \overline{X} \to \operatorname{Spec} k$ be the projection map and let $\omega_{\overline{X}} = \pi^! \mathbf{Z}_\ell$ denote the dualizing sheaf on $\overline{X}$. Let

$$\delta_X : X \to X \times X \qquad \delta_{\overline{X}} : \overline{X} \to \overline{X} \times \overline{X}$$

denote the diagonal maps, and let $\pi_1 : \overline{X} \times \overline{X} \to \overline{X}$ be the projection onto the first factor. Using Proposition 4.6.7, we obtain a canonical equivalence $\pi_2^! \, \mathcal{G} \simeq \omega_{\overline{X}} \boxtimes \mathcal{G}$ for each object $\mathcal{G} \in \mathrm{Shv}_\ell(\overline{X})$. Applying the functor $\delta_{\overline{X}}^!$, we obtain an equivalence

$$\omega_{\overline{X}} \otimes^! \mathcal{G} \simeq \delta_{\overline{X}}^! (\omega_{\overline{X}} \boxtimes \mathcal{G}) \simeq \delta_{\overline{X}}^! \pi_2^! \, \mathcal{G} \simeq \mathcal{G}\,.$$

For any object $\mathcal{F} \in \mathrm{Shv}_\ell^!(X)$, we have canonical equivalences

$$\begin{aligned} \mathcal{F} &\simeq j^* j_* \mathcal{F} \\ &\simeq j^*(\omega_{\overline{X}} \otimes^! j_* \mathcal{F}) \\ &\simeq j^* \delta_{\overline{X}}^! (\omega_{\overline{X}} \boxtimes j_* \mathcal{F}) \\ &\overset{\alpha}{\simeq} \delta_X^! (j \times j)^* (\omega_{\overline{X}} \boxtimes j_* \mathcal{F}) \\ &\simeq \delta_X^! (j^* \omega_{\overline{X}} \boxtimes j^* j_* \mathcal{F}) \\ &\simeq \delta_X^! (j^* \omega_{\overline{X}} \boxtimes \mathcal{F}) \\ &= j^* \omega_{\overline{X}} \otimes^! \mathcal{F}, \end{aligned}$$

where the equivalence α is obtained by applying Proposition 4.5.12 to the pullback square

$$\begin{array}{ccc} X & \xrightarrow{\delta_X} & X \times X \\ \downarrow & & \downarrow{\scriptstyle j \times j} \\ \overline{X} & \xrightarrow{\delta_{\overline{X}}} & \overline{X} \times \overline{X}. \end{array}$$

It follows that $j^*\omega_{\overline{X}}$ is a unit for the tensor product $\otimes^!$. $\square$

Remark 4.6.14. In the situation of Proposition 4.6.13, the object $\mathcal{E} \in \mathrm{Shv}_\ell(X)$ is determined uniquely up to equivalence. To see this, we note that if $\mathcal{E}$ and $\mathcal{E}'$ are two objects of $\mathrm{Shv}_\ell(X)$ which satisfy the requirements of Proposition 4.6.13, then we have equivalences

$$\mathcal{E}' \simeq \mathcal{E} \otimes^! \mathcal{E}' \simeq \mathcal{E}' \otimes^! \mathcal{E} \simeq \mathcal{E}\,.$$

Notation 4.6.15. Let X be a quasi-projective k-scheme. We let ω_X denote a unit with respect to the !-tensor product on $\mathrm{Shv}_\ell(X)$. We will refer to ω_X as the *dualizing sheaf* on X. The proof of Proposition 4.6.13 shows that this notation is compatible with Definition 4.5.21 in the special case where X is projective.

Remark 4.6.16. Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Using the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

and Corollary 4.6.8, we obtain for every pair of ℓ-adic sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(Y)$ a chain of equivalences

$$\begin{aligned} f^!(\mathcal{F} \otimes^! \mathcal{G}) &= f^! \delta_Y^! (\mathcal{F} \boxtimes \mathcal{G}) \\ &\simeq \delta_X^! (f \times f)^! (\mathcal{F} \boxtimes \mathcal{G}) \\ &\simeq \delta_X^! (f^! \mathcal{F} \boxtimes f^! \mathcal{G}) \\ &\simeq (f^! \mathcal{F}) \otimes^! (f^! \mathcal{G}). \end{aligned}$$

In fact, much more is true: one can refine $f^!$ to a symmetric monoidal functor between the ∞-categories $\mathrm{Shv}_\ell(Y)$ and $\mathrm{Shv}_\ell(X)$; see Corollary 5.5.22.

Let X be a quasi-projective k-scheme. The !-tensor product functor $\otimes^!$ factors as a composition

$$\mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \xrightarrow{\boxtimes} \mathrm{Shv}_\ell(X \times X) \xrightarrow{\delta^!} \mathrm{Shv}_\ell(X).$$

The functor $\delta^!$ commutes with all limits (because it is a right adjoint), but the external tensor product functor $\boxtimes$ does not commute with limits in general. Nevertheless, the !-tensor product $\otimes^!$ can be shown to commute with limits in good cases. For later reference, we record one result to this effect:

Proposition 4.6.17. *Let X and Y be quasi-projective k-schemes. Suppose that $\mathcal{F}^\bullet$ is a cosimplicial object of $\mathrm{Shv}_\ell(X)_{\leq 0}$ and that $\mathcal{G}^\bullet$ is a cosimplicial object of $\mathrm{Shv}_\ell(Y)_{\leq 0}$. Then the canonical map*

$$\mathrm{Tot}(\mathcal{F}^\bullet) \boxtimes \mathrm{Tot}(\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$$

is an equivalence in $\mathrm{Shv}_\ell(X \times Y)$.

Lemma 4.6.18. *Let X be a quasi-projective k-scheme and let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(X)_{\leq 0}$. Then $\mathcal{F} \otimes \mathcal{G} \in \mathrm{Shv}_\ell(X)_{\leq 2}$.*

Remark 4.6.19. With more effort, one can show that the tensor product functor carries $\mathrm{Shv}_\ell(X)_{\leq 0} \times \mathrm{Shv}_\ell(X)_{\leq 0}$ into $\mathrm{Shv}_\ell(X)_{\leq 1}$, but Lemma 4.6.18 will be sufficient for our purposes.

Remark 4.6.20. Let X be a quasi-projective k-scheme and let Λ be a field. Then the tensor product functor $\otimes : \mathrm{Shv}(X; \Lambda) \times \mathrm{Shv}(X; \Lambda) \to \mathrm{Shv}(X; \Lambda)$ is left t-exact: that is, it carries $\mathrm{Shv}(X; \Lambda)_{\leq 0} \times \mathrm{Shv}(X; \Lambda)_{\leq 0}$ into $\mathrm{Shv}(X; \Lambda)_{\leq 0}$. This follows from Remark 4.1.15, since the tensor product $\otimes_\Lambda : \mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda$ carries $(\mathrm{Mod}_\Lambda)_{\leq 0} \times (\mathrm{Mod}_\Lambda)_{\leq 0}$ into $(\mathrm{Mod}_\Lambda)_{\leq 0}$.

Proof of Lemma 4.6.18. Since $\mathrm{Shv}_\ell(X)_{\leq 1}$ is closed under filtered colimits and the tensor product $\otimes$ preserves filtered colimits separately in each variable, we may assume without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are constructible, so that $\mathcal{F} \otimes \mathcal{G}$ is likewise constructible. Set $\mathcal{F}_1 = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ and $\mathcal{G}_1 = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{G}$. Using Remark 4.4.6 we see that $\mathcal{F}_1, \mathcal{G}_1 \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq 1}$. Using Remark 4.6.20, we conclude that the tensor product

$$(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{F}_1 \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \mathcal{G}_1$$

belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq 2}$, so that $\mathcal{F} \otimes \mathcal{G}$ belongs to $\mathrm{Shv}^c_\ell(X)_{\leq 2}$ by Remark 4.4.6. □

Lemma 4.6.21. *Let X be a quasi-projective k-scheme, let $\mathcal{F} \in \mathrm{Shv}_\ell(X)_{\leq 0}$, and let $\mathcal{G}^\bullet$ be a cosimplicial object of $\mathrm{Shv}_\ell(X)_{\leq 0}$. Then the canonical map*

$$\theta : \mathcal{F} \otimes \mathrm{Tot}(\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F} \otimes \mathcal{G}^\bullet)$$

is an equivalence in $\mathrm{Shv}_\ell(X)$.

Proof. For each $n \geq 0$, let $\mathrm{Tot}^n(\mathcal{G}^\bullet)$ denote the nth stage of the Tot-tower of $\mathcal{G}^\bullet$ (that is, the limit of the restriction of $\mathcal{G}$ to the category $\mathbf{\Delta}_{\leq n}$ of simplices of dimension $\leq n$). The construction $\mathcal{G}^\bullet \mapsto \mathrm{Tot}^n(\mathcal{G}^\bullet)$ is given by a finite limit, and therefore commutes with any exact functor. It follows that θ can be identified with the composition

$$\begin{array}{rcl} \mathcal{F} \otimes \mathrm{Tot}(\mathcal{G}^\bullet) & \simeq & \mathcal{F} \otimes \varprojlim \mathrm{Tot}^n(\mathcal{G}^\bullet) \\ & \xrightarrow{\theta'} & \varprojlim (\mathcal{F} \otimes \mathrm{Tot}^n(\mathcal{G}^\bullet)) \\ & \simeq & \varprojlim \mathrm{Tot}^n(\mathcal{F} \otimes \mathcal{G}^\bullet) \\ & \simeq & \mathrm{Tot}(\mathcal{F} \otimes \mathcal{G}^\bullet). \end{array}$$

We are therefore reduced to proving that θ' is an equivalence. Since $\mathrm{Shv}_\ell(X)$ is right complete, it will suffice to show that the fiber of θ' belongs to $\mathrm{Shv}_\ell(X)_{\leq -m}$ for each integer m. For $n \geq m+2$, let $\mathcal{H}_n$ denote the cofiber of the natural map $\mathrm{Tot}^n(\mathcal{G}^\bullet) \to \mathrm{Tot}^{m+2}(\mathcal{G}^\bullet)$, so that we have a pushout square

$$\begin{array}{ccc} \mathcal{F} \otimes \varprojlim \mathrm{Tot}^n(\mathcal{G}^\bullet) & \xrightarrow{\theta'} & \varprojlim_{n' \geq n} (\mathcal{F} \otimes \mathrm{Tot}^n(\mathcal{G}^\bullet)) \\ \downarrow & & \downarrow \\ \mathcal{F} \otimes \varprojlim \mathcal{H}_n & \xrightarrow{\theta''} & \varprojlim \mathcal{F} \otimes \mathcal{H}_n \end{array}$$

Since each $\mathcal{G}^q$ belongs to $\mathrm{Shv}_\ell(X)_{\leq 0}$, the cofibers $\mathcal{H}_n$ belong to $\mathrm{Shv}_\ell(X)_{\leq -m-2}$. Using Lemma 4.6.18, we deduce that the domain and codomain of θ'' belong to $\mathrm{Shv}_\ell(X)_{\leq -m}$, so that $\mathrm{fib}(\theta') \simeq \mathrm{fib}(\theta'')$ belongs to $\mathrm{Shv}_\ell(X)_{\leq -m}$ as desired. □

Proof of Proposition 4.6.17. Embedding X and Y into projective space, we may assume without loss of generality that X and Y are smooth. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ denote the projection maps onto the first and second factor, respectively. Unwinding the definitions, we wish to show that the composite map

$$\begin{array}{rcl} \mathrm{Tot}(\mathcal{F}^\bullet) \boxtimes \mathrm{Tot}(\mathcal{G}^\bullet) & \simeq & p^* \mathrm{Tot}(\mathcal{F}^\bullet) \otimes q^* \mathrm{Tot}(\mathcal{G}^\bullet) \\ & \xrightarrow{\theta} & \mathrm{Tot}(p^* \mathcal{F}^\bullet) \otimes \mathrm{Tot}(q^* \mathcal{G}^\bullet) \\ & \xrightarrow{\theta'} & \mathrm{Tot}(p^* \mathcal{F}^\bullet \otimes \mathrm{Tot}(q^* \mathcal{G}^\bullet)) \\ & \xrightarrow{\theta''} & \mathrm{Tot}(\mathrm{Tot}(p^* \mathcal{F}^\bullet \otimes q^* \mathcal{G}^\bullet)) \\ & \simeq & \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet) \end{array}$$

is an equivalence. The map θ is an equivalence by Proposition 4.3.39, and the maps θ' and θ'' are equivalences by virtue of Lemma 4.6.21. □

5. The Product Formula

Let k be an algebraically closed field, let ℓ be a prime number which is invertible in k, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. In §3, we proved that if the generic fiber of G is semisimple and simply connected, then the forgetful functor $\mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$ induces an isomorphism of ℓ-adic homology groups

$$\mathrm{H}_*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell) \to \mathrm{H}_*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

(Corollary 3.2.12). We can regard this as a sort of local-to-global principle for computing the ℓ-adic (co)homology of the moduli stack $\mathrm{Bun}_G(X)$: rather than contemplating arbitrary G-bundles on X, it suffices to consider G-bundles which are "supported" on finite subsets of X. In the special case where X and G are defined over some finite subfield $\mathbf{F}_q \subseteq k$, we would like to use this principle to compute the trace of the (arithmetic) Frobenius automorphism of $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$. For this, it is useful to contemplate the diagram of prestacks

$$\begin{array}{ccccc} & & \mathrm{Ran}_G(X) & & \\ & \swarrow^{\phi} & & \searrow^{\psi} & \\ \mathrm{Bun}_G(X) & & & & \mathrm{Ran}(X). \end{array}$$

For purposes of motivation, let us suppose that the formalism of ℓ-adic sheaves has been extended to the setting of prestacks, and let $\underline{\mathbf{Z}}_\ell$ denote the constant ℓ-adic sheaf on $\mathrm{Ran}_G(X)$. Then Corollary 3.2.12 supplies isomorphisms

$$\begin{array}{rcl} \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) & \xrightarrow{\phi^*} & \mathrm{H}^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell) \\ & \simeq & \mathrm{H}^*(\mathrm{Ran}(X); \mathcal{A}), \end{array}$$

where $\mathcal{A}$ denotes the ℓ-adic sheaf on $\mathrm{Ran}(X)$ given by $\psi_* \underline{\mathbf{Z}}_\ell$.

To understand the structure of the sheaf $\mathcal{A}$, we need to understand the map $\psi : \mathrm{Ran}_G(X) \to \mathrm{Ran}(X)$. Let us begin by describing the fibers of π. Let Gr_G denote the fiber product

$$\mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} X \simeq \mathrm{Ran}_G(X) \times_{\mathrm{Fin}^s} \{\langle 1 \rangle\}.$$

We will refer to Gr_G as the *affine Grassmannian* of G. Unwinding the definitions, we see that Gr_G is a prestack whose R-valued points can be identified with triples $(s, \mathcal{P}, \gamma)$ where $s \in X(R)$, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over the open set $X_R - |s|$, where $|s|$ denotes the image of the map $\mathrm{Spec}\, R \to X_R$ determined by s. The construction $(s, \mathcal{P}, \gamma) \mapsto s$

determines a morphism of prestacks $\mathrm{Gr}_G \to X$. For each k-valued point $x \in X(k)$, we let Gr_G^x denote the fiber product $\mathrm{Gr}_G \times_X \{x\}$.

Fix a point $x \in X(k)$. Let $\mathcal{O}_x$ denote the completed local ring of X at the point x, and let K_x denote its fraction field. Unwinding the definitions, we see that the fiber product $\mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} \mathrm{Spec}\, k$ is a prestack whose k-points are given by pairs $(\mathcal{P}, \gamma)$, where $\mathcal{P}$ is a G-bundle on X and γ is a trivialization of $\mathcal{P}$ over the open set $X - \{x\}$. Since k is algebraically closed and G is smooth, we can always choose a trivialization of $\mathcal{P}$ over the formal completion of X at x. It follows that $\mathcal{P}$ is obtained by "regluing": that is, it can be obtained by gluing the G-bundle on $X - \{x\}$ to the trivial G-bundle on $\mathrm{Spec}\, \mathcal{O}_x$ using some isomorphism over $\mathrm{Spec}\, K_x$, which we can identify with an element of $G(K_x)$. Here the trivialization of $\mathcal{P}$ over $X - \{x\}$ is given as part of the data, but we are free to modify the trivialization on $\mathrm{Spec}\, \mathcal{O}_x$: consequently, the k-points of the fiber $\mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} \mathrm{Spec}\, k$ can be identified with the quotient $G(K_x)/G(\mathcal{O}_x)$.

We will denote the fiber product $\mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} \mathrm{Spec}\, k$ by Gr_G^x, and refer to it as the *affine Grassmannian* of G at the point x. It is generally not representable by a scheme, but one can show that it is an Ind-scheme: more precisely, it can be written as a the direct limit of a sequence

$$Y(0) \hookrightarrow Y(1) \hookrightarrow Y(2) \hookrightarrow \cdots$$

of quasi-projective k-schemes, where each of the morphisms is a closed embedding. If G is reductive, then each of the k-schemes $Y(m)$ is actually projective. In fact, even more is true: if G is reductive, then the projection map $\psi : \mathrm{Ran}_G(X) \to \mathrm{Ran}(X)$ itself is Ind-proper, so that one has base change and Künneth equivalences

$$\mathcal{A}_{(S)} \xrightarrow{\sim} C^*(\mathrm{Gr}_G^S; \mathbf{Z}_\ell) \xleftarrow{\sim} \bigotimes_{x \in S} C^*(\mathrm{Gr}_G^x; \mathbf{Z}_\ell).$$

We may therefore regard the chain complex $C^*(\mathrm{Ran}(X); \mathcal{A})$ as a sort of continuous tensor product of the chain complexes $C^*(\mathrm{Gr}_G^x; \mathbf{Z}_\ell)$, so that Corollary 3.2.12 supplies a version of the equivalence

$$\bigotimes_{x \in X} C^*(\mathrm{Gr}_G^x; \mathbf{Z}_\ell) \simeq C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

appearing in Example 1.4.11.

Remark 5.0.1. Suppose that k is the field $\mathbf{C}$ of complex numbers, and that G is the split reductive group scheme over $\mathbf{C}$ corresponding to a reductive algebraic group G_0 over $\mathbf{C}$, so that $G_0(\mathbf{C})$ is a complex Lie group. In this case, we can view the $\mathbf{C}$-points of the affine Grassmannian of G (at any chosen point $x \in X$) as a topological space, which is given (as a set) by the quotient $G_0(\mathbf{C}((t)))/G_0(\mathbf{C}[[t]])$. Here $G_0(\mathbf{C}((t)))$ has the homotopy type of the free loop space of $G_0(\mathbf{C})$, while $G_0(\mathbf{C}[[t]])$ has the homotopy type of $G_0(\mathbf{C})$ itself. The quotient $\mathrm{Gr}_{G,x}$ has the homotopy type of the based loop space $\Omega G_0(\mathbf{C}) \simeq \Omega^2 \mathrm{B}\, G_0(\mathbf{C})$.

Recall that if Y is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell^c(Y)$ is a constructible ℓ-adic sheaf on Y, and the pair $(Y, \mathcal{F})$ is defined over a finite field $\mathbf{F}_q \subseteq k$, then the trace of the geometric Frobenius automorphism Frob on the compactly supported cohomology $\mathrm{H}_c^*(Y; \mathcal{F})$ can be computed using the Grothendieck-Lefschetz trace formula

$$\mathrm{Tr}(\mathrm{Frob} \,|\, \mathrm{H}_c^*(Y; \mathcal{F})) = \sum_{y \in Y(\mathbf{F}_q)} \mathrm{Tr}(\mathrm{Frob} \,|\, \mathrm{H}^*(\mathcal{F}_y))$$

Taking $\mathcal{F}$ to be the Verdier dual of another ℓ-adic sheaf $\mathcal{G}$, we can rewrite this formula as

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(Y; \mathcal{G})) = \sum_{y \in Y(\mathbf{F}_q)} \mathrm{Tr}(\mathrm{Frob} \,|\, \mathrm{H}^*(\mathbf{D}(\mathcal{G})_y)). \tag{10}$$

Roughly speaking, we would like to prove Theorem 1.3.5 by applying a version of this formula where Y is replaced by the Ran space $\mathrm{Ran}(X)$, and $\mathcal{G}$ is replaced by the ℓ-adic sheaf $\mathcal{A}$.

Unfortunately, things are not so simple. The Ran space $\mathrm{Ran}(X)$ is an infinite-dimensional algebro-geometric object and the sheaf $\mathcal{A}$ is not constructible, so the usual theory of Verdier duality is not directly applicable. To address this difficulty, we will need to modify the notion of sheaf. In §5.2, we will define ∞-category $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ whose objects we refer to as *!-sheaves on* $\mathrm{Ran}(X)$. Roughly speaking, an object $\mathcal{F}$ of the ∞-category $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ consists of a collection of ℓ-adic sheaves $\{\mathcal{F}^{(T)} \in \mathrm{Shv}_\ell(X^T)\}$ (here T ranges over all nonempty finite sets), equipped with a coherent family of identifications $\delta^!_{T/T'}\,\mathcal{F}^{(T)} \simeq \mathcal{F}^{(T')}$ (where $\delta_{T/T'} : X^{T'} \to X^T$ denotes the diagonal map associated to a surjection of finite sets $T \to T'$). Heuristically, we can think of $\mathcal{F}$ as a sheaf on $\mathrm{Ran}(X)$ which is generated by its compactly supported sections, and $\mathcal{F}^{(T)}$ as the !-restriction of $\mathcal{F}$ along the map $X^T \to \mathrm{Ran}(X)$. In particular, if T is given as a finite subset of $X(k)$, then we have a canonical point $\eta : \operatorname{Spec} k \to X^T$, and we can consider the *costalk* $\mathcal{F}_{(T)} = \eta^!\,\mathcal{F}^{(T)} \in \mathrm{Mod}_{\mathbf{Z}_\ell}$.

If $\pi : Z \to Y$ is a map of quasi-projective k-schemes, then we let $[Z]_Y$ denote the ℓ-adic sheaf given by $\pi_*\pi^*\omega_Y$. In §5.1 we will extend the definition of $[Z]_Y$ to the case where Z is an arbitrary prestack. In §5.4, we apply this general constructions to produce a !-sheaf $\mathcal{B} \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$, whose costalks at a point $\nu : T \to X(k)$ is given by

$$\nu^!\,\mathcal{B}^{(T)} = \bigotimes_{x\in\nu(T)} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell).$$

The !-sheaf $\mathcal{B}$ can be regarded as a sort of Koszul dual to $\mathcal{A}$ (more precisely, we will prove in §9 that certain "reduced" versions of $\mathcal{A}$ and $\mathcal{B}$ differ by a covariant version of Verdier duality). In §5.4, we will construct a canonical map

$$\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell);$$

here $\int \mathcal{B}$ denotes the complex of compactly supported cochains on $\mathrm{Ran}(X)$ with coefficients in $\mathcal{B}$ (or the *chiral homology* of $\mathcal{B}$), which we will study in §5.3. The second main result of this paper (Theorem 5.4.5) asserts that the map ρ is a quasi-isomorphism.

Suppose that T is a union of nonempty finite sets T' and T'', and that $\nu : T \to X(k)$ has the property that $\nu(T')$ and $\nu(T'')$ are disjoint. Then we have a canonical equivalence

$$(\nu|_{T'})^!\,\mathcal{B}^{(T')} \otimes (\nu|_{T''})^!\,\mathcal{B}^{(T'')} \simeq \nu^!\,\mathcal{B}^{(T)}$$

This equivalence behaves well as ν varies: in fact, it is given by a map of ℓ-adic sheaves

$$\mathcal{B}^{(T')} \boxtimes \mathcal{B}^{(T'')} \to \mathcal{B}^{(T)}.$$

Letting T' and T'' vary, we can view these maps as defining a multiplication

$$m : \mathcal{B} \star \mathcal{B} \to \mathcal{B},$$

where $\star$ denotes the *convolution product* on !-sheaves which we study in §5.5. We will show that this multiplication $\mathcal{B}$ with the structure of a *commutative factorization algebra*, so that $\mathcal{B}$ can be functorially recovered from the ℓ-adic sheaf $\mathcal{B}^{(1)} = [\mathrm{BG}]_X$ (Theorem 5.6.4); in concrete terms, the costalk of $\mathcal{B}$ at a point given *injective* map $\nu : T \to X(k)$ is given by $\nu^!\,\mathcal{B}^{(T)} \bigotimes_{t\in T} \nu(t)^!\,\mathcal{B}^{(1)}$. In §5.7, we will use this observation to give a reformulation of Theorem 5.4.5 which expresses the cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ as a "continuous tensor product" $\bigotimes_{x\in X} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ (Theorem 5.7.1); compare with Theorem 1.4.9.

5.1. **The Cohomology Sheaf of a Morphism.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let X be a quasi-projective k-scheme and let ω_X denote its dualizing sheaf (Notation 4.6.15). Given any morphism $f : Y \to X$ of quasi-projective k-schemes, we let $[Y]_X \in \mathrm{Shv}_\ell(X)$ denote the ℓ-adic sheaf given by $f_* f^* \omega_X$. We will refer to the ℓ-adic sheaf $[Y]_X$ as the *cohomology sheaf of the morphism* f.

Remark 5.1.1. We will primarily be interested in the construction $Y \mapsto [Y]_X$ in the special case where Y is *smooth* over X. In this case, for any proper morphism of quasi-projective k-schemes $g : X' \to X$, Variant 4.5.5 and Proposition 4.5.12 supply an equivalence of ℓ-adic sheaves

$$[Y \times_X X']_{X'} \simeq g^! [Y]_X.$$

Taking $X' = \operatorname{Spec} k$, we obtain the following informal description of $[Y]_X$: it is the ℓ-adic sheaf whose costalk at a point $\eta \in X(k)$ can be identified with the cochain complex $C^*(Y_\eta; \mathbf{Z}_\ell)$, where Y_η denotes the fiber $Y \times_X \operatorname{Spec} k$ of f over the point η.

Our goal in this section is to generalize the construction $Y \mapsto [Y]_X$ to the case where Y is a prestack. For the purpose of setting up the definitions, it will be convenient to consider a further generalization which depends on a choice of ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$.

Construction 5.1.2. Let X be a quasi-projective k-scheme and let $\mathcal{C}$ be a prestack equipped with a map $\pi : \mathcal{C} \to X$. For each object $\eta \in \mathcal{C}$, we let R_η denote its image in Ring_k, so that π determines a map of k-schemes $\operatorname{Spec} R_\eta \to X$ which we will (by abuse of notation) denote by η.

For each ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we let $[\mathcal{C}]_{\mathcal{F}}$ denote the inverse limit

$$\varprojlim_{\eta \in \mathcal{C}} \eta_* \eta^* \mathcal{F} \in \mathrm{Shv}_\ell(X).$$

Example 5.1.3. Let $\pi : Y \to X$ be a morphism of quasi-projective k-schemes. For each object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we can identify $[Y]_{\mathcal{F}}$ with the pushforward $\pi_* \pi^* \mathcal{F}$. In particular, if $\mathcal{F} \in \mathrm{Shv}_\ell(X)_{\leq n}$ for some integer n, then $[Y]_{\mathcal{F}} \in \mathrm{Shv}_\ell(X)_{\leq n}$.

Notation 5.1.4. Let X be a quasi-projective k-scheme and let $\mathcal{C}$ be a prestack equipped with a map $\pi : \mathcal{C} \to X$. We let $[\mathcal{C}]_X$ denote the sheaf $[\mathcal{C}]_{\omega_X} \in \mathrm{Shv}_\ell(X)$, where ω_X is the dualizing sheaf of X.

Example 5.1.5. Let $X = \operatorname{Spec} k$. Then for every prestack $\mathcal{C}$, we have a canonical equivalence

$$[\mathcal{C}]_X \simeq C^*(\mathcal{C}; \mathbf{Z}_\ell).$$

Remark 5.1.6. Let X be a quasi-projective k-scheme and let $\mathcal{C}$ be a prestack with a map $\mathcal{C} \to X$. Then the ℓ-adic sheaf $[\mathcal{C}]_X$ is ℓ-complete. When $\mathcal{C}$ is a quasi-projective k-scheme this follows from Remark 4.3.35 (since $[\mathcal{C}]_X$ is constructible), and the general case follows from the observation that the collection of ℓ-complete objects of $\mathrm{Shv}_\ell(X)$ is closed under limits.

Remark 5.1.7 (Functoriality)**.** Let X be a quasi-projective k-scheme, and suppose we are given a commutative diagram of prestacks

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

For every ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the morphism f induces a pullback map

$$f^* : [\mathcal{D}]_{\mathcal{F}} \to [\mathcal{C}]_{\mathcal{F}}.$$

We can summarize the situation informally by saying that the ℓ-adic sheaf $[\mathcal{C}]_{\mathcal{F}}$ depends functorially on $\mathcal{C}$. We will discuss this functoriality in more detail below and in §A.5.

Remark 5.1.8. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell(X)$. Let $\mathcal{C}$ be a prestack equipped with a map $\pi : \mathcal{C} \to X$. Suppose that $\mathcal{C}$ can be realized as a filtered colimit of prestacks $\{\mathcal{C}_\alpha\}$. Then the canonical map $[\mathcal{C}]_{\mathcal{F}} \to \varprojlim_\alpha [\mathcal{C}_\alpha]_{\mathcal{F}}$ is an equivalence in $\mathrm{Shv}_\ell(X)$.

We will typically be interested in the special case of Construction 5.1.2 where $\mathcal{C}$ is an Artin stack. In this case, we do not need to use the *entire* category $\mathcal{C}$ to compute the limit $[\mathcal{C}]_{\mathcal{F}} = \varprojlim_{\eta \in \mathcal{C}} \eta_* \eta^* \mathcal{F}$.

We now establish a generalization of Remark 5.1.1:

Proposition 5.1.9. *Let $f : X' \to X$ be a proper morphism of quasi-projective k-schemes, let $\mathcal{C}$ be an Artin stack with affine diagonal which is equipped with a morphism $\pi : \mathcal{C} \to X$, and let $\mathcal{C}' = \mathcal{C} \times_X X'$. Then for every ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, there is a canonical map $[\mathcal{C}']_{f^! \mathcal{F}} \to f^! [\mathcal{C}]_{\mathcal{F}}$, which is an equivalence when π is smooth. In particular (taking $\mathcal{F} = \omega_X$), when π is smooth there is a canonical equivalence $[\mathcal{C}']_{X'} \simeq f^! [\mathcal{C}]_X$.*

Remark 5.1.10. The assumption that $\mathcal{C}$ have affine diagonal is not really needed; however, it is satisfied in all of our applications and allows for a slightly simpler proof. Similarly, the hypothesis that f is proper can be removed given a more robust theory of the exceptional inverse image functor $f^!$.

Lemma 5.1.11. *Let X be a quasi-projective k-scheme and let $\mathcal{C}$ be an Artin stack with affine diagonal equipped with a map $\mathcal{C} \to X$. Let U_0 be a quasi-projective k-scheme equipped with a surjective map $U_0 \to \mathcal{C}$, and let $U_\bullet$ be the simplicial scheme given by the iterated fiber powers of U_0 over $\mathcal{C}$. For every object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the canonical map*

$$[\mathcal{C}]_{\mathcal{F}} \simeq \mathrm{Tot}[U_\bullet]_{\mathcal{F}}$$

is an equivalence in $\mathrm{Shv}_\ell(X)$.

Proof. For each object $\eta \in \mathcal{C}$, we can identify η with a map of prestacks $\operatorname{Spec} R_\eta \to \mathcal{C}$. Then $U_\bullet$ can be identified with a cosimplicial object of $\mathcal{C}$, given by a map $\rho : \mathbf{\Delta} \to \mathcal{C}$. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the full subcategory spanned by those objects $\eta \in \mathcal{C}$ for which the map $\operatorname{Spec} R_\eta \to \mathcal{C}$ factors through U_0. Note for each object $\eta \in \mathcal{C}$, the fiber product $\mathbf{\Delta} \times_{\mathcal{C}} \mathcal{C}_{/\eta}$ is empty if $\eta \notin \mathcal{C}_0$, and weakly contractible otherwise. It follows that ρ induces a right cofinal map $\mathbf{\Delta} \to \mathcal{C}_0$. For any quasi-projective k-scheme X, any morphism $\pi : \mathcal{C} \to X$, and any object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we can regard $[U_\bullet]_{\mathcal{F}}$ as a cosimplicial object of $\mathrm{Shv}_\ell(X)$, whose totalization is equivalent to $[\mathcal{C}_0]_{\mathcal{F}}$. The desired result now follows from the observation that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ induces an equivalence after étale sheafication (by virtue of our assumption that the map $U_0 \to \mathcal{C}$ is surjective). □

Proof of Proposition 5.1.9. Elementary considerations of functoriality supply a natural comparison map

$$[\mathcal{C}']_{f^! \mathcal{F}} \to f^! [\mathcal{C}]_{\mathcal{F}}$$

(see §A.5 for a detailed discussion); we will prove that this map is an equivalence. Writing $\mathcal{C}$ as a union of quasi-compact open substacks, we can reduce to the case where $\mathcal{C}$ is quasi-compact (see Remark 5.1.8). Choose a smooth surjection $\rho : U_0 \to \mathcal{C}$, where V_0 is an affine scheme. Let

$U_\bullet$ be the simplicial (affine) scheme given by the nerve of ρ. For each integer i, let $\pi(i)$ denote the composite map $V_i \to \mathcal{C} \xrightarrow{\pi} X$, and form a pullback diagram

$$\begin{array}{ccc} U'_i & \xrightarrow{g_i} & U_i \\ \downarrow{\scriptstyle \pi'(i)} & & \downarrow{\scriptstyle \pi(i)} \\ X' & \xrightarrow{f} & X. \end{array}$$

Note that each of the maps $\pi(i)$ is smooth. We may therefore identify θ with the composition

$$\begin{aligned} f^![\mathcal{C}]_{\mathcal{F}} &\simeq f^! \varprojlim_{[i]\in\mathbf{\Delta}} \pi(i)_*\pi(i)^*\,\mathcal{F} \\ &\simeq \varprojlim_{[i]\in\mathbf{\Delta}} f^!\pi(i)_*\pi(i)^*\,\mathcal{F} \\ &\simeq \varprojlim_{[i]\in\mathbf{\Delta}} \pi'(i)_*g_i^!\pi(i)^*\,\mathcal{F} \\ &\simeq \varprojlim_{[i]\in\mathbf{\Delta}} \pi'(i)_*\pi'(i)^*f^!\,\mathcal{F} \\ &\simeq [\mathcal{C}']_{f^!\,\mathcal{F}} \end{aligned}$$

of equivalences supplied by Proposition 4.5.12 and Remark 5.1.11. □

We will also need a slight variation on Proposition 5.1.9.

Definition 5.1.12. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of prestacks in groupoids. We will say that f is an *open immersion* if the following conditions is satisfied: for every quasi-projective k-scheme X and every map $X \to \mathcal{D}$, the fiber product $\mathcal{C} \times_{\mathcal{D}} X$ is representable by an open subscheme $U \subseteq X$.

Proposition 5.1.13. *Let $f : Y' \to Y$ be a proper morphism between quasi-projective k-schemes, let $\pi : \mathcal{D} \to \mathrm{Ring}_k$ be a prestack in groupoids, let $j : \mathcal{C} \to Y \times_{\mathrm{Spec}\, k} \mathcal{D}$ be an open immersion, and form a pullback diagram*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y. \end{array}$$

Then for every object $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, the canonical map $u : [\mathcal{C}']_{f^!\,\mathcal{F}} \to f^![\mathcal{C}]_{\mathcal{F}}$ is an equivalence in $\mathrm{Shv}_\ell(Y')$. In particular, we have an equivalence $[\mathcal{C}']_{Y'} \simeq f^![\mathcal{C}]_Y$.

Remark 5.1.14. Propsition 5.1.13 is valid more generally if the map $j : \mathcal{C} \to Y \times_{\mathrm{Spec}\, k} \mathcal{D}$ is a smooth relative Artin stack; similarly, the hypothesis that the morphism f be proper can be removed given a more general theory of the exceptional inverse image functor $f^!$.

Lemma 5.1.15. *Let $f : Y' \to Y$ be a proper morphism of quasi-projective k-schemes, let Z be a quasi-projective k-scheme, let U be an open subset of $Y \times Z$, and form a pullback diagram*

$$\begin{array}{ccc} U' & \xrightarrow{f'} & U \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y. \end{array}$$

Then, for every object $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, the natural map $\gamma : \pi'^* f^!\,\mathcal{F} \to f'^! \pi^*\,\mathcal{F}$ is an equivalence in $\mathrm{Shv}_\ell(U')$.

Proof. The assertion is local on U (see Proposition 4.5.12). Enlarging U, we may assume without loss of generality that $U = Y \times Z$, in which case the desired result is a special case of Proposition 4.6.7. □

Proof of Proposition 5.1.13. For each object $D \in \mathcal{D}$, let U_D denote the open subscheme of $Y \times_{\operatorname{Spec} k} \operatorname{Spec} \pi(D)$ given by the fiber product $\operatorname{Spec} \pi(D) \times_{\mathcal{D}} \mathcal{C}$, and form a pullback square

$$\begin{array}{ccc} U'_D & \longrightarrow & U_D \\ \downarrow{\scriptstyle \pi'_D} & & \downarrow{\scriptstyle \pi_D} \\ Y' & \xrightarrow{f} & Y. \end{array}$$

A simple cofinality argument shows that u can be identified with a limit of maps

$$u_D : \pi'_{D*} \pi'^{*}_{D} f^{!} \mathcal{F} \to f^{!} \pi_{D*} \pi^{*}_{D} \mathcal{F}.$$

The desired result now follows by combining Variant 4.5.7 with Lemma 5.1.15. □

Example 5.1.16. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes and let $\mathcal{C}$ be an arbitrary prestack. Then, for each $\mathcal{F} \in \operatorname{Shv}_\ell(Y)$, the canonical map $\theta : [\mathcal{C} \times_{\operatorname{Spec} k} X]_{f^! \mathcal{F}} \to f^! [\mathcal{C} \times_{\operatorname{Spec} k} Y]_{\mathcal{F}}$ is an equivalence in $\operatorname{Shv}_\ell(X)$.

Example 5.1.17. Let Y be a projective k-scheme, let X be a quasi-projective k-scheme, and let $\mathcal{C}$ be a prestack. Applying Example 5.1.16 to the projection map $X \times Y \to X$, we obtain an equivalence

$$[\mathcal{C} \times_{\operatorname{Spec} k} (X \times Y)]_{X \times Y} \simeq [\mathcal{C} \times_{\operatorname{Spec} k} X]_X \boxtimes \omega_Y.$$

In particular, if $X = \operatorname{Spec} k$, we obtain an equivalence

$$[\mathcal{C} \times_{\operatorname{Spec} k} Y]_Y \simeq C^*(\mathcal{C}; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \omega_Y.$$

We conclude this section with an elaboration on Remark 5.1.7 and Proposition 5.1.9, which supplies a more complete description of the functorial dependence of the ℓ-adic sheaf $[\mathcal{C}]_X$ on both X and $\mathcal{C}$.

Informal Definition 5.1.18. We define an ∞-category $\operatorname{Shv}^!_\ell$ informally as follows:

- The objects of $\operatorname{Shv}^!_\ell$ are pairs $(X, \mathcal{F})$, where X is a quasi-projective k-scheme and $\mathcal{F} \in \operatorname{Shv}_\ell(X)$ is an ℓ-adic sheaf on X.
- A morphism from $(X, \mathcal{F})$ to $(X', \mathcal{F}')$ in $\operatorname{Shv}^!_\ell$ consists of a proper morphism of quasi-projective k-schemes $f : X \to X'$ together with a map $f_* \mathcal{F} \to \mathcal{F}'$ of ℓ-adic sheaves on X' (or, equivalently, a map $\mathcal{F} \to f^! \mathcal{F}'$ of ℓ-adic sheaves on X).

Informal Definition 5.1.19. We define a 2-category $\operatorname{AlgStack}^!$ informally as follows:

- The objects of $\operatorname{AlgStack}^!$ are pairs $(X, \mathcal{C})$, where X is a quasi-projective k-scheme and $\mathcal{C}$ is a quasi-compact Artin stack with affine diagonal equipped with a smooth morphism $\mathcal{C} \to X$.
- A morphism from $(X, \mathcal{C})$ to $(X', \mathcal{C}')$ in $\operatorname{AlgStack}^!$ is a proper morphism of k-schemes $f : X \to X'$ together with a map $X \times_{X'} \mathcal{C}' \to \mathcal{C}$ of Artin stacks over X.

We regard $\operatorname{AlgStack}^!$ as a symmetric monoidal ∞-category with tensor product given by

$$(X, \mathcal{C}) \otimes (X', \mathcal{C}') = (X \times X', \mathcal{C} \times \mathcal{C}').$$

The following result refines Remark 5.1.7 and Proposition 5.1.9:

Proposition 5.1.20. *The construction $(X, \mathcal{C}) \mapsto (X, [\mathcal{C}]_X)$ determines a functor Φ from $\operatorname{AlgStack}^!$ to $\operatorname{Shv}^!_\ell$.*

For precise definitions of the ∞-categories $\mathrm{Shv}^!_\ell$ and $\mathrm{AlgStack}^!$ and a proof of Proposition 5.1.20, we refer the reader to §A.5.

5.2. !-Sheaves on $\mathrm{Ran}(X)$.

Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. If X is a quasi-projective k-scheme, we let $\mathrm{Ran}(X)$ denote the prestack introduced in Definition 2.4.9, so that the R-valued points of X can be identified with pairs (T, ν) where T is a nonempty finite set and $\nu : T \to X(R)$ is any map. Our goal in this section is to introduce the notion of a *!-sheaf* on $\mathrm{Ran}(X)$.

Definition 5.2.1. Let $\mathrm{Sch}^{\mathrm{pr}}_k$ denote the category whose objects are quasi-projective k-schemes and whose morphisms are proper maps, let $\mathrm{Shv}^!_\ell$ be the ∞-category of Definition 5.1.18, and let $\phi : \mathrm{Shv}^!_\ell \to \mathrm{Sch}^{\mathrm{pr}}_k$ be the forgetful functor given by $\phi(X, \mathcal{F}) = X$. If X is a quasi-projective k-scheme, then the construction $T \mapsto X^T$ determines a functor $\rho : \mathrm{Fin}^{\mathrm{sop}} \to \mathrm{Sch}^{\mathrm{pr}}_k$. We define a *lax !-sheaf* on $\mathrm{Ran}(X)$ to be a functor $\overline{\rho} : (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathrm{Sch}^{\mathrm{pr}}_k$ which fits into a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Shv}^!_\ell \\ & \overset{\mathcal{F}}{\nearrow} & \downarrow{\scriptstyle \phi} \\ (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} & \xrightarrow{\rho} & \mathrm{Sch}^{\mathrm{pr}}_k . \end{array}$$

We define a *!-sheaf* on $\mathrm{Ran}(X)$ to be a lax !-sheaf for which $\mathcal{F}$ carries each morphism in $\mathrm{Fin}^{\mathrm{s}}$ to a ϕ-Cartesian morphism in $\mathrm{Shv}^!_\ell$. We let $\mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$ denote the ∞-category $\mathrm{Fun}_{\mathrm{Sch}^{\mathrm{pr}}_k}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Shv}^!_\ell)$ whose objects are lax !-sheaves on $\mathrm{Ran}(X)$, and we let $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ denote the full subcategory of $\mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$ whose objects are !-sheaves on $\mathrm{Ran}(X)$.

Notation 5.2.2. Let X be a quasi-projective k-scheme, and let $\mathcal{F} \in \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$. For every nonempty finite set T, we let $\mathcal{F}^{(T)} \in \mathrm{Shv}_\ell(X^T)$ denote the ℓ-adic sheaf obtained by applying $\mathcal{F}$ to T. If $T = \{1, \ldots, n\}$ for some positive integer n, we will denote $\mathcal{F}^{(T)}$ by $\mathcal{F}^{(n)}$.

Remark 5.2.3. More informally, a lax !-sheaf on $\mathrm{Ran}(X)$ is given by the following:

- (a) For every nonempty finite set T, an ℓ-adic sheaf $\mathcal{F}^{(T)} \in \mathrm{Shv}_\ell(X^T)$.
- (b) For every surjection of nonempty finite sets $\pi : T \to T'$, with associated diagonal map $\delta_{T/T'} : X^{T'} \to X^T$, a morphism $\alpha_\pi : \mathcal{F}^{(T')} \to \delta^!_{T/T'} \mathcal{F}^{(T)}$ of ℓ-adic sheaves on $X^{T'}$ (or, equivalently, a morphism $\beta_\pi : \delta_{T/T'*} \mathcal{F}^{(T')} \to \mathcal{F}^{(T)}$ of ℓ-adic sheaves on X^T).
- (c) Additional coherence data expressing the idea that construction $\pi \mapsto \alpha_\pi$ is compatible with composition.

A !-sheaf on $\mathrm{Ran}(X)$ is a lax !-sheaf for which the morphisms α_π appearing in (b) are equivalences.

Example 5.2.4. Let X be a quasi-projective k-scheme. Then the construction $T \mapsto \omega_{X^T}$ determines a !-sheaf on $\mathrm{Ran}(X)$, which we will denote by $\omega_{\mathrm{Ran}(X)}$. We will refer to $\omega_{\mathrm{Ran}(X)}$ as the *dualizing sheaf* on $\mathrm{Ran}(X)$.

Remark 5.2.5. Let X be a quasi-projective k-scheme. Then the construction $T \mapsto \mathrm{Shv}_\ell(X^T)$ determines a functor from the category $\mathrm{Fin}^{\mathrm{s}}$ to the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories, which assigns to each surjection $T \to T'$ the exceptional inverse image functor $\delta^!_{T/T'} : \mathrm{Shv}_\ell(X^T) \to \mathrm{Shv}_\ell(X^{T'})$. The ∞-category $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ is an explicit realization of the (homotopy) inverse limit $\varprojlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \mathrm{Shv}_\ell(X^T)$.

Example 5.2.6. Let $X = \operatorname{Spec} k$. Then the functor $T \mapsto \mathrm{Shv}_\ell(X^T)$ is constant. Since the simplicial set $\mathrm{N}(\mathrm{Fin}^{\mathrm{s}})$ is weakly contractible, it follows that the evaluation functor $\mathcal{F} \mapsto \mathcal{F}^{(1)}$ induces an equivalence of ∞-categories $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$.

Definition 5.2.7. Let X be a quasi-projective k-scheme. We let $\mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X))$ denote the full subcategory of $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ spanned by those commutative diagrams

$$\begin{array}{ccc} & & \mathrm{Shv}_\ell^! \\ & \overset{\mathcal{F}}{\nearrow} & \downarrow \phi \\ (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} & \xrightarrow{\rho} & \mathrm{Sch}_k^{\mathrm{pr}}. \end{array}$$

for which $\mathcal{F}$ carries each morphism in $\mathrm{Fin}^{\mathrm{s}}$ to a ϕ-coCartesian morphism in $\mathrm{Shv}_\ell^!$.

Remark 5.2.8. More informally, a lax !-sheaf $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ belongs to $\mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X))$ if and only if, for each surjection of finite sets $\pi : T \to T'$, the associated map $\beta_\pi : \delta_{T/T'*}\mathcal{F}^{(T')} \to \mathcal{F}^{(T)}$ is an equivalence of ℓ-adic sheaves on X^T.

Remark 5.2.9. The category $\mathrm{Fin}^{\mathrm{s}}$ has a final object, given by the one-element set $T = \{1\}$. It follows that a lax !-sheaf $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ belongs to $\mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X))$ if and only if it is a ϕ-left Kan extension of its restriction to the full subcategory $\{T\} \subseteq (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ (see §HTT.4.3 for a discussion of relative Kan extensions). In particular, the construction $\mathcal{F} \mapsto \mathcal{F}^{(1)}$ induces an equivalence of ∞-categories $\mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X)$.

Remark 5.2.10. Let X be a quasi-projective k-scheme. Then the full subcategory

$$\mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X)) \subseteq \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$$

is contained in $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$. To prove this, consider an arbitrary surjection $\pi : T' \to T$ of nonempty finite sets, so that we have a commutative diagram σ :

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow \delta_T & & \downarrow \delta_{T'} \\ X^T & \xrightarrow{\delta_{T/T'}} & X^{T'}. \end{array}$$

We wish to show that, for each object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the canonical map

$$\delta_{T*}\,\mathrm{id}^!\,\mathcal{F} \to \delta_{T'/T}^!\delta_{T'*}\,\mathcal{F}$$

is an equivalence. This follows from Theorem 4.5.4, since σ is a pullback square and the map $\delta_{T'/T}$ is proper.

Let X be a quasi-projective k-scheme. Then the full subcategory

$$\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \subseteq \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$$

is closed under limits and colimits. It follows from the adjoint functor theorem (Corollary HTT.5.5.2.9) that the inclusion $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \hookrightarrow \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ admits a left adjoint (it also admits a right adjoint, but we will not need this). In other words, we can regard $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ as a localization of $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$. Our next result describes this localization more explicitly.

Proposition 5.2.11. *Let X be a quasi-projective k-scheme, let*

$$L : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$$

denote a left adjoint to the inclusion functor, and let $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$. Then $L(\mathcal{F}) \simeq 0$ if and only if the following condition is satisfied for every nonempty finite set T:

($*_T$) *Let $\overset{\circ}{X}{}^T$ denote the open subset of X^T whose k-valued points are injective maps $\nu : T \to X(k)$. Then the colimit*

$$\varinjlim_{T' \to T} (\delta^!_{T'/T}\, \mathcal{F}^{(T')})|_{\overset{\circ}{X}{}^T}$$

vanishes in $\mathrm{Shv}_\ell(\overset{\circ}{X}{}^T)$. Here the colimit is indexed by (the opposite of) the category $(\mathrm{Fin}^{\mathrm{s}})_{/T}$.

The proof of Proposition 5.2.11 will require some preliminaries.

Remark 5.2.12. Let X be a quasi-projective k-scheme. For every nonempty finite set T, the evaluation functor

$$e_T : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X^T)$$
$$\mathcal{F} \mapsto \mathcal{F}^{(T)}$$

admits a left adjoint e_T^L, given by relative left Kan extension along the functor $\{T\} \to \mathrm{Fin}^{\mathrm{s}}$. More concretely, we have

$$(e_T^L\, \mathcal{G})^{(T')} = \bigoplus_{\alpha : T' \to T} \delta_{\alpha *}\, \mathcal{G},$$

where the direct sum is indexed by all surjections $\alpha : T' \to T$ and $\delta_\alpha : X^T \to X^{T'}$ denotes the associated diagonal map.

For every object $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$, we have a canonical equivalence

$$\mathcal{F} \simeq \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} e_T^L(\mathcal{F}^{(T)}).$$

In particular, the ∞-category $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ is generated under small colimits by the essential images of the functors $\{e_T^L\}_{T \in \mathrm{Fin}^{\mathrm{s}}}$.

Remark 5.2.13. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$. Then $\mathcal{F}$ is a !-sheaf if and only if for every surjection of nonempty finite sets $\alpha : T' \to T$ and every object $\mathcal{G} \in \mathrm{Shv}_\ell(X^T)$, the canonical map

$$e_{T'}^L(\delta_{T'/T*}\, \mathcal{G}) \to e_T^L(\mathcal{G})$$

induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))}(e_T^L(\mathcal{G}), \mathcal{F}) \to \mathrm{Map}_{\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))}(e_{T'}^L(\delta_{T'/T*}\, \mathcal{G}), \mathcal{F}).$$

Proof of Proposition 5.2.11. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ spanned by those objects $\mathcal{F}$ for which $L(\mathcal{F}) \simeq 0$. We first show that every object of $\mathcal{C}$ satisfies condition ($*_T$) for every nonempty finite set T. Using Remark 5.2.13, we see that $\mathcal{C}$ is generated under small colimits by cofibers of maps

$$e_{S'}^L(\delta_{S'/S*}\, \mathcal{G}) \to e_S^L(\mathcal{G})$$

where $S' \to S$ is a surjection of nonempty finite sets and $\mathcal{G} \in \mathrm{Shv}_\ell(X^S)$. It will therefore suffice to show that every such cofiber satisfies ($*_T$). Equivalently, we must show that the canonical map

$$\theta : \varinjlim_{T' \to T} \delta^!_{T'/T} e_{S'}^L(\delta_{S'/S*}\, \mathcal{G})^{(T')} \to \varinjlim_{T' \to T} \delta^!_{T'/T} e_S^L(\mathcal{G})^{(T')}$$

is an equivalence over the open set $\overset{\circ}{X}{}^T \subseteq X^T$.

Appealing to the definition of e_S^L, we can identify the codomain of θ with

$$\varinjlim_{T' \to T} \bigoplus_{T' \to S} \delta^!_{T'/T} \delta_{T'/S*} \mathcal{G}.$$

Note that if we are given surjections $\alpha : T' \to T$, $\beta : T' \to S$, then the ℓ-adic sheaf $(\delta^!_{T'/T}\delta_{T'/S*}\mathcal{G})|_{\overset{\circ}{X}{}^T}$ vanishes unless α factors as a composition

$$T' \xrightarrow{\beta} S \to T,$$

and is otherwise equivalent to $(\delta^!_{S/T}\mathcal{G})|_{\overset{\circ}{X}{}^T}$. It follows that over the open set $\overset{\circ}{X}{}^T$, we can identify the codomain of θ with $(\bigoplus_{S \to T} \delta^!_{T/S}\mathcal{G})|_{\overset{\circ}{X}{}^T}$. Similarly, over the open set $\overset{\circ}{X}{}^T$ we can identify the domain of θ with $(\bigoplus_{S' \to T} \delta^!_{T/S}\mathcal{G})|_{\overset{\circ}{X}{}^T}$. It now suffices to observe that if $S' \to T$ is a surjection which does not factor through the map $S' \to S$, then $\delta^!_{T/S}\mathcal{G})|_{\overset{\circ}{X}{}^T} \simeq 0$.

We now prove the converse: suppose that $\mathcal{F} \in \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$ satisfies condition $(*_T)$ for every nonempty finite set T. The fiber of the canonical map $\mathcal{F} \to L\,\mathcal{F}$ belongs to the ∞-category $\mathcal{C}$ and therefore also satisfies condition $(*_T)$ for every nonempty finite set T. It follows that $L\,\mathcal{F}$ also satisfies condition $(*_T)$ for each T. Since $L\,\mathcal{F}$ is a !-sheaf, we have

$$\varinjlim_{T' \to T} \delta^!_{T'/T}(L\,\mathcal{F})^{(T')} \simeq (L\,\mathcal{F})^{(T)},$$

so that $(L\,\mathcal{F})^{(T)}|_{\overset{\circ}{X}{}^T} \simeq 0$ for each $T \in \mathrm{Fin}^{\mathrm{s}}$. It follows by induction that each of the ℓ-adic sheaves $(L\,\mathcal{F})^{(T)}$ vanishes, so that $L\,\mathcal{F} \simeq 0$ as desired. □

Remark 5.2.14. Let X be a quasi-projective k-scheme and let

$$L : \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X)) \to \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$$

denote a left adjoint to the inclusion. For each object $\mathcal{F} \in \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$ and each nonempty finite set T, Proposition 5.2.11 shows that the canonical map

$$\varinjlim_{T' \to T} \delta^!_{T'/T}\,\mathcal{F}^{(T')} \to \varinjlim_{T' \to T} \delta^!_{T'/T}(L\,\mathcal{F})^{(T')} \simeq (L\,\mathcal{F})^{(T)}$$

is an equivalence when restricted to the open set $\overset{\circ}{X}{}^T \subseteq X^T$.

Our next goal is produce some nontrivial examples of !-sheaves on $\mathrm{Ran}(X)$.

Definition 5.2.15. Let X be a quasi-projective k-scheme. We define a $\mathrm{Ran}(X)$-*prestack* to be a category $\mathcal{C}$ equipped with a coCartesian fibration $\mathcal{C} \to \mathrm{Ran}(X)$.

Notation 5.2.16. Let $\mathcal{C}$ be a $\mathrm{Ran}(X)$-prestack. For every nonempty finite set T, let $\mathcal{C}^{(T)}$ denote the fiber product $\mathcal{C} \times_{\mathrm{Fin}^{\mathrm{s}}} \{T\}$, so that the map $\rho : \mathcal{C} \to \mathrm{Ran}(X)$ induces a map of prestacks $\rho^{(T)} : \mathcal{C}^{(T)} \to X^T$. Suppose we are given a surjection of finite sets $\alpha : T \to T'$. We can identify $\mathcal{C}^{(T)} \times_{X^T} X^{T'}$ with the category comprised of those objects $C \in \mathcal{C}$ for which $\rho(C) \in \mathrm{Ran}(X)$ has the form (R, T, ν), where $\nu : T \to X(R)$ is provided with a factorization $T \xrightarrow{\alpha} T' \xrightarrow{\nu'} X(R)$. Since π is a coCartesian fibration, the natural map $(R, T, \nu) \mapsto (R, T', \nu')$ in $\mathrm{Ran}(X)$ can be lifted to a ρ-coCartesian morphism $C \to C'$ in $\mathcal{C}$. The construction $C \mapsto C'$ determines a functor

$$F_\alpha : \mathcal{C}^{(T)} \times_{X^T} X^{T'} \to \mathcal{C}^{(T')}.$$

Remark 5.2.17. In the situation of Notation 5.2.16, the construction $T \mapsto (X^T, \mathcal{C}^{(T)})$ determines a functor from $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ to the 2-category $\mathrm{RelStack}^!$ of Construction A.5.14. Conversely, any functor $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathrm{RelStack}^!$ is equivalent to one which arises in this way, for an essentially unique $\mathrm{Ran}(X)$-prestack $\mathcal{C}$.

Example 5.2.18. If X is an algebraic curve and G is a smooth affine group scheme over X, then the Beilinson-Drinfeld Grassmannian $\mathrm{Ran}_G(X)$ (see Definition 3.2.3) is a $\mathrm{Ran}(X)$-prestack.

Construction 5.2.19. Let X be a quasi-projective k-scheme and let $\rho : \mathcal{C} \to \mathrm{Ran}(X)$ be a $\mathrm{Ran}(X)$-prestack. We let $[\mathcal{C}]_{\mathrm{Ran}(X)}$ denote the lax !-sheaf on $\mathrm{Ran}(X)$ given objectwise by the formula

$$[\mathcal{C}]^{(T)}_{\mathrm{Ran}(X)} = [\mathcal{C}^{(T)}]_{X^T} \in \mathrm{Shv}_\ell(X^T),$$

where $\mathcal{C}^{(T)}$ is defined as in Notation 5.2.16 and $[\mathcal{C}^{(T)}]_{X^T}$is defined as in §5.1. More formally, $[\mathcal{C}]_{\mathrm{Ran}(X)}$ is defined by composing the functor $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathrm{RelStack}^!$ of Remark 5.2.17 with the functor $\Psi^! : \mathrm{RelStack}^! \to \mathrm{Shv}^!_\ell$ of Remark A.5.24. We will refer to $[\mathcal{C}]_{\mathrm{Ran}(X)}$ as the *cohomology sheaf* of the morphism ρ.

Example 5.2.20. The identity map $\mathrm{id} : \mathrm{Ran}(X) \to \mathrm{Ran}(X)$ exhibits $\mathrm{Ran}(X)$ as a $\mathrm{Ran}(X)$-prestack, and its relative cohomology sheaf $[\mathrm{Ran}(X)]_{\mathrm{Ran}(X)}$ can be identified with the dualizing sheaf $\omega_{\mathrm{Ran}(X)}$ of Example 5.2.4.

5.3. **Chiral Homology.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let X be a quasi-projective k-scheme. In §5.2, we defined the ∞-category $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ of !-sheaves on the Ran space of X. In this section, we will introduce a functor

$$\int : \mathrm{Shv}^!_\ell(\mathrm{Ran}(X)) \to \mathrm{Mod}_{\mathbf{Z}_\ell},$$

which we will refer to as the *chiral homology functor*. Heuristically, we can think of an object $\mathcal{F} \in \mathrm{Shv}^!_\ell(X)$ as a rule which assigns to each "compact subset" K of $\mathrm{Ran}(X)$ a space of sections supported on K, and $\int \mathcal{F}$ can be described as the direct limit of these spaces as the size of K increases.

Remark 5.3.1. While the chiral homology functor $\int : \mathrm{Shv}^!_\ell(\mathrm{Ran}(X)) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ can be defined for any quasi-projective k-scheme X, we will consider only the case where X is projective (for our applications in this paper, we are only interested in the case where X is a projective algebraic curve).

We begin with some general remarks.

Construction 5.3.2. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then f determines proper morphisms $f^T : X^T \to Y^T$, depending functorially on $T \in \mathrm{Fin}^{\mathrm{s}}$. We let

$$\mathrm{Ran}(f)^! : \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(Y)) \to \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$$

denote the functor given on objects by the formula $(\mathrm{Ran}(f)^! \,\mathcal{F})^{(T)} = f^{T!}\,\mathcal{F}^{(T)}$. We let $\mathrm{Ran}(f)_*$ denote a left adjoint to $\mathrm{Ran}(f)^!$, given on objects by the formula $(\mathrm{Ran}(f)_* \,\mathcal{F})^{(T)} = f^T_*\,\mathcal{F}^{(T)}$.

Remark 5.3.3. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then the functor $\mathrm{Ran}(f)^!$ restricts to a functor $\mathrm{Shv}^!_\ell(\mathrm{Ran}(Y)) \to \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$, which we will also denote by $\mathrm{Ran}(f)^!$. However, the functor $\mathrm{Ran}(f)_*$ generally does *not* carry $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ into

$\mathrm{Shv}_\ell^!(\mathrm{Ran}(Y))$: for a surjection of finite sets $T \to T'$, the induced diagram

$$\begin{array}{ccc} X^{T'} & \longrightarrow & X^T \\ \downarrow & & \downarrow \\ Y^{T'} & \longrightarrow & Y^T \end{array}$$

is generally not a pullback square.

Example 5.3.4. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then the functor $\mathrm{Ran}(f)^!$ carries the dualizing sheaf $\omega_{\mathrm{Ran}(Y)}$ to the dualizing sheaf $\omega_{\mathrm{Ran}(X)}$.

Definition 5.3.5. Let X be a projective k-scheme, and let $\pi : X \to \operatorname{Spec} k$ be the projection map. We let $\int : \mathrm{Shv}_!^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ denote a left adjoint to the composite functor

$$\mathrm{Mod}_{\mathbf{Z}_\ell} \simeq \mathrm{Shv}_\ell^!(\mathrm{Ran}(\operatorname{Spec} k)) \xrightarrow{\mathrm{Ran}(\pi)^!} \mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \subseteq \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)).$$

If $\mathcal{F}$ is a lax !-sheaf on $\mathrm{Ran}(X)$, we will refer to $\int \mathcal{F}$ as the *chiral homology* of $\mathcal{F}$.

Remark 5.3.6. We will generally abuse notation by not distinguishing between the chiral homology functor $\int$ and its restriction to the full subcategory $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ of !-sheaves on $\mathrm{Ran}(X)$, which can be identified with a left adjoint to the composite functor

$$\mathrm{Mod}_{\mathbf{Z}_\ell} \simeq \mathrm{Shv}_\ell^!(\mathrm{Ran}(\operatorname{Spec} k)) \xrightarrow{\mathrm{Ran}(\pi)^!} \mathrm{Shv}_\ell^!(\mathrm{Ran}(X)).$$

Remark 5.3.7. For any projective k-scheme X, we can regard the ∞-categories $\mathrm{Shv}_\ell(X)$ and $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ as tensored over the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$. Unwinding the definitions, we see that the chiral homology functor $\mathcal{F} \mapsto \int \mathcal{F}$ is left adjoint to the functor

$$\mathrm{Mod}_{\mathbf{Z}_\ell} \to \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$$

$$M \mapsto M \otimes_{\mathbf{Z}_\ell} \omega_{\mathrm{Ran}(X)}.$$

In particular, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\int \mathcal{F}, \mathbf{Z}_\ell) \simeq \mathrm{Map}_{\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))}(\mathcal{F}, \omega_X).$$

Remark 5.3.8. Let X be a projective k-scheme. Then the right adjoint of the chiral homology functor $\int$ can be identified with the composition

$$\begin{array}{rcl} \mathrm{Mod}_{\mathbf{Z}_\ell} & \hookrightarrow & \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell}) \\ & \simeq & \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(\operatorname{Spec} k)) \\ & \xrightarrow{\mathrm{Ran}(\pi)^!} & \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)). \end{array}$$

These functors each admit left adjoints, given by $\mathrm{Ran}(\pi)_*$ and $\varinjlim : \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell}) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$, respectively. It follows that the chiral homology functor $\int$ is given on objects by the formula

$$\int \mathcal{F} = \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C^*(X^T; \mathcal{F}^{(T)}).$$

Example 5.3.9. Let X be a projective k-scheme. Then we have

$$\begin{aligned}\int \omega_{\mathrm{Ran}(X)} &\simeq \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C^*(X^T; \omega_{X^T}) \\ &\simeq \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C_*(X^T; \mathbf{Z}_\ell) \\ &\simeq C_*(\mathrm{Ran}(X); \mathbf{Z}_\ell).\end{aligned}$$

If X is connected, then Theorem 2.4.5 supplies an equivalence $\int \omega_{\mathrm{Ran}(X)} \simeq \mathbf{Z}_\ell$.

Example 5.3.10. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes and let $\mathcal{C}$ be an arbitrary prestack. Then we can regard $\mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$ and $\mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(Y)$ as $\mathrm{Ran}(X)$ and $\mathrm{Ran}(Y)$-prestacks, respectively. Using Example 5.1.16, we can identify the relative cohomology sheaf $[\mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)]_{\mathrm{Ran}(X)}$ with the exceptional inverse image

$$\mathrm{Ran}(f)^! [\mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(Y)]_{\mathrm{Ran}(Y)}.$$

In particular, if X is projective, we can take $Y = \mathrm{Spec}\, k$ to obtain an equivalence

$$[\mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)]_{\mathrm{Ran}(X)} \simeq \mathrm{Ran}(f)^! (C^*(\mathcal{C}; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \omega_{\mathrm{Ran}(\mathrm{Spec}\, k)}) \simeq C^*(\mathcal{C}; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \omega_{\mathrm{Ran}(X)}.$$

It follows that the chiral homology $\int [\mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)]_{\mathrm{Ran}(X)}$ can be identified with $C^*(\mathcal{C}; \mathbf{Z}_\ell)$.

Let X be a projective k-scheme. To analyze the chiral homology functor

$$\int : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Mod}_{\mathbf{Z}_\ell},$$

it is convenient to consider the natural filtration on $\mathrm{Ran}(X)$ given by measuring the cardinalities of finite subsets of X.

Construction 5.3.11. For each integer $d \geq 0$, we $\mathrm{Fin}^s_{\leq d}$ denote the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by the nonempty finite sets having cardinality $\leq d$. If X is a projective k-scheme and $\mathcal{F}$ is a lax !-sheaf on $\mathrm{Ran}(X)$, we let $\int^{(d)} \mathcal{F}$ denote the colimit $\varinjlim_{T \in \mathrm{Fin}^s_{\leq d}} C^*(X^T; \mathcal{F}^{(T)})$. We regard the construction $\mathcal{F} \mapsto \int^{(d)} \mathcal{F}$ as a functor from $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ to $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 5.3.12. The category $\mathrm{Fin}^{\mathrm{s}}$ can be identified with the (filtered) colimit of the sequence of full subcategories

$$\emptyset = \mathrm{Fin}^s_{\leq 0} \subset \mathrm{Fin}^s_{\leq 1} \subset \mathrm{Fin}^s_{\leq 2} \subset \cdots$$

It follows that the chiral homology functor $\int : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ is a colimit of the sequence of functors $\{\int^{(d)}\}_{d \geq 0}$.

The individual functors $\int^{(d)}$ have the following convenient property (which is not shared by the chiral homology functor $\int = \varinjlim \int^{(d)}$ itself):

Proposition 5.3.13. *Let X be a projective k-scheme and let $d \geq 0$ be an integer. Then the functor $\int^{(d)} : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ preserves limits when restricted to $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$.*

The proof of Proposition 5.3.13 will require some preliminaries.

Lemma 5.3.14. *Let X be a projective k-scheme and let $\Delta \subseteq X^2$ be the closed subscheme given by the image of the diagonal map $X \to X^2$. For each $d \geq 1$, let $\Delta^{(d)} \subseteq X^d$ denote the "fat diagonal" given by the union of the closed subschemes $\{p_{ij}^{-1} \Delta\}_{1 \leq i,j \leq d}$, where $p_{ij} : X^d \to X^2$*

denotes the projection onto the ith and jth factors. Let $\overset{\circ}{X}{}^d \subseteq X^d$ *be the complement of* $\Delta^{(d)}$*. Then for each* $\mathcal{F} \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$*, we have a canonical fiber sequence*

$$\int^{(d-1)} \mathcal{F} \to \int^{(d)} \mathcal{F} \to C^*(\overset{\circ}{X}{}^d, \mathcal{F}^{(d)}|_{\overset{\circ}{X}{}^d})_{\Sigma_d},$$

in the ∞*-category* $\mathrm{Mod}_{\mathbf{Z}_\ell}$*.*

Lemma 5.3.15. *Let* Y *be a quasi-projective* k*-scheme equipped with a free action of a finite group* G*, so that* G *also acts on the* ∞*-category* $\mathrm{Shv}_\ell(Y)$*. Let* $\mathcal{F}$ *be a* G*-equivariant object of* $\mathrm{Shv}_\ell(Y)$*, so that* $C^*(Y;\mathcal{F})$ *is a* G*-equivariant object of* $\mathrm{Mod}_{\mathbf{Z}_\ell}$ *(where the group* G *acts trivially on* $\mathrm{Mod}_{\mathbf{Z}_\ell}$*). Then the norm map* $\mathrm{Nm} : C^*(Y;\mathcal{F})_G \to C^*(Y;\mathcal{F})^G$ *is an equivalence (for a definition of group actions and norm maps in the* ∞*-categorical setting, we refer the reader to §HA.6.1.6).*

Proof of Proposition 5.3.13. We proceed by induction on d. If $d = 0$, then $\int^{(d)}$ can be identified with the constant functor taking the value $0 \in \mathrm{Mod}_{\mathbf{Z}_\ell}$, and there is nothing to prove. To carry out the inductive step, we note that Lemma 5.3.14 supplies a fiber sequence

$$\int^{(d-1)} \mathcal{F} \to \int^{(d)} \mathcal{F} \to C^*(\overset{\circ}{X}{}^d; \mathcal{F}^{(d)}|_{\overset{\circ}{X}{}^d})_{\Sigma_d}$$

depending functorially on $\mathcal{F} \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$. It will therefore suffice to show that the functor $\mathcal{F} \mapsto C^*(\overset{\circ}{X}{}^d; \mathcal{F}^{(d)}|_{\overset{\circ}{X}{}^d})_{\Sigma_d}$ preserves limits when restricted to $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$. This follows immediately from Lemma 5.3.15, since the symmetric group Σ_d acts freely on $\overset{\circ}{X}{}^d$. □

We now turn to the proofs of Lemmas 5.3.14 and 5.3.15.

Proof of Lemma 5.3.14. Let $\mathrm{Fin^s}_+$ denote the category whose objects are finite (possibly empty) sets and whose morphisms are surjections. The construction $T \mapsto X^T$ determines a functor $\mathrm{Fin}_+^{\mathrm{sop}} \to \mathrm{Sch}_k^{\mathrm{pr}}$. We let $\mathcal{C}_+$ denote the fiber product $(\mathrm{Fin^s}_+)^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{pr}}} \mathrm{Shv}_!^\ell$. More informally, we can identify $\mathcal{C}_+$ with the ∞-category whose objects are pairs $(T, \mathcal{F})$, where T is a finite set and $\mathcal{F} \in \mathrm{Shv}_\ell(X^T)$. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{C}_+$ spanned by those pairs $(T, \mathcal{F})$ where T is nonempty, and let $q : \mathcal{C} \to (\mathrm{Fin^s})^{\mathrm{op}}$ denote the projection map. Then $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ can be identified with the ∞-category of all sections of q, and $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ with the full subcategory spanned by the Cartesian sections of q.

Suppose that $\mathcal{F} \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$. Let $\mathcal{F}' \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ be a q-left Kan extension of the restriction $\mathcal{F}|_{(\mathrm{Fin}^s_{\leq d-1})^{\mathrm{op}}}$, so that we have an evident map $v : \mathcal{F}' \to \mathcal{F}$ which induces an equivalence $\int^{(d)} \mathcal{F}' \simeq \int^{(d-1)} \mathcal{F}$. It follows that the cofiber of the canonical map $\int^{(d-1)} \mathcal{F} \to \int^{(d)} \mathcal{F}$ can be identified with $\int^{(d)} \mathcal{F}''$, where $\mathcal{F}'' = \mathrm{cofib}(v)$. By construction, we have $\mathcal{F}''^{(T)} \simeq 0$ if the cardinality of T is $< d$. It follows that $\mathcal{F}''|_{(\mathrm{Fin}^s_{\leq d})^{\mathrm{op}}}$ is a left Kan extension of $\mathcal{F}''|_{(\mathrm{Fin}^s_{=d})^{\mathrm{op}}}$, where $\mathrm{Fin}^s_{=d}$ denotes the full subcategory of $\mathrm{Fin^s}$ spanned by those finite sets having cardinality exactly d. Note that $\mathrm{Fin}^s_{=d}$ is equivalent to a category having a single object, with automorphism group Σ_d. We therefore obtain an equivalence

$$\int^{(d)} \mathcal{F}'' = \varinjlim_{T \in \mathrm{Fin}^s_{\leq d}} C^*(X^T; \mathcal{F}''^{(T)}) \simeq \varinjlim_{T \in \mathrm{Fin}^s_{=d}} C^*(X^T; \mathcal{F}''^{(T)}) \simeq C^*(X^d; \mathcal{F}''^{(d)})_{\Sigma_d}.$$

Note that we have Σ_d-equivariant maps

$$C^*(X^d; \mathcal{F}''^{(d)}) \xrightarrow{\alpha} C^*(\overset{\circ}{X}{}^d; \mathcal{F}''^{(d)}|_{\overset{\circ}{X}{}^d}) \xleftarrow{\beta} C^*(\overset{\circ}{X}{}^d; \mathcal{F}^{(d)}|_{\overset{\circ}{X}{}^d}).$$

We will complete the proof by showing that α and β are equivalences.

To show that β is an equivalence, it will suffice to show that $\mathcal{F}'^{(d)}$ vanishes on $\overset{\circ}{X}{}^d$. Unwinding the definitions, we see that for each nonempty finite set T, the sheaf $\mathcal{F}'^{(T)} \in \mathrm{Shv}_\ell(X^T)$ is given by the formula

$$\varinjlim_{E} \delta(E)_* \mathcal{F}^{T/E},$$

where the colimit is taken over all equivalence relations E on T such that T/E has cardinality $< d$, and $\delta(E) : X^{T/E} \to X^T$ denotes the corresponding diagonal map. It now suffices to observe that when $T = \{1, \ldots, d\}$, each of the maps $\delta(E)$ has image contained in the big diagonal $\Delta^{(d)} \subseteq X^d$.

To prove that α is an equivalence, it will suffice to show that the sheaf $\mathcal{F}''^{(d)} \in \mathrm{Shv}_\ell(X^d)$ is the pushforward of a sheaf on $\overset{\circ}{X}{}^d$. Note that the complement $\Delta^{(d)}$ of $\overset{\circ}{X}{}^d$ is the union of the images of the closed embeddings $\delta(E)$, where E ranges over all equivalence relations on $T = \{1, \ldots, d\}$ such that T/E has cardinality $< d$. It will therefore suffice to show that $\delta(E)^! \mathcal{F}''^{(d)}$ vanishes for every such equivalence relation E. Unwinding the definitions, we see that $\delta(E)^! \mathcal{F}''^{(d)}$ is the cofiber of the canonical map

$$\theta : \varinjlim_{E'} \delta(E)^! \delta(E')_* \mathcal{F}^{(T/E')} \to \delta(E)^! \mathcal{F}^{(T)},$$

where the colimit is taken over all equivalence relations E' on T such that T/E' has cardinality $< d$. For every such equivalence relation E', let EE' denote the equivalence relation on S generated by E and E', so that we have a pullback diagram of schemes

$$\begin{array}{ccc} X^{T/EE'} & \xrightarrow{\delta(E,E')} & X^{T/E} \\ \downarrow{\scriptstyle \delta(E',E)} & & \downarrow{\scriptstyle \delta(E)} \\ X^{T/E'} & \xrightarrow{\delta(E')} & X^T. \end{array}$$

Using the proper base change theorem (Theorem 4.5.4), we can identify θ with the canonical m ap

$$\varinjlim_{E'} \delta(E,E')_* \delta(E',E)^! \mathcal{F}^{(T/E')} \to \delta(E)^! \mathcal{F}^{(T)}.$$

Invoking our assumption that $\mathcal{F}$ is a !-sheaf, we are reduced to proving that the canonical map

$$\varinjlim_{E'} \delta(E,E')_* \mathcal{F}^{(T/EE')} \to \mathcal{F}^{T/E}$$

is an equivalence. Let P denote the partially ordered set of all equivalence relations E' on S such that $E' = EE'$. The construction $E' \mapsto \delta(E,E')_* \mathcal{F}^{(T/EE')}$ factors through the map $E' \mapsto EE'$ and is therefore a left Kan extension of its restriction to P. We are therefore reduced to showing that the canonical map

$$\varinjlim_{E' \in P} \delta(E,E')_* \mathcal{F}^{(T/EE')} \to \mathcal{F}^{(T/E)}$$

is an equivalence. This is clear, since the equivalence relation E itself is a final object of P^{op}. $\square$

Proof of Lemma 5.3.15. Let Z denote the quotient Y/G, let $q : Y \to Z$ denote the projection map, and let $\Gamma : \mathrm{Shv}_\ell(Z) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ denote the global sections functor. Let $\mathcal{G} = q_* \mathcal{F}$, so that we can identify $\mathcal{G}$ with a G-equivariant object $\mathrm{Shv}_\ell(Z)$, and we have a G-equivariant equivalence $C^*(Y; \mathcal{F}) \simeq \Gamma(\mathcal{G})$. The functor Γ preserves limits (since it is right adjoint to pullback along the projection map $Z \to \operatorname{Spec} k$) and colimits (since it is defined as the Ind-extension of the global sections functor on constructible sheaves). It follows that the norm map Nm can be identified with the image under Γ of the natural map $\mathrm{Nm}_{\mathcal{G}} : \mathcal{G}_G \to \mathcal{G}^G$. We are therefore reduced to proving that $\mathrm{Nm}_{\mathcal{G}}$ is an equivalence. By virtue of Corollary 4.3.42, it will suffice to prove that $q^* \mathrm{Nm}_{\mathcal{G}}$ is an equivalence in $\mathrm{Shv}_\ell(Y)$. The functor q^* admits left and right adjoints $q_!, q_* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(Z)$, and therefore preserves limits and colimits and so commutes with the formation of norm maps. We are therefore reduced to proving that the norm map $\mathrm{Nm}_{q^* \mathcal{G}} : (q^* \mathcal{G})_G \to (q^* \mathcal{G})^G$ is an equivalence in $\mathrm{Shv}_\ell(Y)$. This follows from the observation that the counit map $q^* \mathcal{G} = q^* q_* \mathcal{F} \to \mathcal{F}$ exhibits $q^* \mathcal{G}$ as an induced representation of G. □

If we restrict our attention to !-sheaves satisfying appropriate boundedness hypotheses, then we can prove an analogue of Proposition 5.3.13 for the chiral homology functor $\mathcal{F} \mapsto \int \mathcal{F}$ itself.

Corollary 5.3.16. *Let $\phi : \mathbf{Z}_{\geq 0} \to \mathbf{Z}$ be a function which is nondecreasing and unbounded, and let $\mathcal{C}_\phi \subseteq \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ denote the full subcategory spanned by those !-sheaves $\mathcal{F}$ which satisfy the following additional condition:*

- $(*)$ *For each integer $d \geq 1$, $C^*(\overset{\circ}{X}{}^d; \mathcal{F}^{(d)}|_{\overset{\circ}{X}{}^d})$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -\phi(d)}$.*

Then the functor $\mathcal{F} \mapsto \int \mathcal{F}$ preserves limits when restricted to $\mathcal{C}_\phi$.

Remark 5.3.17. For any nondecreasing unbounded function $\phi : \mathbf{Z}_{\geq 0} \to \mathbf{Z}$, the full subcategory $\mathcal{C}_\phi \subseteq \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ appearing in the statement of Corollary 5.3.16 is closed under limits.

Proof of Corollary 5.3.16. For each integer e, let $T_e : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq -e}$ be the functor given by $T_e(\mathcal{F}) = \tau_{\geq -e} \int \mathcal{F}$. We will complete the proof by showing that each of the functors T_e commutes with limits when restricted to $\mathcal{C}_\phi$. To prove this, choose an integer d such that $\phi(d') > e$ for $d' > d$. It follows from Proposition 5.3.13 that the functor $\mathcal{F} \mapsto \int^{(d)} \mathcal{F}$ commutes with limits. It will therefore suffice to show that the map $\tau_{\geq -e} \int^{(d)} \mathcal{F} \to \tau_{\geq -e} \int \mathcal{F}$ is an equivalence for $\mathcal{F} \in \mathcal{C}_\phi$. In fact, we claim that for each $d' \geq d$, the map

$$\tau_{\geq -e} \int^{(d)} \mathcal{F} \to \tau_{\geq -e} \int^{(d')} \mathcal{F}$$

is an equivalence. Using induction on d', we are reduced to proving that the natural map

$$\tau_{\geq -e} \int^{(d'-1)} \mathcal{F} \to \tau_{\geq -e} \int^{(d')} \mathcal{F}$$

is an equivalence for $d' > d$. Using Lemma 5.3.14, we are reduced to proving that

$$C^*(U(d'); \mathcal{F}^{(d')}|_{U(d')})_{\Sigma_{d'}}$$

belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -e-1}$. Lemma 5.3.15 supplies an equivalence

$$C^*(U(d'); \mathcal{F}^{(d')}|_{U(d')})_{\Sigma_{d'}} \simeq C^*(U(d'); \mathcal{F}^{(d')}|_{U(d')})^{\Sigma_{d'}}.$$

It will therefore suffice to show that $C^*(U(d'); \mathcal{F}^{(d')}|_{U(d')})^{\Sigma_{d'}} \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -e-1}$. Because $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -e-1}$ is closed under limits in $\mathrm{Mod}_{\mathbf{Z}_\ell}$, we are reduced to proving that

$$C^*(U(d'); \mathcal{F}^{(d')}|_{U(d')}) \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -e-1},$$

which follows from $(*)$ since $f(d') > e$. □

5.4. **The Product Formula: First Formulation.** Throughout this section, we let k denote an algebraically closed field and ℓ a prime number which is invertible in k. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. For each point $x \in X$, let G_x denote the fiber product $G \times_X \{x\}$, and let BG_x denote the classifying stack of G_x. Our goal is to formulate an algebro-geometric version of Theorem 1.4.9, which expresses the cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ as a "continuous direct limit" of cochain complexes of the form $C^*(\prod_{x \in T} \mathrm{BG}_x; \mathbf{Z}_\ell)$ where T ranges over all finite subsets of X.

Construction 5.4.1. We define a category $\mathrm{Ran}^G(X)$ as follows:

- The objects of $\mathrm{Ran}^G(X)$ are quadruples $(R, T, \nu, \mathcal{P})$ where R is a finitely generated k-algebra, T is a nonempty finite set, $\nu : T \to X(R)$ is a map of sets, and $\mathcal{P}$ is a G-bundle on the divisor $|\nu(T)| \subseteq X_R$ determined by ν.
- A morphism from $(R, T, \nu, \mathcal{P})$ to $(R', T', \nu', \mathcal{P}')$ in the category $\mathrm{Ran}^G(X)$ consists of a morphism $(R, T, \nu) \to (R', T', \nu')$ in $\mathrm{Ran}(X)$, together with a G-bundle isomorphism $\mathcal{P}' \simeq |\nu'(T')| \times_{|\nu(T)|} \mathcal{P}$.

The construction $(R, T, \nu, \mathcal{P}) \mapsto (R, T, \nu)$ determines a forgetful functor $\mathrm{Ran}^G(X) \to \mathrm{Ran}(X)$, which exhibits $\mathrm{Ran}^G(X)$ as a $\mathrm{Ran}(X)$-prestack. For each nonempty finite set T, we can identify the prestack $\mathrm{Ran}^G(X)^{(T)}$ with the classifying stack for the group scheme over X^T given by the Weil restriction of $G \times_X D$ along the map $D \to X^T$, where $D \subseteq X \times_{\mathrm{Spec}\, k} X^T$ denotes the "incidence divisor" determined by the natural maps from X^T into X. In particular, for each point $x \in X(k)$, the fiber product $\mathrm{Ran}^G(X) \times_{\mathrm{Ran}(G)} \mathrm{Spec}\, k$ can be identified with the classifying stack BG_x.

Notation 5.4.2. We let $\mathcal{B} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ denote the lax !-sheaf given by the formula $\mathcal{B} = [\mathrm{Ran}^G(X)]_{\mathrm{Ran}(X)}$. In situations where it is necessary to emphasize the dependence of $\mathcal{B}$ on the group scheme G (which we will encounter in §7.1), we will denote $\mathcal{B}$ by $\mathcal{B}_G$.

Proposition 5.4.3. *The lax !-sheaf $\mathcal{B}$ of Notation 5.4.2 is a !-sheaf on* $\mathrm{Ran}(X)$.

Proof. For every nonempty finite set T, let $\Delta_T \subseteq X^T \times_{\mathrm{Spec}\, k} X$ be the incidence divisor (consisting of those points $(\{x_t\}_{t \in T}, y)$ where $y = x_t$ for some t), and let $G(T)$ denote the scheme given by the Weil restriction of $G \times_X \Delta_T$ along the finite flat map $\Delta_T \to X^T$. Then the fiber $\mathrm{Ran}^G(X)^T = \mathrm{Ran}^G(X) \times_{\mathrm{Fin}^s} \{T\}$ can be identified with the classifying stack of $G(T)$ (where we regard $G(T)$ as a group scheme over X^T). For each integer n, we let $G(T)_n$ denote the nth fiber poewr of $G(T)$ over X^T.

Let $T' \to T$ be a surjection of nonempty finite sets, and let $\delta : X^T \to X^{T'}$ be the associated diagonal map. We wish to show that the canonical map $\beta : [\mathrm{Ran}^G(X)^{(T)}]_{X^T} \to \delta^! [\mathrm{Ran}^G(X)^{(T')}]_{X^{T'}}$ is an equivalence in $\mathrm{Shv}_\ell(X^T)$. Using Proposition 5.1.9, we can identify the codomain of β with $[\mathrm{Ran}^G(X)^{(T')} \times_{X^{T'}} X^T]_{X^T}$. It follows that β can be identified with the map of totalizations induced by a morphism of cosimplicial ℓ-adic sheaves

$$[G(T)_\bullet]_{X^T} \to [G(T')_\bullet \times_{X^{T'}} X^T]_{X^T}.$$

To complete the proof, it will suffice to show that the natural map

$$\beta^n : [G(T)_n]_{X^T} \to [G(T')_n \times_{X^{T'}} X^T]_{X^T}$$

is an equivalence for each $n \geq 0$. Consider the diagram

$$G(T')_n \times_{X^{T'}} X^T \xrightarrow{\phi} G(T)_n \xrightarrow{\psi} X^T.$$

Unwinding the definitions, we see that β^n is given by the map

$$\psi_* \psi^* \omega_{X^T} \to \psi_* \phi_* \phi^* \psi^* \omega_{X^T}$$

determined by the unit transformation $u : \mathrm{id} \to \phi_* \phi^*$. To complete the proof, it will suffice to show that u is an equivalence. This follows from the observation that, Zariski-locally on X^T, u can be identified with the projection map

$$\mathbf{A}^d \times G(T)_n \to G(T)_n$$

for some integer $d \geq 0$ (since the kernel of the map $G(T') \times_{X^{T'}} X^T \to G(T)$ is an extension of vector bundles over X^T). □

Remark 5.4.4. Let $\iota : \operatorname{Spec} k \to \operatorname{Ran}(X)$ be the point classifying an inclusion $\nu : T \hookrightarrow X(k)$, so that the fiber product $\operatorname{Ran}^G(X) \times_{\operatorname{Ran}(X)} \operatorname{Spec} k$ can be identified with the product stack $\prod_{t \in T} \mathrm{BG}_{\nu(t)}$. Using Proposition 5.1.9, we see that the costalk $\iota^! \mathcal{B}$ can be identified with the cochain complex $C^*(\prod_{t \in T} \mathrm{BG}_{\nu(t)}; \mathbf{Z}_\ell)$.

Any G-bundle on the entire curve X can be restricted to any divisor on X. The formation of restrictions determines a morphism of $\operatorname{Ran}(X)$-prestacks

$$\operatorname{Ran}(X) \times_{\operatorname{Spec} k} \operatorname{Bun}_G(X) \to \operatorname{Ran}^G(X),$$

hence a map of !-sheaves

$$[\operatorname{Ran}^G(X)]_{\operatorname{Ran}(X)} \to [\operatorname{Ran}(X) \times_{\operatorname{Spec} k} \operatorname{Bun}_G(X)]_{\operatorname{Ran}(X)}.$$

We can now formulate our algebro-geometric analogue of Theorem 1.4.9:

Theorem 5.4.5 (Product Formula)**.** *Let G be a smooth affine group scheme over X with connected fibers whose generic fiber is semisimple and simply connected. Then the composite map*

$$\begin{aligned} \int \mathcal{B} &= \int [\operatorname{Ran}^G(X)]_{\operatorname{Ran}(X)} \\ &\to \int [\operatorname{Ran}(X) \times_{\operatorname{Spec} k} \operatorname{Bun}_G(X)]_{\operatorname{Ran}(X)} \\ &\simeq C^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell) \end{aligned}$$

is a quasi-isomorphism (here the last equivalence is supplied by Example 5.3.10).

We will give the proof of Theorem 5.4.5 in §9.

5.5. **Convolution of !-Sheaves.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let X be a quasi-projective k-scheme. In §5.2, we introduced the ∞-category $\operatorname{Shv}_\ell^{\mathrm{lax}}(X)$ of lax !-sheaves on the Ran space $\operatorname{Ran}(X)$. In this section, we will study an operation

$$\star : \operatorname{Shv}_\ell^{\mathrm{lax}}(\operatorname{Ran}(X)) \times \operatorname{Shv}_\ell^{\mathrm{lax}}(\operatorname{Ran}(X)) \to \operatorname{Shv}_\ell^{\mathrm{lax}}(\operatorname{Ran}(X))$$

called the *convolution product*, given on objects by the formula

$$(\mathcal{F}' \star \mathcal{F}'')^{(T)} = \bigoplus_{T = T' \amalg T''} \mathcal{F}'^{(T')} \boxtimes \mathcal{F}''^{(T'')}.$$

Here the direct sum is taken over the collection of all decompositions of T as a disjoint union into disjoint *nonempty* $T', T'' \subseteq T$. The main result of this section is to show that the convolution product induces (nonunital) symmetric monoidal structures on the full subcategories

$$\operatorname{Shv}_\ell^!(\operatorname{Ran}(X)), \operatorname{Shv}_\ell^{\mathrm{diag}}(\operatorname{Ran}(X)) \subseteq \operatorname{Shv}_\ell^{\mathrm{lax}}(\operatorname{Ran}(X))$$

(Propositions 5.5.14 and 5.5.19). In the latter case, this symmetric monoidal structure can be identified with the !-tensor product on $\operatorname{Shv}_\ell(X)$ introduced in §4.6.

We begin with some general categorical remarks.

Construction 5.5.1 (Day Convolution Product)**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits finite coproducts and that the tensor product

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

preserves finite coproducts separately in each variable.

Given a pair of functors

$$F', F'' : (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathcal{C},$$

one can define a new functor

$$F' \star F'' : (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathcal{C},$$

given on objects by the formula

$$(F' \star F'')(T) = \coprod_{T = T' \amalg T''} F'(T') \otimes F''(T''),$$

where the coproduct is taken over all decompositions of T as a disjoint union of subsets $T', T'' \subseteq T$. More formally, $F' \star F''$ is obtained from the composite functor

$$(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \times (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \xrightarrow{F' \times F''} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

by left Kan extension along the disjoint union functor

$$\amalg : (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \times (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}.$$

We will refer to $F' \star F''$ as the *Day convolution product* of F' and F''.

The Day convolution product is commutative and associative up to coherent homotopy. More precisely, the ∞-category $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C})$ inherits a nonunital symmetric monoidal structure whose underlying tensor product is given by Day convolution

$$\star : \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C}) \times \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C}).$$

Remark 5.5.2. In the situation of Construction 5.5.1, there is generally no unit object for the Day convolution product on $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C})$. We could correct this problem by enlarging the category $\mathrm{Fin}^{\mathrm{s}}$ to include the empty set, but for our applications it will be more convenient not to do so.

Remark 5.5.3. Let $\mathcal{C}$ be as in Construction 5.5.1 and let $F : (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathcal{C}$ be a functor. Then the data of a multiplication map $m : F \star F \to F$ is equivalent to the data of family of maps $m_{T',T''} : F(T') \otimes F(T'') \to F(T' \amalg T'')$, depending functorially on T' and T''. Elaborating on this observation, one can show that F has the structure of a lax nonunital symmetric monoidal functor (where we regard $\mathrm{Fin}^{\mathrm{s}}$ as a nonunital symmetric monoidal category via the formation of disjoint unions) if and only if it has the structure of a commutative algebra object of the ∞-category $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C})$ (with respect to the convolution product). More precisely, we have an equivalence of ∞-categories

$$\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C})) \simeq \mathrm{Fun}^{\mathrm{lax}}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C}),$$

where $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C}))$ denotes the ∞-category of nonunital commutative algebra objects of $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{Fun}^{\mathrm{lax}}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C})$ denotes the ∞-category of lax nonunital symmetric monoidal functors from $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ into $\mathcal{C}$.

Remark 5.5.4. Construction 5.5.1 is functorial. Suppose we are given a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$, where the ∞-categories $\mathcal{C}$ and $\mathcal{D}$ admit finite coproducts and the tensor product functors

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

preserve coproducts separately in each variable. Then composition with F induces a symmetric monoidal functor

$$\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathcal{D}),$$

where we regard each side as equipped with the Day convolution product of Construction 5.5.1.

Example 5.5.5. Let $\mathrm{Sch}_k^{\mathrm{pr}}$ denote the category whose objects are quasi-projective k-schemes and whose morphisms are proper maps. We can regard $\mathrm{Sch}_k^{\mathrm{pr}}$ as a symmetric monoidal category with respect to the formation of Cartesian products. Note that $\mathrm{Sch}_k^{\mathrm{pr}}$ admits finite coproducts (given by disjoint unions of k-schemes), and that the Cartesian product functor

$$\times : \mathrm{Sch}_k^{\mathrm{pr}} \times \mathrm{Sch}_k^{\mathrm{pr}} \to \mathrm{Sch}_k^{\mathrm{pr}}$$

preserves coproducts separately in each variable. Applying Construction 5.5.1, we can regard the $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Sch}_k^{\mathrm{pr}})$ as a symmetric monoidal category via the Cartesian product. For every quasi-projective k-scheme X, the construction

$$T \mapsto X^T$$

determines a symmetric monoidal functor from $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ into $\mathrm{Sch}_k^{\mathrm{pr}}$, which we can view as a nonunital commutative algebra object of $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Sch}_k^{\mathrm{pr}})$ (Remark 5.5.3).

Example 5.5.6. Let $\mathrm{Shv}_\ell^!$ denote the ∞-category introduced in Definition 5.1.18: the objects of $\mathrm{Shv}_\ell^!$ are pairs $(X, \mathcal{F})$ where X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ is an ℓ-adic sheaf, and a morphism from $(X, \mathcal{F})$ to $(X', \mathcal{F}')$ is a proper map of k-schemes $f : X \to X'$ together with a morphism $f_* \mathcal{F} \to \mathcal{F}'$ in $\mathrm{Shv}_\ell(X')$. We regard $\mathrm{Shv}_\ell^!$ as a symmetric monoidal ∞-category, with tensor product given by

$$(X, \mathcal{F}) \otimes (X', \mathcal{F}') = (X \times X', \mathcal{F} \boxtimes \mathcal{F}')$$

(see §A.5). The ∞-category $\mathrm{Shv}_\ell^!$ also admits finite coproducts: the coproduct of $(X, \mathcal{F})$ and $(X', \mathcal{F}')$ is $(X \amalg X', \mathcal{F}'')$, where $\mathcal{F}''|_X = \mathcal{F}$ and $\mathcal{F}''|_{X'} = \mathcal{F}'$. Moreover, the tensor product

$$\otimes : \mathrm{Shv}_\ell^! \times \mathrm{Shv}_\ell^! \to \mathrm{Shv}_\ell^!$$
$$((X, \mathcal{F}), (X', \mathcal{F}')) \mapsto (X \times X', \mathcal{F} \boxtimes \mathcal{F}')$$

preserves finite coproducts in each variable. It follows that we may regard the ∞-category $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Shv}_\ell^!)$ as a nonunital symmetric monoidal ∞-category with respect to the Day convolution product. Note that the forgetful functor

$$\pi : \mathrm{Shv}_\ell^! \to \mathrm{Sch}_k^{\mathrm{pr}}$$
$$(X, \mathcal{F}) \mapsto X$$

is symmetric monoidal and preserves coproducts, and therefore induces a nonunital symmetric monoidal functor

$$\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Shv}_\ell^!) \to \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Sch}_k^{\mathrm{pr}})$$

(see Remark 5.5.4).

For our applications, we will need a relative version of Construction 5.5.1. We begin with a general remark.

Construction 5.5.7. Let $\pi : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration of ∞-categories. For each object $D \in \mathcal{D}$, set

$$\mathcal{C}_D = \mathcal{C} \times_{\mathcal{D}} \{D\} \qquad \mathcal{C}_{/D} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D},$$

so that we have an evident inclusion map $\mathcal{C}_D \hookrightarrow \mathcal{C}_{/D}$. Since π is a coCartesian fibration, every morphism $e : D \to D'$ in $\mathcal{D}$ induces a functor $e_! : \mathcal{C}_D \to \mathcal{C}_{D'}$. Then, for every object $D \in \mathcal{D}$, the inclusion map $\mathcal{C}_D \hookrightarrow \mathcal{C}_{/D}$ admits a left adjoint $L_D : \mathcal{C}_{/D} \to \mathcal{C}_D$. Concretely, if we identify objects of $\mathcal{C}_{/D}$ with pairs (C, α) where $C \in \mathcal{C}$ and $\alpha : \pi(C) \to D$ is a morphism in $\mathcal{D}$, then L_D is given by the formula $L_D(C, \alpha) = \alpha_!(C)$.

Now suppose that the ∞-categories $\mathcal{C}$ and $\mathcal{D}$ are equipped with nonunital symmetric monoidal structures, and that π is a symmetric monoidal functor. For each nonunital commutative algebra object $A \in \mathcal{D}$, the overcategory $\mathcal{D}_{/A}$ inherits the structure of a symmetric monoidal ∞-category, where the tensor product of objects $u : M \to A$ and $v : N \to A$ is given by the composite map

$$M \otimes N \xrightarrow{u \otimes v} A \otimes A \xrightarrow{m} A,$$

where m denotes the multiplication on A (see §HA.2.2.2). It follows that $\mathcal{C}_{/A}$ also inherits a nonunital symmetric monoidal structure. If the collection of π-coCartesian morphisms in $\mathcal{D}$ is closed under tensor products, then the tensor product on $\mathcal{C}_{/A}$ is compatible with the localization functor L_A defined above, so that the fiber $\mathcal{C}_A$ inherits a symmetric monoidal structure which is determined (up to equivalence) by the requirement that L_A is a symmetric monoidal functor (see §HA.2.2.1). In this case, the inclusion $\mathcal{C}_A \hookrightarrow \mathcal{C}_{/A}$ is a lax symmetric monoidal functor. Concretely, the tensor product $\otimes_A : \mathcal{C}_A \times \mathcal{C}_A \to \mathcal{C}_A$ is given by the formula

$$C \otimes_A C' = m_!(C \otimes C').$$

Remark 5.5.8. In the situation of Construction 5.5.7, suppose that we are given a map $e : A \to B$ between nonunital commutative algebra objects of $\mathcal{D}$. Then the associated map $e_! : \mathcal{C}_A \to \mathcal{C}_B$ inherits the structure of a nonunital symmetric monoidal functor.

Construction 5.5.9. Let $\pi : \mathrm{Shv}_\ell^! \to \mathrm{Sch}_k^{\mathrm{pr}}$ be the forgetful functor of Example 5.5.6. Then π induces a symmetric monoidal functor

$$\overline{\pi} : \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Shv}_\ell^!) \to \mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Sch}_k^{\mathrm{pr}}).$$

Let X be a quasi-projective k-scheme and let $A_X : (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathrm{Sch}_k^{\mathrm{pr}}$ be the functor given by $T \mapsto X^T$, so that the inverse image $\overline{\pi}^{-1}\{A_X\}$ can be identified with the ∞-category $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ of lax !-sheaves on $\mathrm{Ran}(X)$ introduced in Definition 5.2.1. It follows from Example 5.5.5 that we can regard the functor A_X as a nonunital commutative algebra object of $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}, \mathrm{Sch}_k^{\mathrm{pr}})$. Applying Construction 5.5.7, we see that $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ inherits a nonunital symmetric monoidal structure. We will denote the underlying tensor product by

$$\star : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \times \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)).$$

We will refer to this product as the *convolution product* on $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$.

Remark 5.5.10. Unwinding the definitions, we see that the convolution product is given by the formula

$$(\mathcal{F}' \star \mathcal{F}'')^{(T)} = \bigoplus_{T = T' \amalg T''} \mathcal{F}'^{(T')} \boxtimes \mathcal{F}''^{(T'')}.$$

Warning 5.5.11. For every pair of objects $\mathcal{F}, \mathcal{F}' \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$, the ℓ-adic sheaf

$$(\mathcal{F} \star \mathcal{F}')^{(1)} \in \mathrm{Shv}_\ell(X)$$

vanishes (a set with one element cannot be decomposed as a union of two nonempty subsets). It follows that the ∞-category $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ does not have a unit object with respect to the convolution product.

Remark 5.5.12. Let X be a quasi-projective k-scheme and let $\mathcal{F}$ be a lax !-sheaf on $\mathrm{Ran}(X)$. Using Remark 5.5.3, we see that the following data are equivalent:

- Nonunital commutative algebra structures on $\mathcal{F}$ (with respect to the convolution product).

- Nonunital lax symmetric monoidal structures on the underlying functor $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathrm{Shv}_\ell^!$ (that is, a collection of multiplication
$$(X^T, \mathcal{F}^{(T)} \otimes (X^{T'}, \mathcal{F}^{(T')}) \to (X^{T \amalg T'}, \mathcal{F}^{(T \amalg T')})$$
in $\mathrm{Shv}_\ell^!$ which are coherently commutative and and associative) which are compatible with the lax symmetric monoidal structure on underlying map $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathrm{Sch}_k$ given by $T \mapsto X^T$.

Example 5.5.13. Let X be a quasi-projective k-scheme. Then the dualizing sheaf $\omega_{\mathrm{Ran}(X)}$ can be regarded as a nonunital commutative algebra object of $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$.

The full subcategory $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \subseteq \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ is not closed under the convolution product of Construction 5.5.9. However, we will prove the following:

Proposition 5.5.14. *Let X be a quasi-projective k-scheme and let $L : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ denote a left adjoint to the inclusion functor (see Proposition 5.2.11). Then there exists an essentially unique nonunital symmetric monoidal structure on $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ for which L can be promoted to a nonunital symmetric monoidal functor.*

Notation 5.5.15. Let X be a quasi-projective k-scheme and regard $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ as equipped with the nonunital symmetric monoidal structure described in Proposition 5.5.14. We will denote the underlying tensor product on $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ by
$$\odot : \mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \times \mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell^!(\mathrm{Ran}(X)),$$
and refer to it as the *convolution product* on $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$.

Remark 5.5.16. Let X be a quasi-projective k-scheme. Then the inclusion $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \hookrightarrow \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ is lax symmetric monoidal. Moreover, it induces a fully faithful embedding
$$\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)))$$
whose essential image consists of those nonunital commutative algebras $\mathcal{F}$ of $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ which are !-sheaves.

Remark 5.5.17. Let X be a quasi-projective k-scheme. Then the convolution product on $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ is given by the formula $(\mathcal{F} \odot \mathcal{F}') = L(\mathcal{F} \star \mathcal{F}')$, where $L : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ is a left adjoint to the inclusion. Using the description of L given in Remark 5.2.14, we see that the convolution product can be described concretely by the formula
$$(\mathcal{F}_0 \odot \mathcal{F}_1)^{(T)}|_{\mathring{X}^T} = \varinjlim_{\alpha : T_0 \amalg T_1 \to T} \delta_\alpha^! (\mathcal{F}_0^{(T_0)} \boxtimes \mathcal{F}_1^{(T_1)})|_{\mathring{X}^T},$$
where $\mathring{X}^T \subseteq X^T$ denotes the open subset whose k-valued points are injective maps $\nu : T \to X(k)$, the colimit is taken over the category
$$\mathcal{C} = (\mathrm{Fin}^{\mathrm{s}} \times \mathrm{Fin}^{\mathrm{s}}) \times_{\mathrm{Fin}^{\mathrm{s}}} (\mathrm{Fin}^{\mathrm{s}})_{/T}$$
whose objects are surjections $\alpha : T_0 \amalg T_1 \to T$, and $\delta_\alpha : X^T \to X^{T_0} \times X^{T_1}$ is the associated diagonal map. Let $\mathcal{C}'$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $\alpha : T_0 \amalg T_1 \to T$ where $\alpha|_{T_0}$ and $\alpha|_{T_1}$ are injective. If $\mathcal{F}_0$ and $\mathcal{F}_1$ are !-sheaves, then the construction
$$\alpha \mapsto \delta_\alpha^! (\mathcal{F}_0^{(T_0)} \boxtimes \mathcal{F}_1^{(T_1)})$$

determines a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Shv}_\ell(X^T)$ which is a left Kan extension of its restriction to $\mathcal{C}'^{\mathrm{op}}$. It follows that the convolution product can be described more simply by the formula

$$(\mathcal{F}_0 \odot \mathcal{F}_1)^{(T)}|_{\overset{\circ}{X}{}^T} = (\bigoplus_{T = T_0 \cup T_1} \delta^!_{T_0, T_1}(\mathcal{F}_0^{(T_0)} \boxtimes \mathcal{F}_1^{(T_1)})|_{\overset{\circ}{X}{}^T},$$

where the direct sum is taken over all pairs of nonempty (not necessarily disjoint) subsets $T_0, T_1 \subseteq T$ such that $T_0 \cup T_1 = T$, and $\delta_{T_0, T_1} : X^T \to X^{T_0} \times X^{T_1}$ denotes the associated diagonal map.

Example 5.5.18. Let X be a quasi-projective k-scheme and let $\mathcal{F}_0, \mathcal{F}_1 \in \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$. Then we have

$$(\mathcal{F}_0 \odot \mathcal{F}_1)^{(1)} \simeq \mathcal{F}_0^{(1)} \otimes^! \mathcal{F}_1^{(1)},$$

where the !-tensor product $\otimes^!$ is defined as in §4.6.

Proof of Proposition 5.5.14. Let $\mathcal{C} \subseteq \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$ denote the full subcategory spanned by those objects $\mathcal{F}$ such that $L(\mathcal{F}) \simeq 0$. It will suffice to show that $\mathcal{C}$ is an ideal with respect to the convolution product: that is, that the convolution product carries $\mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X)) \times \mathcal{C}$ into $\mathcal{C}$ (see §HA.2.2.1). Arguing as in the proof of Proposition 5.2.11, we see that $\mathcal{C}$ is generated under colimits by cofibers of maps $\mathcal{C}$ is generated under small colimits by cofibers of maps

$$e^L_{T'}(\delta_{T'/T*}\, \mathcal{G}) \to e^L_T(\mathcal{G})$$

(see Remark 5.2.13), where $T' \to T$ is a surjection of nonempty finite sets and $\mathcal{G} \in \mathrm{Shv}_\ell(X^T)$. Similarly, $\mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$ is generated under small colimits by objects of the form $e^L_S(\mathcal{F})$ for $\mathcal{F} \in \mathrm{Shv}_\ell(X^S)$ (see Remark 5.2.12). It will therefore suffice to show that the functor L carries every morphism of the form

$$\theta : e^L_S(\mathcal{F}) \star e^L_{T'}(\delta_{T'/T*}\, \mathcal{G}) \to e^L_S(\mathcal{F}) \star e^L_T(\mathcal{G})$$

to an equivalence in $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$. This follows from Remark 5.2.13, since we can identify θ with the canonical map

$$e^L_{S \amalg T'}((\delta_{S \amalg T'/S \amalg T})_*(\mathcal{F} \boxtimes \mathcal{G})) \to e^L_{S \amalg T}(\mathcal{F} \boxtimes \mathcal{G}).$$

□

We now discuss the relationship between the convolution product for sheaves on $\mathrm{Ran}(X)$ and the !-tensor product of ℓ-adic sheaves on X.

Proposition 5.5.19. *Let X be a quasi-projective k-scheme. Then the ∞-category $\mathrm{Shv}_\ell(X)$ admits a symmetric monoidal structure whose underlying tensor product is the functor*

$$\otimes^! : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$$

constructed in §4.6. Moreover, the construction $\mathcal{F} \mapsto \mathcal{F}^{(1)}$ determines a nonunital symmetric monoidal functor $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X)$.

Proof. Let $U : \mathrm{Shv}^!_\ell(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X)$ be the functor given by $U(\mathcal{F}) = \mathcal{F}^{(1)}$. The functor U admits a fully faithful left adjoint, whose essential image is the full subcategory

$$\mathrm{Shv}^{\mathrm{diag}}_\ell(\mathrm{Ran}(X)) \subseteq \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$$

introduced in Definition 5.2.7. We may therefore regard U as a colocalization functor on the ∞-category $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$. Using Example 5.5.18, we see that U is compatible with the convolution product so that $\mathrm{Shv}_\ell(X)$ admits an essential unique nonunital symmetric monoidal structure for which the functor U is nonunital symmetric monoidal (see §HA.2.2.1). Example

5.5.18 also shows that the tensor product underlying this nonunital symmetric monoidal structure agrees with the !-tensor product of §4.6. To complete the proof, it will suffice to show that this nonunital symmetric monoidal structure on $\mathrm{Shv}_\ell(X)$ can be promoted to a symmetric monoidal structure. By virtue of Corollary HA.5.4.4.7, it will suffice to show that there exists a quasi-unit for the !-tensor product: that is, an object $\mathcal{E}$ for which the functor $\mathcal{F} \mapsto \mathcal{F} \otimes^! \mathcal{E}$ is equivalent to the identity. This follows from Proposition 4.6.13. □

We now discuss the functorial behavior of some of the preceding constructions. Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. It follows from Remark 5.5.8 that the induced map

$$\mathrm{Ran}(f)_* : \mathrm{Shv}_\ell^{\mathrm{lax}}(X) \to \mathrm{Shv}_\ell^{\mathrm{lax}}(Y)$$

is a nonunital symmetric monoidal functor. It follows that the right adjoint

$$\mathrm{Ran}(f)^! : \mathrm{Shv}_\ell^{\mathrm{lax}}(Y) \to \mathrm{Shv}_\ell^{\mathrm{lax}}(X)$$

has the structure of a nonunital lax symmetric monoidal functor. More concretely, for every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(Y)$, we obtain a natural map

$$(\mathrm{Ran}(f)^! \mathcal{F}) \star (\mathrm{Ran}(f)^! \mathcal{G}) \to \mathrm{Ran}(f)^!(\mathcal{F} \star \mathcal{G}).$$

Unwinding the definitions (using the formula for the convolution product given in Remark 5.5.10), we see that this map is an equivalence. This proves the following:

Proposition 5.5.20. *Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then the functors*

$$\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \underset{\mathrm{Ran}(f)^!}{\overset{\mathrm{Ran}(f)_*}{\rightleftarrows}} \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(Y))$$

commute with convolution products. More precisely, we can regard $\mathrm{Ran}(f)_*$ and $\mathrm{Ran}(f)^!$ as (an adjoint pair of) nonunital symmetric monoidal functors.

In the situation of Proposition 5.5.20, the functor $\mathrm{Ran}(f)^!$ carries !-sheaves on $\mathrm{Ran}(Y)$ to !-sheaves on $\mathrm{Ran}(X)$, and can therefore be regarded as a lax nonunital symmetric monoidal functor from $\mathrm{Shv}_\ell^!(Y)$ to $\mathrm{Shv}_\ell^!(X)$. More concretely, for every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell^!(Y)$, we obtain a canonical map

$$(\mathrm{Ran}(f)^! \mathcal{F}) \odot (\mathrm{Ran}(f)^! \mathcal{G}) \to \mathrm{Ran}(f)^!(\mathcal{F} \odot \mathcal{G}).$$

It follows easily from the description of the functor $\odot$ given in Remark 5.5.17 that this map is an equivalence. We therefore obtain the following variant of Proposition 5.5.20:

Corollary 5.5.21. *Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then the functor $\mathrm{Ran}(f)^! : \mathrm{Shv}_\ell^!(\mathrm{Ran}(Y)) \to \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ preserves convolution products: that is, it can be regarded as a nonunital symmetric monoidal functor from $\mathrm{Shv}_\ell^!(\mathrm{Ran}(Y))$ to $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$.*

Corollary 5.5.22. *Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then we can regard $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ as a symmetric monoidal functor, where $\mathrm{Shv}_\ell(X)$ and $\mathrm{Shv}_\ell(Y)$ are equipped with the symmetric monoidal structure of Proposition 5.5.19.*

Proof. Let $T_X : \mathrm{Shv}^!_\ell(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X)$ denote the functor given by $T_X(\mathcal{F}) = \mathcal{F}^{(1)}$ and define T_Y similarly, so that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}^!_\ell(\mathrm{Ran}(Y)) & \xrightarrow{T_Y} & \mathrm{Shv}_\ell(Y) \\ \downarrow{\scriptstyle \mathrm{Ran}(f)^!} & & \downarrow{\scriptstyle f^!} \\ \mathrm{Shv}^!_\ell(\mathrm{Ran}(X)) & \xrightarrow{T_X} & \mathrm{Shv}_\ell(X). \end{array}$$

The functor $\mathrm{Ran}(f)^!$ is nonunital symmetric monoidal by Corollary 5.5.21 and the functor T_X is nonunital symmetric monoidal by construction. It follows that $T_X \circ \mathrm{Ran}(f)^! \simeq f^! \circ T_Y$ can be regarded as a nonunital symmetric monoidal functor. Since T_Y is a nonunital symmetric monoidal colocalization, the functor $f^!$ inherits a nonunital symmetric monoidal structure. To complete the proof, it will suffice to show that $f^!$ is quasi-unital: that is, that it carries unit objects for the !-tensor product on $\mathrm{Shv}_\ell(Y)$ to unit objects for the !-tensor product on $\mathrm{Shv}_\ell(X)$.

Since f is proper, we can choose a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle j_Y} \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{Y}, \end{array}$$

where $\overline{X}$ and $\overline{Y}$ are projective k-schemes, and the vertical maps are open immersions. The proof of Proposition 4.6.13 shows $\omega_Y = j_Y^* \omega_{\overline{Y}}$ is a unit object of $\mathrm{Shv}_\ell(Y)$. We complete the proof by noting that Proposition 4.5.12 supplies an equivalence

$$f^! j_Y^* \omega_{\overline{Y}} \simeq j_X^* \overline{f}^! \omega_{\overline{Y}} \simeq j_X^* \omega_{\overline{X}},$$

which is a unit object of $\mathrm{Shv}_\ell(X)$ (as in the proof of Proposition 4.6.13). □

Remark 5.5.23. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. It follows from Corollary 5.5.22 that the pushforward $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ can be regarded as a colax symmetric monoidal functor with respect to the !-tensor product. In particular, for every pair of ℓ-adic sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(X)$, we have a canonical map

$$f_*(\mathcal{F} \otimes^! \mathcal{G}) \to (f_* \mathcal{F}) \otimes^! (f_* \mathcal{G}).$$

If f is a closed immersion, then this map is an equivalence (see Remark 4.6.12). In this case, we can view f_* as a nonunital symmetric monoidal functor (though it generally does not preserve unit objects).

In the situation of Corollary 5.5.21, the functor $\mathrm{Ran}(f)_*$ generally does not carry !-sheaves on $\mathrm{Ran}(X)$ to !-sheaves on $\mathrm{Ran}(Y)$. Nevertheless, the functor $\mathrm{Ran}(f)^! : \mathrm{Shv}^!_\ell(\mathrm{Ran}(Y)) \to \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ admits a left adjoint, given by composing $\mathrm{Ran}(f)_*$ with a left adjoint L_Y to the inclusion $\mathrm{Shv}^!_\ell(\mathrm{Ran}(Y)) \to \mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(Y))$. It follows from Propositions 5.5.20 and 5.5.14 that this construction is compatible with convolution products:

Corollary 5.5.24. *Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then the functor $\mathrm{Ran}(f)^! : \mathrm{Shv}^!_\ell(\mathrm{Ran}(Y)) \to \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ admits a nonunital symmetric monoidal left adjoint $\mathrm{Ran}(f)_{⊛} : \mathrm{Shv}^!_\ell(\mathrm{Ran}(X)) \to \mathrm{Shv}^!_\ell(\mathrm{Ran}(Y))$.*

In the special case where $Y = \mathrm{Spec}\, k$, the functor $\mathrm{Ran}(f)_{⊛}$ of Corollary 5.5.24 can be identified with the functor of chiral homology. This proves the following:

Corollary 5.5.25. *Let X be a projective k-scheme. Then the functor of chiral homology*

$$\int : \mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$$

is nonunital symmetric monoidal: that is, it carries convolution products to tensor products.

Remark 5.5.26. In fact, something stronger is true: the functor of chiral homology is nonunital symmetric monoidal on the entire ∞-category $\mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ of lax !-sheaves. In concrete terms, this follows from the calculation

$$\begin{aligned}
\int \mathcal{F} \star \mathcal{G} &= \varinjlim_{T} C^*(X^T; (\mathcal{F} \star \mathcal{G})^{(T)}) \\
&\simeq \varinjlim_{T \in \mathrm{Fin}^s} \bigoplus_{T = T' \amalg T''} C^*(X^T; \mathcal{F}^{(T')} \boxtimes \mathcal{G}^{(T'')}) \\
&\simeq \varinjlim_{T', T'' \in \mathrm{Fin}^s} C^*(X^{T'}; \mathcal{F}^{(T')}) \otimes_{\mathbf{Z}_\ell} C^*(X^{T''}; \mathcal{G}^{(T'')}) \\
&\simeq (\varinjlim_{T' \in \mathrm{Fin}^s} C^*(X^{T'}; \mathcal{F}^{(T')})) \otimes_{\mathbf{Z}_\ell} (\varinjlim_{T'' \in \mathrm{Fin}^s} C^*(X^{T''}; \mathcal{G}^{(T'')})) \\
&\simeq (\int \mathcal{F}) \otimes_{\mathbf{Z}_\ell} (\int \mathcal{G}).
\end{aligned}$$

5.6. **Commutative Factorization Algebras.** Throughout this section, let us fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let X be a quasi-projective k-scheme. In §5.5 we defined the convolution product on $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ and proved that the restriction map

$$\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X)$$

carries convolution products to !-tensor products of ℓ-adic sheaves on X (Proposition 5.5.19). Our goal in this section is to study the relationship between (nonunital) commutative algebras in $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ and $\mathrm{Shv}_\ell(X)$. We begin by introducing some terminology.

Definition 5.6.1. Let X be a quasi-projective k-scheme. A *commutative factorization algebra* on X is a nonunital commutative algebra object $\mathcal{A}$ of the ∞-category $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ which has the following additional property:

(∗) Let T and T' be nonempty finite sets, and let $(X^T \times X^{T'})_{\mathrm{disj}} \subseteq X^T \times X^{T'}$ be the open subset whose k-valued points correspond to maps $\nu : T \amalg T' \to X(k)$ with $\nu(T) \cap \nu(T') = \emptyset$. Then the multiplication on $\mathcal{A}$ induces an equivalence of ℓ-adic sheaves

$$(\mathcal{A}^{(T)} \boxtimes \mathcal{A}^{(T')})|_{(X^T \times X^{T'})_{\mathrm{disj}}} \to \mathcal{A}^{(T \amalg T')}|_{(X^T \times X^{T'})_{\mathrm{disj}}}.$$

Remark 5.6.2. Let X be a quasi-projective k-scheme and let $\mathcal{A}_\bullet$ be a simplicial object of the ∞-category $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)))$. If each $\mathcal{A}_n$ is a commutative factorization algebra on X, then the geometric realization $|\mathcal{A}_\bullet|$ is also a commutative factorization algebra on X.

Remark 5.6.3. Let X be a quasi-projective k-scheme and let $\mathcal{A}$ be a nonunital commutative algebra object of $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$. Then $\mathcal{A}$ is a commutative factorization algebra if and only if, for every integer $n > 0$, the canonical map

$$(\mathcal{A}^{(1)})^{\boxtimes n} \to \mathcal{A}^{(n)}$$

is an equivalence of ℓ-adic sheaves over the open subset $\overset{\circ}{X}{}^n \subseteq X^n$ whose k-valued points are n-tuples of *distinct* elements of $X(k)$.

Our main goal in this section is to prove the following:

Theorem 5.6.4. *Let X be a quasi-projective k-scheme and let*

$$G : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X))$$

be the the functor given by $\mathcal{A} \mapsto \mathcal{A}^{(1)}$. Then the functor G admits a fully faithful left adjoint $\mathrm{Fact} : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}^!_\ell(\mathrm{Ran}(X)))$, whose essential image is the full subcategory of $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}^!_\ell(\mathrm{Ran}(X)))$ spanned by the commutative factorization algebras on X.

Example 5.6.5. Let X be a quasi-projective k-scheme. Then the dualizing sheaf ω_X is the unit object of $\mathrm{Shv}_\ell(X)$ (with respect to the !-tensor product), and therefore inherits the structure of a nonunital commutative algebra object of $\mathrm{Shv}_\ell(X)$. The corresponding commutative factorization algebra is given by $\omega_{\mathrm{Ran}(X)} \in \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ (see Example 5.5.13).

Apart from the description of the essential image of F, Theorem 5.6.4 is subsumed by the following general category-theoretic principle:

Proposition 5.6.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital symmetric monoidal ∞-categories. Assume that $\mathcal{C}$ and $\mathcal{D}$ are presentable and that the tensor product functors*

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

preserve colimits separately in each variable. Suppose we are given a pair of adjoint functors

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$$

where g is nonunital lax symmetric monoidal, so that g induces a functor $G : \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D}) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$. Then the functor G admits a left adjoint $F : \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})$.

Assume further that the functor g is nonunital symmetric monoidal and preserves colimits. If f is fully faithful, then F is fully faithful.

Proof. Since the functor G preserves limits and filtered colimits, the existence of F follows from the adjoint functor theorem (Corollary HTT.5.5.2.9). Suppose now that g is a nonunital symmetric monoidal functor which preserves colimits. The adjoint functor theorem implies that g admits a right adjoint $h : \mathcal{C} \to \mathcal{D}$, which then inherits the structure of a lax symmetric monoidal functor. Since the left adjoint to g is fully faithful, the right adjoint h is also fully faithful (see §HA.2.2.1). It follows that h induces a fully faithful embedding $H : \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})$ which is right adjoint to G. Since the right adjoint to G is fully faithful, it follows that the left adjoint to G is also fully faithful. □

Example 5.6.7. In the situation of Proposition 5.6.6, if the functor f is nonunital symmetric monoidal, then F is simply the functor induced by f at the level of nonunital commutative algebras: in other words, the diagram

$$\begin{array}{ccc} \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) & \xrightarrow{F} & \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

commutes (up to preferred homotopy).

Example 5.6.8. Let $f : X \to Y$ be a closed immersion of quasi-projective k-schemes. Then the pushforward functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ is nonunital symmetric monoidal (with respect to the !-tensor product; see Remark 5.5.23) and therefore induces a map $f_* : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(Y))$ which is left adjoint to $f^! : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(Y)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X))$.

Example 5.6.9. Let X be a projective k-scheme and let $\pi : X \to \operatorname{Spec} k$ be the projection map. Let us regard $\mathrm{Shv}_\ell(X)$ as a symmetric monoidal ∞-category with respect to the !-tensor product (Proposition 5.5.19) and consider the symmetric monoidal functor $\pi^! : \mathrm{Mod}_{\mathbf{Z}_\ell} \simeq \mathrm{Shv}_\ell(\operatorname{Spec} k) \to \mathrm{Shv}_\ell(X)$ given by $\pi^! M = M \otimes \omega_X$. It follows from Proposition 5.6.6 that the induced map

$$\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell}) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X))$$

admits a left adjoint $\pi_\star^{\mathrm{nu}} : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell})$. Note that $\pi^!$ factors as a composition

$$\mathrm{Mod}_{\mathbf{Z}_\ell} \simeq \mathrm{Shv}_\ell^!(\mathrm{Ran}(\operatorname{Spec} k)) \xrightarrow{\mathrm{Ran}(\pi)^!} \mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X).$$

Using Theorem 5.6.4, Example 5.6.7, and Corollary 5.5.25, we see that $\pi_\star^{\mathrm{nu}}$ factors as a composition

$$\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) \xrightarrow{\mathrm{Fact}} \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \xrightarrow{\int} \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell}).$$

In other words, if $\mathcal{A}$ is a commutative factorization algebra on X, then we have a canonical equivalence $\pi_\star^{\mathrm{nu}}(\mathcal{A}^{(1)}) \simeq \int \mathcal{A}$ in the ∞-category $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell})$.

Many variations on Proposition 5.6.6 are possible. For example, if $\mathcal{C}$ and $\mathcal{D}$ admit unit objects (and g is a lax symmetric monoidal functor), then the same argument shows that the induced map $\mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C})$ admits a left adjoint. We will be interested in situations where this left adjoint is compatible with the functor F described in Proposition 5.6.6.

Proposition 5.6.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories with unit objects $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$. Assume that $\mathcal{C}$ and $\mathcal{D}$ are presentable and that the tensor product functors*

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

preserve colimits separately in each variable. Suppose we are given a pair of adjoint functors

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$$

where g is lax symmetric monoidal and let $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})$ be as in Proposition 5.6.6. Suppose that the unit map $\mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{D}})$ induces an equivalence $\alpha : F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$. Then:

1. *The functor $G_+ : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C})$ determined by g admits a left adjoint F_+.*
2. *The diagram*

$$\begin{array}{ccc} \mathrm{CAlg}(\mathcal{C}) & \xrightarrow{F_+} & \mathrm{CAlg}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) & \xrightarrow{F} & \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D}) \end{array}$$

commutes (up to canonical homotopy).

Remark 5.6.11. Assertion (1) follows from the proof of Proposition 5.6.6 and does not require the assumption that α is an equivalence.

Example 5.6.12. Let X be a projective k-scheme, let $\pi : X \to \operatorname{Spec} k$ denote the projection map, and let $\pi_\star^{\mathrm{nu}} : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell})$ be as in Example 5.6.9. If X is connected, then the acyclicity of the Ran space implies that the natural map

$$\pi_\star^{\mathrm{nu}} \omega_X \to \mathbf{Z}_\ell$$

is an equivalence (see Example 5.3.9). Applying Proposition 5.6.10, we deduce that the functor $\pi^! : \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell}) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(X))$ admits a left adjoint $\pi_\star : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) & \xrightarrow{\pi_\star} & \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell}) \\ \downarrow & & \downarrow \\ \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) & \xrightarrow{\pi_\star^{\mathrm{nu}}} & \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell}). \end{array}$$

In particular, if $\mathcal{A}$ is a commutative factorization algebra on X having the property that $\mathcal{A}^{(1)} \in \mathrm{Shv}_\ell(X)$ admits a unit, then we have a canonical equivalence

$$\pi_\star \mathcal{A}^{(1)} \simeq \int \mathcal{A}$$

in $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell})$.

Proof of Proposition 5.6.10. If B is a nonunital commutative algebra object of $\mathcal{D}$, we say that a map $u : \mathbf{1}_\mathcal{D} \to B$ is a *quasi-unit* for B if the composite map

$$B \simeq \mathbf{1}_\mathcal{D} \otimes B \xrightarrow{u \otimes \mathrm{id}} B \otimes B \xrightarrow{m} B$$

is homotopic to the identity, where m denotes the multiplication on B. By virtue of Theorem HA.5.4.4.5, we can identify $\mathrm{CAlg}(\mathcal{D})$ with the subcategory of $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})$ whose objects are those nonunital commutative algebras which admit quasi-units and whose morphisms are maps which preserve quasi-units. It will therefore suffice to prove the following:

- (a) If A is a commutative algebra object of $\mathcal{C}$, then $F(A) \in \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})$ admits a quasi-unit.
- (b) If A is a commutative algebra object of $\mathcal{C}$ and B is a commutative algebra object of $\mathcal{D}$, then a map of nonunital commutative algebras $F(A) \to B$ preserves quasi-units if and only if the associated map $A \to G(B)$ preserves quasi-units.

To prove (a), let $u_0 : \mathbf{1}_\mathcal{C} \to A$ denote the unit map of A, so that we can regard u_0 as a map of nonunital commutative algebra objects of A. Let $u : \mathbf{1}_\mathcal{D} \to F(A)$ be the map given by the composition

$$\mathbf{1}_\mathcal{D} \xrightarrow{\alpha^{-1}} F(\mathbf{1}_\mathcal{C}) \xrightarrow{F(u_0)} F(A).$$

We claim that u is a quasi-unit for $F(A)$. To prove that, it will suffice to show that the lower square in the diagram

$$\begin{array}{ccc} F(\mathbf{1}_\mathcal{C} \otimes A) & \xrightarrow{F(u \otimes \mathrm{id})} & F(A \otimes A) \\ \downarrow s & & \downarrow \\ F(\mathbf{1}_\mathcal{C}) \otimes F(A) & \xrightarrow{F(u) \otimes \mathrm{id}} & F(A) \otimes F(A) \\ \downarrow \alpha \otimes \mathrm{id} & & \downarrow m \\ \mathbf{1}_\mathcal{D} \otimes F(A) & \longrightarrow & F(A), \end{array}$$

commutes, where m denotes the multiplication on $F(A)$. Since $(\alpha \otimes \mathrm{id}) \circ s$ and α are equivalences, it follows that s is an equivalence. Using the commutativity of the upper square, we are reduced to proving the commutativity of the outer rectangle. Note that if $m' : A \otimes A \to A$ is the multiplication map, then the diagram

$$\begin{array}{ccc} F(\mathbf{1}_{\mathcal{C}} \otimes A) & \xrightarrow{F(u \otimes \mathrm{id})} & F(A \otimes A) \\ \downarrow & & \downarrow{\scriptstyle F(m')} \\ \mathbf{1}_{\mathcal{D}} \otimes F(A) & \longrightarrow & F(A) \end{array}$$

evidently commutes. It will therefore suffice to show that $F(m')$ is homotopic to the composite map

$$F(A \otimes A) \to F(A) \otimes F(A) \xrightarrow{m} F(A).$$

Equivalently, if $v : A \to GFA$ denotes the unit for the adjunction between F and G, it will suffice to show that the diagram

$$\begin{array}{ccccc} A \otimes A & & \xrightarrow{\quad m \quad} & & A \\ \downarrow{\scriptstyle v \otimes v} & & & & \downarrow{\scriptstyle v} \\ GFA \otimes GFA & \longrightarrow & G(FA \otimes FA) & \xrightarrow{G(m)} & GFA \end{array}$$

commutes, which follows immediately from the fact that $v : A \to GFA$ is a map of nonunital commutative algebras. This completes the proof of (a).

We now prove (b). Let $\phi : FA \to B$ be a morphism of nonunital commutative algebra objects of $\mathcal{D}$ and let $\psi : A \to GB$ be the corresponding morphism of nonunital commutative algebra objects of $\mathcal{C}$. We wish to prove that $\psi \circ u_0$ is a quasi-unit for GB if and only if $\phi \circ u$ is a quasi-unit for B. This follows immediately from the observation that $\psi \circ u_0$ and $\phi \circ u$ correspond to one another under the canonical bijection

$$\pi_0 \operatorname{Map}_{\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})}(\mathbf{1}_{\mathcal{D}}, B) \simeq \pi_0 \operatorname{Map}_{\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})}(\mathbf{1}_{\mathcal{C}}, GB).$$

□

Remark 5.6.13. Let $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ and $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})$ be as in Proposition 5.6.10. Suppose that $\mathfrak{m}$ is a nonunital commutative algebra object $\mathcal{C}$, and let $\mathfrak{m} \amalg \mathbf{1}_{\mathcal{C}}$ be the commutative algebra obtained by freely adjoining a unit to $\mathfrak{m}$. Then for every commutative algebra object $A \in \mathcal{D}$, we have canonical homotopy equivalences

$$\begin{aligned} \operatorname{Map}_{\mathrm{CAlg}(\mathcal{D})}(F(\mathfrak{m}) \amalg \mathbf{1}_{\mathcal{D}}, A) &\simeq \operatorname{Map}_{\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{D})}(F(\mathfrak{m}), A) \\ &\simeq \operatorname{Map}_{\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})}(\mathfrak{m}, G(A)) \\ &\simeq \operatorname{Map}_{\mathrm{CAlg}(\mathcal{C})}(\mathfrak{m} \amalg \mathbf{1}_{\mathcal{C}}, G_+(A)) \end{aligned}$$

where $G_+ : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C})$ denotes the functor determined by G. It follows that the functor $F_+ : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{D})$ left adjoint to G_+ carries $\mathfrak{m} \oplus \mathbf{1}_{\mathcal{C}}$ to $F(\mathfrak{m}) \oplus \mathbf{1}_{\mathcal{D}}$.

We now turn to the proof of Theorem 5.6.4. We will need a few preliminaries.

Lemma 5.6.14. *Let X be a quasi-projective k-scheme and let $f : \mathcal{A} \to \mathcal{A}'$ be a morphism of commutative factorization algebras on X. Then f is an equivalence if and only if the induced map $\mathcal{A}^{(1)} \to \mathcal{A}'^{(1)}$ is an equivalence in* $\mathrm{Shv}_\ell(X)$.

Proof. The "only if" direction is clear. Conversely, suppose that f induces an equivalence $\mathcal{A}^{(1)} \to \mathcal{A}'^{(1)}$. We wish to show that f induces an equivalence $\mathcal{A}^{(n)} \to \mathcal{A}'^{(n)}$ for each integer $n \geq 1$. We proceed by induction on n. Since both $\mathcal{A}$ and $\mathcal{A}'$ are !-sheaves, it follows from the inductive hypothesis that for every diagonal inclusion $i : X^m \hookrightarrow X^n$ where $m < n$, the induced map $i^! : \mathcal{A}^{(n)} \to \mathcal{A}'^{(n)}$ is an equivalence of ℓ-adic sheaves on X^m. It will therefore suffice to prove that f induces an equivalence $\mathcal{A}^{(n)}|_{\overset{\circ}{X}{}^n} \to \mathcal{A}'^{(n)}|_{\overset{\circ}{X}{}^n}$, where $\overset{\circ}{X}{}^n \subseteq X^n$ denotes the open subset whose k-valued points are n-tuples of distinct elements of $X(k)$. We have a commutative diagram

$$\begin{array}{ccc} (\mathcal{A}^{(1)})^{\boxtimes n} & \longrightarrow & (\mathcal{A}'^{(1)})^{\boxtimes n} \\ \downarrow & & \downarrow \\ \mathcal{A}^{(n)} & \longrightarrow & \mathcal{A}'^{(n)} \end{array}$$

of ℓ-adic sheaves on X^n. The upper horizontal map is an equivalence by assumption, and the vertical maps are equivalences when restricted to $\overset{\circ}{X}{}^n$ by virtue of our hypothesis that $\mathcal{A}$ and $\mathcal{A}'$ are commutative factorization algebras (Remark 5.6.3). It follows that the lower horizontal map is also an equivalence when restricted to $\overset{\circ}{X}{}^n$, as desired. □

Lemma 5.6.15. *Let X be a quasi-projective k-scheme, let $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X))$, and let $\mathcal{A} = \mathrm{Sym}^{>0}\,\mathcal{F}$ denote the free nonunital commutative algebra object of $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ generated by $\mathcal{F}$. Then $\mathcal{A}$ is a commutative factorization algebra on X.*

Proof. For each integer $n > 0$, let $\overset{\circ}{X}{}^n \subseteq X^n$ denote the open subset whose k-valued points are n-tuples of distinct elements of $X(k)$. Using the description of the convolution product on $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ given in Remark 5.5.17, we compute

$$(\mathcal{F}^{\odot m})^{(n)}|_{\overset{\circ}{X}{}^n} \simeq \bigoplus_{\alpha} (\mathcal{F}^{\boxtimes n})|_{\overset{\circ}{X}{}^n},$$

where the sum is indexed by all bijections $\alpha : \{1, \ldots, m\} \simeq \{1, \ldots, n\}$. It follows that

$$\mathcal{A} \simeq \bigoplus_{m>0} (\mathcal{F}^{\odot m})_{\Sigma_m}$$

is described by the formula $\mathcal{A}^{(n)}|_{\overset{\circ}{X}{}^n} \simeq \mathcal{F}^{\boxtimes n}|_{\overset{\circ}{X}{}^n}$. In particular, we have $\mathcal{A}^{(1)} \simeq \mathcal{F}$, so that $\mathcal{A}^{(n)}|_{\overset{\circ}{X}{}^n} \simeq (\mathcal{A}^{(1)})^{\boxtimes n}|_{\overset{\circ}{X}{}^n}$. It is easy to verify that this identification is induced by the multiplication on $\mathcal{A}$, so that $\mathcal{A}$ is a commutative factorization algebra by virtue of Remark 5.6.3. □

Proof of Theorem 5.6.4. Let $\mathrm{Fact} : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)))$ be a left adjoint to the restriction map. It follows from Proposition 5.6.6 that the functor Fact exists and is fully faithful. We next prove that for each object $\mathcal{A} \in \mathrm{Shv}_\ell(X)$, the image $\mathrm{Fact}(\mathcal{A})$ is a commutative factorization algebra on X. Using Proposition HA.4.7.4.14, we can write $\mathcal{A}$ as the geometric realization of a simplicial object $\mathcal{A}_\bullet$ of $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X))$, where each $\mathcal{A}_n$ is the free nonunital commutative algebra $\mathrm{Sym}^{>0}\,\mathcal{F}_n$ for some ℓ-adic sheaf $\mathcal{F}_n \in \mathrm{Shv}_\ell(X)$. Since the functor Fact preserves colimits, we have $\mathrm{Fact}(\mathcal{A}) \simeq |\,\mathrm{Fact}(\mathcal{A}_\bullet)|$. By virtue of Remark 5.6.2, it will suffice to show that each $\mathrm{Fact}(\mathcal{A}_n)$ is a commutative factorization algebra. Let $\mathcal{F}'_n \in \mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X))$ denote the image of $\mathcal{F}_n$ under the equivalence of ∞-categories $\mathrm{Shv}_\ell(X) \simeq \mathrm{Shv}_\ell^{\mathrm{diag}}(\mathrm{Ran}(X))$. Unwinding the definitions, we see that $\mathrm{Fact}(\mathcal{A}_n)$ can be identified with the free nonunital commutative algebra generated by $\mathcal{F}'_n$, and is therefore a commutative factorization algebra by virtue of Lemma 5.6.15. This completes the proof that the functor Fact takes values in commutative factorization algebras.

We now prove the converse. Let $\mathcal{A}$ be a commutative factorization algebra on X, so that $\mathcal{A}^{(1)} \in \mathrm{Shv}_\ell(X)$ is a nonunital commutative algebra with respect to the !-tensor product. We wish to prove that the counit map $v : \mathrm{Fact}(\mathcal{A}^{(1)}) \to \mathcal{A}$ is an equivalence in $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$.

Since the functor Fact is fully faithful, it follows immediately that v induces an equivalence $\mathrm{Fact}(\mathcal{A}^{(1)})^{(1)} \to \mathcal{A}^{(1)}$. The desired result now follows from Lemma 5.6.14, since both $\mathrm{Fact}(\mathcal{A}^{(1)})$ and $\mathcal{A}$ are commutative factorization algebras. □

5.7. **The Product Formula: Second Formulation.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible over k, and an algebraic curve X over k. Let G be a smooth affine group scheme over X and let $\mathcal{B} \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ be the !-sheaf introduced in Notation 5.4.2. If the fibers of G are connected and the generic fiber of G is semisimple and simply connected, then Theorem 5.4.5 supplies a quasi-isomorphism

$$\int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell).$$

If ν is a k-valued point ν of X^T which can be identified with an *injective* map $\nu : T \to X(k)$, then the costalk $\nu^! \mathcal{B}^{(T)}$ can be identified with the tensor product $\bigotimes_{t \in T} C^*(\mathrm{BG}_{\nu(t)}; \mathbf{Z}_\ell)$. We therefore have a natural map

$$f_\nu : \bigotimes_{t \in T} C^*(\mathrm{BG}_{\nu(t)}; \mathbf{Z}_\ell) \to C^*(X^T; \mathcal{B}^{(T)}) \to \int \mathcal{B}.$$

Heuristically, we can think of the chiral homology $\int \mathcal{B}$ as a continuous tensor product

$$\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$$

obtained by assembling the domains of the maps f_ν as ν varies. Since each factor $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ can be regarded as a commutative algebra object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$, we can regard the tensor product $\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ as a "continuous colimit" of the commutative algebras $\{C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)\}_{x \in X}$ in the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$. Our goal in this section is to make this heuristic precise by establishing the following reformulation of Theorem 5.4.5:

Theorem 5.7.1. *The ℓ-adic sheaf $[\mathrm{BG}]_X$ can be regarded as a commutative algebra object of the ∞-category $\mathrm{Shv}_\ell(X)$ (where we regard $\mathrm{Shv}_\ell(X)$ as endowed with the symmetric monoidal structure given by the !-tensor product; see Proposition 5.5.19), and the canonical map $\phi : [\mathrm{BG}]_X \to \pi^! C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ can be regarded as a morphism of commutative algebra objects of $\mathrm{CAlg}(\mathrm{Shv}_\ell(X))$.*

Let A be a commutative algebra object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$. If the fibers of G are connected and the generic fiber of G is semisimple and simply connected, then composition with ϕ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathbf{Z}_\ell)}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), A) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \pi^! A).$$

Remark 5.7.2. The first part of Theorem 5.7.1 is a special case of a much more general fact: if $\mathcal{C}$ is any Artin stack equipped with a smooth map $\pi : \mathcal{C} \to X$ to a quasi-projective k-scheme X, then the diagonal $\mathcal{C} \to \mathcal{C} \times_X \mathcal{C}$ induces a map of ℓ-adic sheaves

$$[\mathcal{C}]_X \otimes^! [\mathcal{C}]_X \to [\mathcal{C}]_X$$

which exhibits $[\mathcal{C}]_X$ as a commutative algebra with respect to the !-tensor product on $\mathrm{Shv}_\ell(X)$. However, the situation of Theorem 5.7.1 is particularly convenient because Construction 5.4.1 supplies an explicit geometric description of the associated commutative factorization algebra on X.

Construction 5.7.3. For every nonempty finite set T, let $\mathrm{Ran}^G(X)^{(T)}$ denote the prestack $\mathrm{Ran}^G(X) \times_{\mathrm{Ran}(X)} X^T$ (see Construction 5.4.1). For every pair of nonempty finite sets $T, T' \in \mathrm{Fin}^{\mathrm{s}}$, restriction of G-bundles defines a canonical map

$$m_{T,T'} : \mathrm{Ran}^G(X)^{T \amalg T'} \to \mathrm{Ran}^G(X)^T \times \mathrm{Ran}^G(X)^{T'}.$$

Pullback along these restriction maps determines maps of ℓ-adic sheaves

$$\begin{aligned} \mathcal{B}^{(T)} \boxtimes \mathcal{B}^{(T')} &= [\mathrm{Ran}^G(X)^T]_{X^T} \boxtimes [\mathrm{Ran}^G(X)^{T'}]_{X^{T'}} \\ &\simeq [\mathrm{Ran}^G(X)^T \times \mathrm{Ran}^G(X)^{T'}]_{X^{T \amalg T'}} \\ &\to [\mathrm{Ran}^G(X)^{T \amalg T'}]_{X^{T \amalg T'}} \\ &= \mathcal{B}^{(T \amalg T')} \end{aligned}$$

which exhibit $\mathcal{B}$ as a nonunital commutative algebra object of $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$.

More formally, we note that the maps $m_{T,T'}$ (together with the evident commutativity and associativity constraints) exhibit the map $T \mapsto (X^T, \mathrm{Ran}^G(X)^T)$ as a nonunital lax symmetric monoidal functor from $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ to the 2-category $\mathrm{AlgStack}^!$ of Definition 5.1.19. Composing with the symmetric monoidal functor $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ of Construction A.5.26, we can regard the construction

$$T \mapsto (X^T, [\mathrm{Ran}^G(X)^T]_{X^T}) = (X^T, \mathcal{B}^{(T)})$$

as a nonunital lax symmetric monoidal functor from $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ to $\mathrm{Shv}_\ell^!$, so that $\mathcal{B}$ can be regarded as a nonunital commutative algebra object of $\mathrm{Shv}_\ell^!$ by virtue of Remark 5.5.12.

Proposition 5.7.4. *For any smooth affine group scheme G over X, Construction 5.7.3 exhibits $\mathcal{B}$ as a commutative factorization algebra on X.*

Proof. The lax !-sheaf $\mathcal{B}$ is a !-sheaf by virtue of Proposition 5.4.3. It will therefore suffice to show that if T and T' are nonempty finite sets and $U \subseteq X^T \times X^{T'}$ is the open set whose k-valued points correspond to maps $\nu : T \amalg T' \to X(k)$ with $\nu(T) \cap \nu(T') = \emptyset$, then the multiplication on $\mathcal{B}$ induces an equivalence

$$(\mathcal{B}^{(T)} \boxtimes \mathcal{B}^{(T')})|_U \to \mathcal{B}^{(T \amalg T')}|_U$$

of ℓ-adic sheaves on U. This follows immediately from the observation that the map

$$\mathrm{Ran}^G(X)^{T \amalg T'} \times_{X^{T \amalg T'}} U \to (\mathrm{Ran}^G(X)^T \times \mathrm{Ran}^G(X)^{T'}) \times_{X^{T \amalg T'}} U$$

is an equivalence of prestacks. □

Since the construction $\mathcal{F} \mapsto \mathcal{F}^{(1)}$ determines a lax symmetric monoidal functor

$$\mathrm{Shv}_\ell^!(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell(X)$$

(see Proposition 5.5.19), it follows from Construction 5.7.3 that we can regard $\mathcal{B}^{(1)} = [\mathrm{BG}]_X$ as a nonunital commutative algebra object of $\mathrm{Shv}_\ell(X)$. Concretely, the multiplication

$$[\mathrm{BG}]_X \otimes^! [\mathrm{BG}]_X \to [\mathrm{BG}]_X$$

is given by pullback along the diagonal map $\mathrm{BG} \to \mathrm{BG} \times_X \mathrm{BG}$, as indicated in Remark 5.7.2.

Proposition 5.7.5. *Let G be a smooth affine group scheme over X. Then the nonunital commutative algebra $[\mathrm{BG}]_X \in \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X))$ can be promoted (in an essentially unique way) to a commutative algebra structure on $[\mathrm{BG}]_X$.*

Proof. By virtue of Theorem HA.5.4.4.5, it will suffice to show that there exists a map $u : \omega_X \to [\mathrm{BG}]_X$ which is a quasi-unit for the multiplication m on $[\mathrm{BG}]_X$: that is, for which the composite map

$$[\mathrm{BG}]_X \simeq \omega_X \otimes^! [\mathrm{BG}]_X \xrightarrow{u} [\mathrm{BG}]_X \otimes^! [\mathrm{BG}]_X \xrightarrow{m} [\mathrm{BG}]_X$$

is homotopic to the identity. Unwinding the definitions, we see that m is given by the composition

$$\begin{aligned} [\mathrm{BG}]_X \otimes^! [\mathrm{BG}]_X &= \delta^!([\mathrm{BG}]_X \boxtimes [\mathrm{BG}]_X) \\ &\simeq \delta^![\mathrm{BG} \times_{\mathrm{Spec}\, k} \mathrm{BG}]_{X \times X} \\ &\simeq [\mathrm{BG} \times_X \mathrm{BG}]_X \\ &\xrightarrow{\phi} [\mathrm{BG}]_X, \end{aligned}$$

where $\delta : X \times X \times X$ denotes the diagonal map and ϕ is given by pullback along the relative diagonal $\mathrm{BG} \to \mathrm{BG} \times_X \mathrm{BG}$. From this description, it is easy to see that the map $u : \omega_X = [X]_X \to [\mathrm{BG}]_X$ given by pullback along the projection $\mathrm{BG} \to X$ has the desired property. □

To compare $[\mathrm{BG}]_X$ with $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$, we consider another nonunital commutative algebra object of $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ (which is *not* a commutative factorization algebra):

Construction 5.7.6. Let $\mathcal{C}$ be a smooth Artin stack over k which is quasi-compact and has affine diagonal. For every pair of nonempty finite sets T and T', the diagonal of $\mathcal{C}$ determines a map

$$m'_{T,T'} : (\mathcal{C} \times_{\mathrm{Spec}\, k} X^{T \amalg T'}) \to (\mathcal{C} \times_{\mathrm{Spec}\, k} X^T) \times_{\mathrm{Spec}\, k} (\mathcal{C} \times_{\mathrm{Spec}\, k} X^{T'}).$$

As T and T' vary, these maps exhibit the construction $T \mapsto (X^T, \mathcal{C} \times_{\mathrm{Spec}\, k} X^T)$ as a nonunital lax symmetric monoidal functor from $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ to $\mathrm{AlgStack}^!$. Composing with the symmetric monoidal functor $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ of Construction A.5.26, we can regard the construction

$$T \mapsto (X^T, [\mathcal{C} \times_{\mathrm{Spec}\, k} X^T]_{X^T}) = (X^T, C^*(\mathcal{C}; \mathbf{Z}_\ell) \otimes \omega_{X^T})$$

as a nonunital lax symmetric monoidal functor from $(\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}}$ to $\mathrm{Shv}_\ell^!$, so that $C^*(\mathcal{C}; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}$ can be regarded as a nonunital commutative algebra object of $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ by virtue of Remark 5.5.12. Note that this nonunital commutative algebra can be defined more directly by the formula

$$C^*(\mathcal{C}; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} = \mathrm{Ran}(\pi)^! C^*(\mathcal{C}; \mathbf{Z}_\ell),$$

where $\pi : X \to \mathrm{Spec}\, k$ is the projection map.

Let U be a quasi-compact open substack of $\mathrm{Bun}_G(X)$. For any nonempty finite set T, we have an evident evaluation map

$$U \times_{\mathrm{Spec}\, k} X^T \hookrightarrow \mathrm{Bun}_G(X) \times_{\mathrm{Spec}\, k} X^T \to \mathrm{Ran}^G(X)^T.$$

These evaluation maps are compatible with the multiplications of Construction 5.7.3 and 5.7.6 and therefore induce a map

$$\mathcal{B} \to C^*(U; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}$$

of nonunital commutative algebras in $\mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$. Passing to the inverse limit as U varies, we obtain a map of nonunital commutative algebras

$$\mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}.$$

Passing to chiral homology (and invoking Example 5.3.9), we deduce that the canonical map $\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ can be regarded as a map of nonunital commutative algebra objects of $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proposition 5.7.7. *Let G be a smooth affine group scheme over X. Then the construction described above determines a map $\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ of commutative algebra objects of* $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proof. Proposition 5.7.5 implies that the algebra $[\mathrm{BG}]_X = \mathcal{B}^{(1)}$ admits a unit, which we can identify with a map of nonunital commutative algebras $u_0 : \omega_X \to [\mathrm{BG}]_X$ in $\mathrm{Shv}_\ell(X)$. Let $u : \omega_{\mathrm{Ran}(X)} \to \mathcal{B}$ be the induced map of commutative factorization algebras on X. Then the induced map

$$\mathbf{Z}_\ell \simeq \int \omega_{\mathrm{Ran}(X)} \to \int \mathcal{B}$$

is a quasi-unit for the nonunital commutative algebra structure on $\int \mathcal{B}$ (see the proof of Proposition 5.6.10). By virtue of Theorem HA.5.4.4.5, it will suffice to show that the composite map

$$\mathbf{Z}_\ell \simeq \int \omega_{\mathrm{Ran}(X)} \xrightarrow{\int u} \int \mathcal{B} \xrightarrow{\rho} C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

is homotopic to the unit map $v : \mathbf{Z}_\ell \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$. Since X is connected, the composite map

$$\omega_{\mathrm{Ran}(X)} \xrightarrow{u} \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}.$$

is homotopic to the tensor product of some map $v : \mathbf{Z}_\ell \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ with the identity on $\omega_{\mathrm{Ran}(X)}$; we wish to prove that v is homotopic to v'. This assertion can be tested after passing to the costalk at any point $x \in X(k)$, in which case the desired result follows from the description of u_0 given in the proof of Proposition 5.7.5. □

Let $\pi : X \to \mathrm{Spec}\, k$ denote the projection map. It follows from Example 5.6.12 that the functor $\pi^! : \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell}) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(X))$ admits a left adjoint $\pi_\star : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$, and that the functor $\pi_\star$ carries $[\mathrm{BG}]_X$ to the chiral homology $\int \mathcal{B}$. Combining this observation with Proposition 5.7.7, we obtain the following:

Proposition 5.7.8. *Let G be a smooth affine group scheme over X. Then:*

- *The chiral homology $\int \mathcal{B}$ can be regarded as a commutative algebra object of* $\mathrm{Mod}_{\mathbf{Z}_\ell}$.
- *The map $\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ can be regarded as a map of commutative algebra objects of $\mathcal{C}$.*
- *There is a canonical map of commutative algebras $\alpha : [\mathrm{BG}]_X \to \int \mathcal{B}$ with the following universal property: for any commutative algebra object $A \in \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$, composition with α induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(\int \mathcal{B}, A) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \pi^! A).$$

Proof of Theorem 5.7.1. Combine Proposition 5.7.8 with Theorem 5.4.5. □

Remark 5.7.9. It follows from Proposition 5.7.8 that Theorems 5.4.5 and 5.7.1 are equivalent to one another. In other words, if G is a smooth affine group scheme over X, then the canonical map $\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ is an equivalence if and only if the natural map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathbf{Z}_\ell)}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), A) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \pi^! A)$$

is a homotopy equivalence for each $A \in \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$.

6. Calculation of the Trace

Let X_0 be an algebraic curve defined over a finite field $\mathbf{F}_q$, and let G_0 be a smooth affine group scheme over a X_0 with connected fibers. Let $X = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} X_0$ and $G = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} G_0$ denote the $\overline{\mathbf{F}}_q$-schemes associated to X_0 and G_0, respectively. Let ℓ be a prime number which is invertible in $\mathbf{F}_q$, and suppose we have fixed an embedding $\mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. Recall that our goal in this paper is to compute the trace

$$\operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)).$$

Let $\mathcal{B}$ denote the !-sheaf on $\operatorname{Ran}(X)$ defined in §5.4. If the generic fiber of G is semisimple and simply connected, then Theorem 5.4.5 asserts that the canonical map

$$\int \mathcal{B} = \varinjlim_{T} C^*(X^T; \mathcal{B}^{(T)}) \to C^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$$

is an equivalence. Let us assume this result for the moment, and see how it relates to the calculation of Tamagawa numbers.

For every positive integer n, let $\overset{\circ}{X}{}^n \subseteq X^n$ denote the open subset whose $\overline{F}_q$-points are n-tuples of *distinct* points in X, and let $\overset{\circ}{X}{}^{(n)}$ denote the quotient of $\overset{\circ}{X}{}^n$ by the (free) action of the symmetric group Σ_n. The restriction $\mathcal{B}^{(n)}|_{\overset{\circ}{X}{}^n}$ descends to an ℓ-adic sheaf $\mathcal{B}_n$ on $\overset{\circ}{X}{}^{(n)}$, and we have a canonical equivalence

$$C^*(\overset{\circ}{X}{}^{(n)}; \mathcal{B}_n) \simeq C^*(\overset{\circ}{X}{}^n; \mathcal{B}^{(n)}|_{\overset{\circ}{X}{}^n})^{\Sigma_n}.$$

Using Lemmas 5.3.14 and 5.3.15, we see that the chiral homology $\int \mathcal{B}$ admits a filtration

$$0 \to \int^{(1)} \mathcal{B} \to \int^{(2)} \mathcal{B} \to \cdots$$

whose successive quotients can be identified with $C^*(\overset{\circ}{X}{}^{(n)}; \mathcal{B}_n)$. Modulo issues of convergence, this leads to a formula

$$\operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)) = \sum_{n>0} \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\overset{\circ}{X}{}^{(n)}; \mathcal{B}_n)).$$

Each $\overset{\circ}{X}{}^{(n)}$ is a quasi-projective $\overline{F}_q$-scheme which is defined over $\mathbf{F}_q$. Invoking the Grothendieck-Lefschetz trace formula (in its Verdier dual incarnation, expressed as Theorem 1.3.2 in the case of constant coefficients), we expect the individual terms to be given by

$$\operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\overset{\circ}{X}{}^{(n)}; \mathcal{B}_n)) = \sum_{\eta} \operatorname{Tr}(\operatorname{Frob}^{-1} | \eta^! \mathcal{B}_n),$$

where the sum is taken over the (finite) set of all maps $\eta : \operatorname{Spec} \overline{\mathbf{F}}_q \to \overset{\circ}{X}{}^{(n)}$ which are defined over $\mathbf{F}_q$. Unwinding the definitions, we can identify such points with subsets $T = \{y_1, \ldots, y_n\} \subseteq X(\overline{\mathbf{F}}_q)$ which have cardinality exactly n and are invariant under the action of the Galois group $\operatorname{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$. The latter condition holds if and only if we can write T as the inverse image of a finite set $\{x_1, \ldots, x_m\}$ of closed points of X_0. In this case, we have $n = |T| = \deg(x_1) + \cdots + \deg(x_m)$. Moreover, the group scheme $G_T = G_{y_1} \times \cdots \times G_{y_n}$ can be written as a product $\prod_{1 \leq i \leq m} G_{x_i}$, where each G_{x_i} denotes the base change to $\overline{\mathbf{F}}_q$ of the group scheme given by the

Weil restriction of the fiber G_{0x_i} along the map $\operatorname{Spec}\kappa(x_i) \to \operatorname{Spec}\mathbf{F}_q$. We therefore expect the formula

$$\begin{aligned}\operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell) &= \sum_{\{x_1,\ldots,x_m\}\subseteq X_0} \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\prod \mathrm{BG}_{x_i}; \mathbf{Z}_\ell)) \\ &= \sum_{\{x_1,\ldots,x_m\}\subseteq X_0} \prod_{1\leq i\leq m} \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\mathrm{BG}_{x_i}; \mathbf{Z}_\ell)) \\ &= -1 + \prod_{x\in X_0} (1 + \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Z}_\ell))).\end{aligned}$$

Unfortunately, this heuristic calculation is nonsensical: the infinite product (which is taken over all closed points of the curve X_0) does not converge, since the individual factors $1 + \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\mathrm{BG}_{x_i}; \mathbf{Z}_\ell))$ accumulate to 2.

One strategy for resolving the issue is to carry out a "reduced" version of the preceding discussion. In §8, we will introduce a new !-sheaf $\mathcal{B}_{\mathrm{red}} \in \operatorname{Shv}^!_\ell(\operatorname{Ran}(X))$ whose costalk at a point $T = \{y_1, \ldots, y_n\}$ of $\operatorname{Ran}(X)$ can be identified with the tensor product

$$\bigotimes_{1\leq i\leq n} C^*_{\mathrm{red}}(\mathrm{BG}_{y_i}; \mathbf{Z}_\ell),$$

where $C^*_{\mathrm{red}}(\mathrm{BG}_{y_i}; \mathbf{Z}_\ell)$ denotes the *reduced* ℓ-adic cochain complex of the classifying stack BG_{y_i}. Repeating the preceding calculations, we obtain an expectation

$$\begin{aligned}\operatorname{Tr}(\operatorname{Frob}^{-1} \mid \int \mathcal{B}_{\mathrm{red}}) &= -1 + \prod_{x\in X_0} (1 + \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{red}}(\mathrm{BG}_x; \mathbf{Z}_\ell))) \\ &= -1 + \prod_{x\in X_0} \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Z}_\ell)).\end{aligned} \tag{11}$$

Using Theorem 5.4.5, we will show that the chiral homology $\int \mathcal{B}_{\mathrm{red}}$ can be identified with the reduced cochain complex $C^*_{\mathrm{red}}(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$ (Theorem 8.2.14), from which we can repeat the above reasoning to obtain the desired product formula

$$\operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell) = \prod_{x\in X_0} \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Z}_\ell).$$

However, we still do not know how to make this heuristic calculation precise. The problem is that the spaces $\overset{\circ}{X}{}^{(n)}$ become increasingly complicated as n grows, so it is hard to verify the convergence of expressions like

$$\sum_{n>0} \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\overset{\circ}{X}{}^{(n)}; \mathcal{B}_n)).$$

To circumvent this difficulty, we will consider a *different* filtration of the ℓ-adic cochain complex $C^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$. Roughly speaking, the idea is to regard the reduced cochain complex $\mathfrak{m} = C^*_{\mathrm{red}}(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$ as an "ideal" in $C^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$ and to filter $C^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$ by the powers of $\mathfrak{m}$. More generally, suppose that A is any commutative algebra object of $\operatorname{Mod}_{\mathbf{Z}_\ell}$ which is equipped with an augmentation $\epsilon : A \to \mathbf{Z}_\ell$, and let $\mathfrak{m}_A$ denote the fiber $\operatorname{fib}(\epsilon)$. In §6.1, we will construct a tower

$$\cdots \to \mathfrak{m}_A^{(3)} \to \mathfrak{m}_A^{(2)} \to \mathfrak{m}_A^{(1)} = \mathfrak{m}_A$$

which can be regarded as a "derived" $\mathfrak{m}_A$-adic filtration of A. Moreover, we show that successive quotients appearing in this filtration can be identified with the symmetric algebra on the cofiber $\operatorname{cofib}(\mathfrak{m}_A^{(2)} \to \mathfrak{m}_A)$ (Proposition 6.1.17), which we will refer to as the *cotangent fiber* of A and denote by $\cot(A)$ (the homologies of the chain complex $\cot(A)$ are also referred to in the literature as the *topological André-Quillen homology* or the Γ-*homology* of A).

Let us now specialize to the case where $A = C^*(\mathcal{C}; \mathbf{Z}_\ell)$ for some prestack $\mathcal{C}$ over k. We will show that if $\mathrm{H}^0(\mathcal{C}; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$ (a condition which is satisfied when $\mathcal{C} = \mathrm{Bun}_G(X)$, by virtue of Theorem 8.3.1), then the $\mathfrak{m}$-adic filtration of A converges after inverting ℓ: in other words, the inverse limit of the tower

$$\cdots \to \mathfrak{m}_A^{(3)}[\ell^{-1}] \to \mathfrak{m}_A^{(2)}[\ell^{-1}] \to \mathfrak{m}_A^{(1)}[\ell^{-1}]$$

vanishes (see Proposition 6.1.18). It follows that the cohomology of $A[\ell^{-1}]$ can be computed by means of a spectral sequence whose second page can be identified with the symmetric algebra on $\cot(A)[\ell^{-1}]$. If $\mathcal{C}$ is defined over a finite field $\mathbf{F}_q \subseteq k$, we can use this spectral sequence to calculation the trace of the arithmetic Frobenius Frob^{-1} on $\mathrm{H}^*(\mathcal{C}; \mathbf{Z}_\ell)$ in terms of the trace of its powers Frob^{-n} on $\cot(A)$ (Proposition 6.3.4). We will describe this calculation (and treat the relevant convergence issues) in §6.3.

In §6.4, we specialize further to the case where $A = C^*(\mathrm{BH}; \mathbf{Z}_\ell)$ for some connected algebraic group H defined over k. In this case, the cotangent fiber $\cot(A)[\ell^{-1}]$ can be identified (up to a Tate twist) with the *motive* of the group H, as introduced in [21] (at least for reductive groups H). In the special case where H is defined over a finite field $\mathbf{F}_q \subseteq k$, we can use Proposition 6.3.4 to infer a version of the Grothendieck-Lefschetz trace formula for the classifying stack BH (Proposition 6.4.12).

The formalism of §6.1 is quite general, and can be applied to augmented commutative algebras in ∞-categories other than $\mathrm{Mod}_{\mathbf{Z}_\ell}$. For example, if BG denotes the classifying stack of G (regarded as a group scheme over X), then the relative cohomology sheaf $[\mathrm{BG}]_X$ can be regarded as a commutative algebra object of the ∞-category $\mathrm{Shv}_\ell(X)$ (equipped with the !-tensor product of §4.6), so that we can study the cotangent fiber $\cot[\mathrm{BG}]_X \in \mathrm{Shv}_\ell(X)$. In §6.2, we construct a canonical map

$$\rho_0 : C^*(X, \cot[\mathrm{BG}]_X) \to \cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell),$$

which we prove to be an equivalence using Theorem 5.7.1 (Theorem 6.2.4). As we explain in §6.2, this can be regarded as an analogue of the Atiyah-Bott calculation of $\mathrm{H}^*(\mathrm{Bun}_{\mathrm{GL}_n}(\Sigma); \mathbf{Q})$ when Σ is a compact Riemann surface; consequently, we will refer to Theorem 6.2.4 as the *Atiyah-Bott formula*. When combined with the results of §6.3 and the connectivity of $\mathrm{Bun}_G(X)$, the Atiyah-Bott formula leads immediately to a calculation of $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$, as we explain in §6.5.

6.1. **The Cotangent Fiber.** Let $X = \operatorname{Spec} A$ be an affine algebraic variety over a field k and let $x \in X(k)$ be a k-valued point of X, corresponding to a k-algebra homomorphism $\epsilon : A \to k$. We let $\mathfrak{m}_x = \ker(\epsilon)$, so that $\mathfrak{m}_x$ is a maximal ideal of A. The *Zariski cotangent space* of X at the point x is defined to be the k-vector space given by the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$. For each $n \geq 0$, there is an evident surjective map

$$\mathrm{Sym}^n(\mathfrak{m}_x/\mathfrak{m}_x^2) \to \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1},$$

which is an isomorphism if X is smooth at the point x. Consequently, the structure of the completed local ring $\widehat{A} = \varprojlim A/\mathfrak{m}_x^n$ is in some sense controlled by the finite-dimensional vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. Then one can consider commutative algebra objects $A \in \mathcal{C}$ equipped with an augmentation $\epsilon : A \to \mathbf{1}$ (here $\mathbf{1}$ denotes the unit object of $\mathcal{C}$). For every such pair (A, ϵ), one can consider an analogue of the Zariski cotangent space, which we will refer to as the *cotangent fiber* of A and denote by $\cot(A)$ (Definition 6.1.6). Our goal in this section is to review some elementary properties of the construction $A \mapsto \cot(A)$.

Notation 6.1.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category which we regard as fixed throughout this section. We will assume that $\mathcal{C}$ is stable, presentable, and that the tensor product

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

preserves colimits separately in each variable. Let $\mathrm{CAlg}(\mathcal{C})$ denote the ∞-category of commutative algebra object of $\mathcal{C}$ and let $\mathbf{1}$ denote the unit object of $\mathcal{C}$, which we identify with the initial object of $\mathrm{CAlg}(\mathcal{C})$.

For $A \in \mathrm{CAlg}(\mathcal{C})$, we define an *augmentation* on A to be a map of commutative algebra objects $\epsilon : A \to \mathbf{1}$. An *augmented commutative algebra object of* $\mathcal{C}$ is a pair (A, ϵ), where A is a commutative algebra object of $\mathcal{C}$, and ϵ is an augmentation on A. The collection of augmented commutative algebra objects of $\mathcal{C}$ can be organized into an ∞-category $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C})_{/\mathbf{1}}$.

If $\mathcal{C}$ admits a unit object and (A, ϵ) is an augmented commutative algebra object of $\mathcal{C}$, we let $\mathfrak{m}_A$ denote the fiber of the augmentation map $\epsilon : A \to \mathbf{1}$. We will refer to $\mathfrak{m}_A$ as the *augmentation ideal* of A. Note that $\mathfrak{m}_A$ inherits the structure of a nonunital commutative algebra object of $\mathcal{C}$. Moreover, the construction $A \mapsto \mathfrak{m}_A$ determines an equivalence from the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$ of augmented commutative algebra objects of $\mathcal{C}$ to the ∞-category $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$ of nonunital commutative algebra objects of $\mathcal{C}$ (Proposition HA.5.4.4.10).

Definition 6.1.2. Let $\mathrm{Fin}^{\mathrm{s}}$ denote the category whose objects are finite sets and whose morphisms are surjective maps of nonempty finite sets. For each integer $n \geq 0$, we let $\mathrm{Fin}^s_{\geq n}$ denote the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by those finite sets which have cardinality $\geq n$.

Suppose that A is an augmented commutative algebra object of $\mathcal{C}$. Then the construction $S \mapsto \mathfrak{m}_A^{\otimes S}$ determines a functor $\mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$. For each integer $n > 0$, we let $\mathfrak{m}_A^{(n)}$ denote the colimit

$$\varinjlim_{S \in \mathrm{Fin}^s_{\geq n}} \mathfrak{m}^{\otimes S}.$$

By convention, we set $\mathfrak{m}_A^{(0)} = A$.

Example 6.1.3. The category $\mathrm{Fin}^s_{\geq 1}$ has a final object, given by a 1-element set. It follows that for every augmented commutative algebra object A of $\mathcal{C}$, we have a canonical equivalence $\mathfrak{m}_A^{(1)} \simeq \mathfrak{m}$.

Example 6.1.4. Let k be a field and let A be an augmented commutative algebra over k (which we regard as a chain complex concentrated in degree zero), with augmentation ideal $\mathfrak{m}_A$. Then we can regard $\mathfrak{m}_A$ as a nonunital commutative algebra object of the symmetric monoidal ∞-category Mod_k. Then we can think of the object $\mathfrak{m}_A^{(n)} \in \mathrm{Mod}_k$ as a “derived version” of the usual nth power ideal $\mathfrak{m}_A^n \subseteq A$. Multiplication in A determines a compatible family of maps $\mathfrak{m}_A^{\otimes S} \to \mathfrak{m}_A^n$, which can be amalgamated to give a map $\mathfrak{m}_A^{(n)} \to \mathfrak{m}_A^n$. One can show that this map is an equivalence if k is of characteristic zero and A is smooth over k (this follows from Proposition 6.1.17 below).

Remark 6.1.5. Let A be an augmented commutative algebra object of $\mathcal{C}$. Then the inclusions of categories

$$\cdots \mathrm{Fin^s}_{\geq 3} \hookrightarrow \mathrm{Fin^s}_{\geq 2} \hookrightarrow \mathrm{Fin^s}_{\geq 1}$$

determine maps

$$\cdots \to \mathfrak{m}_A^{(3)} \to \mathfrak{m}_A^{(2)} \to \mathfrak{m}_A^{(1)} \simeq \mathfrak{m}_A,$$

depending functorially on A.

Definition 6.1.6. Let A be an augmented commutative algebra object of $\mathcal{C}$. We let $\cot(A)$ denote the cofiber of the canonical map $\mathfrak{m}_A^{(2)} \to \mathfrak{m}_A^{(1)} \simeq \mathfrak{m}_A$. We will refer to $\cot(A)$ as the *cotangent fiber* of A.

Remark 6.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable stable symmetric monoidal ∞-categories for which the tensor product functors

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

preserve colimits separately in each variable, and let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Then F carries augmented commutative algebra objects A of $\mathcal{C}$ to augmented commutative algebra objects $F(A)$ of $\mathcal{D}$. If F preserves colimits, then we have a canonical equivalence $\cot(F(A)) \simeq F(\cot(A))$ for each $A \in \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$.

Example 6.1.8. Let V be an object of $\mathcal{C}$ and let $\mathrm{Sym}^*(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$ denote the free commutative algebra object of $\mathcal{C}$ generated by V. The zero map $V \to \mathbf{1}$ determines an augmentation $\epsilon : \mathrm{Sym}^*(V) \to \mathbf{1}$, whose fiber is given by $\mathrm{Sym}^{>0}(V) \simeq \bigoplus_{n>0} \mathrm{Sym}^n(V)$. For any finite set S, we can identify $\mathrm{Sym}^{>0}(V)^{\otimes S}$ with the colimit $\varinjlim_{f:T \to S} V^{\otimes T}$, where the colimit is taken over all surjections $f : T \to S$. For $n > 0$, we compute

$$\begin{aligned}
\mathrm{Sym}^{>0}(V)^{(n)} &\simeq \varinjlim_{|S| \geq n} \mathrm{Sym}^{>0}(V)^{\otimes S} \\
&\simeq \varinjlim_{|S| \geq n} \varinjlim_{f:T \to S} V^{\otimes T} \\
&\simeq \varinjlim_{T} \varinjlim_{f:T \to S, |S| \geq n} V^{\otimes T} \\
&\simeq \varinjlim_{T} \begin{cases} V^{\otimes T} & \text{if } |T| \geq n \\ 0 & \text{if } |T| < n. \end{cases} \\
&\simeq \bigoplus_{m \geq n} \mathrm{Sym}^m(V).
\end{aligned}$$

Here in each colimit, we allow T to range over the category of finite sets and bijections and f to range over all surjections. In particular, we have a canonical equivalence $\cot(\mathrm{Sym}^*(V)) \simeq V$.

Proposition 6.1.9. *The formation of cotangent fibers determines a functor* $\cot : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ *which preserves colimits.*

Proof. To show that the functor cot preserves all colimits, it will suffice to show that it preserves sifted colimits and finite coproducts. Since the tensor product on $\mathcal{C}$ preserves colimits separately in each variable, the functor $V \mapsto V^{\otimes S}$ preserves sifted colimits for every finite set S. It follows that the construction $A \mapsto \cot(\mathfrak{m}_A^{(n)})$ commutes with sifted colimits for each n, so that $A \mapsto \cot(A)$ commutes with sifted colimits. Since the functor cot clearly preserves initial objects, we are reduced to showing that it preserves pairwise coproducts. Let A and B be augmented commutative algebra objects of $\mathcal{C}$; we wish to show that the canonical map

$$\cot(A) \oplus \cot(B) \to \cot(A \otimes B)$$

is an equivalence. Resolving the augmentation ideal $\mathfrak{m}_A$ by free augmented commutative algebras, we can reduce to the case where $A \simeq \mathrm{Sym}^*(V)$ for some object $V \in \mathcal{C}$. Similarly, we may suppose that $B \simeq \mathrm{Sym}^*(W)$ for some $W \in \mathcal{C}$. In this case, the desired result follows from Example 6.1.8. $\square$

6.1.1. *Cotangent Fibers and Square-Zero Extensions.* For each object $V \in \mathcal{C}$, let $\mathbf{1} \oplus V$ denote the trivial square-zero extension of $\mathbf{1}$ by V. The construction $V \mapsto \mathbf{1} \oplus V$ determines a functor $\Omega^\infty : \mathcal{C} \to \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$.

Proposition 6.1.10. *The construction* $\cot : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ *is left adjoint to the formation of trivial square-zero extensions* $V \mapsto \mathbf{1} \oplus V$*. In other words, for every augmented commutative algebra* $A \in \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$ *and every object* $V \in \mathcal{C}$*, we have a canonical homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})}(A, \mathbf{1} \oplus V) \simeq \mathrm{Map}_{\mathcal{C}}(\cot(A), V).$$

Proof. Theorem HA.7.3.4.13 implies that the functor Ω^∞ exhibits $\mathcal{C}$ as a stabilization of the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$. In particular, we have an adjunction

$$\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{C},$$

where $\Sigma^\infty : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ denotes the absolute cotangent complex functor introduced in Definition HA.7.3.2.14. The functor Σ^∞ is universal among colimit-preserving functors from $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$ to stable ∞-categories. It follows from Proposition 6.1.9 that the formation of cotangent fibers factors as a composition

$$\cot : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \xrightarrow{\Sigma^\infty} \mathcal{C} \xrightarrow{\lambda} \mathcal{C},$$

where λ is some functor from $\mathcal{C}$ to itself. Using Example 6.1.8, we obtain equivalences of functors

$$\mathrm{id}_{\mathcal{C}} \simeq \cot \circ \mathrm{Sym}^* \simeq \lambda \circ (\Sigma^\infty \circ \mathrm{Sym}^*) \simeq \lambda,$$

so that λ is equivalent to the identity functor. □

Remark 6.1.11. Let k be a field and let A be an augmented commutative algebra object of Mod_k. We can identify $I = \mathrm{H}^*(\mathfrak{m}_A)$ with a maximal ideal in the graded-commutative ring $\mathrm{H}^*(A)$. We may therefore consider the (purely algebraic) Zariski cotangent space I/I^2. Note that I^2 is contained in the image of the map

$$\mathrm{H}^*(\mathfrak{m}_A^{\otimes 2}) \to \mathrm{H}^*(\mathfrak{m}_A) = I,$$

and therefore also in the kernel of the map $\mathrm{H}^*(\mathfrak{m}_A) \to \mathrm{H}^*(\cot(A))$. We therefore obtain a canonical comparison map

$$I/I^2 \to \mathrm{H}^*(\cot(A)).$$

Proposition 6.1.12. *Let k be a field of characteristic zero, let A be an augmented commutative algebra object of* CAlg_k*, and suppose that the cohomology* $\mathrm{H}^*(A)$ *is a graded polynomial ring (that is,* $\mathrm{H}^*(A)$ *is a tensor product of a polynomial ring on generators of even degree and an exterior algebra on generators of odd degree). Then the comparison map* $I/I^2 \to \mathrm{H}^*(\cot(A))$ *of Remark 6.1.11 is an isomorphism.*

Proof. Choose homogeneous polynomial generators $\{t_i\}_{i \in I}$ of $\mathrm{H}^*(A)$ which are annihilated by the augmentation map $\epsilon : A \to k$. Let V denote the graded vector space freely generated by homogeneous elements $\{T_i\}_{i \in I}$ with $\deg(T_i) = \deg(t_i)$ and regard V as a chain complex with trivial differential. Then we can choose a map of chain complexes $\phi_0 : V \to \mathfrak{m}_A$ which carries each T_i to a cycle representing the homology class t_i. Then ϕ_0 extends to a map of augmented commutative algebras $\phi : \mathrm{Sym}^*(V) \to A$. The assumption that k has characteristic zero guarantees that the cohomology of $\mathrm{Sym}^*(V)$ is a graded polynomial ring on the generators T_i, so that ϕ is an equivalence. It follows from Example 6.1.8 that ϕ determines an equivalence

$$V \simeq \cot(\mathrm{Sym}^*(V)) \to \cot(A)$$

in Mod_k. □

Example 6.1.13. Let k be a field of characteristic zero (or, more generally, any $\mathbf{Q}$-algebra), and let A be an augmented commutative k-algebra with maximal ideal $\mathfrak{m}_A$. The *cotangent complex* $L_{A/k}$ is a chain complex of A-modules, obtained from the simplicial A-module $A \otimes_{P^\bullet} \Omega_{P^\bullet/k}$, where $P^\bullet$ is a simplicial resolution of A by free k-algebras. One can show that the cotangent fiber $\cot(A)$ is given by the (derived) tensor product $k \otimes_A L_{A/k}$. This follows from Proposition 6.1.12 when A is a free algebra over k, and the general case can be reduced to the case of free algebras using Proposition 6.1.9 below.

More generally, if k is arbitrary commutative ring and $A \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Mod}_k)$, then the cotangent fiber $\cot(A)$ can be identified with the tensor product $k \otimes_A L^t_{A/k}$, where $L^t_{A/k}$ denotes the complex of *topological* André-Quillen chains of A over k (Proposition 6.1.10).

Example 6.1.14 (Rational Homotopy Theory)**.** Let X be a simply connected topological space, and assume that the cohomology ring $\mathrm{H}^*(X;\mathbf{Q})$ is finite-dimensional in each degree. Let $x \in X$ be a base point, so that x determines an augmentation $C^*(X;\mathbf{Q}) \to C^*(\{x\};\mathbf{Q}) \simeq \mathbf{Q}$ of commutative algebra objects of the ∞-category $\mathrm{Mod}_{\mathbf{Q}}$. We will denote the augmentation ideal by $C^*_{\mathrm{red}}(X;\mathbf{Q})$. Then the cotangent fiber of $C^*_{\mathrm{red}}(X;\mathbf{Q})$ is a cochain complex M. One can show that the cohomologies of this chain complex are given by

$$\mathrm{H}^n(M) = \mathrm{Hom}(\pi_n(X,x), \mathbf{Q}).$$

Remark 6.1.15. Let k be a field of characteristic zero, and let A be an augmented commutative algebra object of Mod_k. One can show that the shifted dual $\Sigma^{-1}\cot(A)^\vee$ of the cotangent fiber $\cot(A)$ is quasi-isomorphic to the underlying chain complex of a differential graded Lie algebra which depends functorially on A. In other words, the construction

$$A \mapsto \Sigma^{-1}\cot(A)^\vee$$

determines a contravariant functor from the ∞-category of augmented commutative algebra objects of Mod_k to the ∞-category of differential graded Lie algebras over k. This construction is adjoint to the functor $\mathfrak{g} \mapsto C^*(\mathfrak{g})$ which carries a differential graded Lie algebra $\mathfrak{g}$ to the Chevalley-Eilenberg complex which computes the Lie algebra cohomology of $\mathfrak{g}$. See §SAG.4.2 for more details.

6.1.2. *The* $\mathfrak{m}$*-adic Filtration.* Let k be a field of characteristic zero and let A be an augmented commutative algebra object of Mod_k. In good cases, one can recover the tower $\{\mathrm{cofib}(\mathfrak{m}_A^{(n)} \to A)\}_{n\geq 1}$ from the cotangent fiber $\cot(A)$ together with the Lie algebra structure on $\Sigma^{-1}\cot(A)^\vee$. However, for our applications in this paper, it will be sufficient to describe the *successive quotients* of the filtration $\mathfrak{m}_A^{(n)}$. This does not require us to consider Lie algebra structure at all, and works without any restrictions on A or $\mathcal{C}$:

Construction 6.1.16. Let A be an augmented commutative algebra object of $\mathcal{C}$, so that its cotangent fiber is given by

$$\cot(A) \simeq \mathrm{cofib}(\varinjlim_{|T|\geq 2} \mathfrak{m}_A^{\otimes T} \to \varinjlim_{|T|\geq 1} \mathfrak{m}_A^{\otimes T}).$$

An easy calculation shows that for every finite set S, we can identify $\cot(\mathfrak{m}_A)^{\otimes S}$ with the cofiber

$$\mathrm{cofib}(\varinjlim_{f:T \overset{\not\sim}{\to} S} \mathfrak{m}^{\otimes T} \to \varinjlim_{f:T\twoheadrightarrow S} \mathfrak{m}_A^{\otimes T}),$$

where the colimits are taken over the category of all finite sets T equipped with a surjection $f: T \to S$ (which, on the left hand side, is required to be non-bijective).

Let $\mathcal{J}$ denote the category whose objects are finite sets T equipped with an equivalence relation E such that $|T/E| = n$, where a morphism from (T,E) to (T',E') is a surjection of finite sets $\alpha : T \to T'$ such that xEy if and only if $\alpha(x)E'\alpha(y)$. Let $\mathcal{J}_0$ denote the full subcategory of $\mathcal{J}$ spanned by those pairs (T, E) where $|T| > n$. Then the above considerations determine an equivalence

$$\mathrm{Sym}^n \cot(\mathfrak{m}_A) \simeq \mathrm{cofib}(\varinjlim_{(T,E)\in\mathcal{J}_0} \mathfrak{m}_A^{\otimes T} \to \varinjlim_{(T,E)\in\mathcal{J}} \mathfrak{m}_A^{\otimes T}).$$

We have an evident commutative diagram

$$\begin{array}{ccc} \mathcal{J}_0 & \longrightarrow & \mathcal{J} \\ \downarrow & & \downarrow \\ \mathrm{Fin}^s_{\geq n+1} & \longrightarrow & \mathrm{Fin}^s_{\geq n}, \end{array}$$

which determines a map

$$\theta : \mathrm{Sym}^n \cot(\mathfrak{m}_A) \to \mathrm{cofib}(\mathfrak{m}_A^{(n+1)} \to \mathfrak{m}_A^{(n)}).$$

Proposition 6.1.17. *Let A be an augmented commutative algebra object of $\mathcal{C}$. Then, for each integer $n \geq 0$, Construction 6.1.16 determines an equivalence $\mathrm{Sym}^n \cot(A) \to \mathrm{cofib}(\mathfrak{m}_A^{(n+1)} \to \mathfrak{m}_A^{(n)})$. In other words, we have a fiber sequence*

$$\mathfrak{m}_A^{(n+1)} \to \mathfrak{m}_A^{(n)} \to \mathrm{Sym}^n \cot(\mathfrak{m}_A).$$

Proof. The case $n = 0$ follows immediately from our convention $\mathfrak{m}_A^{(0)} = A$. We will therefore assume $n > 0$. Let $F : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$ denote the functor given by $F(S) = \mathfrak{m}_A^S$. For every category $\mathcal{I}$ equipped with a forgetful functor $\mathcal{I} \to \mathrm{Fin}^{\mathrm{s}}$, we let $F|_{\mathcal{I}}$ denote the restriction of F to $\mathcal{I}$, and $\varinjlim(F|_{\mathcal{I}})$ the colimit of $F|_{\mathcal{I}}$ (regarded as a diagram in $\mathcal{C}$). Unwinding the definitions, we wish to prove that the diagram σ :

$$\begin{array}{ccc} \varinjlim(F|_{\mathcal{J}_0}) & \longrightarrow & \varinjlim(F|_{\mathcal{J}}) \\ \downarrow & & \downarrow \\ \varinjlim(F|_{\mathrm{Fin}^s_{\geq n+1}}) & \longrightarrow & \varinjlim(F|_{\mathrm{Fin}^s_{\geq n}}) \end{array}$$

is a pushout diagram in the ∞-category $\mathcal{C}$. We will show that this holds for *any* functor $F : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$.

Let $F' : \mathrm{Fin}^s_{\geq n} \to \mathcal{C}$ be a left Kan extension of the functor $F|_{\mathrm{Fin}^s_{\geq n+1}}$ along the inclusion

$$\mathrm{Fin}^s_{\geq n+1} \hookrightarrow \mathrm{Fin}^s_{\geq n}.$$

Let $U : \mathcal{J} \to \mathrm{Fin}^s_{\geq n}$ denote the forgetful functor. Note that for every object $(T, E) \in \mathcal{J}$, the functor U induces an equivalence of categories $\mathcal{J}_{/(T,E)} \to (\mathrm{Fin}^s_{\geq n})_{/T}$. It follows that $F' \circ U$ is a left Kan extension of $F|_{\mathcal{J}_0}$ along the inclusion $\mathcal{J}_0 \hookrightarrow \mathcal{J}$. We may therefore identify σ with the commutative diagram

$$\begin{array}{ccc} \varinjlim_{(T,E)\in\mathcal{J}} F'(T) & \longrightarrow & \varinjlim_{(T,E)\in\mathcal{J}} F(T) \\ \downarrow & & \downarrow \\ \varinjlim_{T\in\mathrm{Fin}^s_{\geq n}} F'(T) & \longrightarrow & \varinjlim_{T\in\mathrm{Fin}^s_{\geq n}} F(T). \end{array}$$

For $T \in \mathrm{Fin}^s_{\geq n}$, let $F''(T)$ denote the cofiber of the canonical map $F'(T) \to F(T)$. Unwinding the definitions, we are reduced to proving that the map

$$\theta : \varinjlim_{(T,E) \in \mathcal{J}} F''(T) \to \varinjlim_{T \in \mathrm{Fin}^s_{\geq n}} F''(T)$$

is an equivalence. Let $\mathrm{Fin}^s_{=n}$ denote the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by those sets having cardinality n, and let $\mathcal{J}_{=n} \subseteq \mathcal{J}$ denote the inverse image of $\mathrm{Fin}^s_{=n}$ under U. Note that $F''(T) \simeq 0$ if $|T| > n$, so that F'' is a left Kan extension of its restriction to $\mathrm{Fin}^s_{=n}$ and $F'' \circ U$ is a left Kan extension of its restriction to $\mathcal{J}_{=n}$. We may therefore identify θ with the canonical map

$$\varinjlim_{(T,E) \in \mathcal{J}_{=n}} F''(T) \to \varinjlim_{T \in \mathrm{Fin}^s_{=n}} F''(T).$$

This map is an equivalence because U induces an equivalence of categories $\mathcal{J}_{=n} \to \mathrm{Fin}^s_{=n}$. $\square$

6.1.3. *Convergence.* Let A be an augmented commutative algebra object of $\mathcal{C}$. It follows from Proposition 6.1.17 that the successive quotients of the filtration

$$\cdots \to \mathfrak{m}_A^{(3)} \to \mathfrak{m}_A^{(2)} \to \mathfrak{m}_A^{(1)} \to \mathfrak{m}_A^{(0)} = A$$

can be functorially recovered from the cotangent fiber $\cot(A)$. We next study a condition which guarantees that this filtration is convergent, so that information about the cotangent fiber $\cot(A)$ gives information about the algebra A itself.

Proposition 6.1.18. *Suppose that $\mathcal{C} = \mathrm{Mod}_k$, where k is a field of characteristic zero. Let A be an augmented commutative algebra object of $\mathcal{C}$ whose augmentation ideal $\mathfrak{m}_A$ belongs to $(\mathrm{Mod}_k)_{\leq -1}$. Then, for every integer $n > 0$, the object $\mathfrak{m}_A^{(n)}$ belongs to $(\mathrm{Mod}_k)_{\leq -n}$. In particular, the inverse limit $\varprojlim \mathfrak{m}_A^{(n)}$ vanishes in $\mathcal{C}$.*

The proof of Proposition 6.1.18 depends on the following elementary combinatorial fact about t-structures:

Lemma 6.1.19. *Suppose that the ∞-category $\mathcal{C}$ is equipped with a t-structure which is compatible with filtered colimits (that is, the full subcategory $\mathcal{C}_{\leq 0}$ is closed under filtered colimits). Let P be a partially ordered set, let $\lambda : P \to \mathbf{Z}_{\geq 0}$ be a strictly monotone function, and suppose we are given a functor $G : \mathrm{N}(P)^{\mathrm{op}} \to \mathcal{C}$ such that $G(x) \in \mathcal{C}_{\leq -n-\lambda(x)}$ for each $x \in P$. Then the colimit $\varinjlim G$ belongs to $\mathcal{C}_{\leq -n}$.*

Proof of Proposition 6.1.18. We define a category $\mathcal{I}$ as follows:

- The objects of $\mathcal{I}$ are diagrams

$$S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_d} S_d,$$

  where each S_i is a finite set of cardinality $\geq n$, and each of the maps ϕ_i is surjective but not bijective.
- Let $\vec{S} = (S_0 \to S_1 \to \cdots \to S_d)$ and $\vec{T} = (T_0 \to T_1 \to \cdots \to T_e)$ be objects of $\mathcal{I}$. A morphism from $\vec{S}$ to $\vec{T}$ in $\mathcal{I}$ consists of a map $\alpha : \{0, \ldots, e\} \to \{0, \ldots, d\}$, together with a collection of bijections $S_{\alpha(i)} \simeq T_i$ for which the diagrams

$$\begin{array}{ccc} S_{\alpha(i)} & \longrightarrow & S_{\alpha(i+1)} \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & T_{i+1} \end{array}$$

  commute.

We have an evident forgetful functor $\rho : \mathcal{I} \to \mathrm{Fin}^s_{\geq n}$, given by $(S_0 \to \cdots \to S_d) \mapsto S_0$. We first prove:

$(*)$ The functor ρ is left cofinal.

Fix a finite set T of cardinality $\geq n$, and let $\mathcal{I}_{T/}$ denote the fiber product $\mathcal{I} \times_{\mathrm{Fin}^s_{\geq n}} (\mathrm{Fin}^s_{\geq n})_{T/}$. To prove $(*)$, we must show that each of the categories $\mathcal{I}_{T/}$ has weakly contractible nerve. Unwinding the definitions, we can identify objects of $\mathcal{I}_{T/}$ with chains of surjections

$$T \xrightarrow{\psi} S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_d} S_d,$$

where the maps ϕ_i are not bijective. Let $\mathcal{I}^\circ_{T/}$ denote the full subcategory of $\mathcal{I}_{T/}$ spanned by those objects for which ψ is bijective. Since the inclusion $\mathcal{I}^\circ_{T/} \hookrightarrow \mathcal{I}_{T/}$ admits a right adjoint, it will suffice to prove that the category $\mathcal{I}^\circ_{T/}$ has weakly contractible nerve. This is clear, since $\mathcal{I}^\circ_{T/}$ contains a final object (given by the map $T \xrightarrow{\mathrm{id}} T$).

Let $F : \mathcal{I} \to \mathcal{C}$ denote the functor given by the formula

$$F(S_0 \to \cdots \to S_d) = \mathfrak{m}_A^{\otimes S_0}.$$

It follows from $(*)$ that we can identify $\mathfrak{m}^{(d)}$ with the colimit $\varinjlim(F)$. Let P denote the set of all finite subsets of $\mathbf{Z}_{\geq n}$, partially ordered by inclusion. The construction

$$(S_0 \to S_1 \to \cdots \to S_d) \mapsto \{|S_0|, |S_1|, \ldots, |S_d|\}$$

determines a functor $\rho' : \mathcal{I} \to P^{\mathrm{op}}$. Let $G : \mathrm{N}(P)^{\mathrm{op}} \to \mathcal{C}$ denote a left Kan extension of F along ρ', so that $\mathfrak{m}^{(n)} \simeq \varinjlim_{J \in P} G(J)$.

Fix a finite subset $J \subseteq \mathbf{Z}_{\geq n}$. Since ρ' is a coCartesian fibration, G is given by the formula $G(J) = \varinjlim F|_{\mathcal{J} \times_{P^{\mathrm{op}}} \{J\}}$. For every object $\vec{S} = (S_0 \to \cdots \to S_d)$ in $\mathcal{J} \times_{P^{\mathrm{op}}} \{J\}$, the set S_0 has cardinality $\geq d + |S_d| \geq d + n = |J| + n - 1$. The category $\mathcal{J} \times_{P^{\mathrm{op}}} \{J\}$ is a groupoid in which every object has a finite automorphism group. It follows that $G(J)$ can be written as a direct sum of objects of the form $(\mathfrak{m}_A^{\otimes T})_\Gamma$, where T is a finite set of cardinality $|J| + n - 1$ and Γ is a finite group acting on $\mathfrak{m}_A^{\otimes T}$ via permutations of T. Since k has characteristic zero, it follows that $G(J) \in (\mathrm{Mod}_k)_{\leq 1-n-|J|}$. The desired result now follows from Lemma 6.1.19 (take $\lambda : P \to \mathbf{Z}_{\geq 0}$ to be the function given by $\lambda(J) = |J| - 1$). □

Proof of Lemma 6.1.19. We will prove the following more general assertion: for every simplicial subset $K \subseteq \mathrm{N}(P)^{\mathrm{op}}$, the colimit $\varinjlim(G|_K)$ belongs to $\mathcal{C}_{\leq -n}$. Writing K as a filtered colimit of finite simplicial sets, we may reduce to the case where K is finite. We proceed by induction on the number of nondegenerate simplices of K. If K is empty, there is nothing to prove. Otherwise, we can choose a pushout diagram

$$\begin{array}{ccc} \partial \Delta^m & \longrightarrow & K_0 \\ \downarrow & & \downarrow \\ \Delta^m & \longrightarrow & K. \end{array}$$

Since $\varinjlim(G|_{K_0}) \in \mathcal{C}_{\leq -n}$ by the inductive hypothesis, it will suffice to prove that the cofiber of the canonical map $\theta : \varinjlim(G|_{K_0}) \to \varinjlim(G|_K)$ belongs to $\mathcal{C}_{\leq -n}$. For this, we may replace K by Δ^m and K_0 by $\partial \Delta^m$. Let $x \in P^{\mathrm{op}}$ denote the image of the final vertex $\{m\} \in \Delta^m$. We will prove that $\mathrm{cofib}(\theta) \in \mathcal{C}_{\leq -n-\lambda(x)}$. The proof proceeds by induction on m. If $m = 0$, then $\mathrm{cofib}(\theta) = G(x)$ and there is nothing to prove. If $m > 0$, then the inclusion $\Lambda^m_m \hookrightarrow \Delta^m$ is right anodyne and therefore left cofinal. It follows that the composite map

$$\varinjlim(G|_{\Lambda^m_m}) \xrightarrow{\theta'} \varinjlim(G|_{\partial \Delta^m}) \xrightarrow{\theta} \varinjlim(G_{\Delta^m})$$

is an equivalence, so that $\operatorname{cofib}(\theta) \simeq \Sigma \operatorname{cofib}(\theta')$. Using the pushout diagram of simplicial sets

$$\begin{array}{ccc} \partial \Delta^{m-1} & \longrightarrow & \Delta^{m-1} \\ \downarrow & & \downarrow \iota \\ \Lambda^m_m & \longrightarrow & \partial \Delta^m, \end{array}$$

we can identify $\operatorname{cofib}(\theta')$ with the cofiber of the induced map $\theta'' : \varinjlim(G|_{\partial \Delta^{m-1}}) \to \varinjlim(G|_{\Delta^{m-1}})$. Let $y \in P$ denote the image of the final vertex of Δ^{m-1}. Then $y > x$. Since λ is monotone, we have $\lambda(y) > \lambda(x)$. Using the inductive hypothesis, we deduce that

$$\operatorname{cofib}(\theta'') \in \mathcal{C}_{\leq -n-\lambda(y)} \subseteq \mathcal{C}_{\leq -n-1-\lambda(x)},$$

so that $\operatorname{cofib}(\theta) \simeq \Sigma \operatorname{cofib}(\theta') \simeq \Sigma \operatorname{cofib}(\theta'') \in \mathcal{C}_{\leq -n-\lambda(x)}$, as desired. □

6.2. **The Atiyah-Bott Formula.** Let X be a compact Riemann surface of genus g and let $\operatorname{Bun}_{\mathrm{GL}_n}(X)$ be the moduli stack of rank n vector bundles on X. Then we can write $\operatorname{Bun}_{\mathrm{GL}_n}(X)$ as a disjoint union of connected components

$$\amalg_{d \in \mathbf{Z}} \operatorname{Bun}^d_{\mathrm{GL}_n}(X),$$

where $\operatorname{Bun}^d_{\mathrm{GL}_n}(X)$ denotes the moduli stack of vector bundles on X of rank n and degree d. Each $\operatorname{Bun}^d_{\mathrm{GL}_n}(X)$ determines a complex-analytic stack which has a well-defined homotopy type. In [4], Atiyah and Bott show that the cohomology ring $\mathrm{H}^*(\operatorname{Bun}^d_{\mathrm{GL}_n}(X); \mathbf{Z})$ is isomorphic to the free graded-commutative algebra on homogeneous generators

$$\begin{array}{ccc} \{x_i\}_{2 \leq i \leq n} & \{y_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq 2g} & \{z_i\}_{1 \leq i \leq n} \\ \deg(x_i) = 2i-2 & \deg(y_{i,j}) = 2i-1 & \deg(z_i) = 2i. \end{array}$$

The Atiyah-Bott calculation admits a straightforward generalization to algebraic groups other than GL_n. For simplicity, we will restrict our attention to the case of simply connected groups. Fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let G_0 be a simply connected semisimple algebraic group over k and let BG_0 denote its classifying stack. Then the ℓ-adic cohomology $\mathrm{H}^*(\mathrm{BG}_0; \mathbf{Q}_\ell)$ is isomorphic to a polynomial ring

$$\mathbf{Q}_\ell[t_1, \ldots, t_r]$$

where r is the rank of G_0 and each generator t_i is a homogeneous element of some even degree $d_i \geq 4$. The integers $d_1, \ldots, d_r$ are called the *exponents* of the group G_0. Let X be an algebraic curve of genus g over k and let $G = G_0 \times X$ be the associated constant group scheme over X. One can show that the cohomology ring $\mathrm{H}^*(\operatorname{Bun}_G(X); \mathbf{Q}_\ell)$ is isomorphic to a free graded-commutative algebra on homogenous generators

$$\begin{array}{ccc} \{x_i\}_{1 \leq i \leq r} & \{y_{i,j}\}_{1 \leq i \leq r, 1 \leq j \leq 2g} & \{z_i\}_{1 \leq i \leq r} \\ \deg(x_i) = d_i - 2 & \deg(y_{i,j}) = d_i - 1 & \deg(z_i) = d_i. \end{array}$$

The proof of this assertion can be broken into two steps:

(a) Let us regard the cochain complex $C^*(\operatorname{Bun}_G(X); \mathbf{Q}_\ell)$ as a commutative algebra object of $\operatorname{Mod}_{\mathbf{Q}_\ell}$, with an augmentation

$$\epsilon : C^*(\operatorname{Bun}_G(X); \mathbf{Q}_\ell) \to C^*(\operatorname{Spec} k; \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell$$

given by the base point of $\operatorname{Bun}_G(X)$. Then one can show that the cotangent fiber $\cot(C^*(\operatorname{Bun}_G(X); \mathbf{Q}_\ell))$ is equivalent to the tensor product

$$C_*(X; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \cot(C^*(\mathrm{BG}_0; \mathbf{Q}_\ell)).$$

It follows from Proposition 6.1.12 that the homology of $\cot(C^*(\mathrm{BG}_0; \mathbf{Q}_\ell))$ admits a basis $\{T_i\}_{1 \le i \le r}$ consisting of homogeneous elements of (cohomological) degree $\deg(T_i) = d_i$. One can then calculate that $\cot(C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell))$ is quasi-isomorphic to a graded vector space V on homogeneous generators

$$\{x_i\}_{1 \le i \le r} \qquad \{y_{i,j}\}_{1 \le i \le r, 1 \le j \le 2g} \qquad \{z_i\}_{1 \le i \le r}$$
$$\deg(x_i) = d_i - 2 \qquad \deg(y_{i,j}) = d_i - 1 \qquad \deg(z_i) = d_i.$$

(*b*) According to Remark 6.1.15, the dual

$$\Sigma^{-1} \cot(C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell))^\vee \simeq \Sigma^{-1} V^\vee$$

is quasi-isomorphic to the underlying chain complex of a canonically determined differential graded Lie algebra $\mathfrak{g}$. One can show that this Lie algebra is equivalent to an abelian one and that the natural map

$$C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell) \to C^*(\mathfrak{g})$$

is a quasi-isomorphism, where

$$C^*(\mathfrak{g}) \simeq \prod_{n \ge 0} (\Sigma \mathfrak{g})^\vee \simeq \mathrm{Sym}^*(V)$$

denotes the cohomological Chevalley-Eilenberg complex of $\mathfrak{g}$. Passing to cohomology, it follows that $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ is isomorphic to a polynomial ring on generators

$$\{x_i\}_{1 \le i \le r} \qquad \{y_{i,j}\}_{1 \le i \le r, 1 \le j \le 2g} \qquad \{z_i\}_{1 \le i \le r}$$
$$\deg(x_i) = d_i - 2 \qquad \deg(y_{i,j}) = d_i - 1 \qquad \deg(z_i) = d_i.$$

Remark 6.2.1. We will not discuss assertion (*b*) in this paper. However, it should really be regarded as a formal consequence of (*a*). Set $\mathfrak{g} = \Sigma^{-1} \cot(C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell))$ and $\mathfrak{g}_0 = \Sigma^{-1} \cot(C^*(\mathrm{BG}_0; \mathbf{Q}_\ell))$. The constructions outlined below can be used to produce a canonical map

$$\theta : \mathfrak{g} \to C^*(X; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathfrak{g}_0,$$

and the content of assertion (*a*) is that this map is a quasi-isomorphism. Given a suitably robust formalism for Koszul duality (which we do not discuss here), one can realize θ as a morphism of differential graded Lie algebras. Since the cohomology $\mathrm{H}^*(\mathrm{BG}_0; \mathbf{Q}_\ell)$ is a polynomial ring, the Lie algebra $\mathfrak{g}_0$ can be chosen to be abelian, so that the Lie algebra structure on $\mathfrak{g}$ can likewise be chosen abelian.

Remark 6.2.2. Combining assertion (*a*) with Propositions 6.1.18 and 6.1.17, it follows that there exists a spectral sequence with second page $E_2^{**} \simeq \mathrm{Sym}^*(V)$ which converges to the cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$. It follows from (*b*) that this spectral sequence degenerates. For our purposes, this is irrelevant: our goal is to compute the trace of the arithmetic Frobenius Frob^{-1} on the cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ (in the special case where X is defined over a finite field $\mathbf{F}_q$). Modulo issues of convergence (which we will discuss in §6.5), the existence of the spectral sequence E_*^{**} shows that this is equal to the trace of Frob^{-1} on the symmetric algebra $\mathrm{Sym}^*(V)$, and can therefore be computed directly from the Frobenius eigenvalues of $\mathrm{H}^*(X; \mathbf{Q}_\ell)$ and $\mathrm{H}^*(\mathrm{BG}_0; \mathbf{Q}_\ell)$.

Remark 6.2.3. In [4], Atiyah and Bott actually show that the *integral* cohomology of the moduli stacks $\mathrm{Bun}^d_{\mathrm{GL}_n}(X)$ are graded polynomial rings with the structure indicated above. However, this description is specific to the case of the group GL_n: in general, the cohomology of the moduli stack $\mathrm{Bun}_G(X)$ has torsion.

We now formulate an analogue (a) for the case of a group scheme G over X which is not assumed to be constant (Theorem 6.2.4). At this level of generality, we do not expect the analogue of (b) to hold.

Theorem 6.2.4. *Let k be an algebraically closed field, let $\pi : X \to \operatorname{Spec} k$ exhibit X as an algebraic curve over k, let G be a smooth affine group scheme over X with connected fibers whose generic fiber is semisimple and simply connected, and let ℓ be a prime number which is invertible in k. Let us regard $[\mathrm{BG}]_X$ as a commutative algebra object of $\mathrm{Shv}_\ell(X)$ as in §5.7, with augmentation $[\mathrm{BG}]_X \to \omega_X$ given by the map $X \to \mathrm{BG}$ classifying the trivial G-bundle on X. Then the canonical map*

$$[\mathrm{BG}]_X \to [X \times_{\operatorname{Spec} k} \mathrm{Bun}_G(X)]_X = \pi^! C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

induces an map $\cot[\mathrm{BG}]_X \to \pi^! \cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ which determines an equivalence

$$\theta : \pi_* \cot[\mathrm{BG}]_X \to \cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

in $\mathrm{Shv}_\ell(\operatorname{Spec} k) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$.

Let $C^*(X; \cot[\mathrm{BG}]_X)$ denote the image of $\pi_* \cot[\mathrm{BG}]_X$ under the equivalence of ∞-categories $\mathrm{Shv}_\ell(\operatorname{Spec} k) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$. We have formulated Theorem 6.2.4 as a result about the *integral* ℓ-adic cohomology of $\mathrm{Bun}_G(X)$. However, we will be primarily interested in the following consequence (obtained by passing to global sections and inverting ℓ):

Corollary 6.2.5. *Suppose that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected, and regard $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}]$ as an augmented commutative algebra object of $\mathrm{Mod}_{\mathbf{Q}_\ell}$. Then we have a canonical equivalence*

$$\cot(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}]) \simeq C^*(X; \cot[\mathrm{BG}]_X)[\ell^{-1}].$$

Remark 6.2.6. One can show that for any prestack $\mathcal{C}$ and any base point $\eta : \operatorname{Spec} k \to \mathcal{C}$, the cotangent fiber $\cot C^*(\mathcal{C}; \mathbf{Z}/\ell\mathbf{Z})$ vanishes (when regarded as an augmented commutative algebra object of $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$. It follows from this that multiplication by ℓ induces equivalences

$$\ell : \cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to \cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \qquad \ell : \cot[\mathrm{BG}]_X \to \cot[\mathrm{BG}]_X,$$

so that $\cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ and $C^*(X, \cot[\mathrm{BG}]_X)$ already admit the structure of $\mathbf{Q}_\ell$-modules. In other words, Theorem 6.2.4 and Corollary 6.2.5 are equivalent to one another.

Remark 6.2.7. Let $\mathcal{C}$ be an arbitrary prestack. Then there are (at least) two reasonable definitions for cohomology of $\mathcal{C}$ with coefficients in $\mathbf{Q}_\ell$: one can consider the rational cohomology ring $\mathrm{H}^*(\mathcal{C}; \mathbf{Q}_\ell)$ introduced in §2.3, or consider the integral cohomology ring $\mathrm{H}^*(\mathcal{C}; \mathbf{Z}_\ell)$ and invert ℓ. These are related by a canonical map

$$\mathrm{H}^*(\mathcal{C}; \mathbf{Z}_\ell)[\ell^{-1}] \to \mathrm{H}^*(\mathcal{C}; \mathbf{Q}_\ell).$$

This map is an isomorphism if $\mathcal{C}$ is an algebraic stack of finite type over k (Remark 2.3.30), but not in general (since the operation of inverting ℓ does not commute with inverse limits; see Warning 2.3.29). For our purposes, it will be more convenient work with the second definition. Consequently, the results of this section will be phrased in terms of the $\mathbf{Q}_\ell$-modules $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}]$ and $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}]$. In fact, this makes no difference: one can show that the natural map

$$\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}] \to \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$$

is an isomorphism (at least when the generic fiber of G is semisimple and simply connected; it can fail when $\mathrm{Bun}_G(X)$ has many connected components, for example if $G = \mathrm{GL}_n$), but we will not need this fact.

Example 6.2.8. In the special case where $G = X \times G_0$ is a constant group scheme on X, Corollary 6.2.5 supplies an equivalence

$$\begin{aligned} \cot(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}]) &\simeq C^*(X, \cot[\mathrm{BG}]_X)[\ell^{-1}] \\ &\simeq C^*(X, \cot(C^*(\mathrm{BG}_0; \mathbf{Z}_\ell) \otimes \omega_X))[\ell^{-1}] \\ &\simeq C^*(X, (\cot C^*(\mathrm{BG}_0; \mathbf{Z}_\ell)) \otimes \omega_X)[\ell^{-1}] \\ &\simeq (\cot C^*(\mathrm{BG}_0; \mathbf{Z}_\ell)) \otimes_{\mathbf{Z}_\ell} C^*(X; \omega_X))[\ell^{-1}] \\ &\simeq (\cot C^*(\mathrm{BG}_0; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C_*(X; \mathbf{Z}_\ell))[\ell^{-1}] \\ &\simeq \cot(C^*(\mathrm{BG}_0; \mathbf{Z}_\ell)[\ell^{-1}]) \otimes_{\mathbf{Q}_\ell} C_*(X; \mathbf{Q}_\ell) \end{aligned}$$

as promised in the introduction to this section.

Proof of Theorem 6.2.4. Fix an object $M \in \mathrm{Mod}_{\mathbf{Z}_\ell}$; we wish to show that composition with θ induces a homotopy equivalence

$$\theta_M : \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), M) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\pi_* \cot[\mathrm{BG}]_X, M).$$

Using the universal property of the cotangent fiber (Proposition 6.1.10), we obtain homotopy equivalences

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), M) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), \mathbf{Z}_\ell \oplus M)$$

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\pi_* \cot[\mathrm{BG}]_X, M) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \omega_X \oplus \pi^! M).$$

It will therefore suffice to show that the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), \mathbf{Z}_\ell \oplus M) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \omega_X \oplus \pi^! M) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), \mathbf{Z}_\ell) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \omega_X) \end{array}$$

is a homotopy pullback square. In fact, the horizontal maps in this diagram are homotopy equivalences by virtue of Theorem 5.7.1. □

6.3. **Summable** Frob**-Modules.** Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. Suppose that X and G are defined over a finite field $\mathbf{F}_q \subseteq k$, so that the moduli stack $\mathrm{Bun}_G(X)$ is equipped with a geometric Frobenius map $\mathrm{Frob} : \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X)$. To prove Theorem 1.3.5, we need to compute the trace $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$. However, this requires some care: typically the cohomology groups $\mathrm{H}^n(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ are nonzero for infinitely many values of n. We therefore devote this section to a discussion of some of the convergence issues which arise when forming infinite sums such as

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)) = \sum_{n \geq 0} (-1)^n \, \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^n(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)).$$

Our main result (Proposition 6.3.4) allows us to deduce the convergence of these sums from the the structure of the cotangent fiber $\cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$, which we will discuss in §6.2.

Throughout this section, we fix a prime number ℓ and an embedding of fields $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$.

Definition 6.3.1. Let V^* be a graded vector space over $\mathbf{Q}_\ell$ and F an endomorphism of V^*. We will say that (V^*, F) is *summable* if the following conditions are satisfied:

(1) The vector space V^m is finite dimensional for every integer m.

(2) For each $\lambda \in \mathbf{C}$ and every integer m, let $d_{\lambda,m}$ denote the dimension of the generalized λ-eigenspace of F on the complex vector space $\mathbf{C} \otimes_{\mathbf{Q}_\ell} V^m$. Then the sum

$$\sum_{m,\lambda} d_{\lambda,m} |\lambda|$$

is convergent.

If (V^*, F) is summable, we let $|V^*|_F$ denote the nonnegative real number $\sum_{m,\lambda} d_{\lambda,m}|\lambda|$; we will refer to $|V^*|_F$ as the *norm* of the pair (V^*, F). We let $\mathrm{Tr}(F|V^*)$ denote the complex number $\sum_{m,\lambda} (-1)^m d_{\lambda,m}\lambda$. Note that this sum converges absolutely, and we have $|\mathrm{Tr}(F|V^*)| \leq |V^*|_F$.

Remark 6.3.2. The definition of a summable pair (V^*, F), and the trace $\mathrm{Tr}(F|V^*)$ depend on a choice of embedding $\iota : \mathbf{Q}_\ell \to \mathbf{C}$. However, for the pairs (V^*, F) of interest to us, the traces $\mathrm{Tr}(F|V^*)$ can be shown to be independent of ι.

Remark 6.3.3. Suppose we are given graded $\mathbf{Q}_\ell$ vector spaces V'^*, V^*, and V''^* equipped with endomorphisms F', F, and F'' respectively, together with a long exact sequence

$$\cdots \to V''^{n-1} \to V'^n \to V^n \to V''^n \to V'^{n+1} \to \cdots$$

compatible with the actions of F, F', and F''. If (V'^*, F') and (V''^*, F'') are summable, then (V^*, F) is also summable. Moreover, in this case we have

$$|V^*|_F \leq |V'^*|_{F'} + |V''^*|_{F''} \qquad \mathrm{Tr}(F|V^*) = \mathrm{Tr}(F'|V'^*) + \mathrm{Tr}(F''|V''^*).$$

Proposition 6.3.4. *Let A be an augmented commutative algebra object of $\mathrm{Mod}_{\mathbf{Q}_\ell}$ equipped with an automorphism F. We let $V = \cot(A)$ denote the cotangent fiber of A, so that F determines an automorphism of V (which we will also denote by F). Suppose that the following conditions are satisfied:*

(1) *The augmentation ideal $\mathfrak{m}_A$ belongs to $(\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -1}$.*
(2) *The graded $\mathbf{Q}_\ell$-vector space $\mathrm{H}^*(V)$ is finite-dimensional.*
(3) *For every integer i and every eigenvalue λ of F on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} \mathrm{H}^i(V)$, we have $|\lambda| < 1$.*

Then $(\mathrm{H}^*(A); F)$ is summable. Moreover, we have

$$\mathrm{Tr}(F|\,\mathrm{H}^*(A)) = \exp(\sum_{n>0} \frac{1}{n} \mathrm{Tr}(F^n|\,\mathrm{H}^*(V))),$$

where the sum on the right hand side is absolutely convergent.

Remark 6.3.5. Proposition 6.3.4 asserts that, under mild hypotheses, the trace of F on the cohomology of A is equal to the trace of F on the cohomology of the symmetric algebra $\mathrm{Sym}^*(V)$.

Remark 6.3.6. Let V^* be a finite-dimensional graded $\mathbf{Q}_\ell$-vector space equipped with an automorphism F. We define the *L-function* of the pair (V^*, F) to be the rational function of one variable t given by the formula

$$L_{V^*,F}(t) = \prod_{m \in \mathbf{Z}} \det(1 - tF|V^m)^{(-1)^{m+1}}.$$

An easy calculation yields

$$L_{V^*,F}(t) = \exp(\sum_{n>0} \frac{t^n}{n} \mathrm{Tr}(F^n|\,\mathrm{H}^*(V))).$$

for $|t| < C$, where $C = \sup\{\frac{1}{|\lambda|}\}$ where λ ranges over the eigenvalues of F. In particular, if all of the eigenvalues of F have complex absolute values < 1, then we have

$$L_{V^*,F}(1) = \exp(\sum_{n>0} \frac{1}{n} \operatorname{Tr}(F^n | \operatorname{H}^*(V))).$$

In the situation of Proposition 6.3.4, we can rewrite the conclusion as

$$\operatorname{Tr}(F | \operatorname{H}^*(A)) = L_{\operatorname{H}^*(\cot A),F}(1) = \prod_{m>0} \det(1 - F | \operatorname{H}^m(V))^{(-1)^{m+1}}.$$

Proof of Proposition 6.3.4. Write the graded vector space $\operatorname{H}^*(V)$ as a direct sum $\operatorname{H}^{\text{even}}(V) \oplus \operatorname{H}^{\text{odd}}(V)$. Let $\{\lambda_1, \ldots, \lambda_m\}$ denote the eigenvalues of F on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} \operatorname{H}^{\text{even}}(V)$ (counted with multiplicity), and let $\{\mu_1, \ldots, \mu_{m'}\}$ denote the eigenvalues of F on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} \operatorname{H}^{\text{odd}}(V)$ (again counted with multiplicity). For every integer $n \geq 0$, we set

$$s_n = \sum_{n=n_1+\cdots+n_m+|S|, S \subseteq \{1,\ldots,m'\}} (\prod_{1 \leq i \leq m} |\lambda_i|^{n_i} \prod_{j \in S} |\mu_j|)$$

$$\sigma_n = \sum_{n=n_1+\cdots+n_m+|S|, S \subseteq \{1,\ldots,m'\}} (\prod_{1 \leq i \leq m} \lambda_i^{n_i} \prod_{j \in S} -\mu_j).$$

It follows from (3) that the sum $s_0 + s_1 + s_2 + \cdots$ converges to $\prod_{1 \leq i \leq m} \frac{1}{1-|\lambda_i|} \prod_{1 \leq j \leq m'} (1 + |\mu_j|)$, so that the sum $\sigma_0 + \sigma_1 + \sigma_2 + \cdots$ converges absolutely to $\prod_{1 \leq i \leq m} \frac{1}{1-|\lambda_i|} \prod_{1 \leq j \leq m'} (1 - \mu_j)$.

For each $n \geq 0$, we can identify $\operatorname{H}^*(\operatorname{Sym}^n(V))$ with the nth symmetric power of $\operatorname{H}^*(V)$ (in the category of graded vector spaces with the usual sign convention). It follows that

$$|\operatorname{H}^*(\operatorname{Sym}^n(V))|_F = s_n \qquad \operatorname{Tr}(F | \operatorname{H}^*(\operatorname{Sym}^n(V))) = \sigma_n.$$

Let $\mathfrak{m}_A$ denote the augmentation ideal of A. For each integer $n \geq 1$, let $\mathfrak{m}_A^{(n)}$ be as in Definition 6.1.2, and set $\mathfrak{m}_A^{(0)} = A$. For every pair of integers $i \leq j$, let $Q_{i,j}$ denote the cofiber of the map $\mathfrak{m}_A^{(j)} \to \mathfrak{m}_A^{(i)}$. If $i < j$, then Proposition 6.1.17 supplies a fiber sequence

$$Q_{i+1,j} \to Q_{i,j} \to \operatorname{Sym}^i(V).$$

Applying Remark 6.3.3 repeatedly, we deduce that each pair $(\operatorname{H}^*(Q_{i,j}), F)$ is summable, with

$$|\operatorname{H}^*(Q_{i,j})|_F \leq s_i + \cdots + s_{j-1} \qquad \operatorname{Tr}(F | H^*(Q_{i,j})) = \sigma_i + \cdots + \sigma_{j-1}.$$

We next prove the following:

$(*)$ For each integer n, the pair $(\operatorname{H}^*(\mathfrak{m}_A^{(n)}), F)$ is summable, with

$$|\operatorname{H}^*(\mathfrak{m}_A^{(n)})|_F \leq s_n + s_{n+1} + \cdots.$$

For every integer $d \geq 0$, set $W(d)^* = \bigoplus_{i \leq d} \operatorname{H}^i(\mathfrak{m}_A^{(n)})$. To prove $(*)$, it will suffice to show that each of the pairs $(W(d)^*, F)$ is summable with $|W(d)^*|_F \leq s_n + s_{n+1} + \cdots$. Without loss of generality we may assume that $d > n$. It follows from Proposition 6.1.18 that the map $\operatorname{H}^i(\mathfrak{m}_A^{(n)}) \to \operatorname{H}^i(Q_{n,d+2})$ is an isomorphism for $i \leq d$, so that

$$|W(d)^*|_F \leq |\operatorname{H}^*(Q_{n,d+2})|_F \leq s_n + \cdots + s_{d+1} \leq \sum_{n' \geq n} s_{n'} < \infty,$$

as desired.

Applying $(*)$ when $n = 0$, we deduce that $(\operatorname{H}^*(A), F)$ is summable. Moreover, for every integer n, applying Remark 6.3.3 to the fiber sequence

$$\mathfrak{m}_A^{(n)} \to A \to Q_{0,n}$$

gives an inequality

$$\begin{aligned}
|\operatorname{Tr}(F|\operatorname{H}^*(A)) - \sigma_0 - \cdots - \sigma_{n-1}| &= |\operatorname{Tr}(F|\operatorname{H}^*(A)) - \operatorname{Tr}(F|\operatorname{H}^*(Q_{0,n}))| \\
&= |\operatorname{Tr}(F|\operatorname{H}^*(\mathfrak{m}_A^{(n)}))| \\
&\leq |\operatorname{H}^*(\mathfrak{m}_A^{(n)})|_F \\
&\leq s_n + s_{n+1} + \cdots .
\end{aligned}$$

It follows that $\operatorname{Tr}(F|\operatorname{H}^*(A))$ is given by the absolutely convergent sum

$$\sum_{n\geq 0} \sigma_n = \prod_{1\leq i\leq m} \frac{1}{1-\lambda_i} \prod_{1\leq j\leq m'} (1-\mu_j).$$

In particular, we have

$$\begin{aligned}
\log \operatorname{Tr}(F|\operatorname{H}^*(A)) &= \sum_{1\leq i\leq m} \log \frac{1}{1-\lambda_i} - \sum_{1\leq j\leq m'} \log \frac{1}{1-\mu_i} \\
&= \sum_{1\leq i\leq m} \sum_{n>0} \frac{1}{n}\lambda_i^n - \sum_{1\leq j\leq m'} \sum_{n>0} \frac{1}{n}\mu_j^n \\
&= \sum_{n>0} \frac{1}{n}\Big(\sum_{1\leq i\leq m} \lambda_i^n - \sum_{1\leq j\leq m'} \mu_j^n \Big) \\
&= \sum_{n>0} \frac{1}{n} \operatorname{Tr}(F^n|\operatorname{H}^*(V)).
\end{aligned}$$

□

6.4. **The Trace Formula for** BG**.** Let $\mathbf{F}_q$ denote a finite field, let Y be a smooth algebraic stack of dimension d over $\mathbf{F}_q$, and let $\overline{Y} = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} Y$ be the associated algebraic stack over $\overline{\mathbf{F}}_q$. Fix a prime number ℓ which is invertible in $\mathbf{F}_q$ and an embedding $\mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. Pullback along the geometric Frobenius map $\operatorname{Frob} : \overline{Y} \to \overline{Y}$ induces an automorphism of the ℓ-adic cohomology $\operatorname{H}^*(\overline{Y}; \mathbf{Q}_\ell)$, which we will also denote by Frob. If Y is quasi-compact and quasi-separated, then Behrend proved that $(\operatorname{H}^*(\overline{Y}; \mathbf{Q}_\ell); \operatorname{Frob}^{-1})$ is summable and that its trace satisfies the following analogue of Theorem 1.3.2:

$$\operatorname{Tr}(\operatorname{Frob}^{-1} | \operatorname{H}^*(\overline{Y}; \mathbf{Q}_\ell)) = q^{-d} \sum_{\eta \in Y(\mathbf{F}_q)} \frac{1}{|\operatorname{Aut}(\eta)|}.$$

Here the sum is taken over all isomorphism classes of objects in the groupoid $Y(\mathbf{F}_q)$; see [6] for more details.

Let us now specialize to the case where $Y = \mathrm{BG}$, where G is a connected algebraic group over $\mathbf{F}_q$. In this case, it follows from Lang's theorem that every G-bundle on $\operatorname{Spec} \mathbf{F}_q$ is trivial, so that the sum on the right hand side contains only one term. We therefore obtain the formula

$$\operatorname{Tr}(\operatorname{Frob}^{-1} | \operatorname{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) = \frac{q^{\dim(G)}}{|G(\mathbf{F}_q)|}.$$

Our goal in this section is to give a different proof of this identity using a method which can be adapted to the case where BG is replaced by the moduli stack of bundles on an algebraic curve.

We begin with some general remarks.

Notation 6.4.1. Let k be a field, let ℓ be a prime number which is invertible in k, and let $\overline{k}$ be an algebraic closure of k. Let Y be an Artin stack over k, so that $\overline{Y} = Y \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ can be regarded as an Artin stack over the algebraically closed field $\overline{k}$. We let $C^*_{\mathrm{geom}}(Y)$ denote the cochain complex $C^*(\overline{Y}; \mathbf{Z}_\ell)[\ell^{-1}] \in \mathrm{Mod}_{\mathbf{Q}_\ell}$. We will refer to $C^*_{\mathrm{geom}}(Y)$ as the *geometric cochain complex* of Y. We let $\mathrm{H}^*_{\mathrm{geom}}(Y)$ denote the cohomology of the cochain complex $C^*_{\mathrm{geom}}(Y)$; we will refer to $\mathrm{H}^*_{\mathrm{geom}}(Y)$ as the *geometric cohomology of* Y. Note that the cochain complex $C^*_{\mathrm{geom}}(Y)$ and its cohomology $\mathrm{H}^*_{\mathrm{geom}}(Y)$ equipped with an action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$.

Remark 6.4.2. The definition of the geometric cohomology $\mathrm{H}^*_{\mathrm{geom}}(Y)$ depends on a choice of algebraic closure $\overline{k}$ of k and on a choice of prime number ℓ which is invertible in k. However, to avoid making the exposition too burdensome, we will often neglect to mention these choices explicitly.

Warning 6.4.3. In the situation of Notation 6.4.1, the geometric cohomology $\mathrm{H}^*_{\mathrm{geom}}(Y)$ comes equipped with a canonical map

$$\theta : \mathrm{H}^*_{\mathrm{geom}}(Y) \to \mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell).$$

This map is an isomorphism if Y is of finite type over k (Remark 2.3.30), but not in general. For example, if X is a disjoint union of countably many copies of $\operatorname{Spec} \mathbf{F}_q$, then θ is given by the canonical monomorphism $(\prod_{i \geq 0} \mathbf{Z}_\ell)[\ell^{-1}] \to \prod_{i \geq 0} \mathbf{Q}_\ell)$.

Remark 6.4.4. Let Y be a smooth algebraic variety of dimension d over a finite field $\mathbf{F}_q$, so that the geometric cohomology $\mathrm{H}^*_{\mathrm{geom}}(Y)$ is equipped with a geometric Frobenius automorphism Frob. Since $\mathrm{H}^*_{\mathrm{geom}}(Y)$ is a finite-dimensional vector space over $\mathbf{Q}_\ell$, the pair $(\mathrm{H}^*_{\mathrm{geom}}(Y), \mathrm{Frob}^{-1})$ is automatically summable. Moreover, the Grothendieck-Lefschetz trace formula yields an equality $\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(Y)) = q^{-d}|Y(\mathbf{F}_q)|$ (see Theorem 1.3.2).

Definition 6.4.5. Let G be a connected algebraic group defined over a field k and let $I = \mathrm{H}^{>0}_{\mathrm{geom}}(G)$ denote the (two-sided) ideal in $\mathrm{H}^*_{\mathrm{geom}}(G)$ generated by homogeneous elements of positive degree. We define the *motive of* G to be the quotient I/I^2, which we regard as a representation of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$.

Remark 6.4.6. For a reductive group G over a field k, the motive $M(G)$ was introduced by Gross in [21]. Definition 6.4.5 appears in [59]. Beware that our conventions differ from those of [21] and [59] by a Tate twist (the motive of G is defined in [21] to be the tensor product $\mathbf{Q}_\ell(1) \otimes_{\mathbf{Q}_\ell} M(G)$; see Remark 6.5.4 below).

Remark 6.4.7. Let $\phi : G \to H$ be an isogeny between connected algebraic groups over a field k. Then ϕ induces an isomorphism $\phi^* : \mathrm{H}^*_{\mathrm{geom}}(H) \to \mathrm{H}^*_{\mathrm{geom}}(G)$, which restricts to a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism $M(H) \simeq M(G)$.

Proposition 6.4.8. *Let G be a connected algebraic group defined over a field k and let $I = \mathrm{H}^{>0}_{\mathrm{geom}}(G)$. Then the canonical map*

$$M(G) = I/I^2 \to \mathrm{H}^*(\cot C^*_{\mathrm{geom}}(G))$$

(see Remark 6.1.11) is an isomorphism.

Proof. The group law $m : G \times_{\operatorname{Spec} k} G \to G$ induces a comultiplication

$$m^* : \mathrm{H}^*_{\mathrm{geom}}(G) \to \mathrm{H}^*_{\mathrm{geom}}(G) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^*_{\mathrm{geom}}(G),$$

which endows $\mathrm{H}^*_{\mathrm{geom}}(G)$ with the structure of a finite-dimensional graded-commutative Hopf algebra over $\mathbf{Q}_\ell$. Since G is connected, it follows that $\mathrm{H}^*_{\mathrm{geom}}(G)$ is isomorphic to an exterior algebra on finitely many generators $x_1, \ldots, x_r$ of odd degrees $d_1, \ldots, d_r$ (see [40]). The desired result now follows from Proposition 6.1.12. □

Remark 6.4.9. Let G be a connected algebraic group over a field k. We have a pullback diagram of algebraic stacks

$$\begin{array}{ccc} G & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathrm{BG}\,. \end{array}$$

Applying Lemma 7.1.7, we obtain a pushout square

$$\begin{array}{ccc} C^*_{\mathrm{geom}}(G) & \longleftarrow & C^*_{\mathrm{geom}}(\operatorname{Spec} k) \\ \uparrow & & \uparrow \\ C^*_{\mathrm{geom}}(\operatorname{Spec} k) & \longleftarrow & C^*_{\mathrm{geom}}(\mathrm{BG}) \end{array}$$

of augmented commutative algebra objects of $\mathrm{Mod}_{\mathbf{Q}_\ell}$, hence a pushout diagram of cotangent fibers

$$\begin{array}{ccc} \cot C^*_{\mathrm{geom}}(G) & \longleftarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \cot C^*_{\mathrm{geom}}(\mathrm{BG}) \end{array}$$

see Proposition 6.1.9 (here we regard BG as equipped with the base point $\operatorname{Spec} k \to \mathrm{BG}$ corresponding to the trivial G-bundle, and G as equipped with the base point $\operatorname{Spec} k \to G$ given by the identity section). In other words, we can identify $\cot C^*_{\mathrm{geom}}(G)$ with the suspension $\Sigma \cot C^*_{\mathrm{geom}}(\mathrm{BG})$. In particular, we obtain a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism $M(G) \simeq \mathrm{H}^*(\cot C^*_{\mathrm{geom}}(\mathrm{BG}))$ (which shifts the grading by 1).

Remark 6.4.10. Let G be a connected algebraic group over a field k. One can show that the geometric cohomology ring $\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG})$ is a polynomial ring on generators of even degree. It follows from Proposition 6.1.12 and Remark 6.4.9 that the motive $M(G)$ can be identified with the quotient J/J^2, where $J = \mathrm{H}^{>0}_{\mathrm{geom}}(\mathrm{BG})$ is the ideal generated by elements of positive degree.

Remark 6.4.11. Let G be a reductive algebraic group over a field k and let G' be a quasi-split inner form of G. Then there exists a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism $M(G) \simeq M(G')$. To prove this, we may assume without loss of generality that G and G' are adjoint (Remark 6.4.7). In this case, the classifying stacks BG and BG′ are equivalent to one another, so the desired result follows from the characterization of $M(G)$ and $M(G')$ given in Remark 6.4.10.

Since G' is quasi-split, we can choose a Borel subgroup $B' \subseteq G'$. Let $T' \subseteq B'$ be a maximal torus and let Λ be the character lattice of $\operatorname{Spec} \overline{k} \times_{\operatorname{Spec} k} T'$. The Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on Λ preserving a system of positive roots, and there is a canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism

$$\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BT}') \simeq \mathrm{Sym}^*(\mathbf{Q}_\ell(-1) \otimes_{\mathbf{Z}} \Lambda).$$

One can show that the restriction map

$$\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG}') \to \mathrm{H}^*_{\mathrm{geom}}(\mathrm{BT}')$$

is injective, and its image consists of those elements of $\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BT}')$ which are invariant under the action of the Weyl group $(N(T')/T')(\overline{k})$. Combining this observation with Remark 6.4.10, we obtain a very explicit description of the motive $M(G) \simeq M(G')$, which agrees with the definition given in [21] (up to a Tate twist); see [59] for a more detailed explanation.

Proposition 6.4.12. *Let G be a connected algebraic group of dimension d defined over a finite field $\mathbf{F}_q$ and let BG denote its classifying stack. Assume that ℓ is invertible in $\mathbf{F}_q$. Then $(\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG}), \mathrm{Frob}^{-1})$ is summable, and we have*

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG})) = \frac{q^d}{|G(\mathbf{F}_q|}.$$

Moreover, both sides are equal to

$$\exp(\sum_{n>0} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} | M(G))) = \det(1 - \mathrm{Frob}^{-1} | M(G))^{-1},$$

where $M(G)$ denotes the motive of G.

Lemma 6.4.13. *Let G be a connected algebraic group over a finite field $\mathbf{F}_q$. Then each eigenvalue of the Frobenius automorphism* Frob *on the motive $M(G)$ has complex absolute value $\geq q$. If G is semisimple, then each eigenvalue has complex absolute value $\geq q^2$.*

Proof. Since $\mathbf{F}_q$ is perfect, the unipotent radical U of G is defined over $\mathbf{F}_q$. Replacing G by the quotient G/U, we may reduce to the case where G is reductive. In this case, the assertion follows immediately from the explicit description of $M(G)$ supplied by Remark 6.4.11. □

Proof of Proposition 6.4.12. Proposition 6.4.8 and Remark 6.4.9 supply Frobenius-equivariant isomorphisms

$$\mathrm{H}^*(\cot C^*_{\mathrm{geom}}(\mathrm{BG})) \simeq M(G) \simeq \mathrm{H}^*(\cot C^*_{\mathrm{geom}}(G)),$$

where the groups on the left hand side are concentrated in even degrees and the groups on the right hand side are concentrated in odd degrees. Applying Proposition 6.3.4 to the augmented commutative algebras $C^*_{\mathrm{geom}}(\mathrm{BG})$ and $C^*_{\mathrm{geom}}(G)$ and using Remark 6.4.4, we obtain

$$\begin{aligned}
\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG})) &= \exp(\sum_{n>0} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} | M(G))) \\
&= \exp(\sum_{n>0} \frac{-1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} | M(G)))^{-1} \\
&= \mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(G))^{-1} \\
&= (q^{-d} |G(\mathbf{F}_q)|)^{-1} \\
&= \frac{q^d}{|G(\mathbf{F}_q)|}.
\end{aligned}$$

□

Remark 6.4.14. We can rewrite the final assertion of Proposition 6.4.12 as a formula

$$|G(\mathbf{F}_q)| = q^d \det(1 - \mathrm{Frob}^{-1} | M(G)) = q^d \det(1 - \mathrm{Frob} | M(G)^\vee)$$

for the order of the finite group $G(\mathbf{F}_q)$; this formula is due originally to Steinberg ([53]).

6.5. **Calculation of the Trace.** Throughout this section, we fix a finite field $\mathbf{F}_q$, an algebraic curve X over $\mathbf{F}_q$, and a smooth affine group scheme G over X with connected fibers whose generic fiber is semisimple and simply connected. We also fix an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$, a prime number ℓ which is invertible in $\mathbf{F}_q$, and an embedding $\mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. Our goal is to verify Theorem 1.3.5 by establishing the following:

Theorem 6.5.1. *The pair* $(\mathrm{H}^*_{\mathrm{geom}}(\mathrm{Bun}_G(X)); \mathrm{Frob}^{-1})$ *is summable. Moreover, we have*

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(\mathrm{Bun}_G(X))) = \prod_{x \in X} \frac{|\kappa(x)|^{\dim(G)}}{|G(\kappa(x))|},$$

where the product on the right hand side is absolutely convergent.

We will deduce Theorem 6.5.1 by combining the Atiyah-Bott formula (Theorem 6.2.4), Steinberg's formula (Proposition 6.4.12) and the Grothendieck-Lefschetz trace formula.

Construction 6.5.2. Let $\overline{X} = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} X$ and let $\overline{G} = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} G$. We will regard $\mathrm{Shv}_\ell(X)$ as a symmetric monoidal ∞-category with respect to the !-tensor product of §4.6. Let $\overline{\mathrm{BG}}$ denote the classifying stack of $\overline{G}$ so that we can regard the relative cohomology sheaf $[\overline{\mathrm{BG}}]_{\overline{X}}$ as an augmented commutative algebra object of $\mathrm{Shv}_\ell(\overline{X})$, as in §6.2. We define $\mathcal{M}(G) \in \mathrm{Shv}_\ell(\overline{X})$ by the formula

$$\mathcal{M}(G) = (\cot[\overline{\mathrm{BG}}]_{\overline{X}})[\ell^{-1}].$$

We will refer to $\mathcal{M}(G)$ as the *motive of G relative to X*.

The relative motive $\mathcal{M}(G)$ is closely related to the motives defined in §6.4.

Remark 6.5.3. Let $x \in \overline{X}(\overline{\mathbf{F}}_q)$. Using Remark 6.1.7 and Proposition 5.1.9, we obtain equivalences

$$\begin{aligned} x^! \,\mathcal{M}(G) &= x^!(\cot[\overline{\mathrm{BG}}]_{\overline{X}})[\ell^{-1}] \\ &\simeq \cot(x^![\overline{\mathrm{BG}}]_{\overline{X}}[\ell^{-1}]) \\ &\simeq \cot(C^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell)). \end{aligned}$$

in the ∞-category $\mathrm{Mod}_{\mathbf{Q}_\ell}$. In particular, we can identify the cohomology of the chain complex $x^! \,\mathcal{M}(G)$ with the motive $M(\overline{G}_x)$ (see Remark 6.4.9).

Remark 6.5.4. Let U be the largest open subset of $\overline{X}$ over which the group $\overline{G}$ is semisimple. Then we can choose a surjective étale morphism $V \to U$ and an equivalence

$$V \times_{\overline{X}} \overline{G} \simeq V \times_{\operatorname{Spec} \overline{\mathbf{F}}_q} H,$$

where H is a semisimple algebraic group over $\overline{\mathbf{F}}_q$. We then have

$$\begin{aligned} \mathcal{M}(G)|_V &= \cot([\overline{\mathrm{BG}}]_{\overline{X}})[\ell^{-1}]|_V \\ &\simeq \cot([V \times_{\operatorname{Spec} \overline{\mathbf{F}}_q} \mathrm{BH}]_V)[\ell^{-1}] \\ &\simeq \cot(C^*(\mathrm{BH}; \mathbf{Z}_\ell) \otimes \omega_V)[\ell^{-1}] \\ &\simeq \cot(C^*(\mathrm{BH}; \mathbf{Q}_\ell)) \otimes \omega_V. \end{aligned}$$

It follows that the ℓ-adic sheaf $\mathcal{M}(G)$ is lisse when restricted to U (in fact, it is even locally constant: after base change to V, it is equivalent to a direct sum of finitely many shifted copies of $\omega_V[\ell^{-1}]$). In particular, for any point $x \in U(\overline{\mathbf{F}}_q)$ we have a canonical equivalence

$$x^* \,\mathcal{M}(G) \simeq (x^! \,\mathcal{M}(G)) \otimes_{\mathbf{Q}_\ell} \Sigma^2 \,\mathbf{Q}_\ell(1)$$

so that the cohomology of $x^* \mathcal{M}(G)$ can be identified with the Tate-twisted motive $M(\overline{G}_x) \otimes_{\mathbf{Q}_\ell} \mathbf{Q}_\ell(1)$.

Proposition 6.5.5. *The cohomology* $\mathrm{H}^*(\overline{X}; \mathcal{M}(G))$ *is a finite-dimensional vector space over* $\mathbf{Q}_\ell$. *Moreover, each eigenvalue of the Frobenius map* Frob *on* $\mathrm{H}^*(\overline{X}; \mathcal{M}(G))$ *has complex absolute value* $\geq q$.

Proof. This can be deduced from Deligne's work on the Weil conjectures ([14]). However, we will proceed in a more elementary way. Let H denote a split form of the generic fiber of G, regarded as an algebraic group over $\mathbf{F}_q$. Choose a finite generically étale map $X' \to X$, where X' is a smooth connected curve over $\mathbf{F}_q$ (not necessarily geometrically connected) and the groups $H \times_{\operatorname{Spec} \mathbf{F}_q} X'$ and $G \times_X X'$ are isomorphic at the generic point of X'. Then there exists a dense open subset $U' \subseteq X'$ and an isomorphism

$$\alpha : H \times_{\operatorname{Spec} \mathbf{F}_q} U' \simeq G \times_X U'$$

of group schemes over U'. Shrinking U' if necessary, we may assume that U' is the inverse image of a dense open subset $U \subseteq X$, and that the map $U' \to U$ is finite étale.

Let $\{x_1, \ldots, x_n\}$ be the set of closed points of X which do not belong to U. Replacing $\mathbf{F}_q$ by a finite extension if necessary, we may assume that each x_i is defined over $\mathbf{F}_q$. Let $f_i : \operatorname{Spec} \overline{\mathbf{F}}_q \to \overline{X}$ denote the map determined by x_i and let $\overline{U} = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} U$, so that we have an exact sequence

$$\bigoplus_{1 \leq i \leq n} \mathrm{H}^*(f_i^! \mathcal{M}(G)) \to \mathrm{H}^*(\overline{X}; \mathcal{M}(G)) \to \mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_U).$$

It will therefore suffice to prove the following:

(a) For $1 \leq i \leq n$, the cohomology $\mathrm{H}^*(f_i^! \mathcal{M}(G))$ is finite-dimensional and each eigenvalue of Frob on $\mathrm{H}^*(f_i^! \mathcal{M}(G))$ has complex absolute value $\geq q$.

(b) The cohomology $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ is finite-dimensional and each eigenvalue of Frob on $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ has complex absolute value $\geq q$.

Assertion (a) follows immediately from Lemma 6.4.13 and the identification $\mathrm{H}^*(f_i^! \mathcal{M}(G)) \simeq M(G_{x_i})$ supplied by Remark 6.5.3. To prove (b), let $\overline{H} = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} H$, let $\overline{U}' = \operatorname{Spec} \overline{\mathbf{F}}_q \times_{\operatorname{Spec} \mathbf{F}_q} U'$ and let $\pi : \overline{U}' \to \overline{U}$ denote the projection map. Then $\mathcal{M}(G)|_{\overline{U}}$ is a direct summand of $\pi_* \pi^* \mathcal{M}(G)|_{\overline{U}}$, so that $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ is a direct summand of

$$\begin{aligned} \mathrm{H}^*(\overline{U}'; \mathcal{M}(G)|_{\overline{U}'}) &\simeq \mathrm{H}^*(\overline{U}'; \cot(C^*(\overline{\mathrm{BH}}; \mathbf{Q}_\ell)) \otimes \omega_{\overline{U}'}) \\ &\simeq M(H) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{*+2}(\overline{U}'; \mathbf{Q}_\ell(1)). \end{aligned}$$

The finite-dimensionality of $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ follows immediately. To prove the assertion about Frobenius eigenvalues, we note that each eigenvalue of Frob on $\mathrm{H}^*(\overline{U}'; \mathbf{Q}_\ell)$ has complex absolute value ≥ 1 and therefore each eigenvalue of Frob on $\mathrm{H}^*(\overline{U}'; \mathbf{Q}_\ell(1))$ has complex absolute value $\geq q^{-1}$. We are therefore reduced to proving that each eigenvalue of Frob on $M(H)$ has complex absolute value $\geq q^2$, which follows from Lemma 6.4.13. □

We will also need the following assertion, whose proof will be given in §8.3 (see Theorem 8.3.1):

Proposition 6.5.6. *The moduli stack* $\mathrm{Bun}_G(X)$ *is connected.*

Remark 6.5.7. We will prove a stronger version of Proposition 6.5.6 in §8.3 using Theorem 5.4.5 (which we prove in §7). However, it is possible to deduce Proposition 6.5.6 directly from Theorem 3.2.9 together with some basic facts about the affine Grassmannian of G. Note first that if $D \subseteq X$ is an effective divisor, then the map $\mathrm{Bun}_G(X, D) \to \mathrm{Bun}_G(X)$ is surjective; it will therefore suffice to verify the connectedness of $\mathrm{Bun}_G(X, D)$. We can then use Theorem 3.2.9 to reduce to proving the connectedness of $\mathrm{Ran}_G(X - D)$. If D is sufficiently large (so that G is reductive over the open set $X - D \subseteq X$), then the projection map $\pi : \mathrm{Ran}_G(X - D) \to \mathrm{Ran}(X-D)$ is Ind-proper (see Lemma 8.5.8). Using the connectedness of $\mathrm{Ran}(X-D)$ (Theorem 2.4.5), we are reduced to showing that the fibers of π are connected. Note that if μ is a k-valued point of $\mathrm{Ran}(X - D)$ corresponding to a finite subset $S \subseteq X(k)$, then the fiber of π over the point μ can be identified with the product $\prod_{x \in S} \mathrm{Gr}_G^x$. It will therefore suffice to show that each of the Ind-schemes Gr_G^x is connected, which follows from the fact that G is semisimple and simply connected at the point x (since $x \notin D$); see Lemma 9.5.9.

Using Corollary 6.2.5, we can identify the cotangent fiber $\cot C^*_{\mathrm{geom}}(\mathrm{Bun}_G(X))$ with the chain complex $C^*(\overline{X}; \mathcal{M}(G))$. It follows from Proposition 6.5.5 that the cohomologies of this chain complex are finite dimensional and that the eigenvalues of Frob^{-1} have complex absolute value < 1. Proposition 6.5.6 implies that the group $\mathrm{H}^0_{\mathrm{geom}}(\mathrm{Bun}_G(X))$ is isomorphic to $\mathbf{Q}_\ell$. Applying Proposition 6.3.4, we obtain the following preliminary version of Theorem 6.5.1:

Corollary 6.5.8. *The pair* $(\mathrm{H}^*_{\mathrm{geom}}(\mathrm{Bun}_G(X)); \mathrm{Frob}^{-1})$ *is summable. Moreover, we have*

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{geom}}(\mathrm{Bun}_G(X))) = \exp(\sum_{n>0} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} \,|\, \mathrm{H}^*(\overline{X}; \mathcal{M}(G)))).$$

In particular, the sum on the right is absolutely convergent.

6.5.1. *Application of the Trace Formula.* For each integer $n > 0$, the Grothendieck-Lefschetz trace formula and Steinberg's formula (Proposition 6.4.12) supply equalities

$$\begin{aligned} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} \,|\, \mathrm{H}^*(\overline{X}; \mathcal{M}(G))) &= \frac{1}{n} \sum_{\eta \in X(\mathbf{F}_{q^n})} \mathrm{Tr}(\mathrm{Frob}^{-n} \,|\, \mathrm{H}^*(\eta^! \,\mathcal{M}(G))) \\ &= \sum_{\eta \in X(\mathbf{F}_{q^n})} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} \,| M(\overline{G}_\eta)) \\ &= \sum_{n = e \deg(x)} \frac{1}{e} \mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)) \end{aligned}$$

where the latter sum is taken over all closed points $x \in X$ whose degree divides n, and Frob_x denotes the geometric Frobenius at the point x. Combining this with Corollary 6.5.8, we obtain an equality

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{geom}}(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)) = \exp(\sum_{n>0} \sum_{e \deg(x) = n} \frac{1}{e} \mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)). \tag{12}$$

Proposition 6.5.9. *The double summation appearing in formula (12) is absolutely convergent.*

Proof. For each closed point $x \in X$, let $\lambda_{x,1}, \ldots, \lambda_{x,m_x} \in \mathbf{C}$ denote the eigenvalues of Frob_x on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} M(G_x)$, so that $\mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)) = \sum_{1 \leq i \leq m_x} \lambda_{x,i}^{-e}$. We will show that the triple sum

$$\sum_{n>0} \sum_{e \deg(x) = n} \frac{1}{e} \sum_{1 \leq i \leq m_x} |\lambda_{x,i}^{-e}|$$

is convergent.

For each integer d, set

$$C_d = \sum_{\deg(x)=d} \sum_{e>0} \sum_{1 \leq i \leq m_x} \frac{1}{e} |\lambda_{x,i}^{-e}|;$$

we wish to show that that each C_d is finite and that the sum $\sum_{d>0} C_d$ is convergent. Let g denote the genus of the curve X, so that we have an inequality $|X(\mathbf{F}_{q^d})| \leq q^d + 2gq^{\frac{d}{2}} + 1$. It follows that the number of closed points of X_0 having degree exactly d is bounded above by $d^{-1}(q^d + 2gq^{\frac{d}{2}} + 1)$. Let H be a split form of the generic fiber of G and let r denote the dimension of $M(H)$ as a vector space over $\mathbf{Q}_\ell$ (the number r is equal to the rank of the generic fiber of G, but we will not need to know this). For all but finitely many closed points $x \in X$, the motive $M(G_x)$ is isomorphic to $M(H)$ as a $\mathbf{Q}_\ell$-vector space (see Remark 6.5.4) so that $m_x = r$. In this case, each of the eigenvalues $\lambda_{x,i}$ has complex absolute value $\geq q^2$ (Lemma 6.4.13). For $d \gg 0$, we have

$$\begin{aligned} C_d &\leq \frac{q^d + 2gq^{\frac{d}{2}} + 1}{d} r \sum_{e>0} \frac{1}{e} q^{-2de} \\ &\leq (2g+2) q^d r \sum_{e>0} q^{-2de} \\ &\leq (2g+2) q^d r \frac{q^{-2d}}{1 - q^{-2d}} \\ &\leq \frac{(2g+2)r}{1-q^{-2}} q^{-d}. \end{aligned}$$

It follows that the series $\sum_{d>0} C_d$ is dominated (apart from finitely many terms) by the geometric series $\sum_{d>0} \frac{(2g+2)r}{1-q^{-2}} q^{-d}$ and is therefore convergent. □

Proof of Theorem 6.5.1. By virtue of Proposition 6.5.9, we are free to rearrange the order of summation appearing in formula (12). We therefore obtain

$$\begin{aligned} \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(\operatorname{Bun}_G(X))) &= \exp(\sum_{n>0} \sum_{e \deg(x)=n} \frac{1}{e} \operatorname{Tr}(\operatorname{Frob}_x^{-e} \mid M(G_x)) \\ &= \exp(\sum_{x \in X} \sum_{e>0} \frac{1}{e} \operatorname{Tr}(\operatorname{Frob}_x^{-e} \mid M(G_x)) \\ &= \prod_{x \in X} \exp(\sum_{e>0} \frac{1}{e} \operatorname{Tr}(\operatorname{Frob}_x^{-e} \mid M(G_x)) \\ &= \prod_{x \in X_0} \frac{|\kappa(x)|^{\dim(G)}}{|G_x(\kappa(x))|}, \end{aligned}$$

where the last equality follows from Proposition 6.4.12. □

Remark 6.5.10. To the relative motive $\mathcal{M}(G)$ we can associate an L-function

$$L_{\mathcal{M}(G), \operatorname{Frob}^{-1}}(t) = \det(1 - t \operatorname{Frob}^{-1} \mid \mathrm{H}^*(\overline{X}; \mathcal{M}(G)))^{-1},$$

which is a rational function of t. The proof of Proposition 6.5.9 shows that this L-function admits an Euler product expansion

$$L_{\mathcal{M}(G), \operatorname{Frob}^{-1}}(t) = \prod_{x \in X} L_{M(G_x), \operatorname{Frob}_x^{-1}}(t)$$

where the product on the right hand side converges absolutely for $|t| < q$. Combining this observation with Steinberg's formula (Proposition 6.4.12), we obtain

$$L_{\mathcal{M}(G),\mathrm{Frob}^{-1}}(1) = \prod_{x \in X} \frac{|\kappa(x)|^{\dim(G)}}{|G(\kappa(x))|}.$$

The right hand side of this formula is given by

$$q^{-\dim \mathrm{Bun}_G(X)} \tau(G)^{-1} \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|},$$

where $\tau(G) = \mu_{\mathrm{Tam}}(G(K_X) \backslash G(\mathbf{A}))$ denotes the Tamagawa number of G (see the discussion preceding Conjecture 1.2.18). We therefore obtain an equality

$$\tau(G) L_{\mathcal{M}(G),\mathrm{Frob}^{-1}}(1) = q^{-\dim \mathrm{Bun}_G(X)} \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|},$$

which we can regard as a function field analogue of Theorem 9.9 of [21].

7. Decomposition of the Sheaf $\mathcal{B}$

Let k be an algebraically closed field, let ℓ be a prime number which is invertible in k, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected, and let $\mathcal{B} \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ denote the !-sheaf introduced in §5.4.2. In this section, we begin our proof of Theorem 5.4.5, which asserts that the canonical map

$$\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

is an equivalence (modulo a technical assertion about the compatibility of chiral homology with inverse limits, which we will establish in §9).

In order to describe our strategy of proof, it will be convenient to first assume for simplicity that the group scheme G is everywhere reductive. Let $\mathrm{Ran}_G(X)$ denote the Beilinson-Drinfeld Grassmannian of G (see Definition 3.2.3). It follows from Theorem 3.2.9 that the forgetful functor $\mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$ is a universal homology equivalence, and therefore induces a quasi-isomorphism $\theta : C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell)$. To prove Theorem 5.4.5, it will suffice to show that the composite map

$$\int \mathcal{B} \xrightarrow{\rho} C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \xrightarrow{\theta} C^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell)$$

is a quasi-isomorphism.

The prestack $\mathrm{Ran}_G(X)$ admits a Cartesian fibration $\psi : \mathrm{Ran}_G(X) \to \mathrm{Fin}^{\mathrm{s}}$. For every nonempty finite set S, let $\mathrm{Ran}_G(X)_S$ denote the fiber $\psi^{-1}\{S\}$, so that we have a canonical equivalence

$$C^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell) \simeq \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell).$$

Our strategy will be to find a corresponding decomposition of $\mathcal{B}$ as an inverse limit $\varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_S$ for certain !-sheaves $\{\mathcal{B}_S \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))\}_{S \in \mathrm{Fin}^{\mathrm{s}}}$, so that the composite map $\theta \circ \rho$ can be written as a composition

$$\begin{aligned}
&(13) \qquad \int \mathcal{B} \xrightarrow{\alpha} \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \\
&(14) \qquad \xrightarrow{\beta} \varprojlim_{S \in \mathrm{Fin}^s} \int \mathcal{B}_S \\
&(15) \qquad \xrightarrow{\gamma} \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell) \\
&(16) \qquad \simeq C^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell).
\end{aligned}$$

Let us now outline the contents of this section. We begin in §7.2 by giving a definition of the !-sheaves $\mathcal{B}_S$ and the maps which appear in diagram (13). For the moment, let us give an informal summary:

- (i) If S is a nonempty finite set, then $\mathcal{B}_S$ can be described informally as follows: its !-fiber at a point $\nu : T \to X(k)$ is the cochain complex of the prestack which parametrizes G-bundles which are defined on an open subset of X containing the divisor $|\nu(T)|$, and trivialized away from the image of some map $\mu : S \to X$.
- (ii) Any G-bundle defined on an open neighborhood of the divisor $|\nu(T)|$ determines a G-bundle on the divisor $|\nu(T)|$ itself. This observation determines a map of !-sheaves $\mathcal{B} \to \mathcal{B}_S$ which depends functorially on S. Passing to the limit over S, we obtain a map $\alpha : \mathcal{B} \to \varprojlim_S \mathcal{B}_S$.
- (iii) The prestack described in (i) contains $\mathrm{Ran}_G(X)_S$ as a full subcategory (comprised of those objects which correspond to G-bundles which are defined on the entire curve X). This observation induces a map

$$\gamma_S : \int \mathcal{B}_S \to C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell).$$

Passing to the inverse limit over S, we obtain a map

$$\gamma : \varprojlim_{S \in \mathrm{Fin}^s} \int \mathcal{B}_S \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell) \simeq C^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell).$$

To prove Theorem 5.4.5, it will suffice to show that the maps α, β, and γ are quasi-isomorphisms. In this section, we will take the first steps towards this proof by showing that the map α is a quasi-isomorphism (Theorem 7.2.10). At the level of costalks, this asserts that the moduli stack parametrizing G-bundles on the divisor $|\nu(T)|$ has the same ℓ-adic cohomology as the prestack parametrizing G-bundles defined on an open neighborhood of $|\nu(T)|$, and trivialized away from a finite set which does not meet $|\nu(T)|$. We will outline the proof of this statement in §7.3, and carry out the details in §7.4, §7.5, and §7.6.

It is relatively straightforward to show that for every nonempty finite set S, the map $\gamma_S : \int \mathcal{B}_S \to C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell)$ described in (iii) is a quasi-isomorphism (Theorem 7.2.11). Granting this, the proof of Theorem 5.4.5 can be reduced to showing that that the comparison map

$$\beta : \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \to \varprojlim_{S \in \mathrm{Fin}^s} \int \mathcal{B}_S$$

is a quasi-isomorphism. However, this is not a formality: the definition of chiral homology involves an infinite direct limit, and does not commute with inverse limits in general (in fact, the map β generally fails to be a quasi-isomorphism when the generic fiber of G is not simply connected). The proof of Theorem 5.4.5 which we give in §9 will actually proceed along somewhat different lines, and does not make direct use of the statement that γ is a quasi-isomorphism. We nevertheless include a proof of this fact in §7.7, which the reader can omit if desired.

Our analysis of the map α depends crucially on the fact that for every finite set S, the Beilinson-Drinfeld Grassmannian $\mathrm{Ran}_G(X)_S$ can be written as a direct limit of algebraic varieties which are proper over X^S. This statement generally fails when G is not everywhere reductive. To circumvent this difficulty, we will choose an effective divisor $Q \subseteq X$ such that G is reductive over $X - Q$, and therefore each Beilinson-Drinfeld Grassmannian $\mathrm{Ran}_G(X - Q)_S$ is Ind-proper over $(X - Q)^S$ (see Lemma 8.5.8). According to the noncompact version of nonabelian Poincare duality (Theorem 3.2.9), the prestack $\mathrm{Ran}_G(X - Q)$ has the same ℓ-adic cohomology as $\mathrm{Bun}_G(X, Q)$. If the fibers G_x are vector group for $x \in Q$, then $\mathrm{Ran}_G(X - Q)$ has the same ℓ-adic cohomology as $\mathrm{Bun}_G(X)$ and we can use it as a replacement for $\mathrm{Ran}_G(X)$ in the argument outlined above. This can always be arranged by replacing G by an appropriate dilitation. It will therefore be useful to know that the validity of Theorem 5.4.5 depends only on the generic fiber of the group scheme G, which we will prove in §7.1 (Proposition 7.1.1).

Remark 7.0.1. Since the theory of Tamagawa numbers depends only on the generic fiber G_0 of G, we are free to choose any integral model that we like for the purposes of proving the equality

$$\prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|} = q^{-\dim \mathrm{Bun}_G(X)} \sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}.$$

of Conjecture 1.2.18. Consequently, for purposes of proving Weil's conjecture, the results of §7.1 are not needed: it suffices to show that there exists an integral model G' of G_0 for which both of the equalities

$$\prod_{x \in X} \frac{|\kappa(x)|^d}{|G'(\kappa(x))|} = \mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*(\overline{\mathrm{Bun}}_{G'}(X); \mathbf{Z}_\ell)) = q^{-\dim \mathrm{Bun}_{G'}(X)} \sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}$$

are valid, and these equalities requires only that we prove Theorem 5.4.5 for the group scheme G'. We include §7.1 nevertheless, since it may be of independent interest to know that the equivalence $\int \mathcal{B} \simeq C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ is valid for *any* integral model of G_0 (provided that it is smooth, affine, and has connected fibers).

7.1. **Independence of G.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, and an algebraic curve X over k. For every smooth affine group scheme G over X, let $\mathcal{B}_G \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ be defined as in Notation 5.4.2, and let $\rho_G : \int \mathcal{B}_G \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ denote the map appearing in the statement of Theorem 5.4.5. Our goal is to prove the following:

Proposition 7.1.1. *If the fibers of G are connected, then the statement that $\rho_G : \int \mathcal{B}_G \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ is an equivalence depends only on the generic fiber of G. In other words, if G and G' are smooth affine group schemes over X with connected fibers and the generic fibers of G and G' are isomorphic, then ρ_G is an equivalence if and only if $\rho_{G'}$ is an equivalence.*

We will deduce Proposition 7.1.1 from the pair of lemmas:

Lemma 7.1.2. *Let G be a smooth affine group scheme over X, let x be a closed point of X, let e_x denote the identity element of the algebraic group G_x, and let G' denote the smooth affine group scheme over X obtained by dilitation of G at the point e_x (see §A.3). Suppose that G_x is connected. Then G satisfies the conclusion of Theorem 5.4.5 if and only if G' satisfies the conclusion of Theorem 5.4.5. That is, the canonical map*

$$\rho_G : \int \mathcal{B}_G \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

is an equivalence if and only if the canonical map

$$\rho_{G'} : \int \mathcal{B}_{G'} \to C^*(\mathrm{Bun}_{G'}(X); \mathbf{Z}_\ell)$$

is an equivalence.

Lemma 7.1.3. *In the situation of Lemma 7.1.2, suppose that the fiber G_x is a vector group. Then the canonical maps*

$$\int \mathcal{B}_G \to \int \mathcal{B}_{G'} \qquad C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Bun}_{G'}(X); \mathbf{Z}_\ell)$$

are equivalences.

Remark 7.1.4. Let G and G' be as in Lemma 7.1.2. Then the canonical map $G' \to G$ induces the trivial map $G'_x \to G_x$, so that we have a commutative diagram of algebraic stacks

$$\begin{array}{ccc} \mathrm{Bun}_{G'}(X) & \longrightarrow & \mathrm{Spec}\, k \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G(X) & \longrightarrow & \mathrm{BG}_x . \end{array}$$

It is not hard to see that this diagram is a pullback square: that is, $\mathrm{Bun}_{G'}(X)$ can be identified with principal G_x-bundle $\mathrm{Bun}_G(X, \{x\})$ over $\mathrm{Bun}_G(X)$ which classifies G-bundles on X which are equipped with a trivialization at the point x.

Proof of Proposition 7.1.1. Since the generic fibers of G and G' are isomorphic, we can choose an nonempty open set $U \subseteq X$ and an isomorphism

$$\alpha : U \times_X G' \to U \times_X G$$

of group schemes over U. Suppose that $\rho_{G'}$ is an equivalence; we will show that ρ_G is an equivalence. Using Lemma 7.1.2 repeatedly, we can replace G by the group scheme obtained from G by dilitation at the points of $X - U$, and thereby reduce to the case where G_x is a vector group for $x \in X - U$.

Using Lemma 7.1.2 and Proposition A.3.11, we can replace G' by the group scheme obtained from G' by finitely many dilitations at the points of $X - U$ and thereby reduce to the case where α extends to a morphism $\overline{\alpha} : G' \to G$ of group schemes over X. Similarly, there exists a group scheme G'' obtained from G by finitely many dilitations at the points of $X - U$ so that α^{-1} extends to a map of group schemes $\beta : G'' \to G'$. We then have a commutative diagram

$$\begin{array}{ccccc} \int \mathcal{B}_G & \longrightarrow & \int \mathcal{B}_{G'} & \longrightarrow & \int \mathcal{B}_{G''} \\ \downarrow{\scriptstyle \rho_G} & & \downarrow{\scriptstyle \rho_{G'}} & & \downarrow{\scriptstyle \rho_{G''}} \\ C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) & \longrightarrow & C^*(\mathrm{Bun}_{G'}(X); \mathbf{Z}_\ell) & \longrightarrow & C^*(\mathrm{Bun}_{G''}(X); \mathbf{Z}_\ell). \end{array}$$

The horizontal composite maps are equivalences by virtue of Lemma 7.1.3 and Remark 7.1.4, so that this diagram exhibits ρ_G as a retract of $\rho_{G'}$. Since $\rho_{G'}$ is an equivalence, it follows that ρ_G is also an equivalence. □

The proofs of Lemmas 7.1.2 and 7.1.3 will require a few purely algebraic results whose proofs will be given at the end of this section.

Lemma 7.1.5. *Let A be an associative algebra object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$ equipped with an augmentation $\epsilon : A \to \mathbf{Z}_\ell$. Suppose that ϵ induces an isomorphism $\mathrm{H}_0(A) \to \mathbf{Z}_\ell$, that $\mathrm{H}_i(A) \simeq 0$ for $i > 0$, and that $\mathrm{H}_{-1}(A)$ is a flat $\mathbf{Z}_\ell$-module. Let $\phi : M \to N$ be a morphism of left A-module objects of $\mathrm{Mod}_{\mathbf{Z}_\ell}$, and suppose that $\mathrm{H}_i(M) \simeq \mathrm{H}_i(N) \simeq 0$ for $i > 0$. Then ϕ is an equivalence if and only if the induced map $\mathbf{Z}_\ell \otimes_A M \to \mathbf{Z}_\ell \otimes_A N$ is an equivalence.*

Lemma 7.1.6. *Let $A^\bullet$ be a cosimplicial object of $\mathrm{Alg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$. Suppose we are given a cosimplicial right $A^\bullet$-module $M^\bullet$ and a cosimplicial left $A^\bullet$-module $N^\bullet$ satisfying the following requirements:*

- (*a*) *For each integer $n \geq 0$, the homology groups $\mathrm{H}_*(M^n)$, $\mathrm{H}_*(N^n)$, and $\mathrm{H}_*(A^n)$ vanish for $* > 0$.*
- (*b*) *For each integer $n \geq 0$, the unit map $\mathbf{Z}_\ell \to \mathrm{H}_0(A^n)$ is an isomorphism.*
- (*c*) *For each integer $n \geq 0$, the homology group $\mathrm{H}_{-1}(A^n)$ is torsion-free.*

Then the canonical map

$$\theta : \mathrm{Tot}(M^\bullet) \otimes_{\mathrm{Tot}(A^\bullet)} \mathrm{Tot}(N^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_{A^\bullet} N^\bullet)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Lemma 7.1.7. *Let H be a connected algebraic group over k, let BH denote the classifying stack of H, let $\mathcal{C}$ be a prestack equipped with a map $\pi : \mathcal{C} \to \mathrm{BH}$, and form a pullback square*

$$\begin{array}{ccc} \overline{\mathcal{C}} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathrm{BH}. \end{array}$$

Then the associated diagram of cochain complexes

$$\begin{array}{ccc} C^*(\overline{\mathcal{C}}; \mathbf{Z}_\ell) & \longleftarrow & C^*(\mathcal{C}; \mathbf{Z}_\ell) \\ \uparrow & & \uparrow \\ C^*(\operatorname{Spec} k; \mathbf{Z}_\ell) & \longleftarrow & C^*(\mathrm{BH}; \mathbf{Z}_\ell) \end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$.

Proof. Let $U_0 = \operatorname{Spec} k$, and let $U_\bullet$ denote the simplicial scheme given by the nerve of the smooth map $U_0 \to \mathrm{BH}$ (so that $U_d \simeq H^d$). For each integer $d \geq 0$, the pullback diagram σ_d:

$$\begin{array}{ccc} \overline{\mathcal{C}} \times_{\mathrm{BH}} U_d & \longrightarrow & \mathcal{C} \times_{\mathrm{BH}} U_d \\ \downarrow & & \downarrow \\ \operatorname{Spec} k \times_{\mathrm{BH}} U_d & \longrightarrow & U_d \end{array}$$

can be rewritten as

$$\begin{array}{ccc} \overline{\mathcal{C}} \times_{\operatorname{Spec} k} H^{d+1} & \longrightarrow & \overline{\mathcal{C}} \times_{\operatorname{Spec} k} H^d \\ \downarrow & & \downarrow \\ H^{d+1} & \longrightarrow & H^d. \end{array}$$

Using Corollary 2.3.43, we deduce that σ_d determines a pushout square

$$\begin{array}{ccc} C^*(\overline{\mathcal{C}} \times_{\mathrm{BH}} U_d; \mathbf{Z}_\ell) & \longleftarrow & C^*(\mathcal{C} \times_{\mathrm{BH}} U_d; \mathbf{Z}_\ell) \\ \uparrow & & \uparrow \\ C^*(\operatorname{Spec} k \times_{\mathrm{BH}} U_d; \mathbf{Z}_\ell) & \longleftarrow & C^*(U_d; \mathbf{Z}_\ell) \end{array}$$

in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. We may therefore identify $C^*(\overline{\mathcal{C}})$ with $\mathrm{Tot}(M^\bullet \otimes_{A^\bullet} N^\bullet)$, where $A^\bullet = C^*(U_\bullet; \mathbf{Z}_\ell)$, $M^\bullet = C^*(\mathcal{C} \times_{\mathrm{BH}} U_\bullet; \mathbf{Z}_\ell)$, and $N^\bullet = C^*(\operatorname{Spec} k \times_{\mathrm{BH}} U_\bullet; \mathbf{Z}_\ell)$. To prove Lemma 7.1.7, we must show that the canonical map

$$\theta : \mathrm{Tot}(M^\bullet) \otimes_{\mathrm{Tot}(A^\bullet)} \mathrm{Tot}(N^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_{A^\bullet} N^\bullet)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. For this, it will suffice to show that $A^\bullet$, $M^\bullet$, and $N^\bullet$ satisfy the hypotheses of Lemma 7.1.6. Hypothesis (a) is obvious, and hypotheses (b) and (c) follows from our assumption that H is connected. □

Proof of Lemmas 7.1.2 and 7.1.3. Let us identify x with a closed immersion $\operatorname{Spec} k \to X$. Set $\mathcal{A} = \omega_X \oplus x_* C^*_{\mathrm{red}}(\mathrm{BG}_x; \mathbf{Z}_\ell)$. Using Example 5.6.8 and Remark 5.6.13, we see that $\mathcal{A}$ can be regarded as a commutative algebra object of $\mathrm{Shv}_\ell(X)$ with the following universal property: for every commutative algebra object $\mathcal{A}'$ of $\mathrm{Shv}_\ell(X)$, the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}(\mathcal{A}, \mathcal{A}') \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(C^*(\mathrm{BG}_x; \mathbf{Z}_\ell), x^! \mathcal{A}')$$

is a homotopy equivalence. In particular, we have a canonical map $\mathcal{A} \to \mathcal{B}_G$ for which the composite map $\mathcal{A} \to \mathcal{B}_G \to \mathcal{B}_{G'}$ factors through ω_X. We claim that the diagram σ :

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B}_G \\ \downarrow & & \downarrow \\ \omega_X & \longrightarrow & \mathcal{B}_{G'} \end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Shv}_\ell(X))$. To prove this, it suffices to show that $x^!(\sigma)$ is a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$ and that $j^*(\sigma)$ is a pushout square in $\mathrm{CAlg}(\mathrm{Shv}_\ell(X - \{x\}))$, where $j : X - \{x\} \hookrightarrow X$ is the inclusion map. This is clear, since the horizontal maps in the diagram $x^!(\sigma)$ are equivalences and the vertical maps in $j^*(\sigma)$ are equivalences.

Let $\pi_\star : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$ be as in Example 5.6.12. It follows immediately from the universal property of $\mathcal{A}$ that $\pi_\star \mathcal{A} \simeq C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$. We have a commutative diagram σ :

$$\begin{array}{ccccc} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell) & \longrightarrow & \int \mathcal{B}_G & \xrightarrow{\rho_G} & C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Z}_\ell & \longrightarrow & \int \mathcal{B}_{G'} & \xrightarrow{\rho_{G'}} & C^*(\mathrm{Bun}_{G'}(X); \mathbf{Z}_\ell) \end{array}$$

in the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$, where the left square is given by $\pi_\star(\sigma)$ (and is therefore a pushout square) and the outer square is obtained from the pullback diagram of algebraic stacks

$$\begin{array}{ccc} \mathrm{Bun}_{G'}(X) & \longrightarrow & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathrm{BG}_x, \end{array}$$

and is therefore a pushout diagram by virtue of Lemma 7.1.7 (see Remark 7.1.4). If G_x is a vector group, then the left vertical map $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell) \to \mathbf{Z}_\ell$ is an equivalence, so that vertical maps

$$\int \mathcal{B}_G \to \int \mathcal{B}_{G'} \qquad C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Bun}_{G'}(X); \mathbf{Z}_\ell)$$

are equivalences; this proves Lemma 7.1.3.

In the general case, we conclude that the right square in the diagram σ is also a pushout, so that we can identify $\rho_{G'}$ with the induced map

$$\mathbf{Z}_\ell \otimes_{C^*(\mathrm{BG}_x;\mathbf{Z}_\ell)} \int \mathcal{B}_G \to \mathbf{Z}_\ell \otimes_{C^*(\mathrm{BG}_x;\mathbf{Z}_\ell)} C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

induced by ρ_G. It follows immediately that if ρ_G is an equivalence, then $\rho_{G'}$ is also an equivalence. The converse follows from Lemma 7.1.5 (applied to the algebra $A = C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$), since the cohomologies of $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ and $\int \mathcal{B}_G$ are concentrated in nonnegative degrees. In the first case this is obvious, and in the second it follows from Theorem 8.2.18 and Corollary 8.3.6. This completes the proof of Lemma 7.1.2. □

We now turn to the proof of Lemmas 7.1.5 and 7.1.6.

Lemma 7.1.8. *Let A be an associative algebra object of the ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$ equipped with an augmentation $\epsilon : A \to \mathbf{Z}_\ell$ and let M be a left A-module in $\mathbf{Z}_\ell$. Suppose that ϵ induces an isomorphism $\mathrm{H}_0(A) \to \mathbf{Z}_\ell$, that $\mathrm{H}_i(A) \simeq \mathrm{H}_i(M) \simeq 0$ for $i > 0$, and that $\mathrm{H}_{-1}(A)$ is a flat $\mathbf{Z}_\ell$-module. Then the groups $\mathrm{H}_n(\mathbf{Z}_\ell \otimes_A M)$ vanish for $n > 0$, and the canonical map $\mathrm{H}_0(M) \to \mathrm{H}_0(\mathbf{Z}_\ell \otimes_A M)$ is injective.*

Proof. We first construct a sequence of right A-modules

$$N_0 \to N_1 \to N_2 \to \cdots$$

equipped with a compatible collection of maps $\epsilon_j : N_j \to \mathbf{Z}_\ell$. Set $N_0 = A$ and $\epsilon_0 = \epsilon$. Assuming that N_j and ϵ_j have been defined, let $K_j = \mathrm{fib}(\epsilon_j)$, and regard K_j as an object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$. Then the canonical map $K_j \to N_j$ extends to a right A-module morphism $\theta_j : K_j \otimes_{\mathbf{Z}_\ell} A \to N_j$, whose composition with ϵ_j is canonically nullhomotopic. We define $N_{j+1} = \mathrm{cofib}(\theta_j)$, and we let ϵ_{j+1} be the extension of ϵ_j determines by the canonical nullhomotopy of $\epsilon_j \circ \theta_j$. Note that the fiber of the map ϵ_{j+1} can be identified with $K_j \otimes_{\mathbf{Z}_\ell} \Sigma K_0$. Since $\mathbf{Z}_\ell$ is a principal ideal domain, we have exact sequences

$$0 \to \bigoplus_{p+q=n} \mathrm{Tor}_0^{\mathbf{Z}_\ell}(\mathrm{H}_p(K_j), \mathrm{H}_q(K_0)) \to \mathrm{H}_{n-1}(K_{j+1}) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathrm{H}_p(K_j), \mathrm{H}_q(K_0)) \to 0.$$

It follows by induction on j that $\mathrm{H}_n(K_j) \simeq 0$ for $n \geq 0$ and that $\mathrm{H}_{-1}(K_j)$ is flat as a $\mathbf{Z}_\ell$-module.

By construction, each of the maps $\mathrm{fib}(\epsilon_j) \to \mathrm{fib}(\epsilon_{j+1})$ is nullhomotopic as a map of $\mathbf{Z}_\ell$-modules, so that $\varinjlim \mathrm{fib}(\epsilon_i) \simeq 0$ and therefore $\mathbf{Z}_\ell$ is equivalent to the colimit $\varinjlim N_j$. We may therefore compute

$$\begin{aligned} \mathrm{H}_n(\mathbf{Z}_\ell \otimes_A M) &\simeq \mathrm{H}_n(\varinjlim N_j \otimes_A \varinjlim M) \\ &\simeq \varinjlim \mathrm{H}_n(N_j \otimes_A M). \end{aligned}$$

It will therefore suffice to show that the maps $\mathrm{H}_n(N_j \otimes_A M) \to \mathrm{H}_n(N_{j+1} \otimes_A M)$ are bijective for $n > 0$ and injective when $n = 0$. We have a fiber sequence

$$(K_j \otimes_{\mathbf{Z}_\ell} A) \otimes_A M \to M \otimes_A N_j \to M \otimes_A N_{j+1}$$

which determines a long exact sequence of abelian groups

$$\mathrm{H}_n(K_j \otimes_{\mathbf{Z}_\ell} A) \to \mathrm{H}_n(N_j \otimes_A M) \to \mathrm{H}_n(N_{j+1} \otimes_A M) \to \mathrm{H}_{n-1}(K_j \otimes_{\mathbf{Z}_\ell} M)$$

We are therefore reduced to proving that the groups $\mathrm{H}_n(M \otimes_{\mathbf{Z}_\ell} K_j)$ vanish for $n \geq 0$. This follows immediately from the existence of an exact sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{Tor}_0^{\mathbf{Z}_\ell}(\mathrm{H}_p(M), \mathrm{H}_q(K_j)) \to \mathrm{H}_n(M \otimes_{\mathbf{Z}_\ell} K_j) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathrm{H}_p(M), \mathrm{H}_q(K_j)) \to 0.$$

□

Example 7.1.9. Let $\mathcal{C}$ be a prestack. Then $A = C^*(\mathcal{C}; \mathbf{Z}_\ell)$ can be regarded as a commutative algebra object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$. Writing A as the $\mathbf{Z}_\ell$-linear dual of $C_*(\mathcal{C}; \mathbf{Z}_\ell)$, we obtain exact sequences

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}_\ell}(\mathrm{H}_{1-n}(\mathcal{C}; \mathbf{Z}_\ell), \mathbf{Z}_\ell) \to \mathrm{H}_n(A) \to \mathrm{Ext}^0_{\mathbf{Z}_\ell}(\mathrm{H}_{-n}(\mathcal{C}; \mathbf{Z}_\ell), \mathbf{Z}_\ell).$$

If $\mathrm{H}_0(\mathcal{C}; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$, this gives isomorphisms

$$\mathrm{H}_n(A) \simeq \begin{cases} 0 & \text{if } n > 0 \\ \mathbf{Z}_\ell & \text{if } n = 0 \\ \mathrm{Ext}^0_{\mathbf{Z}_\ell}(\mathrm{H}_1(\mathcal{C}; \mathbf{Z}_\ell), \mathbf{Z}_\ell) & \text{if } n = -1. \end{cases}$$

It follows that A satisfies the hypotheses of Lemma 7.1.8.

Proof of Lemma 7.1.5. Let $K = \mathrm{cofib}(\phi)$; we wish to show that if $\mathbf{Z}_\ell \otimes_A K \simeq 0$, then $K \simeq 0$. Note that $\mathrm{H}_i(K) \simeq 0$ for $i > 0$. If $K \neq 0$, then there exists some largest integer i such that $\mathrm{H}_i(K) \neq 0$. Applying Lemma 7.1.8, we see that the canonical map $\mathrm{H}_i(K) \to \mathrm{H}_i(\mathbf{Z}_\ell \otimes_A K)$ is injective, contradicting our assumption that $\mathbf{Z}_\ell \otimes_A K \simeq 0$. □

Lemma 7.1.10. *Let $M^\bullet$ be a cosimplicial object of $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$ and let $N \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$. Then the canonical map*

$$\mathrm{Tot}(M^\bullet) \otimes_{\mathbf{Z}_\ell} N \to \mathrm{Tot}(M^\bullet \otimes_{\mathbf{Z}_\ell} N)$$

is an equivalence.

Proof. For each integer $p \geq 0$, let $K(p)$ denote the pth partial totaliation of $M^\bullet$. Since the operation of tensoring with N is exact, we can identify $K(p) \otimes_{\mathbf{Z}_\ell} N$ with the pth partial totalization of $M^\bullet \otimes_{\mathbf{Z}_\ell} N$. It will therefore suffice to show that the canonical map

$$\theta : (\varprojlim K(p)) \otimes_{\mathbf{Z}_\ell} N \to \varprojlim (K(p) \otimes_{\mathbf{Z}_\ell} N)$$

is an equivalence. Note that for each $q \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} (\varprojlim K(p)) \otimes_{\mathbf{Z}_\ell} N & \xrightarrow{\theta} & \varprojlim (K(p) \otimes_{\mathbf{Z}_\ell} N) \\ & {\scriptstyle\phi} \searrow \quad \swarrow {\scriptstyle\psi} & \\ & K(q) \otimes_{\mathbf{Z}_\ell} N. & \end{array}$$

where the fibers of ϕ and ψ belong to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -q}$. It follows that the fiber of θ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -q}$ for all q, so that θ is an equivalence. □

Lemma 7.1.11. *Let $M^\bullet$ and $N^\bullet$ be cosimplicial objects of $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$. Then the canonical map*

$$\theta : \mathrm{Tot}(M^\bullet) \otimes_{\mathbf{Z}_\ell} \mathrm{Tot}(N^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_{\mathbf{Z}_\ell} N^\bullet)$$

is an equivalence.

Proof. Let $\mathbf{\Delta}$ denote the category whose objects are the nonempty linearly ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms are nondecreasing maps. Then the diagonal map $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$ is right cofinal (Lemma HTT.5.5.8.4), so that we can identify θ with the natural map

$$(\varprojlim_{[m] \in \mathbf{\Delta}} M^m) \otimes_{\mathbf{Z}_\ell} (\varprojlim_{[n] \in \mathbf{\Delta}} N^n) \to \varprojlim_{[m],[n] \in \mathbf{\Delta}} (M^m \otimes_{\mathbf{Z}_\ell} N^n).$$

This follows from two applications of Lemma 7.1.10. □

Proof of Lemma 7.1.6. If A is an associative algebra object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$ equipped with a right A-module M and a left A-module N, then the tensor product $M \otimes_A N$ can be computed as the geometric realization of a simplicial object $\mathrm{Bar}_A(M, N)_\bullet$ with

$$\mathrm{Bar}_A(M, N)_q \simeq M \otimes_{\mathbf{Z}_\ell} A^{\otimes q} \otimes_{\mathbf{Z}_\ell} N^\bullet.$$

For each integer d, we let $B^d_A(M, N)$ denote the realization of the d-skeleton of this simplicial object, so we have a sequence

$$M \otimes_{\mathbf{Z}_\ell} N \simeq B^0_A(M, N) \to B^1_A(M, N) \to \cdots$$

with colimit $M \otimes_A N$. Moreover, if we let $\overline{A}$ denote the cofiber of the unit map $\mathbf{Z}_\ell \to A$, then we have cofiber sequences

$$B^{d-1}_A(M, N) \to B^d_A(M, N) \to M \otimes_{\mathbf{Z}_\ell} (\Sigma \overline{A})^{\otimes d} \otimes_{\mathbf{Z}_\ell} N.$$

If $A^\bullet$, $M^\bullet$, and $N^\bullet$ are as in the statement of the Proposition, then assumption (a) and Lemma 7.1.11 supply equivalences

$$B^d_{\mathrm{Tot}(A^\bullet)}(\mathrm{Tot}(M^\bullet), \mathrm{Tot}(N^\bullet)) \simeq \mathrm{Tot}(B^d_{A^\bullet}(M^\bullet, N^\bullet))$$

for each integer $d \geq 0$. We may therefore identify θ with the canonical map

$$\varinjlim_d \mathrm{Tot}(B^d_{A^\bullet}(M^\bullet, N^\bullet)) \to \mathrm{Tot}(\varinjlim_d B^d_{A^\bullet}(M^\bullet, N^\bullet)).$$

To prove that this map is an equivalence, it will suffice to show that there exists an integer k such that $B^d_{A^p}(M^p, N^p)$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq k}$ for all $p, d \geq 0$. We claim that this is satisfied for $k = 1$. Using the cofiber sequence above, we are reduced to proving that

$$M^p \otimes_{\mathbf{Z}_\ell} (\Sigma \overline{A}^p)^{\otimes d} \otimes_{\mathbf{Z}_\ell} N^p$$

belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 1}$ for all p and all $d \geq 0$. It follows immediately from (a) that $M^p \otimes_{\mathbf{Z}_\ell} N^p$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 1}$. To complete the proof, it suffices to show that $\overline{A}^p$ has Tor-amplitude ≤ -1 for all $p \geq 0$, which follows from assumptions (a), (b), and (c). □

7.2. **Construction of the Sheaves $\mathcal{B}_S$.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which invertible in k, an algebraic curve X over k, and a smooth affine group scheme G over X. We will assume that each fiber of G is connected and that the generic fiber of G is semisimple and simply connected. We also fix an effective divisor $Q \subseteq X$ (in practice, we will take Q to be the set of points at which the group scheme G fails to be reductive; see Definition 7.2.9 below).

Let $\mathcal{B} = [\mathrm{Ran}^G(X)]_{\mathrm{Ran}(X)}$ denote the !-sheaf introduced in §5.4.2. Our goal in this section is to outline a proof of Theorem 5.4.5 by analyzing the composite map

$$\int \mathcal{B} \xrightarrow{\rho} C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Bun}_G(X, Q); \mathbf{Z}_\ell) \to C^*(\mathrm{Ran}_G(X - Q); \mathbf{Z}_\ell).$$

Let us identify $C^*(\mathrm{Ran}_G(X - Q); \mathbf{Z}_\ell)$ with the chiral homology of the !-sheaf $[\mathrm{Ran}_G(X - Q) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)]_{\mathrm{Ran}(X)}$ (where we regard $\mathrm{Ran}_G(X - Q) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$ as a $\mathrm{Ran}(X)$-prestack via projection onto the second factor). Then the map in question arises from a map of $\mathrm{Ran}(X)$-prestacks

$$\mathrm{Ran}_G(X-Q)\times_{\mathrm{Spec}\,k}\mathrm{Ran}(X)\to\mathrm{Ran}^G(X).$$

Unwinding the definitions, we can identify R-valued points of the left hand side with quadruples $(\mu:S\to(X-Q)(R),\nu:T\to X(R),\mathcal{P},\gamma)$ where S and T are nonempty finite sets, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over the open set $X_R-|\mu(S)|$. With respect to this identification, the map f is given by

$$(\mu:S\to(X-Q)(R),\nu:T\to X(R),\mathcal{P},\gamma)\mapsto(\nu:T\to X(R),\mathcal{P}\,|_{|\nu(T)|})$$

in other words, it takes generically trivialized G-bundles which are defined on the entire curve X and replaces them by their restrictions to the divisor $|\nu(T)|$ (ignoring their generic trivializations). The basic observation that we would like to exploit is the following: to form the restriction $\mathcal{P}\,|_{|\nu(T)|}$ on the divisor $|\nu(T)|$, it is not necessary that $\mathcal{P}$ be defined on the entire curve X: it is sufficient that $\mathcal{P}$ be defined on any open set which contains $\nu(T)$. In particular, if K is a subset of S such that $|\mu(K)|\cap|\nu(T)|=\emptyset$, then any G-bundle on the open set $X_R-|\mu(K)|$ can be restricted to the divisor $|\nu(T)|$.

Definition 7.2.1. Let S be a nonempty finite set. We define a category $\mathrm{Ran}^{\dagger}_G(X-Q)_S$ as follows:

- The objects of $\mathrm{Ran}^{\dagger}_G(X-Q)_S$ are septuples
$$(R,K_-,K_+,\mu:S\to(X-Q)(R),\nu:T\to X(R),\mathcal{P},\gamma)$$
where R is a finitely generated k-algebra, K_- and K_+ are subsets of S with $K_-\subseteq K_+$, T is a nonempty finite set, $\mu:S\to(X-Q)(R)$ and $\nu:T\to X(R)$ are maps of sets such that $|\mu(K_+)|\cap|\nu(T)|=\emptyset$, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over the open set $X_R-|\mu(S)|\subseteq X_R$.
- Given a pair of objects $C=(R,K_-,K_+,\mu:S\to(X-Q)(R),\nu:T\to X(R),\mathcal{P},\gamma)$ and $C'=(R',K'_-,K'_+,\mu':S\to(X-Q)(R'),\nu':T'\to X(R'),\mathcal{P}',\gamma')$ in $\mathrm{Ran}^{\dagger}_G(X-Q)_S$, there are no morphisms from C to C' unless $K'_-\subseteq K_-$ and $K_+\subseteq K'_+$. If both of these inclusions hold, then a morphism from C to C' consists of a k-algebra homomorphism $\phi:R\to R'$ which carries μ to μ', a surjection of finite sets $\lambda:T\to T'$ fitting into a commutative diagram
$$\begin{array}{ccc} T & \xrightarrow{\lambda} & T' \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R'), \end{array}$$
and a G-bundle isomorphism between $X_{R'}\times_{X_R}\mathcal{P}$ and $\mathcal{P}'$ over the inverse image of $X_R-|\mu(K_-)|$ which carries γ to γ'.

Remark 7.2.2. The isomorphism class of an object $(R,K_-,K_+,\mu:S\to X(R),\nu:T\to X(R),\mathcal{P},\gamma)\in\mathrm{Ran}^{\dagger}_G(X-Q)_S$ does not depend on the entire G-bundle $\mathcal{P}$, only on its restriction $\mathcal{P}_0=\mathcal{P}\,|_{X_R-|\mu(K_-)|}$. Consequently, it may be useful to think of the objects of $\mathrm{Ran}^{\dagger}_G(X-Q)_S$ as septuples $(R,K_-,K_+,\mu,\nu,\mathcal{P}_0,\gamma)$ where $\mathcal{P}_0$ is a G-bundle on $X_R-|\mu(K_-)|$. However, it is important that we consider only those G-bundles on $X_R-|\mu(K_-)|$ which can be extended to G-bundles on all of X_R (this condition is automatic when $R=k$, but not in general).

Remark 7.2.3. In the situation of Definition 7.2.1, the conditions $K_-\subseteq K_+$ and $|\mu(K_+)|\cap|\nu(T)|=\emptyset$ guarantee that the divisor $|\nu(T)|$ is contained in the open set $X_R-|\mu(K_-)|$, which is the locus of definition of the G-bundle $\mathcal{P}_0$ of Remark 7.2.2.

Remark 7.2.4. Let S be a nonempty finite set. We can describe the prestack $\mathrm{Ran}^{\dagger}_{G}(X-Q)_S$ informally as follows: it parametrizes pairs of maps

$$\mu : S \to X - Q \qquad \nu : T \to X$$

together with G-bundles that are defined on the open set $(X-\mu(S)) \cup \nu(T)$ and trivialized on $X - \mu(S)$.

Notation 7.2.5. Let S be a nonempty finite set. Then the construction

$$(R, K_-, K_+, \mu : S \to X(R), \nu : T \to X(R), \mathcal{P}, \gamma) \mapsto (R, \nu : T \to X(R))$$

determines a coCartesian fibration $\mathrm{Ran}^{\dagger}_{G}(X-Q)_S \to \mathrm{Ran}(X)$. We may therefore regard $\mathrm{Ran}^{\dagger}_{G}(X-Q)_S$ as a $\mathrm{Ran}(X)$-prestack in the sense of Definition 5.2.15. Let $\mathcal{B}_S \in \mathrm{Shv}^{\mathrm{lax}}_{\ell}(\mathrm{Ran}(X))$ denote the lax !-sheaf given by $[\mathrm{Ran}^{\dagger}_{G}(X-Q)_S]_{\mathrm{Ran}(X)}$ (see Definition 5.2.19).

Remark 7.2.6. Let S be a nonempty finite set, and let $\mathrm{Ran}_G(X-Q)_S$ denote the fiber $\mathrm{Ran}_G(X-Q) \times_{\mathrm{Fin}^{\mathrm{s}}} \{S\}$. There is an evident fully faithful embedding

$$\iota : \mathrm{Ran}_G(X-Q)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \to \mathrm{Ran}^{\dagger}_{G}(X-Q)_S$$

given by the formula

$$((R, \mu : S \to X(R), \mathcal{P}, \gamma), (R, \nu : T \to X(R))) \mapsto (R, \emptyset, \emptyset, \mu : S \to X(R), \nu : T \to X(R), \mathcal{P}, \gamma).$$

The essential image of this embedding consists of those objects

$$(R, K_-, K_+, \mu : S \to X(R), \nu : T \to X(R), \mathcal{P}, \gamma) \in \mathrm{Ran}^{\dagger}_{G}(X-Q)_S$$

for which $K_- = K_+ = \emptyset$. Note that ι is a map of $\mathrm{Ran}(X)$-prestacks, and therefore determines a map of relative cohomology sheaves

$$\begin{aligned} \mathcal{B}_S &= [\mathrm{Ran}^{\dagger}_{G}(X-Q)_S]_{\mathrm{Ran}(X)} \\ &\to [\mathrm{Ran}_G(X-Q)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)]_{\mathrm{Ran}(X)} \\ &\simeq C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}. \end{aligned}$$

Remark 7.2.7. Let S be a nonempty finite set. By virtue of Remark 7.2.3, we have a forgetful functor $\mathrm{Ran}^{\dagger}_{G}(X-Q)_S \to \mathrm{Ran}^{G}(X)$ given on objects by

$$(R, K_-, K_+, \mu : S \to X(R), \nu : T \to X(R), \mathcal{P}, \gamma) \mapsto (R, \nu : T \to X(R), \mathcal{P}\,|_{|\nu(T)|}).$$

Passing to relative cohomology sheaves, we obtain a morphism of lax !-sheaves

$$\mathcal{B} = [\mathrm{Ran}^{G}(X)]_{\mathrm{Ran}(X)} \to [\mathrm{Ran}^{\dagger}_{G}(X-Q)_S]_{\mathrm{Ran}(X)} = \mathcal{B}_S\,.$$

Remark 7.2.8. Let S be a nonempty finite set and let $Q \subseteq X$ be an effective divisor. The constructions described in Remarks 7.2.6 and 7.2.7 determine a commutative diagram of $\mathrm{Ran}(X)$-prestacks

$$\begin{array}{ccc} \mathrm{Ran}_G(X-Q)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) & \longrightarrow & \mathrm{Bun}_G(X) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \\ \downarrow & & \downarrow \\ \mathrm{Ran}^{\dagger}_{G}(X-Q)_S & \longrightarrow & \mathrm{Ran}^{G}(X). \end{array}$$

Passing to relative cohomology sheaves, we obtain a commutative diagram of lax !-sheaves

$$\begin{array}{ccc} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} & \longleftarrow & C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} \\ \uparrow & & \uparrow \\ \mathcal{B}_S & \longleftarrow & \mathcal{B}\,. \end{array}$$

It follows from Remark 7.2.8 that the composite map

$$\int \mathcal{B} \xrightarrow{\rho} C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \simeq C^*(\mathrm{Ran}_G(X-Q); \mathbf{Z}_\ell)$$

admits another factorization

$$\int \mathcal{B} \xrightarrow{\alpha} \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \xrightarrow{\beta} \varprojlim_{S \in \mathrm{Fin}^s} \int \mathcal{B}_S \xrightarrow{\gamma} \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell).$$

To prove Theorem 5.4.5, it will suffice to show that the maps α and $\gamma \circ \beta$ are equivalences. To prove this, we wil need to make some additional assumptions on Q.

Definition 7.2.9. Let $Q \subseteq X$ be an effective divisor. We will say that the group scheme G is *Q-adapted* if it satisfies the following conditions:

(a) There exists a simply connected semisimple algebraic group G_0 over k, a finite group Γ which acts on G_0 preserving a pinning $(B_0, T_0, \{u_\alpha\})$, an algebraic curve $\widetilde{X}$ with an action of Γ, an isomorphism of algebraic curves $\widetilde{X}/\Gamma \simeq X$, and a Γ-equivariant homomorphism
$$\widetilde{X} \times_X G \to \widetilde{X} \times_{\mathrm{Spec}\, k} G_0$$
of group schemes over $\widetilde{X}$ which is an isomorphism over the open set $\widetilde{X} \times_X (X-Q)$. Moreover, the projection map $\widetilde{X} \to X$ is étale over $X - Q$.

(b) For each point $x \in Q$, the fiber G_x is a vector group (that is, it is isomorphic to a finite product of copies of the additive group $\mathbf{G}_a$.

We can now state our main results:

Theorem 7.2.10. *Assume that the group scheme G is Q-adapted. Then the canonical map*

$$[\mathrm{Ran}^G(X)]_{\mathrm{Ran}(X)} \to \varprojlim_{S \in \mathrm{Fin}^s} [\mathrm{Ran}^\dagger_G(X-Q)_S]_{\mathrm{Ran}(X)}$$

is an equivalence of lax !-sheaves on $\mathrm{Ran}(X)$*. In particular, the induced map*

$$\alpha : \int [\mathrm{Ran}^G(X)]_{\mathrm{Ran}(X)} \to \int \varprojlim_{S \in \mathrm{Fin}^s} [\mathrm{Ran}^\dagger_G(X-Q)_S]_{\mathrm{Ran}(X)}$$

is a quasi-isomorphism.

Theorem 7.2.11. *For every nonempty finite set S, the inclusion*

$$\mathrm{Ran}_G(X-Q)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X) \hookrightarrow \mathrm{Ran}^\dagger_G(X-Q)_S$$

of Remark 7.2.6 induces a quasi-isomorphism

$$\gamma_S : \int \mathcal{B}_S \to \int C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} \simeq C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell).$$

In particular, these maps induce a quasi-isomorphism

$$\gamma : \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \simeq C^*(\mathrm{Ran}_G(X-Q); \mathbf{Z}_\ell).$$

Most of this section is devoted to the proof of Theorem 7.2.10; we will give an outline in §7.3, and carry out the details in §7.4, §7.5, and §7.6. Theorem 7.2.11 is a relatively straightforward application of the acyclicity of the Ran space; we give a proof in §7.7. However, we include a proof of this statement only to highlight the significance of the sheaves $\mathcal{B}_S$ introduced above: the proof of Theorem 5.4.5 that we give in §9 will not make direct use of Theorem 7.2.11.

Since Theorem 7.2.10 requires the group scheme G to be Q-adapted, it will be useful to know that this can always be arranged:

Proposition 7.2.12. *Let G be a smooth affine group scheme over X whose generic fiber is semisimple and simply connected. Then there exists an effective divisor $Q \subseteq X$ and a map $G' \to G$ of group schemes over X which is an isomorphism over the open set $X - Q$, where G' is Q-adapted.*

Remark 7.2.13. Using Theorems 3.2.9, 7.2.10, 7.2.11, and Proposition 7.2.12, we can *almost* complete the proof of Theorem 5.4.5. Let G be an arbitrary smooth affine group scheme over X with connected fibers whose generic fiber is semisimple and simply connected. We wish to show that the canonical map $\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ is a quasi-isomorphism. Using Proposition 7.2.12, we can choose an effective divisor $Q \subseteq X$, a Q-adapted group scheme G' over X, and a map of group schemes $G' \to G$ which is an isomorphism over the open set $X - Q$. Using Proposition 7.1.1, we can replace G by G' and thereby reduce to the case where G itself is Q-adapted. In this case, the projection map $\mathrm{Bun}_G(X, Q) \to \mathrm{Bun}_G(X)$ is an affine space bundle and therefore induces an isomorphism on ℓ-adic homology. We are therefore reduced to showing that the composite map

$$\int \mathcal{B} \xrightarrow{\rho} C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Bun}_G(X, Q); \mathbf{Z}_\ell)$$

is a quasi-isomorphism. Using Theorems 7.2.10, 7.2.11, and 3.2.9, we can factor this composite map as a composition

$$\begin{aligned}
\int \mathcal{B} \quad & \simeq \quad \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \\
& \xrightarrow{\beta} \varprojlim_{S \in \mathrm{Fin}^s} \int \mathcal{B}_S \\
& \simeq \quad \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X - Q)_S; \mathbf{Z}_\ell) \\
& \simeq \quad C^*(\mathrm{Ran}_G(X - Q); \mathbf{Z}_\ell) \\
& \simeq \quad C^*(\mathrm{Bun}_G(X, Q); \mathbf{Z}_\ell),
\end{aligned}$$

and thereby reduce to proving that β is an equivalence. Unfortunately, this is not so easy: we will therefore use a slightly different strategy in §9.

Proof of Proposition 7.2.12. Let G_0 denote the split form of the generic fiber of G (regarded as an algebraic group over k). Since the generic fiber of G is quasi-split, we can choose a finite Galois extension L of the fraction field K_X with Galois group $\Gamma = \mathrm{Gal}(L/K_X)$, an action of Γ on G_0 which preserves a pinning, and a Γ-equivariant isomorphism $\alpha : \mathrm{Spec}\, L \times_X G \simeq \mathrm{Spec}\, L \times_{\mathrm{Spec}\, k} G_0$. Let $\widetilde{X}$ denote the algebraic curve over k with fraction field L, so that the group Γ acts on $\widetilde{X}$ with quotient $\widetilde{X}/\Gamma \simeq X$. Let H denote the Weil restriction of the group scheme $\widetilde{X} \times_{\mathrm{Spec}\, k} G_0$ along the map $\widetilde{X} \to X$, and let H^Γ denote the fixed points for the evident action of Γ on H. Then α induces an isomorphism

$$\beta_0 : \mathrm{Spec}\, K_X \times_X G \simeq \mathrm{Spec}\, K_X \times_X H^\Gamma.$$

of algebraic groups over K_X. By a direct limit argument, we can choose a finite subset $Q \subseteq X$ such that β_0 extends to an isomorphism

$$\beta : (X - Q) \times_X G \to (X - Q) \times_X H^\Gamma.$$

Enlarging Q if necessary, we may assume that the map $\widetilde{X} \to X$ is étale over the open set $X - Q$. Fix $N > 0$, and let G' denote the group scheme over X obtained from G by applying an Nth order dilitation (along the identity section) at each point of Q (see §A.3). It is then clear that the fiber G'_x is a vector group for each $x \in Q$, and that the projection map $G' \to G$ is an isomorphism over $X - Q$. Using Proposition A.3.11, we see that for sufficiently large N, the map β extends (uniquely) to a map of group schemes $G' \to H^{\Gamma}$ which we can identify with a Γ-equivariant map $\widetilde{X} \times_X G' \to \widetilde{X} \times_{\operatorname{Spec} k} G_0$. By construction, this map is an isomorphism when restricted to the open subset $\widetilde{X} \times_X (X - Q)$. □

7.3. **The Limit of the Sheaves $\mathcal{B}_S$.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, a finite subset $Q \subseteq X$, and a smooth affine group scheme G which is Q-adapted (see Definition 7.2.9). Our goal in this section is to outline our strategy for proving Theorem 7.2.10, which asserts that the canonical map

$$\alpha : [\operatorname{Ran}^G(X)]_{\operatorname{Ran}(X)} \to \varprojlim_{S \in \operatorname{Fin}^s} [\operatorname{Ran}^{\dagger}_G(X - Q)_S]_{\operatorname{Ran}(X)}$$

is an equivalence of lax !-sheaves on $\operatorname{Ran}(X)$.

We begin by noting that the statement that α is an equivalence can be tested *locally* on the Ran space $\operatorname{Ran}(X)$. This motivates the following:

Notation 7.3.1. Let T be a (possibly empty) finite set. We define a category $\operatorname{Ran}^G(X)^T$ as follows:

- The objects of $\operatorname{Ran}^G(X)^T$ are triples $(R, \nu, \mathcal{P})$, where R is a finitely generated k-algebra, $\nu : T \to X(R)$ is a map of sets, and $\mathcal{P}$ is a G-bundle on the divisor $|\nu(T)|$.
- A morphism from $(R, \nu, \mathcal{P})$ to $(R', \nu', \mathcal{P}')$ in $\operatorname{Ran}^G(X)^T$ is a k-algebra homomorphism $R \to R'$ for which the composite map

  $$T \xrightarrow{\nu} X(R) \to X(R')$$

  coincides with ν', together with an isomorphism

  $$\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{P} \simeq \mathcal{P}'$$

  of G-bundles on the divisor $|\nu'(T)| \subseteq X_{R'}$.

The forgetful functor $(R, \nu, \mathcal{P}) \mapsto R$ is a coCartesian fibration which exhibits $\operatorname{Ran}^G(X)^T$ as a prestack, and the map $(R, \nu, \mathcal{P}) \mapsto (R, \nu)$ determines a map of prestacks $\operatorname{Ran}^G(X)^T \to X^T$. Note that if T is nonempty, then we can identify $\operatorname{Ran}^G(X)^T$ with the fiber product $\operatorname{Ran}^G(X) \times_{\operatorname{Ran}(X)} X^T$.

Notation 7.3.2. Let S and T be finite sets, where S is nonempty. We define a category $\operatorname{Ran}^{\dagger}_G(X - Q)^T_S$ as follows:

- The objects of $\operatorname{Ran}^{\dagger}_G(X - Q)^T_S$ are septuples

  $$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$$

  where R is a finitely generated k-algebra, K_- and K_+ are subsets of S with $K_- \subseteq K_+$, $\mu : S \to (X - Q)(R)$ and $\nu : T \to X(R)$ are maps of sets such that $|\mu(K_+)| \cap |\nu(T)| = \emptyset$, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu(S)| \subseteq X_R$.
- Given a pair of objects $C = (R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ and $C' = (R', K'_-, K'_+, \mu', \nu', \mathcal{P}', \gamma')$ in $\operatorname{Ran}^{\dagger}_G(X - Q)^T_S$, there are no morphisms from C to C' unless $K'_- \subseteq K_-$ and $K_+ \subseteq K'_+$. If both of these inclusions hold, then a morphism from C to C' consists of a k-algebra homomorphism $\phi : R \to R'$ which carries μ to μ' and ν to ν', together with a G-bundle isomorphism between $X_{R'} \times_{X_R} \mathcal{P}$ and $\mathcal{P}'$ over the inverse image of $X_R - |\mu(K_-)|$ which carries γ to γ'.

The construction $(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma) \mapsto R$ determines a coCartesian fibration

$$\mathrm{Ran}^{\dagger}_G(X - Q)^T_S \to \mathrm{Ring}_k,$$

so that we can regard $\mathrm{Ran}^{\dagger}_G(X - Q)^T_S$ as a prestack. Moreover, the construction

$$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma) \mapsto (R, \nu)$$

determines a map of prestacks $\mathrm{Ran}^{\dagger}_G(X - Q)^T_S \to X^T$. If T is nonempty, we have a canonical isomorphism

$$\mathrm{Ran}^{\dagger}_G(X - Q)^T_S \simeq \mathrm{Ran}^{\dagger}_G(X - Q)_S \times_{\mathrm{Ran}(X)} X^T.$$

If S and T are as in Notation 7.3.2, then the construction

$$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma) \mapsto (R, \nu, \mathcal{P}|_{|\nu(T)|})$$

determines a map of prestacks

$$\mathrm{Ran}^{\dagger}_G(X - Q)^T_S \to \mathrm{Ran}^G(X)^T,$$

which depends functorially on S. Theorem 7.2.10 is an immediate consequence of the following:

Proposition 7.3.3. *Let T be a finite set. Then the canonical map*

$$[\mathrm{Ran}^G(X)^T]_{X^T} \to \varprojlim_{S \in \mathrm{Fin}^s} [\mathrm{Ran}^{\dagger}_G(X - Q)^T_S]_{X^T}$$

is an equivalence in $\mathrm{Shv}_\ell(X^T)$.

Remark 7.3.4. Theorem 7.2.10 is equivalent to the assertion that Proposition 7.3.3 holds for every *nonempty* finite set T. However, our method of proof will require that we also treat the case where T is empty (this does not really pose any additional difficulties: when T is empty, Proposition 7.3.3 is a formal consequence of the acyclicity of $\mathrm{Ran}(X)$; see the argument following the statement of Proposition 7.3.12 below).

We will deduce Proposition 7.3.3 from the following stronger assertion:

Proposition 7.3.5. *Let T be a finite set and let Y be a quasi-projective k-scheme equipped with a map $Y \to X^T$. Then the canonical map*

$$\alpha_Y : [\mathrm{Ran}^G(X)^T \times_{X^T} Y]_Y \to \varprojlim_{S \in \mathrm{Fin}^s} [\mathrm{Ran}^{\dagger}_G(X - Q)^T_S \times_{X^T} Y]_Y$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$.

The virtue the formulation given in Proposition 7.3.5 is that it will allow us to apply a devissage to the scheme Y. Suppose that we are given a proper morphism of quasi-projective k-schemes $f : Y' \to Y$. We then have a commutative diagram

$$\begin{array}{ccc} [\mathrm{Ran}^G(X)^T \times_{X^T} Y']_{Y'} & \xrightarrow{\alpha_{Y'}} & \varprojlim_{S \in \mathrm{Fin}^s} [\mathrm{Ran}^{\dagger}_G(X - Q)^T_S \times_{X^T} Y']_{Y'} \\ \downarrow & & \downarrow \\ f^![\mathrm{Ran}^G(X)^T \times_{X^T} Y]_Y & \xrightarrow{f^!\alpha_Y} & \varprojlim_{S \in \mathrm{Fin}^s} f^![\mathrm{Ran}^{\dagger}_G(X - Q)^T_S \times_{X^T} Y]_Y \end{array}$$

in the ∞-category $\mathrm{Shv}_\ell(Y')$. Since $\mathrm{Ran}^G(X)^T$ is a smooth Artin stack with affine diagonal over X^T, it follows from Proposition 5.1.9 that the left vertical map in this diagram is an equivalence. The right vertical map is also an equivalence:

Proposition 7.3.6. *Let T be a finite set and suppose we are given a commutative diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & X^T, & \end{array}$$

where f is proper. For every nonempty finite set S, the canonical map

$$\phi : [\mathrm{Ran}^{\dagger}_{G}(X-Q)^{T}_{S} \times_{X^T} Y']_{Y'} \to f^{!}[\mathrm{Ran}^{\dagger}_{G}(X-Q)^{T}_{S} \times_{X^T} Y]_{Y}$$

is an equivalence in $\mathrm{Shv}_{\ell}(Y')$.

Proof. Let P denote the collection of all pairs (K_-, K_+), where K_- and K_+ are subsets of S satisfying $K_- \subseteq K_+$. We will regard P as a partially ordered set with $(K_-, K_+) \leq (K'_-, K'_+)$ if and only if $K'_- \subseteq K_- \subseteq K_+ \subseteq K'_+$. The construction

$$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma) \mapsto (K_-, K_+)$$

determines a Cartesian fibration $\pi : \mathrm{Ran}^{\dagger}_{G}(X-Q)^{T}_{S} \to P$ whose fibers are prestacks. The map ϕ can be written as an inverse limit of maps

$$\phi_{K_-,K_+} : [\pi^{-1}\{(K_-, K_+)\} \times_{X^T} Y']_{Y'} \to f^{!}[\pi^{-1}\{(K_-, K_+)\} \times_{X^T} Y]_{Y}.$$

It will therefore suffice to show that each ϕ_{K_-,K_+} is an equivalence in $\mathrm{Shv}_{\ell}(Y')$. This follows from Proposition 5.1.13, because each fiber $\pi^{-1}\{K_-, K_+\}$ admits an open immersion to the product prestack $\mathrm{Ran}_G(X-Q)_S \times_{\mathrm{Spec}\, k} X^T$. □

Corollary 7.3.7. *Let $f : Y' \to Y$ be a proper morphism between quasi-projective k-schemes. Suppose we are given a finite set T and a map $Y \to X^T$, and let α_Y and $\alpha_{Y'}$ be defined as in Proposition 7.3.5. Then $\alpha_{Y'}$ can be identified with the image of α_Y under the functor* $f^{!} : \mathrm{Shv}_{\ell}(Y) \to \mathrm{Shv}_{\ell}(Y')$.

Corollary 7.3.8. *Let T be a finite set and let Y be a quasi-projective k-scheme equipped with a map $Y \to X^T$. Let $i : Y' \to Y$ be a closed immersion and let $j : U \hookrightarrow Y$ be the complementary open immersion. If α_U and $\alpha_{Y'}$ are equivalences, then α_Y is also an equivalence.*

Proof. It follows from Corollary 7.3.7 that we have a fiber sequence $i_*(\alpha_{Y'}) \to \alpha_Y \to j_*(\alpha_U)$ in the ∞-category of morphisms in $\mathrm{Shv}_{\ell}(Y)$. □

To prove Proposition 7.3.5, we proceed by Noetherian induction on Y: that is, we may assume without loss of generality that $\theta_{Y'}$ is an equivalence for every closed subscheme $Y' \subsetneq Y$. If Y is non-reduced, we can complete the proof by taking $Y' = Y_{\mathrm{red}}$. Let us assume that Y is nonempty (otherwise, there is nothing to prove). By virtue of Corollary 7.3.8, it will suffice to prove Proposition 7.3.5 after replacing Y by an arbitrary nonempty open subset of Y.

Definition 7.3.9. Let T be a finite set and let $f : Y \to X^T$ be a map of quasi-projective k-schemes, which we identify with a finite set of maps $\{f_t : Y \to X\}_{t \in T}$. We will say that f is *Q-adapted* if, for every element $t \in T$, one of the following conditions holds:

- (a) The map f_t factors as a composition $Y \to \mathrm{Spec}\, k \xrightarrow{q} X$, where $q \in Q \subseteq X(k)$.
- (b) The map $f_t : Y \to X$ factors through $X - Q$.

Remark 7.3.10. Let $f : Y \to X^T$ be as in Definition 7.3.9. If Y is nonempty and reduced, then there exists a nonempty open subscheme $U \subseteq Y$ such that $f|_U$ is Q-adapted.

Using Remark 7.3.10, we see that it will suffice to prove Proposition 7.3.5 in the special case where the map $f : Y \to X^T$ is Q-adapted. In this case, we can write $T = T_0 \amalg T_1$, where $f = \{f_t\}_{t \in T}$ has the property that f_t is the constant map corresponding to some $q_t \in Q$ for $t \in T_0$, and f_t factors through $X - Q$ for $t \in T_1$.

Lemma 7.3.11. *Let $f : Y \to X^T \simeq X^{T_0} \times X^{T_1}$ be as above. Set $Y' = Y$, but regard Y' as equipped with the map $f' : Y' \to X^{T_1}$ given by composing f with the projection $X^T \to X^{T_1}$. Then α_Y is an equivalence if and only if $\alpha_{Y'}$ is an equivalence.*

Proof. It follows immediately from the definitions that we have equivalences of prestacks

$$\mathrm{Ran}_G^{\dagger}(X)^T_S \times_{X^T} Y \simeq \mathrm{Ran}_G^{\dagger}(X)^{T_1}_S \times_{X^{T_1}} Y'.$$

Under these equivalences, we can identify the natural map

$$\mathrm{Ran}_G^{\dagger}(X)^{T_1}_S \times_{X^{T_1}} Y' \to \mathrm{Ran}^G(X)^{T_1} \times_{X^{T_1}} Y'$$

with the composite map

$$\mathrm{Ran}_G^{\dagger}(X)^T_S \times_{X^T} Y \to \mathrm{Ran}^G(X)^T \times_{X^T} Y \xrightarrow{\theta} \mathrm{Ran}^G(X)^{T_1} \times_{X^{T_1}} Y',$$

where the map θ assigns to each G-bundle on a divisor $|\nu(T)|$ its restriction to the smaller divisor $|\nu(T_1)|$. It follows that $\alpha_{Y'}$ factors as a composition

$$[\mathrm{Ran}^G(X)^{T_1} \times_{X^{T_1}} Y']_{Y'} \xrightarrow{\theta^*} [\mathrm{Ran}^G(X)^T \times_{X^T} Y]_Y \xrightarrow{\alpha_Y} \varprojlim_S [\mathrm{Ran}_G^{\dagger}(X)^T_S \times_{X^T} Y]_Y,$$

where θ^* is the map given by pullback along θ. It will therefore suffice to show that the map θ^* is an equivalence in $\mathrm{Shv}_\ell(Y) = \mathrm{Shv}_\ell(Y')$.

Note that if $\nu : T \to X(R)$ determines a map $\mathrm{Spec}\, R \to X^T$ which factors through f, then the divisor $|\nu(T)| \subseteq X_R$ can be written as a disjoint union of divisors $|\nu(T_0)|$ and $|\nu(T_1)|$. It follows that every G-bundle on $|\nu(T_1)|$ extends canonically to a G-bundle on $|\nu(T)|$ (by taking that extension to be trivial on $|\nu(T_0)|$). This construction determines a map

$$\theta' : \mathrm{Ran}^G(X)^{T_1} \times_{X^{T_1}} Y' \to \mathrm{Ran}^G(X)^T \times_{X^T} Y$$

which is left inverse to θ. It will therefore suffice to show that θ' induces an equivalence

$$\theta'^* : [\mathrm{Ran}^G(X)^T \times_{X^T} Y]_Y \to [\mathrm{Ran}^G(X)^{T_1} \times_{X^{T_1}} Y']_{Y'}.$$

For $t \in T_0$, the map $f_t : Y \to X$ takes some constant value $q_t \in Q$. Let $D \subseteq X$ be the effective divisor given by the sum $\sum_{t \in T_0} q_t$, and let H denote the affine group scheme over k given by the Weil restriction of $G \times_X D$ along the projection map $D \to \mathrm{Spec}\, k$. Since the fiber $G \times_X \{q\}$ is a vector group for $q \in Q$, the group scheme H admits a finite filtration by vector groups and is therefore isomorphic (as a scheme) to an affine space $\mathbf{A}^d$ for some $d \geq 0$. We now complete the proof by observing that the map θ' exhibits $\mathrm{Ran}^G(X)^{T_1} \times_{X^{T_1}} Y'$ as an H-torsor over $\mathrm{Ran}^G(X)^T \times_{X^T} Y$, and is therefore a fiber bundle (locally trivial with respsect to the étale topology) with fiber $\mathbf{A}^d$. □

By virtue of Lemma 7.3.11, it will suffice to prove Proposition 7.3.5 in the special case where the map $Y \to X^T$ factors through $(X - Q)^T$. Passing to a dense open subset of Y if necessary, we may assume that Y is smooth and affine. Note that the domain and codomain of α_Y are ℓ-complete (Remark 5.1.6). Consequently, to prove that α_Y is an equivalence it will suffice to show that for every étale morphism $u : U \to Y$, the induced map

$$C^*(U; u^*[\mathrm{Ran}^G(X)^T \times_{X^T} Y]_Y) \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(U; u^*[\mathrm{Ran}_G^{\dagger}(X - Q)^T_S \times_{X^T} Y]_Y)$$

is an equivalence (Proposition 4.3.37). Replacing Y by U (and using the fact that the dualizing sheaf ω_Y is equivalent to $\Sigma^{2d}\underline{\mathbf{Z}_\ell(d)}$ for $d = \dim(Y)$, we are reduced to proving that the natural map

$$C^*(\mathrm{Ran}^G(X)^T \times_{X^T} Y; \mathbf{Z}_\ell) \to \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathrm{Ran}_G^\dagger(X-Q)_S^T \times_{X^T} Y; \mathbf{Z}_\ell)$$

is an equivalence. In fact, we will prove a slightly stronger statement at the level of homology:

Proposition 7.3.12. *Let T be a finite set and let Y be a smooth affine k-scheme equipped with a a map $Y \to (X-Q)^T$. Then the canonical map*

$$\varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C_*(\mathrm{Ran}_G^\dagger(X-Q)_S^T \times_{X^T} Y; \mathbf{Z}_\ell) \to C_*(\mathrm{Ran}^G(X)^T \times_{X^T} Y; \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proof of Proposition 7.3.12 when $T = \emptyset$. Using the Künneth formula (Proposition 2.3.40), we may assume without loss of generality that $Y = \operatorname{Spec} k$: that is, we wish to show that the canonical map

$$\varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C_*(\mathrm{Ran}_G^\dagger(X-Q)_S^\emptyset; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} k; \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. Note that this map factors as a composition

$$\varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C_*(\mathrm{Ran}_G^\dagger(X-Q)_S^\emptyset; \mathbf{Z}_\ell) \to \varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C_*(\mathrm{Ran}(X-Q)_S; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} k; \mathbf{Z}_\ell),$$

where the second map is an equivalence by virtue of the acyclicty of $\mathrm{Ran}(X-Q)$ (Corollary 2.4.13). We will complete the proof by showing that for each $S \in \mathrm{Fin}^{\mathrm{s}}$, the canonica; map

$$\theta_S : \mathrm{Ran}_G^\dagger(X-Q)_S^\emptyset \to \mathrm{Ran}(X-Q)_S$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology.

For the remainder of the proof, we fix a nonempty finite set S and let P be the partially ordered set introduced in the proof of Proposition 7.3.6. The construction

$$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma) \mapsto (K_-, K_+)$$

determines a Cartesian fibration $\mathrm{Ran}_G^\dagger(X-Q)_S^\emptyset \to P$; let us denote the fiber over an object (K_-, K_+) by $\mathrm{Ran}_G^\dagger(X-Q)_{K_-,K_+}^\emptyset$. Unwinding the definitions, we see that the forgetful functor $\mathrm{Ran}_G^\dagger(X-Q)_{S,S}^\emptyset \to \mathrm{Ran}(X-Q)_S$ is an equivalence of prestacks. Consequently, the composite map

$$\begin{aligned} C_*(\mathrm{Ran}_G^\dagger(X-Q)_{S,S}^\emptyset; \mathbf{Z}_\ell) &\xrightarrow{\rho} \\ \varinjlim_{(K_-,K_+) \in P} C_*(\mathrm{Ran}_G^\dagger(X-Q)_{K_-,K_+}^\emptyset; \mathbf{Z}_\ell) & \\ &\simeq C_*(\mathrm{Ran}_G^\dagger(X-Q)_S^\emptyset; \mathbf{Z}_\ell) \\ &\to C_*(\mathrm{Ran}(X-Q)_S; \mathbf{Z}_\ell). \end{aligned}$$

is an equivalence. To prove that θ_S induces an isomorphism on $\mathbf{Z}_\ell$-homology, we are reduced to proving that the map ρ is an equivalence.

For each $(K_-, K_+) \in P$, let $F(K_-, K_+)$ denote the chain complex $C_*(\mathrm{Ran}_G^\dagger(X-Q)_{K_-,K_+})^\emptyset$, so that F determines a functor $P^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$. We wish to show that the canonical map $\rho : F(S,S) \to \varinjlim_{(K_-,K_+) \in P} F(K_-, K_+)$ is an equivalence. Let $P_0 \subseteq P$ be the subset consisting of those pairs (K_-, K_+) where $K_+ = S$. Then P_0 contains (S,S) as a least element, so we can identify the domain of ρ with $\varinjlim_{(K_-,K_+) \in P_0} F(K_-, K_+)$. Consequently, to show that ρ is an equivalence, it will suffice to show that the functor F is a left Kan extension of its restriction to P_0^{op}. This is equivalent to the assertion that for each $(K_-, K_+) \in P$, the canonical map $F(K_-, S) \to F(K_-, K_+)$ is an equivalence. This is clear: since $T = \emptyset$, the inclusion $K_+ \hookrightarrow S$ induces an equivalence of prestacks $\mathrm{Ran}_G^{\dagger}(X - Q)^{\emptyset}_{K_-,S} \to \mathrm{Ran}_G^{\dagger}(X - Q)^{\emptyset}_{K_-,K_+}$. □

To prove Proposition 7.3.12 when T is nonempty, we will need some auxiliary constructions.

Notation 7.3.13. Let R be a finitely generated k-algebra, let $\mathcal{P}$ and $\mathcal{P}'$ be G-bundles on X_R. If $U \subseteq X_R$ is an open subscheme, we let $\mathrm{Iso}_U(\mathcal{P}, \mathcal{P}')$ denote the set of G-bundle isomorphisms between $\mathcal{P}|_U$ and $\mathcal{P}'|_U$. If $D \subseteq X_R$ is a closed subscheme, we let $\mathrm{Iso}_D^{\mathrm{germ}}(\mathcal{P}, \mathcal{P}') = \varinjlim_U \mathrm{Iso}_U(\mathcal{P}, \mathcal{P}')$, where the direct limit is taken over all open subschemes $U \subseteq X_R$ which contain D. We will refer to $\mathrm{Iso}_D^{\mathrm{germ}}(\mathcal{P}, \mathcal{P}')$ as the set of *germs of isomorphisms* of $\mathcal{P}$ with $\mathcal{P}'$ around D.

Definition 7.3.14. Let T be a nonempty finite set. We define a category $\mathrm{Ran}^G_{\mathrm{germ}}(X)^T$ as follows:

- (a) The objects of $\mathrm{Ran}^G_{\mathrm{germ}}(X)^T$ are triples $(R, \nu, \mathcal{P})$ where R is a finitely generated k-algebra, $\nu : T \to X(R)$ is a map, and $\mathcal{P}$ is a G-bundle on X_A.
- (b) A morphism from $(R, \nu, \mathcal{P})$ to $(R', \nu', \mathcal{P}')$ is a k-algebra homomorphism $R \to R'$ such that ν' coincides with the composite map $T \xrightarrow{\nu} X(R) \to X(R')$, together with a germ of isomorphisms between $X_{R'} \times_{X_R} \mathcal{P}$ around the divisor $|\nu'(T)| \subseteq X_{R'}$.

Remark 7.3.15. It may be helpful to think of the objects of $\mathrm{Ran}^G_{\mathrm{germ}}(X)^T$ as triples $(R, \nu, \mathcal{P})$ where $\mathcal{P}$ is a *germ* of G-bundles defined on an open subset of X_R containing the divisor $|\nu(T)|$. However, we consider only germs which can be extended to the entire curve X_R.

Restriction of G-bundles determines morphisms of prestacks

$$\mathrm{Ran}_G^{\dagger}(X - Q)^T_S \xrightarrow{\phi_S} \mathrm{Ran}^G_{\mathrm{germ}}(X)^T \to \mathrm{Ran}^G(X)^T,$$

where ϕ_S depends functorially on the nonempty finite set S. Proposition 7.3.12 is an immediate consequence of the following two assertions:

Proposition 7.3.16. *Let T be a nonempty finite set and let Y be an affine k-scheme equipped with a map $Y \to X^T$. Then the morphisms ϕ_S above induce an equivalence*

$$\varinjlim_{S \in \mathrm{Fin^s}} C_*(\mathrm{Ran}_G^{\dagger}(X - Q)^T_S \times_{X^T} Y; \mathbf{Z}_\ell) \to C_*(\mathrm{Ran}^G_{\mathrm{germ}}(X)^T \times_{X^T} Y; \mathbf{Z}_\ell).$$

Proposition 7.3.17. *Let T be a nonempty finite set and let Y be an affine k-scheme equipped with a map $f : Y \to (X - Q)^T$. Then the induced map*

$$\mathrm{Ran}^G_{\mathrm{germ}}(X)^T \times_{X^T} Y \to \mathrm{Ran}^G_{\mathrm{form}}(X)^T \times_{X^T} Y$$

is a universal homology equivalence.

Proposition 7.3.16 is a formal consequence of nonabelian Poincare duality (more specifically, of Theorem 3.3.1); the deduction is essentially an elaborate combinatorial exercise which we will carry out in §7.4. The geometric core of our proof is in the verification of Proposition 7.3.17, which we prove in §7.6 (using a calculation which we carry out in §7.5): essentially, this result expresses the idea that there is not much difference between G-bundles defined on a divisor $D \subseteq X - Q$ and G-bundles defined on an open neighborhood of D.

7.4. **Germs of G-Bundles.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, a finite subset $Q \subseteq X$, and a smooth affine group scheme G over X which is Q-adapted (Definition 7.2.9). Let T be a nonempty finite set and let $Y = \operatorname{Spec} R$ be an affine k-scheme of finite type equipped with a map of k-schemes $Y \to X^T$, which we will identify with a map $\nu : T \to X(R)$. Let $D = |\nu(T)| \subseteq X_R$ denote the associated divisor in X_R. For every finitely generated R-algebra A, we let D_A denote the inverse image of D in X_A.

Our goal is to prove Proposition 7.3.16, which asserts that the canonical map

$$\varinjlim_{S \in \mathrm{Fin}^s} C_*(\mathrm{Ran}^\dagger_G(X-Q)^T_S \times_{X^T} Y; \mathbf{Z}_\ell) \to C_*(\mathrm{Ran}^G_{\mathrm{germ}}(X)^T \times_{X^T} Y; \mathbf{Z}_\ell)$$

is a quasi-isomorphism. As a first step, we will identify the left hand side with the chain complex of a single prestack.

Construction 7.4.1. We define a prestack $\mathcal{C}$ as follows:

- The objects of $\mathcal{C}$ are septuples $(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma)$ where S is a nonempty finite set, $K_- \subseteq K_+ \subseteq S$, A is a finitely generated R-algebra, $\mu : S \to (X-Q)(A)$ is a map of sets for which $|\mu(K_+)|$ does not intersect D_A, $\mathcal{P}$ is a G-bundle on X_A, and γ is a trivialization of $\mathcal{P}$ on $X_A - |\mu(S)|$.
- A morphism from $(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma)$ to $(S', K'_-, K'_+, A', \mu', \mathcal{P}', \gamma')$ in the category $\mathcal{C}$ consists of a surjection of finite sets $\alpha : S \to S'$ satisfying $\alpha^{-1}(K'_-) \subseteq K_- \subseteq K_+ \subseteq \alpha^{-1}(K'_+)$, an R-algebra homomorphism $A \to A'$ for which the diagram

$$\begin{array}{ccc} S & \xrightarrow{\mu} & (X-Q)(A) \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ S' & \xrightarrow{\mu'} & (X-Q)(A') \end{array}$$

  commutes, and an isomorphism between the pullbacks of $\mathcal{P}$ and $\mathcal{P}'$ to $X_A \times_{X_A} (X_A - |\mu(K_-)|)$ which carries γ to γ'.

Note that the construction $(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma) \mapsto S$ determines a Cartesian fibration of categories $\mathcal{C} \to \mathrm{Fin}^s$, whose fiber over an object $S \in \mathrm{Fin}^s$ can be identified with the prestack $Y \times_{X^T} \mathrm{Ran}^\dagger_G(X-Q)^T_S$. It follows that we have a canonical equivalence

$$C_*(\mathcal{C}; \mathbf{Z}_\ell) \simeq \varinjlim_{S \in \mathrm{Fin}^s} C_*(Y \times_{X^T} \mathrm{Ran}^\dagger_G(X)^T_S; \mathbf{Z}_\ell).$$

We can therefore reformulate Proposition 7.3.16 as follows:

Theorem 7.4.2. *The forgetful functor*

$$\theta : \mathcal{C} \to Y \times_{X^T} \mathrm{Ran}^G_{\mathrm{germ}}(X)^T$$

$$(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma) \mapsto (A, \mathcal{P})$$

induces an isomorphism $\mathrm{H}_*(\mathcal{C}; \mathbf{Z}_\ell) \to \mathrm{H}_*(Y \times_{X^T} \mathrm{Ran}^G_{\mathrm{germ}}(X)^T; \mathbf{Z}_\ell)$.

Proof. We will show that the functor θ factors as a composition

$$\mathcal{C} \xrightarrow{\theta_0} \mathcal{C}_0 \xrightarrow{\theta_1} \mathcal{C}_1 \xrightarrow{\theta_2} \mathcal{C}_2 \xrightarrow{\theta_3} \mathcal{C}_3 \xrightarrow{\theta_4} \mathcal{C}_4 \xrightarrow{\theta_5} \mathcal{C}_5 \xrightarrow{\theta_6} Y \times_{X^T} \mathrm{Ran}^G_{\mathrm{germ}}(X)^T,$$

where each $\mathcal{C}_i$ is a prestack via some forgetful functor $\pi_i : \mathcal{C}_i \to \mathrm{Ring}_k$ and each θ_i is a morphism of prestacks which induces an isomorphism on ℓ-adic homology. Most of the steps in our argument will be completely formal: geometric input will be needed only in our proof that

θ_3 is an equivalence (which depends on the acyclicity of the Ran space) and the proof that θ_4 is an equivalence (which depends on Theorem 3.3.1).

- The category $\mathcal{C}_0$ is defined as follows:
  - The objects of $\mathcal{C}_0$ are septuples $(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma)$ where S is a nonempty finite set, $K_- \subseteq K_+ \subseteq S$, A is a finitely generated R-algebra, $\mu : S \to (X - Q)(A)$ is a map of sets for which $|\mu(K_+)|$ does not intersect D_A, $\mathcal{P}$ is a G-bundle on X_A, and γ is a trivialization of $\mathcal{P}$ on $X_A - |\mu(S)|$.
  - A morphism from $(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma)$ to $(S', K'_-, K'_+, A', \mu', \mathcal{P}', \gamma')$ in the category $\mathcal{C}_0$ consists of a surjection of finite sets $\alpha : S \to S'$ satisfying $K'_- \subseteq \alpha(K_-) \subseteq \alpha(K_+) \subseteq K'_+$, an R-algebra homomorphism $A \to A'$ for which the diagram

$$\begin{array}{ccc} S & \xrightarrow{\mu} & (X-Q)(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ S' & \xrightarrow{\mu'} & (X-Q)(A') \end{array}$$

    commutes, and an isomorphism between the pullbacks of $\mathcal{P}$ and $\mathcal{P}'$ to $X_{A'} \times_{X_A} (X_A - |\mu(K_-)|)$ which carries γ to γ'.

We will regard $\mathcal{C}_0$ as a prestack via the forgetful functor

$$\pi_0 : \mathcal{C}_0 \to \mathrm{Ring}_k$$

$$\pi_0(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma) = A.$$

By construction, we can identify $\mathcal{C}$ with a (non-full) subcategory of $\mathcal{C}_0$. Let $\theta_0 : \mathcal{C} \hookrightarrow \mathcal{C}_0$ be the inclusion map. We claim that θ_0 induces an isomorphism on $\mathbf{Z}_\ell$-homology. To prove this, it will suffice to show that the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}_0$ is right cofinal when regarded as a functor between ∞-categories.

Let $\mathcal{J}$ denote the category whose objects are given by triples (S, K_-, K_+) where S is a nonempty finite set and $K_- \subseteq K_+ \subseteq S$, where a morphism from (S, K_-, K_+) to (S', K'_-, K'_+) is a surjection $\alpha : S \to S'$ such that $K'_- \subseteq \alpha(K_-) \subseteq \alpha(K_+) \subseteq K'_+$. Let $\mathcal{J}' \subseteq \mathcal{J}$ be the subcategory containing all objects, whose morphisms are required to satisfy the stronger condition that $\alpha^{-1} K'_- \subseteq K_-$. We have a pullback diagram of categories

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}_0 \\ \downarrow & & \downarrow \\ \mathcal{J}' & \longrightarrow & \mathcal{J}, \end{array}$$

where the vertical maps are Cartesian fibrations. Consequently, it will suffice to show that the inclusion $\mathcal{J}' \hookrightarrow \mathcal{J}$ is right cofinal (Remark HTT.4.1.2.10 and Proposition HTT.4.1.2.15). Fix an object $(S, K_-, K_+) \in \mathcal{J}$, and let $\mathcal{E}$ denote the fiber product $\mathcal{J}' \times_{\mathcal{J}} \mathcal{J}_{/(S, K_-, K_+)}$; we wish to prove that the simplicial set $\mathrm{N}(\mathcal{E})$ is weakly contractible. Note that the projection map $\mathcal{E} \to \mathrm{Fin}^{\mathrm{s}}{}_{/S}$ is a Cartesian fibration of categories. It will therefore suffice to show that for each surjection $\alpha : S' \to S$, the fiber product $\mathcal{E}_{S'} = \mathcal{E} \times_{\mathrm{Fin}^{\mathrm{s}}{}_{/S}} \{S'\}$ has weakly contractible nerve. Unwinding the definition, we see that $\mathcal{E}_{S'}$ can be identified with the partially ordered set of ordered pairs (K'_-, K'_+) of subsets of S' satisfying $K_- \subseteq \alpha(K'_-)$, $K'_- \subseteq K'_+$, and $K'_+ \subseteq \alpha^{-1} K_+$. Let $\mathcal{E}^0_{S'}$ denote the full subcategory of $\mathcal{E}_{S'}$ spanned by those objects where $K'_+ = \alpha^{-1} K_+$. Then the inclusion $\mathcal{E}^0_{S'} \hookrightarrow \mathcal{E}_{S'}$ admits a left adjoint given by $(K'_-, K'_+) \mapsto (K'_-, \alpha^{-1} K_+)$. We are therefore reduced to proving that $\mathrm{N}(\mathcal{E}^0_{S'})$ is weakly contractible. This is clear, since $\mathrm{N}(\mathcal{E}^0_{S'})$ has an initial object (given by the pair $(\alpha^{-1}K_+, \alpha^{-1}K_+)$).

- Let $\mathcal{C}_1$ denote the full subcategory of $\mathcal{C}_0$ spanned by those objects $(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma)$ where K_+ is the largest subset of S for which $|\mu(K_+)|$ does not intersect D_A. Note that the inclusion functor $\mathcal{C}_1 \hookrightarrow \mathcal{C}_0$ admits a left adjoint $\theta_1 : \mathcal{C}_0 \to \mathcal{C}_1$, given on objects by the formula

$$\theta_1(S, K_-, K_+, A, \mu, \mathcal{P}, \gamma) = (S, K_-, K'_+, A, \mu, \mathcal{P}, \gamma),$$

where K'_+ is the largest subset of S such that $|\mu(K'_+)|$ does not intersect the inverse image of D. The projection map $\pi_0 : \mathcal{C}_0 \to \mathrm{Ring}_k$ restricts to a coCartesian fibration $\pi_1 : \mathcal{C}_1 \to \mathrm{Ring}_k$ which exhibits $\mathcal{C}_1$ as a prestack and θ_1 as a morphism of prestacks. Using Remark 2.3.32, we deduce that θ_1 induces an isomorphism on ℓ-adic homology (with inverse induced by the weak morphism of prestacks $\mathcal{C}_1 \hookrightarrow \mathcal{C}_0$).

- We define a category $\mathcal{C}_2$ as follows:
  - The objects of $\mathcal{C}_2$ are tuples $(S, A, \mu, I, \mathcal{P}, \gamma)$, where S is a nonempty finite set, A is a finitely generated R-algebra, $\mu : S \to (X - Q)(A)$ is a map of sets, I is a subset of $\mu(S) \subseteq X(A)$ such that $|I|$ does not intersection D_A, $\mathcal{P}$ is a G-bundle on X_A, and γ is a trivialization of $\mathcal{P}$ on $X_A - |\mu(S)|$.
  - A morphism from $(S, A, \mu, I, \mathcal{P}, \gamma)$ to $(S', A', \mu', I', \mathcal{P}', \gamma')$ in $\mathcal{C}_2$ consists of a surjection of finite sets $S \to S'$, a map of R-algebras $A \to A'$ such that I' is contained in the image of I under the induced map $X(A) \to X(A')$ and the diagram

$$\begin{array}{ccc} S & \xrightarrow{\mu} & (X-Q)(A) \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\mu'} & (X-Q)(A') \end{array}$$

  commutes, together with an isomorphism between the pullbacks of $\mathcal{P}$ and $\mathcal{P}'$ to $X_{A'} \times_{X_A} (X_A - |I|)$ which carries γ to γ'.

  We regard $\mathcal{C}_2$ as a prestack via the map $\pi_2 : \mathcal{C}_2 \to \mathrm{Ring}_k$ given on objects by $(S, A, \mu, I, \mathcal{P}, \gamma) \mapsto A$. There is an evident forgetful functor $\theta_2 : \mathcal{C}_1 \to \mathcal{C}_2$, given by

$$(S, K_-, K_+, \mu, A, \mathcal{P}, \gamma) \mapsto (S, \mu, A, \mu(K_-), \mathcal{P}, \gamma).$$

  This functor admits a left adjoint, given by

$$(S, \mu, A, I, \mathcal{P}, \gamma) \mapsto (S, K_-, K_+, \mu, A, \mathcal{P}, \gamma),$$

  where $K_- = \mu^{-1}(I)$ and K_+ is the largest subset of S such that $|\mu(K_+)|$ does not intersect D_A. Invoking Remark 2.3.32, we see that θ_2 induces an isomorphism on ℓ-adic homology.

- We define a category $\mathcal{C}_3$ as follows:
  - The objects of $\mathcal{C}_3$ are tuples $(A, I, J, \mathcal{P}, \gamma)$ where A is a finitely generated R-algebra, J is a (possibly empty) finite subset of $(X - Q)(A)$, I is a subset of J such that $|I|$ does not intersect the inverse image of D_A, $\mathcal{P}$ is a G-bundle on X_A, and γ is a trivialization of $\mathcal{P}$ on $X_A - |J|$.
  - A morphism from $(A, I, J, \mathcal{P}, \gamma)$ to $(A', I', J', \mathcal{P}', \gamma')$ in $\mathcal{C}_3$ consists of an R-algebra homomorphism $A \to A'$ for which the induced map $(X - Q)(A) \to (X - Q)(A')$ carries J into J', I to a subset of $X(A')$ which contains I', together with an isomorphism between the pullbacks of $\mathcal{P}$ and $\mathcal{P}'$ to $X_{A'} \times_{X_A} (X_A - |I|)$ which carries γ to γ'.

We regard $\mathcal{C}_3$ as a prestack via the forgetful functor $\pi_3 : \mathcal{C}_3 \to \mathrm{Ring}_k$ given by $(A, I, J, \mathcal{P}, \gamma) \mapsto A$. We have an evident map of prestacks $\theta_3 : \mathcal{C}_2 \to \mathcal{C}_3$, given on objects by $\theta_3(S, A, \mu, I, \mathcal{P}, \gamma) = (A, I, \mu(S), \mathcal{P}, \gamma)$. We observe that θ_3 fits into a pullback diagram of prestacks

$$\begin{array}{ccc} \mathcal{C}_2 & \xrightarrow{\theta_3} & \mathcal{C}_3 \\ \downarrow & & \downarrow \\ \mathrm{Ran}(X-Q) & \longrightarrow & \mathrm{Ran}^+(X-Q). \end{array}$$

where the vertical maps are coCartesian fibrations. It follows from Theorem 2.5.19 that the map $\mathrm{Ran}(X - Q) \to \mathrm{Ran}^+(X - Q)$ is a universal homology equivalence. Applying Proposition 2.5.11, we deduce that θ_3 is also a universal homology equivalence.

- We define a category $\mathcal{C}_4$ as follows:
  - The objects of $\mathcal{C}_4$ are triples $(A, I, \mathcal{P})$ where A is a finitely generated R-algebra, I is a finite subset of $(X - Q)(A)$ such that $|I| \cap D_A = \emptyset$, and $\mathcal{P}$ is a G-bundle on X_A.
  - A morphism from $(A, I, \mathcal{P})$ to $(A', I', \mathcal{P}')$ in $\mathcal{C}_4$ consists of an R-algebra homomorphism $A \to A'$ for which I' is contained in the image of the composite map $I \hookrightarrow X(A) \to X(A')$, together with an isomorphism between the pullbacks of $\mathcal{P}$ and $\mathcal{P}'$ to $X_{A'} \times_{X_A} (X_A - |I|)$.

We regard $\mathcal{C}_4$ as a prestack via the forgetful functor $\pi_4 : \mathcal{C}_4 \to \mathrm{Ring}_k$ given by $(A, I, \mathcal{P}) \mapsto A$. Let $\theta_4 : \mathcal{C}_3 \to \mathcal{C}_4$ denote the map of prestacks given by $(A, I, J, \mathcal{P}, \gamma) \mapsto (A, I, \mathcal{P})$. We claim that θ_4 is a universal homology equivalence. To prove this, consider an object $C = (A, I, \mathcal{P}) \in \mathcal{C}_4$, and let $\mathcal{D} = \mathcal{C}_3 \times_{\mathcal{C}_4} (\mathcal{C}_4)_{C/}$. We wish to show that the canonical map $C_*(\mathcal{D}; \mathbf{Z}_\ell) \to C_*(\mathrm{Spec}\, A; \mathbf{Z}_\ell)$ is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Unwinding the definitions, we can identify objects of $\mathcal{D}$ with tuples $(B, I_B, J, \mathcal{P}_B, \gamma_B)$, where B is a finitely generated A-algebra, I_B is a subset of $X(B)$ which is contained in the image of I, J is a finite subset of $X(B)$ containing I_B, $\mathcal{P}_B$ is a G-bundle on $X_B - |I_B|$ compatible with $\mathcal{P}|_{X_A - |I|}$, and γ_B is a trivialization of $\mathcal{P}_B$ over $X_B - |J|$. Let $\mathcal{D}'$ denote the full subcategory of $\mathcal{D}$ spanned by those tuples where J contains the image of I in $X(B)$. The inclusion $\mathcal{D}' \to \mathcal{D}$ admits a left adjoint in the 2-category of prestacks, and therefore induces an isomorphism on ℓ-adic homology by Remark 2.3.32. Let $\mathcal{D}'' \subseteq \mathcal{D}'$ be the full subcategory spanned by those tuples $(B, I_B, J, \mathcal{P}_B, \gamma)$ where I_B coincides with the image of I in $X(B)$. The inclusion $\mathcal{D}'' \hookrightarrow \mathcal{D}'$ admits a right adjoint in the 2-category of prestacks, and therefore also induces an isomorphism on homology.

Let Z denote the A-scheme given by the product

$$\prod_{x \in Q} \mathcal{P} \times_X \{x\}.$$

The construction

$$(B, I_B, J, \mathcal{P}_B, \gamma_B) \mapsto \gamma_B|_{Q_B}$$

determines a morphism of prestacks $\rho : \mathcal{D}'' \to Z$. Since we have assumed that G_x is a vector group for $x \in Q$ and $\mathrm{Spec}\, R$ is affine, Z is isomorphic to an affine space $\mathbf{A}^d \times \mathrm{Spec}\, R$ over $\mathrm{Spec}\, R$. It will therefore suffice to show that ρ induces an isomorphism in ℓ-adic homology. In fact, we will show that ρ is a universal homology equivalence. Fix a map $\mathrm{Spec}\, B \to Z$, so that B is a finitely generated A-algebra and we are given a trivialization γ_0 of $\mathcal{P}$ over the divisor Q_B. Let $\mathrm{Sect}_{Q_B}(\mathcal{P}_B)$ be the prestack introduced in §3.3 whose objects are given by triples (B', S, γ) where B' is a finitely generated

B-algebra, S is a nonempty finite subset of $(X-Q)(B')$, and γ is a trivialization of $\mathcal{P}$ over $X_B - |S|$ which is compatible with γ_0. Unwinding the definitions, we can identify $\mathcal{D}'' \times_Z \operatorname{Spec} B$ with the full subcategory of $\operatorname{Sect}_{Q_B}(\mathcal{P})$ spanned by those triples (B', S, γ) where S contains the image of I. Note that the inclusion $\mathcal{D}'' \times_Z \operatorname{Spec} B \hookrightarrow \operatorname{Sect}_{Q_B}(\mathcal{P})$ admits a left adjoint (in the 2-category of prestacks), and therefore induces an isomorphism on $\mathbf{Z}_\ell$-homology (Remark 2.3.32). We are therefore reduced to proving that the projection map $\operatorname{Sect}_{Q_B}(\mathcal{P}) \to \operatorname{Spec} B$ induces an isomorphism in ℓ-adic homology, which is a special case of Theorem 3.3.1.

- We define a category $\mathcal{C}_5$ as follows:
  - The objects of $\mathcal{C}_5$ are pairs $(A, \mathcal{P})$, where $\mathcal{P}$ is a finitely generated R-algebra and $\mathcal{P}$ is a G-bundle on X_A.
  - A morphism from $(A, \mathcal{P})$ to $(A', \mathcal{P}')$ is an R-algebra homomorphism $A \to A'$, together with an element $\alpha \in \varinjlim_{U \subseteq X_{A'}} \operatorname{Iso}_U(\operatorname{Spec} A' \times_{\operatorname{Spec} A} \mathcal{P}, \mathcal{P}')$; here the direct limit is taken over all open subsets $U \subseteq X_{A'}$ of the form $X_{A'} - |J|$, where J is a finite subset of $X(A')$ for which $|J|$ does not intersect the inverse image of D.

  We regard $\mathcal{C}_5$ as a prestack via the forgetful functor $\pi_5 : \mathcal{C}_5 \to \operatorname{Ring}_k$ given by $(A, \mathcal{P}) \mapsto A$. The construction $(A, I, \mathcal{P}) \mapsto (A, \mathcal{P})$ determines a map of prestacks $\theta_5 : \mathcal{C}_4 \to \mathcal{C}_5$. We claim that the functor π_5 is right cofinal and therefore induces an equivalence $C_*(\mathcal{C}_5; \mathbf{Z}_\ell) \to C_*(\mathcal{C}_4; \mathbf{Z}_\ell)$. To prove this, it will suffice to show that for every object $C = (A, \mathcal{P})$ in $\mathcal{C}_5$, the category $\mathcal{E} = \mathcal{C}_4 \times_{\mathcal{C}_5} (\mathcal{C}_5)_{/C}$ has weakly contractible nerve. Note that every object of $\mathcal{E}$ determines in particular a finitely generated R-algebra B together with a map of R-algebras $\psi : B \to A$.

  Let $\mathcal{E}_0$ denote the subcategory of $\mathcal{E}$ spanned by those objects where $B = A$ and $\psi = \mathrm{id}_A$. The inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ admits a left adjoint, and therefore induces a weak homotopy equivalence $\mathrm{N}(\mathcal{E}_0) \to \mathrm{N}(\mathcal{E})$. It will therefore suffice to prove that $\mathrm{N}(\mathcal{E}_0)$ is weakly contractible. Unwinding the definitions, we can identify objects of $\mathcal{E}_0$ with triples $(I, \mathcal{Q}, \alpha)$, where I is a finite subset of $(X-Q)(A)$ such that $|I| \cap D_A = \emptyset$, $\mathcal{Q}$ is a G-bundle on $X_A - |I|$ which can be extended to a G-bundle on X_A, and $\alpha \in \varinjlim_U \operatorname{Iso}_U(\mathcal{P}, \mathcal{Q})$; a morphism from $(I, \mathcal{Q}, \alpha)$ to $(I', \mathcal{Q}', \alpha')$ is an inclusion $I' \subseteq I$ together with an isomorphism of $\mathcal{Q}$ with $\mathcal{Q}'|_{X_A - |I|}$ which carries α to α'. The weak contractibility of $\mathrm{N}(\mathcal{E}_0)$ follows from the observation that the opposite category $\mathcal{E}_0^{\mathrm{op}}$ is filtered.

- Unwinding the definitions, we can identify the fiber product $Y \times_{X^T} \operatorname{Ran}^G_{\mathrm{germ}}(X)^T$ with the prestack whose objects are pairs $(A, \mathcal{P})$, where A is a finitely generated R-algebra and $\mathcal{P}$ is a G-bundle on X_A, where a morphism from $(A, \mathcal{P})$ to $(A', \mathcal{P}')$ is an R-algebra homomorphism from A to A' together with an element $\alpha \in \operatorname{Iso}^{\mathrm{germ}}_{D_{A'}}(X_{A'} \times_{X_A} \mathcal{P}, \mathcal{P}')$. We can therefore identify $\mathcal{C}_5$ with a (non-full) subcategory of $Y \times_{X^T} \operatorname{Ran}^G_{\mathrm{germ}}(X)^T$; let $\theta_6 : \mathcal{C}_5 \to Y \times_{X^T} \operatorname{Ran}^G_{\mathrm{germ}}(X)^T$ denote the inclusion map.

  We claim that θ_6 is a universal homology equivalence. To prove this, fix an object of $C = (A, \mathcal{P}) \in Y \times_{X^T} \operatorname{Ran}^G_{\mathrm{germ}}(X)^T$ and let $\mathcal{E}$ denote the fiber product

$$\mathcal{C}_5 \times_{Y \times_{X^T} \operatorname{Ran}^G_{\mathrm{germ}}(X)^T} (Y \times_{X^T} \operatorname{Ran}^G_{\mathrm{germ}}(X)^T)_{C/}.$$

  We wish to show that the canonical map $\psi : \mathcal{E} \to \operatorname{Spec} A$ induces an isomorphism on ℓ-adic homology. Unwinding the definitions, we can identify objects of $\mathcal{E}$ with triples $(B, \mathcal{Q}, \alpha)$, where B is a finitely generated A-algebra, $\mathcal{Q}$ is a G-bundle on X_B, and α is a germ of isomorphisms of $\operatorname{Spec} B \times_{\operatorname{Spec} A} \mathcal{P}$ with $\mathcal{Q}$ near the divisor D_B; a morphism from $(B, \mathcal{Q}, \alpha)$ to $(B', \mathcal{Q}', \alpha')$ in $\mathcal{E}$ is given by an A-algebra homomorphism from B to

B' with the property that the germ

$$\mathcal{Q}' \xrightarrow{\alpha'^{-1}} \operatorname{Spec} B' \times_{\operatorname{Spec} A} \mathcal{P} \xrightarrow{\alpha} \operatorname{Spec} B' \times_{\operatorname{Spec} B} \mathcal{Q},$$

which is *a priori* defined on some open subset of $X_{B'}$ containing $D_{B'}$, can be extended over an open set of the form $X_{B'} - |I|$ for some finite set $I \subseteq X(B')$ with $|I| \cap D_{B'} = \emptyset$. The construction $B \mapsto (B, \operatorname{Spec} B \times_{\operatorname{Spec} A} \mathcal{P}, \mathrm{id})$ determines a map $s : \operatorname{Spec} A \to \mathcal{E}$ which is a section of ψ. To prove that ψ induces an isomorphism on ℓ-adic homology, it will suffice to show that s is a universal homology equivalence. To prove this, fix an object $C' = (B, \mathcal{P}', \alpha) \in \mathcal{E}$ and set $\mathcal{E}' = \operatorname{Spec} B \times_{\mathcal{E}} \operatorname{Spec} A$. Note that $\mathcal{E}'$ can be identified with the full subcategory of Ring_B spanned by those finitely generated B-algebras B' which satisfy the following condition:

- $(*)$ There exists a finite subset $J \subseteq X(B')$ such that $D_{B'} \subseteq X_{B'} - |J|$ and the germ α extends over the open set U.

Using Corollary A.2.10, we see that $\mathcal{Y}$ is a covering sieve of $\operatorname{Ring}_{B'}$ with respect to the fppf topology; the desired result now follows from Proposition 2.5.15.

□

7.5. **Digression: Germs of Equivariant Maps.** Our goal in this section is to prove a somewhat technical result (Theorem 7.5.2) which will be needed in §7.6. Fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve $\widetilde{X}$ over k, and a finite group Γ acting faithfully on $\widetilde{X}$. Let X denote the quotient $\widetilde{X}/\Gamma$ and let Q be a finite set of closed points of X such that the quotient map $\widetilde{X} \to X$ is étale over the open set $X - Q \subseteq X$. Let R be a finitely generated k-algebra and let $D \subseteq X_R$ be an effective divisor that is contained in the open curve $(X - Q)_R$.

Notation 7.5.1. If Y and Z are k-schemes equipped with actions of Γ, we let $\operatorname{Map}_\Gamma(Z, Y)$ denote the set of Γ-equivariant maps from Z into Y. For every finitely generated R-algebra A, we let $\operatorname{Map}_\Gamma^{\mathrm{germ}}(\widetilde{X}_A, Y)$ denote the direct limit $\varinjlim_U \operatorname{Map}_\Gamma(U, Y)$, where the limit is taken over all Γ-invariant open subsets of $\widetilde{X}$ which contain the divisor $\widetilde{D}_A = D \times_{X_R} \widetilde{X}_A$. Let $\operatorname{Eq}_{\mathrm{germ}}(Y)$ denote the prestack whose objects are pairs (A, ϕ), where A is a finitely generated R-algebra and $\phi \in \operatorname{Map}_\Gamma^{\mathrm{germ}}(\widetilde{X}_A, Y)$, and we let $\operatorname{Eq}(Y)$ denote the prestack whose objects are pairs (A, ϕ_0) where A is a finitely generated R-algebra and $\phi_0 \in \operatorname{Map}_\Gamma(\widetilde{D}_A, Y)$. Note that the construction $(A, \phi) \mapsto (A, \phi|_{\widetilde{D}_A})$ determines a map of prestacks $\operatorname{Eq}_{\mathrm{germ}}(Y) \to \operatorname{Eq}(Y)$.

The main result of this section can be stated as follows:

Theorem 7.5.2. *Let G be a semisimple simply connected algebraic group over k, and suppose that we are given an action of Γ on G which preserves a pinning. Then the restriction map $\operatorname{Eq}_{\mathrm{germ}}(G) \to \operatorname{Eq}(G)$ is a universal homology equivalence.*

For the proof of Theorem 7.5.2, it will be convenient to introduce a bit of terminology.

Definition 7.5.3. Let Y be a k-scheme equipped with an action of the finite group Γ. We will say that Y *has the equivariant approximation property* if the restriction map $\operatorname{Eq}_{\mathrm{germ}}(Y) \to \operatorname{Eq}(Y)$ is a universal homology equivalence.

We will deduce Theorem 7.5.2 from the following:

Theorem 7.5.4. *Let G be a semisimple simply connected algebraic group over k, let Γ be a finite group with an action on G which preserves a pinning $(B, T, \{u_\alpha\})$, let B_- be the unique Borel subgroup of G satisfying $B_- \cap B = T$, let $V = B_- B$ be the corresponding "big cell" of the*

Bruhat decomposition of G, and let $W = \bigcup gV$ where g ranges over all k-valued points of the identity component of G^{Γ}. Then W has the equivariant approximation property.

Remark 7.5.5. Suppose that G is simple. Using Propositions A.4.2 and A.4.4, we see that $W = G$ unless the field k has characteristic 2, the group G is isomorphic to SL_{2n+1} for some integer n, and the group Γ acts nontrivially on G.

We first explain how to deduce Theorem 7.5.2 from Theorem 7.5.4:

Lemma 7.5.6. *Let $E \subseteq X$ be an effective divisor, let $\widetilde{E} \subseteq \widetilde{X}$ be the inverse image of E, and let $\phi : \widetilde{E} \to G$ be any Γ-equivariant map. Then there exists a Γ-invariant open subset $U \subseteq \widetilde{X}$ containing $\widetilde{E}$ and a Γ-equivariant map $h : U \to G$ with the following property: for every k-valued point x of E, the product $h(x)\phi(x)$ belongs to the subset $W(k) \subseteq G(k)$ appearing in the statement of Theorem 7.5.4.*

Remark 7.5.7. In the proof of Theorem 7.5.2, we need only the special case of Lemma 7.5.6 where E is disjoint from Q. This can be used to slightly simplify the proof given below.

Proof of Lemma 7.5.6. Write G as a product of simple factors $\prod_{i \in I} G_i$. Factoring G as a product if necessary, we may reduce to the case where I forms a single orbit under the action of Γ. Choose an index $i \in I$ and let $\Gamma_i \subseteq \Gamma$ denote its stabilizer. Replacing G by G_i (and X by the quotient $\widetilde{X}/\Gamma_i$), we may reduce to the case where G is simple. Let Γ_0 be the kernel of the action of Γ on G. Replacing $\widetilde{X}$ by $\widetilde{X}/\Gamma_0$ (and Γ by Γ/Γ_0), we may assume without loss of generality that Γ acts faithfully on G. If $W = G$, then we can choose h to be a constant map. We may therefore assume without loss of generality that the field k has characteristic 2, the group G is isomorphic to SL_{2n+1} for some $n \geq 1$, and that $\Gamma \simeq \mathbf{Z}/2\mathbf{Z}$ acts nontrivially on G via an involution $\sigma : G \to G$ (Remark 7.5.5).

Fix a pinning $(B, T, \{u_\alpha\})$ of G which is invariant under the action of Γ. Since $G = \mathrm{SL}_{2n+1}$, we can enumerate the simple roots of G as $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_{2n}\}$, where α_i is adjacent to α_j in the Dynkin diagram of G if and only if $|i - j| = 1$. We first claim that there exists a rational map $\lambda : \mathbf{P}^1 \to B$ with the following properties:

- The map λ is regular on the open set $\mathbf{P}^1 - \{0, 1\}$.
- The map λ carries $\infty \in \mathbf{P}^1$ to the identity element of G and carries each point of $\mathbf{P}^1 - \{0, 1, \infty\}$ to a regular unipotent element of G.
- The map λ satisfies $\lambda(a+1) = \sigma(\lambda(a))$.

To prove this, consider the map $\iota : \mathrm{SL}_3 \to \mathrm{SL}_{2n+1}$ corresponding to the pair of adjacent roots $\{\alpha_n, \alpha_{n+1}\}$, and let $\lambda_0 : \mathbf{P}^1 \to \mathrm{SL}_3$ be the map given by

$$\lambda_0(a) = \begin{pmatrix} 1 & \frac{1}{a} & \frac{1}{a} \\ 0 & 1 & \frac{1}{a+1} \\ 0 & 0 & 1 \end{pmatrix}.$$

A simple calculation now shows that the map

$$\lambda(a) = \iota(\lambda_0(a)) \prod_{1 \leq i < n} (u_{\alpha_i}(\frac{1}{a}) u_{\alpha_{2n+1-i}}(\frac{1}{a+1}))$$

has the desired properties (for any choice of ordering of the product).

For each integer $m \geq 0$, let $K_m \subseteq G$ denote the scheme-theoretic image of the map

$$\theta_m : (\mathbf{P}^1 - \{0, 1\})^m \to G$$

$$(a_1, \ldots, a_m) \mapsto \lambda(a_1) \ldots \lambda(a_m)$$

Each K_m is an irreducible reduced closed subscheme of G, so the ascending chain

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$$

must eventually stabilize to a closed subgroup $K \subseteq G$. Note that K is contained in the unipotent radical of B; in particular, it is a connected solvable subgroup of B. Moreover, we can choose an integer m_0 for which K is the scheme-theoretic image of θ_{m_0}.

Let τ be the involution of $\widetilde{X}$ given by the nontrivial element of Γ. Let K_X and $K_{\widetilde{X}}$ denote the function fields of the curves X and $\widetilde{X}$, respectively. Then $K_{\widetilde{X}}$ is a quadratic Galois extension of K_X. Since K_X has characteristic 2, it follows that $K_{\widetilde{X}}$ is an Artin-Schreier extension of K_X. We may therefore choose a nonzero rational function $f : \widetilde{X} \to \mathbf{P}^1$ satisfying the identity $f(\tau x) = f(x) + 1$. Note that the poles of f are precisely the fixed points for τ. Write $\widetilde{E}$ as a disjoint union $\widetilde{E}_- \amalg \widetilde{E}_+$, where the points of $\widetilde{E}_-$ are fixed by τ and the group Γ acts freely on $\widetilde{E}_+$. For each point $x \in \widetilde{E}_+$, consider the regular map

$$\mu_x : (\mathbf{P}^1 - \{f(x), f(x)+1\})^{m_0} \to G$$

$$(a_1, \ldots, a_{m_0}) \mapsto \lambda(a_1 + f(x)) \ldots \lambda(a_{m_0} + f(x)).$$

Note that the Zariski closure Z_x of the set $\{\mu_x(a_1, \ldots, a_{m_0})\phi(x)B\}$ in G/B is a closed subset with an action of the connected solvable group K. It follows that there exists a fixed point for the action of K on Z_x. Since K contains a regular unipotent element of B, the only point of G/B fixed by the action of K is the identity coset. It follows that $\{\mu_x(\vec{a})\phi(x)B\}$ intersects every open neighborhood of the base point of G/B. In particular, we can choose a point $\vec{a} \in (\mathbf{P}^1 - \{f(x), f(x)+1\})^{m_0}$ for which $\mu_x(\vec{a})\phi(x)$ belongs to the big cell of G.

For each point x of $\widetilde{E}_+$, let V_x denote the open subset of $(\mathbf{P}^1 - \{f(x), f(x)+1\})^{m_0}$ consisting of those points $\vec{a}$ such that $\mu_x(\vec{a})\phi(x)$ belongs to the big cell of G. Then each V_x is a nonempty open subset of $(\mathbf{P}^1)^{m_0}$. Since $(\mathbf{P}^1)^{m_0}$ is irreducible, we can choose a sequence $\vec{a} = (a_1, \ldots, a_{m_0})$ which belongs to each V_x. We now define

$$U = \{x \in \widetilde{X} : \lambda(x), \lambda(x) + 1 \notin \{\lambda(a_1), \lambda(a_2), \ldots, \lambda(a_{m_0})\}$$

$$h(x) = \lambda(a_1 + f(x)) \ldots \lambda(a_{m_0} + f(x)).$$

It follows immediately from the construction that for $x \in \widetilde{E}_+$, the product $h(x)\phi(x)$ belongs to the big cell of G (and in particular to the open set $W \subseteq G$). If $x \in \widetilde{E}_-$, then x is fixed by τ so that $f(x) = \infty$ and therefore $h(x)$ is the identity element of $G(k)$. Consequently, to prove that $h(x)\phi(x) \in W(k)$ we must show that $\phi(x) \in W(k)$, which follows from Proposition A.4.4 (note that $\tau(x) = x \Rightarrow \phi(x) = \sigma(\phi(x))$). □

Proof of Theorem 7.5.2. It will suffice to show that for every finitely generated R-algebra A and every map of R-schemes $f : \operatorname{Spec} A \to \operatorname{Eq}(G)$, the induced map

$$\theta_A : \mathrm{H}_*(\mathrm{Eq}_{\mathrm{germ}}(G) \times_{\mathrm{Eq}(G)} \operatorname{Spec} A; \mathbf{Z}_\ell) \to \mathrm{H}_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$$

is an isomorphism.

Let $\phi : \widetilde{D}_A \to G$ denote the Γ-equivariant map determined by f. For every k-valued point η of $\operatorname{Spec} A$, let D_η denote the fiber product $\operatorname{Spec} k \times_{\operatorname{Spec} R} D$, let $\widetilde{D}_\eta$ denote its inverse image in $\widetilde{X}$, and let $\phi_\eta : \widetilde{D}_\eta \to G$ be the restriction of ϕ. Let W be the open subset appearing in the statement of Theorem 7.5.4. Using Lemma 7.5.6, we can choose a Γ-invariant open subset $U \subseteq \widetilde{X}$ containing $\widetilde{D}_x$ and a Γ-invariant map $h_\eta : U \to G$ such that $h_\eta(x)\phi_\eta(x) \in W$ for every k-valued point x of $\widetilde{D}_\eta$. Since the map $\widetilde{D}_A \to \operatorname{Spec} A$ is proper, there exists an open subset $V_\eta \subseteq \operatorname{Spec} A$ containing the point η such that $\widetilde{D} \times_{\operatorname{Spec} R} V_\eta$ is contained in the open set

$U \times_{\operatorname{Spec} k} V_\eta \subseteq \widetilde{X}_R \times_{\operatorname{Spec} R} V_\eta$. Shrinking V_η if necessary, we may further assume $h_\eta(x)\phi(x) \in W$ for every k-valued point x of $\widetilde{D} \times_{\operatorname{Spec} R} V_\eta$.

Since $\operatorname{Spec} A$ is quasi-compact, we can choose finitely many k-valued points $\eta_1, \ldots, \eta_n$ for which the open sets V_{η_i} form an open covering of $\operatorname{Spec} A$. For each subset $I \subseteq \{1, \ldots, n\}$, let $V_I = \bigcap V_{\eta_i}$. We then have a commutative diagram

$$\begin{array}{ccc} \varinjlim_{I \neq \emptyset} C_*(\operatorname{Eq}_{\mathrm{germ}}(G) \times_{\operatorname{Eq}(G)} V_I; \mathbf{Z}_\ell) & \longrightarrow & C_*(\operatorname{Eq}_{\mathrm{germ}}(G) \times_{\operatorname{Eq}(G)} \operatorname{Spec} A; \mathbf{Z}_\ell) \\ \downarrow & & \downarrow \theta_A \\ \varinjlim_{I \neq \emptyset} C_*(V_I; \mathbf{Z}_\ell) & \longrightarrow & C_*(\operatorname{Spec} A, \mathbf{Z}_\ell) \end{array}$$

where the horizontal morphisms are equivalences (by virtue of Zariski descent). Consequently, to show that θ_A is an isomorphism, it will suffice to show that each of the maps

$$\operatorname{Eq}_{\mathrm{germ}}(G) \times_{\operatorname{Eq}(G)} V_I \to \operatorname{Eq}_{\mathrm{form}}(G) \times_{\operatorname{Eq}(G)} V_I$$

induces an isomorphism on ℓ-adic homology. This follows from Theorem 7.5.4, since multiplication by $h_{\eta_i}^{-1}$ (for any $i \in I$) determines a commutative diagram

$$\begin{array}{ccc} \operatorname{Eq}_{\mathrm{germ}}(W) \times_{\operatorname{Eq}(W)} V_I & \longrightarrow & \operatorname{Eq}_{\mathrm{form}}(W) \times_{\operatorname{Eq}(W)} V_I \\ \downarrow & & \downarrow \\ V_I & \xrightarrow{\mathrm{id}} & V_I. \end{array}$$

where the vertical maps are equivalences by virtue of Theorem 7.5.4. □

We now turn to the proof of Theorem 7.5.4.

Lemma 7.5.8. *Let Y be a k-scheme with an action of Γ, and let $U \subseteq Y$ be a Γ-invariant open subset. If Y has the equivariant approximation property, then so does U.*

Proof. This follows from Corollary 2.5.12, since the diagram

$$\begin{array}{ccc} \operatorname{Eq}_{\mathrm{germ}}(U) & \longrightarrow & \operatorname{Eq}_{\mathrm{germ}}(Y) \\ \downarrow & & \downarrow \\ \operatorname{Eq}(U) & \longrightarrow & \operatorname{Eq}(Y) \end{array}$$

is a pullback square of prestacks in groupoids. □

Lemma 7.5.9. *Let Y be a k-scheme equipped with an action of Γ. Suppose that there exists a collection of Γ-invariant open subsets $\{U_\alpha \subseteq Y\}$ with the following property: for every finite set $S \subseteq Y(k)$, there exists an index α such that $S \subseteq U_\alpha(k)$. If each U_α has the approximation property, then so does Y.*

Proof. Fix an integer N, a finitely generated k-algebra A, and a map $f : \operatorname{Spec} A \to \operatorname{Eq}(Y)$. We wish to show that the induced map

$$C_*(\operatorname{Spec} A \times_{\operatorname{Eq}(Y)} \operatorname{Eq}_{\mathrm{germ}}(Y); \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} A; \mathbf{Z}_\ell)$$

is a quasi-isomorphism. This assertion can be tested locally with respect to the Zariski topology on $\operatorname{Spec} A$. For each closed point $x \in \operatorname{Spec} A$, f determines a map $\{x\} \times_{\operatorname{Spec} A} \widetilde{D}_A \to Y$. Since the domain is a finite k-scheme, the image of this map is contained in some subset $U_\alpha \subseteq Y$.

Shrinking Spec A if necessary, we may suppose that f factors through $\mathrm{Eq}(U_\alpha)$. In this case, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A \times_{\mathrm{Eq}(U_\alpha)} \mathrm{Eq}_{\mathrm{germ}}(U_\alpha) & \longrightarrow & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} A \times_{\mathrm{Eq}(Y)} \mathrm{Eq}_{\mathrm{germ}}(Y) & \longrightarrow & \operatorname{Spec} A \end{array}$$

where the vertical maps are equivalences. Since U_α has the equivariant approximation property, the upper horizontal map induces an isomorphism on $\mathbf{Z}_\ell$-homology. It follows that the lower horizontal map also induces an isomorphism on $\mathbf{Z}_\ell$-homology. □

Lemma 7.5.10. *Let Y be a k-scheme with an action of the group Γ, let V be a finite-dimensional linear representation of Γ, and let $\mathcal{E}$ be a Γ-equivariant V-torsor over Y. If Y has the equivariant approximation property, then so does $\mathcal{E}$.*

Proof of Lemma 7.5.10. The map $\mathrm{Eq}_{\mathrm{germ}}(\mathcal{E}) \to \mathrm{Eq}(\mathcal{E})$ factors as a composition

$$\mathrm{Eq}_{\mathrm{germ}}(\mathcal{E}) \xrightarrow{\theta} \mathrm{Eq}(\mathcal{E}) \times_{\mathrm{Eq}(Y)} \mathrm{Eq}_{\mathrm{germ}}(Y) \xrightarrow{\theta'} \mathrm{Eq}(\mathcal{E}),$$

where θ' is a pullback of the map $\mathrm{Eq}_{\mathrm{germ}}(Y) \to \mathrm{Eq}(Y)$ and is therefore a universal homology equivalence by virtue of our assumption that Y has the equivariant approximation property (and Corollary 2.5.12). It will therefore suffice to show that θ is a universal homology equivalence. Fix a map $u : \operatorname{Spec} A \to \mathrm{Eq}(\mathcal{E}) \times_{\mathrm{Eq}(Y)} \mathrm{Eq}_{\mathrm{germ}}(Y)$; we wish to show that the projection map $\pi : \mathcal{C} \to \operatorname{Spec} A$ induces an isomorphism in $\mathbf{Z}_\ell$-homology, where $\mathcal{C}$ denotes the fiber product

$$\mathrm{Eq}_{\mathrm{germ}}(\mathcal{E}) \times_{\mathrm{Eq}(\mathcal{E}) \times_{\mathrm{Eq}(Y)} \mathrm{Eq}_{\mathrm{germ}}(Y)} \operatorname{Spec} A.$$

Replacing R by A, we may assume that $A = R$. In this case, we can identify u with a pair (F_0, f), where $F_0 : \widetilde{D} \to \mathcal{E}$ is an equivariant map and $f : \widetilde{U} \to Y$ is a Γ-equivariant extension of the composite map

$$f_0 : \widetilde{D} \xrightarrow{F_0} \mathcal{E} \to Y$$

to a Γ-equivariant open set $\widetilde{U} \subseteq \widetilde{X}_R$ which contains the divisor $\widetilde{D}$. Shrinking the domain of f if necessary, we may assume that $\widetilde{U}$ is contained in $\widetilde{X - Q_R}$. Working étale locally on Spec R, we may assume that the complement of $\widetilde{U}$ is contained in a relative divisor $D' \subseteq \widetilde{X}_R$ of positive degree which does not intersect D (Proposition A.2.6). Replacing D' by the sum $\sum_{\gamma \in \Gamma} \gamma(D')$ if necessary, we may assume that D' is Γ-equivariant. We may then replace $\widetilde{U}$ by the compleent $\widetilde{X}_R - D'$ and thereby reduce to the case where $\widetilde{U}$ is affine.

Since $\widetilde{U}$ is Γ-equivariant, it is the inverse image of an open subset $U \subseteq X_R$. Writing $U = \widetilde{U}/\Gamma$, we see that U is affine. The product $\widetilde{U} \times_{\operatorname{Spec} k} V$ is a Γ-equivariant vector bundle over $\widetilde{U}$. Since $\widetilde{U}$ is contained in $\widetilde{X - Q_R}$, the action of Γ on $\widetilde{U}$ is free so that we can write $\widetilde{U} \times_{\operatorname{Spec} k} V$ as the pullback of a vector bundle $\mathcal{F}$ on $U \subseteq X_R$. The Γ-equivariant map $f : \widetilde{U} \to Y$ determines a Γ-equivariant V-torsor $\mathcal{E} \times_Y \widetilde{U}$, which we can descend to a $\mathcal{F}$-torsor $\mathcal{E}'$ on U. Note that the map F_0 determines a trivialization η_0 of $\mathcal{E}'$ over the divisor $D \subseteq U$. Since U is affine, the cohomology group $\mathrm{H}^1(U; \mathcal{F}(-D))$ vanishes. It follows that η_0 can be lifted to a trivialization η of $\mathcal{E}'$ over the entire affine open set $U \subseteq X_R$. Using η, we can identify the prestack $\mathcal{C}$ with the category whose objects are pairs (B, x), where B is a finitely generated R-algebra and x is an element of the direct limit

$$\varinjlim_{W \subseteq U_B} \mathrm{H}^0(W; \mathcal{F}(-D)|_W),$$

where the direct limit is taken over all open subsets

$$W \subseteq U_B = U \times_{X_R} X_B$$

which contain the divisor D_B.

The projection map $\pi : \mathcal{C} \to \operatorname{Spec} R$ has a left inverse $e : \operatorname{Spec} R \to \mathcal{C}$, given on objects by the construction

$$(B \in \operatorname{Ring}_R) \mapsto (B, 0).$$

To show that π induces an isomorphism on homology, it will suffice to show that the composite map $e \circ \pi : \mathcal{C} \to \mathcal{C}$ induces a map $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathcal{C}; \mathbf{Z}_\ell)$ which is homotopic to the identity. In fact, we claim that $e \circ \pi$ is $\mathbf{A}^1$-homotopic to the identity: that is, there exists a map of prestacks

$$h : \mathcal{C} \times_{\operatorname{Spec} k} \mathbf{A}^1 \to \mathcal{C}$$

such that $h|_{\mathcal{C} \times \{0\}} \simeq e \circ \pi$ and $h|_{\mathcal{C} \times \{1\}} \simeq \operatorname{id}_{\mathcal{C}}$. This is clear: at the level of objects, we can take h to be given by

$$((B, x) \in \mathcal{C}, t \in B) \mapsto (B, tx).$$

□

Proof of Theorem 7.5.4. Let $(B, T, \{u_\alpha\})$ be a pinning of G, let B' denote the unique Borel subgroup of G satisfying $B' \cap B = T$, and let U' and U denote the unipotent radicals of B_- and B, respectively. Let $V = U'TU$ be the big cell of the Bruhat decomposition of G, let G_0 denote the identity component of G^Γ, and let $W = \bigcup_{g \in G_0(k)} gV$. For each point $h \in W(k)$, the set $\{g \in G_0(k) : h \in gV(k)\}$ is nonempty and Zariski-open. Because G_0 is irreducible, it follows that for every finite set of points $h_1, \ldots, h_n \in W(k)$, the set

$$\{g \in G_0(k) : h_1, \ldots, h_n \in gV(k)\}$$

is nonempty. By virtue of Lemma 7.5.9, it will suffice to show that each gV has the equivariant approximation property. Since each gV is Γ-equivariantly isomorphic to V, we are reduced to proving that V has the equivariant approximation property.

Choose a Γ-invariant filtration of U by normal subgroups

$$\{1\} = U_m \subseteq U_{m-1} \subseteq \cdots \subseteq U_0 = U,$$

here each quotient U_{i-1}/U_i is isomorphic to a vector group with a linear action of Γ, and choose

$$\{1\} = U'_m \subseteq U'_{m-1} \subseteq \cdots \subseteq U'_0 = U'$$

similarly. For $0 \leq i \leq m$, let V_i denote the double quotient $U'_i \backslash V / U_i$. We will prove by induction on i that each V_i has the equivariant approximation property. This will complete the proof, since $V_m \simeq V$.

We first treat the case where $i = 0$, so that $V_i \simeq T$. Since G is simply connected, we can identify T with the product $\prod_{s \in S} \mathbf{G}_m$, where the product is indexed by the finite set S of fundamental weights of G. In particular, there is a Γ-equivariant open immersion $V_0 \hookrightarrow \mathbf{A}^r$, where r is the rank of G, and the group Γ acts on $\mathbf{A}^r$ by permuting the coordinates. By virtue of Lemma 7.5.8, it will suffice to show that $\mathbf{A}^r$ has the equivariant approximation property, which follows immediately from Lemma 7.5.10.

We now carry out the inductive step. Assume that $i > 0$ and that V_{i-1} has the equivariant approximation property. We note that the projection map $V_i \to V_{i-1}$ is a torsor for the vector group $(U_{i-1}/U_i) \times (U'_{i-1}/U'_i)$. Applying Lemma 7.5.10, we deduce that V_i also has the equivariant approximation property. □

7.6. **From Divisors to Open Neighborhoods.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, a finite subset $Q \subseteq X$, and a smooth affine group scheme G over X which is Q-adapted (Definition 7.2.9). Let T be a nonempty finite set and let R be a finitely generated k-algebra equipped with a map $f : \operatorname{Spec} R \to (X - Q)^T$ (see Definition 7.3.9), which we can identify with a map $\nu : T \to X(R)$. Let $D = |\nu(T)|$ denote the associated divisor in X_R (note that D is actually contained in the open subscheme $(X - Q)_R$). Our goal is to prove Theorem 7.3.17, which asserts that the forgetful functor

$$\operatorname{Ran}^G_{\text{germ}}(X)^T \times_{X^T} \operatorname{Spec} R \to \operatorname{Ran}^G(X)^T \times_{X^T} \operatorname{Spec} R$$

is a universal homology equivalence.

For every finitely generated R-algebra A, we let D_A denote the fiber product $D \times_{\operatorname{Spec} R} \operatorname{Spec} A$, which we regard as an effective divisor in the relative curve X_A. The main ingredient we will need is the following:

Lemma 7.6.1. *Let A be a finitely generated R-algebra and let $\mathcal{P}$ be a G-bundle on X_A. Then there exists a faithfully flat étale morphism $A \to A'$ such that the G-bundle $\mathcal{P}' = \mathcal{P} \times_{X_A} X_{A'}$ is trivial on an open subset of $X_{A'}$ which contains the divisor $D_{A'}$.*

Remark 7.6.2. When $R = k$, Lemma 7.6.1 follows from Theorem 3.3.6. In general, neither result implies the other: in Lemma 7.6.1 we allow "variable" divisors $D \subseteq X_R$ (not necessarily arising from a fixed divisor in the curve X itself), but we do not allow the fibers of D to intersect the locus $Q \subseteq X$ where G fails to be reductive (which is permitted in Theorem 3.3.6).

Proof of Lemma 7.6.1. The assertion is local with respect to the étale topology on $\operatorname{Spec} R$. We may therefore assume without loss of generality that there exists an affine open subset $V \subseteq X$ such that the divisor D is contained in V_R (for example, if $Q \neq \emptyset$, then we can take $U = X - Q$).

Since G is Q-adapted, we can choose a semisimple simply connected algebraic group G_0 over k, a finite group Γ which acts on G_0 by automorphisms which preserve a pinning $(B_0, T_0, \{\phi_\alpha : \mathbf{G}_a \to B_0\})$ of G_0, an algebraic curve $\widetilde{X}$ with an action of Γ, an isomorphism $\widetilde{X}/\Gamma \simeq X$ for which the induced map $\widetilde{X} \to X$ is étale over $X - Q$, and a Γ-equivariant homomorphism

$$\beta : \widetilde{X} \times_X G \to \widetilde{X} \times_{\operatorname{Spec} k} G_0$$

of group schemes over $\widetilde{X}$ which is an isomorphism over the inverse image of $X - Q$.

Note that B_0 determines a Borel subgroup B_{X-Q} of G over the open set $X - Q$. Let B denote the scheme-theoretic closure of B_{X-Q} in G, as in §3.7. Applying Theorem 3.7.1 (to the case of an empty divisor), we may assume (after passing to an étale cover of $\operatorname{Spec} A$) that $\mathcal{P}$ admits a reduction to a B-bundle $\mathcal{Q}$. Note that β restricts to a map of group schemes

$$\beta_B : \widetilde{X} \times_X B \to \widetilde{X} \times_{\operatorname{Spec} k} B_0,$$

so that $\mathcal{Q}$ determines a Γ-equivariant B_0-bundle $\mathcal{Q}_0$ on the curve $\widetilde{X}_A$.

Since G_0 is simply connected, we can identify the split torus T_0 with $\mathbf{G}_m^I$, where I denotes the finite set of fundamental weights of G_0. Under this identification, the action of Γ on T_0 comes from a permutation action of Γ on I. The algebraic group B_0 fits into a Γ-equivariant exact sequence

$$0 \to U_0 \to B_0 \xrightarrow{\psi} \mathbf{G}_m^I \to 0,$$

where U_0 denotes the unipotent radical of B_0. Let $\mathcal{Q}'_0$ denote the Γ-equivariant $\mathbf{G}_m^I$-torsor on $\widetilde{X}_A$ obtained from $\mathcal{Q}_0$ using the homomorphism ψ. Then we can identify $\mathcal{Q}'_0$ with a Γ-equivariant line bundle on the product $\widetilde{X}_A \times I$, which we can in turn identify with a line bundle $\mathcal{L}$ on $\overline{X}_A$ where $\overline{X}$ denotes the quotient $(\widetilde{X} \times I)/\Gamma$.

Let $\overline{D}_A$ and $\overline{V}_A$ denote the inverse images of D_A and V_A in the relative curve $\overline{X}_A$. The map $\overline{D}_A \to \operatorname{Spec} A$ is finite. Passing to a Zariski covering of $\operatorname{Spec} A$ if necessary, we may assume that the line bundle $\mathcal{L}$ is trivial over the divisor $\overline{D}_A$. Since $\overline{V}_A$ is affine, any trivialization of $\mathcal{L}$ over $\overline{D}_A$ can be extended to a section of $\mathcal{L}$ over $\overline{V}_A$, which is a trivialization of $\mathcal{L}$ over an affine open subset $\overline{W} \subseteq \overline{V}_A$. Let $\widetilde{W}$ denote the inverse image of $\overline{W}$ in $\widetilde{X}_A \times I$, which we can identify with a collection of affine open subsets $\{\widetilde{W}_i \subseteq \widetilde{X}\}_{i \in I}$. The intersection $\bigcap_{i\in I} \widetilde{W}_i$ is a Γ-equivariant open subset of $\widetilde{X}_A \times_X V$ and is therefore the inverse image of an affine open subset $W \subseteq V_A$. By construction, the affine open set W contains the divisor D_A and the B_{X-Q}-bundle $\mathcal{Q}|_W$ arises from a U_{X-Q}-bundle on W, where U_{X-Q} denotes the unipotent radical of B_{X-Q}. Since U_0 admits a Γ-equivariant filtration by vector groups equipped with linear actions of Γ, the group scheme U_{X-Q} admits a finite filtration by group schemes associated to vector bundles over $X - Q$. It follows that any U_{X-Q}-bundle on an affine scheme is trivial. In particular, $\mathcal{Q}|_W$ is trivial and therefore $\mathcal{P}|_W$ is also trivial. □

Proof of Theorem 7.3.17. We wish to prove that the canonical map

$$\operatorname{Ran}^G_{\mathrm{germ}}(X)^T \times_{X^T} \operatorname{Spec} R \to \operatorname{Ran}^G(X)^T \times_{X^T} \operatorname{Spec} R$$

is a universal homology equivalence. Since both sides are prestacks in groupoids, it will suffice to show that for every map $u : \operatorname{Spec} A \to \operatorname{Ran}^G(X)^T \times_{X^T} \operatorname{Spec} R$, the projection map

$$\theta : \operatorname{Ran}^G_{\mathrm{germ}}(X)^T \times_{\operatorname{Ran}^G(X)^T} \operatorname{Spec} A \to \operatorname{Spec} A$$

induces an equivalence on $\mathbf{Z}_\ell$-homology. Replacing R by A, we may assume without loss of generality that $A = R$. In this case, the map u classifies a G-bundle $\mathcal{P}_0$ on the divisor D. The assertion that θ induces an isomorphism on homology can be tested locally with respect to the étale topology (Proposition 2.3.34); we may therefore assume without loss of generality that the G-bundle $\mathcal{P}_0$ is trivial.

Let $\operatorname{Ran}^G_{\mathrm{germ}}(X)^T_0 \subseteq \operatorname{Ran}^G_{\mathrm{germ}}(X)^T$ denote the full subcategory spanned by those triples $(R, \nu, \mathcal{P})$ where the G-bundle $\mathcal{P}$ is trivial. It follows from Lemma 7.6.1 that the inclusion of prestacks $\operatorname{Ran}^G_{\mathrm{germ}}(X)^T_0 \hookrightarrow \operatorname{Ran}^G_{\mathrm{germ}}(X)^T$ induces an equivalence after stackification with respect to the étale topology. In particular, the induced map

$$\operatorname{Ran}^G_{\mathrm{germ}}(X)^T_0 \times_{\operatorname{Ran}^G(X)^T} \operatorname{Spec} R \hookrightarrow \operatorname{Ran}^G_{\mathrm{germ}}(X)^T \times_{\operatorname{Ran}^G(X)^T} \operatorname{Spec} R$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology. We are therefore reduced to proving that the composite map

$$\operatorname{Ran}^G_{\mathrm{germ}}(X)^T_0 \times_{\operatorname{Ran}^G(X)^T} \operatorname{Spec} R \to \operatorname{Spec} R$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology.

Let H denote the group-valued functor $\operatorname{Ring}_R \to \mathcal{S}\mathrm{et}$ given by

$$H(A) = \varinjlim_{V \subseteq X_A} G(V),$$

where the direct limit is taken over all open subsets $V \subseteq X_A$ which contain the divisor D_A. Let $H_0 : \operatorname{Ring}_R \to \mathcal{S}\mathrm{et}$ be the group-valued functor given by $H_0(A) = G(D_A)$ (in other words, H_0 is obtained from $G \times_X D$ by Weil restriction along the finite flat map $D \to \operatorname{Spec} R$; in particular, H_0 is representable by an affine group scheme over $\operatorname{Spec} R$). We will regard H and H_0 as prestacks (equipped with maps to $\operatorname{Spec} R$). For each $n \geq 0$, let H^n denote the nth fiber power of H over $\operatorname{Spec} R$, and define H^n_0 similarly (by convention, we set $H^{-1} = H^{-1}_0 = \operatorname{Spec} R$).

There is an evident restriction map $r : H \to H_0$, which determines an action of H on H_0. Unwinding the definitions, we can identify the fiber product $\operatorname{Ran}^G_{\mathrm{germ}}(X)^T_0 \times_{\operatorname{Ran}^G(X)^T} \operatorname{Spec} R$ with the category-theoretic quotient of H_0 by the action of H (in other words, for every finitely generated R-algebra A, we can identify the groupoid of A-valued points of $\mathrm{Ran}^G_{\mathrm{germ}}(X)^T_0 \times_{\mathrm{Ran}^G(X)^T} \mathrm{Spec}\, R$ with the groupoid whose objects are the elements of $H_0(A)$, where a morphism from $x \in H_0(A)$ to $y \in H_0(A)$ is an element $g \in H(A)$ such that $x = r(g)y$). Similarly, we can identify $\mathrm{Spec}\, R$ itself with the category-theoretic quotient of H_0 by the translation action of itself. It follows that the map

$$C_*(\mathrm{Ran}^G_{\mathrm{germ}}(X)^T_0 \times_{\mathrm{Ran}^G(X)^T} \mathrm{Spec}\, R; \mathbf{Z}_\ell) \to C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell)$$

can be obtained via geometric realization from a map of simplicial objects of $\mathrm{Mod}_{\mathbf{Z}_\ell}$, whose nth term is given by

$$C_*(H_0 \times_{\mathrm{Spec}\, R} H^{n-1}; \mathbf{Z}_\ell) \to C_*(H_0 \times_{\mathrm{Spec}\, R} H_0^{n-1}; \mathbf{Z}_\ell)$$

We claim that each of these maps is a quasi-isomorphism. To prove this, it will suffice to establish that the restriction map $H \to H_0$ is a universal homology equivalence.

Since G is Q-adapted, we can choose a semisimple simply connected algebraic group G_0 over k, a finite group Γ which acts on G_0 by automorphisms which preserve a pinning $(B_0, T_0, \{\phi_\alpha : \mathbf{G}_a \to B_0\})$ of G_0, an algebraic curve $\widetilde{X}$ with an action of Γ, an isomorphism $\widetilde{X}/\Gamma \simeq X$ for which the induced map $\widetilde{X} \to X$ is étale over $X - Q$, and a Γ-equivariant homomorphism

$$\beta : \widetilde{X} \times_X G \to \widetilde{X} \times_{\mathrm{Spec}\, k} G_0$$

of group schemes over $\widetilde{X}$ which is an isomorphism over the inverse image of $X - Q$. Unwinding the definitions, we can identify the restriction map $H \to H_0$ with the map $\mathrm{Eq}_{\mathrm{germ}}(G_0) \to \mathrm{Eq}(G_0)$ studied in §7.5. The desired result now follows from Theorem 7.5.2. □

7.7. **The Chiral Homology of the Sheaves $\mathcal{B}_S$.** Throughout this section, we fix an algebraically closed field k, a prime numbe ℓ which is invertible in k, an algebraic curve X over k, a finite subset $Q \subseteq X$, a smooth affine group scheme G over X, and a nonempty finite set S. Let $\mathrm{Ran}^\dagger_G(X - Q)_S$ be the $\mathrm{Ran}(X)$-prestack introduced in Definition 7.2.1, and let us regard $\mathrm{Ran}_G(X - Q)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$ as a full subcategory of $\mathrm{Ran}^\dagger_G(X - Q)_S$ as in Remark 7.2.6. Our goal in this section is to prove Theorem 7.2.11, which asserts that the inclusionof $\mathrm{Ran}_G(X - Q)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$ into $\mathrm{Ran}^\dagger_G(X - Q)_S$ induces a quasi-isomorphism

$$\begin{aligned}
\int \mathcal{B}_S &= \int [\mathrm{Ran}^\dagger_G(X - Q)_S]_{\mathrm{Ran}(X)} \\
&\to \int [\mathrm{Ran}_G(X - Q)_S \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)]_{\mathrm{Ran}(X)} \\
&\simeq \int C^*(\mathrm{Ran}_G(X - Q)_S; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} \\
&\simeq C^*(\mathrm{Ran}_G(X - Q)_S; \mathbf{Z}_\ell).
\end{aligned}$$

Remark 7.7.1. The results of this section will not be used in the proof of Theorem 5.4.5 that we give in §9, and are therefore logically unrelated to the remainder of this paper. We include this section only to clarify the role played by the sheaves $\mathcal{B}_S$ in our analysis of $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$; it may be safely skipped by a reader who is interested in taking the most efficient road to the proof of Weil's conjecture.

Notation 7.7.2. Let P denote the partially ordered set introduced in the proof of Proposition 7.3.6. The construction

$$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \alpha) \mapsto (K_-, K_+)$$

induces a Cartesian fibration

$$\theta : \mathrm{Ran}_G^\dagger(X - Q)_S \to P.$$

For every pair $(K_-, K_+) \in P$, we let $\mathrm{Ran}_G^\dagger(X - Q)_{K_-,K_+}$ denote the fiber $\theta^{-1}\{(K_-, K_+)\}$. Then $\mathrm{Ran}_G^\dagger(X - Q)_{K_-,K_+}$ is a $\mathrm{Ran}(X)$-prestack; we let $\mathcal{B}_{K_-,K_+}$ denote the lax !-sheaf on $\mathrm{Ran}(X)$ given by the formula

$$\mathcal{B}_{K_-,K_+} = [\mathrm{Ran}_G^\dagger(X - Q)_{K_-,K_+}]_{\mathrm{Ran}(X)} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)).$$

Unwinding the definitions, we have

$$\mathrm{Ran}_G^\dagger(X - Q)_{\emptyset,\emptyset} \simeq \mathrm{Ran}_G(X - Q) \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$$

$$\mathcal{B}_{\emptyset,\emptyset} \simeq C^*(\mathrm{Ran}_G(X - Q); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}$$

$$\mathcal{B}_S \simeq \varprojlim_{(K_-,K_+)\in P} \mathcal{B}_{K_-,K_+}.$$

Remark 7.7.3. For $K_- \subseteq K_+ \subseteq S$, we can describe the prestack $\mathrm{Ran}_G^\dagger(X - Q)_{K_-,K_+}$ informally as follows: its k-valued points are given by quadruples $(\mu, \nu, \mathcal{P}, \gamma)$ where $\mu : S \to (X - Q)(k)$ and $\nu : T \to X(k)$ are maps satisfying $\mu(K_+) \cap \nu(T) = \emptyset$, $\mathcal{P}$ is a G-bundle on the open curve $X - \mu(K_-)$ (which does not meet the divisor $\nu(T)$), and γ is a trivialization of $\mathcal{P}$ over the smaller open curve $X - \mu(S)$.

Theorem 7.2.11 is an immediate consequence of the following assertion:

Proposition 7.7.4. *Let K_- and K_+ be subsets of S satisfying $K_- \subseteq K_+$. If K_+ is nonempty, then the chiral homology $\int \mathcal{B}_{K_-,K_+}$ vanishes.*

Proof of Theorem 7.2.11 from Proposition 7.7.4. The construction $(K_-, K_+) \to \int \mathcal{B}_{K_-,K_+}$ determines a functor $F : P \to \mathrm{Mod}_{\mathbf{Z}_\ell}$. Since chiral homology commutes with finite limits, Theorem 7.2.11 is equivalent to the assertion that the evaluation map

$$\varprojlim_{(K_-,K_+)\in P} F(K_-, K_+) \to F(\emptyset, \emptyset)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$ (see Notation 7.7.2).

Let $P_0 \subseteq P$ be the subset consisting of those pairs (K_-, K_+) where $K_- = \emptyset$. Then P_0 contains $(\emptyset, \emptyset)$ as an initial object, so that the evaluation map

$$\varprojlim_{(K_-,K_+)\in P_0} F(K_-, K_+) \to F(\emptyset, \emptyset)$$

is an equivalence. Consequently, to complete the proof, it will suffice to show that the functor F is a right Kan extension of the restriction $F|_{P_0}$. In other words, it will suffice to show that for each $(K_-, K_+) \in P$, the canonical map

$$\phi : F(K_-, K_+) \to \varprojlim_{K_+ \subseteq K'_+} F(\emptyset, K'_+) \simeq F(\emptyset, K_+)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. If $K_+ = \emptyset$, then $K_- = \emptyset$ and the desired result is a tautology. If $K_+ \neq \emptyset$, then ϕ is an equivalence because the domain and codomain of ϕ both vanish (by virtue of Proposition 7.7.4). □

The remainder of this section is devoted to the proof of Proposition 7.7.4. Let us therefore fix subsets $K_- \subseteq K_+ \subseteq S$ where K_+ is nonempty.

Notation 7.7.5. Let $V \subseteq (X-Q)^S \times_{\operatorname{Spec} k} \operatorname{Ran}(X)$ denote the substack whose R-valued points are pairs of maps $(\mu : S \to (X-Q)(R), \nu : T \to X(R))$ which satisfy $|\mu(K_+)| \cap |\nu(T)| = \emptyset$. For every nonempty finite set T, we let V_T denote the fiber product $V \times_{\operatorname{Ran}(X)} X^T$, which we regard as an open subscheme of the product $(X-Q)^S \times X^T$ (namely, the open subscheme whose k-valued points are pairs of maps $\mu : S \to (X-Q)(k)$, $\nu : T \to X(k)$ satisfying $\mu(K_+) \cap \nu(T) = \emptyset$. We let

$$\phi_T : V_T \to X^T \qquad \psi_T : V_T \to (X-Q)^S$$

denote the projection maps.

For any ℓ-adic sheaf $\mathcal{F} \in \operatorname{Shv}_\ell((X-Q)^S)$, the construction

$$T \mapsto \phi_{T*}(\mathcal{F} \boxtimes \omega_{X^T})|_{V_T}$$

determines a lax !-sheaf $\mathcal{F}^+$ on $\operatorname{Ran}(X)$. We will deduce Proposition 7.7.4 from the following assertion:

Lemma 7.7.6. *For each $\mathcal{F} \in \operatorname{Shv}_\ell((X-Q)^S)$, the chiral homology*

$$\int \mathcal{F}^+ \simeq \varinjlim_{T \in \operatorname{Fin}^s} C^*(V_T; (\mathcal{F} \boxtimes \omega_{X^T})|_{V_T})$$

vanishes.

Proof of Proposition 7.7.4 from Lemma 7.7.6. We define a prestack $\mathcal{C}$ as follows:

- The objects of $\mathcal{C}$ are quadruples $(R, \mu, \mathcal{P}, \gamma)$ where R is a finitely generated K-algebra, $\mu : S \to (X-Q)(R)$ is a map of sets, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu(S)| \subseteq X_R$.
- A morphism from an object $C = (R, \mu, \mathcal{P}, \gamma)$ and $C' = (R', \mu', \mathcal{P}', \gamma')$ in the category $\mathcal{C}$ consists of a k-algebra homomorphism $R \to R'$ which carries μ to μ' and a G-bundle isomorphism between $X_{R'} \times X_R \mathcal{P}$ and $\mathcal{P}'$ over the open set $X_{R'} - |\mu'(K_-)|$ which carries γ to γ'.

The construction

$$(R, \mu, \mathcal{P}, \gamma) \mapsto (R, \mu)$$

determines a morphism of prestacks $\mathcal{C} \to (X-Q)^S$. Unwinding the definitions, we have an equivalence of $\operatorname{Ran}(X)$-prestacks

$$\operatorname{Ran}_G^\dagger(X-Q)_{K_-,K_+} \simeq \mathcal{C} \times_{(X-Q)^S} V,$$

where $V \subseteq (X-Q)^S \times_{\operatorname{Spec} k} \operatorname{Ran}(X)$ is defined as in Notation 7.7.5. Applying Proposition 5.1.9, we obtain

$$\begin{aligned} \mathcal{B}^{(T)}_{K_-,K_+} &\simeq [\mathcal{C} \times_{(X-Q)^S} V_T]_{X^T} \\ &\simeq \phi_{T*}[\mathcal{C} \times_{(X-Q)^S} V_T]_{\phi_T^* \omega_{X^T}} \\ &\simeq \phi_{T*}([\mathcal{C} \times_{\operatorname{Spec} k} X^T]_{\underline{\mathbf{Z}}_\ell \boxtimes \omega_{X^T}})|_{V_T} \\ &\simeq \phi_{T*}([\mathcal{C}]_{\underline{\mathbf{Z}}_\ell} \boxtimes \omega_{X^T})|_{V_T} \end{aligned}$$

where $\underline{\mathbf{Z}}_\ell$ denotes the constant sheaf on $(X-Q)^S$. This identification depends functorially on T and yields an equivalence of lax !-sheaves $\mathcal{B}_{K_-,K_+} \simeq \mathcal{F}^+$, where $\mathcal{F} = [\mathcal{C}]_{\underline{\mathbf{Z}}_\ell} \in \operatorname{Shv}_\ell((X-Q)^S)$. The vanishing of the chiral homology $\int \mathcal{B}_{K_-,K_+}$ is now a special case of Lemma 7.7.6. □

We will deduce Lemma 7.7.6 from a more general statement. Suppose that Y is a quasi-projective k-scheme equipped with a map $f : Y \to (X-Q)^S$. For each $T \in \mathrm{Fin}^{\mathrm{s}}$, we let $V_{T,Y}$ denote the fiber product $Y \times_{(X-Q)^S} V_T \subseteq Y \times X^T$, and we let

$$\phi_{T,Y} : V_{T,Y} \to X^T \qquad \psi_{T,Y} : V_{T,Y} \to Y$$

denote the projection maps. For each $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, we let $\mathcal{F}^+ \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ denote the lax !-sheaf given by

$$T \mapsto \phi_{T,Y*}(\mathcal{F} \boxtimes \omega_Y)|_{V_{T,Y}}.$$

We will prove:

Lemma 7.7.7. *For every map of quasi-projective k-schemes $f : Y \to (X-Q)^S$ and every $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, the chiral homology*

$$\int \mathcal{F}^+ \simeq \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C^*(V_{T,Y}; (\mathcal{F} \boxtimes \omega_{X^T})|_{V_{T,Y}})$$

vanishes.

Note that Lemma 7.7.6 is just the special case of Lemma 7.7.7 in which the map $f : Y \to (X-Q)^S$ is an isomorphism. The virtue of the more general formulation is that it permits us to apply devissage to the k-scheme Y. More precisely, suppose we are given a closed subscheme $Y' \subseteq Y$ with open complement U. Let $i : Y' \hookrightarrow Y$ and $j : U \hookrightarrow Y$ be the corresponding closed and open immersions. For each $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, we have a canonical fiber sequence

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F},$$

in $\mathrm{Shv}_\ell(Y)$, which in turn yields a fiber sequence

$$(i^! \mathcal{F})^+ \to \mathcal{F}^+ \to (j^* \mathcal{F})^+$$

of lax !-sheaves on $\mathrm{Ran}(X)$. Consequently, to verify the conclusion of Lemma 7.7.7 for Y, it will suffice to verify Lemma 7.7.7 for the subschemes Y' and U. Proceeding by Noetherian induction, we may assume that Lemma 7.7.7 is valid for every *proper* closed subscheme $Y' \subseteq Y$. In particular, we may assume that Y is reduced (otherwise, we can apply the above argument $Y' = Y_{\mathrm{red}}$). To complete the proof, it will suffice to show that there exists a nonempty open set $U \subseteq Y$ such that $f|_U$ satisfies the conclusions of Lemma 7.7.7. Replacing Y by an open set, we may assume that Y is irreducible.

Let us regard the map f as given by a collection of maps

$$\{f_s : Y \to X - Q\}_{s \in S}.$$

Let S' denote the quotient of S by the equivalence relation

$$(s \sim s') \text{ if } f_s = f_{s'}.$$

Replacing S by S' (and the subsets $K_-, K_+ \subseteq S$ by their images in S'), we can reduce to the case where $S' = S$, so that $f_s \neq f_{s'}$ for $s \neq s'$. Since Y is reduced and irreducible, it follows that there exists a dense open subset $W \subseteq Y$ such that $f_s(y) \neq f_{s'}(y)$ for $y \in W(k)$. Replacing Y by W, we may assume that the maps $\{f_s\}_{s \in S}$ have disjoint graphs. Let $U \subseteq Y \times (X - Q)$ denote the open set obtained by removing the graphs of the morphisms $\{f_s\}_{s \in K_+}$. Unwinding the definitions, we can identify each $V_{T,Y}$ with the T-fold fiber power of U over Y.

Lemma 7.7.8. *Let $f : Y \to (X-Q)^S$ be as above (so that the morphisms $\{f_s\}_{s \in S}$ have disjoint graphs). Then, for every nonempty finite set T and each $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, the canonical map*

$$\theta : \mathcal{F} \otimes \psi_{T,Y*} \phi_{T,Y}^* \omega_{X^T} \to \psi_{T,Y*}(\mathcal{F} \boxtimes \omega_{X^T})|_{V_{T,Y}}$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$.

Proof. Proceeding by induction on the number of elements of T, we can reduce to the case where T has a single element. In this case, we have $V_{T,Y} = U$, where $U \subseteq Y \times (X - Q)$ is the open subset defined above. Let $\overline{\psi} : Y \times X \to Y$ be the projection onto the first factor and let ψ be the restriction of $\overline{\psi}$ to the open set U. Note that the complement of U in $Y \times X$ is the disjoint union of finitely many closed subschemes $\{Z_i\}_{i \in I}$ which are the images of sections of $\overline{\psi}$. Let ψ_i denote the restriction of ψ to Z_i. It follows that the map θ fits into a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \bigoplus_{i \in I}(\mathcal{F} \otimes \psi_{i*}\underline{\mathbf{Z}}_{\ell Z_i}) & \longrightarrow & \mathcal{F} \otimes \overline{\psi}_*(\underline{\mathbf{Z}}_{\ell Y} \boxtimes \omega_X) & \longrightarrow & \mathcal{F} \otimes \psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T} \\ \downarrow & & \downarrow & & \downarrow \theta \\ \bigoplus_{i \in I}(\psi_{i*}\psi_i^* \mathcal{F}) & \longrightarrow & \overline{\psi}_*(\mathcal{F} \boxtimes \omega_X) & \longrightarrow & \psi_{T,Y*}(\mathcal{F} \boxtimes \omega_{X^T})|_{V_{T,Y}}. \end{array}$$

It follows from the projection formula (Corollary 4.5.10) that the vertical maps on the left and middle are equivalences, so that θ is an equivalence as well. □

Let us now return to the proof of Lemma 7.7.7. We wish to prove that the direct limit

$$\varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C^*(V_{T,Y}; (\mathcal{F} \boxtimes \omega_{X_T})|_{V_{T,Y}})$$

vanishes. Using the projection $\psi_{T,Y} : V_{T,Y} \to Y$ and invoking Lemma 7.7.8, we can rewrite this direct limit as

$$\begin{aligned} \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C^*(Y, \psi_{T,Y*}(\mathcal{F} \boxtimes \omega_{X^T})|_{V_{T,Y}}) &\simeq \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C^*(Y, \mathcal{F} \otimes \psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T}) \\ &\simeq C^*(Y, \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} (\mathcal{F} \otimes \psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T})) \\ &\simeq C^*(Y, \mathcal{F} \otimes \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T}). \end{aligned}$$

It will therefore suffice to prove the following assertion at the level of sheaves:

Lemma 7.7.9. *Let $f : Y \to (X - Q)^S$ be as in Lemma 7.7.8. Then the direct limit*

$$\varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T}$$

vanishes in $\mathrm{Shv}_\ell(Y)$.

Proof. Let $\mathcal{G}_Y = \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T} \in \mathrm{Shv}_\ell(Y)$; we wish to show that $\mathcal{G}_Y$ vanishes. For each integer $n \geq -1$, let $\mathcal{G}_Y^{\leq n} = \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}_{\leq n}} \psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T} \in \mathrm{Shv}_\ell(Y)$, so that we have a sequence

$$0 = \mathcal{G}_Y^{\leq -1} \to \mathcal{G}_Y^{\leq 0} \to \mathcal{G}_Y^{\leq 1} \to \cdots$$

whose colimit is $\mathcal{G}_Y$. For each $n \geq 0$, form a fiber sequence

$$\mathcal{G}_Y^{\leq n-1} \to \mathcal{G}_Y^{\leq n} \to \mathcal{G}_Y^{=n}.$$

Let $U \subseteq Y \times (X - Q)$ be the open subset defined prior to the statement of Lemma 7.7.8 and let U^n_Y denote the n-fold fiber power of U over Y. We let $\mathring{U}^n_Y$ denote the open subset of U^n_Y whose k-valued points are sequences $(y, x_1, \ldots, x_n)$ where each (y, x_i) is a k-valued point of $U \subseteq Y \times X$ and the points $\{x_i\}_{1 \leq i \leq n}$ are pairwise distinct. The symmetric group Σ_n acts freely on $\mathring{U}^n_Y$, and we let $\mathring{U}^{(n)}_Y$ denote the quotient $\mathring{U}^n_Y / \Sigma_n$. Note that the projection map

$$\rho : \mathring{U}^{(n)}_Y \to Y$$

is a smooth morphism of relative dimension n; we denote its relative dualizing sheaf by $\omega_{\overset{\circ}{U}{}^{(n)}_Y/Y}$.

Arguing as in the proof of Lemma 5.3.14, we obtain an equivalence

$$\mathcal{G}_Y^{=n} \simeq \rho_* \omega_{\overset{\circ}{U}{}^{(n)}_Y/Y} \in \mathrm{Shv}_\ell(Y).$$

In particular, the sheaf $\mathcal{G}_Y^{=n} \in \mathrm{Shv}_\ell(Y)$ is constructible. It follows by induction that each of the sheaves $\mathcal{G}_Y^{\leq n}$ is likewise constructible.

We next prove the following:

(∗) For each integer $n \geq 0$, the sheaf $\mathcal{G}_Y^{=n} \in \mathrm{Shv}_\ell(Y)$ belongs to $\mathrm{Shv}_\ell(Y)_{\geq n}$.

Since $\mathcal{G}_Y^{=n}$ is constructible, it will suffice to show that for each map $i : \operatorname{Spec} k \to Y$, the stalk $i^* \mathcal{G}_Y^{=n}$ belongs to $\mathrm{Shv}_\ell(\operatorname{Spec} k)_{\geq n} \simeq (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq n}$ (Corollary 4.4.11). It follows from Lemma 7.7.7 that the construction formation of the direct image $\psi_{T,Y*}\phi^*_{T,Y}\omega_{X^T}$ is compatible with base change along i for each $T \in \mathrm{Fin}^{\mathrm{s}}$. It follows that we have canonical equivalences

$$i^* \mathcal{G}_Y^{\leq n} \simeq \mathcal{G}_{\operatorname{Spec} k}^{\leq n} \qquad i^* \mathcal{G}_Y^{=n} \simeq \mathcal{G}_{\operatorname{Spec} k}^{\leq n}.$$

Consequently, to prove (∗), we may assume without loss of generality that $Y = \operatorname{Spec} k$, so that we can regard U as an open subset of X. Note that $U \neq X$ (since $K_+ \neq \emptyset$), so that U is affine and therefore $\overset{\circ}{U}{}^{(n)}_Y$ is likewise affine. We then have

$$\mathrm{H}^*(\mathcal{G}_{\operatorname{Spec} k}^{=n}) \simeq \mathrm{H}^*(\overset{\circ}{U}{}^{(n)}_Y; \omega_{\overset{\circ}{U}{}^{(n)}_Y}) \simeq \mathrm{H}^{*+2n}(\overset{\circ}{U}{}^{(n)}_Y; \mathbf{Z}_\ell(n)),$$

which is trivial for $* \geq -n$ by virtue of Artin's vanishing theorem.

It follows from assertion (∗) that the canonical map

$$\tau_{\leq n} \mathcal{G}_Y^{\leq m} \to \tau_{\leq n} \mathcal{G}_Y^{\leq m+1}$$

is an equivalence for $m > n$. In particular, the map $\tau_{\leq n} \mathcal{G}_Y^{\leq n+1} \to \tau_{\leq n} \mathcal{G}_Y$ is an equivalence. To prove that $\mathcal{G}_Y \simeq 0$, it will suffice to show that each truncation $\tau_{\leq n} \mathcal{G}_Y$ vanishes (Proposition 4.4.19), which is equivalent to the vanishing of $\tau_{\leq n} \mathcal{G}_Y^{\leq n+1}$. Since $\tau_{\leq n} \mathcal{G}_Y^{\leq n+1}$ is constructible, this vanishing can be checked stalkwise: that is, it will suffice to show that $\tau_{\leq n} \mathcal{G}_Y^{\leq n+1}$ vanishes in the special case $Y = \operatorname{Spec} k$. By virtue of the fact that the map $\tau_{\leq n} \mathcal{G}_Y^{n+1} \to \tau_{\leq n} \mathcal{G}_Y$ is an equivalence also for $Y = \operatorname{Spec} k$, we are reduced to proving that $\mathcal{G}_Y \simeq 0$ in the special case $Y = \operatorname{Spec} k$.

For the remainder of the proof, we will identify U with an open curve contained in X. Let Q' denote the set of points of X which do not belong to U, so that Q' is the union of the subset $Q \subseteq X$ with the image of K_+ under the map $S \to (X - Q)(k)$ determined by $f : \operatorname{Spec} k = Y \to (X - Q)^S$. For each $T \in \mathrm{Fin}^{\mathrm{s}}$, let $Z(T)$ denote the complement of U^T in X^T, which we regard as a reduced closed subscheme of X^T. Let $\mathrm{Ran}(X)' \subseteq \mathrm{Ran}(X)$ denote the full subcategory of $\mathrm{Ran}(X)$ spanned by those pairs $(R, \nu : T \to X(R))$ for which ν corresponds to a map $\operatorname{Spec} R \to X^T$ which factors through $Z(T)$. (more informally, $\mathrm{Ran}(X)'$ is the prestack which classifies finite subsets of X which have nonempty intersection with Q').

For each $T \in \mathrm{Fin}^{\mathrm{s}}$, there is a canonical fiber sequence

$$C^*(Z(T); \omega_{Z(T)}) \to C^*(X^T; \omega_{X^T}) \to C^*(U^T; \omega_{X^T}|_{U^T}).$$

Since $Z(T)$ and X^T are proper over $\operatorname{Spec} k$, we can identify the first two terms in this fiber sequence with $C_*(Z(T); \mathbf{Z}_\ell)$ and $C_*(X^T; \mathbf{Z}_\ell)$, respectively. Passing to the colimit as T varies, we obtain a fiber sequence

$$\varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C_*(Z(T); \mathbf{Z}_\ell) \to \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C_*(X^T; \mathbf{Z}_\ell) \to \mathcal{G}_{\operatorname{Spec} k},$$

where we identify $\mathcal{G}_{\operatorname{Spec} k}$ with its image under the equivalence of ∞-categories $\operatorname{Shv}_\ell(\operatorname{Spec} k) \simeq \operatorname{Mod}_{\mathbf{Z}_\ell}$. Consequently, the assertion that $\mathcal{G}_{\operatorname{Spec} k}$ vanishes is equivalent to the assertion that the inclusion $\operatorname{Ran}'(X) \hookrightarrow \operatorname{Ran}(X)$ induces a quasi-isomorphism

$$C_*(\operatorname{Ran}'(X); \mathbf{Z}_\ell) \to C_*(\operatorname{Ran}(X); \mathbf{Z}_\ell).$$

This follows from Example 2.5.21. □

8. The Reduced Cohomology of $\operatorname{Bun}_G(X)$

Let k be an algebraically closed field, let X be an algebraic curve over k, let G be a reductive group scheme over X, and let ℓ be a prime number which is invertible in k. In §3.2 and §5.4, we outlined two different approaches to describing the cochain complex $C^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$. Roughly speaking, one can construct an ℓ-adic sheaf $\mathcal{A}$ on $\operatorname{Ran}(X)$ (given as the direct image of the constant sheaf on $\operatorname{Ran}_G(X)$) and an ℓ-adic !-sheaf $\mathcal{B}$ on $\operatorname{Ran}(X)$ (see Notation 5.4.2), together with natural maps

$$C^*_c(\operatorname{Ran}(X); \mathcal{B}) \xrightarrow{\rho} C^*(\operatorname{Bun}_G(X); \mathbf{Z}_\ell) \xrightarrow{\theta} C^*(\operatorname{Ran}(X); \mathcal{A}).$$

If the generic fiber of G is semisimple and simply connected, we proved that the map θ is an equivalence (Theorem 3.2.9). In §7, we outlined a strategy to deduce from this that ρ is also an equivalence. In order to implement this strategy, it is useful to consider "reduced" versions of the sheaves $\mathcal{A}$ and $\mathcal{B}$ on $\operatorname{Ran}(X)$.

Fix a closed point $x \in X$ which we will identify with a k-valued point of the Ran space $\operatorname{Ran}(X)$. Then we have canonical equivalences

$$x^* \mathcal{A} \to C^*(\operatorname{Gr}_G^x; \mathbf{Z}_\ell) \qquad x^! \mathcal{B} \simeq C^*(\operatorname{BG}_x; \mathbf{Z}_\ell).$$

In this section, we will consider reduced versions $\mathcal{A}_{\text{red}}$ and $\mathcal{B}_{\text{red}}$ of $\mathcal{A}$ and $\mathcal{B}$ respectively, whose stalks and costalks are given by the formulae

$$x^* \mathcal{A}_{\text{red}} = C^*_{\text{red}}(\operatorname{Gr}_G^x; \mathbf{Z}_\ell) \qquad x^! \mathcal{B}_{\text{red}} \simeq C^*_{\text{red}}(\operatorname{BG}_x; \mathbf{Z}_\ell).$$

Our main goal is to show that these modifications do not substantially change the cohomologies of the sheaves $\mathcal{A}$ and $\mathcal{B}$. More precisely, we will show that θ and ρ admit reduced versions

$$\rho_{\text{red}} : C^*_c(\operatorname{Ran}(X); \mathcal{B}_{\text{red}}) \to C^*_{\text{red}}(\operatorname{Bun}_G(X); \mathbf{Z}_\ell)$$
$$\theta_{\text{red}} : C^*_{\text{red}}(\operatorname{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\operatorname{Ran}(X); \mathcal{A}_{\text{red}})$$

which are equivalences if and only if ρ and θ are equivalences.

Let us now outline the contents of this section. We begin in §8.1 by formulating a reduced version of nonabelian Poincare duality. In our presentation, we have opted to avoid the language of ℓ-adic sheaves on $\operatorname{Ran}(X)$ and instead adopt the slightly more pedestrian approach of §3, phrasing our result in terms of ℓ-adic cohomology of prestacks with constant coefficients (Theorem 8.1.11).

In §8.2, we introduce the notion of an *augmentation* on a !-sheaf on the Ran space $\operatorname{Ran}(X)$. The collection of all augmented (lax) !-sheaves can be organized into an ∞-category which we will denote by $\operatorname{Shv}_\ell^{\text{aug}}(\operatorname{Ran}(X))$ (Construction 8.2.2). This ∞-category is equipped with a pair of forgetful functors

$$\operatorname{Shv}_\ell^{\text{aug}}(\operatorname{Ran}(X)) \to \operatorname{Shv}_\ell^{\text{lax}}(\operatorname{Ran}(X))$$
$$\mathcal{F} \mapsto \mathcal{F}_{\text{und}} \qquad \mathcal{F} \mapsto \mathcal{F}_{\text{red}}$$

which correspond to "passing to the underlying !-sheaf" and "passing to the augmentation ideal," respectively. Using an "augmented" version of Construction 5.4.1, we produce an augmented version of the !-sheaf $\mathcal{B}$, whose augmentation ideal we denote by $\mathcal{B}_{\text{red}}$. We will construct a canonical map $\mathbf{Z}_\ell \oplus \int \mathcal{B}_{\text{red}} \to \int \mathcal{B}$ and assert that it is an equivalence (Theorem 8.2.18).

One virtue of our reduced versions of Theorems 3.2.9 and 5.4.5 is that they give a much more efficient description of the cohomology $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$. In §8.3, we will illustrate the usefulness of these results by using them to show that the first two cohomology groups of $\mathrm{Bun}_G(X)$ are given by

$$\mathrm{H}^0(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell \qquad \mathrm{H}^1(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \simeq 0,$$

which was stated without proof in §6.5.

The proofs of Theorems 8.1.11 and 8.2.18 are formally very similar. Both involve some elementary geometric considerations combined with some elaborate formal manipulations. In §8.4, we will formulate a general principle (Theorem 8.4.9) which formalizes the combinatorial essence of all three arguments (and which will be needed in §9). In §8.5, we will apply Theorem 8.4.9 (together with the proper base change theorem and the Ind-projectivity of the Beilinson-Drinfeld Grassmannian) to give a proof of Theorem 8.1.11; in §8.6, we will apply Theorem 8.4.9 (together with the smooth base change theorem) to give a proof of Theorem 8.2.18.

8.1. **Reduced Nonabelian Poincare Duality.** Throughout this section, we fix an algebraically closed field k, an algebraic curve X over k, an effective divisor $D \subseteq X$, a prime number ℓ which is invertible in k, and a smooth affine group scheme G over X. We will assume that the generic fiber of G is semisimple and simply connected. Theorem 3.2.9 then supplies an equivalence

$$(17)\quad C^*(\mathrm{Bun}_G(X,D); \mathbf{Z}_\ell) \simeq C^*(\mathrm{Ran}_G(X-D); \mathbf{Z}_\ell) \simeq \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X-D)_S; \mathbf{Z}_\ell).$$

Here $\mathrm{Ran}_G(X-D)_S$ denotes the Beilinson-Drinfeld Grassmannian $\mathrm{Ran}_G(X-D) \times_{\mathrm{Fin}^s} \{S\}$ which parametrizes maps $\mu : S \to X - D$ together with G-bundles on X equipped with a trivialization outside of the divisor $|\mu(S)|$.

Fix a nonempty finite set S, and consider the projection map

$$\pi : \mathrm{Ran}_G(X-D)_S \to \mathrm{Ran}(X-D)_S = (X-D)^S.$$

The ℓ-adic cochain complex $C^*(\mathrm{Ran}_G(X-D)_S; \mathbf{Z}_\ell)$ can be computed as $C^*((X-D)^S; \mathcal{A}^{(S)})$, where $\mathcal{A}^{(S)} \in \mathrm{Shv}_\ell((X-D)^S)$ is obtained by pushforward of the constant sheaf along the map π, defined by the formula

$$\mathcal{A}^{(S)} = [\mathrm{Ran}_G(X-D)_S]_{\underline{\mathbf{Z}}_{\ell X^S}} \simeq \omega_{X^S}^{-1} \otimes [\mathrm{Ran}_G(X-D)_S]_{X^S};$$

see §5.1 and §A.5. If the group scheme G is reductive over the open set $X - D$, then the map π is Ind-proper (see Lemma 8.5.8). One can then use the proper base change theorem to show that the stalk of $\mathcal{A}^{(S)}$ at a k-valued point $\mu \in (X-D)^S$ is given by

$$\begin{aligned} \mu^* \mathcal{A}^{(S)} &\simeq C^*(\mathrm{Ran}_G(X-D) \times_{\mathrm{Ran}(X-D)} \{\mu\}; \mathbf{Z}_\ell) \\ &\simeq \otimes_{x \in \mu(S)} C^*(\mathrm{Gr}_G^x; \mathbf{Z}_\ell), \end{aligned}$$

where Gr_G^x denotes the affine Grassmannian of G at x (that is, the prestack parametrizing G-bundles on X equipped with a trivialization on the open subset $X - \{x\}$).

In §8.5, we will show that (assuming that the restriction of G to $X - D$ is reductive) there is an analogous formula for the *reduced* cohomology $C^*_{\mathrm{red}}(\mathrm{Bun}_G(X,D); \mathbf{Z}_\ell)$, using "reduced" versions $\mathcal{A}^{(S)}_{\mathrm{red}}$ of the ℓ-adic sheaves $\mathcal{A}^{(S)}$ whose stalks are given by

$$\mu^* \mathcal{A}^{(S)}_{\mathrm{red}} \simeq \otimes_{x \in \mu(S)} C^*_{\mathrm{red}}(\mathrm{Gr}_G^x; \mathbf{Z}_\ell),$$

Our first goal is to formulate this result more precisely.

Construction 8.1.1. Let S be a (possibly empty) finite set and let S_0 be a subset of S. We define a category $\mathrm{Ran}_G(X-D)_{S_0\subseteq S}$ as follows:

- The objects of $\mathrm{Ran}_G(X-D)_{S_0\subseteq S}$ are quadruples $(R,\mu,\mathcal{P},\alpha)$ where R is a finitely generated k-algebra, $\mu : S\to (X-D)(R)$ is a map of sets, $\mathcal{P}$ is a G-bundle on X_R, and α is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu|$.
- A morphism from $(R,\mu,\mathcal{P},\alpha)$ to $(R',\mu',\mathcal{P}',\alpha')$ consists of a k-algebra homomorphism $\phi : R\to R'$ such that μ' is given by the composition $S\xrightarrow{\mu}(X-D)(R)\xrightarrow{X(\phi)}(X-D)(R')$) and an isomorphism of $\mathcal{P}'$ with the pullback $\mathrm{Spec}\,R'\times_{\mathrm{Spec}\,R}\mathcal{P}$ over the open set $X_R-|\mu(S_0)|$ which carries α to α'.

It is easy to see that the forgetful functor $(R,\mu,\mathcal{P},\alpha)\mapsto R$ determines a coCartesian fibration $\mathrm{Ran}_G(X)_{S_0\subseteq S}\to \mathrm{Ring}_k$, so that we can regard $\mathrm{Ran}_G(X)_{S_0\subseteq S}$ as a prestack.

Remark 8.1.2. We can describe $\mathrm{Ran}_G(X-D)_{S_0\subseteq S}$ as the prestack which parametrizes maps $\mu : S\to X-D$ together with a G-bundle $\mathcal{P}$ which is defined on $X-|\mu(S_0)|$ and trivialized over $X-|\mu(S)|$. This description is slightly misleading: we always require our G-bundles to admit an extension to the complete curve X, but we identify bundles which are isomorphic away from the image of S_0.

Example 8.1.3. In the situation of Construction 8.1.1, if the set S_0 is empty and S is not, then the prestack $\mathrm{Ran}_G(X-D)_{S_0\subseteq S}$ can be identified with $\mathrm{Ran}_G(X-D)_S$. If $S_0=S$, then the forgetful functor $\mathrm{Ran}_G(X-D)_{S_0\subseteq S}\to (X-D)^S$ is an equivalence of categories.

Remark 8.1.4. In the situation of Construction 8.1.1, there is an evident projection map $\pi : \mathrm{Ran}_G(X-D)_{S_0\subseteq S}\to (X-D)^S$. Suppose we are given a map of sets $\mu : S\to (X-D)(k)$ which we can identify with a k-valued point of $(X-D)^S$. Then $\mu(S)-\mu(S_0)$ can be identified with a finite set of closed points $\{x_1,\ldots,x_d\}$ of the curve X. In this case, the fiber

$$\mathrm{Ran}_G(X-D)_{S_0\subseteq S}\times_{(X-D)^S}\{\mu\}$$

can be identified with the product $\prod_{1\leq i\leq d}\mathrm{Gr}_G^{x_i}$. In §8.5, we will see that π behaves in many respects as if it were a proper map, so that the cochain complex $C^*(\mathrm{Ran}_G(X-D)_{S_0\subseteq S};\mathbf{Z}_\ell)$ can be computed as the global sections of a sheaf $\mathcal{A}^{(S_0\subseteq S)}$ on $(X-D)^S$ whose stalks are given by

$$\mathcal{A}_\mu^{(S_0\subseteq S)}\simeq\bigotimes_{1\leq i\leq d} C^*(\mathrm{Gr}_G^{x_i},\mathbf{Z}_\ell)$$

(beware that the number of factors d depends on the point μ and is not constant on $(X-D)^S$).

Remark 8.1.5. In the situation of Construction 8.1.1, if $S_0\subseteq S_0'\subseteq S$, then we can regard $\mathrm{Ran}_G(X-D)_{S_0\subseteq S}$ as a subcategory of $\mathrm{Ran}_G(X-D)_{S_0'\subseteq S}$, having the same set of objects. Geometrically, it is better to think of $\mathrm{Ran}_G(X-D)_{S_0'\subseteq S}$ as the quotient of $\mathrm{Ran}_G(X-D)_{S_0\subseteq S}$ obtained by identifying those bundles which have the same behavior away from the image of the set S_0'.

Let S be a nonempty finite set. Roughly speaking, we would like to consider the ℓ-adic sheaf $\mathcal{A}_{\mathrm{red}}^{(S)}$ on $(X-D)^S$ given by the cofiber of the canonical map

$$\varinjlim_{\emptyset\neq S_0\subseteq S}\mathcal{A}^{(S_0\subseteq S)}\to\mathcal{A}^{(\emptyset\subseteq S)}=\mathcal{A}^{(S)}.$$

The formula

$$\mu^*\mathcal{A}_{\mathrm{red}}^{(S)}\quad\simeq\quad\otimes_{x\in\mu(S)}C^*_{\mathrm{red}}(\mathrm{Gr}_G^x;\mathbf{Z}_\ell),$$

would then follow from the heuristic description of the stalks of $\mathcal{A}^{(S_0\subseteq S)}$ supplied by Remark 8.1.4. Note that the cochain complex $C^*((X-D)^S; \mathcal{A}^{(S)}_{\mathrm{red}})$ admits a direct description as the cofiber

$$\mathrm{cofib}(\varinjlim_{\emptyset\neq S_0\subseteq S} C^*(\mathrm{Ran}_G(X-D)_{S_0\subseteq S}; \mathbf{Z}_\ell) \to C^*(\mathrm{Ran}_G(X-D)_{\emptyset\subseteq S}; \mathbf{Z}_\ell)),$$

which does not require us to study (or even consider) the ℓ-adic sheaves $\mathcal{A}^{(S_0\subseteq S)} \in \mathrm{Shv}_\ell((X-D)^S)$.

We now introduce a bit of notation which will be helpful for organizing Construction 8.1.1 and its relatives.

Notation 8.1.6. We define a category $\Theta_\blacksquare$ as follows:

- The objects of $\Theta_\blacksquare$ are pairs $(S_0 \subseteq S)$, where S is a nonempty finite set and S_0 is a (possibly empty) subset of S.
- A morphism from $(S_0 \subseteq S)$ to $(S'_0 \subseteq S')$ in $\Theta_\blacksquare$ is a surjection of finite sets $\alpha : S \to S'$ such that $S'_0 \subseteq \alpha(S_0)$.

Let $\Theta_\square$ denote the full subcategory of $\Theta_\blacksquare$ spanned by those objects $(S_0 \subseteq S)$ where S_0 is nonempty.

The construction $(S_0 \subseteq S) \mapsto \mathrm{Ran}_G(X)_{S_0\subseteq S}$ determines a functor from $\Theta_\blacksquare^{\mathrm{op}}$ to the 2-category of prestacks. Consequently, we may regard the construction

$$(S_0 \subseteq S) \mapsto C^*(\mathrm{Ran}_G(X-D)_{S_0\subseteq S}; \mathbf{Z}_\ell)$$

as a functor from $\Theta_\blacksquare$ to the ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$. We now introduce some formal constructions which will be useful when studying such functors.

Construction 8.1.7. The construction $(S_0 \subseteq S) \mapsto S$ determines forgetful functors

$$\nu_\blacksquare : \Theta_\blacksquare \to \mathrm{Fin}^{\mathrm{s}} \qquad \nu_\square : \Theta_\square \to \mathrm{Fin}^{\mathrm{s}}.$$

Let $\mathcal{C}$ be a stable ∞-category and let $V : \Theta_\blacksquare \to \mathcal{C}$ be a functor. We let

$$V_\blacksquare, V_\square : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$$

be the functors obtained by left Kan extension of V along the projection maps $\nu_\blacksquare$ and $\nu_\square$, respectively. Since $\nu_\blacksquare$ and $\nu_\square$ are coCartesian fibrations, these functors can be described more concretely by the formulae

$$V_\blacksquare(S) = \varinjlim_{S_0\subseteq S} V(S_0 \subseteq S) = V(\emptyset \subseteq S) \qquad V_\square(S) = \varinjlim_{\emptyset\neq S_0\subseteq S} V(S_0 \subseteq S).$$

The inclusion $\Theta_\square \hookrightarrow \Theta_\blacksquare$ induces a natural transformation $V_\square \to V_\blacksquare$. We will denote the cofiber of this map by $V_{\mathrm{prim}} : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$.

If the stable ∞-category $\mathcal{C}$ admits limits, we let $\varprojlim_{\mathrm{prim}} V(S_0 \subseteq S)$ denote the inverse limit

$$\varprojlim_{S\in\mathrm{Fin}^{\mathrm{s}}} V_{\mathrm{prim}}(S).$$

We will refer to $\varprojlim_{\mathrm{prim}}(V)$ as the *primitive limit* of V.

Example 8.1.8. Let $V : \Theta_\blacksquare \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ be the functor given by

$$V(S_0 \subseteq S) = C^*(\mathrm{Ran}_G(X-D)_{S_0\subseteq S}; \mathbf{Z}_\ell).$$

Then the functor $V_{\mathrm{prim}} : \mathrm{Fin}^{\mathrm{s}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ is given (heuristically) by the formula $V_{\mathrm{prim}}(S) = C^*((X-D)^S; \mathcal{A}^{(S)}_{\mathrm{red}})$.

Variant 8.1.9. There is a (nearly constant) contravariant functor F from the category $\Theta_{\blacksquare}$ to the 2-category of prestacks, given on objects by the formula

$$F(S_0 \subseteq S) = \begin{cases} \mathrm{Bun}_G(X, D) & \text{if } S_0 = \emptyset \\ \operatorname{Spec} k & \text{if } S_0 \neq \emptyset. \end{cases}$$

We may therefore define a functor $W : \Theta_{\blacksquare} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ by the formula

$$W(S_0 \subseteq S) = C^*(F(S_0 \subseteq S); \mathbf{Z}_\ell) = \begin{cases} C^*(\mathrm{Bun}_G(X, D); \mathbf{Z}_\ell) & \text{if } S_0 = \emptyset \\ C^*(\operatorname{Spec} k; \mathbf{Z}_\ell) & \text{if } S_0 \neq \emptyset. \end{cases}$$

Then $W_{\mathrm{prim}} : \mathrm{Fin}^{\mathrm{s}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ can be identified with the constant functor

$$S \mapsto C^*_{\mathrm{red}}(\mathrm{Bun}_G(X, D); \mathbf{Z}_\ell).$$

Remark 8.1.10. For every nonempty finite set S, there is an evident morphism of prestacks

$$\mathrm{Ran}_G(X - D)_S \to \mathrm{Bun}_G(X, D).$$

These morphisms assemble to give a family of maps

$$\mathrm{Ran}_G(X - D)_{S_0 \subseteq S} \to F(S_0 \subseteq S)$$

where F is the functor described in Variant 8.1.9, depending functorially on $S_0 \subseteq S$. Passing to cohomology, we obtain a natural transformation $W \to V$ of functors from $\Theta_{\blacksquare}$ to $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

We can now formulate our main result:

Theorem 8.1.11 (Nonabelian Poincare Duality, Reduced Version)**.** *Suppose that G is reductive over the open set $X - D \subseteq X$. Then the natural transformation $W \to V$ of Remark 8.1.10 induces an equivalence*

$$\theta_{red} : C^*_{red}(\mathrm{Bun}_G(X, D); \mathbf{Z}_\ell) \simeq \varprojlim_{\mathrm{prim}} W(S_0 \subseteq S) \to \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X)_{S_0 \subseteq S}; \mathbf{Z}_\ell).$$

Let us now outline our proof of Theorem 8.1.11. Note that the natural transformation $W \to V$ induces a map of fiber sequences

$$\begin{array}{ccccc} \mathbf{Z}_\ell & \longrightarrow & C^*(\mathrm{Bun}_G(X, D); \mathbf{Z}_\ell) & \longrightarrow & C^*_{\mathrm{red}}(\mathrm{Bun}_G(X, D); \mathbf{Z}_\ell) \\ \downarrow & & \downarrow & & \downarrow \\ V_{\square}(S) & \longrightarrow & V_{\blacksquare}(S) & \longrightarrow & V_{\mathrm{prim}}(S), \end{array}$$

depending functorially on S. It follows from Theorem 3.2.9 that the vertical maps assemble to an equivalence

$$C^*(\mathrm{Bun}_G(X, D); \mathbf{Z}_\ell) \to \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathrm{Ran}_G(X - D)_S; \mathbf{Z}_\ell) = \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} V_{\blacksquare}(S).$$

Consequently, Theorem 8.1.11 is equivalent to the assertion that the left vertical maps induce an equivalence

$$\alpha : \mathbf{Z}_\ell \simeq \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} W_{\square}(S) \to \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} V_{\square}(S).$$

We can study *this* map by means of the commutative diagram

$$\begin{array}{ccc} \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} W(S \subseteq S) & \xrightarrow{\beta} & \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} W_{\square}(S) \\ \downarrow \alpha' & & \downarrow \alpha \\ \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} V(S \subseteq S) & \xrightarrow{\beta'} & \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} V_{\square}(S). \end{array}$$

It will therefore suffice to show that the maps α', β, and β' are equivalences. In the first two cases, this is relatively easy:

- The morphism β can be written as an inverse limit of morphisms
$$\beta_S : W(S \subseteq S) \to \varinjlim_{\emptyset \neq S_0 \subseteq S} W(S_0 \subseteq S).$$
Each of these maps is an equivalence because the partially ordered set of nonempty subsets of S has weakly contractible nerve (by construction, the functor $S_0 \mapsto W(S_0 \subseteq S)$ is constant for $S_0 \neq \emptyset$).
- Using Example 8.1.3, we can identify α' with the canonical map
$$C^*(\operatorname{Spec} k; \mathbf{Z}_\ell) \to \varprojlim_{S \in \operatorname{Fin}^s} C^*((X-D)^S; \mathbf{Z}_\ell) \simeq C^*(\operatorname{Ran}(X-D); \mathbf{Z}_\ell).$$
Since $X - D$ is connected, this map is an equivalence by virtue of Corollary 2.4.13.

We may therefore reformulate Theorem 8.1.11 as follows:

Theorem 8.1.12. *Let* $V : \Theta_{\blacksquare} \to \operatorname{Mod}_{\mathbf{Z}_\ell}$ *denote the functor given by*
$$V(S_0 \subseteq S) = C^*(\operatorname{Ran}_G(X)_{S_0 \subseteq S}; \mathbf{Z}_\ell).$$
Then the canonical map
$$\varprojlim_{S \in \operatorname{Fin}^s} V(S \subseteq S) \to \varprojlim_{S \in \operatorname{Fin}^s} \varinjlim_{\emptyset \neq S_0 \subseteq S} V(S_0 \subseteq S)$$
is an equivalence in $\operatorname{Mod}_{\mathbf{Z}_\ell}$.

The proof of Theorem 8.1.12 will be given in §8.5 as an application of some general machinery which we will develop in §8.4.

8.2. **The Reduced Product Formula.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. In §5.2, we introduced the ∞-category $\operatorname{Shv}_\ell^!(\operatorname{Ran}(X))$ of !-sheaves on the Ran space $\operatorname{Ran}(X)$, and in §5.4 we constructed an object $\mathcal{B} \in \operatorname{Shv}_\ell^!(\operatorname{Ran}(X))$ whose costalk at a k-valued point $x \in X \subseteq \operatorname{Ran}(X)$ can be identified with the ℓ-adic cochain complex $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$, where BG_x denotes the classifying stack of the smooth affine group $G_x = \{x\} \times_X G$. The classifying stack BG_x has a canonical base point (given by the trivial G-bundle on $\{x\} = \operatorname{Spec} k$), which determines an augmentation map
$$C^*(\mathrm{BG}_x; \mathbf{Z}_\ell) \to C^*(\operatorname{Spec} k; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell.$$
These augmentation maps can be regarded as an additional structure on the !-sheaf $\mathcal{B}$. Our goal in this section is to describe this additional structure more precisely. To accomplish this, we will introduce the notion of an *augmented* !-sheaf on the Ran space $\operatorname{Ran}(X)$. The collection of augmented !-sheaves on $\operatorname{Ran}(X)$ can be organized into an ∞-category which we will denote by $\operatorname{Shv}_\ell^{\mathrm{aug}}(\operatorname{Ran}(X))$. This ∞-category is equipped with a forgetful functor
$$\operatorname{Shv}_\ell^{\mathrm{aug}}(\operatorname{Ran}(X)) \to \operatorname{Shv}_\ell^{\mathrm{lax}}(\operatorname{Ran}(X)).$$
We will show that the !-sheaf $\mathcal{B}$ has a preimage under this forgetful functor, which we will denote by $\mathcal{B}_{\mathrm{aug}}$. We will also introduce a second functor
$$\mathrm{red} : \operatorname{Shv}_\ell^{\mathrm{aug}}(\operatorname{Ran}(X)) \to \operatorname{Shv}_\ell^{\mathrm{lax}}(\operatorname{Ran}(X)),$$
which we call the *reduction functor*. This functor carries $\mathcal{B}_{\mathrm{aug}}$ to a !-sheaf $\mathcal{B}_{\mathrm{red}} \in \operatorname{Shv}_\ell^!(\operatorname{Ran}(X))$ whose stalk at a point $x \in X \subseteq \operatorname{Ran}(X)$ can be identified with the *reduced* cochain complex $C^*_{\mathrm{red}}(\mathrm{BG}_x; \mathbf{Z}_\ell)$. The !-sheaf $\mathcal{B}_{\mathrm{red}}$ is equipped with a canonical map $\mathcal{B}_{\mathrm{red}} \to \mathcal{B}$. The main result

of this section (which we will prove in §8.6) asserts that this map is *almost* an isomorphism on chiral homology (Theorem 8.2.14).

Notation 8.2.1. Let $\Theta_\blacksquare$ be the category introduced in Notation 8.1.6. In this section, we will denote objects of $\Theta_\blacksquare$ by pairs $(T_0 \subseteq T)$, where T is a nonempty finite set and T_0 is a subset of T.

Construction 8.2.2. Let $\mathrm{Sch}_k^{\mathrm{pr}}$ denote the category whose objects are quasi-projective k-schemes and whose morphisms are proper maps. We let $\mathrm{Shv}_\ell^!$ denote the ∞-category introduced in Construction A.5.11: the objects of $\mathrm{Shv}_\ell^!$ are pairs $(X, \mathcal{F})$ where X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ is an ℓ-adic sheaf, and a morphism from $(X, \mathcal{F})$ to $(X', \mathcal{F}')$ is a proper morphism of quasi-projective k-schemes $f : X \to X'$ together with a map $f_* \mathcal{F} \to \mathcal{F}'$.

For each quasi-projective k-scheme X, the construction $(T_0 \subseteq T) \to X^T$ determines a functor $\Theta_\blacksquare^{\mathrm{op}} \to \mathrm{Sch}_k^{\mathrm{pr}}$. We let $\mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$ denote the ∞-category $\mathrm{Fun}_{\mathrm{Sch}_k^{\mathrm{pr}}}(\Theta_\blacksquare^{\mathrm{op}}, \mathrm{Shv}_\ell^!)$ whose objects are functors $\mathcal{F} : \Theta_\blacksquare^{\mathrm{op}} \to \mathrm{Shv}_\ell^!$ which fit into a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Shv}_\ell^! \\ & \overset{\mathcal{F}}{\nearrow} & \downarrow \\ \Theta_\blacksquare^{\mathrm{op}} & \longrightarrow & \mathrm{Sch}_k^{\mathrm{op}}. \end{array}$$

We will refer to $\mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$ as the *∞-category of augmented !-sheaves on* $\mathrm{Ran}(X)$.

Remark 8.2.3. More informally, an object $\mathcal{F}$ of $\mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$ consists of a family of ℓ-adic sheaves $\mathcal{F}^{(T_0 \subseteq T)} \in \mathrm{Shv}_\ell(X^T)$ equipped with transition maps $\mathcal{F}^{(T_0' \subseteq T')} \to \delta^!_{T/T'} \mathcal{F}^{(T_0 \subseteq T)}$ for every surjection $\alpha : T \to T'$ having the property that $T_0' \subseteq \alpha(T_0)$. Here $\delta_{T/T'} : X^{T'} \to X^T$ denote the diagonal map determined by α.

Remark 8.2.4. The construction $T \mapsto (\emptyset \subseteq T)$ determines a fully faithful embedding $\iota : \mathrm{Fin}^{\mathrm{s}} \hookrightarrow \Theta_\blacksquare$. For every quasi-projective k-scheme X, composition with ι determines a forgetful functor $\mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$. For each object $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$, we let $\mathcal{F}^{\mathrm{und}}$ denote the image of $\mathcal{F}$ under this forgetful functor; we will refer to $\mathcal{F}_{\mathrm{und}}$ as the *underlying lax !-sheaf* of $\mathcal{F}$. Concretely, it is given by the formula $\mathcal{F}_{\mathrm{und}}^{(T)} = \mathcal{F}^{(\emptyset \subseteq T)}$.

Remark 8.2.5. Let X be a quasi-projective k-scheme. The functor $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{und}}$ of Remark 8.2.4 admits a right adjoint Q, given by relative right Kan extension along the fully faithful embedding $\iota : \mathrm{Fin}^{\mathrm{s}} \hookrightarrow \Theta_\blacksquare$. Unwinding the definitions, we see that Q is given concretely by the formula

$$Q(\mathcal{F})^{(T_0 \subseteq T)} = \begin{cases} \mathcal{F}^{(T)} & \text{if } T_0 = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Construction 8.2.6. Let X be a quasi-projective k-scheme. Then the functor

$$Q : \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$$

preserves colimits and therefore admits a right adjoint (Corollary HTT.5.5.2.9). For each object $\mathcal{F} \in \mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$, we will denote the image of $\mathcal{F}$ under this right adjoint by $\mathcal{F}_{\mathrm{red}}$. We will refer to $\mathcal{F}_{\mathrm{red}}$ as the *reduced part* of $\mathcal{F}$. Concretely, $\mathcal{F}_{\mathrm{red}}$ can be described by the formula

$$\mathcal{F}_{\mathrm{red}}^{(T)} = \mathrm{fib}(\mathcal{F}^{(\emptyset \subseteq T)} \to \varprojlim_{\emptyset \neq T_0 \subseteq T} \mathcal{F}^{(T_0 \subseteq T)}) \in \mathrm{Shv}_\ell(X^T).$$

Remark 8.2.7. The construction $T \mapsto (T \subseteq T)$ determines a fully faithful embedding ι' : $\mathrm{Fin}^{\mathrm{s}} \to \Theta_{\blacksquare}$. For any quasi-projecitve k-scheme X, composition with ι' determines a forgetful functor

$$\mathrm{Shv}_{\ell}^{\mathrm{aug}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X))$$

which we will denote by $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{triv}}$. We will refer to $\mathcal{F}_{\mathrm{triv}}$ as the *trivial part* of the augmented !-sheaf $\mathcal{F}$. The identity maps $\mathrm{id}_T : T \to T$ determine morphisms

$$(T \subseteq T) \to (\emptyset \subseteq T)$$

in the category $\Theta_{\blacksquare}$, which induce a map of lax !-sheaves $\mathcal{F}_{\mathrm{und}} \to \mathcal{F}_{\mathrm{triv}}$ (which depends functorially on $\mathcal{F}$).

Example 8.2.8. Let X be any quasi-projective k-scheme and let $Q : \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X)) \to \mathrm{Shv}_{\ell}^{\mathrm{aug}}(\mathrm{Ran}(X))$ be the functor described in Remark 8.2.5. Then we have $Q(\mathcal{F})_{\mathrm{triv}} \simeq 0$ for each $\mathcal{F} \in \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X))$.

Construction 8.2.9. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_{\ell}^{\mathrm{aug}}(\mathrm{Ran}(X))$ be an augmented !-sheaf on $\mathrm{Ran}(X)$. Applying Remark 8.2.7 to the counit map $Q(\mathcal{F}^{\mathrm{red}}) \to \mathcal{F}$, we obtain a commutative diagram of lax !-sheaves σ :

$$\begin{array}{ccc} (Q(\mathcal{F}_{\mathrm{red}}))_{\mathrm{und}} & \longrightarrow & \mathcal{F}_{\mathrm{und}} \\ \downarrow & & \downarrow \\ (Q(\mathcal{F}_{\mathrm{red}}))_{\mathrm{triv}} & \longrightarrow & \mathcal{F}_{\mathrm{triv}} . \end{array}$$

For every lax !-sheaf $\mathcal{G}$ on $\mathrm{Ran}(X)$, the counit map $Q(\mathcal{G})_{\mathrm{und}} \to \mathcal{G}$ is an equivalence. Combining this observation with Example 8.2.8, we can identify σ with a diagram

$$\begin{array}{ccc} \mathcal{F}_{\mathrm{red}} & \longrightarrow & \mathcal{F}_{\mathrm{und}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}_{\mathrm{triv}} . \end{array}$$

In other words, we have a (not necessarily exact) triangle of lax !-sheaves

$$\mathcal{F}_{\mathrm{red}} \to \mathcal{F}_{\mathrm{und}} \to \mathcal{F}_{\mathrm{triv}}$$

on $\mathrm{Ran}(X)$.

Definition 8.2.10. Let X be a quasi-projective k-scheme. We let $\mathrm{Ran}_{\mathrm{aug}}(X)$ denote the prestack given by the fiber product $\mathrm{Ran}(X) \times_{\mathrm{Fin}^{\mathrm{s}}} \Theta_{\blacksquare}$. We will refer to $\mathrm{Ran}_{\mathrm{aug}}(X)$ as the *augmented Ran space* of X. We can identify objects of $\mathrm{Ran}_{\mathrm{aug}}(X)$ with quadruples $(R, T_0 \subseteq T, \nu)$ where R is a finitely generated k-algebra, T is a nonempty finite set, $\nu : T \to X(R)$ is a map of sets, and T_0 is a (possibly empty) subset of T. We will generally abuse notation by identifying $\mathrm{Ran}(X)$ with the full subcategory of $\mathrm{Ran}_{\mathrm{aug}}(X)$ spanned by those objects $(R, T_0 \subseteq T, \nu)$ where $T_0 = \emptyset$.

An *augmented* $\mathrm{Ran}(X)$*-prestack* is a category $\mathcal{C}$ equipped with a coCartesian fibration $\mathcal{C} \to \mathrm{Ran}_{\mathrm{aug}}(X)$ (see Definition 5.2.15). If $\mathcal{C}$ is a $\mathrm{Ran}(X)$-prestack, then for each object $(T_0 \subseteq T)$ of $\Theta_{\blacksquare}$ we let $\mathcal{C}_{(T_0 \subseteq T)}$ denote the prestack given by the fiber product

$$\mathcal{C} \times_{\Theta_{\blacksquare}} \{(T_0 \subseteq T)\} \simeq \mathcal{C} \times_{\mathrm{Ran}_{\mathrm{aug}}(X)} X^T.$$

Note that if we are given surjection of finite sets $\alpha : T \to T'$ and T'_0 is a subset of $\alpha(T_0) \subseteq T'$, then α induces a morphism of prestacks $\mathcal{C}_{(T_0 \subseteq T)} \times_{X^T} X^{T'} \to \mathcal{C}_{(T'_0 \subseteq T')}$. It follows that the construction

$$(T_0 \subseteq T) \mapsto [\mathcal{C}_{(T_0 \subseteq T)}]_{X^T}$$

determines an augmented !-sheaf on $\mathrm{Ran}(X)$, which we will denote by $[\mathcal{C}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}$ (more precisely, we let $[\mathcal{C}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}$ denote the augmented !-sheaf obtained by composing the functor $(T_0 \subseteq T) \mapsto \mathcal{C}_{T_0 \subseteq T}$ with the functor $\Phi : \mathrm{RelStack}^! \to \mathrm{Shv}_\ell^!$ of Construction A.5.26).

Remark 8.2.11. Let $\mathcal{C}$ be an augmented Ran prestack on a quasi-projective k-scheme X, and let $\mathcal{C}_{\mathrm{und}} = \mathcal{C} \times_{\mathrm{Ran}_{\mathrm{aug}}(X)} \mathrm{Ran}(X)$ denote the underlying $\mathrm{Ran}(X)$-prestack. Then we have a canonical equivalence of lax !-sheaves

$$([\mathcal{C}]_{\mathrm{Ran}_{\mathrm{aug}}(X)})_{\mathrm{und}} \simeq [\mathcal{C}_{\mathrm{und}}]_{\mathrm{Ran}(X)}.$$

Construction 8.2.12. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. We define a category $\mathrm{Ran}^G_{\mathrm{aug}}(X)$ as follows:

- The objects of $\mathrm{Ran}^G_{\mathrm{aug}}(X)$ are tuples $(R, T_0 \subseteq T, \nu, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, T is a nonempty finite set, T_0 is a subset of T, $\nu : T \to X(R)$ is a map of sets, $\mathcal{P}$ is a G-bundle on the divisor $|\nu(T)| \subseteq X_R$, and γ is a trivialization of $\mathcal{P}$ over the closed subscheme $|\nu(T_0)| \subseteq |\nu(T)|$.
- A morphism from $(R, T_0 \subseteq T, \nu, \mathcal{P}, \gamma)$ to $(R', T'_0 \subseteq T', \nu', \mathcal{P}', \gamma')$ consists a morphism from $(R, T_0 \subseteq T, \nu)$ to $(R', T'_0 \subseteq T', \nu')$ in $\mathrm{Ran}_{\mathrm{aug}}(X)$ together with a G-bundle isomorphism of $\mathcal{P}'$ with $\mathcal{P} \times_{|\nu(T)|} |\nu'(T')|$ which is compatible with the trivializations γ and γ' over the relative divisor $|\nu'(T'_0)|$.

It is not difficult to see that the construction $(R, T_0 \subseteq T, \nu, \mathcal{P}, \gamma) \mapsto (R, T_0 \subseteq T, \nu)$ determines a coCartesian fibration of categories

$$\mathrm{Ran}^G_{\mathrm{aug}}(X) \to \mathrm{Ran}_{\mathrm{aug}}(X),$$

so that we can regard $\mathrm{Ran}^G_{\mathrm{aug}}(X)$ as an augmented $\mathrm{Ran}(X)$-prestack. We let $\mathcal{B}_{\mathrm{aug}}$ denote the augmented lax !-sheaf on $\mathrm{Ran}(X)$ given

$$\mathcal{B}_{\mathrm{aug}} = [\mathrm{Ran}^G_{\mathrm{aug}}(X)]_{\mathrm{Ran}_{\mathrm{aug}}(X)}.$$

Using Remark 8.2.11, we see that the underlying lax !-sheaf of $\mathcal{B}_{\mathrm{aug}}$ can be identified with the !-sheaf $\mathcal{B}$ of Notation 5.4.2. We let $\mathcal{B}_{\mathrm{red}}$ denote the image of $\mathcal{B}_{\mathrm{aug}}$ under the reduction functor $\mathrm{red} : \mathrm{Shv}^{\mathrm{aug}}_\ell(\mathrm{Ran}(X)) \to \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$.

Remark 8.2.13. Let X and G be as in Construction 8.2.12. If $T_0 = T$, then the projection map

$$\mathrm{Ran}^G_{\mathrm{aug}}(X)_{(T_0 \subseteq T)} \to X^T$$

is an equivalence. It follows that that the unit map

$$\omega_{\mathrm{Ran}(X)} \to (\mathcal{B}_{\mathrm{aug}})_{\mathrm{triv}}$$

is an equivalence of lax !-sheaves on $\mathrm{Ran}(X)$.

The following result describes the relationship between $\mathcal{B}$ and $\mathcal{B}_{\mathrm{red}}$:

Theorem 8.2.14. *Let X be an algebraic curve over k, let G be a smooth affine group scheme over X, and let $\mathcal{B}_{\mathrm{aug}}$ be the augmented !-sheaf on* $\mathrm{Ran}(X)$ *given by Construction 8.2.12. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{B}_{red} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \omega_{\mathrm{Ran}(X)} \end{array}$$

of Construction 8.2.9 determines a pullback square

$$\begin{array}{ccc} \int \mathcal{B}_{red} & \longrightarrow & \int \mathcal{B} \\ \downarrow & & \downarrow \\ \int 0 & \longrightarrow & \int \omega_{\mathrm{Ran}(X)}. \end{array}$$

in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. *Identifying* $\int \omega_{\mathrm{Ran}(X)}$ *with* $\mathbf{Z}_\ell$*, we obtain an exact triangle*

$$\int \mathcal{B}_{red} \to \int \mathcal{B} \to \mathbf{Z}_\ell.$$

Remark 8.2.15. We will give a proof of Theorem 8.2.14 in §8.6 by applying the machinery of §8.4. However, let us briefly indicate an alternate proof which uses the machinery developed in §5. Using a variant of Construction 5.7.3, one can define a multiplication map $\mathcal{B}_{\mathrm{red}} \star \mathcal{B}_{\mathrm{red}} \to \mathcal{B}_{\mathrm{red}}$ which exhibits $\mathcal{B}_{\mathrm{red}}$ as a commutative factorization algebra over X. Moreover, the commutative diagram

$$\begin{array}{ccc} \mathcal{B}_{\mathrm{red}} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \omega_{\mathrm{Ran}(X)} \end{array}$$

appearing in Theorem 8.2.14 is a diagram of commutative factorization algebras, corresponding to a commutative diagram

$$\begin{array}{ccc} \mathcal{B}^{(1)}_{\mathrm{red}} & \longrightarrow & \mathcal{B}^{(1)} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \omega_X \end{array}$$

of nonunital commutative algebra objects of $\mathrm{Shv}_\ell(X)$ (see Theorem 5.6.4). It follows that $\mathcal{B}^{(1)}$ is the commutative algebra obtained from $\mathcal{B}^{(1)}_{\mathrm{red}}$ by formally adjoining a unit object to $\mathcal{B}^{(1)}_{\mathrm{red}}$. Let

$$\pi_\star^{\mathrm{nu}} : \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{\mathbf{Z}_\ell})$$

$$\pi_\star : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$$

be as in Examples 5.6.9 and 5.6.12. Then Remark 5.6.13 implies that $\pi_\star^{\mathrm{nu}}(\mathcal{B}_{\mathrm{red}}) \simeq \int \mathcal{B}_{\mathrm{red}}$ can be identified with the augmentation ideal in $\pi_\star(\mathcal{B}) \simeq \int \mathcal{B}$.

Let us conclude this section by describing an application of Theorem 8.2.14. For this, we need a variant of Construction 8.2.12:

Construction 8.2.16. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. We define a category $\overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X)$ as follows:

- The objects of $\overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X)$ are tuples $(R, T_0 \subseteq T, \nu, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, T is a nonempty finite set, T_0 is a subset of T, $\nu : T \to X(R)$ is a map of sets, $\mathcal{P}$ is a G-bundle on the divisor X_R, and γ is a trivialization of $\mathcal{P}$ over the open set

$$U = \begin{cases} \emptyset & \text{if } T_0 = \emptyset \\ X_R & \text{otherwise.} \end{cases} \subseteq X_R.$$

- A morphism from $(R, T_0 \subseteq T, \nu, \mathcal{P}, \gamma)$ to $(R', T'_0 \subseteq T', \nu', \mathcal{P}', \gamma')$ consists a morphism from $(R, T_0 \subseteq T, \nu)$ to $(R', T'_0 \subseteq T', \nu')$ in $\mathrm{Ran}_{\mathrm{aug}}(X)$ together with a G-bundle isomorphism of $\mathcal{P}'$ with $\mathcal{P} \times_{X_R} X_{R'}$ which is compatible with the trivializations γ and γ'.

The construction $(R, T_0 \subseteq T, \nu, \mathcal{P}, \gamma) \mapsto (R, T_0 \subseteq T, \nu)$ determines a coCartesian fibration $\overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X) \to \mathrm{Ran}_{\mathrm{aug}}(X)$, so that we can regard $\overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X)$ as an augmented $\mathrm{Ran}(X)$-prestack.

Remark 8.2.17. In the situation of Construction 8.2.16, we have canonical equivalences

$$\overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X)_{(T_0 \subseteq T)} \simeq \begin{cases} \mathrm{Bun}_G(X) \times_{\mathrm{Spec}\, k} X^T & \text{if } T_0 = \emptyset \\ X^T & \text{if } T_0 \neq \emptyset. \end{cases}$$

Setting $\overline{\mathcal{B}} = [\overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X)]_{\mathrm{Ran}_{\mathrm{aug}}(X)}$, we obtain equivalences

$$\overline{\mathcal{B}}_{\mathrm{red}} \simeq C^*_{\mathrm{red}}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} \quad \overline{\mathcal{B}}_{\mathrm{und}} \simeq C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)} \quad \overline{\mathcal{B}}_{\mathrm{triv}} \simeq \omega_{\mathrm{Ran}(X)}.$$

From this description, it is easy to see that the augmented !-sheaf $\overline{\mathcal{B}}_{\mathrm{aug}}$ satisfies the analogue of Theorem 8.2.14: that is, the triangle

$$\int \overline{\mathcal{B}}_{\mathrm{red}} \to \int \overline{\mathcal{B}}_{\mathrm{und}} \to \int \overline{\mathcal{B}}_{\mathrm{triv}}$$

is exact (it can be identified with the exact triangle $C^*_{\mathrm{red}}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to \mathbf{Z}_\ell$ determined by the base point of $\mathrm{Bun}_G(X)$).

In the situation of Construction 8.2.16, there is an evident forgetful functor

$$\theta : \overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X) \to \mathrm{Ran}^G_{\mathrm{aug}}(X),$$

given on objects by the formula

$$(R, T_0 \subseteq T, \nu, \mathcal{P}, \gamma) \mapsto (R, T_0 \subseteq T, \nu, \mathcal{P}\,|_{|\nu(T)|}, \gamma|_{|\nu(T_0)|}).$$

This forgetful functor determines a morphism of augmented !-sheaves $\mathcal{B}_{\mathrm{aug}} \to \overline{\mathcal{B}}$. Note that θ determines an equivalence of prestacks

$$\overline{\mathrm{Ran}}^G_{\mathrm{aug}}(X)_{(T_0 \subseteq T)} \to \mathrm{Ran}^G_{\mathrm{aug}}(X)_{(T_0 \subseteq T)}$$

whenever $T_0 = T$ and therefore induces an equivalence of trivial parts $(\mathcal{B}_{\mathrm{aug}})_{\mathrm{triv}} \to \overline{\mathcal{B}}_{\mathrm{triv}}$. We have a commutative diagram of triangles

$$\begin{array}{ccccc} \int \mathcal{B}_{\mathrm{red}} & \longrightarrow & \int \mathcal{B} & \longrightarrow & \int \omega_X \\ \downarrow & & \downarrow & & \downarrow \\ \int \overline{\mathcal{B}}_{\mathrm{red}} & \longrightarrow & \int \overline{\mathcal{B}}_{\mathrm{und}} & \longrightarrow & \int \overline{\mathcal{B}}_{\mathrm{triv}}. \end{array}$$

It follows from Theorem 8.2.14 and Remark 8.2.17 that these triangles are exact, and the right vertical map is an equivalence. It follows that the left square is a pullback. Combining this observation with the identifications

$$\int \overline{\mathcal{B}}_{\mathrm{red}} \simeq C^*_{\mathrm{red}}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \qquad \int \overline{\mathcal{B}}_{\mathrm{und}} \simeq C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell),$$

we obtain the following result:

Theorem 8.2.18. *Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. Then the preceding construction supplies a pullback square*

$$\begin{array}{ccc} \int \mathcal{B}_{red} & \longrightarrow & \int \mathcal{B} \\ \downarrow{\scriptstyle \rho_{red}} & & \downarrow{\scriptstyle \rho} \\ C^*_{red}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) & \longrightarrow & C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \end{array}$$

in $\mathrm{Mod}_{\mathbf{Z}_\ell}$*, where* ρ *is the morphism appearing in the statement of Theorem 5.4.5. In particular,* ρ *is an equivalence if and only if* ρ_{red} *is an equivalence.*

8.3. **Application: The Cohomology** $\mathrm{Bun}_G(X)$ **in Low Degrees.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, and a smooth affine group scheme G over X. Our goal in this section is to prove the following stronger form of Proposition 6.5.6 (which was asserted without proof in §6.5):

Theorem 8.3.1. *Suppose that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then the cohomology groups* $\mathrm{H}^i(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ *vanish for* $i \leq 1$*. In particular, the moduli stack* $\mathrm{Bun}_G(X)$ *is connected.*

Warning 8.3.2. Theorem 8.3.1 does not assert that $\mathrm{Bun}_G(X)$ is simply connected: for example, it does not rule out the existence of nontrivial $\mathbf{Z}/p\mathbf{Z}$-torsors over $\mathrm{Bun}_G(X)$ when the field k has characteristic p.

Remark 8.3.3. The proof of Theorem 8.3.1 that we give in this section depends on Theorem 5.4.5. The proof Theorem 5.4.5 in turn depends on Proposition 8.3.5. However, no circularity results: the proof of Proposition 8.3.5 given in this section is completely independent of the results of §7.

Let $\mathcal{B}_{\mathrm{red}} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$ be defined as in Construction 8.2.12. Theorem 8.3.1 is an easy consequence of the following pair of results:

Proposition 8.3.4. *The lax !-sheaf* $\mathcal{B}_{red}$ *is a !-sheaf on* $\mathrm{Ran}(X)$*.*

Proposition 8.3.5. *Let $n \geq 1$ and let $U \subseteq X^n$ be the open subset consisting of n-tuples of distinct points of X. If the fibers of G are connected, then* $C^*(U; (\mathcal{B}_{red})^{(n)}|_U) \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$*. If, in addition, the generic fiber of G is semisimple and simply connected, then* $C^*(U; (\mathcal{B}_{red})^{(n)}|_U) \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2n}$*.*

Corollary 8.3.6. *If the fibers of G are connected, then the chiral homology* $\int \mathcal{B}_{red}$ *belongs to* $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$*. If, in addition, the generic fiber of G is semisimple and simply connected, then* $\int \mathcal{B}_{red} \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2}$*.*

Proof. We will prove the second assertion (since it is what we need for Theorem 8.3.1); the proof of the first assertion is similar. Since $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2}$ is closed under filtered colimits, it will suffice to show that each $\int^{(n)} \mathcal{B}_{\mathrm{red}}$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2}$. We proceed by induction on n, the case $n = 0$ being trivial. Since $\mathcal{B}_{\mathrm{red}}$ is a !-sheaf (Proposition 8.3.4), Lemma 5.3.14 supplies a fiber sequence

$$\int^{(n-1)} \mathcal{B}_{\mathrm{red}} \to \int^{(n)} \mathcal{B}_{\mathrm{red}} \to C^*(U; \mathcal{B}^{(n)}_{\mathrm{red}}|_U)_{\Sigma_n},$$

where U is as in the statement of Proposition 8.3.5. It will therefore suffice to show that the chain complexes $C^*(U; \mathcal{B}^{(n)}_{\mathrm{red}}|_U)_{\Sigma_n}$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2}$. Using Lemma 5.3.15, we see that the norm map

$$C^*(U; \mathcal{B}^{(n)}_{\mathrm{red}}|_U)_{\Sigma_n} \to C^*(U; \mathcal{B}^{(n)}_{\mathrm{red}}|_U)^{\Sigma_n}$$

is an equivalence. It will therefore suffice to show that $C^*(U; \mathcal{B}^{(n)}_{\mathrm{red}}|_U)^{\Sigma_n}$ is contained in the ∞-category $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2}$. This follows from Proposition 8.3.5, since $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2}$ is closed under limits. □

Proof of Theorem 8.3.1. Using Theorems 5.4.5 and 8.2.18, we can identify $C^*_{\mathrm{red}}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ with the chiral homology $\int \mathcal{B}_{\mathrm{red}}$; the desired result now follows from Corollary 8.3.6. □

Remark 8.3.7. The methods described above can be used to obtain more precise information about the cohomology of $\mathrm{Bun}_G(X)$. Assume that the fibers of G are connected and that the generic fiber is semisimple and simply connected. Let BG denote the classifying stack of G, so that we have a direct sum decomposition

$$[\mathrm{BG}]_X = \mathcal{B}^{(1)} \simeq \mathcal{B}^{(1)}_{\mathrm{red}} \oplus \omega_X.$$

Passing to global sections, we obtain a direct sum decomposition

$$\begin{aligned} C^*(\mathrm{BG}; \Sigma^2 \mathbf{Z}_\ell(1)) &\simeq C^*(X; [\mathrm{BG}]_X) \\ &\simeq C^*(X; \omega_X) \oplus \int^{(1)} \mathcal{B}_{\mathrm{red}} \\ &\simeq C_{-*}(X; \mathbf{Z}_\ell) \oplus \int^{(1)} \mathcal{B}_{\mathrm{red}}. \end{aligned}$$

In particular, for $n > 0$, we have a canonical isomorphism

$$\mathrm{H}^n(\int^{(1)} \mathcal{B}_{\mathrm{red}}) \simeq \mathrm{H}^{n+2}(\mathrm{BG}; \mathbf{Z}_\ell(1)).$$

The proof of Theorem 8.3.1 shows that the cofiber of the canonical map

$$\int^{(1)} \mathcal{B}_{\mathrm{red}} \to \int \mathcal{B}_{\mathrm{red}} \simeq C^*_{\mathrm{red}}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -4}$. Passing to cohomology, we obtain maps

$$\mathrm{H}^{n+2}(\mathrm{BG}; \mathbf{Z}_\ell(1)) \to \mathrm{H}^n(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

which are bijective for $0 < n < 4$ and injective when $n = 4$.

We now turn to the proofs of Propositions 8.3.4 and 8.3.5. We will deduce Proposition 8.3.4 from the following general criterion:

Lemma 8.3.8. *Let $\mathcal{F} \in \mathrm{Shv}^{\mathrm{aug}}_\ell(\mathrm{Ran}(X))$ be an augmented !-sheaf on $\mathrm{Ran}(X)$. Suppose that $\mathcal{F}$ satisfies the following condition:*

- $(*)$ *For every surjection of nonempty finite sets $\alpha : T \to T'$ and every subset $T_0 \subseteq T$, the canonical map*
$$\mathcal{F}^{(\alpha(T_0) \subseteq T')} \to \delta^!_{T/T'} \mathcal{F}^{(T_0 \subseteq T)}$$
is an equivalence in $\mathrm{Shv}_\ell(X^{T'})$; here $\delta_{T/T'} : X^{T'} \to X^T$ denotes the diagonal embedding determined by α.

Then $\mathcal{F}_{red}$ is a !-sheaf on $\mathrm{Ran}(X)$.

Proof. Let $\alpha : T \to T'$ be a surjection of nonempty finite sets; we wish to prove that the canonical map $\theta : \mathcal{F}^{(T')}_{\mathrm{red}} \to \delta^!_{T/T'} \mathcal{F}^{(T')}_{\mathrm{red}}$ is an equivalence in $\mathrm{Shv}_\ell(X^{T'})$. Let $P(T)$ denote the collection of all nonempty subsets of T and let $P(T')$ denote the collection of all nonempty subsets of T'.

We have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \mathcal{F}^{(T')}_{\mathrm{red}} & \longrightarrow & \mathcal{F}^{(\emptyset \subseteq T')} & \longrightarrow & \varprojlim_{T'_0 \in P(T')} \mathcal{F}^{(T'_0 \subseteq T')} \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta''} \\ \delta^!_{T/T'} \mathcal{F}^{(T)}_{\mathrm{red}} & \longrightarrow & \delta^!_{T/T'} \mathcal{F}^{(\emptyset \subseteq T)} & \longrightarrow & \delta^!_{T/T'} \varprojlim_{T_0 \in P(T)} \mathcal{F}^{(T_0 \subseteq T)} \end{array}$$

(see Construction 8.2.6). It follows immediately from $(*)$ that θ' is an equivalence. We are therefore reduced to proving that θ'' is an equivalence. The morphism θ'' factors as a composition

$$\varprojlim_{T'_0 \in P(T')} \mathcal{F}^{(T'_0 \subseteq T')} \to \varprojlim_{T_0 \in P(T)} \mathcal{F}^{(\alpha(T_0) \subseteq T')} \to \delta^!_{T/T'} \varprojlim_{T_0 \in P(T)} \mathcal{F}^{(T_0 \subseteq T)},$$

where the second map is an equivalence by virtue of $(*)$. To show that the first map is an equivalence, it suffices to show that the construction $T_0 \mapsto \alpha(T_0)$ determines a right cofinal functor $P(T) \to P(T')$. This is clear, since the functor has a right adjoint (given by $T'_0 \mapsto \alpha^{-1}(T'_0)$). □

Lemma 8.3.9. *Let T be a nonempty finite set, let E be an equivalence relation on T, and let Y be a scheme equipped with a map $u : Y \to X^{T/E}$. Then for every subset $T_0 \subseteq T$, the canonical map*

$$[\mathrm{Ran}^G(X)_{(T_0/E \subseteq T/E)} \times_{X^{T/E}} Y]_Y \to [\mathrm{Ran}^G(X)_{(T_0 \subseteq T)} \times_{X^T} Y]_Y$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$.

Proof of Proposition 8.3.4. Combine Lemma 8.3.8 with Lemma 8.3.9. □

Proof of Lemma 8.3.9. The assertion is local on Y. We may therefore assume without loss of generality that $Y = \operatorname{Spec} R$ is affine. Then u determines a map $\beta' : T/E \to X(R)$. Let β denote the composition of β' with the quotient map $T \to T/E$, so that β and β'

$$D_0 = |\beta(T_0)| \qquad D = |\beta(T)|$$

$$D'_0 = |\beta'(T_0/E)| \qquad D' = |\beta'(T/E)|$$

in the relative curve X_R. Let H_0 denote the Weil restriction of the group scheme $D_0 \times_X G$ along the finite flat map $D_0 \to \operatorname{Spec} R$, and define H'_0, H, and H' similarly. The restriction maps

$$H \to H_0 \qquad H' \to H'_0$$

are smooth surjections of group schemes over R. It follows from Remark 8.6.5 that we can identify $Y \times_{X^T} \mathrm{Ran}^G(X)_{(T_0 \subseteq T)}$ and $Y \times_{X^{T/E}} \mathrm{Ran}^G(X)_{(T_0/E \subseteq T/E)}$ with the (stack theoretic) quotients of H_0 by H_1 and H'_0 by H'_1, respectively. For each integer $d \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} H_0 \times_Y H^d & \xrightarrow{g^d} & H'_0 \times_Y H'^d \\ & {\scriptstyle f^d} \searrow \quad \swarrow {\scriptstyle f'^d} & \\ & Y. & \end{array}$$

Then we can identify μ with the totalization of a cosimplicial object $\mu^\bullet$ of $\mathrm{Fun}(\Delta^1, \mathrm{Shv}_\ell(Y))$, where each μ^d is given by the canonical map

$$f'^d_* f'^{d*} \omega_{Y'} \to f^d_* f^{d*} \omega_{Y'}.$$

We are therefore reduced to proving that the unit map $\mathrm{id} \to g^d_* g^{d*}$ is an equivalence of functors from $\mathrm{Shv}_\ell(H'_0 \times_Y H'^d)$ to itself. This follows from the observation that the natural maps $H_0 \to H'_0$ and $H \to H'$ admit factorizations

$$H_0 = \mathcal{E}_0(m) \to \mathcal{E}_0(m-1) \to \cdots \to \mathcal{E}_0(0) = H'_0$$

$$H = \mathcal{E}_1(n) \to \mathcal{E}_1(n-1) \to \cdots \to \mathcal{E}_1(0) = H',$$

where each $\mathcal{E}_i(j)$ is the total space of a vector bundle over $\mathcal{E}_i(j-1)$. □

The proof of Proposition 8.3.5 depends on some elementary facts about the étale cohomology of algebraic groups.

Lemma 8.3.10. *Let H be a simply connected semisimple algebraic group over k. Then the cohomology groups $\mathrm{H}^i(H; \mathbf{Z}/\ell\mathbf{Z})$ vanish for $i = 1$ and $i = 2$.*

Proof. Choose a Borel subgroup $B \subseteq H$ and a maximal torus $T \subseteq B$. Let $\pi : H \to H/B$ denote the projection map. Let $\mathcal{F} = \pi_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_H \in \mathrm{Shv}(H/B; \mathbf{Z}/\ell\mathbf{Z})$. Note that the action of B on H determines a pullback diagram

$$\begin{array}{ccc} B \times H & \longrightarrow & H \\ \downarrow & & \downarrow \pi \\ H & \xrightarrow{\pi} & H/B. \end{array}$$

Using the smooth base change theorem, we deduce that $\pi^* \mathcal{F}$ is the constant sheaf with value $C^*(B; \mathbf{Z}/\ell\mathbf{Z})$. Since B is connected, it follows that each $\pi_i \mathcal{F} \in \mathrm{Shv}(H/B; \mathbf{Z}/\ell\mathbf{Z})^\heartsuit$ is the constant sheaf associated to the cohomology group $\mathrm{H}^{-i}(B; \mathbf{Z}/\ell\mathbf{Z})$. We therefore obtain a Leray-Serre spectral sequence

$$\mathrm{H}^p(H/B; \mathrm{H}^q(T; \mathbf{Z}/\ell\mathbf{Z})) \Rightarrow \mathrm{H}^{p+q}(H; \mathbf{Z}/\ell\mathbf{Z}).$$

The flag variety H/B admits a Bruhat decomposition into cells isomorphic to affine spaces so that its cohomologies are concentrated in even degrees. We therefore obtain an exact sequence of low degree terms

$$0 \to \mathrm{H}^1(H; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^1(B; \mathbf{Z}/\ell\mathbf{Z}) \xrightarrow{d} \mathrm{H}^2(H/B; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^2(H; \mathbf{Z}/\ell\mathbf{Z})$$

We first claim that d is an isomorphism. To prove this, it suffices to show that the dual map $d^\vee : \mathrm{H}_2(H/B; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}_1(B; \mathbf{Z}/\ell\mathbf{Z})$ is an isomorphism. Let $\mu_\ell(k)$ denote the group of ℓth roots of unity (which we regard as a 1-dimensional vector space over $\mathbf{Z}/\ell\mathbf{Z}$), let Λ denote the coweight lattice of T, and let $\Lambda_0 \subseteq \Lambda$ denote the sublattice generated by coroots. The Bruhat decomposition of H/B supplies a canonical isomorphism of $\mathrm{H}_2(H/B; \mathbf{Z}/\ell\mathbf{Z})$ with a direct sum of copies of μ_ℓ indexed by the simple coroots of H, which we will identify with $\Lambda_0 \otimes_{\mathbf{Z}} \mu_\ell(k)$. Note that B is isomorphic to the product of T with an affine space, so that the Künneth formula supplies a canonical isomorphism $\mathrm{H}_1(B; \mathbf{Z}/\ell\mathbf{Z}) \simeq \mathrm{H}_1(T; \mathbf{Z}/\ell\mathbf{Z}) \simeq \Lambda \otimes_{\mathbf{Z}} \mathrm{H}_1(\mathbf{G}_m; \mathbf{Z}/\ell\mathbf{Z}) \simeq \Lambda \otimes_{\mathbf{Z}} \mu_\ell(k)$. A simple calculation shows that under these isomorphisms, the map $d^\vee$ is obtained by tensoring the identity map from $\mu_\ell(k)$ to itself with the inclusion of lattices $\Lambda_0 \hookrightarrow \Lambda$. Our assumption that H is simply connected guarantees that $\Lambda_0 = \Lambda$, so that $d^\vee$ and d are isomorphisms. It follows that $\mathrm{H}^1(H; \mathbf{Z}/\ell\mathbf{Z}) \simeq 0$ and that the pullback map $\mathrm{H}^2(H/B; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^2(H; \mathbf{Z}/\ell\mathbf{Z})$ vanishes. Since $\mathrm{H}^1(H/B; \mathrm{H}^1(T; \mathbf{Z}/\ell\mathbf{Z})) \simeq 0$ as well, the Leray-Serre spectral sequence supplies an identification of $\mathrm{H}^2(H; \mathbf{Z}/\ell\mathbf{Z})$ with a subgroup of the kernel of the first differential

$$d' : \mathrm{H}^2(B; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^2(H/B; \mathrm{H}^1(B; \mathbf{Z}/\ell\mathbf{Z})) \simeq \mathrm{H}^2(H/B; \mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \mathrm{H}^1(B; \mathbf{Z}/\ell\mathbf{Z}).$$

Note that if $x_1, \ldots, x_r$ is a basis for $\mathrm{H}^1(B; \mathbf{Z}/\ell\mathbf{Z})$, then the cup products $\{x_i \cup x_j\}_{i<j}$ form a basis for $\mathrm{H}^2(B; \mathbf{Z}/\ell\mathbf{Z})$. Using the Leibniz rule for the differential d', we compute

$$d'(x_i \wedge x_j) = dx_i \otimes x_j - dx_j \otimes x_i.$$

Since d is an isomorphism, the elements $dx_i \otimes x_j - dx_j \otimes x_i$ are linearly independent in the tensor product $\mathrm{H}^2(H/B; \mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \mathrm{H}^1(B; \mathbf{Z}/\ell\mathbf{Z})$. It follows that d' is a monomorphism and therefore $\mathrm{H}^2(H; \mathbf{Z}/\ell\mathbf{Z}) \simeq 0$. □

Lemma 8.3.11. *Let H be a connected algebraic group over k and let* BH *denote its classifying stack. Then the reduced homology groups $\mathrm{H}_i^{red}(\mathrm{BH}; \mathbf{Z}_\ell)$ vanish for $i < 2$. If H is semisimple and simply connected, then the reduced cohomology groups $\mathrm{H}_i^{red}(\mathrm{BH}; \mathbf{Z}_\ell)$ vanish for $i < 4$.*

Proof. Note that BH can be represented by the simplicial scheme $[n] \mapsto H^n$. It follows that the ℓ-adic chain complex $C_*(\mathrm{BH}; \mathbf{Z}_\ell)$ is given by the geometric realization of the simplicial object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$ given by $[n] \mapsto C_*(H^n; \mathbf{Z}_\ell) \simeq C_*(H; \mathbf{Z}_\ell)^{\otimes n}$. The skeletal filtration of this geometric realization has successive quotients of the form $\Sigma^n C_*^{\mathrm{red}}(H; \mathbf{Z}_\ell)^{\otimes n}$. To show that each of these objects belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq d}$ for $n \geq 1$, it will suffice to show that $C_*^{\mathrm{red}}(H; \mathbf{Z}_\ell)$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq d-1}$. Since H is a quasi-projective variety, $C_*^{\mathrm{red}}(H; \mathbf{Z}_\ell)$ is perfect; it will therefore suffice to show that the chain complex $C_*^{\mathrm{red}}(H; \mathbf{Z}/\ell\mathbf{Z})$ belongs to $(\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}})_{\geq d-1}$. When $d = 2$, this follows from our assumption that H is connected; when $d = 4$ it follows from Lemma 8.3.10 (under the additional assumption that H is semisimple and simply connected). □

Proof of Proposition 8.3.5. Let G_0 be the generic fiber of G, which we regard as an algebraic group over the fraction field K_X. Choose a finite (possibly inseparable) extension field L of K_X such that G_0 splits over L: that is, the group scheme $\operatorname{Spec} L \times_X G$ fits into an exact sequence

$$0 \to U \to \operatorname{Spec} L \times_X G \to \operatorname{Spec} L \times_{\operatorname{Spec} k} H \to 0$$

where H is a reductive algebraic group over k and U is a successive extension of finitely many copies of $\mathbf{G}_a$. Then L is the fraction field of an algebraic curve $\widetilde{X}$ which is finite over X. Then the projection map $\widetilde{X} \to X$ factors as a composition

$$\widetilde{X} \xrightarrow{\alpha} \overline{X} \xrightarrow{\beta} X$$

where β is finite étale and α is purely inseparable. Choose an open subset $V \subseteq X$ such that β is étale over V, and let $\overline{V}$ and $\widetilde{V}$ denote the inverse images of V in $\overline{X}$ and $\widetilde{X}$, respectively. Shrinking V if necessary, we may assume that $\widetilde{V} \times_X G$ fits into an exact sequence

$$0 \to U' \to \widetilde{V} \times_X G \to \widetilde{V} \times_{\operatorname{Spec} k} H \to 0$$

where U' is a successive extension of finitely many copies of the additive group over $\widetilde{V}$.

Let $\{x_1, \ldots, x_d\}$ be the set of k-valued points of X which do not belong to V. Set $T = \{1, \ldots, n\}$. For each function $\phi : T \to \{0, 1, \ldots, d\}$, let Y_ϕ denote the reduced closed subscheme of X^T whose k-valued points are monomorphisms $\nu : T \to X(k)$ such that $\nu(t) \in V(k)$ if $\phi(t) = 0$ and $\nu(t) = x_{\phi(i)}$ otherwise. Let $\overline{Y}_\phi = \prod_{t \in T} \begin{cases} X & \text{if } \phi(t) = 0 \\ \{x_{\phi(t)}\} & \text{otherwise.} \end{cases}$ denote the closure of Y_ϕ in X^T and let $i_\phi : \overline{Y}_\phi \hookrightarrow X^T$ denote the corresponding closed embedding. The locally closed subschemes Y_ϕ comprise a stratification of $U \subseteq X^T$, so that $C^*(U; \mathcal{B}_{\mathrm{red}}^{(n)}|_U)$ admits a finite filtration whose successive quotients have the form $C^*(Y_\phi; i_\phi \mathcal{B}_{\mathrm{red}}^{(n)}|_{Y_\phi})$. It will therefore suffice to show that each of the chain complexes $C^*(Y_\phi; (i_\phi^! \mathcal{B}_{\mathrm{red}}^{(n)})|_{Y_\phi})$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq d}$, where $d = -2n$ if the generic fiber of G is semisimple and simply connected and $d = 0$ otherwise. In fact, we will prove a slightly stronger assertion: each of the ℓ-adic sheaves $(i_\phi^! \mathcal{B}_{\mathrm{red}}^{(n)})|_{Y_\phi}$ belongs to $\mathrm{Shv}_\ell(Y_\phi)_{\leq d}$ (the desired assertion then follows from the fact that the global sections functor $\mathcal{F} \mapsto C^*(Y_\phi; \mathcal{F})$ is left t-exact).

Fix a map $\phi : T \to \{0, \ldots, d\}$ and set

$$\widetilde{Y}_\phi = Y_\phi \times_{\prod_{\phi(t)=0} V} \prod_{\phi(t)=0} \widetilde{V}$$

$$\overline{Y}_\phi = Y_\phi \times_{\prod_{\phi(t)=0} V} \prod_{\phi(t)=0} \overline{V}$$

Since the evident projection map $\overline{Y}_\phi \to Y_\phi$ is an étale surjection, it will suffice to show that the ℓ-adic sheaf $(i_\phi^! \mathcal{B}_{\mathrm{red}}^{(n)})|_{\overline{Y}_\phi}$ belongs to $\mathrm{Shv}_\ell(\overline{Y}_\phi)_{\leq d}$. The map $\widetilde{Y}_\phi \to \overline{Y}_\phi$ is surjective, finite, and radicial, and therefore induces a t-exact equivalence of ∞-categories $\mathrm{Shv}_\ell(\overline{Y}_\phi) \to \mathrm{Shv}_\ell(\widetilde{Y}_\phi)$. We are therefore reduced to showing that the restriction $(i_\phi^! \mathcal{B}_{\mathrm{red}}^{(n)})|_{\widetilde{Y}_\phi}$ belongs to $\mathrm{Shv}_\ell(\widetilde{Y}_\phi)_{\leq d}$

For each subset $T_0 \subseteq T$, let let $\mathcal{F}(T_0)$ denote the ℓ-adic sheaf

$$[(\{(T_0 \subseteq T)\} \times_{\Theta_\blacksquare} \mathrm{Ran}^G_{\mathrm{aug}}(X)) \times_{X^T} \widetilde{Y}_\phi]_{\widetilde{Y}_\phi} \mathrm{Shv}_\ell(\widetilde{Y}_\phi).$$

Since each $\{(T_0 \subseteq T)\} \times_{\Theta_\blacksquare} \mathrm{Ran}^G_{\mathrm{aug}}(X))$ is an Artin stack which is smooth over X^T, Proposition 5.1.9 supplies an identification

$$(i_\phi^! \mathcal{B}_{\mathrm{red}}^{(n)})|_{\widetilde{Y}_\phi} \simeq \mathrm{fib}(\mathcal{F}(\emptyset) \to \varprojlim_{\emptyset \neq T_0 \subseteq T} \mathcal{F}(T_0)).$$

For each $T_0 \subseteq T$, let $Z(T_0)$ denote the prestack given by the product

$$\prod_{t \in T - T_0} \begin{cases} \mathrm{BG}_{x_{\phi(t)}} & \text{if } \phi(t) > 0 \\ \mathrm{BH} & \text{if } \phi(t) = 0. \end{cases}$$

By construction, each fiber product

$$\mathrm{Ran}^G(X)_{T_0 \subseteq T \subseteq T} \times_{X^T} \widetilde{Y}_\phi$$

is equivalent to a product $\widetilde{Y}_\phi \times_{\mathrm{Spec}\, k} Z(T_0)$. Invoking Proposition 5.1.9 again, we can identify each $\mathcal{F}(T_0)$ with the tensor product $C^*(Z(T_0); \mathbf{Z}_\ell) \otimes \omega_{\widetilde{Y}_\phi}$, so that $(i_\phi^! \mathcal{B}_{\mathrm{red}}^{(n)})|_{\widetilde{Y}_\phi}$ is the tensor product of $\omega_{\widetilde{Y}_\phi}$ with the fiber

$$K = \mathrm{fib}(C^*(Z(\emptyset); \mathbf{Z}_\ell) \to \varprojlim_{\emptyset \neq T_0 \subseteq T} C^*(Z(T_0); \mathbf{Z}_\ell)).$$

Let $e = |\phi^{-1}\{0\}|$, so that $\widetilde{Y}_\phi$ is a smooth k-scheme of dimension e and therefore $\omega_{\widetilde{Y}_\phi} \simeq \Sigma^{2e} \mathbf{Z}_\ell(e)$. To prove that $(i_\phi^! \mathcal{B}_{\mathrm{red}}^{(n)})|_{\widetilde{Y}_\phi} \simeq K \otimes \omega_{\widetilde{Y}_\phi}$ belongs to $\mathrm{Shv}_\ell(\widetilde{Y}_\phi)_{\leq d}$, it will suffice to show that K belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq d-2e}$.

Note that K can be identified with the $\mathbf{Z}_\ell$-linear dual of the cofiber

$$K' = \mathrm{cofib}(\varinjlim_{\emptyset \neq T_0 \subseteq T} C_*(Z(T_0); \mathbf{Z}_\ell) \to C_*(Z(\emptyset); \mathbf{Z}_\ell)).$$

It will therefore suffice to show that K' belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 2e-d}$. Using the Künneth formula, we obtain an equivalence

$$K' \simeq \bigotimes_{t \in T} \begin{cases} C_*^{\mathrm{red}}(\mathrm{BG}_{x_{\phi(t)}}; \mathbf{Z}_\ell) & \text{if } \phi(t) > 0 \\ C_*^{\mathrm{red}}(\mathrm{BH}; \mathbf{Z}_\ell) & \text{if } \phi(t) = 0. \end{cases}$$

Note that each tensor factor belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 2}$ (Lemma 8.3.11), so that we automatically have $K' \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 2n} \subseteq (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 2e}$. If the generic fiber of G is semisimple and simply connected, then Lemma 8.3.11 implies that $C^{\mathrm{red}}_*(\mathrm{BH}; \mathbf{Z}_\ell)$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 4}$, so that

$$K' \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 4e+2(n-e)} = (\mathrm{Mod}_{\mathbf{Z}_\ell})_{2e+2n},$$

as desired. □

8.4. **Normalization.** Let k be a field, and let A be an associative algebra over k equipped with an augmentation $\epsilon : A \to k$ and augmentation ideal $\mathfrak{m} = \ker(\epsilon)$. Using ϵ, we can regard k as either a right or a left module over A. The Tor-groups $\mathrm{Tor}^A_*(k, k)$ can be computed by the bar complex

$$\cdots \to A \otimes_k A \otimes_k A \xrightarrow{d} A \otimes_k A \xrightarrow{d} A \xrightarrow{d} k,$$

where the differential d carries a tensor product $a_1 \otimes \cdots \otimes a_n$ to the sum

$$\epsilon(a_1)a_2 \otimes \cdots \otimes a_n + (\sum_{0<i<n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n \epsilon(a_n) a_1 \otimes \cdots \otimes a_{n-1}.$$

However, it is often more convenient to work with the *reduced* bar complex: that is, the subcomplex

$$\cdots \to \mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m} \xrightarrow{d} \mathfrak{m} \otimes \mathfrak{m} \xrightarrow{d} \mathfrak{m} \xrightarrow{d} k,$$

whose differential can be written more simply as

$$d(a_1 \otimes \cdots \otimes a_n) = \sum_{0<i<n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

One can show that this subcomplex is quasi-isomorphic to the entire complex, and can therefore also be used to compute the groups $\mathrm{Tor}^A_*(k, k)$.

The chain complexes

$$C^*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell) \simeq \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathrm{Ran}_G(X)_S; \mathbf{Z}_\ell)$$

$$\int \mathcal{B} \simeq \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} C^*(X^T; \mathcal{B}^{(T)})$$

studied in §3 and §5 can be viewed as loosely analogous to the bar complex of an associative algebra. Our goal in this section is to describe a systematic procedure for selecting "reduced" versions of these constructions which give (essentially) the same result. Theorem 8.1.12 provides a basic prototype for what such a statement might look like. We are therefore led to ask the following:

Question 8.4.1. Let $\Theta_{\blacksquare}$ be the category described in Notation 8.1.6, let $\mathcal{C}$ be a stable ∞-category which admits limits, and let $V : \Theta_{\blacksquare} \to \mathcal{C}$ be a functor. Is the canonical map

$$\varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} V(S \subseteq S) \to \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \varinjlim_{\emptyset \neq S_0 \subseteq S} V(S_0 \subseteq S)$$

is an equivalence in $\mathcal{C}$?

The answer to Question 8.4.1 is negative in general. However, there are combinatorial conditions on V which guarantee a positive answer which can be verified in many important special cases, including the case of the functor

$$(S_0 \subseteq S) \mapsto C^*(\mathrm{Ran}_G(X - D)_{S_0 \subseteq S}; \mathbf{Z}_\ell)$$

studied in §8.1. To formulate these conditions, we need to introduce a bit of notation.

Notation 8.4.2. We define a category Θ as follows:

- The objects of Θ are triples $(S_0 \subseteq S_1 \subseteq S)$, where S is a nonempty finite set, S_1 is a (possibly empty) subset of S, and S_0 is a (possibly empty) subset of S_1.
- A morphism from $(S_0 \subseteq S_1 \subseteq S)$ to $(S'_0 \subseteq S'_1 \subseteq S')$ in Θ is a map of finite sets $\alpha : S \to S'$ such that $S'_0 \subseteq \alpha(S_0)$ and $S'_1 \subseteq \alpha(S_1)$.

The construction $(S_0 \subseteq S) \mapsto (S_0 \subseteq S \subseteq S)$ determines a faithful functor $\iota : \Theta_{\blacksquare} \to \Theta$, where $\Theta_{\blacksquare}$ is the category introduced in Notation 8.1.6. In what follows, we will generally abuse notation by identifying $\Theta_{\blacksquare}$ with its image in Θ.

Notation 8.4.3. Let $\mathcal{C}$ be a stable ∞-category and let $V : \Theta \to \mathcal{C}$ be a functor. We define

$$V_{\blacksquare} = W_{\blacksquare} \qquad V_{\square} = W_{\square} \qquad V_{\mathrm{red}} = W_{\mathrm{red}},$$

where $W = V|_{\Theta_{\blacksquare}}$ is the functor obtained by restricting V to the subcategory $\Theta_{\blacksquare} \subseteq \Theta$ (see Construction 8.1.7).

Notation 8.4.4. Let S be a finite set. We let $\mathrm{Equiv}(S)$ denote the set of all equivalence relations on S, partially ordered by refinement (so that $E \leq E'$ if xEy implies $xE'y$). If E is an equivalence relation on S, we let S/E denote the quotient of S by E. For each subset $S' \subseteq S$, we let S'/E denote the image of S' in the quotient S/E. We let $\mathrm{Equiv}^{\circ}(S)$ denote the subset of $\mathrm{Equiv}(S)$ consisting of non-discrete equivalence relations (that is, equivalence relations E such that S/E is smaller than S).

Definition 8.4.5. Let $\mathcal{C}$ be an ∞-category which admits limits, and let $V : \Theta \to \mathcal{C}$ be a functor. We will say that V is *unital* if it satisfies the following pair of conditions:

$(U1)$ Let S be a nonempty finite set and let $S_1 \subseteq S$. Then the canonical map

$$V(\emptyset \subseteq S_1 \subseteq S) \to \varprojlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} V(\emptyset \subseteq S_1 \subseteq S \amalg S')$$

is an equivalence in $\mathcal{C}$.

$(U2)$ Let $(S_0 \subseteq S_1 \subseteq S)$ be an object of Θ, let $s \in S$ be an element which is not contained in S_1, and set $S'_0 = S_0 \cup \{s\}$, $S'_1 = S_1 \cup \{s\}$. Then the diagram

$$\begin{array}{ccc} V(S'_0 \subseteq S'_1 \subseteq S) & \longrightarrow & V(S_0 \subseteq S_1 \subseteq S) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} V(S'_0/E \subseteq S'_1/E \subseteq S/E) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} V(S_0/E \subseteq S_1/E \subseteq S/E) \end{array}$$

is a pullback square in $\mathcal{C}$.

Example 8.4.6. Let X, G, and D be as in Theorem 8.1.11. Then the construction

$$(S_0 \subseteq S_1 \subseteq S) \mapsto C^*(\mathrm{Ran}_G(X-D)_{S_0 \subseteq S_1} \times_{(X-D)^{S_1}} (X-D)^S$$

determines a functor $V : \Theta \to \mathrm{Mod}_{\mathbf{Z}_\ell}$. In §8.5, we will show that V satisfies conditions $(U1)$ and $(U2)$ of Definition 8.4.5 (Propositions 8.5.3 and 8.5.4).

Example 8.4.7. Let X, G, and $\mathcal{B}_{\mathrm{aug}}$ be as in Theorem 8.2.14. Then the construction

$$(T_0 \subseteq T_1 \subseteq T) \mapsto C^*(X^T; \pi^![X^{T_0} \times_{\mathrm{Ran}^G(X)^{(T_0)}} \mathrm{Ran}^G(X)^{(T_1)}]_{X^{T_1}})$$

determines a functor $V : \Theta \to \mathrm{Mod}_{\mathbf{Z}_\ell}^{\mathrm{op}}$. In §8.6, we will show that this functor satisfies conditions $(U1)$ and $(U2)$ of Definition 8.4.5 (Propositions 8.6.6 and 8.6.7).

Remark 8.4.8. In Examples 8.4.6 and 8.4.7 (and other examples we will encounter later in this paper), the verification of conditions $(U1)$ reduces to the acyclicity of the Ran space $\mathrm{Ran}(X)$.

We can now state our main result.

Theorem 8.4.9. *Let $\mathcal{C}$ be a stable ∞-category which admits limits and let $V : \Theta_{\blacksquare} \to \mathcal{C}$ be a functor. Suppose that V can be extended to a unital functor $\overline{V} : \Theta \to \mathcal{C}$. Then the canonical map*

$$\varprojlim_{S \in \mathrm{Fin}^s} V(S \subseteq S) \to \varprojlim_{S \in \mathrm{Fin}^s} \varinjlim_{\emptyset \neq S_0 \subseteq S} V(S_0 \subseteq S)$$

is an equivalence in $\mathcal{C}$.

The main ingredient in our proof of Theorem 8.4.9 is the following result, whose proof we defer for the moment:

Theorem 8.4.10. *Let $\mathcal{C}$ be a stable ∞-category which admits limits and let $V : \Theta \to \mathcal{C}$ be a unital functor. Then the canonical maps*

$$\varprojlim_{S \in \mathrm{Fin}^s} V(\emptyset \subseteq \emptyset \subseteq S) \leftarrow \varprojlim_{S \in \mathrm{Fin}^s} V_{\blacksquare}(S) \to \varprojlim_{S \in \mathrm{Fin}^s} V_{red}(S)$$

exhibit $\varprojlim_{S \in \mathrm{Fin}^s} V_{\blacksquare}(S)$ as a product of $\varprojlim_{S \in \mathrm{Fin}^s} V(\emptyset \subseteq \emptyset \subseteq S)$ with $\varprojlim_{S \in \mathrm{Fin}^s} V_{red}(S)$ in the ∞-category $\mathcal{C}$ (see Notation 8.4.3).

Lemma 8.4.11. *Let $\mathcal{C}$ be an ∞-category which admits limits and let $V : \Theta \to \mathcal{C}$ be a unital functor. Then, for every nonempty finite set S, the canonical map*

$$V(S \subseteq S \subseteq S) \to V(\emptyset \subseteq \emptyset \subseteq S)$$

is an equivalence in $\mathcal{C}$.

Proof. We show more generally that for $S_0 \subseteq S_0' \subseteq S$, the canonical map

$$V(S_0' \subseteq S_0' \subseteq S) \to V(S_0 \subseteq S_0 \subseteq S)$$

is an equivalence. The proof proceeds by induction on the cardinality of S. Without loss of generality, we may assume that S_0' is obtained from S_0 by adjoining a single element. In this case, condition $(U2)$ of Definition 8.4.5 guarantees that we have a pullback diagram

$$\begin{array}{ccc} V(S_0' \subseteq S_0' \subseteq S) & \longrightarrow & V(S_0 \subseteq S_0 \subseteq S) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} V(S_0'/E \subseteq S_0'/E \subseteq S/E) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} V(S_0/E \subseteq S_0/E \subseteq S/E). \end{array}$$

It follows from the inductive hypothesis that the bottom horizontal map is an equivalence in $\mathcal{C}$, so that the upper horizontal map is an equivalence as well. □

Proof of Theorem 8.4.9. Let $V : \Theta_{\blacksquare} \to \mathcal{C}$ be a functor which extends to a unital functor $\overline{V} : \Theta \to \mathcal{C}$. It follows from Lemma 8.4.11 that the composite map

$$\varprojlim_{S \in \mathrm{Fin}^s} V(S \subseteq S) \xrightarrow{\alpha} \varprojlim_{S \in \mathrm{Fin}^s} V_{\square}(S) \xrightarrow{\beta} \varprojlim_{S \in \mathrm{Fin}^s} \overline{V}(\emptyset \subseteq \emptyset \subseteq S)$$

is an equivalence. Since β is an equivalence by virtue of Theorem 8.4.10, it follows that α is an equivalence as desired. □

The proof of Theorem 8.4.10 will require a number of preliminaries.

Notation 8.4.12. We let Θ° denote the full subcategory of Θ spanned by those objects $(S_0 \subseteq S_1 \subseteq S)$ where $S_0 = \emptyset$. We will think of objects of Θ° as pairs $(S_1 \subseteq S)$. Let Θ^s denote the subcategory of Θ° containing all objects, where a morphism from $(S_1 \subseteq S)$ to $(S'_1 \subseteq S')$ in Θ^s is a *surjective* map $\alpha : S \to S'$ such that $S'_1 \subseteq \alpha(S_1)$. We will identify $\mathrm{Fin}^{\mathrm{s}}$ with the full subcategory of Θ^s spanned by those objects $(S_1 \subseteq S)$ where $S_1 = S$.

Let $\mathcal{C}$ be an ∞-category which admits limits. We will say that a functor $V : \Theta^\circ \to \mathcal{C}$ is *weakly unital* it is satisfies the following condition:

($*$) Let S be a nonempty finite set and let $S_1 \subseteq S$. Then the canonical map
$$V(S_1 \subseteq S) \to \varprojlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} V(S_1 \subseteq S \amalg S')$$
is an equivalence in $\mathcal{C}$.

Lemma 8.4.13. *Let $\mathcal{C}$ be an ∞-category which admits limits and let $V : \Theta^\circ \to \mathcal{C}$ be a weakly unital functor. Let $\iota : \Theta^s \hookrightarrow \Theta^\circ$ be the inclusion map. Then V is a right Kan extension of $V|_{\Theta^s}$ along ι.*

We will defer the proof of Lemma 8.4.13 until the end of this section.

Proposition 8.4.14. *Let $\mathcal{C}$ be an ∞-category which admits limits and let $V : \Theta^\circ \to \mathcal{C}$ be a weakly unital functor. Then the*
$$\varprojlim(V) \to \varprojlim(V|_{\Theta^s}) \to \varprojlim(V|_{\mathrm{Fin}^{\mathrm{s}}})$$
are equivalences in $\mathcal{C}$.

Proof. The assertion that the map $\varprojlim(V) \to \varprojlim(V|_{\Theta^s})$ is an equivalence, follows immediately from Lemma 8.4.13. To show that the restriction map $\varprojlim(V|_{\Theta^s}) \to \varprojlim(V|_{\mathrm{Fin}^{\mathrm{s}}})$ is an equivalence, it suffices to show that the inclusion $j : \mathrm{Fin}^{\mathrm{s}} \hookrightarrow \Theta^s$ is right cofinal. This is clear, since j admits a right adjoint, given on objects by $(S_1 \subseteq S) \mapsto S$. □

Notation 8.4.15. For each integer $n \geq 0$, let $\langle n \rangle$ denote the finite set $\{1, 2, \ldots, n\}$, and let $\mathrm{Fin}^s_{\leq n}$ denote the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by those sets S having cardinality $\leq n$.

Proposition 8.4.16. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $V : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$ be a functor. For $n > 0$, the diagram*
$$\begin{array}{ccc} \varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} V(S) & \longrightarrow & \varprojlim_{S \in \mathrm{Fin}^s_{\leq n-1}} V(S) \\ \downarrow & & \downarrow \\ V(\langle n \rangle)^{\Sigma_n} & \longrightarrow & (\varprojlim_{E \in \mathrm{Equiv}^\circ(\langle n \rangle)} V(\langle n \rangle / E))^{\Sigma_n} \end{array}$$
is a pullback diagram in $\mathcal{C}$.

We will need a slightly fancier version of Proposition 8.4.16, which applies to diagrams indexed by the category Θ°.

Notation 8.4.17. Let $n \geq 0$ be an integer. We let $\Theta^\circ_{\leq n}$ denote the full subcategory of Θ° spanned by those objects $(S_1 \subseteq S)$ where S_1 has cardinality $\leq n$.

Let $\mathcal{T}(n)$ denote the partially ordered set of pairs (E, S_1), where $E \in \mathrm{Equiv}(\langle n \rangle)$ and S_1 is a subset of $\langle n \rangle / E$ having cardinality $< n$ (we set $(E, S_1) \leq (E', S'_1)$ if $E \leq E'$ and $S'_1 \subseteq S_1/E$). The partially ordered set $\mathcal{T}(n)$ carries an action of the symmetric group Σ_n, and the construction $(E, S_1) \mapsto (S_1 \subseteq \langle n \rangle / E)$ determines a Σ_n-equivariant functor from $\mathcal{T}(n)$ to $\Theta^\circ_{\leq n-1}$.

Proposition 8.4.18. *Let $\mathcal{C}$ be an ∞-category which admits limits, and let $V : \Theta^\circ \to \mathcal{C}$ be a functor. For each integer $n > 0$, the diagram*

$$\begin{array}{ccc} \varprojlim(V|_{\Theta^\circ_{\leq n}}) & \longrightarrow & \varprojlim(V|_{\Theta^\circ_{\leq n-1}}) \\ \downarrow & & \downarrow \\ V(\langle n \rangle \subseteq \langle n \rangle)^{\Sigma_n} & \longrightarrow & \varprojlim(V|_{\mathcal{J}(n)})^{\Sigma_n} \end{array}$$

is a pullback square in $\mathcal{C}$.

We defer the proofs of Propositions 8.4.16 and 8.4.18 until the end of this section.

Remark 8.4.19. Let $\mathcal{C}$ be a stable ∞-category, and suppose we are given a diagram

$$\begin{array}{ccccc} & & X' & & \\ & & \downarrow{\scriptstyle f} & & \\ Y' & \xrightarrow{f'} & X & \xrightarrow{g'} & Y'' \\ & & \downarrow{\scriptstyle g} & & \\ & & X'' & & \end{array}$$

where the row and the column are fiber sequences. Then $g' \circ f$ is an equivalence if and only if $g \circ f'$ is an equivalence. Indeed, both of these conditions are equivalent to the requirement that f and f' induce an equivalence $X' \oplus Y' \to X$.

Lemma 8.4.20. *Let $\mathcal{C}$ be a stable ∞-category, let $n \geq 0$ be an integer, and let P denote the partially ordered set of pairs (S_0, S_1), where S_0 and S_1 are subsets of $\langle n \rangle$ satisfying $S_0 \subseteq S_1$. Suppose we are given a functor $U : P^{\text{op}} \to \mathcal{C}$ satisfying the following condition:*

- $(*)$ *Let $(S_0, S_1) \in P$, and suppose that $s \in \langle n \rangle - S_1$. Then the canonical map $U(S_0 \cup \{s\}, S_1 \cup \{s\}) \to U(S_0, S_1)$ is an equivalence.*

Then the composite map

$$\varinjlim_{\emptyset \neq S_0 \subseteq \langle n \rangle} U(S_0, \langle n \rangle) \to U(\emptyset, \langle n \rangle) \to \varprojlim_{S_1 \subsetneq \langle n \rangle} U(\emptyset, S_1).$$

is an equivalence.

Proof of Theorem 8.4.10. For each $n \geq 0$, let $W(n)$ denote the inverse limit

$$\varprojlim_{(S_1 \subseteq S) \in \Theta^\circ_{\leq n}} V(\emptyset \subseteq S_1 \subseteq S),$$

so that we have a tower of objects

$$\cdots \to W(2) \to W(1) \to W(0)$$

of the ∞-category $\mathcal{C}$. Let $W(\infty) = \varprojlim_n W(n)$, so that we have a commutative diagram

$$\begin{array}{ccc} W(\infty) & \longrightarrow & W(0) \\ \downarrow & & \downarrow \\ \varprojlim_{S \in \text{Fin}^s} V_{\blacksquare}(S) & \longrightarrow & \varprojlim_{S \in \text{Fin}^s} V(\emptyset \subseteq \emptyset \subseteq S) \end{array}$$

Proposition 8.4.14 implies that the left vertical map is an equivalence. Lemma 8.4.13 shows that the restriction of V to $\Theta^\circ_{\leq 0}$ is a right Kan extension of its restriction $\Theta^\circ_{\leq 0} \cap \Theta^s$, so that the right vertical map is also an equivalence. It will therefore suffice to show that the canonical map

$$\theta : W(\infty) \to W(0) \times \varprojlim_{S \in \mathrm{Fin}^s} V_{\mathrm{red}}(S)$$

is an equivalence in $\mathcal{C}$.

Unwinding the definitions, we see that θ is the limit of a tower of maps

$$\theta_n : W(n) \to W(0) \times \varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} V_{\mathrm{red}}(S).$$

It will therefore suffice to show that each of the maps θ_n is an equivalence. We proceed by induction on n. If $n = 0$, then $\mathrm{Fin}^s_{\leq 0}$ is empty and the result is obvious. To handle the inductive step, it will suffice to show that for $n > 0$, the upper square in the diagram

$$\begin{array}{ccc}
W(n) & \longrightarrow & W(n-1) \\
\downarrow & & \downarrow \\
\varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} V_{\mathrm{red}}(S) & \longrightarrow & \varprojlim_{S \in \mathrm{Fin}^s_{\leq n-1}} V_{\mathrm{red}}(S) \\
\downarrow & & \downarrow \\
V_{\mathrm{red}}(\langle n \rangle)^{\Sigma_n} & \longrightarrow & (\varprojlim_{E \in \mathrm{Equiv}^{\circ}(\langle n \rangle)} V_{\mathrm{red}}(\langle n \rangle / E))^{\Sigma_n}
\end{array}$$

is a pullback square. The lower square is a pullback by virtue of Proposition 8.4.16. It will therefore suffice to prove that the outer rectangle is a pullback square.

Let $\mathcal{T}(n)$ be defined as in Notation 8.4.17 so that we have a commutative diagram

$$\begin{array}{ccc}
W(n) & \longrightarrow & W(n-1) \\
\downarrow & & \downarrow \\
V_{\blacksquare}(\langle n \rangle)^{\Sigma_n} & \longrightarrow & (\varprojlim V|_{\mathcal{T}(n)})^{\Sigma_n} \\
\downarrow & & \downarrow \\
V_{\mathrm{red}}(\langle n \rangle)^{\Sigma_n} & \longrightarrow & (\varprojlim_{E \in \mathrm{Equiv}^{\circ}(\langle n \rangle)} V_{\mathrm{red}}(\langle n \rangle / E))^{\Sigma_n}.
\end{array}$$

The upper square is a pullback diagram by Proposition 8.4.18. We are therefore reduced to proving that the lower square is a pullback diagram. Since the collection of pullback diagrams in $\mathcal{C}$ is closed under limits, it will suffice to show that the diagram

$$\begin{array}{ccc}
V_{\blacksquare}(\langle n \rangle) & \xrightarrow{\phi} & (\varprojlim V|_{\mathcal{T}(n)}) \\
\downarrow & & \downarrow \\
V_{\mathrm{red}}(\langle n \rangle) & \xrightarrow{\psi} & \varprojlim_{E \in \mathrm{Equiv}^{\circ}(\langle n \rangle)} V_{\mathrm{red}}(\langle n \rangle / E)
\end{array}$$

is a pullback square in $\mathcal{C}$. Equivalently, we wish to show that the canonical map $\mathrm{fib}(\phi) \to \mathrm{fib}(\psi)$ is an equivalence.

Let $\mathcal{T}'(n)$ denote the full subcategory of $\mathcal{T}(n)$ spanned by those pairs (E, S_1) where $E \in \mathrm{Equiv}^{\circ}(\langle n \rangle)$, and let $\phi' : \varprojlim(V|_{\mathcal{T}(n)}) \to \varprojlim(V|_{\mathcal{T}'(n)})$ denote the canonical map. Note that there is a right cofinal map $\mathrm{Equiv}^{\circ}(\langle n \rangle) \to \mathcal{T}'(n)$, given by $E \mapsto (E, \langle n \rangle / E)$. We may therefore identify $\phi' \circ \phi$ with the canonical map $V_{\blacksquare}(\langle n\rangle) \to \varprojlim_{E \in \mathrm{Equiv}^\circ(\langle n\rangle)} V_{\blacksquare}(\langle n\rangle/E)$. We have a commutative diagram σ :

$$\begin{array}{ccccc} & & \mathrm{fib}(\phi) & & \\ & & \downarrow & & \\ \mathrm{fib}(\mu) & \longrightarrow & \mathrm{fib}(\phi' \circ \phi) & \longrightarrow & \mathrm{fib}(\psi) \\ & & \downarrow & & \\ & & \mathrm{fib}(\phi'), & & \end{array}$$

where μ denotes the canonical map $V_{\square}(\langle n\rangle) \to \varprojlim_{E \in \mathrm{Equiv}^\circ(\langle n\rangle)} V_{\square}(\langle n\rangle/E)$, where the row and column of σ are fiber sequences. Using Remark 8.4.19, we are reduced to proving that the composite map

$$\xi : \mathrm{fib}(\mu) \to \mathrm{fib}(\phi' \circ \phi) \to \mathrm{fib}(\phi')$$

is an equivalence.

To identify the fiber of ϕ', let $V'' : \mathcal{T}(n) \to \mathcal{C}$ denote a right Kan extension of $V|_{\mathcal{T}'(n)}$ along the inclusion $\mathcal{T}'(n) \hookrightarrow \mathcal{T}(n)$, and let $V' : \mathcal{T}'(n) \to \mathcal{C}$ denote the fiber of the canonical map $V|_{\mathcal{T}(n)} \to V''$. Then $\mathrm{fib}(\phi')$ can be identified with the limit of the diagram V'. Note that V' vanishes on $\mathcal{T}'(n)$, so that V' is a right Kan extension of its restriction to the subset of $\mathcal{T}(n)$ spanned by those objects (E_0, S_1) where E_0 denotes the discrete equivalence relation on $\langle n\rangle$ (this subset is equivalent to the opposite of the partially ordered set of subsets $S_1 \subseteq \langle n\rangle$). Moreover, for such an object (E_0, S_1), the category $\mathcal{D} = \mathcal{T}'(n) \times_{\mathcal{T}(n)} \mathcal{T}(n)_{(E_0,S_1)/}$ contains a full subcategory $\mathcal{D}_0$ consisting of those maps $(E_0, S_1) \to (E', S_1')$ where S_1' is the image of S_1, and the inclusion of this full subcategory is right cofinal. It follows that V'' is given by the formula $V''(E_0, S_1) = \varprojlim_{E \in \mathrm{Equiv}^\circ(\langle n\rangle)} V(\emptyset \subseteq S_1/E \subseteq \langle n\rangle/E)$, so that $V'(E_0, S_1)$ is the fiber of the canonical map $V(\emptyset S_1 \subseteq \langle n\rangle) \to \varprojlim_{E \in \mathrm{Equiv}^\circ(\langle n\rangle)} V(\emptyset \subseteq S_1/E \subseteq \langle n\rangle/E)$. Passing to the limit, we can identify $\mathrm{fib}(\phi')$ with the fiber of the canonical map

$$\varprojlim_{S_1 \subsetneq \langle n\rangle} V(\emptyset \subseteq S_1 \subseteq \langle n\rangle) \to \varprojlim_{S_1 \subsetneq \langle n\rangle} \varprojlim_{E \in \mathrm{Equiv}^\circ(\langle n\rangle)} V(\emptyset \subseteq S_1/E \subseteq \langle n\rangle/E).$$

Let P be as in Lemma 8.4.20. For each $(S_0, S_1) \in P$, we let $U(S_0, S_1)$ denote the fiber of the canonical map

$$V(S_0 \subseteq S_1 \subseteq \langle n\rangle) \to \varprojlim_{E \in \mathrm{Equiv}^\circ(\langle n\rangle)} V(S_0/E \subseteq S_1/E \subseteq \langle n\rangle/E).$$

Unwinding the definitions, we see that ξ is given by the composition

$$\varinjlim_{\emptyset \neq S_0 \subseteq \langle n\rangle} U(S_0, \langle n\rangle) \to U(\emptyset, \langle n\rangle) \to \varprojlim_{S_1 \subsetneq \langle n\rangle} U(\emptyset, S_1).$$

By virtue of Lemma 8.4.20, it will suffice to show that for $S_0 \subseteq S_1 \subseteq \langle n\rangle$ and $s \in \langle n\rangle - S_1$, the canonical map $U(S_0 \cup \{s\}, S_1 \cup \{s\}) \to U(S_0, S_1)$ is an equivalence, which follows immediately from Condition $(U2)$ of Definition 8.4.5. □

We now turn the proofs of Lemma 8.4.13, Lemma 8.4.20, Proposition 8.4.16, and Proposition 8.4.18.

Lemma 8.4.21. *Let $\mathcal{C}$ be an ∞-category which admits limits and let S be a nonempty finite set. Let $\mathcal{J}$ be the category whose objects are pairs $(\overline{S}, \beta)$, where $\overline{S}$ is a finite set and $\beta : S \to \overline{S}$ is an injection; a morphism from $(\overline{S}, \beta)$ to $(\overline{S}', \beta')$ in $\mathcal{J}$ is a surjection of finite sets $\alpha : \overline{S} \to \overline{S}'$*

such that $\beta' = \alpha \circ \beta$. If $V : \Theta^{\circ} \to \mathcal{C}$ is weakly unital, then for every subset $S_1 \subseteq S$ the canonical map

$$\theta : V(S_1 \subseteq S) \to \varprojlim_{(\overline{S}, \beta) \in \mathcal{J}} V(\beta(S_1) \subseteq \overline{S})$$

is an equivalence in $\mathcal{C}$.

Proof. We have a commutative diagram

$$\begin{array}{ccc}
V(S_1 \subseteq S) & \xrightarrow{\theta} & \varprojlim_{(\overline{S},\beta)\in\mathcal{J}} V(\beta(S_1) \subseteq \overline{S}) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
V(S_1 \subseteq S) & \xrightarrow{\theta'} & \varprojlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} V(S_1 \subseteq S \amalg S') \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
V(S_1 \subseteq S) & \longrightarrow & \varprojlim_{(\overline{S},\beta)\in\mathcal{J}} V(S_1 \subseteq S \amalg \overline{S}) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
V(S_1 \subseteq S) & \xrightarrow{\theta} & \varprojlim_{(\overline{S},\beta)\in\mathcal{J}} V(\beta(S_1) \subseteq \overline{S})
\end{array}$$

where the composite of the left vertical maps is an equivalence. It follows that θ is a retract of θ', which is an equivalence by virtue of our assumption that V is weakly unital. $\square$

Proof of Lemma 8.4.13. We must show that for every object $(S_1 \subseteq S) \in \Theta^{\circ}$, the canonical map

$$V(S_1 \subseteq S) \to \varprojlim_{(S_1' \subseteq S') \in \mathcal{D}} V(S_1' \subseteq S')$$

is an equivalence, where $\mathcal{D}$ denotes the fiber product $\Theta^s \times_{\Theta^{\circ}} (\Theta^{\circ})_{(S_1 \subseteq S)/}$. Let $\mathcal{D}_0$ denote the full subcategory of $\mathcal{D}$ spanned by those maps $(S_1 \subseteq S) \to (S_1' \subseteq S')$ for which the map $S \to S'$ is injective, and S_1' is the image of S_1. Then $\mathcal{D}_0$ is equivalent to the category $\mathcal{J}$ appearing in Lemma 8.4.21. Since V is weakly unital Lemma 8.4.21 implies that the canonical map

$$V(S_1 \subseteq S) \to \varprojlim_{(S_1' \subseteq S') \in \mathcal{D}_0} V(S_1' \subseteq S')$$

is an equivalence. It will therefore suffice to show that the inclusion $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ is right cofinal. To this end, choose an object $D \in \mathcal{D}$, given by a morphism $\alpha : (S_1 \subseteq S) \to (S_1'' \subseteq S'')$ in Θ°. We wish to prove that the fiber product $\mathcal{E} = \mathcal{D}_{/D} \times_{\mathcal{D}} \mathcal{D}_0$ has weakly contractible nerve. Unwinding the definitions, we can identify $\mathcal{E}$ with the category whose objects are factorizations

$$S \xrightarrow{\alpha'} S' \xrightarrow{\alpha''} S''$$

of α, where α' is injective and α'' is surjective; the morphisms in $\mathcal{E}$ are given by commutative diagrams

$$\begin{array}{ccccc}
 & & S_0' & & \\
 & {\scriptstyle \alpha_0'} \nearrow & & \searrow {\scriptstyle \alpha_0''} & \\
S & & \downarrow{\scriptstyle \rho} & & S'' \\
 & {\scriptstyle \alpha_1'} \searrow & & \nearrow {\scriptstyle \alpha_1''} & \\
 & & S_1' & &
\end{array}$$

where ρ is surjective.

For each $x \in S''$, let S_x denote the inverse image $\alpha^{-1}\{s\}$. Then $\mathcal{E}$ is equivalent to the product $\prod_{x \in S''} \mathcal{E}_x$, where $\mathcal{E}_x$ is the category whose objects are nonempty finite sets T equipped with an injective map $\alpha'_x : S_x \to T$, and whose morphisms are surjective maps. We are therefore reduced to proving that each of the categories $\mathcal{E}_x$ has weakly contractible nerve. We consider two cases:

(a) The set S_x is empty. In this case, $\mathcal{E}_x$ has a final object (given by taking T to consist of a single element).

(b) The set S_x is nonempty. In this case, we argue as in the proof of Theorem 2.4.5. Note first that the simplicial set $\mathrm{N}(\mathcal{E}_x)$ is connected (since every object $T \in \mathcal{E}_x$ admits a map $T \to S_x$). We can equip the category $\mathcal{E}_x$ with a monoidal structure $\star$, given by $T \star T' = T \amalg_{S_x} T'$. The unit object of $\mathcal{E}_x$ determines a base point $q \in \mathrm{N}(\mathcal{E}_x)$.

This monoidal structure induces a map $m : \mathrm{N}(\mathcal{E}_x) \times \mathrm{N}(\mathcal{E}_x) \to \mathrm{N}(\mathcal{E}_x)$. For each $T \in \mathcal{E}_x$, we have an natural map $T \star T \to T$, which determines a homotopy from the composite map

$$\mathrm{N}(\mathcal{E}_x) \xrightarrow{\delta} \mathrm{N}(\mathcal{E}_x) \times \mathrm{N}(\mathcal{E}_x) \xrightarrow{m} \mathrm{N}(\mathcal{E}_x)$$

to the identity. It follows that $g^2 = g$ for each $g \in \pi_i(\mathrm{N}(\mathcal{E}_x), q)$, so that the group $\pi_i(\mathrm{N}(\mathcal{E}_x), q)$ is trivial.

□

Proof of Lemma 8.4.20. We proceed by induction on n, the case $n = 0$ being trivial. Define a functor $U' : P^{\mathrm{op}} \to \mathcal{C}$ by the formula $U'(S_0, S_1) = U(S_0 \cup \{n\}, S_1 \cup \{n\})$, and let $U'' : P^{\mathrm{op}} \to \mathcal{C}$ be the cofiber of the canonical map $U' \to U$. We have a fiber sequence of functors

$$U' \to U \to U''.$$

It will therefore suffice to prove the analogous statements for the functors U' and U'':

- Let P' denote the subset of P consisting of those pairs (S_0, S_1) where $S_1 = \langle n \rangle$ and $S_0 \neq \emptyset$, and let $P'' \subseteq P$ denote the subset consisting of those pairs where (S_0, S_1) where $S_1 = \langle n \rangle$ and $n \in S_0$. Note that the functor U' factors through the construction $(S_0, S_1) \mapsto (S_0 \cup \{n\}, S_1 \cup \{n\})$ so that $U'|_{P'^{\mathrm{op}}}$ is a left Kan extension of $U'|_{P''^{\mathrm{op}}}$. Since P'' contains $\{n\}$ as a smallest element, it follows that the canonical maps
$$U'(\{n\}, \langle n \rangle) \to \varinjlim_{(S_0,S_1) \in P''} U'(S_0, S_1) \to \varinjlim_{\emptyset \neq S_0 \subseteq \langle n \rangle} U'(S_0, \langle n \rangle)$$
is an equivalence. The same argument shows that the canonical map
$$\varprojlim_{S_1 \subsetneq \langle n \rangle} U'(\emptyset, S_1) \to U'(\emptyset, \langle n \rangle - \{n\})$$
is an equivalence. We are therefore reduced to proving that the composite map
$$U'(\{n\}, \langle n \rangle) \to U'(\emptyset, \langle n \rangle) \to U'(\emptyset, \langle n \rangle - \{n\})$$
is an equivalence, which follows immediately from the definition of U'.
- By construction, the functor U'' vanishes on pairs (S_0, S_1) where $n \in S_0$. Using $(*)$, we see that U'' also vanishes on pairs (S_0, S_1) where $n \notin S_1$. It follows that the restriction of U'' to P'^{op} is a left Kan extension of its restriction to $(P' - P'')^{\mathrm{op}}$, so that the canonical map
$$\varinjlim_{\emptyset \neq S_0 \subseteq \langle n-1 \rangle} U''(S_0, \langle n \rangle) \to \varinjlim_{\emptyset \neq S_0 \subseteq \langle n \rangle} U''(S_0, \langle n \rangle)$$

is an equivalence. Similar reasoning shows that the canonical map

$$\varprojlim_{S_1 \subsetneq \langle n \rangle} U''(\emptyset, S_1) \to \varprojlim_{n \in S_1 \subsetneq \langle n \rangle} U''(\emptyset, S_1)$$

is an equivalence. We are therefore reduced to showing that the composite map

$$\varinjlim_{\emptyset \neq S_0 \subseteq \langle n-1 \rangle} U''(S_0, \langle n \rangle) \to U''(\emptyset, \langle n \rangle) \to \varprojlim_{n \in S_1 \subseteq \langle n \rangle} U''(\emptyset, S_1)$$

is an equivalence, which follows from the inductive hypothesis.

□

Proof of Proposition 8.4.16. We argue as in the proof of Lemma 5.3.14. Let $\mathrm{Fin}^s_{=n}$ denote the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by those finite sets having cardinality exactly n. Let

$$i : \mathrm{Fin}^s_{=n} \hookrightarrow \mathrm{Fin}^s_{\leq n} \qquad j : \mathrm{Fin}^s_{\leq n-1} \hookrightarrow \mathrm{Fin}^s_{\leq n}$$

be the inclusion functors. We let

$$i^* : \mathrm{Fun}(\mathrm{Fin}^s_{\leq n}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fin}^s_{=n}, \mathcal{C}) \qquad j^* : \mathrm{Fun}(\mathrm{Fin}^s_{\leq n}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fin}^s_{\leq n-1}, \mathcal{C})$$

denote the functors given by composition with i and j, respectively. Let

$$i_* : \mathrm{Fun}(\mathrm{Fin}^s_{=n}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fin}^s_{\leq n}, \mathcal{C}) \qquad j_* : \mathrm{Fun}(\mathrm{Fin}^s_{\leq n-1}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fin}^s_{\leq n}, \mathcal{C})$$

denote their right adjoints (given by right Kan extension along i and j). Let $U = V|_{\mathrm{Fin}^s_{\leq n}}$. We first claim that the diagram τ :

$$\begin{array}{ccc} U & \longrightarrow & j_* j^* U \\ \downarrow & & \downarrow \\ i_* i^* U & \longrightarrow & i_* i^* j_* j^* U \end{array}$$

is a pullback square in $\mathrm{Fun}(\mathrm{Fin}^s_{\leq n}, \mathcal{C})$. To prove this, it will suffice to show that τ yields a pullback square in $\mathcal{C}$ when evaluated at any nonempty set S of cardinality $\leq n$. If $|S| < n$, this follows from the fact that the horizontal maps in the diagram τ are equivalences (the lower horizontal map in τ is an equivalence, since it is a morphism between final objects of $\mathcal{C}$). If $|S| = n$, then the vertical maps in the diagram τ are equivalences.

Extracting inverse limits, we obtain a pullback diagram in $\mathcal{C}$:

$$\begin{array}{ccc} \varprojlim(V|_{\mathrm{Fin}^s_{\leq n}}) & \longrightarrow & \varprojlim(V|_{\mathrm{Fin}^s_{\leq n-1}}) \\ \downarrow & & \downarrow \\ \varprojlim(V|_{\mathrm{Fin}^s_{=n}}) & \longrightarrow & \varprojlim(i^* j_* j^* U). \end{array}$$

Note that $\mathrm{Fin}^s_{=n}$ is equivalent to the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by the object $\langle n \rangle$, whose endomorphism monoid coincides with the symmetric group Σ_n. We therefore have canonical equivalences

$$\varprojlim(V|_{\mathrm{Fin}^s_{=n}}) \simeq V(\langle n \rangle)^{\Sigma_n} \qquad \varprojlim(i^* j_* j^* U) \simeq (j_* j^* U)(\langle n \rangle)^{\Sigma_n}.$$

To complete the proof, we observe that $(j_* j^* U)(\langle n \rangle)$ can be identified with the inverse limit $\varprojlim(V|_{\mathcal{D}})$, where $\mathcal{D}$ denotes the full subcategory of $\mathrm{Fin}^s_{\langle n \rangle /}$ spanned by those surjective maps $\langle n \rangle \to S$ where S has cardinality $< n$. This category is evidently equivalent to the partially ordered set $\mathrm{Equiv}^{\circ}(\langle n \rangle)$. □

Proof of Proposition 8.4.18. We follow the basic outline of the proof of Proposition 8.4.16. Let $\Theta^\circ_n \subseteq \Theta^\circ$ denote the full subcategory spanned by those objects (S, S_1) where S_1 has cardinality exactly n. Let

$$i : \Theta^\circ_n \hookrightarrow \Theta^\circ_{\leq n} \qquad j : \Theta^\circ_{\leq n-1} \hookrightarrow \Theta^\circ_{\leq n}$$

denote the inclusion functors. We let

$$i^* : \mathrm{Fun}(\Theta^\circ_{\leq n}, \mathcal{C}) \to \mathrm{Fun}(\Theta^\circ_n, \mathcal{C}) \qquad j^* : \mathrm{Fun}(\Theta^\circ_{\leq n}, \mathcal{C}) \to \mathrm{Fun}(\Theta^\circ_{\leq n-1}, \mathcal{C})$$

denote the functors given by composition with i and j, respectively, and let

$$i_* : \mathrm{Fun}(\Theta^\circ_n, \mathcal{C}) \to \mathrm{Fun}(\Theta^\circ_{\leq n}, \mathcal{C}) \qquad j_* : \mathrm{Fun}(\Theta^\circ_{\leq n-1}, \mathcal{C}) \to \mathrm{Fun}(\Theta^\circ_{\leq n}, \mathcal{C})$$

denote their right adjoints (given by right Kan extension along i and j). Let $U = V|_{\Theta^\circ_{\leq n}}$. We first claim that the diagram τ

$$\begin{array}{ccc} U & \longrightarrow & j_* j^* U \\ \downarrow & & \downarrow \\ i_* i^* U & \longrightarrow & i_* i^* j_* j^* U \end{array}$$

is a pullback square in $\mathrm{Fun}(\Theta^\circ_{\leq n}, \mathcal{C})$. To prove this, it will suffice to show that τ yields a pullback square in $\mathcal{C}$ when evaluated at any object $(S_1 \subseteq S) \in \Theta^\circ_{\leq n}$. If $|S_1| < n$, this follows from the fact that the horizontal maps in the diagram τ are equivalences (the lower horizontal map in τ is an equivalence when evaluated at $(S_1 \subseteq S)$ because it is a morphism between final objects of $\mathcal{C}$). If $|S_1| = n$, then the vertical maps in the diagram τ are equivalences.

Extracting projective limits, we obtain a pullback diagram in $\mathcal{C}$:

$$\begin{array}{ccc} \varprojlim(V|_{\Theta^\circ_{\leq n}}) & \longrightarrow & \varprojlim(V|_{\Theta^\circ_{\leq n-1}}) \\ \downarrow & & \downarrow \\ \varprojlim(V|_{\Theta^\circ_n}) & \longrightarrow & \varprojlim((i^* j_* j^* U)). \end{array}$$

Let $\mathcal{E}$ denote the full subcategory of Θ°_n spanned by the single object $(\langle n \rangle \subseteq \langle n \rangle)$. The set of morphisms from this object to itself can be identified with the symmetric group Σ_n. The inclusion $\mathcal{E} \hookrightarrow \Theta^\circ_n$ admits a right adjoint and is therefore right cofinal. It follows that the canonical maps

$$\varprojlim(V|_{\Theta^\circ_n}) \to V(\langle n \rangle \subseteq \langle n \rangle)^{\Sigma_n} \qquad \varprojlim((i^* j_* j^* U)) \to (j_* j^* U)(\langle n \rangle, \langle n \rangle)^{\Sigma_n}$$

are equivalences. To complete the proof, we observe that $(j_* j^* U)(\langle n \rangle, \langle n \rangle)$ can be identified with the inverse limit $\varprojlim(V|_{\mathcal{D}})$, where $\mathcal{D}$ denotes the full subcategory of $(\Theta^\circ)_{(\langle n \rangle \subseteq \langle n \rangle)/}$ spanned by those maps $(\langle n \rangle, \langle n \rangle) \to (S, S_1)$ where $|S_1| < n$. Unwinding the definitions, we see that $\mathcal{T}(n)$ can be identified with the full subcategory of $\mathcal{D}$ spanned by those maps $(\langle n \rangle \subseteq \langle n \rangle) \to (S_1 \subseteq S)$ for which the underlying map $\langle n \rangle \to S$ is surjective. The resulting inclusion $\mathcal{T}(n) \hookrightarrow \mathcal{D}$ admits a right adjoint and is therefore right cofinal, so that we obtain an equivalence $(j_* j^* U)(\langle n \rangle, \langle n \rangle) \simeq \varprojlim(V|_{\mathcal{T}(n)})$. □

8.5. **Proof of Theorem 8.1.12.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, and a smooth affine group scheme G over X whose generic fiber is semisimple and simply connected. Let $D \subseteq X$ be an effective divisor for which G is reductive over the open set $X - D \subseteq X$. Our goal is to prove Theorem 8.1.12, which asserts that the canonical map

$$\varprojlim_{S \in \mathrm{Fin}^s} C^*((X-D)^S; \mathbf{Z}_\ell) \to \varprojlim_{S \in \mathrm{Fin}^s} \varinjlim_{\emptyset \neq S_0 \subseteq S} C^*(\mathrm{Ran}_G(X-D)_{S_0 \subseteq S}; \mathbf{Z}_\ell)$$

is an equivalence. By virtue of Theorem 8.4.9, it will suffice to show that the construction

$$(S_0 \subseteq S) \mapsto C^*(\mathrm{Ran}_G(X-D)_{S_0 \subseteq S}; \mathbf{Z}_\ell)$$

can be extended to a unital functor $V : \Theta \to \mathrm{Mod}_{\mathbf{Z}_\ell}$. We will exhibit such an extension by an explicit geometric construction.

Notation 8.5.1. Let S be a nonempty finite set and let $S_0 \subseteq S_1$ be (possibly empty) subsets of S. We define an object $V(S_0 \subseteq S_1 \subseteq S) \in \mathrm{Mod}_{\mathbf{Z}_\ell}$ by the formula

$$V(S_0 \subseteq S_1 \subseteq S) = C^*(\mathrm{Ran}_G(X-D)_{S_0 \subseteq S_1} \times_{(X-D)^{S_1}} (X-D)^S; \mathbf{Z}_\ell).$$

Note that the construction

$$(S_0 \subseteq S_1 \subseteq S) \mapsto \mathrm{Ran}_G(X-D)_{S_0 \subseteq S_1} \times_{(X-D)^{S_1}} (X-D)^S$$

determines a contravariant functor from the category Θ to the 2-category of prestacks, so that we can regard V as a functor from Θ into $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 8.5.2. Let S be a nonempty finite set. For every subset $S_0 \subseteq S$, the projection map

$$\mathrm{Ran}_G(X-D)_{S_0 \subseteq S_0} \times_{(X-D)^{S_0}} (X-D)^S \to (X-D)^S$$

is an equivalence of prestacks, which induces an equivalence

$$V(S_0 \subseteq S_0 \subseteq S) \to V(\emptyset \subseteq \emptyset \subseteq S).$$

In particular, the canonical map

$$\varprojlim_S V(S \subseteq S \subseteq S) \to \varprojlim_S V(\emptyset \subseteq \emptyset \subseteq S)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

To complete the proof of Theorem 8.1.12, it will suffice to show that the functor V of Notation 8.5.1 conditions $(U1)$ and $(U2)$ of Definition 8.4.5. This is the content of the following two assertions:

Proposition 8.5.3. *Let S be a nonempty finite set and let $S_1 \subseteq S$. Then the canonical map*

$$V(\emptyset \subseteq S_1 \subseteq S) \to \varprojlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} V(\emptyset \subseteq S_1 \subseteq S \amalg S')$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proposition 8.5.4. *Let $(S_0 \subseteq S_1 \subseteq S)$ be an object of Θ, let $s \in S$ be an element which is not contained in S_1, and set $S_0' = S_0 \cup \{s\}$, $S_1' = S_1 \cup \{s\}$. Then the diagram*

$$\begin{array}{ccc} V(S_0' \subseteq S_1' \subseteq S) & \longrightarrow & V(S_0 \subseteq S_1 \subseteq S) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} V(S_0'/E \subseteq S_1'/E \subseteq S) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} V(S_0/E \subseteq S_1/E \subseteq S/E) \end{array}$$

is a pullback square in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proof of Proposition 8.5.3. Set $\mathcal{C} = \mathrm{Ran}_G(X-D)_{S_1} \times_{(X-D)^{S_1}} (X-D)^S$. Unwinding the definitions, we wish to prove that the canonical map

$$C^*(\mathcal{C}; \mathbf{Z}_\ell) \to \varprojlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathcal{C} \times_{\mathrm{Spec}\, k} X^{S'}; \mathbf{Z}_\ell)$$

is an equivalence. In fact, we will prove the stronger assertion that the predual

$$\varinjlim_{S' \in \mathrm{Fin}^s} C_*(\mathcal{C} \times_{\operatorname{Spec} k} (X - D)^{S'}; \mathbf{Z}_\ell) \to C_*(\mathcal{C}; \mathbf{Z}_\ell)$$

is an equivalence. Using the Künneth formula (Corollary 2.3.43), we are reduced to proving that the natural map

$$\varinjlim_{S' \in \mathrm{Fin}^s} C_*((X - D)^{S'}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} k; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$$

is an equivalence, which follows from the acyclicity of $\mathrm{Ran}(X - D)$ (Corollary 2.4.13). □

Let us now sketch our proof of Proposition 8.5.4. The main idea is to reduce to a local statement about $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$-valued sheaves on $(X - D)^S$, which can be checked after passing to stalks at any k-valued point μ of $(X - D)^S$ (which we can identify with a map from S into $(X - D)(k)$). To compute the relevant stalks, we will need to establish a version of the proper base change theorem for the projection maps $\mathrm{Ran}_G(X - D)_{S_0 \subseteq S} \to (X - D)^S$ (Lemma 8.5.6). We will use combinatorial arguments to reduce to the case where the map $\mu : S \to (X - D)(k)$ is injective, in which case the desired result is a consequence of the following simple observation:

Lemma 8.5.5. *Let $(S_0 \subseteq S_1 \subseteq S)$ be an object of Θ, let $s \in S$ be an element which is not contained in S_1, and let $U \subseteq (X - D)^S$ be the open subset whose k-valued points are injective maps $\mu : S \to (X - D)(k)$. Then the canonical map*

$$\phi : \mathrm{Ran}_G(X - D)_{S_0 \subseteq S_1} \times_{(X-D)^{S_1}} U \to \mathrm{Ran}_G(X - D)_{S_0 \cup \{s\} \subseteq S_1 \cup \{s\}} \times_{(X-D)^{S_1 \cup \{s\}}} U$$

is an equivalence of categories.

Proof. Let $(R, \mathcal{P}, \mu, \gamma)$ be an object of

$$\mathrm{Ran}_G(X - D)_{S_0 \cup \{s\} \subseteq S_1 \cup \{s\}} \times_{(X-D)^{S_1 \cup \{s\}}} U,$$

so that $|\mu(s)|$ and $|\mu(S_1)|$ are disjoint closed subsets of X_R. Let $\mathcal{P}'$ be the G-bundle on X_R obtained by gluing $\mathcal{P}|_{X_R - |\mu(s)|}$ to the trivial bundle on $\mathcal{P}|_{X_R - |\mu(S_1)|}$, where the gluing data is provided by the trivialization γ. Then γ extends to a trivialization γ' of $\mathcal{P}$ on the open set $X_R - |\mu(S_1)|$. The construction

$$(R, \mathcal{P}, \mu, \gamma) \mapsto (R, \mathcal{P}', \mu, \gamma')$$

determines a homotopy inverse to the functor ϕ. □

Lemma 8.5.6. *Let S be a (possibly empty) finite set, let $S_0 \subseteq S$ be a subset, and let $q : \mathrm{Ran}_G(X - D)_{S_0 \subseteq S} \to (X - D)^S$ be the projection map. Let $\mathcal{F} \in \mathrm{Shv}((X - D)^S; \mathbf{Z}/\ell\mathbf{Z})$ be the sheaf given by the formula*

$$\mathcal{F}(U) = C^*(U \times_{(X-D)^S} \mathrm{Ran}_G(X - D)_{S_0 \subseteq S}; \mathbf{Z}/\ell\mathbf{Z}).$$

Fix a point $\eta : \operatorname{Spec} k \to (X - D)^S$. Then the natural map

$$\eta^* \mathcal{F} \to C^*(\operatorname{Spec} k \times_{(X-D)^S} \mathrm{Ran}_G(X - D)_{S_0 \subseteq S}; \mathbf{Z}/\ell\mathbf{Z})$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$.

We defer the proof of Lemma 8.5.6 until the end of this section.

Proof of Proposition 8.5.4. Let $(S_0 \subseteq S_1 \subseteq S)$ be an object of Θ, let $s \in S$ be an element which is not contained in S_1, and set $S'_0 = S_0 \cup \{s\}$, $S'_1 = S_1 \cup \{s\}$. We wish to prove that the diagram σ :

$$\begin{array}{ccc} V(S'_0 \subseteq S'_1 \subseteq S) & \longrightarrow & V(S_0 \subseteq S_1 \subseteq S) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} V(S'_0/E \subseteq S'_1/E \subseteq S) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} V(S_0/E \subseteq S_1/E \subseteq S/E) \end{array}$$

is a pullback square in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. Since each entry in the diagram is ℓ-complete, it will suffice to prove that σ is a pullback diagram after tensoring with $\mathbf{Z}/\ell\mathbf{Z}$. For each equivalence relation $E \in \mathrm{Equiv}(S)$, let $\mathcal{F}_E, \mathcal{F}'_E \in \mathrm{Shv}((X-D)^S; \mathbf{Z}/\ell\mathbf{Z})$ denote the sheaves given by the formulae

$$\mathcal{F}_E(U) = C^*(U \times_{(X-D)^{S_1}} \mathrm{Ran}_G(X-D)_{S_0/E \subseteq S_1/E}; \mathbf{Z}/\ell\mathbf{Z})$$

$$\mathcal{F}'_E(U) = C^*(U \times_{(X-D)^{S_1}} \mathrm{Ran}_G(X-D)_{S'_0/E \subseteq S'_1/E}; \mathbf{Z}/\ell\mathbf{Z}).$$

Let E_0 denote the discrete equivalence relation on S (so that $S/E_0 \simeq S$). Unwinding the definitions, we must show that the diagram

$$\begin{array}{ccc} C^*((X-D)^S; \mathcal{F}'_{E_0}) & \longrightarrow & C^*((X-D)^S; \mathcal{F}_{E_0}) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} C^*((X-D)^S; \mathcal{F}'_E) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} C^*((X-D)^S; \mathcal{F}_E) \end{array}$$

is a pullback square in $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$. We will prove this by showing that the diagram

$$\begin{array}{ccc} \mathcal{F}'_{E_0} & \longrightarrow & \mathcal{F}_{E_0} \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} \mathcal{F}'_E & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} \mathcal{F}_E \end{array}$$

is a pullback square in $\mathrm{Shv}((X-D)^S; \mathbf{Z}/\ell\mathbf{Z})$.

Fix a point $\eta : \mathrm{Spec}\, k \to (X-D)^S$, which we will identify with a map $\mu : S \to X(k)$. Using Proposition 4.1.11, we are reduced to proving that the diagram σ' :

$$\begin{array}{ccc} \eta^* \mathcal{F}'_{E_0} & \longrightarrow & \eta^* \mathcal{F}_{E_0} \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} \eta^* \mathcal{F}'_E & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} \eta^* \mathcal{F}_E \end{array}$$

is a pullback square in $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$. Let $E_1 \in \mathrm{Equiv}^\circ(S)$ be the equivalence relation determined by μ (so that sE_1s' if and only if $\mu(s) = \mu(s')$). Note that that if $E \in \mathrm{Equiv}(S)$ is an equivalence relation such that $E \nleq E_1$, then the sheaves $\mathcal{F}_E$ and $\mathcal{F}'_E$ are supported on a closed subset of $(X-D)^S$ which does not contain η, so that $\eta^* \mathcal{F}_E \simeq 0 \simeq \eta^* \mathcal{F}'_E$. We may therefore identify σ'

with the diagram σ'':

$$\begin{array}{ccc} \eta^* \mathcal{F}'_{E_0} & \longrightarrow & \eta^* \mathcal{F}_{E_0} \\ \downarrow & & \downarrow \\ \varprojlim_{E_0 < E \leq E_1} \eta^* \mathcal{F}'_E & \longrightarrow & \varprojlim_{E_0 < E \leq E_1} \eta^* \mathcal{F}_E \end{array}$$

We now distinguish two cases:

(a) Suppose that $E_0 \neq E_1$. Then the partially ordered set $\{E \in \mathrm{Equiv}(S) : E_0 < E \leq E_1\}$ contains a largest element, and therefore has weakly contractible nerve. Using Lemma 8.5.6, we deduce that for $E \leq E_1$, the restriction maps

$$\eta^* \mathcal{F}_{E_0} \to \eta^* \mathcal{F}_E \qquad \eta^* \mathcal{F}'_{E_0} \to \eta^* \mathcal{F}'_E$$

are equivalences. It follows that the vertical maps in the diagram σ'' are equivalences, so that σ'' is a pullback square.

(b) Suppose that $E_0 = E_1$: that is, the map $\eta : S \to X(k)$ is injective. Then the bottom horizontal map in the diagram σ'' is an equivalence (since it is a map between zero objects of $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$). It will therefore suffice to show that the top horizontal map is an equivalence, which follows immediately from Lemma 8.5.5.

□

We now turn to the proof of Lemma 8.5.6. We wish to prove a base change theorem for the map of prestacks

$$\pi : \mathrm{Ran}_G(X - D)_{S_0 \subseteq S} \to (X - D)^S.$$

Note that the desired result is trivial if S is empty (in this case, π is an equivalence); we will therefore assume henceforth that $S \neq \emptyset$. Roughly speaking, the idea of the proof is to show that that π behaves like a proper morphism.

Definition 8.5.7. Let Z be a quasi-projective k-scheme and let $\pi : \mathcal{C} \to Z$ be a morphism of prestacks. We will say that $\mathcal{C}$ is Ind-*projective over* Z it is equivalent to a colimit

$$K_0 \xrightarrow{f_0} K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} \cdots$$

where each K_i is a projective Z-scheme and each of the morphisms f_i is a closed immersion.

Lemma 8.5.8. *Let S be a nonempty finite set and let $S_0 \subseteq S$. For each integer n, let Y_n denote the prestack given by the $(n+1)$st fiber power of* $\mathrm{Ran}_G(X-D)_S$ *over* $\mathrm{Ran}_G(X-D)_{S_0 \subseteq S}$*. Then there exists an étale surjection $V \to (X-D)^S$ such that $V \times_{(X-D)^S} Y_n$ is* Ind-*projective over $(X-D)^S$.*

Remark 8.5.9. Lemma 8.5.8 can be strengthened: the prestack Y_n is already Ind-projective over $(X - D)^S$. However, we will not need this.

Remark 8.5.10. Lemma 8.5.8 depends in an essential way on our assumption that G is reductive on the open curve $X - D \subseteq X$ (the proof can be generalized to the case of parahoric reduction, at least in good cases, but we will not need this).

We will also need a technical regularity result, which guarantees that π becomes a locally trivial fibration after passing to a suitable stratification of the base:

Lemma 8.5.11. *Let S be a nonempty finite set, let $S_0 \subseteq S$, and let $f : Z \to X^S$ be a morphism of quasi-projective k-schemes. Let $n \geq 0$ and let Y_n be defined as in Lemma 8.5.8. Then there exists a stratification of Z by locally closed subschemes Z_α with the following property:*

- *For each index α, there exists a finite group Γ_α and a pullback diagram of prestacks*

$$\begin{array}{ccc} Y_n \times_{(X_D)^S} Z_\alpha & \longrightarrow & \mathcal{C}_\alpha \\ \downarrow & & \downarrow \\ Z_\alpha & \longrightarrow & \mathrm{B}\,\Gamma_\alpha. \end{array}$$

Here $\mathrm{B}\,\Gamma_\alpha$ *denotes the classifying stack of the finite group* Γ_α*.*

Proof of Lemma 8.5.6. Let $Y_\bullet$ be the simplicial prestack whose nth term is given by the $(n+1)$st fiber power of $\mathrm{Ran}_G(X-D)_S$ over $\mathrm{Ran}_G(X-D)_{S_0 \subseteq S}$. Define a cosimplicial object $\mathcal{F}^\bullet$ of $\mathrm{Shv}((X-D)^S; \mathbf{Z}/\ell\mathbf{Z})$ by the formula

$$\mathcal{F}^\bullet(U) = C^*(U \times_{(X-D)^S} Y_\bullet; \mathbf{Z}/\ell\mathbf{Z}).$$

Since the forgetful functor $\mathrm{Ran}_G(X-D)_S \to \mathrm{Ran}_G(X-D)_{S_0 \subseteq S}$ is essentially surjective, we obtain equivalences

$$\mathcal{F} \simeq \varprojlim \mathcal{F}^\bullet$$
$$C^*(\mathrm{Spec}\, k \times_{(X-D)^S} \mathrm{Ran}_G(X-D)_{S_0 \subseteq S}; \mathbf{Z}/\ell\mathbf{Z}) \simeq \varprojlim C^*(\mathrm{Spec}\, k \times_{(X-D)^S} Y^\bullet; \mathbf{Z}/\ell\mathbf{Z}).$$

It will therefore suffice to verify the following pair of assertions:

(a) The canonical map

$$\eta^* \varprojlim \mathcal{F}^\bullet \to \varprojlim \eta^* \mathcal{F}^\bullet$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$.

(b) For each $m \geq 0$, the canonical map

$$\theta_m : \eta^* \mathcal{F}^m \to C^*(\mathrm{Spec}\, k \times_{X^S} Y_m; \mathbf{Z}/\ell\mathbf{Z})$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$.

Assertion (a) follows formally from the right completeness of $\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$, since the functor η^* is t-exact and the sheaves $\mathcal{F}^n$ belong to $\mathrm{Shv}((X-D)^S; \mathbf{Z}/\ell\mathbf{Z})_{\leq 0}$ (see Lemma HA.1.3.3.11). To prove (b), choose an étale surjection $V \to (X-D)^S$ such that $V \times_{(X-D)^S} Y^m$ is Ind-projective over V and a point $\overline{\eta} \in V(k)$ lying over η. We can then write $V \times_{(X-D)^S} Y^m$ as the filtered colimit of a sequence

$$K_0 \hookrightarrow K_1 \hookrightarrow K_2 \hookrightarrow \cdots,$$

of projective V-schemes. For each $i \geq 0$, let $\mathcal{F}(i)$ denote the direct image of the constant sheaf $\underline{\mathbf{Z}/\ell\mathbf{Z}}_{K_i}$ along the projection map $K_i \to V$. We may then identify θ_m with the compose map

$$\overline{\eta}^* \varprojlim \mathcal{F}(i) \xrightarrow{\theta'} \varprojlim \overline{\eta}^* \mathcal{F}(i) \xrightarrow{\theta''} \varprojlim C^*(\mathrm{Spec}\, k \times_V K_i; \mathbf{Z}/\ell\mathbf{Z})$$

Using the proper base change theorem (Theorem 4.5.4), we deduce that θ'' is an equivalence. We will complete the proof by showing that θ' is an equivalence.

Choose a stratification of V by locally closed subschemes V_α satisfying the conclusions of Lemma 8.5.11. Refining the stratification if necessary, we may suppose that each V_α is smooth. For each index α, let $\mathcal{F}_\alpha(i)$ denote the sheaf on V obtained from the restriction $\mathcal{F}(i)|_{Z_\alpha}$ by extending by zero.

The collection of those towers

$$\cdots \to \mathcal{G}(2) \to \mathcal{G}(1) \to \mathcal{G}(0)$$

of objects of $\mathrm{Shv}(V; \mathbf{Z}/\ell\mathbf{Z})$ for which the canonical map $\eta^* \varprojlim \mathcal{G}(i) \to \varprojlim \eta^* \mathcal{G}(i)$ is an equivalence is closed under extensions. Consequently, to prove that θ' is an equivalence, it will suffice to show that each of the maps

$$\theta'_\alpha : \eta^* \varprojlim \mathcal{F}_\alpha(i) \to \varprojlim \eta^* \mathcal{F}_\alpha(i)$$

is an equivalence.

We may assume without loss of generality that V_α is nonempty (otherwise there is nothing to prove). Choose a point $v \in V_\alpha(k)$. By assumption, there exists a finite group Γ, a prestack $\mathcal{C}$ acted on by Γ, and a pullback diagram

$$\begin{array}{ccc} Y_n \times_{(X-D)^S} V_\alpha & \longrightarrow & \mathcal{C}/\Gamma \\ \downarrow & & \downarrow \\ V_\alpha & \xrightarrow{\rho} & \mathrm{B}\,\Gamma. \end{array}$$

In particular, we can identify $\mathcal{C}$ with the fiber product $Y_n \times_{(X-D)^S} \{v\}$, so that it can be realized as the colimit of a sequence

$$W_0 \hookrightarrow W_1 \hookrightarrow \cdots$$

where each W_j is a projective k-scheme and each of the maps $W_j \to W_{j+1}$ is a closed immersion. Enlarging the W_j if necessary, we may suppose that each W_j is invariant under the action of Γ, so that we can write $Y_n \times_{(X-D)^S} V_\alpha$ as a filtered colimit $\varinjlim W_j/\Gamma \times_{\mathrm{B}\,\Gamma} V_\alpha$. Let $\mathcal{G}(j)$ denote the direct image of the constant sheaf $\underline{\mathbf{Z}/\ell\mathbf{Z}}$ along the projection map

$$W_j/\Gamma \times_{\mathrm{B}\,\Gamma} V_\alpha \to V_\alpha.$$

The direct systems $\{W_j/\Gamma \times_{\mathrm{B}\,\Gamma} V_\alpha\}_{j\geq 0}$ and $\{K_i \times_V V_\alpha\}$ are mutually cofinal, so that the towers $\{\mathcal{F}(i)|_{Y_\alpha}\}_{i\geq 0}$ and $\{\mathcal{G}(j)\}_{j\geq 0}$ are equivalent as pro-objects of $\mathrm{Shv}(V_\alpha; \mathbf{Z}/\ell\mathbf{Z})$. Let $\iota_! : \mathrm{Shv}(V_\alpha; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Shv}(V; \mathbf{Z}/\ell\mathbf{Z})$ denote the functor of extension by zero (see Remark 4.2.8). Then θ'_α is an equivalence if and only if the canonical map

$$\xi : \eta^* \varprojlim \iota_! \, \mathcal{G}(j) \to \varprojlim \eta^* \iota_! \, \mathcal{G}(j)$$

is an equivalence.

For each $j \geq 0$, let $\mathcal{H}(j) \in \mathrm{Shv}(\mathrm{B}\,\Gamma; \mathbf{Z}/\ell\mathbf{Z})$ denote the direct image of the constant sheaf $\underline{\mathbf{Z}/\ell\mathbf{Z}}$ along the map $W_j/\Gamma \to \mathrm{B}\,\Gamma$. Using the smooth base change theorem, we obtain equivalences $\mathcal{G}(j) \simeq \rho^* \,\mathcal{H}(j)$. Let $F : \mathrm{Shv}(\mathrm{B}\,\Gamma; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Shv}(V; \mathbf{Z}/\ell\mathbf{Z})$ denote the functor $\iota_! \circ \rho^*$, so that we have a commutative diagram

$$\begin{array}{ccccc} & & \eta^* F(\varprojlim \mathcal{H}(j)) & & \\ & \swarrow & & \searrow & \\ \eta^* \varprojlim F(\mathcal{H}(j)) & & \xrightarrow{\quad\xi\quad} & & \varprojlim \eta^* F(\mathcal{H}(j)). \end{array}$$

To show that ξ is an equivalence, it will suffice to show that the functors F and $\eta^* \circ F$ preserve limits. In the second case, this is clear: the functor $\eta^* \circ F$ either vanishes (if η does not factor through V_α) or is given by pullback along an étale map $\operatorname{Spec} k \xrightarrow{\eta} V_\alpha \to \mathrm{B}\,\Gamma$ (if η does factor through V_α).

We now complete the proof by showing that the functor F preserves limits. Let us abuse notation by identifying the ∞-category $\mathrm{Shv}(\mathrm{B}\,\Gamma; \mathbf{Z}/\ell\mathbf{Z})$ with the ∞-category RMod_A of chain complexes of right A-modules, where $A = (\mathbf{Z}/\ell\mathbf{Z})[\Gamma]$ denotes the group algebra of Γ with coefficients in $\mathbf{Z}/\ell\mathbf{Z}$. For every étale morphism $W \to V$, evaluation at W determines a functor $\mathrm{Shv}(V; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$ which preserves colimits (Proposition 4.1.16). Since the functor F preserves colimits, it follows that the construction $M \mapsto F(M)(W)$ preserves colimits, and is therefore given by $M \mapsto M \otimes_A N_W$ where $N_W = C^*(W; F(A))$. Proposition 4.2.15 implies that N_W is perfect as a $\mathbf{Z}/\ell\mathbf{Z}$-module. Since the functor $M \mapsto F(M)(W)$ is left t-exact, the object N_W has Tor-amplitude ≤ 0 (as an A-module) and is therefore perfect as an A-module

(Proposition HA.7.2.5.23). It follows that the construction $M \mapsto F(M)(W)$ commutes with limits. Since W was chosen arbitrarily, we conclude that F commutes with limits. $\square$

It remains to prove Lemmas 8.5.8 and 8.5.11. The former depends on the following preliminary result:

Lemma 8.5.12. *Let Y be a quasi-projective k-scheme, let S be a nonempty finite set, and let $\mu : S \to \mathrm{Hom}_{\mathrm{Sch}_k}(Y, X - D)$ be a map of sets. For each subset $S_0 \subseteq S$, let $|S_0|$ denote the effective divisor in $X \times Y$ (which we regard as a relative curve over Y) determined by the collection of maps $\{\mu(s) : Y \to X \times Y\}_{s \in S_0}$. Let $S_0 \subseteq S$ be a subset, let $\rho : Z \to X \times Y$ be an affine morphism of finite type, and suppose that ρ admits a section over the open set $X' - |S|$. Then there exists a sequence of closed subschemes $K_0 \subseteq K_1 \subseteq \cdots \subseteq Y$ with the following property: a map $\phi : \operatorname{Spec} R \to Y$ factors through some K_n if and only if the lifting problem*

$$\begin{array}{ccc} X_{R'} - |S|_R & \longrightarrow & Z \\ \big\downarrow & \overset{\overline{s}}{\nearrow} & \big\downarrow \rho \\ X_{R'} - |S_0|_R & \longrightarrow & X \times Y \end{array}$$

admits a solution.

Proof. Write Z as the spectrum of a quasi-coherent sheaf of algebras $\mathcal{A}$ on $X \times Y$. Let $\mathcal{O}$ denote the quasi-coherent sheaf on X' given by the direct image of the structure sheaf of $(X \times Y) - |S|$, and let $\mathcal{O}_0$ be the direct image of the structure sheaf of $(X \times Y) - |S_0|$. Then the quotient $\mathcal{O} / \mathcal{O}_0$ can be written as a direct limit of coherent sheaves $\mathcal{F}_n = \mathcal{O}_{X'}(nS) / \mathcal{O}_{X'}(nS_0)$, each of which is flat (with finite support) over Y. Since ρ is of finite type, we can choose a coherent subsheaf $\mathcal{A}_0 \subseteq \mathcal{A}$ which generates $\mathcal{A}$ as a sheaf of algebras. Then the existence of $\overline{s}$ is equivalent to the vanishing of the composite map

$$\mathcal{A}_0 \subseteq \mathcal{A} \to \mathcal{O} \to \mathcal{O} / \mathcal{O}_0 .$$

Since $\mathcal{A}_0$ is coherent, this composite map factors as a composition $\mathcal{A}_0 \to \mathcal{F}_m \to \varinjlim_n \mathcal{F}_n$ for $m \gg 0$. It now suffices to define K_n to be the vanishing locus of the map $\pi_* \mathcal{A}_0 \to \pi_* \mathcal{F}_m \to \pi_* \mathcal{F}_{m+n}$, where $\pi : X \times Y \to Y$ denotes the projection map. $\square$

Proof of Lemma 8.5.8. Since G is reductive over the open subset $X - D$, there exists an étale surjection $\pi : U \to X - D$ for which the group scheme $G_U = U \times_X G$ is split reductive. Let $V \subseteq U^S$ be the open subset whose k-valued points are given by maps $\mu : S \to U(k)$ for which π is injective on $\mu(S)$. Using Remark 3.2.7, we obtain a pullback diagram of prestacks

$$\begin{array}{ccc} \mathrm{Ran}_{G_U}(U)_S \times_{U^S} V & \longrightarrow & \mathrm{Ran}_G(X - D)_S \\ \big\downarrow \rho & & \big\downarrow \\ V & \longrightarrow & (X - D)^S. \end{array}$$

Note that the composite map $V \subseteq U^S \to (X - D)^S$ is an étale surjection. We will show that V satisfies the requirements of Lemma 8.5.8.

Let S be a nonempty finite set and let $S_0 \subseteq S$. For each integer $n \geq 0$, let Y_n denote the nth fiber power of $\mathrm{Ran}_G(X - D)_S$ over $\mathrm{Ran}_G(X - D)_{S_0 \subseteq S}$ and let Z_n denote the nth fiber power of $\mathrm{Ran}_G(X - D)_S$ over $(X - D)^S$. Then $Y_\bullet$ and $Z_\bullet$ are simplicial prestacks. Moreover, we have an evident map $\iota_n : Y_n \to Z_n$. The objects of Y_n can be identified with tuples $(R, \mathcal{P}_1, \ldots, \mathcal{P}_n, \mu, \gamma_1, \ldots, \gamma_n)$, where R is a finitely generated k-algebra, each $\mathcal{P}_i$ is a G-bundle on X_R, $\alpha : S \to X(R)$ is a map of finite sets, and each γ_i is a trivialization of $\mathcal{P}_i$ over the open set $X_R - |\mu(S)|$. Unwinding the definitions, we see that ι_n is a fully faithful embedding whose essential image consists of those objects $(R, \mathcal{P}_1, \ldots, \mathcal{P}_n, \mu, \gamma_1, \ldots, \gamma_n)$ for which each of the composite maps $\gamma_i \circ \gamma_j^{-1}$ extends to an isomorphism of $\mathcal{P}_j$ with $\mathcal{P}_i$ over $X_R - |\mu(S_0)|$.

Since the group scheme G_U is constant, the prestack $\mathrm{Ran}_{G_U}(U)_S$ is Ind-projective over U^S (see Appendix A.5 of [20]), so that

$$\mathrm{Ran}_G(X - D)_S \times_{(X-D)^S} V \simeq \mathrm{Ran}_{G_U}(U)_S \times_{U^S} V$$

is Ind-projective over V. Since Z_n is the nth fiber power of $\mathrm{Ran}_G(X - D)_S$ over $(X - D)^S$, it follows that $Z_n \times_{(X-D)^S} V$ can be written as the direct limit of a sequence

$$K_n^0 \hookrightarrow K_n^1 \hookrightarrow K_n^2 \hookrightarrow \cdots$$

of closed immersions between projective V-schemes. It follows from Lemma 8.5.12 that each fiber product $Y_n \times_{Z_n} K_n^i$ can be written as a direct limit $\varinjlim L_n^{i,j}$ of closed subschemes of K_n^i. Reindexing these colimits if necessary, we may assume that the maps $K_n^i \to K_n^{i+1}$ carry $L_n^{i,j}$ into $L_n^{i+1,j}$. It follows that $Y_n \times_{(X-D)^S}$ is equivalent to the direct limit of the sequence

$$L_n^{0,0} \hookrightarrow L_n^{1,1} \hookrightarrow \cdots$$

of closed immersions between projective k-schemes, and is therefore Ind-projective over V. $\square$

Proof of Lemma 8.5.11. We can identify f with a collection of maps $\{f_s : Z \to X - D\}_{s \in S}$. Passing to a stratification of Z, we may assume that for $s, s' \in S$, either $f_s = f_{s'}$, or the graphs of f_s and $f_{s'}$ do not intersect (when regarded as closed subschemes of the product $Z \times X$). Let E be the equivalence relation on S given by sEs' if $f_s = f_{s'}$. Then

$$\mathrm{Ran}_G(X - D)_{S_0 \subseteq S} \times_{(X-D)^S} Z \simeq \mathrm{Ran}_G(X - D)_{S_0/E \subseteq S/E} \times_{(X-D)^{S/E}} Z.$$

We may therefore replace S by S/E and thereby reduce to the case where the morphisms $\{f_s\}_{s \in S}$ have disjoint graphs. In this case, the fiber product $Y_n \times_{(X-D)^S} Z$ can be identified with a fiber product of $(n+1)$-copies of each of the prestacks

$$\{\mathrm{Ran}_G(X - D)_{\{s\}} \times_{X-D} Z\}_{s \in S - S_0}$$

over Z. We may therefore assume without loss of generality that $n = 0$, $S_0 = \emptyset$, and S has a single element s, so that f can be identified with a function from Z to $X - D$.

The group scheme G is quasi-split over the generic point of X. It follows that G is quasi-split over a dense open subset $U \subseteq X - D$. Shrinking U if necessary, we may suppose that there exists an étale morphism $\phi : U \to \mathbf{A}^1_k$. Passing to a stratification of Z, we may assume either that $f(Z) \subseteq U$ or that $f(Z)$ is disjoint from U. In the latter case, we can write Z_{red} as a disjoint union of subschemes on which f is constant, in which case the result is obvious. In the former case, we can assume without loss of generality that $Z = U$.

Let G_0 denote the split form of G over the ground field k, and let $G' = G_0 \times_{\mathrm{Spec}\, k} \mathbf{A}^1$ be the associated group scheme over the affine line $\mathbf{A}^1$. Let $\mathrm{Gr}_{G'}$ denote the affine Grassmannian of G' (see Remark 3.2.7). Then $\mathrm{Gr}_{G'}$ is equipped with an action of the additive group $\mathbf{G}_a$ by translations, and the projection map $\mathrm{Gr}_{G'} \to \mathbf{A}^1$ is $\mathbf{G}_a$-equivariant. It follows that $\mathrm{Gr}_{G'}$ splits as a product $\mathbf{A}^1 \times_{\mathrm{Spec}\, k} \mathrm{Gr}_0$, where Gr_0 denotes the fiber $\mathrm{Gr}_{G'} \times_{\mathbf{A}^1} \{0\}$.

Since G is quasi-split over U, there exists a finite group Γ of automorphisms of the Dynkin diagram of G_0, an étale covering $\widetilde{U} \to U$ with Galois group Γ, and an isomorphism $G \times_X U \simeq (G_0 \times \widetilde{U})/\Gamma$. Let $\widetilde{G}$ denote the fiber product $G \times_X \widetilde{U}$. Using Remark 3.2.7, we see that the diagram of étale morphisms

$$X \leftarrow \widetilde{U} \xrightarrow{\phi} \mathbf{A}^1$$

determines Γ-equivariant equivalences

$$\widetilde{U} \times_X \mathrm{Gr}_G \simeq \mathrm{Gr}_{\widetilde{G}} \simeq \widetilde{U} \times_{\mathbf{A}^1} \mathrm{Gr}_{G'} \simeq \widetilde{U} \times_{\operatorname{Spec} k} \mathrm{Gr}_0 .$$

Dividing by the action of Γ (in the 2-category of étale stacks), we obtain an equivalence

$$U \times_X \mathrm{Gr}_G \simeq U \times_{\mathrm{B}\,\Gamma} (\mathrm{Gr}_0 /\Gamma),$$

where Gr_0 /Γ denotes the stack-theoretic quotient of Gr_0 by the action of Γ. □

8.6. **Proof of Theorem 8.2.14.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible over k, an algebraic curve X over k, and a smooth group scheme G over X. Our goal is to prove that the diagram

$$\begin{array}{ccc} \int \mathcal{B}_{\mathrm{red}} & \longrightarrow & \int \mathcal{B} \\ \downarrow & & \downarrow \\ \int 0 & \longrightarrow & \int \omega_{\mathrm{Ran}(X)}. \end{array}$$

of Theorem 8.2.14 is a pullback square in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Notation 8.6.1. Let T be an arbitrary finite set. We define a category $\mathrm{Ran}^G(X)^{(T)}$ as follows:

- The objects of $\mathrm{Ran}^G(X)^{(T)}$ are triples $(R, \nu : T \to X(R), \mathcal{P})$ where R is a finitely generated k-algebra, ν is a map of sets, and $\mathcal{P}$ is a G-bundle on the divisor $|\nu| \subseteq X_R$.
- A morphism from $(R, \nu, \mathcal{P})$ to $(R', \nu', \mathcal{P}')$ in $\mathrm{Ran}^G(X)^{(T)}$ is a pair (ϕ, γ), where $\phi : R \to R'$ is a map of k-algebras for which ν' is given by the composition $T \xrightarrow{\nu} X(R) \xrightarrow{X(\phi)} X(R')$, and γ is a G-bundle isomorphism of $\mathcal{P}'$ with $\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{P}$.

The construction $(R, \nu : T \to X(R), \mathcal{P}) \mapsto R$ determines a coCartesian fibration $\mathrm{Ran}^G(X)^{(T)} \to \mathrm{Ring}_k$, so that we can regard $\mathrm{Ran}^G(X)^{(T)}$ as a prestack. Moreover, it is equipped with an evident projection map $\mathrm{Ran}^G(X)^{(T)} \to X^T$.

Remark 8.6.2. If T is a nonempty finite set, then $\mathrm{Ran}^G(X)^{(T)}$ can be identified with the fiber product $\mathrm{Ran}^G(X) \times_{\mathrm{Ran}(X)} X^T \simeq \mathrm{Ran}^G(X) \times_{\mathrm{Fin}^s} \{T\}$. If T is empty, then the projection map $\mathrm{Ran}^G(X)^{(T)} \to \operatorname{Spec} k$ is an equivalence.

Remark 8.6.3. To every finite set T, we can associate an "incidence correspondence" $D \subseteq X^T \times_{\operatorname{Spec} k} X$, given by the union of the sections of the projection map $X^T \times_{\operatorname{Spec} k} X \to X^T$. Then D is an effective divisor in the relative curve $X^T \times X$ over X^T (whose degree is the cardinality of T). Let H_T denote the Weil restriction of the group scheme $D \times_X G$ along the finite flat map $D \to X^T$. Then H_T is a smooth affine group scheme over X^T. For any map $Y \to X^T$, we can identify H_T-bundles on Y with G-bundles on the product $Y \times_{X^T} D$. It follows that $\mathrm{Ran}^G(X)^{(T)}$ can be identified with the classifying stack of H_T (regarded as a group scheme over X^T). In particular, $\mathrm{Ran}^G(X)^{(T)}$ is a smooth algebraic stack over X^T which is quasi-compact and has affine diagonal.

Construction 8.6.4. Let T be a nonempty finite set, and suppose we are given subsets $T_0 \subseteq T_1 \subseteq T$. We let Given a pair of subsets $T_0 \subseteq T_1 \subseteq T$, we let $\mathrm{Ran}^G(X)^{(T_0 \subseteq T_1 \subseteq T)}$ denote the fiber product

$$X^{T_0} \times_{\mathrm{Ran}^G(X)^{(T_0)}} \mathrm{Ran}^G(X)^{(T_1)} \times_{X^{T_1}} X^T,$$

where the map $X^{T_0} \to \mathrm{Ran}^G(X)^{T_0}$ is given by the formation of trivial G-bundles. In other words, the objects of $\mathrm{Ran}^G(X)^{(T_0 \subseteq T_1 \subseteq T)}$ are given by quadruples $(R, \nu, \mathcal{P}, \xi)$ where R is a finitely generated k-algebra, $\nu : T \to X(R)$ is a map of sets, $\mathcal{P}$ is a G-bundle on $|\nu(T_1)|$, and ξ is a trivialization of $\mathcal{P}$ on $|\nu(T_0)|$. Using Remark 8.6.3, we see that $\mathrm{Ran}^G(X)^{(T_0 \subseteq T_1 \subseteq T)}$ is a smooth Artin stack over X^T. We define an ℓ-adic sheaf $\mathcal{B}^{(T_0 \subseteq T_1 \subseteq T)} \in \mathrm{Shv}_\ell(X^T)$ by the formula

$$\mathcal{B}^{(T_0 \subseteq T_1 \subseteq T)} = [\mathrm{Ran}^G(X)^{T_0 \subseteq T_1 \subseteq T}]_{X^T}.$$

Let Θ be the category introduced in Notation 8.4.2. In this section, we will denote objects of Θ by triples $(T_0 \subseteq T_1 \subseteq T)$ where T is a nonempty finite set and $T_0 \subseteq T_1$ are (possibly empty) subsets of T. The construction

$$(T_0 \subseteq T_1 \subseteq T) \mapsto (\mathrm{Ran}^G(X)_{T_0 \subseteq T_1 \subseteq T} \to X^T)$$

determines a functor from Θ^{op} to the 2-category $\mathrm{AlgStack}^!$ of Notation A.5.25. Applying the functor Φ of Construction A.5.26, we see that the construction

$$(T_0 \subseteq T_1 \subseteq T) \mapsto (X^T, \mathcal{B}^{(T_0 \subseteq T_1 \subseteq T)})$$

can be regarded as a functor from Θ^{op} to the ∞-category $\mathrm{Shv}_\ell^!$ of Construction A.5.11. We let $V : \Theta^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ denote the functor given by

$$V(T_0 \subseteq T_1 \subseteq T) = C^*(X^T; \mathcal{B}^{(T_0 \subseteq T_1 \subseteq T)}).$$

Remark 8.6.5. Let $T_0 \subseteq T_1 \subseteq T$ be as in Construction 8.6.4. Then $\mathrm{Ran}^G(X)^{(T_0)}$ and $\mathrm{Ran}^G(X)^{(T_1)}$ can be identified with the classifying stacks of smooth affine group schemes H_{T_0} and H_{T_1} over X^{T_0} and X^{T_1}, respectively (see Remark 8.6.3). It follows that $\mathrm{Ran}_G(X)^{(T_0 \subseteq T_1 \subseteq T)}$ can be identified with the stack-theoretic quotient of $H_{T_0} \times_{X^{T_0}} X^T$ by the action of $H_{T_1} \times_{X^{T_1}} X^T$. In particular, $\mathrm{Ran}_G(X)^{(T_0 \subseteq T_1 \subseteq T)}$ is a smooth algebraic stack over X^T which is quasi-compact and has affine diagonal.

Let $V_{\mathrm{red}}, V_{\blacksquare}, V_{\square} : (\mathrm{Fin}^{\mathrm{s}})^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ be the functors defined in Notation 8.4.3 (where we identify V with a functor from Θ to the stable ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}^{\mathrm{op}}$). Unwinding the definitions, we have

$$V_{\mathrm{red}}(T) = C^*(X^T; \mathcal{B}^{(T)}_{\mathrm{red}}) \quad V_{\blacksquare}(T) = C^*(X^T; \mathcal{B}^{(T)}) \quad V_{\square}(T) = \varprojlim_{\emptyset \neq T_0 \subseteq T} C^*(X^T; \mathcal{B}^{(T_0 \subseteq T \subseteq T)}).$$

There is an evident fiber sequence of functors

$$V_{\mathrm{red}} \to V_{\blacksquare} \to V_{\square}$$

which induces a fiber sequence

$$\int \mathcal{B}_{\mathrm{red}} \to \int \mathcal{B} \to \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} V_{\square}(T).$$

We may therefore reformulate Theorem 8.2.14 as the assertion that the canonical map

$$\varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} V_{\square}(T) \to \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} V(T \subseteq T \subseteq T) = \int \omega_{\mathrm{Ran}(X)}$$

is an equivalence. By virtue of Theorem 8.4.9 (applied to the "opposite" functor $V^{\mathrm{op}} : \Theta \to \mathrm{Mod}_{\mathbf{Z}_\ell}^{\mathrm{op}}$), it will suffice to prove the following pair of assertions:

Proposition 8.6.6. *Let T be a nonempty finite set. Then, for every subset $T_1 \subseteq T$, the canonical map*

$$\varinjlim_{T' \in \mathrm{Fin}^{\mathrm{s}}} V(\emptyset \subseteq T_1 \subseteq T \amalg T') \to V(\emptyset \subseteq T_1 \subseteq T)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proposition 8.6.7. *Let $T_0 \subseteq T_1 \subseteq T$ be an object of Θ, let $t \in T$ be an element which is not contained in T_1, and set $T_0' = T_0 \cup \{t\}$, $T_1' = T_1 \cup \{t\}$. Then the diagram*

$$\begin{array}{ccc} \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} V(T_0/E \subseteq T_1/E \subseteq T/E) & \longrightarrow & \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} V(T_0'/E \subseteq T_1'/E \subseteq T/E) \\ \downarrow & & \downarrow \\ V(T_0 \subseteq T_1 \subseteq T) & \longrightarrow & V(T_0' \subseteq T_1' \subseteq T) \end{array}$$

is a pushout square in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proof of Proposition 8.6.6. Set $\mathcal{C} = \mathrm{Ran}^G(X)^{(T_1)} \times_{X^{T_1}} X^T$, and let $\pi : \mathcal{C} \to X^T$ denote the projection onto the second factor. For every finite set T', we can identify $\mathrm{Ran}^G(X)_{\emptyset \subseteq T_1 \subseteq T \amalg T'}$ with the product $\mathcal{C} \times_{\mathrm{Spec}\, k} X^{T'}$. Using Remark 8.6.5 and Proposition 5.1.9, we obtain equivalences

$$\mathcal{B}^{(\emptyset \subseteq T_1 \subseteq T \amalg T')} \simeq [\mathcal{C}]_{X^T} \boxtimes \omega_{T'}.$$

Applying Corollary 4.6.5, we compute

$$\begin{aligned} V(\emptyset \subseteq T_1 \subseteq T \amalg T') &\simeq C^*(X^T \times X^{T'}; [\mathcal{C}]_{X^T} \boxtimes \omega_{X^{T'}}) \\ &\simeq C^*(X^T; [\mathcal{C}]_{X^T}) \otimes_{\mathbf{Z}_\ell} C^*(X^{T'}; \omega_{X^{T'}}) \\ &\simeq V(\emptyset \subseteq T_1 \subseteq T) \otimes_{\mathbf{Z}_\ell} C_*(X^{T'}; \mathbf{Z}_\ell). \end{aligned}$$

Consequently, the canonical map

$$\theta : \varinjlim_{T' \in \mathrm{Fin}^{\mathrm{s}}} V(\emptyset \subseteq T_1 \subseteq T \amalg T') \to V(\emptyset \subseteq T_1 \subseteq T)$$

can be identified with the tensor product of the identity map on $V(\emptyset \subseteq T_1 \subseteq T)$ with the equivalence

$$\varinjlim_{T' \in \mathrm{Fin}^{\mathrm{s}}} C_*(X^{T'}; \mathbf{Z}_\ell) \to \mathbf{Z}_\ell$$

(see Theorem 2.4.5). □

Proof of Proposition 8.6.7. Let Y be a quasi-projective k-scheme equipped with a map $Y \to X^T$. For each equivalence relation $E \in \mathrm{Equiv}(T)$, we let $Y(E)$ denote the fiber product $Y \times_{X^T} X^{T/E}$. Define prestacks $Z(E)$ and $Z'(E)$ by the formulae

$$Z(E) = \mathrm{Ran}^G(X)^{(T_0/E \subseteq T_1/E \subseteq T/E)} \times_{X^T} Y \qquad Z'(E) = \mathrm{Ran}^G(X)_{T_0'/E \subseteq T_1'/E \subseteq T/E} \times_{X^T} Y,$$

so that we have projection maps $q_E : Z(E) \to Y(E)$ and $q'_E : Z'(E) \to Y(E)$, and let $i_E : Y(E) \to Y$ denote the inclusion map. We define objects $\mathcal{F}_E^Y, \mathcal{F}_E'^Y \in \mathrm{Shv}_\ell(Y)$ by the formulae

$$\mathcal{F}_E^Y(U) = i_{E*}[Z(E)]_{Y(E)} \qquad \mathcal{F}_E'^Y(U) = i_{E*}[Z'(E)]_{Y(E)}.$$

Let E_0 be the trivial equivalence relation on T (so that $T/E_0 \simeq T$). We will prove that the diagram σ_Y:

$$\begin{array}{ccc} \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \mathcal{F}_E^Y & \longrightarrow & \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \mathcal{F}_E'^Y \\ \downarrow & & \downarrow \\ \mathcal{F}_{E_0}^Y & \longrightarrow & \mathcal{F}_{E_0}'^Y \end{array}$$

is a pushout square in $\mathrm{Shv}_\ell(Y)$. Taking $Y = X^T$ and passing to global sections, we will obtain a proof of Proposition 8.6.7.

It follows from Remark 8.6.5 that $Z(E)$ and $Z'(E)$ are quasi-compact algebraic stacks which are smooth with affine diagonal over $Y(E)$, for each $E \in \mathrm{Equiv}(T)$. Using Proposition 5.1.9, we deduce that if $i : Y' \to Y$ is a closed immersion, then we have canonical equivalences

$$\mathcal{F}_E^{Y'} \simeq i^! \, \mathcal{F}_E^Y \qquad \mathcal{F}_E'^{Y'} \simeq i^! \, \mathcal{F}_E^Y,$$

so that $\sigma_{Y'}$ can be identified with the image of σ_Y under the functor $i^!$. Proceeding by Noetherian induction, we may assume that $i^! \sigma_Y$ is a pullback square for every closed subscheme $Y' \subsetneq Y$. If Y is nonreduced, then we can take $Y' = Y_{\mathrm{red}}$ to complete the proof. Let us therefore assume that Y is reduced.

Let $Y' \subsetneq Y$ be a closed subscheme, and let U be the complement of Y' in Y. Then we have a fiber sequence of diagrams

$$i_* i^! \sigma_Y \to \sigma_Y \to j_* j^* \sigma_Y.$$

It will therefore suffice to prove that $j^* \sigma_Y \simeq \sigma_U$ is a pullback square in $\mathrm{Shv}_\ell(U)$. We are therefore free to replace Y by any nonempty open subscheme $U \subseteq Y$. In particular, we may assume that Y is smooth, affine, and that there is a fixed equivalence relation $E_1 \in \mathrm{Equiv}(T)$ such that the map $Y \to X^T$ factors through X^{T/E_1}, but the fiber product $Y \times_{X^T} X^{T/E} = \emptyset$ for $E > E_1$.

The proof now breaks into two cases. Suppose first that $E_0 = E_1$. Then $\mathcal{F}_E^Y \simeq \mathcal{F}_E'^Y \simeq 0$ for $E \neq E_0$. We are therefore reduced to proving that the canonical map $\mathcal{F}_{E_0}^Y \to \mathcal{F}_{E_0}'^Y$ is an equivalence in $\mathrm{Shv}_\ell(Y)$. This is clear, since the restriction map $Y \times_{X^T} \mathrm{Ran}^G(X)_{T_0' \subseteq T_1' \subseteq T} \to Y \times_{X^T} \mathrm{Ran}^G(X)^{(T_0 \subseteq T_1 \subseteq T)}$ is an equivalence of prestacks.

We now consider the case where $E_0 \neq E_1$. In this case, we will complete the proof by showing that the maps

$$\theta : \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \mathcal{F}_E^Y \to \mathcal{F}_{E_0}^Y$$

$$\theta' : \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \mathcal{F}_E'^Y \to \mathcal{F}_{E_0}'^Y$$

are equivalences in $\mathrm{Shv}_\ell(Y)$. We will give the proof for θ; the proof for θ' is the same. Using Lemma 8.3.9, we deduce the following:

(∗) Let $E \in \mathrm{Equiv}(T)$, and let E' be the equivalence relation on T generated by E and E_1. Then the canonical map $\mu : \mathcal{F}_{E'}^Y \to \mathcal{F}_E^Y$ is an equivalence in $\mathrm{Shv}_\ell(Y)$.

It follows immediately from (∗) that the diagram $\{\mathcal{F}_E^Y\}_{E \in \mathrm{Equiv}^\circ(T)}$ is a left Kan extension of the diagram $\{\mathcal{F}_E^Y\}_{E \geq E_1}$, so that we have equivalences

$$\varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \mathcal{F}_E^Y \simeq \varinjlim_{E \geq E_1} \mathcal{F}_E^Y \simeq \mathcal{F}_{E_1}^Y$$

and the assertion that θ is an equivalence follows immediately from (∗). □

9. Proof of the Product Formula

Let k be an algebraically closed field, let ℓ be a prime number which is invertible in k, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected, and let $\mathcal{B} \in \mathrm{Shv}_\ell^!(\mathrm{Ran}(X))$ denote the !-sheaf introduced in §5.4.2. We wish to prove Theorem 5.4.5, which asserts that the canonical map

$$\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

is an equivalence. Let us assume that G is Q-adapted for some effective divisor $Q \subseteq X$ (Definition 7.2.9). In §7.2, we showed that the map ρ fits into a commutative diagram

$$\begin{array}{ccc} \int \mathcal{B} & \xrightarrow{\rho} & C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \\ \downarrow & & \downarrow \\ \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_S & \xrightarrow{\beta} & \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathrm{Ran}_G(X - Q)_S; \mathbf{Z}_\ell) \end{array}$$

where the vertical maps are equivalences (for the left vertical map this follows from Theorem 7.2.10, and for the right vertical map it follows from Theorem 3.2.9 together with the fact that the forgetful functor $\mathrm{Bun}_G(X, Q) \to \mathrm{Bun}_G(X)$ is an affine space bundle and therefore induces an isomorphism on ℓ-adic cohomology). We are therefore reduced to proving that the map β is an equivalence. Unfortunately, it is quite difficult to prove this directly: the primary obstacle is that we cannot interchange the formation of chiral homology with the limit $\varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_S$. To circumvent this difficulty, we will need to consider "reduced" versions of each of the objects which appears in the above diagram. Most of the relevant constructions have already been supplied in §8, where we established a reduced version of nonabelian Poincare duality (Theorem 8.1.12) and the equivalence of Theorem 5.4.5 with its reduced analogue (Theorem 8.2.14). In this section, we will complete the proof of Theorem 5.4.5 in three steps:

- (a) In §9.1, we will introduce "reduced" analogues of the !-sheaves $\mathcal{B}_S$, which we will denote by $\mathcal{B}_{S,\mathrm{red}}$.
- (b) In §9.3, we will prove a "reduced" version of Theorem 7.2.10, which asserts that a certain natural map $\mathcal{B}_{\mathrm{red}} \to \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}}$ which induces an equivalence on chiral homology (Proposition 9.1.4). The proof relies on a relative version of the acyclicity of the Ran space, which we establish in §9.2.
- (c) The map β has a "reduced" analogue

$$\beta_{\mathrm{red}} : \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} \to \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X - Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell),$$

  which we will prove to be an equivalence (Proposition 9.1.5). The proof of this fact will be carried out in §9.5, 9.6, and 9.7. It is based on a notion of Verdier duality for $\mathrm{Ran}(X)$, which we describe in §9.4.

We will give a more detailed outline of our approach in §9.1.

9.1. **Reduction to a Nonunital Statement.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, and a smooth affine group scheme G over X. We further assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. We wish to prove Theorem 5.4.5 which asserts that the canonical map

$$\rho : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

is a quasi-isomorphism.

Let us first assume that G is Q-adapted for some effective divisor $Q \subseteq X$ (Definition 7.2.9). The map ρ fits into a commutative diagram σ :

$$\begin{array}{ccc}
\int \mathcal{B}_{\mathrm{red}} & \xrightarrow{\rho_{\mathrm{red}}} & C^*_{\mathrm{red}}(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \\
\downarrow & & \downarrow \\
\int \mathcal{B} & \xrightarrow{\rho} & C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \\
\downarrow & & \downarrow \\
\varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) & \xleftarrow{\theta} & C^*(\mathrm{Bun}_G(X,Q); \mathbf{Z}_\ell) \\
\downarrow & & \downarrow \\
\varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell) & \xleftarrow{\theta_{\mathrm{red}}} & C^*_{\mathrm{red}}(\mathrm{Bun}_G(X,Q); \mathbf{Z}_\ell).
\end{array}$$

The upper square is a pullback by virtue of Theorem 8.2.18, so that ρ is an equivalence if and only if ρ_{red} is an equivalence. Since G is Q-adapted, the fiber G_x is a vector group for $x \in Q$, so that the projection map $\mathrm{Bun}_G(X,Q) \to \mathrm{Bun}_G(X)$ is an affine space bundle and therefore induces an isomorphism on ℓ-adic cohomology. It follows that the right vertical composition in σ is an equivalence. The map θ_{red} is an equivalence by virtue of Theorem 8.1.11. This proves the following:

Lemma 9.1.1. *In the situation described above, the map ρ is an equivalence if and only if the composite map*

$$\int \mathcal{B}_{red} \to \int \mathcal{B} \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \to \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell)$$

is an equivalence.

In §7.2 we introduced a $\mathrm{Ran}(X)$ prestack $\mathrm{Ran}^\dagger_G(X-Q)_S$: roughly speaking, if $\nu : T \to X(k)$ is a k-valued point of the Ran space $\mathrm{Ran}(X)$, then the fiber $\mathrm{Ran}^\dagger_G(X-Q)_S \times_{\mathrm{Ran}(X)} \{\nu\}$ parametrizes maps $\mu : S \to X - Q$ together with G-bundles that are defined on $(X - \mu(S)) \cup \nu(T)$ and trivialized on $X - \mu(S)$ (see Definition 7.2.1). We will need an augmented version of this prestack, which additionally tracks information about a subset $T_0 \subseteq T$ for which $\nu(T_0)$ is contained in the locus $X - \mu(S)$ where our G-bundles are trivial.

Construction 9.1.2. Fix an effective divisor $Q \subseteq X$ and a finite set S. We define a category $\mathrm{Ran}^\dagger_G(X-Q)_{S,\mathrm{aug}}$ as follows:

- The objects of $\mathrm{Ran}^\dagger_G(X-Q)_{S,\mathrm{aug}}$ are tuples $(R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, K_+ is a subset of S, K_- is a subset of K_+, T is a nonempty finite set, T_0 is a subset of T, $\mu : S \to (X-Q)(R)$ and $\nu : T \to X(R)$ are maps of sets, $\mathcal{P}$ is a G-bundle on X_R, γ is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu(S)|$, and we have $|\mu(K_+)| \cap |\nu(T)| = \emptyset = |\mu(S)| \cap |\nu(T_0)|$.
- There are no morphisms from an object $C = (R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \alpha)$ to another object $C' = (R', K'_-, K'_+, T', T'_0, \mu', \nu', \mathcal{P}', \alpha')$ unless $K'_- \subseteq K_-$ and $K_+ \subseteq K'_+$. If these conditions are satisfied, then a morphism from C to C' consists of a k-algebra

homomorphism $\phi : R \to R'$, a surjection of finite sets $\lambda : T \to T'$ for which the diagrams

$$\begin{array}{ccc} T & \xrightarrow{\lambda} & T' \\ \downarrow{\nu} & & \downarrow{\nu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R') \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\mathrm{id}} & S \\ \downarrow{\mu} & & \downarrow{\mu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R') \end{array}$$

commute and $T'_0 \subseteq \lambda(T_0)$, together with a G-bundle isomorphism between $\mathcal{P}'$ and $\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathcal{P}$ over the open set $X_{R'} - |\mu'(K'_-)|$, which carries γ to γ'.

The construction $(R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \gamma) \mapsto (R, T, T_0, \nu)$ determines a forgetful functor

$$\operatorname{Ran}^{\dagger}_G(X - Q)_{S,\mathrm{aug}} \to \operatorname{Ran}_{\mathrm{aug}}(X).$$

This map is a coCartesian fibration and therefore exhibits $\operatorname{Ran}^{\dagger}_G(X - Q)_{S,\mathrm{aug}}$ as an augmented $\operatorname{Ran}(X)$-prestack (see Definition 8.2.10). We let $\mathcal{B}_{S,\mathrm{aug}}$ denote the augmented !-sheaf on $\operatorname{Ran}(X)$ given by

$$\mathcal{B}_{S,\mathrm{aug}} = [\operatorname{Ran}^{\dagger}_G(X - Q)_{S,\mathrm{aug}}]_{\operatorname{Ran}_{\mathrm{aug}}(X)}.$$

We let $\mathcal{B}_{S,\mathrm{red}}$ denote the lax !-sheaf given by $(\mathcal{B}_{S,\mathrm{aug}})_{\mathrm{red}}$.

Remark 9.1.3. We can identify the prestack $\operatorname{Ran}^{\dagger}_G(X - Q)_S$ of Definition 7.2.1 with the full subcategor of $\operatorname{Ran}^{\dagger}_G(X - Q)_{S,\mathrm{aug}}$ spanned by those objects $(R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \gamma)$ where $T_0 = \emptyset$. Consequently, the underlying lax !-sheaf of $\mathcal{B}_{S,\mathrm{aug}}$ can be identified with the lax !-sheaf $\mathcal{B}_S$ of Notation 7.2.5.

Note that for every nonempty finite set S, restriction of G-bundles determines a map of augmented $\operatorname{Ran}(X)$-prestacks

$$\operatorname{Ran}^{\dagger}_G(X - Q)_{S,\mathrm{aug}} \to \operatorname{Ran}^G_{\mathrm{aug}}(X)$$

which induces maps

$$\mathcal{B}_{\mathrm{aug}} \to \mathcal{B}_{S,\mathrm{aug}} \qquad \mathcal{B}_{\mathrm{red}} \to \mathcal{B}_{S,\mathrm{red}}\,.$$

Passing to the limit over S, we obtain a commutative diagram

$$\begin{array}{ccc} \int \mathcal{B}_{\mathrm{red}} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} \\ \downarrow & & \downarrow \\ \int \mathcal{B} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_S \\ \downarrow & \swarrow & \\ \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\operatorname{Ran}_G(X - Q)_S; \mathbf{Z}_\ell) & & \\ \downarrow & & \\ \varprojlim_{\mathrm{prim}} C^*(\operatorname{Ran}_G(X - Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell). & & \end{array}$$

We will deduce Theorem 5.4.5 from the following assertions:

Proposition 9.1.4. *Let Q be an effective divisor for which G is Q-adapted. Then the diagram*

$$\begin{array}{ccc} \int \mathcal{B}_{red} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,red} \\ \downarrow & & \downarrow \\ \int \mathcal{B} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \end{array}$$

is a pullback square.

Proposition 9.1.5. *Let Q be an effective divisor for which G is Q-adapted. Then the composite map*

$$\begin{aligned} \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,red} & \to \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \\ & \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \\ & \to \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell) \end{aligned}$$

is a quasi-isomorphism.

Remark 9.1.6. Note that since the map $\mathcal{B} \to \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S$ is an equivalence of !-sheaves on $\mathrm{Ran}(X)$ (Theorem 7.2.10), Proposition 9.1.4 is equivalent to the assertion that the map

$$\int \mathcal{B}_{\mathrm{red}} \to \int \varprojlim_S \mathcal{B}_{S,\mathrm{red}}$$

is an equivalence. One strategy for proving this would be to generalize Theorem 7.2.10 by showing that the natural map $\mathcal{B}_{\mathrm{aug}} \to \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,\mathrm{aug}}$ is an equivalence of augmented !-sheaves on $\mathrm{Ran}(X)$. However, to adapt our proof of Theorem 7.2.10 we would need a relative version of Theorem 3.3.1 which applies to a family of divisors, rather than a fixed divisor $D \subseteq X$. We will therefore use a different argument: Theorem 8.2.14 shows that the left vertical map in the diagram

$$\begin{array}{ccc} \int \mathcal{B}_{\mathrm{red}} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,\mathrm{red}} \\ \downarrow & & \downarrow \\ \int \mathcal{B} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \end{array}$$

is *almost* an equivalence: it has cofiber equivalent to $\mathbf{Z}_\ell$. We will prove Proposition 9.1.4 in §9.3 by showing that the right vertical map has the same property.

Assuming Propositions 9.1.4 and 9.1.5 for the moment, we can complete the proof of Theorem 5.4.5:

Proof of Theorem 5.4.5. Let G be an arbitrary smooth affine group scheme over X with connected fibers whose generic fiber is semisimple and simply connected. We wish to show that the canonical map $\rho_G : \int \mathcal{B} \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ is a quasi-isomorphism. Using Proposition 7.2.12, we can choose an effective divisor $Q \subseteq X$, a Q-adapted group scheme G' over X, and a map of group schemes $G' \to G$ which is an isomorphism over the open set $X - Q$. Using Proposition 7.1.1, we can replace G by G' and thereby reduce to the case where G itself is Q-adapted. By virtue of Lemma 9.1.1, it will suffice to show that the left vertical composition in the diagram

$$\begin{array}{ccc}
\int \mathcal{B}_{\mathrm{red}} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} \\
\downarrow & & \downarrow \\
\int \mathcal{B} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_S \\
\downarrow & \swarrow & \\
\varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) & & \\
\downarrow & & \\
\varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell). & &
\end{array}$$

is an equivalence. It follows from Theorem 7.2.10 and Proposition 9.1.4 that the horizontal maps are equivalences. We are therefore reduced to proving that the composite map

$$\begin{aligned}
\int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} &\to \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_S \\
&\to \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*(\mathrm{Ran}_G(X-Q)_S; \mathbf{Z}_\ell) \\
&\to \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell)
\end{aligned}$$

is an equivalence, which follows from Proposition 9.1.5. □

Remark 9.1.7. The architecture of the preceding proof of Theorem 5.4.5 is somewhat misleading. Our proof of Proposition 9.1.4 will actually show that the cofiber of the map

$$\int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} \to \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_S$$

is equivalent to $\mathbf{Z}_\ell$. Combining this with Theorems 7.2.10 and 8.1.11, we can directly deduce that Theorem 5.4.5 is equivalent to Proposition 9.1.5: that is, Theorem 8.2.18 is logically unnecessary.

9.2. **Digression: Acyclicity of the Ran Space in Families.** Let k be an algebraically closed field, let ℓ be a prime number which is invertible in k, and let X be an algebraic curve over k. In §2.4, we proved that the prestack $\mathrm{Ran}(X)$ is acyclic. More generally, for any divisor $D \subseteq X$, the Ran space $\mathrm{Ran}(X-D)$ is acyclic, so that the cochain complex

$$C^*(\mathrm{Ran}(X-D); \mathbf{Z}_\ell) \simeq \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C^*((X-D)^S; \mathbf{Z}_\ell)$$

is quasi-isomorphic to $\mathbf{Z}_\ell$. Our goal in this section is to prove the following refinement which will be needed in §9.3:

Proposition 9.2.1. *Let R be a finitely generated k-algebra. Suppose we are given a finite set of R-valued points $y_1, \ldots, y_n \in X(R)$ having the property that the induced maps* $\mathrm{Spec}\, R \to X_R$ *have disjoint images, and let $U \subseteq X_R$ denote the complement of those images. For every nonempty finite set S, let U^S denote the S-fold fiber power of U over* $\mathrm{Spec}\, R$*, and let $\pi_S : U^S \to \mathrm{Spec}\, R$ denote the projection map. Then the canonical map*

$$\underline{\mathbf{Z}_\ell}_{\mathrm{Spec}\, R} \to \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \pi_{S*} \underline{\mathbf{Z}_\ell}_{U^S}$$

is an equivalence in $\mathrm{Shv}_\ell(\mathrm{Spec}\, R)$.

The proof of Proposition 9.2.1 will require some preliminaries. We begin by establishing a more quantitative version of Corollary 2.4.13.

Lemma 9.2.2. *Let $n \geq 0$ be an integer, and let $\mathrm{Fin}^s_{\leq n}$ denote the full subcategory of Fin^s spanned by those nonempty finite sets which have cardinality $\leq n$. For every smooth (not necessarily complete) algebraic curve Y over k, the canonical map*

$$\varinjlim_{S \in \mathrm{Fin}^s_{\leq n}} C_*(Y^S; \mathbf{Z}/\ell\mathbf{Z}) \to C_*(\mathrm{Spec}\, k; \mathbf{Z}/\ell\mathbf{Z})$$

induces an isomorphism on homology in degrees $\leq n-2$.

Proof. Using Corollary 2.4.13, we see that the direct limit

$$\varinjlim_{n} \varinjlim_{S \in \mathrm{Fin}^s_{\leq n}} C_*(Y^S; \mathbf{Z}/\ell\mathbf{Z})$$

is equivalent to $C_*(\mathrm{Spec}\, k; \mathbf{Z}/\ell\mathbf{Z})$. It will therefore suffice to show that each of the maps

$$\varinjlim_{S \in \mathrm{Fin}^s_{\leq n-1}} C_*(Y^S; \mathbf{Z}/\ell\mathbf{Z}) \to \varinjlim_{S \in \mathrm{Fin}^s_{\leq n}} C_*(Y^S; \mathbf{Z}/\ell\mathbf{Z})$$

has $(n-2)$-connective fiber (and therefore induces an isomorphism on homology in degrees $\leq n-3$). Set $S' = \{1, \ldots, n\}$. Applying Proposition 8.4.16, we deduce the existence of a pushout square

$$\begin{array}{ccc} (\varinjlim_{E \in \mathrm{Equiv}(S')} C_*(Y^{S'/E}; \mathbf{Z}/\ell\mathbf{Z}))_{\Sigma_n} & \longrightarrow & C_*(Y^{S'}; \mathbf{Z}/\ell\mathbf{Z})_{\Sigma_n} \\ \downarrow & & \downarrow \\ \varinjlim_{S \in \mathrm{Fin}^s_{\leq n-1}} C_*(Y^S; \mathbf{Z}/\ell\mathbf{Z}) & \longrightarrow & \varinjlim_{S \in \mathrm{Fin}^s_{\leq n}} C_*(Y^S; \mathbf{Z}/\ell\mathbf{Z}). \end{array}$$

We are therefore reduced to proving that the map

$$\varinjlim_{E \in \mathrm{Equiv}(S')} C_*(Y^{S'/E}; \mathbf{Z}/\ell\mathbf{Z}) \to C_*(Y^{S'}; \mathbf{Z}/\ell\mathbf{Z})$$

has $(n-2)$-connective fiber. Note that we can identify the domain of this map with $C_*(\Delta; \mathbf{Z}/\ell\mathbf{Z})$, where Δ denotes the closed subscheme of $Y^{S'}$ whose k-valued points are maps $S' \to Y(k)$ which are not injective. We are therefore reduced to proving that the map $C_*(\Delta; \mathbf{Z}/\ell\mathbf{Z}) \to C_*(Y^n; \mathbf{Z}/\ell\mathbf{Z})$ has $(n-2)$-connective fiber, which follows immediately from Proposition A.6.1. □

Lemma 9.2.3. *Let $f : Y \to Z$ be a morphism of quasi-projective k-schemes and let $\mathcal{F}$ be an object of $\mathrm{Shv}(Z; \mathbf{Z}/\ell\mathbf{Z})_{\leq n}$ which is equipped with an action of a finite group G. Then the canonical map $\alpha : f^*(\mathcal{F}^G) \to (f^* \mathcal{F})^G$ is an equivalence in $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$.*

Proof. We will prove that $\mathrm{fib}(\alpha)$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq m}$ for every integer m. Since the t-structure on $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ is right complete, it will follow that $\mathrm{fib}(\alpha) \simeq 0$ so that α is an equivalence. The proof proceeds by descending induction on m. The case $m = n$ is trivial, since $f^*(\mathcal{F}^G)$ and $(f^* \mathcal{G})^G$ both belong to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq n}$. To carry out the inductive step, let $\mathcal{G} = \prod_{g \in G} \mathcal{F}$, equipped with an action of G given by permuting the factors. Then we have a fiber sequence

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

of G-equivariant objects of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$, hence a commutative diagram of fiber sequences

$$\begin{array}{ccccc} f^*(\mathcal{F}^G) & \longrightarrow & f^*(\mathcal{G}^G) & \longrightarrow & f^*(\mathcal{H}^G) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ (f^*\mathcal{F})^G & \longrightarrow & (f^*\mathcal{G})^G & \longrightarrow & (f^*\mathcal{H})^G. \end{array}$$

Note that β is an equivalence (the domain and codomain of β can both be identified with $f^*\mathcal{F}$), so we obtain an equivalence $\mathrm{fib}(\alpha) \simeq \Sigma^{-1}\,\mathrm{fib}(\gamma)$. Invoking the inductive hypothesis, we deduce that $\mathrm{fib}(\gamma) \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq m+1}$, so that $\mathrm{fib}(\alpha) \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq m}$. □

Lemma 9.2.4. *Let $\widetilde{Y}$ be a quasi-projective k-scheme equipped with an action of a finite group G, and suppose we are given a finite G-equivariant map $f : \widetilde{Y} \to Y$ (where G acts trivially on Y). Suppose further that there is an open subscheme $U \subseteq Y$ such that $\widetilde{U} = U \times_Y \widetilde{Y}$ is a G-torsor over U (that is, G acts freely on $\widetilde{U}$ with quotient U as quotient). Let $K \subseteq X$ be the complement of U (regarded as a reduced closed subscheme of X) and set $\widetilde{K} = \widetilde{Y} \times_Y K$, so that we have a commutative diagram*

$$\begin{array}{ccc} \widetilde{K} & \xrightarrow{\ \widetilde{i}\ } & \widetilde{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ K & \xrightarrow{\ i\ } & Y. \end{array}$$

Then the induced diagram

$$\begin{array}{ccc} \underline{\mathbf{Z}/\ell\mathbf{Z}}_Y & \longrightarrow & (f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\widetilde{Y}})^G \\ \downarrow & & \downarrow \\ i_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_K & \longrightarrow & i_*(f'_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\widetilde{K}})^G \end{array}$$

is a pullback square in $\mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$.

Proof. Let $j : U \to Y$ and $\widetilde{j} : \widetilde{U} \to \widetilde{Y}$ be the inclusion maps. Unwinding the definitions, we wish to prove that the natural map $\theta_0 : j_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_U \to (f_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\widetilde{U}})^G$ is an equivalence. This can be checked after passing to stalks at each k-valued point $\eta : \mathrm{Spec}\, k \to X$. Note that since $f_* \widetilde{j}_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\widetilde{U}}$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq 0}$, the canonical map

$$\eta^*((f_* \widetilde{j}_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\widetilde{U}})^G) \to (\eta^* f_* \widetilde{j}_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\widetilde{U}})^G$$

is an equivalence (Lemma 9.2.3). We are therefore reduced to proving that the map

$$\eta^* j_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_U \to (\eta^* f_* \widetilde{j}_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\widetilde{U}})^G$$

is an equivalence. If η does not factor through U, then both sides vanish; otherwise, the desired result follows from our assumption that $\widetilde{U}$ is a G-torsor over U. □

Lemma 9.2.5. *Let R be a finitely generated k-algebra. Suppose we are given a finite set of R-valued points $y_1, \ldots, y_m \in X(R)$. Let $U \subseteq X_R$ denote the open subset complementary to the images of the maps $\mathrm{Spec}\, R \to X_R$ determined by the points y_i. For every nonempty finite set S, let U^S denote the S-fold fiber power of U over $\mathrm{Spec}\, R$, and let $\pi_S : U^S \to \mathrm{Spec}\, R$ denote the projection map. Then, for each integer $n \geq 0$, the object $\varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} \pi_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^S}$ is a constructible object of* $\mathrm{Shv}(\mathrm{Spec}\, R; \mathbf{Z}/\ell\mathbf{Z})$.

Proof. We proceed by induction on n, the case $n = 0$ being trivial. To carry out the inductive step, it will suffice to show that the fiber of the canonical map

$$\theta : \varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} \pi_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^S} \to \varprojlim_{S \in \mathrm{Fin}^s_{\leq n-1}} \pi_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^S}$$

is a constructible object of $\mathrm{Shv}(\mathrm{Spec}\, R; \mathbf{Z}/\ell\mathbf{Z})$. Let $S' = \{1, 2, \ldots, n\}$. Using Proposition 8.4.16, we can identify the fiber of θ with the fiber of the natural map

$$\theta' : (\pi(S')_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^{S'}})^{\Sigma_n} \to (\varprojlim_{E \in \mathrm{Equiv}^0(S')} \pi(S'/E)_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^{S'/E}})^{\Sigma_n}.$$

Let $\Delta \subseteq U^{S'}$ denote the union of the closed subschemes given by $U^{S'/E}$ where $E \in \mathrm{Equiv}^\circ(S)$, and let $\pi : \Delta \to \mathrm{Spec}\, R$ denote the projection map. Unwinding the definitions, we can identify the codomain of θ' with $(\pi_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_\Delta)^{\Sigma_n}$. Let Y denote the quotient of $U^{S'}$ by the action of the symmetric group Σ_n in the category of schemes and let $Y_0 \subseteq Y$ be the closed subscheme of Y given by the quotient of Δ by the action of Σ_n. Let $\phi : Y \to \mathrm{Spec}\, R$ and $\phi_0 : Y_0 \to \mathrm{Spec}\, R$ denote the projection maps. Since the action of Σ_n on $U^{S'}$ is free away from Δ, Lemma 9.2.4 implies that the fiber of θ' is equivalent to the fiber of the restriction map

$$\theta'' : \phi_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_Y \to \phi_{0*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{Y_0}.$$

Since the domain and codomain of θ'' are constructible, it follows that $\mathrm{fib}(\theta'') \simeq \mathrm{fib}(\theta') \simeq \mathrm{fib}(\theta)$ is constructible. □

Lemma 9.2.6. *Suppose we are given a pullback diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{f} & Z \end{array}$$

where g is smooth. Let $K \subseteq Y$ be a closed subset which is flat over Z whose inverse image in each fiber Y_z of the map g is a divisor with normal crossings, let $K' = Y' \times_Y K$, and let $j : Y - K \hookrightarrow Y$ and $j' : Y' - K' \hookrightarrow Y'$ be the corresponding open inclusions. Then the base change morphism $f'^* j_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{Y-K} \to j_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{Y'-K'}$ is an equivalence in $\mathrm{Shv}_\ell(Y')$.

Proof. The assertion can be tested locally with respect to the étale topology on Y and Z. We may therefore assume without loss of generality that Y is isomorphic to a product $Z \times \mathbf{A}^d$ and that $Y - K$ is given by the subset $Z \times \mathbf{G}_m^{d'} \times \mathbf{A}^{d''}$ where $d' + d'' = d$. In this case, the desired result follows immediately from Proposition 4.6.2. □

Lemma 9.2.7. *Let $\phi : \mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ be a morphism between affine schemes of finite type over k. Suppose we are given a finite set of R-valued points $y_1, \ldots, y_n \in X(R)$ having the property that the induced maps $\mathrm{Spec}\, R \to X_R$ have disjoint images, let $U \subseteq X_R$ denote the complement of those images, and let $U' = U \times_{X_R} X_{R'}$ be the resulting open subset of $X_{R'}$. For every nonempty finite set S, let U^S denote the S-fold fiber power of U over $\mathrm{Spec}\, R$ and define U'^S similarly. Let $\pi_S : U^S \to \mathrm{Spec}\, R$ and $\pi'_S : U'^S \to \mathrm{Spec}\, R'$ denote the projection maps. Then, for every integer n, the canonical map*

$$\phi^* \varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} \pi_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^S} \to \varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} \pi'_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U'^S}$$

is an equivalence in $\mathrm{Shv}(\mathrm{Spec}\, R'; \mathbf{Z}/\ell\mathbf{Z})$.

Proof. We proceed by induction on n, the case $n = 0$ being trivial. To carry out the inductive step, it will suffice to show that the diagram

$$\begin{array}{ccc} \phi^* \varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} \pi_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^S} & \longrightarrow & \varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} \pi'_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U'^S} \\ \downarrow & & \downarrow \\ \phi^* \varprojlim_{S \in \mathrm{Fin}^s_{\leq n-1}} \pi_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^S} & \longrightarrow & \varprojlim_{S \in \mathrm{Fin}^s_{\leq n-1}} \pi'_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U'^S} \end{array}$$

is a pullback square in $\mathrm{Shv}(\operatorname{Spec} R'; \mathbf{Z}/\ell\mathbf{Z})$. Let $S' = \{1, \ldots, n\}$. Applying Proposition 8.4.16, we are reduced to showing that the diagram σ :

$$\begin{array}{ccc} \phi^*((\pi_{S'*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^{S'}})^{\Sigma_n}) & \xrightarrow{\theta} & (\pi'_{S'*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U'^{S'}})^{\Sigma_n} \\ \downarrow & & \downarrow \\ \phi^*((\varinjlim_{E \in \mathrm{Equiv}^\circ(S')} \pi_{S'/E*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^{S'/E}})^{\Sigma_n}) & \xrightarrow{\theta'} & (\varinjlim_{E \in \mathrm{Equiv}^\circ(S')} \pi'_{S'/E*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U'^{S'/E}})^{\Sigma_n} \end{array}$$

is a pullback square in $\mathrm{Shv}(\operatorname{Spec} R'; \mathbf{Z}/\ell\mathbf{Z})$. To complete the proof, it will suffice to show that the maps θ and θ' are equivalences. We will prove that θ is an equivalence; the proof for θ' is analogous. Applying Lemma 9.2.3, we are reduced to showing that the canonical map $u : \phi^* \pi_{S'*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^{S'}} \to \pi'_{S'*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U'^{S'}}$ is an equivalence. The map $\pi_{S'}$ and $\pi'_{S'}$ can be factored as compositions

$$U^{S'} \stackrel{j}{\hookrightarrow} X^{S'} \times \operatorname{Spec} R \stackrel{\overline{\pi}_{S'}}{\to} \operatorname{Spec} R$$

$$U'^{S'} \stackrel{j'}{\hookrightarrow} X^{S'} \times \operatorname{Spec} R' \stackrel{\overline{\pi}'_{S'}}{\to} \operatorname{Spec} R'.$$

Let $\psi : X^{S'} \times \operatorname{Spec} R' \to X^{S'} \times \operatorname{Spec} R$ denote the pullback of ϕ. Using the proper base change theorem, we can identify u with $\overline{\pi}'_{S'*} v$ where v denotes the base change morphism $\psi^* j_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^{S'}} \to j'_* \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U'^{S'}}$. It follows from Lemma 9.2.6 that the map v is an equivalence. □

Proof of Proposition 9.2.1. Note that the domain and codomain of the natural map

$$u : \underline{\mathbf{Z}_\ell}_{\operatorname{Spec} R} \to \varprojlim_{S \in \mathrm{Fin}^s} \pi_{S*} \underline{\mathbf{Z}_\ell}_{U^S}$$

are ℓ-complete (this follows from Remark 4.3.36, since the collection of ℓ-complete objects of $\mathrm{Shv}_\ell(\operatorname{Spec} R)$ is closed under limits). It will therefore suffice to show that the image of u in $\mathrm{Shv}(\operatorname{Spec} R; \mathbf{Z}/\ell\mathbf{Z})$ is an equivalence.

For every integer $n \geq 0$, let $\mathcal{F}_n$ denote the inverse limit

$$\varprojlim_{S \in \mathrm{Fin}^s_{\leq n}} \pi_{S*} \underline{\mathbf{Z}/\ell\mathbf{Z}}_{U^S} \in \mathrm{Shv}(\operatorname{Spec} R; \mathbf{Z}/\ell\mathbf{Z}),$$

and let $\mathcal{F}_n^{\mathrm{red}}$ denote the cofiber of the unit map $\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\operatorname{Spec} R} \to \mathcal{F}_n$. We wish to prove that the limit $\varprojlim_n \mathcal{F}_n^{\mathrm{red}}$ is a zero object of $\mathrm{Shv}(\operatorname{Spec} R; \mathbf{Z}/\ell\mathbf{Z})$. In fact, we will prove something stronger: the tower $\{\mathcal{F}_n^{\mathrm{red}}\}$ is trivial as a Pro-object of $\mathrm{Shv}(\operatorname{Spec} R; \mathbf{Z}/\ell\mathbf{Z})$. To prove this, it will suffice to show that the natural map $\mathcal{F}_n^{\mathrm{red}} \to \mathcal{F}_m^{\mathrm{red}}$ vanishes for $n \gg m$. In fact, we claim that the group $\mathrm{Ext}^0_{\mathrm{Shv}(\operatorname{Spec} R; \mathbf{Z}/\ell\mathbf{Z})}(\mathcal{F}_n^{\mathrm{red}}, \mathcal{F}_m^{\mathrm{red}})$ vanishes for $n \gg m$. To prove this, we first note that $\mathcal{F}_m^{\mathrm{red}}$ is constructible. In particular, there exists an integer t such that $\mathcal{F}_m^{\mathrm{red}} \in \mathrm{Shv}(\operatorname{Spec} R; \mathbf{Z}/\ell\mathbf{Z})_{\geq t}$. Using Proposition 4.2.13, we deduce that there exists an integer s such that the group $\mathrm{Ext}^0_{\mathrm{Shv}(\operatorname{Spec} R; \mathbf{Z}/\ell\mathbf{Z})}(\mathcal{F}_m^{\mathrm{red}}, \mathcal{F}_n^{\mathrm{red}})$ vanishes whenever $\mathcal{F}_n^{\mathrm{red}}$ belongs to

$\mathrm{Shv}(\mathrm{Spec}\, R; \mathbf{Z}/\ell\mathbf{Z})_{\leq s}$. It will therefore suffice to show that $\mathcal{F}_n^{\mathrm{red}}$ belongs to $\mathrm{Shv}(\mathrm{Spec}\, R; \mathbf{Z}/\ell\mathbf{Z})_{\leq s}$ for $m \gg 0$. By virtue of Lemma 9.2.7, it will suffice to prove this when $R = \mathrm{Spec}\, k$, in which case it follows from Lemma 9.2.2. $\square$

9.3. **The Proof of Proposition 9.1.4.** Throughout this section, we fix an algebraically closed field k, a prime number ℓ which is invertible in k, an algebraic curve X over k, a smooth affine group scheme G over X, and an effective divisor $Q \subseteq X$. Let $\mathcal{B}$ and $\mathcal{B}_{\mathrm{red}}$ denote the !-sheaves on $\mathrm{Ran}(X)$ given by Notation 5.4.2 and Construction 8.2.12, and for each nonempty finite set S we let $\mathcal{B}_S$ and $\mathcal{B}_{S,\mathrm{red}}$ be defined as in Notation 7.2.5 and Construction 9.1.2. Our goal is to prove Proposition 9.1.4 by showing that the diagram

$$\begin{array}{ccc} \int \mathcal{B}_{\mathrm{red}} & \longrightarrow & \int \mathcal{B} \\ \downarrow & & \downarrow \\ \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,\mathrm{red}} & \longrightarrow & \int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S \end{array}$$

is a pullback square in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. We will prove this by establishing a version of Theorem 8.2.14 for the lower horizontal map. First, we need to introduce a bit of notation.

Construction 9.3.1. Let Θ be the category introduced in Notation 8.4.2. In this section, we will denote the objects of Θ by triples $\vec{T} = (T_0 \subseteq T_1 \subseteq T)$. Given an object $\vec{T} = (T_0 \subseteq T_1 \subseteq T)$ and a nonempty finite set S, we define a category $\mathcal{C}_S(\vec{T})$ as follows:

- The objects of $\mathcal{C}_S(\vec{T})$ are tuples $(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, $S_- \subseteq S_+ \subseteq S$, $\mu : S \to (X - Q)(R)$ and $\nu : T \to X(R)$ are maps of sets satisfying $|\mu(K_+)| \cap |\nu(T_1)| = \emptyset = |\mu(S)| \cap |\nu(T_0)|$, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu(S)|$.
- There are no morphisms from $(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ to $(R', K'_-, K'_+, \mu', \nu', \mathcal{P}', \gamma')$ unless $K'_- \subseteq K_- \subseteq K_+ \subseteq K'_+$. If this condition is satisfied, then a morphism from $(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ to $(R', K'_-, K'_+, \mu', \nu', \mathcal{P}', \gamma')$ consists of a k-algebra homomorphism $\phi : R \to R'$ which makes the diagrams

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{id}} & S \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ (X-Q)(R) & \xrightarrow{X(\phi)} & (X-Q)(R') \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\mathrm{id}} & T \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R') \end{array}$$

  commute, together with an isomorphism between $\mathrm{Spec}\, R' \times_{\mathrm{Spec}\, R} \mathcal{P}$ and $\mathcal{P}'$ over the open set $X_{R'} \times_{X_R} (X_R - |\mu(K_-)|)$ which carries γ to γ'.

The construction $(R, S_-, S_+, \mu, \nu, \mathcal{P}, \alpha) \mapsto (R, \nu)$ determines a map $\mathcal{C}_S(\vec{T}) \to X^T$, which exhibits $\mathcal{C}_S(\vec{T})$ as a prestack. Moreover, every morphism $\vec{T} \to \vec{T}'$ in the category Θ induces a map of prestacks $X^{T'} \times_{X^T} \mathcal{C}_S(\vec{T}) \to \mathcal{C}_S(\vec{T}')$. We may therefore view the construction

$$\vec{T} \mapsto (\mathcal{C}_S(\vec{T}) \to X^T)$$

as a contravariant functor from the category Θ to the ∞-category $\mathrm{RelStack}^!$ (see Construction A.5.14). We therefore obtain a functor

$$W_S : \Theta^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell},$$

given by $W_S(\vec{T}) = C^*(X^T; [\mathcal{C}_S(\vec{T})]_{X^T})$.

Let $W = \varprojlim_S W_S \in \mathrm{Fun}(\Theta^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}_\ell})$, so that the functor W is given by

$$W(T_0 \subseteq T_1 \subseteq T) \simeq \varprojlim_S C^*(X^T; [\mathcal{C}_S(\vec{T})]_{X^T}) \simeq C^*(X^T; \varprojlim_S [\mathcal{C}_S(\vec{T})]_{X^T}).$$

Notation 9.3.2. Let $V : \Theta^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ denote the functor given in Construction 8.6.4, given by the formula

$$V(T_0 \subseteq T_1 \subseteq T) = C^*(X^T; [\mathrm{Ran}^G(X)^{T_0 \subseteq T_1 \subseteq T}]_{X^T}).$$

Note that for each object $\vec{T} = (T_0 \subseteq T_1 \subseteq T) \in \Theta$ and each nonempty finite set S, we have a canonical map of prestacks

$$\mathcal{C}_S(\vec{T}) \to \mathrm{Ran}^G(X)^{T_0 \subseteq T_1 \subseteq T}$$

$$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma) \mapsto (R, \nu, \mathcal{P}\,|_{|\nu(T_1)|}, \gamma|_{|\nu(T_0)|}).$$

These maps depend functorially on S and $\vec{T}$, and therefore induce a natural transformation of functors

$$V \to W = \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} W_S.$$

We will deduce Proposition 9.1.4 from the following three assertions:

Lemma 9.3.3. *For every finite set T, the canonical map*

$$V(\emptyset \subseteq \emptyset \subseteq T) \to W(\emptyset \subseteq \emptyset \subseteq T)$$

is an equivalence.

Lemma 9.3.4. *Let T be a nonempty finite set, and let $\mathcal{J}$ be the category defined in Lemma 8.4.21. Then, for every nonempty finite set T and every subset $T_1 \subseteq T$, the canonical map*

$$\varinjlim_{T' \in \mathrm{Fin}^{\mathrm{s}}} W(\emptyset \subseteq T_1 \subseteq T \amalg T') \to W(\emptyset \subseteq T_1 \subseteq T)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Lemma 9.3.5. *Let $T_0 \subseteq T_1 \subseteq T$ be an object of Θ, let $t \in T$ be an element which is not contained in T_1, and set $T_0' = T_0 \cup \{t\}$, $T_1' = T_1 \cup \{t\}$. Then the diagram*

$$\begin{array}{ccc} \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \varprojlim_S W_S(T_0/E \subseteq T_1/E \subseteq T/E) & \longrightarrow & \varprojlim_S W_S(T_0 \subseteq T_1 \subseteq T) \\ \downarrow & & \downarrow \\ \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \varprojlim_S W_S(T_0'/E \subseteq T_1'/E \subseteq T/E) & \longrightarrow & \varprojlim_S W_S(T_0' \subseteq T_1' \subseteq T) \end{array}$$

is a pushout square in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proof of Proposition 9.1.4. We have a commutative diagram σ :

$$\begin{array}{ccccc} \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} V(\emptyset \subseteq \emptyset \subseteq T) & \longrightarrow & \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} V(\emptyset \subseteq T \subseteq T) & \longleftarrow & \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} V_{\mathrm{red}}(T) \\ \downarrow & & \downarrow & & \downarrow \\ \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} W(\emptyset \subseteq \emptyset \subseteq T) & \longrightarrow & \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} W(\emptyset \subseteq T \subseteq T) & \longleftarrow & \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} W_{\mathrm{red}}(T). \end{array}$$

To prove Proposition 9.1.4, it will suffice to show that the right square in this diagram is a pushout. The functor V is unital (in the sense of Definition 8.4.5) by virtue of Propositions

8.6.6 and 8.6.7, and the functor W is unital by virtue of Lemmas 9.3.4 and 9.3.5. It follows from Theorem 8.4.10 that the horizontal maps in this diagram determine equivalences

$$\varinjlim_{T\in\mathrm{Fin}^s} V(\emptyset \subseteq T \subseteq T) \simeq \varinjlim_{T\in\mathrm{Fin}^s} V(\emptyset \subseteq \emptyset \subseteq T) \oplus \varinjlim_{T\in\mathrm{Fin}^s} V_{\mathrm{red}}(T)$$

$$\varinjlim_{T\in\mathrm{Fin}^s} W(\emptyset \subseteq T \subseteq T) \simeq \varinjlim_{T\in\mathrm{Fin}^s} W(\emptyset \subseteq \emptyset \subseteq T) \oplus \varinjlim_{T\in\mathrm{Fin}^s} W_{\mathrm{red}}(T).$$

It will therefore suffice to show that the left vertical map in the diagram σ is an equivalence, which follows from Lemma 9.3.3. □

Proof of Lemma 9.3.3. For each nonempty finite set S, let $\mathcal{D}_S$ denote the prestack whose objects are tuples $(R, K_-, K_+, \mu, \mathcal{P}, \gamma)$, where R is a finitely generated k-algebra, $\mu : S \to (X-Q)(R)$ is a map of sets, $K_- \subseteq K_+ \subseteq S$, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ on $X_R - |\mu(S)|$, with morphisms defined as in Construction 9.3.1. For every nonempty finite set T, we have a canonical equivalence

$$\mathcal{C}_S(\emptyset \subseteq \emptyset \subseteq T) \simeq X^T \times_{\operatorname{Spec} k} \mathcal{D}_S,$$

which determines equivalences

$$[\mathcal{C}_S(\emptyset \subseteq \emptyset \subseteq T]_T \simeq C^*(\mathcal{D}_S; \mathbf{Z}_\ell) \otimes \omega_{X^T}$$

$$W_S(\emptyset \subseteq \emptyset \subseteq T) \simeq C^*(\mathcal{D}_S; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C_*(X^T; \mathbf{Z}_\ell)$$

$$W(\emptyset \subseteq \emptyset \subseteq T) \simeq (\varprojlim_{S\in\mathrm{Fin}^s} C^*(\mathcal{D}_S; \mathbf{Z}_\ell)) \otimes_{\mathbf{Z}_\ell} C_*(X^T; \mathbf{Z}_\ell)$$

$$\varinjlim_{T\in\mathrm{Fin}^s} W(\emptyset \subseteq \emptyset \subseteq T) \simeq (\varprojlim_{S\in\mathrm{Fin}^s} C^*(\mathcal{D}_S; \mathbf{Z}_\ell)) \otimes_{\mathbf{Z}_\ell} C_*(\mathrm{Ran}(X); \mathbf{Z}_\ell) \simeq (\varprojlim_{S\in\mathrm{Fin}^s} C^*(\mathcal{D}_S; \mathbf{Z}_\ell)).$$

It will therefore suffice to show that the composite map

$$\mathbf{Z}_\ell \to \varprojlim_{S\in\mathrm{Fin}^s} C^*((X-Q)^S; \mathbf{Z}_\ell) \to \varprojlim_{S\in\mathrm{Fin}^s} C^*(\mathcal{D}_S; \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. Corollary 2.4.13 implies that the first map is an equivalence. We are therefore reduced to proving that, for every nonempty finite set S, the canonical map $C^*((X-Q)^S; \mathbf{Z}_\ell) \to C^*(\mathcal{D}_S; \mathbf{Z}_\ell)$ is an equivalence. To prove this, we let $\mathcal{D}'_S$ denote the full subcategory of $\mathcal{D}_S$ spanned by those objects $(R, K_-, K_+, \mu, \mathcal{P}, \gamma)$ where $K_+ = S$, and $\mathcal{D}''_S$ the full subcategory spanned by those objects where $K_- = K_+ = S$. The inclusion map $\mathcal{D}'_S \hookrightarrow \mathcal{D}_S$ admits a left adjoint (in the 2-category of prestacks) and the inclusion $\mathcal{D}''_S \hookrightarrow \mathcal{D}'_S$ admits a right adjoint (again in the 2-category of prestacks). Using Remark 2.3.32, we are reduced to showing that the composite map

$$C^*((X-Q)^S; \Lambda) \to C^*(\mathcal{D}_S; \Lambda) \to C^*(\mathcal{D}'_S; \Lambda) \to C^*(\mathcal{D}''_S; \Lambda)$$

is an equivalence. This is clear, since $\mathcal{D}''_S$ is equivalent to (the prestack represented by) $(X-Q)^S$. □

Proof of Lemma 9.3.4. For every finite set T' and every finite set S, we have a canonical equivalence of prestacks

$$\mathcal{C}_S(\emptyset \subseteq T_1 \subseteq T \amalg T') \simeq X^{T'} \times_{\operatorname{Spec} k} \mathcal{C}_S(\emptyset \subseteq T_1 \subseteq T),$$

which induces an equivalence

$$[\mathcal{C}_S(\emptyset \subseteq T_1 \subseteq T \amalg T')]_{X^{T \amalg T'}} \simeq [\mathcal{C}_S(\emptyset) \subseteq T_1 \subseteq T]_{X^T} \boxtimes \omega_{X^{T'}},$$

in the ∞-category $\mathrm{Shv}_\ell(X^{T\amalg T'})$, hence an equivalence

$$W_S(\emptyset \subseteq T_1 \subseteq T \amalg T') \simeq W_S(\emptyset \subseteq T_1 \subseteq T) \otimes C^*(X^{T'}; \omega_{X^{T'}}).$$

Since $C^*(X^{T'}; \omega_{X^{T'}}) \simeq C_*(X^{T'}; \mathbf{Z}_\ell)$ is a perfect $\mathbf{Z}_\ell$-module, we can take the inverse limit over S to obtain an equivalence

$$W(\emptyset \subseteq T_1 \subseteq T \amalg T') \simeq W(\emptyset \subseteq T_1 \subseteq T) \otimes_{\mathbf{Z}_\ell} C_*(X^{T'}; \mathbf{Z}_\ell).$$

Setting $M = V(\emptyset \subseteq T_1 \subseteq T)$, we are reduced to proving that the canonical map

$$\varinjlim_{T'} M \otimes_{\mathbf{Z}_\ell} C_*(X^{T'}; \mathbf{Z}_\ell) \to M$$

is an equivalence, which follows from the acyclicity of $\mathrm{Ran}(X)$ (Corollary 2.4.13). □

Proof of Lemma 9.3.5. Let Y be a quasi-projective k-scheme equipped with a map $Y \to X^T$. For each equivalence relation $E \in \mathrm{Equiv}(T)$, we let $Y(E)$ denote the fiber product $Y \times_{X^T} X^{T/E}$, and define prestacks $Z_S(E)$ and $Z'_S(E)$ by the formulae

$$Z_S(E) = \mathcal{C}_S(T_0/E \subseteq T_1/E \subseteq T/E) \times_{X^{T/E}} Y(E)$$

$$Z'_S(E) = \mathcal{C}_S(T'_0/E \subseteq T'_1/E \subseteq T/E) \times_{X^{T/E}} Y(E).$$

Let $i(E) : Y(E) \to Y$ denote the inclusion maps, and define sheaves $\mathcal{F}^Y_{E,S}, \mathcal{F}'^Y_{E,S} \in \mathrm{Shv}_\ell(Y)$ by the formulae

$$\mathcal{F}^Y_{E,S} = i(E)_*[Z_S(E)]_{Y^E} \qquad \mathcal{F}'^Y_{E,S} = i(E)_*[Z'_S(E)]_{Y^E}.$$

Let E_0 be the trivial equivalence relation on T (so that $T/E_0 \simeq T$). We will prove that the diagram σ_Y:

$$\begin{array}{ccc} \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \varprojlim_S \mathcal{F}^Y_{E,S} & \longrightarrow & \varinjlim_{E \in \mathrm{Equiv}^\circ(T)} \varprojlim_S \mathcal{F}'^Y_E \\ \downarrow & & \downarrow \\ \varprojlim_S \mathcal{F}^Y_{E_0,S} & \longrightarrow & \varprojlim_S \mathcal{F}'^Y_{E_0,S} \end{array}$$

is a pushout square in $\mathrm{Shv}_\ell(Y)$. Taking $Y = X^T$ and passing to global sections, this will give a proof of Lemma 9.3.5.

We first prove the following:

$(*)$ If $i : Y' \to Y$ is a closed immersion, then the canonical maps

$$\mathcal{F}^{Y'}_{E,S} \to i^! \mathcal{F}^Y_{E,S} \qquad \mathcal{F}'^{Y'}_{E,S} \to i^! \mathcal{F}'^Y_{E,S},$$

are equivalences for every nonempty finite set S and every equivalence relation $E \in \mathrm{Equiv}(T)$.

To prove $(*)$, let us regard E and S as fixed. Let P denote the set of pairs (K_-, K_+), where $K_- \subseteq K_+ \subseteq S$. We regard P as a partially ordered set, with $(K_-, K_+) \leq (K'_-, K'_+)$ if and only if $K'_- \subseteq K_- \subseteq K_+ \subseteq K'_+$. The construction $(R, K_-, K_+, \mu, \nu, \mathcal{P}, \alpha) \mapsto (K_-, K_+)$ determines fibrations of categories

$$\rho : \mathcal{C}_S(T_0 \subseteq T_1 \subseteq T) \to P \qquad \rho' : \mathcal{C}_S(T'_0 \subseteq T'_1 \subseteq T) \to P.$$

For each pair $(K_-, K_+) \in P$, we let $\mathcal{E}_{K_-,K_+}$ and $\mathcal{E}'_{K_-,K_+}$ denote the fibers of ρ and ρ' over (K_-, K_+). Then we can identify the canonical map

$$\mathcal{F}^{Y'}_{E,S} \to i^! \mathcal{F}^Y_{E,S}$$

with an inverse limit of maps

$$i'(E)_*[Y'(E) \times_{X^T} \mathcal{E}_{K_-,K_+}]_{Y'(E)} \to i^! i(E)_*[Y(E) \times_{X^T} \mathcal{E}_{K_-,K_+}]_{Y(E)}$$

as (K_-, K_+) varies (here $i(E) : Y(E) \to Y$ and $i'(E) : Y'(E) \to Y'$ denote the inclusion maps). Each of these maps is an equivalence by virtue of Proposition 5.1.13, so that $\mathcal{F}^{Y'}_{E,S} \simeq i^! \mathcal{F}^{Y}_{E,S}$. A similar argument shows that the natural map $\mathcal{F}'^{Y'}_{E,S} \to i^! \mathcal{F}'^{Y}_{E,S}$, which proves $(*)$.

It follows from $(*)$ that for any closed immersion $i : Y' \to Y$, we can identify $\sigma_{Y'}$ with $i^! \sigma_Y$. Proceeding by Noetherian induction, we may assume that $i^! \sigma_Y$ is a pullback square for every closed subscheme $Y' \subsetneq Y$. If Y is nonreduced, then we can take $Y' = Y_{\text{red}}$ to complete the proof. Let us therefore assume that Y is reduced. Let $Y' \subsetneq Y$ be a closed subscheme, and let U be the complement of Y' in Y. Then we have a fiber sequence of diagrams

$$i_* i^! \sigma_Y \to \sigma_Y \to j_* j^* \sigma_Y.$$

It will therefore suffice to prove that $j^* \sigma_Y \simeq \sigma_U$ is a pullback square in $\text{Shv}_\ell(U)$. We are therefore free to replace Y by any nonempty open subscheme $U \subseteq Y$. In particular, we may assume that Y is smooth, that the map $Y \to X^T$ is Q-adapted (Definition 7.3.9), and that there is a fixed equivalence relation $E_1 \in \text{Equiv}(T)$ such that the map $Y \to X^T$ factors through X^{T/E_1}, but the fiber product $Y \times_{X^T} X^{T/E} = \emptyset$ for $E \geq E_1$.

The proof now breaks into two cases. Suppose first that $E_0 \neq E_1$. Suppose first that $E_0 \neq E_1$. In this case, we will complete the proof by showing that the maps

$$\theta : \varinjlim_{E \in \text{Equiv}^\circ(T)} \varprojlim_{S} \mathcal{F}^{Y}_{E,S} \to \varprojlim_{S \in \text{Fin}^{\text{s}}} \mathcal{F}^{Y}_{E_0,S}$$

$$\theta' : \varinjlim_{E \in \text{Equiv}^\circ(T)} \varprojlim_{S} \mathcal{F}'^{Y}_{E,S} \to \varprojlim_{S \in \text{Fin}^{\text{s}}} \mathcal{F}'^{Y}_{E_0,S}$$

are equivalences in $\text{Shv}_\ell(Y)$. We will give the proof for θ; the proof for θ' is the same. The main point is to prove the following assertion:

- $(*')$ Let $E \in \text{Equiv}(T)$, and let E' be the equivalence relation on T generated by E and E_1. Then, for every nonempty finite set S, the canonical map $\mathcal{F}^{Y}_{E',S} \to \mathcal{F}^{Y}_{E,S}$ is an equivalence in $\text{Shv}_\ell(Y)$.

Assertion $(*')$ follows immediately from the observation that we have an equivalence of prestacks $Z_S(E) \simeq Z_S(E')$. It follows from $(*')$ that the diagram $\{\varprojlim_{S \in \text{Fin}^{\text{s}}} \mathcal{F}^{Y}_{E,S}\}_{E \in \text{Equiv}^\circ(T)}$ is a left Kan extension of the diagram $\{\varprojlim_{S \in \text{Fin}^{\text{s}}} \mathcal{F}^{Y}_{E,S}\}_{E \geq E_1}$, so that we have equivalences

$$\varinjlim_{E \in \text{Equiv}^\circ(T)} \varprojlim_{S \in \text{Fin}^{\text{s}}} \mathcal{F}^{Y}_{E,S} \simeq \varinjlim_{E \geq E_1} \varprojlim_{S \in \text{Fin}^{\text{s}}} \mathcal{F}^{Y}_{E,S} \simeq \varprojlim_{S \in \text{Fin}^{\text{s}}} \mathcal{F}^{Y}_{E_1,S},$$

so that θ can be identified with an inverse limit of maps of the form $\mathcal{F}^{Y}_{E_1,S} \to \mathcal{F}^{Y}_{E_0,S}$ and is therefore an equivalence by virtue of $(*')$.

We now treat the case where $E_0 = E_1$. In this case, for all $E \neq E_0$ we have $Y(E) = \emptyset$ and therefore $\mathcal{F}^{Y}_{E,S} \simeq \mathcal{F}'^{Y}_{E,S} \simeq 0$. We are therefore reduced to proving that the canonical map

$$\varprojlim_{S} \mathcal{F}^{Y}_{E_0,S} \to \varprojlim_{S} \mathcal{F}'^{Y}_{E_0,S}$$

is an equivalence in $\text{Shv}_\ell(Y)$. To prove this, we let $\mathcal{E}_{S,S}$ denote the full subcategory of $\mathcal{C}_S(T_0 \subseteq T_1 \subseteq T)$ spanned by those objects $(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ where $K_- = K_+ = S$, and define $\mathcal{E}'_{S,S} \subseteq \mathcal{C}_S(T'_0 \subseteq T'_1 \subseteq T)$ similarly. Set

$$\mathcal{G}_S = [\mathcal{E}_{S,S} \times_{X^T} Y]_Y \qquad \mathcal{G}'_S = [\mathcal{E}'_{S,S} \times_{X^T} Y]_Y.$$

We have a commutative diagram of prestacks

$$\begin{array}{ccc} Y \times_{X^T} \mathcal{E}'_S & \longrightarrow & Y \times_{X^T} \mathcal{E}_S \\ \downarrow & & \downarrow \\ Z'_S(E_0) & \longrightarrow & Z_S(E_0), \end{array}$$

which induces a commutative diagram τ_S:

$$\begin{array}{ccc} \mathcal{G}'_S & \longleftarrow & \mathcal{G}_S \\ \uparrow & & \uparrow \\ \mathcal{F}'^{Y}_{E_0,S} & \longleftarrow & \mathcal{F}^{Y}_{E_0,S} \end{array}$$

in the ∞-category $\mathrm{Shv}_\ell(Y)$, depending functorially on $S \in \mathrm{Fin}^{\mathrm{s}}$. We will complete the proof by verifying the following:

(a) Each of the diagrams τ_S is a pullback square in $\mathrm{Shv}_\ell(Y)$.
(b) The canonical map $\varprojlim_S \mathcal{G}_S \to \varprojlim_S \mathcal{G}'_S$ is an equivalence in $\mathrm{Shv}_\ell(Y)$.

To prove (b), we observe that there is a commutative diagram

$$\begin{array}{ccc} & \omega_Y & \\ \swarrow & & \searrow \\ \varprojlim_S \mathcal{G}_S & \longrightarrow & \varprojlim_S \mathcal{G}'_S, \end{array}$$

where the vertical maps are equivalences by virtue of Proposition 9.2.1.

We now prove (a). For the remainder of the proof, we regard the set S as fixed. Let P be defined as above, and for each pair $(K_-, K_+) \in P$ we define

$$\mathcal{H}_{K_-,K_+} = [\mathcal{E}_{K_-,K_+} \times_{X^T} Y]_Y \qquad \mathcal{H}'_{K_-,K_+} = [\mathcal{E}'_{K_-,K_+} \times_{X^T} Y]_Y$$

Unwinding the definitions, we can identify $\mathcal{G}_S$ and $\mathcal{G}'_S$ with $\mathcal{H}_{S,S}$ and $\mathcal{H}'_{S,S}$, respectively. We may therefore identify τ_S with the diagram

$$\begin{array}{ccc} \varprojlim_{(K_-,K_+)\in P} \mathcal{H}_{K_-,K_+} & \longrightarrow & \varprojlim_{(K_-,K_+)\in P} \mathcal{H}_{K_-,K_+} \\ \downarrow & & \downarrow \\ \mathcal{H}_{S,S} & \longrightarrow & \mathcal{H}'_{S,S} . \end{array}$$

For each $(K_-, K_+) \in P$, form a fiber sequence

$$\mathcal{H}''_{K_-,K_+} \to \mathcal{H}_{K_-,K_+} \to \mathcal{H}'_{K_-,K_+} .$$

We will regard the construction $(K_-, K_+) \mapsto \mathcal{H}''_{K_-,K_+}$ as a functor $P \to \mathrm{Shv}_\ell(Y)$. To prove ($a$), we must show that the evaluation amp

$$\varprojlim_{(K_-,K_+)\in P} \mathcal{H}''_{K_-,K_+} \to \mathcal{H}''_{S,S}$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$. In fact, we will prove something slightly more general. Let n denote the cardinality of the set S. For each $m \leq n$, we let P_m denote the subset of P consisting of those pairs (K_-, K_+) where $|S_-| \geq m$. Then $P_0 = P$, and $P_n = \{(S, S)\}$. Assertion (a) is a special case of the following:

(a') For each integer $m \leq n$, the restriction map

$$\varprojlim_{(K_-, K_+) \in P} \mathcal{H}''_{K_-, K_+} \to \varprojlim_{(K_-, K_+) \in P_m} \mathcal{H}''_{K_-, K_+}$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$.

The proof of (a') proceeds by induction on m, the case $m = 0$ being obvious. To carry out the inductive step, we let Q denote the subset of P_m consisting of those pairs (K_-, K_+) with $|K_-| \geq m$ and $|K_+| > m$. We claim that the restriction maps

$$\varprojlim_{(K_-, K_+) \in P_m} \mathcal{H}''_{K_-, K_+} \to \varprojlim_{(K_-, K_+) \in Q} \mathcal{H}''_{K_-, K_+} \to \varprojlim_{(K_-, K_+) \in P_{m+1}} \mathcal{H}''_{K_-, K_+}$$

are equivalences. To prove this, it suffices to verify the following pair of assertions:

(i) The inclusion of partially ordered sets $P_{m+1} \hookrightarrow Q$ induces a right cofinal map of simplicial sets $\mathrm{N}(P_{m+1}) \hookrightarrow \mathrm{N}(Q)$.
(ii) If $m < n$, then the functor $\mathcal{H}''|_{P_m}$ is a right Kan extension of its restriction to Q.

To prove (i), we must show that for every element $(K_-, K_+) \in Q$, the partially set $Z = \{(K'_-, K'_+) \in P : (K'_-, K'_+) \leq (K_-, K_+), |K'_-| > m\}$ has weakly contractible nerve. Let Z_0 denote the subset of Z consisting of those pairs (K'_-, K'_+) with $K'_+ = K_+$. The inclusion $Z_0 \hookrightarrow Z$ admits a left adjoint, given by $(K'_-, K'_+) \mapsto (K'_-, K_+)$. We are therefore reduced to proving that the nerve of Z_0 is weakly contractible. This is clear, since Z_0 contains (K_+, K_+) as a least element.

We now prove (ii). Fix an element $(K_-, K_+) \in P_m$; we wish to show that $\mathcal{H}''$ is a right Kan extension of $\mathcal{H}''|_Q$ at (K_-, K_+). This is obvious if (K_-, K_+) belongs to Q. We may therefore assume without loss of generality that $K_- = K_+$ and that $|K_-| = m$. Set $K = K_- = K_+$. Unwinding the definitions, we note that an element $(K'_-, K'_+) \in P_m$ satisfies $(K, K) \leq (K'_-, K'_+)$ if and only if $K'_- = K \subseteq K'_+$. We are therefore reduced to proving that the map

$$\mathcal{H}''_{K,K} \to \varprojlim_{K \subsetneq K'} \mathcal{H}''_{K,K'}$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$. Equivalently, we wish to prove that the diagram

$$\begin{array}{ccc} \mathcal{H}_{K,K} & \longrightarrow & \mathcal{H}'_{K,K} \\ \downarrow & & \downarrow \\ \varprojlim_{K \subsetneq K'} \mathcal{H}_{K,K'} & \longrightarrow & \varprojlim_{K \subsetneq K'} \mathcal{H}'_{K,K'} \end{array}$$

is a pullback square in $\mathrm{Shv}_\ell(Y)$. Note that for $K \subseteq K'$, we can regard $\mathcal{E}_{K,K'}$ and $\mathcal{E}'_{K,K'}$ as open substacks of $\mathcal{E}_{K,K}$. Moreover, we have

$$\mathcal{E}_{K,K'} = \bigcap_{s \in K' - K} \mathcal{E}_{K, K \cup \{s\}} \qquad \mathcal{E}'_{K,K'} = \mathcal{E}'_{K,K} \cap \bigcap_{s \in K' - K} \mathcal{E}_{K, K \cup \{s\}}.$$

Using Zariski descent, we are reduced to proving that the open substacks

$$Y \times_{X^T} \mathcal{E}'_{K,K}, Y \times_{X^T} \mathcal{E}_{K, K \cup \{s\}} \subseteq Y \times_{X^T} \mathcal{E}_{K,K}$$

comprise an open covering of $\mathcal{E}_{K,K}$, which follows from our assumption that $E_0 = E_1$. This completes the proof of (a). □

9.4. **Digression: Verdier Duality on** $\mathrm{Ran}(X)$**.** Let G be a smooth affine group scheme over an algebraic curve X which is Q-adapted for some effective divisor $Q \subseteq X$. Proposition 9.1.5 asserts that the canonical map

$$\int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} \to \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell)$$

is a quasi-isomorphism. This assertion has a natural interpretation in terms of Verdier duality on the Ran space $\mathrm{Ran}(X)$. Our goal in this section is to explain this interpretation, since it is the foundation on which our proof is constructed. For purposes of this paper, we will regard "Verdier duality on $\mathrm{Ran}(X)$" as a heuristic device; the discussion in this section will be somewhat informal and we will generally omit proofs. However, we will use this discussion to motivate the introduction of some auxiliary sheaves which will be precisely defined in Notation 9.4.14 and will use to break the proof of Proposition 9.1.5 into two parts (Proposition 9.4.17 and Proposition 9.4.18), which will be treated in the sections which follow. A reader who is interested in the most direct route to the proof of Theorem 5.4.5 (or who has a low tolerance for informal heuristics) can safely skip most of this section, proceeding directly to Notation 9.4.14 and the discussion which follows.

Remark 9.4.1. For a rigorous treatment of Verdier duality for !-sheaves on $\mathrm{Ran}(X)$ (and an alternative derivation of Theorem 5.4.5), we refer the reader to [19].

9.4.1. *Verdier Duality on Locally Compact Spaces.* Fix an ∞-category $\mathcal{C}$ which admits limits and colimits. Let X be a topological space and let $\mathcal{U}(X)$ denote the partially ordered set of all open subsets of X. A $\mathcal{C}$*-valued sheaf* on X is a functor $\mathcal{F} : \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{C}$ with the following property: for each open cover $\{U_\alpha\}$ of an open set $U \subseteq X$, the canonical map

$$\mathcal{F}(U) \to \varprojlim_{V} \mathcal{F}(V)$$

is an equivalence in $\mathcal{C}$, where V ranges over all open subsets of U which are contained in some U_α. The collection of all $\mathcal{C}$-valued sheaves on X can be organized into an ∞-category, which we will denote by $\mathrm{Shv}_{\mathcal{C}}(X)$.

There is an evident dual notion of $\mathcal{C}$*-valued cosheaf on* X: a functor $\mathcal{F} : \mathcal{U}(X) \to \mathcal{C}$ with the property that for every open cover $\{U_\alpha\}$ of an open set $U \subseteq X$, the canonical map $\varinjlim_V \mathcal{F}(V) \to \mathcal{F}(U)$ is an equivalence in $\mathcal{C}$ (again the colimit is taken over all open subsets of U which are contained in some U_α). The collection of all $\mathcal{C}$-valued cosheaves on X can be organized into an ∞-category which we will denote by $\mathrm{cShv}_{\mathcal{C}}(X)$. Equivalently, we can define the ∞-category $\mathrm{cShv}_{\mathcal{C}}(X)$ by the formula

$$\mathrm{cShv}_{\mathcal{C}}(X) = \mathrm{Shv}_{\mathcal{C}^{\mathrm{op}}}(X)^{\mathrm{op}}.$$

Construction 9.4.2. Let X be a Hausdorff topological space and let $\mathcal{F}$ be a $\mathcal{C}$-valued sheaf on X. For every closed subset $K \subseteq X$, we let $\mathcal{F}_K$ denote the fiber of the natural map $\mathcal{F}(X) \to \mathcal{F}(X-K)$. Note that $\mathcal{F}_K$ is a covariant functor of K.

Given an open set $U \subseteq X$, we let $\mathcal{F}_c(U)$ denote the colimit $\varinjlim_{K \subseteq U} \mathcal{F}_K$, taken over all compact subsets $K \subseteq U$. We will refer to the object $\mathcal{F}_c(U) \in \mathcal{C}$ as the *compactly supported sections of* $\mathcal{F}$ *over* U.

Remark 9.4.3. In the situation of Construction 9.4.2, the assignment $U \mapsto \mathcal{F}_c(U)$ determines a *covariant* functor from the partially ordered set of open subsets of X to the ∞-category $\mathcal{C}$.

Remark 9.4.4. Let $\mathcal{F} \in \mathrm{Shv}_{\mathcal{C}}(X)$ and let U and V be open subsets of X. For every compact set $K \subseteq U$, we have a canonical map

$$\mathcal{F}_K = \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(X-K)) \to \mathcal{F}(X) \to \mathcal{F}(V).$$

Passing to the colimit over K, we obtain a natural map $\mathcal{F}_c(U) \to \mathcal{F}(V)$. By construction, this map vanishes whenever U and V are disjoint.

Theorem 9.4.5 (Covariant Verdier Duality)**.** *Let $\mathcal{C}$ be a stable ∞-category which admits limits and colimits and let X be a Hausdorff space. Then the construction $\mathcal{F} \mapsto \mathcal{F}_c$ carries $\mathcal{C}$-valued sheaves on X to $\mathcal{C}$-valued cosheaves on X. If X is locally compact, then the construction $\mathcal{F} \mapsto \mathcal{F}_c$ determines an equivalence of ∞-categories*

$$\mathrm{Shv}_{\mathcal{C}}(X) \simeq \mathrm{cShv}_{\mathcal{C}}(X).$$

In other words, if X is a locally compact Hausdorff space, then the ∞-categories $\mathrm{Shv}_{\mathcal{C}}(X)$ and $\mathrm{Shv}_{\mathcal{C}^{\mathrm{op}}}(X)$ are canonically opposite to one another. This equivalence is nontrivial, and involves the topology of X in an essential way.

Remark 9.4.6 (Functoriality)**.** Let $f : X \to Y$ be a continuous map of topological spaces. For any ∞-category $\mathcal{C}$, f determines a pushforward functor $f_* : \mathrm{Shv}_{\mathcal{C}}(X) \to \mathrm{Shv}_{\mathcal{C}}(Y)$ by the formula $(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}U)$. Similarly, we have a pushforward operation on cosheaves $f_+ : \mathrm{cShv}_{\mathcal{C}}(X) \to \mathrm{cShv}_{\mathcal{C}}(Y)$, given by $(f_+ \mathcal{F})(U) = \mathcal{F}(f^{-1}U)$.

If $\mathcal{C}$ is a stable ∞-category which admits limits and colimits, one can show that the functor f_* admits a left adjoint f^*, and the functor f_+ admits a right adjoint f^+. If X and Y are locally compact, then Theorem 9.4.5 supplies equivalences of ∞-categories

$$\mathrm{Shv}_{\mathcal{C}}(X) \simeq \mathrm{cShv}_{\mathcal{C}}(X) \qquad \mathrm{Shv}_{\mathcal{C}}(Y) \simeq \mathrm{cShv}_{\mathcal{C}}(Y).$$

Under these equivalences, we can identify f_+ and f^+ with a pair of adjoint functors

$$f_! : \mathrm{Shv}_{\mathcal{C}}(Y) \to \mathrm{Shv}_{\mathcal{C}}(Z) \qquad f^! : \mathrm{Shv}_{\mathcal{C}}(Z) \to \mathrm{Shv}_{\mathcal{C}}(Y).$$

Unwinding the definitions, we see that the functor $f_!$ is characterized by the formula $(f_! \mathcal{F})_c(U) = \mathcal{F}_c(f^{-1}U)$. We will refer to $f_!$ as the functor of *direct image with compact supports*.

Example 9.4.7. Let R be a commutative ring, and let Mod_R denote the stable ∞-category of chain complexes over R (Example 2.1.23). Then R-linear duality defines a contravariant functor from Mod_R to itself, which carries colimits to limits. Consequently, every Mod_R-valued cosheaf $\mathcal{G}$ on a topological space X determines a Mod_R-valued sheaf $\mathcal{G}^\vee$, given by $\mathcal{G}^\vee(U) = (\mathcal{G}(U))^\vee$.

Let $\mathcal{F}$ be a Mod_R-valued sheaf on a Hausdorff space X, and let $\mathcal{F}_c$ be the associated cosheaf. We let $\mathbf{D}(\mathcal{F})$ denote the Mod_R-valued sheaf $\mathcal{F}_c^\vee$. We refer to $\mathbf{D}(\mathcal{F})$ as the *Verdier dual* of $\mathcal{F}$. We let ω_X denote the Verdier dual of the constant sheaf $\underline{R}_X$ on X, which we refer to as the *dualizing sheaf* on X.

Assuming that X is locally compact, for any pair of sheaves $\mathcal{F}$ and $\mathcal{G}$ we have equivalences

$$\mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(X)}(\mathcal{F}, \mathbf{D}(\mathcal{G})) \simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(X)}(\mathcal{G}, \mathbf{D}(\mathcal{F})),$$

since both sides can be identified with the space of maps from $(\mathcal{F} \otimes \mathcal{G})_c(X)$ to R in Mod_R. In particular, we obtain an equivalence

$$\begin{aligned} \mathcal{F}(X) &\simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(X)}(\underline{R}, \mathbf{D}(\mathcal{F})) \\ &\simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(X)}(\mathcal{F}, \mathbf{D}(\underline{R})) \\ &\simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(X)}(\mathcal{F}, \omega_X) \end{aligned}$$

of chain complexes over R. Performing a similar calculation over each open subset of R, we see that $\mathbf{D}(\mathcal{F})$ can be identified with the sheaf which classifies maps from $\mathcal{F}$ into ω_X.

9.4.2. *Verdier Duality on Infinite-Dimensional Spaces.* Let us now consider what Verdier duality can tell us about a topological space X which is not locally compact, such as the Ran space $\mathrm{Ran}(M)$ of a manifold M. For each integer $n \geq 0$, let $\mathrm{Ran}_{\leq n}(M)$ denote the subset of $\mathrm{Ran}(M)$ consisting of finite subsets $S \subseteq M$ which have cardinality $\leq n$. Each $\mathrm{Ran}_{\leq n}(M)$ is a locally compact topological space (which can be identified with a quotient of M^n). Let us regard $\mathrm{Ran}(M)$ as equipped with the direct limit topology, so that a subset $U \subseteq \mathrm{Ran}(M)$ is open if and only if its intersection with each $\mathrm{Ran}_{\leq n}(M)$ is open (beware that this is not quite the same as the topology on $\mathrm{Ran}(M)$ discussed in §2.4).

Proposition 9.4.8. *Let X be a paracompact topological space which is written as a direct limit of closed subsets*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

and let $\mathcal{C}$ be a stable ∞-category which admits limits and colimits. Then $\mathrm{Shv}_{\mathcal{C}}(X)$ can be identified with the inverse limit $\varprojlim \mathrm{Shv}_{\mathcal{C}}(X_n)$.

Remark 9.4.9. For a proof of Proposition 9.4.8 in the special case where $\mathcal{C}$ is the ∞-category of spaces, we refer the reader to Proposition HTT.7.1.5.8.

Let X be as in Proposition 9.4.8, so that giving a $\mathcal{C}$-valued sheaf $\mathcal{F}$ on the space X is equivalent to giving a sequence of $\mathcal{C}$-valued sheaves $\mathcal{F}_n \in \mathrm{Shv}_{\mathcal{C}}(X_n)$, together with equivalences $\mathcal{F}_n \simeq i(n)^* \mathcal{F}_{n+1}$, where $i(n) : X_n \to X_{n+1}$ denotes the inclusion map. Applying the same reasoning to the ∞-category $\mathcal{C}^{\mathrm{op}}$, we see that $\mathrm{cShv}_{\mathcal{C}}(X)$ can be identified with the inverse limit of the tower

$$\cdots \to \mathrm{cShv}_{\mathcal{C}}(X_3) \overset{i(2)^+}{\to} \mathrm{cShv}_{\mathcal{C}}(X_2) \overset{i(1)^+}{\to} \mathrm{cShv}_{\mathcal{C}}(X_1) \overset{i(0)^+}{\to} \mathrm{cShv}_{\mathcal{C}}(X_0).$$

Suppose now that each of the spaces X_i is locally compact. Applying Theorem 9.4.5 to each term in this sequence, we obtain an equivalent tower

$$\cdots \to \mathrm{Shv}_{\mathcal{C}}(X_3) \overset{i(2)^!}{\to} \mathrm{Shv}_{\mathcal{C}}(X_2) \overset{i(1)^!}{\to} \mathrm{Shv}_{\mathcal{C}}(X_1) \overset{i(0)^!}{\to} \mathrm{Shv}_{\mathcal{C}}(X_0).$$

We will denote the inverse limit of this tower by $\mathrm{Shv}^!_{\mathcal{C}}(X)$, and refer to it as the *∞-category of $\mathcal{C}$-valued !-sheaves* on X. More informally, the objects of $\mathrm{Shv}^!_{\mathcal{C}}(X)$ are sequences of sheaves $\mathcal{F}_n \in \mathrm{Shv}_{\mathcal{C}}(X_n)$, together with equivalences $\mathcal{F}_n \simeq i(n)^! \mathcal{F}_{n+1}$. We can summarize our discussion as follows:

Proposition 9.4.10. *Let X be a paracompact topological space which is given as a direct limit of locally compact closed subsets*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

Then the construction above supplies an equivalence of ∞-categories

$$\mathrm{cShv}_{\mathcal{C}}(X) \simeq \mathrm{Shv}^!_{\mathcal{C}}(X).$$

Remark 9.4.11. In the situation of Proposition 9.4.10, the construction $\mathcal{F} \mapsto \mathcal{F}_c$ can be regarded as a functor

$$\Psi : \mathrm{Shv}_{\mathcal{C}}(X) \to \mathrm{Shv}^!_{\mathcal{C}}(X).$$

We will refer to the functor Ψ as *covariant Verdier duality.* It is generally not an equivalence of ∞-categories if the space X is not locally compact: roughly speaking, the !-sheaf $\Psi(\mathcal{F})$ associated to a sheaf $\mathcal{F} \in \mathrm{Shv}_{\mathcal{C}}(X)$ only "remembers" information about sections of $\mathcal{F}$ which are supported on compact subsets of X.

If $\mathcal{F}$ is a $\mathcal{C}$-valued sheaf on X given by system of sheaves $\{\mathcal{F}_n \in \mathrm{Shv}_{\mathcal{C}}(X_n)\}_{n \geq 0}$ with compatibilities $\mathcal{F}_n \simeq i_n^* \mathcal{F}_{n+1}$, then $\Psi(\mathcal{F})$ is a system of sheaves $\{\mathcal{G}_n \in \mathrm{Shv}_{\mathcal{C}}(X_n)\}_{n \geq 0}$ with compatibilities $\mathcal{G}_n \simeq i_n^! \mathcal{G}_{n+1}$, which can be described concretely by the formula $\mathcal{G}_n = \varprojlim_m i_n^! i_{m*} \mathcal{F}_m$.

9.4.3. *Covariant Verdier Duality on* $\mathrm{Ran}(X)$. We now consider an algebro-geometric analogue of the covariant Verdier duality functor described in Remark 9.4.11. Fix an algebraically closed field k, a prime number ℓ which is invertible in k, and a projective k-scheme X.

Construction 9.4.12. Let $\mathcal{F}$ be an ℓ-adic $*$-sheaf on $\mathrm{Ran}(X)$: that is, a collection of sheaves $\{\mathcal{F}^{(S)} \in \mathrm{Shv}_\ell(X^S)\}_{S \in \mathrm{Fin}^s}$ which are equipped with equivalences $\mathcal{F}^{(S)} \simeq \delta^*_{S'/S}\,\mathcal{F}^{(S')}$ for every surjection of nonempty finite sets $S' \to S$, compatible with composition up to coherent homotopy. For every pair of nonempty finite sets S and T, let $\Delta(S,T)$ denote the closed subset of $X^S \times_{\mathrm{Spec}\,k} X^T$ whose k-valued points are pairs of maps $\mu : S \to X(k)$, $\nu : T \to X(k)$ satisfying $\mu(S) = \nu(T)$ (here $\Delta(S,T)$ plays the role of the fiber product of X^S and X^T over the Ran space $\mathrm{Ran}(X)$, though it is not literally given by the fiber product). We have projection maps

$$X^S \xleftarrow{p_{S,T}} \Delta(S,T) \xrightarrow{q_{S,T}} X^T.$$

For every nonempty finite set T, we can define an ℓ-adic sheaf $\Psi(\mathcal{F})^{(T)}$ on X^T by the formula

$$\Psi(\mathcal{F})^{(T)} = \varprojlim_{S \in \mathrm{Fin}^s} q_{S,T*}\,p^!_{S,T}\,\mathcal{F}^{(S)}\,.$$

The collection of sheaves $\{\Psi(\mathcal{F})^{(T)}\}_{T \in \mathrm{Fin}^s}$ can be organized into a !-sheaf on $\mathrm{Ran}(X)$, which we will denote by $\Psi(\mathcal{F})$ and refer to as the *covariant Verdier dual* of $\mathcal{F}$.

Remark 9.4.13. Let $\mathcal{F}$ be as in Definition 9.4.12. For every pair of nonempty finite sets S and T, we have canonical maps

$$C^*(X^T; \Psi(\mathcal{F})^{(T)}) \to C^*(X^T; q_{S,T*}\,p^!_{S,T}\,\mathcal{F}^{(S)}) \simeq C^*(\Delta(S,T); p^!_{S,T}\,\mathcal{F}^{(S)}) \to C^*(X^S; \mathcal{F}^{(S)}).$$

These maps depend functorially on S and T and therefore induce a map

$$\int \Psi(\mathcal{F}) = \varinjlim_{T \in \mathrm{Fin}^s} C^*(X^T; \Psi(\mathcal{F})^{(T)}) \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(X^S; \mathcal{F}).$$

This map is generally not an equivalence: the right hand side can be thought of as the chain complex of global sections of $\mathcal{F}$, while the left hand side can be thought of as the chain complex of compactly supported global sections of $\mathcal{F}$.

9.4.4. *The Covariant Verdier Dual of* $\mathcal{A}_{red}$. Let us now describe the geometric context in which we would like to apply Construction 9.4.12. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. To simplify the discussion, we will assume that G is reductive (but we will relax this assumption for Notation 9.4.14 and the formulations of Proposition 9.4.17 and Proposition 9.4.18). Let $\mathcal{A}_{\mathrm{red}}$ be the $*$-sheaf on $\mathrm{Ran}(X)$ discussed in the introduction to §8: if $\mu : S \to X(k)$ is a k-valued point of $\mathrm{Ran}(X)$ with $\mu(S) = \{x_1, \ldots, x_n\}$, then the stalk of $\mathcal{A}_{\mathrm{red}}$ at the point μ can be identified with the tensor product

$$\bigotimes_{1 \le i \le n} C^*_{\mathrm{red}}(\mathrm{Gr}_G^{x_i}; \mathbf{Z}_\ell).$$

We would like to study the covariant Verdier dual of $\mathcal{A}_{\mathrm{red}}$. To describe this !-sheaf geometrically, it will be convenient to introduce a bit of notation.

Notation 9.4.14. Let S be a nonempty finite set, let $S_0 \subseteq S$ be a subset, and let $Q \subseteq X$ be an effective divisor. We let $\mathrm{Ran}_G(X-Q)^\circ_{S_0 \subseteq S}$ denote the product

$$\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S} \times_{\mathrm{Spec}\,k} \mathrm{Ran}(X),$$

which we regard as a $\mathrm{Ran}(X)$-prestack via projection onto the second factor.

Concretely, the objects of $\mathrm{Ran}_G(X-Q)^\circ_{S_0 \subseteq S}$ are given by tuples $(R, \mu : S \to (X-Q)(R), \nu : T \to X(R), \mathcal{P}, \gamma)$ where $\mathcal{P}$ is a G-bundle on X_R (but we only care about its restriction to

$X_R - |\mu(S_0)|)$ and γ is a trivialization of $\mathcal{P}$ over $X_R - |\mu(S)|$. We let $\mathrm{Ran}_G(X-Q)^{\circ\,\mathrm{off}}_{S_0 \subseteq S}$ denote the subcategory consisting of those objects for which the associated map $\mathrm{Spec}\, R \to (X-Q)^S \times_{\mathrm{Spec}\, k} X^T$ factors through the complement of the closed set $\Delta(S,T)$.

We define lax !-sheaves $\mathcal{B}^{\circ}_{S_0 \subseteq S}, \mathcal{B}^{\circ\,\mathrm{off}}_{S_0 \subseteq S}$ by the formulae

$$\mathcal{B}^{\circ}_{S_0 \subseteq S} = [\mathrm{Ran}_G(X-Q)^{\circ}_{S_0 \subseteq S}]_{\mathrm{Ran}(X)} \qquad \mathcal{B}^{\circ\,\mathrm{off}}_{S_0 \subseteq S} = [\mathrm{Ran}_G(X-Q)^{\circ\,\mathrm{off}}_{S_0 \subseteq S}]_{\mathrm{Ran}(X)}.$$

Remark 9.4.15. Since $\mathrm{Ran}_G(X-Q)^{\circ}_{S_0 \subseteq S}$ factors as a product of $\mathrm{Ran}(X)$ with $\mathrm{Ran}(X-Q)_{S_0 \subseteq S}$, we have a canonical equivalence

$$\mathcal{B}^{\circ}_{S_0 \subseteq S} \simeq C^*(\mathrm{Ran}(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}.$$

In particular, its chiral homology is given by $\int \mathcal{B}^{\circ}_{S_0 \subseteq S} \simeq C^*(\mathrm{Ran}(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell)$.

Remark 9.4.16. To understand the relevance of the sheaves $\mathcal{B}^{\circ}_{S_0 \subseteq S}$ and $\mathcal{B}^{\circ\,\mathrm{off}}_{S_0 \subseteq S}$, let us assume that $Q = \emptyset$. Let S and T be nonempty finite sets, and consider the diagram

$$X^S \overset{p_{S,T}}{\leftarrow} \Delta(S,T) \overset{q_{S,T}}{\rightarrow} X^T.$$

Let $\pi : X^S \times_{\mathrm{Spec}\, k} X^T \to X^T$ be the projection onto the second factor, let $U \subseteq X^S \times_{\mathrm{Spec}\, k} X^T$ be the complement of $\Delta(S,T)$, and let $\pi_U = \pi|_U$. Set

$$\mathcal{A}^{(S_0 \subseteq S)} = [\mathrm{Ran}_G(X)_{S_0 \subseteq S}]_{\underline{\mathbf{Z}_\ell}_{(X)^S}} \in \mathrm{Shv}_\ell(X^S).$$

Unwinding the definitions, we see that $q_{S,T*} p^!_{S,T} \mathcal{A}^{(S_0 \subseteq S)} \in \mathrm{Shv}_\ell(X^T)$ can be computed as the fiber of the map

$$\pi_*(\mathcal{A}^{(S_0 \subseteq S)} \boxtimes \omega_{X^T}) \to \pi_{U*}((\mathcal{A}^{(S_0 \subseteq S)} \boxtimes \omega_{X^T})|_U).$$

The left hand side can be identified with $\mathcal{B}^{\circ(T)}_{S_0 \subseteq S}$, and the right hand side with $\mathcal{B}^{\circ\,\mathrm{off}(T)}_{S_0 \subseteq S}$. It follows that the covariant Verdier dual of $\mathcal{A}_{\mathrm{red}}$ can be identified with the fiber of the canonical map

$$\varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0 \subseteq S} \to \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ\,\mathrm{off}}_{S_0 \subseteq S}.$$

We will deduce Proposition 9.1.5 from the following pair of assertions:

Proposition 9.4.17. *If the group scheme G is Q-adapted, then the morphism*

$$\int \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,red} \to \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell)$$

appearing in Proposition 9.1.5 is induced by a morphism of lax !-sheaves $\varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,red} \to \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0 \subseteq S}$. Moreover, this map fits into a fiber sequence

$$\varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,red} \to \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0 \subseteq S} \to \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ\,\mathrm{off}}_{S_0 \subseteq S}.$$

Proposition 9.4.18. *If the group scheme G is Q-adapted, then the chiral homology*

$$\int \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ\,\mathrm{off}}_{S_0 \subseteq S}$$

vanishes.

We will give the proof of Proposition 9.4.18 in §9.5. The proof of Proposition 9.4.17 is more involved: we will give an outilne in §9.6 and carry out most of the details in §9.7.

Remark 9.4.19. In the case $Q = \emptyset$, Proposition 9.4.17 can be viewed as saying that the inverse limit $\varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{S,\mathrm{red}}$ is covariant Verdier dual to $\mathcal{A}_{\mathrm{red}}$, and Proposition 9.4.18 can be viewed as saying that the canonical map

$$\int \Psi(\mathcal{A}_{\mathrm{red}}) \to \varprojlim_{S \in \mathrm{Fin}^s} C^*(X^S, \mathcal{A}_{\mathrm{red}}^{(S)})$$

is an equivalence: in other words, all sections of $\mathcal{A}_{\mathrm{red}}$ are compactly supported.

Remark 9.4.20. Combining Proposition 9.4.17 with Proposition 9.1.4, we see that the !-sheaf $\mathcal{B}_{\mathrm{red}}$ can be identified with the covariant Verdier dual of the $*$-sheaf $\mathcal{A}_{\mathrm{red}}$ (at least in the case $Q = \emptyset$). Note that the costalk of $\mathcal{B}_{\mathrm{red}}$ at a point $x \in X \subseteq \mathrm{Ran}(X)$ can be identified with $C^*_{\mathrm{red}}(\mathrm{BG}_x; \mathbf{Z}_\ell)$, while the stalk of $\mathcal{A}_{\mathrm{red}}$ at the point x can be identified with $C^*_{\mathrm{red}}(\mathrm{Gr}_G^x; \mathbf{Z}_\ell)$. When $k = \mathbf{C}$, the Verdier duality between the factorizable sheaves $\mathcal{A}_{\mathrm{red}}$ and $\mathcal{B}_{\mathrm{red}}$ corresponds to the Koszul duality between $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ and $C^*(\mathrm{Gr}_G^x; \mathbf{Z}_\ell)$ (where the former is regarded as an E_2-algebra in $\mathrm{Mod}_{\mathbf{Z}_\ell}$ and the latter as an E_2-coalgebra in $\mathrm{Mod}_{\mathbf{Z}_\ell}$). Topologically, this arises from the fact that BG_x is simply connected and can therefore be recovered by performing a double bar construction on the two-fold loop space $\Omega^2 \mathrm{BG}_x \simeq \mathrm{Gr}_G^x$.

9.5. **A Convergence Argument.** Throughout this section, we fix an algebraically closed field k, an algebraic curve X over k, a smooth affine group scheme G over X, and a finite subset $Q \subseteq X$ for which G is Q-adapted. For every finite set S and every subset $S_0 \subseteq S$, we let $\mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}$ denote the lax !-sheaves introduced in Notation 9.4.14. Our goal is to prove Proposition 9.1.5, which asserts that the chiral homology $\int \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}$ vanishes. This is an immediate consequence of the following pair of assertions:

Proposition 9.5.1. *Let S be a nonempty finite set and let $S_0 \subseteq S$. Then the chiral homology $\int \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}$ vanishes.*

Proposition 9.5.2. *The canonical map*

$$\int \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S} \to \varprojlim_{\mathrm{prim}} \int \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

In fact, we will prove a far more general version of Proposition 9.5.1. To state it, we need a bit of notation.

Construction 9.5.3. Let S be a finite set, let $\mathcal{C}$ be a prestack equipped with a map $\mathcal{C} \to (X - Q)^S$, and let $\mathcal{D} = \mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$ (which we regard as a $\mathrm{Ran}(X)$-prestack via projection onto the second factor). Note that every R-valued point of $\mathcal{D}$ determines a map $\mathrm{Spec}\, R \to (X - Q)^S \times_{\mathrm{Spec}\, k} X^T$ for some nonempty finite set T. Let $U(S, T)$ denote the open subset of $(X - Q)^S \times_{\mathrm{Spec}\, k} X^T$ whose k-valued points are pairs $\mu : S \to (X - Q)(k)$, $\nu : T \to X(k)$ satisfying $\mu(S) \neq \nu(T)$, and let $\mathcal{D}^{\mathrm{off}}$ denote the full subcategory of $\mathcal{D}$ consisting of those objects for which the corresponding map $\mathrm{Spec}\, R \to (X - Q)^S \times_{\mathrm{Spec}\, k} X^T$ factors through $U(S, T)$.

Proposition 9.5.4. *Let $\mathcal{C}$ be a prestack equipped with a map $\pi : \mathcal{C} \to (X - Q)^S$, let $\mathcal{D} = \mathcal{C} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X)$, and define $\mathcal{F} = [\mathcal{D}^{\mathrm{off}}]_{\mathrm{Ran}(X)} \in \mathrm{Shv}_\ell^{\mathrm{lax}}(\mathrm{Ran}(X))$. Then the chiral homology $\int \mathcal{F}$ vanishes.*

The main step in the proof of Proposition 9.5.4 is the following:

Lemma 9.5.5. *Let $f : Y \to (X - Q)^S$ be a map of quasi-projective k-schemes and let $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$. For every nonempty finite set T, let $U(S,T) \subseteq (X - Q)^S \times X^T$ denote the open subscheme introduced in Construction 9.5.3, let $U(S,T)_Y$ denote the fiber product $Y \times_{X^S} U(S,T)$, and let $p_T : U(S,T)_Y \to Y$ and $q_T : U(S,T)_Y \to X^T$ be the projection maps. Then the direct limit*

$$\mathcal{F}^+ = \varinjlim_{T \in \mathrm{Fin}^s} p_{T*}(q_T^* \omega_{X^T} \otimes p_T^* \mathcal{F})$$

is a zero object of $\mathrm{Shv}_\ell(Y)$.

Proof of Proposition 9.5.4. For each nonempty finite set T, let $p_T : U(S,T) \to (X - Q)^S, q_T : U(S,T) \to X^T$ denote the projection maps, so that we have a commutative diagram of prestacks

$$\begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{D}^{\mathrm{off}} \times_{\mathrm{Ran}(X)} X^T & & \\ \downarrow{\scriptstyle\rho} & & \downarrow & \searrow & \\ (X-Q)^S & \xleftarrow{p_T} & U(S,T) & \xrightarrow{q_T} & X^T, \end{array}$$

where the square on the left is a pullback. We have canonical equivalences

$$\begin{aligned} C^*(X^T; \mathcal{F}) &= C^*(X^T; [\mathcal{D}^{\mathrm{off}} \times_{\mathrm{Ran}(X)} X^T]_{\omega_{X^T}} \\ &\simeq C^*(X^T; q_{T*}[\mathcal{D}^{\mathrm{off}} \times_{\mathrm{Ran}(X)} X^T]_{q_T^* \omega_{X^T}} \\ &\simeq C^*(U(S,T); [\mathcal{D}^{\mathrm{off}} \times_{\mathrm{Ran}(X)} X^T]_{q_T^* \omega_{X^T}}) \\ &\simeq C^*(U(S,T); q_T^* \omega_{X^T} \otimes [\mathcal{D}^{\mathrm{off}} \times_{\mathrm{Ran}(X)} X^T]_{\underline{\mathbf{Z}}_{\ell\, U(S,T)}}) \\ &\simeq C^*(U(S,T); q_T^* \omega_{X^T} \otimes p_T^* [\mathcal{C}]_{\underline{\mathbf{Z}}_{\ell\,(X-Q)^S}}), \end{aligned}$$

where the last equivalence follows from the smoothness of the projection map $p_T : U(S,T) \to (X - Q)^S$. Set $\mathcal{G} = [\mathcal{C}]_{\underline{\mathbf{Z}}_{\ell\,(X-Q)^S}} \in \mathrm{Shv}_\ell((X-Q)^S)$. Passing to the colimit as T varies, we obtain an equivalence

$$\begin{aligned} \int \mathcal{F} &= \varinjlim_{T \in \mathrm{Fin}^s} C^*(X^T; \mathcal{F}^{(T)}) \\ &\simeq \varinjlim_{T \in \mathrm{Fin}^s} C^*(U(S,T); q_T^* \omega_{X^T} \otimes p_T^* \mathcal{G}) \\ &\simeq \varinjlim_{T \in \mathrm{Fin}^s} C^*((X-Q)^S; p_{T*}(q_T^* \omega_{X^T} \otimes p_T^* \mathcal{G})) \\ &\simeq C^*((X-Q)^S; \varinjlim_{T \in \mathrm{Fin}^s} p_{T*}(q_T^* \omega_{X^T} \otimes p_T^* \mathcal{G})); \end{aligned}$$

(here the last identification follows from the observation that the global sections functor $C^*((X - Q)^S; \bullet)$ commutes with colimits). The desired vanishing now follows from Lemma 9.5.5. □

Proof of Lemma 9.5.5. For each closed immersion $i : Y' \hookrightarrow Y$, we can consider the sheaf $i^! \mathcal{F} \in \mathrm{Shv}_\ell(Y')$ and the associated direct limit

$$(i^! \mathcal{F})^+ = \varinjlim_{T} p'_{T*}((\omega_{X^T} \boxtimes i^! \mathcal{F})|_{U(S,T)_{Y'}}),$$

where $p'_T : U(S,T)_{Y'} \to Y'$ denotes the projection map. Using Theorem 4.5.4 and Proposition 4.5.12, we obtain a canonical equivalence $(i^! \mathcal{F})^+ \simeq i^!(\mathcal{F}^+)$. We will prove that for every closed immersion $i : Y' \hookrightarrow Y$, the sheaf $(i^! \mathcal{F})^+ \simeq i^!(\mathcal{F}^+) \in \mathrm{Shv}_\ell(Y')$ vanishes. Replacing Y by Y' and proceeding by Noetherian induction, we may assume that $i^! \mathcal{F}^+ \simeq 0$ for every closed immersion

$i : Y' \to Y$ whose image is a proper closed subscheme of Y. In particular, we may assume that Y is reduced (otherwise, take $Y' = Y_{\mathrm{red}}$).

We will assume that Y is nonempty (otherwise there is nothing to prove). The map $f : Y \to (X - Q)^S$ can be identified with a collection of maps $\{f_s : Y \to X\}_{s \in S}$. Choose a nonempty open subset $V \subseteq Y$ with the following property:

($*$) For every pair of elements $s, s' \in S$, the maps $f_s|_V$ and $f_{s'}|_V$ either coincide or have disjoint graphs in the product $(X - Q) \times V$.

Let Y' be the complement of V (regarded as a reduced closed subscheme of Y) and let

$$i : Y' \hookrightarrow Y \qquad j : V \hookrightarrow Y$$

be the inclusion maps. Then we have a fiber sequence

$$i_* i^! \mathcal{F}^+ \to \mathcal{F}^+ \to j_* j^* \mathcal{F}^+$$

The first term vanishes by our inductive hypothesis. Consequently, to show that $\mathcal{F}^+$ vanishes, it will suffice to show that $\mathcal{F}^+|_V$ vanishes. Replacing Y by V, we may assume that for each pair $s, s' \in S$, the morphisms f_s and $f_{s'}$ either coincide or have disjoint graphs. Replacing S by a quotient if necessary, we may assume that f_s and $f_{s'}$ have disjoint graphs whenever $s \neq s'$.

For every nonempty finite set T, let $\Delta(S, T) \subseteq (X - Q)^S \times X^T$ be the reduced closed subscheme complementary to $U(S, T)$, and let $\Delta(S, T)_Y$ denote the fiber product $Y \times_{(X-Q)^S} \Delta(S, T)$. Let $\overline{p}_T : Y \times X^T \to Y$ be the projection map, and let $\overline{p}'_T$ denote the restriction of $\overline{p}_T$ to $\Delta(S, T)_Y$. We then have a fiber sequence

$$\overline{p}'_{T*} \overline{p}'^!_T \mathcal{F} \to \overline{p}_{T*} \overline{p}^!_T \mathcal{F} \to p_{T*}(p^*_T \mathcal{F} \otimes q^*_T \omega_{X^T}),$$

which depends functorially on T. Passing to the colimit over T, we obtain a fiber sequence

$$\varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \overline{p}'_{T*} \overline{p}'^!_T \mathcal{F} \xrightarrow{\theta} \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \overline{p}_{T*} \overline{p}^!_T \mathcal{F} \to \mathcal{F}^+$$

It will therefore suffice to show that θ is an equivalence.

Note that θ fits into a commutative diagram

$$\begin{array}{ccc} \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \overline{p}'_{T*} \overline{p}'^!_T \mathcal{F} & \xrightarrow{\theta} & \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \overline{p}_{T*} \overline{p}^!_T \mathcal{F} \\ & {\scriptstyle \alpha} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & \mathcal{F}. & \end{array}$$

Unwinding the definitions, we see that β can be identified with the tensor product of $\mathcal{F}$ with the natural map

$$\varinjlim_{T} C_*(X^T; \mathbf{Z}_\ell) \to \mathbf{Z}_\ell$$

in the ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$, which is an equivalence by virtue of Corollary 2.4.13. We will complete the proof by showing that α is an equivalence.

Using our assumptions that Y is reduced and that the maps $\{f_s\}_{s \in S}$ have disjoint graphs, we see that $\Delta(S, T)$ is isomorphic to a disjoint union of finitely many copies of Y, where the disjoint union is indexed by the set $\mathrm{Hom}(T, S)$ of all surjections from T to S. It follows that the domain of α can be identified with the direct limit

$$\varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}} \bigoplus_{\gamma \in \mathrm{Hom}(T,S)} \mathcal{F} \simeq \varinjlim_{T \in \mathrm{Fin}^{\mathrm{s}}{}_{/S}} \mathcal{F}.$$

Consequently, to prove that α is an equivalence, it will suffice to show that the index category $\mathrm{Fin}^{\mathrm{s}}{}_{/S}$ has weakly contractible nerve. This is clear, since $\mathrm{Fin}^{\mathrm{s}}{}_{/S}$ has a final object (given by the identity map $\mathrm{id} : S \to S$). □

The proof of Proposition 9.5.2 will require some preliminaries.

Notation 9.5.6. For every nonempty finite set S and every subset $S_0 \subseteq S$, we let $V(S_0 \subseteq S) \in \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X))$ denote the fiber of the restriction map

$$\mathcal{B}^{\circ}_{S_0 \subseteq S} \to \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}.$$

Note that we can regard V as a functor from $\Theta_{\blacksquare}$ to $\mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X))$, where $\Theta_{\blacksquare}$ is the category introduced in Notation 8.1.6. We let $V_{\mathrm{prim}} : \mathrm{Fin}^{\mathrm{s}} \to \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X))$ denote the primitive part of V (Construction 8.1.7), given concretely by the formula

$$V_{\mathrm{prim}}(S) = \mathrm{cofib}(\varinjlim_{\emptyset \neq S_0 \subseteq S} V(S, S_0) \to V(S, \emptyset)).$$

Remark 9.5.7. Fix a nonempty finite set S and a subset $S_0 \subseteq S$, and let T be another nonempty finite set. Then the fiber product $\mathrm{Ran}^{\circ}_G(X - Q)_{S_0 \subseteq S, \mathrm{aug}} \times_{\mathrm{Ran}_{\mathrm{aug}}(X)} X^T$ can be identified with an open substack of the product $\mathrm{Ran}_G(X - Q)_{S_0 \subseteq S} \times_{\mathrm{Spec}\, k} X^T$. Moreover, if we are given a surjection of nonempty finite sets $T' \to T$, then we have a pullback diagram of schemes

$$\begin{array}{ccc} U(S,T) & \longrightarrow & X^T \\ \downarrow & & \downarrow \\ U(S,T') & \longrightarrow & X^T \end{array}$$

and therefore a pullback diagram of prestacks

$$\begin{array}{ccc} \mathrm{Ran}^{\circ,\mathrm{off}}_G(X - Q)_{S_0 \subseteq S, \mathrm{aug}} \times_{\mathrm{Ran}_{\mathrm{aug}}(X)} X^T & \longrightarrow & X^T \\ \downarrow & & \downarrow \\ \mathrm{Ran}^{\circ,\mathrm{off}}_G(X - Q)_{S_0 \subseteq S, \mathrm{aug}} \times_{\mathrm{Ran}_{\mathrm{aug}}(X)} X^{T'} & \longrightarrow & X^{T'}. \end{array}$$

Applying Proposition 5.1.13, we deduce that $\mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}$ is a !-sheaf on $\mathrm{Ran}(X)$. Using similar (but easier) reasoning, we see that $\mathcal{B}^{\circ}_{S_0 \subseteq S}$ belongs to $\mathrm{Shv}_{\ell}^{!}(\mathrm{Ran}(X))$ (the latter is equivalent to $C^*(\mathrm{Ran}_G(X - Q)_{S_0 \subseteq S}; \mathbf{Z}_{\ell}) \otimes \omega_{X^T}$). It follows that the functors

$$V : \Theta_{\blacksquare} \to \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X)) \qquad V_{\mathrm{prim}} : \mathrm{Fin}^{\mathrm{s}} \to \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X))$$

of Notation 9.5.6 take values in $\mathrm{Shv}_{\ell}^{!}(\mathrm{Ran}(X)) \subseteq \mathrm{Shv}_{\ell}^{\mathrm{lax}}(\mathrm{Ran}(X))$.

Note that we have a map of fiber sequences

$$\begin{array}{ccccc} \int \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} V_{\mathrm{prim}}(S) & \longrightarrow & \int \varprojlim \mathcal{B}^{\circ}_{S_0 \subseteq S} & \longrightarrow & \int \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S} \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} \\ \varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \int V_{\mathrm{prim}}(S) & \longrightarrow & \varprojlim_{\mathrm{prim}} \int \mathcal{B}^{\circ}_{S_0 \subseteq S} & \longrightarrow & \varprojlim_{\mathrm{prim}} \int \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}. \end{array}$$

Using the identification

$$\mathcal{B}^{\circ}_{S_0 \subseteq S} \simeq C^*(\mathrm{Ran}_G(X - Q)_{S_0 \subseteq S}; \mathbf{Z}_{\ell}) \otimes \omega_{\mathrm{Ran}(X)}$$

(see the discussion following Construction 9.6.4), we see that the map ϕ is an equivalence (both the domain and codomain of ϕ can be identified with $\varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell))$. Consequently, to prove Proposition 9.5.2, it will suffice to show that the map ϕ' is an equivalence. By virtue of Remark 9.5.7 and Corollary 5.3.16, this is an immediate consequence of the following connectivity estimate:

Proposition 9.5.8. *For every positive integer d and every nonempty finite set S, the object $C^*(\overset{\circ}{X}{}^d; V_{\mathrm{prim}}(S)^{(d)}|_{\overset{\circ}{X}{}^d})$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -d}$; here $\overset{\circ}{X}{}^d \subseteq X^d$ denotes the complement of the fat diagonal.*

The essential ingredient in the proof of Proposition 9.5.8 is the following connectivity statement:

Lemma 9.5.9. *Let $x \in X$ be a point for which the fiber G_x is semisimple, and let $\mathrm{Gr}_G^x = \mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} \{x\}$ denote the affine Grassmannian of G at the point x. Then Gr_G^x is connected.*

Proof. Let e denote the base point of Gr_G^x (corresponding to the trivial G-bundle on X equipped with its tautological trivialization on $X - \{x\}$). To prove that Gr_G^x is connected, it will suffice to show that for every k-valued point y of Gr_G^x, there exists a path $h : \mathbf{A}^1 \to \mathrm{Gr}_G^x$ satisfying $h(0) = e$ and $h(1) = y$. Let us regard y as fixed in what follows.

Let $t \in \mathcal{O}_{X,x}$ be a local coordinate on the curve X at the point x. For every finitely generated k-algebra R, we will identify $R[[t]]$ with the ring of functions on the formal completion of X_R along $\{x\} \times \operatorname{Spec} R$. Combining Proposition A.1.6 with the Beauville-Laszlo theorem ([3]), we can identify the set of R-points of Gr_G^x with the category of G-bundles on $\operatorname{Spec} R[[t]]$ which are equipped with a trivialization on the open subscheme $\operatorname{Spec} R((t))$. In particular, we have an embedding

$$G(R((t)))/G(R[[t]]) \hookrightarrow \mathrm{Gr}_G^x(R)$$

whose image consists of those points $\mathrm{Gr}_G^x(R)$ which correspond to G-bundles which are globally trivial on $\operatorname{Spec} R[[t]]$. It follows from Corollary A.1.7 and the smoothness of G that a G-bundle on $\operatorname{Spec} R[[t]]$ is trivial if and only if its restriction to $\operatorname{Spec} R$ is trivial, and this condition is automatically satisfied when $R = k$. It follows that there exist maps

$$G(R((t))) \to \mathrm{Gr}_G^x(R)$$

which depend functorially on R and are surjective when $R = k$. We may therefore lift the k-valued point y of Gr_G^x to an element $\overline{y} \in G(k((t)))$. To complete the proof, it will suffice to show that there exists an element $\overline{h}$ of $G(k[s]((t)))$ which reduces to the identity when we set $s = 0$ and to $\overline{y}$ when we set $s = 1$.

By assumption, the fiber G_x is semisimple. Since k is algebraically closed, the semisimple group scheme G_x is automatically split. It follows that the group scheme G is split reductive when restricted to the formal disk $\operatorname{Spec} k[[t]]$. Let G_0 denote the fiber product $G \times_X \operatorname{Spec} k((t))$. Then G_0 is a split reductive group over $k((t))$. Choose a split maximal torus $T_0 \subseteq G_0$, let Δ denote the collection of roots of G_0 with respect to T_0, and for each $\alpha \in \Delta$ let $u_\alpha : \mathbf{G}_a \to G_0$ be a paramerization of the corresponding root subgroup. Note that G_0 is obtained from the generic fiber of G by extension of scalars from the function field K_X to its completion $k((t))$, and is therefore simply connected. It follows that the group $G_0(k((t))) = G(k((t)))$ is generated by the images of the maps u_α. In particular, we can write

$$\overline{y} = u_{\alpha_1}(\lambda_1) u_{\alpha_2}(\lambda_2) \cdots u_{\alpha_n}(\lambda_n)$$

for some roots $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ and some elements $\lambda_1, \lambda_2, \ldots, \lambda_n \in k((t))$. We now complete the proof by setting

$$\overline{y} = u_{\alpha_1}(s\lambda_1) u_{\alpha_2}(s\lambda_2) \cdots u_{\alpha_n}(s\lambda_n).$$

□

Proof of Proposition 9.5.8. Let C denote the cardinality of the subset $Q \subseteq X(k)$; we will show that C has the desired property. Let S be a nonempty finite set and let $T = \{1, \ldots, d\}$ for some positive integer d. Let

$$(X-Q)^S \xleftarrow{q_S} (X-Q)^S \times X^T \xrightarrow{q_T} X^T$$

denote the projection maps and let q_T^{off} denote the restriction of q_T to the open subset $U(S,T) \subseteq (X-Q)^S \times X^T$.

For each subset $S_0 \subseteq S$, let $\mathcal{F}(S_0) \in \mathrm{Shv}_\ell((X-Q)^S)$ denote the ℓ-adic sheaf given by

$$[\mathrm{Ran}_G(X)_{S_0 \subseteq S}]_{\underline{\mathbf{Z}_\ell}_{(X-Q)^S}}.$$

Using Proposition 5.1.13, we obtain identifications

$$\mathcal{B}^{\circ(T)}_{S_0 \subseteq S} \simeq q_{T*} q_S^! \, \mathcal{F}(S_0)$$

$$\mathcal{B}^{\circ\,\mathrm{off}(T)}_{S_0 \subseteq S} \simeq q_{T*}^{\mathrm{off}} (q_S^! \, \mathcal{F}(S_0))|_{U(S,T)}.$$

Let $\Delta(S,T)$ be as in the proof of Lemma 9.5.5 and let $\iota : \Delta(S,T) \hookrightarrow (X-Q)^S \times X^T$ be the inclusion map, so that we have

$$V(S_0 \subseteq S) \simeq q_{T*} \iota_* \iota^! q_S^! \, \mathcal{F}(S_0).$$

Note that $\Delta(S,T) \times_{X^T} \overset{\circ}{X}{}^d$ can be identified with a disjoint union of finitely many copies of $(X \overset{\circ}{-} Q)^d$, indexed by the set Λ of all surjective maps $\lambda : S \to T$. For each $\lambda \in \Lambda$, let $\delta_\lambda : (X-Q)^T \to (X-Q)^S$ denote the corresponding diagonal map, so that we obtain an equivalence

$$V(S_0 \subseteq S)|_{\overset{\circ}{X}{}^d} \simeq \bigoplus_{\lambda \in \Lambda} j_* (\delta_\lambda^! \, \mathcal{F}(S_0))|_{(X \overset{\circ}{-} Q)^d}.$$

where $j : (X \overset{\circ}{-} Q)^d \hookrightarrow \overset{\circ}{X}{}^d$ is the inclusion map. Set

$$\mathcal{F}_{\mathrm{prim}} = \mathrm{cofib}(\varinjlim_{\emptyset \neq S_0 \subseteq S} \mathcal{F}(S_0) \to \mathcal{F}(\emptyset)),$$

so that we have equivalences

$$C^*(\overset{\circ}{X}{}^d; V_{\mathrm{prim}}(S)|_{(X \overset{\circ}{-} Q)^d}) \simeq \bigoplus_{\lambda \in \Lambda} C^*((X \overset{\circ}{-} Q)^d; \delta_\lambda^! \, \mathcal{F}_{\mathrm{prim}}).$$

It will therefore suffice to show that for each $\lambda \in \Lambda$, the object $C^*((X \overset{\circ}{-} Q)^d; (\delta_\lambda^! \, \mathcal{F}_{\mathrm{prim}})|_{(X \overset{\circ}{-} Q)^d})$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -d}$.

Note that each of the ℓ-adic sheaves $\mathcal{F}(S_0)$ is ℓ-complete. It follows that the ℓ-adic sheaf $\mathcal{F}_{\mathrm{red}}$ and the chain complex $C^*((X \overset{\circ}{-} Q)^d; (\delta_\lambda^! \, \mathcal{F}_{\mathrm{red}})|_{(X \overset{\circ}{-} Q)^d})$ are also ℓ-complete. For each integer $m \geq 0$, let $\mathcal{F}_{\mathrm{red}} / \ell^m$ denote the image of $\mathcal{F}_{\mathrm{red}}$ under the reduction functor $\mathrm{Shv}_\ell((X-Q)^S) \to \mathrm{Shv}((X-Q)^S; \mathbf{Z}/\ell^m \mathbf{Z})$. Then $C^*((X \overset{\circ}{-} Q)^d; (\delta_\lambda^! \, \mathcal{F}_{\mathrm{red}})|_{(X \overset{\circ}{-} Q)^d})$ can be identified with the limit

$$\varprojlim_{m \geq 0} C^*(\overset{\circ}{X}{}^d; (\delta_\lambda^! \, \mathcal{F}_{\mathrm{red}} / \ell^m)|_{\overset{\circ}{X}{}^d}).$$

It will therefore suffice to show that each $C^*(\overset{\circ}{X}{}^d; (\delta_\lambda^! \mathcal{F}_{\mathrm{red}} / \ell^m)|_{\overset{\circ}{X}{}^d})$ belongs to the ∞-category $(\mathrm{Mod}_{\mathbf{Z}/\ell^m \mathbf{Z}})_{\leq -d}$. Proceeding by induction on m, we can reduce to the case $m = 1$ (though this reduction is not really necessary).

Let $U' \subseteq (X - Q)^S$ denote the open subset whose k-valued points are maps $\mu : S \to X(k)$ such that $\mu(s) \neq \mu(s')$ whenever $\lambda(s) \neq \lambda(s')$. We have a commutative diagram

$$\begin{array}{ccc} (X \overset{\circ}{-} Q)^d & \xrightarrow{\delta} & U' \\ \downarrow & & \downarrow \\ (X - Q)^d & \xrightarrow{\delta_\lambda} & (X - Q)^S, \end{array}$$

where the horizontal maps are closed immersions. We may therefore identify the ℓ-adic sheaf $(\delta_\lambda^!(\mathcal{F}_{\mathrm{red}} / \ell))|_{(X \overset{\circ}{-} Q)^d}$ with $\delta^!((\mathcal{F}_{\mathrm{red}} / \ell)|_{U'})$. Since the functor $\delta^!$ is left t-exact, we are reduced to proving that $(\mathcal{F}_{\mathrm{red}} / \ell)|_{U'}$ belongs to $\mathrm{Shv}((X - Q)^S; \mathbf{Z}/\ell\mathbf{Z})_{\leq -d}$. To prove this, fix a point $\eta : \operatorname{Spec} k \to U'$, corresponding to a map of sets $\mu : S \to X(k)$; we will prove that the stalk $\eta^*(\mathcal{F}_{\mathrm{red}} / \ell)$ belongs to $(\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}})_{\leq -d}$.

For each subset $S_0 \subseteq S$, let $\mathcal{C}_{S_0}$ denote the fiber product $\operatorname{Spec} k \times_{(X-Q)^S} \mathrm{Ran}_G(X - Q)_{S_0 \subseteq S}$. Using Lemma 8.5.6, we obtain identifications

$$\eta^*(\mathcal{F}(S_0)/\ell) \simeq C^*(\mathcal{C}_{S_0}; \mathbf{Z}/\ell\mathbf{Z}) \simeq C_*(\mathcal{C}_{S_0}; \mathbf{Z}/\ell\mathbf{Z})^\vee.$$

For each closed point $x \in X(k)$, let Gr_G^x denote the affine Grassmannian $\mathrm{Ran}_G(X) \times_{\mathrm{Ran}(X)} \{x\}$ of G at the point x. Unwinding the definitions, we see that the prestack $\mathcal{C}_{S_0}$ is equivalent to a product

$$\prod_{x \in \mu(S)} \begin{cases} \mathrm{Gr}_G^x & \text{if } x \notin \mu(S_0) \\ \operatorname{Spec} k & \text{if } x \in \mu(S_0). \end{cases}$$

Using the Künneth formula, we compute

$$\begin{aligned} \eta^*(\mathcal{F}_{\mathrm{red}} / \ell) &\simeq \operatorname{cofib}\Big(\varinjlim_{\emptyset \neq S_0 \subseteq S} C_*(\mathcal{C}_{S_0}; \mathbf{Z}/\ell\mathbf{Z})^\vee \to C_*(\mathcal{C}_\emptyset; \mathbf{Z}/\ell\mathbf{Z})^\vee \Big) \\ &\simeq \operatorname{fib}\Big(C_*(\mathcal{C}_\emptyset; \mathbf{Z}/\ell\mathbf{Z}) \to \varprojlim_{\emptyset \neq S_0 \subseteq S} C_*(\mathcal{C}_{S_0}; \mathbf{Z}/\ell\mathbf{Z})\Big)^\vee \\ &\simeq \Big(\bigotimes_{x \in \mu(S)} C_*^{\mathrm{red}}(\mathrm{Gr}_G^x; \mathbf{Z}/\ell\mathbf{Z})\Big)^\vee. \end{aligned}$$

We are therefore reduced to proving that the tensor product $\bigotimes_{x \in \mu(S)} C_*^{\mathrm{red}}(\mathrm{Gr}_G^x; \mathbf{Z}/\ell\mathbf{Z})$ belongs to $(\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}})_{\geq d}$. Since the point η belongs to the open set U', the set $\mu(S) \subseteq X(k)$ has cardinality exactly d. We are therefore reduced to proving that each tensor factor $C_*^{\mathrm{red}}(\mathrm{Gr}_G^x; \mathbf{Z}/\ell\mathbf{Z})$ belongs to $(\mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}})_{\geq 1}$. This follows from the connectivity of the affine Grassmannian Gr_G^x (Lemma 9.5.9) □

Remark 9.5.10. The proof of Proposition 9.5.8 (and, by extension, Propositions 9.5.2 and 9.1.5 depend in an essential way on the assumption that the generic fiber of G is simply connected: for non-simply connected groups, the formation of the primitive limit $\varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}$ does not commute with chiral homology.

Remark 9.5.11. The estimate obtained in Proposition 9.5.8 is not optimal. Using the vanishing of the homology groups $\mathrm{H}_1(\mathrm{Gr}_{G,x}; \mathbf{Z}/\ell\mathbf{Z})$ vanish for each closed point $x \in X - Q$, the proof of Proposition 9.5.8 gives a more precise estimate

$$C^*(\overset{\circ}{X}{}^d; V_{\mathrm{prim}}(S)^{(d)}|_{\overset{\circ}{X}{}^d}) \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -2d}.$$

9.6. **Proposition 9.4.17: Proof Outline.** Throughout this section, we fix an algebraically closed field k, an algebraic curve X over k, a smooth affine group scheme G over X, and a finite subset $Q \subseteq X$ such that G is Q-adapted (Definition 7.2.9). Assume that the fibers of G are connected and let ℓ be a prime number which is invertible in k. Our goal is to outline a proof of Proposition 9.4.17, which asserts the existence of a fiber sequence

$$\varprojlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} \to \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0 \subseteq S} \to \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ\,\mathrm{off}}_{S_0 \subseteq S}.$$

Our proof will require some auxiliary constructions. We begin by introducing a more elaborate version of Construction 9.1.2:

Construction 9.6.1. Fix a finite set S and a subset $S_0 \subseteq S$. We define a category $\mathrm{Ran}^{\dagger}_G(X - Q)_{S_0 \subseteq S,\mathrm{aug}}$ as follows:

- The objects of $\mathrm{Ran}^{\dagger}_G(X - Q)_{S_0 \subseteq S,\mathrm{aug}}$ are tuples $(R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \gamma)$ where R is a finitely generated k-algebra, K_+ is a subset of S, K_- is a subset of $K_+ \cup S_0$, T is a nonempty finite set, T_0 is a subset of T, $\mu : S \to (X - Q)(R)$ and $\nu : T \to X(R)$ are maps of sets, $\mathcal{P}$ is a G-bundle on X_R, γ is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu(S)|$, and we have $|\mu(K_+)| \cap |\nu(T)| = \emptyset = |\mu(S)| \cap |\nu(T_0)|$.
- There are no morphisms from an object $C = (R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \alpha)$ to another object $C' = (R', K'_-, K'_+, T', T'_0, \mu', \nu', \mathcal{P}', \alpha')$ unless $K'_- \subseteq K_-$ and $K_+ \subseteq K'_+$. If these conditions are satisfied, then a morphism from C to C' consists of a k-algebra homomorphism $\phi : R \to R'$, a surjection of finite sets $\lambda : T \to T'$ for which the diagrams

$$\begin{array}{ccc} T & \xrightarrow{\lambda} & T' \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R') \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\mathrm{id}} & S \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ X(R) & \xrightarrow{X(\phi)} & X(R') \end{array}$$

  commute and $T'_0 \subseteq \lambda(T_0)$, together with a G-bundle isomorphism between $\mathcal{P}'$ and $\mathrm{Spec}\, R' \times_{\mathrm{Spec}\, R} \mathcal{P}$ over the open set $X_{R'} - |\mu'(K'_-)|$, which carries γ to γ'.

The construction $(R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \gamma) \mapsto (R, T, T_0, \nu)$ determines a forgetful functor

$$\mathrm{Ran}^{\dagger}_G(X - Q)_{S_0 \subseteq S,\mathrm{aug}} \to \mathrm{Ran}_{\mathrm{aug}}(X).$$

This map is a coCartesian fibration and therefore exhibits $\mathrm{Ran}^{\dagger}_G(X - Q)_{S_0 \subseteq S,\mathrm{aug}}$ as an augmented $\mathrm{Ran}(X)$-prestack (see Definition 8.2.10). We let $\mathcal{B}_{S_0 \subseteq S,\mathrm{aug}}$ denote the augmented !-sheaf on $\mathrm{Ran}(X)$ given by

$$\mathcal{B}_{S_0 \subseteq S,\mathrm{aug}} = [\mathrm{Ran}^{\dagger}_G(X - Q)_{S_0 \subseteq S,\mathrm{aug}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}.$$

We let $\mathcal{B}_{S_0 \subseteq S,\mathrm{red}}$ denote the lax !-sheaf given by $(\mathcal{B}_{S_0 \subseteq S,\mathrm{aug}})_{\mathrm{red}}$.

Remark 9.6.2. Note that when the finite set S_0 is empty, Construction 9.6.1 reduces to Construction 9.1.2. In particular we have equivalences

$$\mathrm{Ran}^{\dagger}_G(X - Q)_{\emptyset \subseteq S,\mathrm{aug}} \simeq \mathrm{Ran}^{\dagger}_G(X - Q)_{S,\mathrm{aug}} \qquad \mathcal{B}_{\emptyset \subseteq S,\mathrm{aug}} \simeq \mathcal{B}_{S,\mathrm{aug}} \qquad \mathcal{B}_{\emptyset \subseteq S,\mathrm{red}} = \mathcal{B}_{S,\mathrm{red}}.$$

Remark 9.6.3. Let $\overline{\nu}$ be a k-valued point of the augmented Ran space $\mathrm{Ran}_{\mathrm{aug}}(X)$, corresponding to a map $\nu : T \to X(k)$ and a subset $T_0 \subseteq T$. Roughly speaking, we can think of the fiber

$$\mathrm{Ran}^{\dagger}_G(X-Q)_{S_0 \subseteq S, \mathrm{aug}} \times_{\mathrm{Ran}_{\mathrm{aug}}(X)} \{\overline{\nu}\}$$

as parametrizing maps $\mu : S \to X - Q$ together with G-bundles on $(X - \mu(S)) \cup (\nu(T) - \mu(S_0))$ which are trivialized on the open set $X - \mu(S)$.

Construction 9.6.4. For each nonempty finite set S and each subset $S_0 \subseteq S$, we let

$$\mathrm{Ran}_G(X-Q)^{\circ}_{S_0 \subseteq S, \mathrm{aug}} \subseteq \mathrm{Ran}^{\dagger}_G(X-Q)_{S_0 \subseteq S, \mathrm{aug}}$$

denote the full subcategory spanned by those objects $(R, K_-, K_+, T, T_0, \mu, \nu, \mathcal{P}, \gamma)$ where $K_- = K_+ = \emptyset$ (see Construction 9.6.1), which we regard as an augmented $\mathrm{Ran}(X)$-prestack. Let $\mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{aug}}$ denote the augmented !-sheaf given by the formula

$$\mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{aug}} = [\mathrm{Ran}_G(X-Q)^{\circ}_{S_0 \subseteq S, \mathrm{aug}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}$$

(see Definition 8.2.10). We let $\mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{red}}$ denote the lax !-sheaf given by

$$\mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{red}} = (\mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{aug}})_{\mathrm{red}}.$$

Remark 9.6.5. Note that when $T_0 = \emptyset$, Construction 9.6.4 reduces to Notation 9.4.14. That is, we have

$$\begin{aligned} \mathrm{Ran}_G(X-Q)^{\circ}_{S_0 \subseteq S, \mathrm{aug}} \times_{\mathrm{Ran}_{\mathrm{aug}}(X)} \mathrm{Ran}(X) &\simeq \mathrm{Ran}_G(X-Q)^{\circ}_{S_0 \subseteq S} \\ &= \mathrm{Ran}_G(X-Q)_{S_0 \subseteq S} \times_{\mathrm{Spec}\, k} \mathrm{Ran}(X). \end{aligned}$$

We may therefore identify

$$\mathcal{B}^{\circ}_{S_0 \subseteq S} \simeq C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}$$

with the underlying lax !-sheaf of $\mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{aug}}$.

Using the acyclicity of the Ran space, we obtain an equivalence

$$\begin{aligned} \int \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0 \subseteq S} &\simeq \int \varprojlim_{\mathrm{prim}} (C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}; \mathbf{Z}_\ell) \otimes \omega_{\mathrm{Ran}(X)}) \\ &\simeq \int (\varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S})) \otimes \omega_{\mathrm{Ran}(X)} \\ &\simeq \varprojlim_{\mathrm{prim}} C^*(\mathrm{Ran}_G(X-Q)_{S_0 \subseteq S}). \end{aligned}$$

Under this equivalence, the map appearing in Proposition 9.1.5 is given by the chiral homology of the morphism from the upper left to the lower right corner in the diagram

$$\begin{array}{ccccccc}
\varprojlim_{S\in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} & & \longrightarrow & & \varprojlim_{S\in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}^{\circ}_{S,\mathrm{red}} & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
& & \varprojlim_{S\in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S} & \longrightarrow & & & \varprojlim_{S\in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}^{\circ}_{S} \\
\varprojlim_{\mathrm{prim}} \mathcal{B}_{S_0\subseteq S,\mathrm{red}} & & \downarrow & \longrightarrow & \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0\subseteq S,\mathrm{red}} & & \downarrow \\
& \searrow & & & & \searrow & \\
& & \varprojlim_{\mathrm{prim}} \mathcal{B}_{S_0\subseteq S} & \longrightarrow & & & \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0\subseteq S},
\end{array}$$

where the horizontal maps are induced by the inclusion

$$\mathrm{Ran}_G(X-Q)^{\circ}_{S_0\subseteq S,\mathrm{aug}} \subseteq \mathrm{Ran}^{\dagger}_G(X-Q)_{S_0\subseteq S,\mathrm{aug}}.$$

In particular, it can be described as a composition

$$\int \varprojlim_{S\in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{S,\mathrm{red}} \xrightarrow{\xi_0} \int \varprojlim_{\mathrm{prim}} \mathcal{B}_{S_0\subseteq S,\mathrm{red}} \xrightarrow{\xi_1} \int \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0\subseteq S,\mathrm{red}} \xrightarrow{\xi_2} \int \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0\subseteq S}.$$

We will deduce Proposition 9.4.17 from the following pair of assertions:

Proposition 9.6.6. *The canonical map*

$$\varprojlim_{S\in \mathrm{Fin}^{\mathrm{s}}} \mathcal{B}_{\emptyset\subseteq S,red} \to \varprojlim_{\mathrm{prim}} \mathcal{B}_{S_0\subseteq S,red}$$

is an equivalence in $\mathrm{Shv}^{\mathrm{lax}}_{\ell}(\mathrm{Ran}(X))$.

Proposition 9.6.7. *There exists a collection of pullback squares*

$$\begin{array}{ccc}
\mathcal{B}_{S_0\subseteq S,red} & \longrightarrow & \mathcal{B}^{\circ}_{S_0\subseteq S} \\
\downarrow & & \downarrow \\
\mathcal{B}^{\mathrm{skw}}_{S_0\subseteq S,red} & \longrightarrow & \mathcal{B}^{\circ,\mathrm{off}}_{S_0\subseteq S},
\end{array}$$

depending functorially on $(S_0\subseteq S)$. *Moreover, the primitive limit of the upper horizontal map agrees with* $\xi_2\circ\xi_1$, *while the primitive limit* $\varprojlim_{\mathrm{prim}} \mathcal{B}^{\mathrm{skw}}_{S_0\subseteq S,red}$ *vanishes in* $\mathrm{Shv}^{\mathrm{lax}}_{\ell}(\mathrm{Ran}(X))$.

The proof of Proposition 9.6.7 will require some auxiliary constructions and will be given in §9.7. We conclude this section by giving a proof of Proposition 9.6.6, using the ideas developed in §8.4.

Notation 9.6.8. Let S be a nonempty finite set. For each pair of subsets $S_0\subseteq S_1\subseteq S$, we let $V(S_0\subseteq S_1\subseteq S)$ denote the augmented !-sheaf on $\mathrm{Ran}(X)$ given by

$$[\mathrm{Ran}^{\dagger}_G(X-Q)_{S_0\subseteq S_1}\times_{(X-Q)^{S_1}}(X-Q)^{S}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}.$$

Note that the construction $(S_0\subseteq S_1\subseteq S)$ determines a functor $\Theta\to \mathrm{Shv}^{\mathrm{aug}}_{\ell}(\mathrm{Ran}(X))$ and that $V(S_0\subseteq S\subseteq S)=\mathcal{B}_{S_0\subseteq S,\mathrm{aug}}$.

We will deduce Proposition 9.6.6 from the following:

Lemma 9.6.9. *The functor $V : \Theta \to \mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$ is unital, in the sense of Definition 8.4.5.*

Proof of Proposition 9.6.6. Combining Lemma 9.6.9 with Theorem 8.4.10, we deduce that the canonical map

$$\varprojlim_{S \in \mathrm{Fin}^s} V(\emptyset \subseteq S \subseteq S) \to \varprojlim_{\mathrm{prim}} V_{\mathrm{red}}(S_0 \subseteq S \subseteq S) \times \varprojlim_{S \in \mathrm{Fin}^s} V(\emptyset \subseteq \emptyset \subseteq S)$$

is an equivalence in $\mathrm{Shv}_\ell^{\mathrm{aug}}(\mathrm{Ran}(X))$. Using the equality $\mathcal{B}_{S_0 \subseteq S, \mathrm{red}} = V(S_0 \subseteq S \subseteq S)_{\mathrm{red}}$, we obtain a product decomposition

$$\varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_{\emptyset \subseteq S, \mathrm{red}} \simeq \varprojlim_{\mathrm{prim}} \mathcal{B}_{S_0 \subseteq S, \mathrm{red}} \times \varprojlim_{S \in \mathrm{Fin}^s} V(\emptyset \subseteq \emptyset \subseteq S)_{\mathrm{red}}.$$

It will therefore suffice to show that for each nonempty finite set S, the lax !-sheaf $V(\emptyset \subseteq \emptyset \subseteq S)_{\mathrm{red}}$ vanishes. Fix a nonempty finite set T; we wish to show that the canonical map

$$\varinjlim_{\emptyset \neq T_0 \subseteq T} V(\emptyset \subseteq \emptyset \subseteq S)^{(T_0 \subseteq T)} \to V(\emptyset \subseteq \emptyset \subseteq S)^{(\emptyset \subseteq T)}$$

is an equivalence in $\mathrm{Shv}_\ell(X^T)$. This is clear, since the inclusions

$$\mathrm{Ran}_G^\dagger(X - Q)^{T_0 \subseteq T}_{S_0 \subseteq S_1, \mathrm{aug}} \hookrightarrow \mathrm{Ran}_G^\dagger(X - Q)^{\emptyset \subseteq T}_{S_0 \subseteq S_1, \mathrm{aug}}$$

are equivalences when $S_1 = \emptyset$. □

We now turn to the proof of Lemma 9.6.9. For the remainder of this section, we fix a nonempty finite set T and a (possibly empty) subset $T_0 \subseteq T$. Let $W : \Theta \to \mathrm{Shv}_\ell(X^T)$ denote the functor given by the formula

$$W(S_0 \subseteq S_1 \subseteq S) = V(S_0 \subseteq S_1 \subseteq S)^{(T_0 \subseteq T)} = [\mathrm{Ran}_G^\dagger(X - Q)^{T_0 \subseteq T}_{S_0 \subseteq S_1, \mathrm{aug}} \times_{(X-Q)^{S_1}} (X - Q)^S]_{X^T}.$$

We must prove the following:

Lemma 9.6.10. *Let S be a nonempty finite set and let $S_1 \subseteq S$ be a subset. Then the canonical map*

$$W(\emptyset \subseteq S_1 \subseteq S) \to \varprojlim_{S' \in \mathrm{Fin}^s} W(\emptyset \subseteq S_1 \subseteq S \amalg S')$$

is an equivalence in $\mathrm{Shv}_\ell(X^T)$.

Lemma 9.6.11. *Let $(S_0 \subseteq S_1 \subseteq S)$ be an object of Θ, let $s \in S$ be an element which is not contained in S_1, and set $S_0' = S_0 \cup \{s\}$, $S_1' = S_1 \cup \{s\}$. Then the diagram*

$$\begin{array}{ccc} W(S_0' \subseteq S_1' \subseteq S) & \longrightarrow & W(S_0 \subseteq S_1 \subseteq S) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} W(S_0'/E \subseteq S_1'/E \subseteq S/E) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^\circ(S)} W(S_0/E \subseteq S_1/E \subseteq S/E) \end{array}$$

is a pullback square in $\mathrm{Shv}_\ell(X^T)$.

Proof of Lemma 9.6.10. Fix an nonempty finite set S and a subset $S_1 \subseteq S$; we wish to show that the canonical map

$$\theta : W(\emptyset \subseteq S_1 \subseteq S) \to \varprojlim_{S' \in \mathrm{Fin}^s} W(\emptyset \subseteq S_1 \subseteq S \amalg S')$$

is an equivalence in $\mathrm{Shv}_\ell(X^T)$. Set $\mathcal{C} = \mathrm{Ran}_G^\dagger(X - Q)^{T_0 \subseteq T}_{\emptyset \subseteq S_1, \mathrm{aug}} \times (X - Q)^{(S - S_0)}$, so we have $W(\emptyset \subseteq S \subseteq S \amalg S') = [\mathcal{C} \times (X - Q)^{S'}]_{X^T}$. Since the domain and codomain of θ are ℓ-complete

(Remark 5.1.6), it will suffice to show that the image of θ is an equivalence in the ∞-category $\mathrm{Shv}(X^T; \mathbf{Z}/\ell\mathbf{Z})$. Tensoring with $\omega_{X^T}^{-1}$, we are reduced to proving that for every étale morphism $U \to X^T$, the canonical map

$$C^*(U \times_{X^T} \mathcal{C}; \mathbf{Z}/\ell\mathbf{Z}) \to \varprojlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} C^*(U \times_{X^T} \mathcal{C} \times (X - Q)^{S'}; \mathbf{Z}/\ell\mathbf{Z})$$

is an equivalence. In fact, we claim that the predual

$$u : \varinjlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} C_*(U \times_{X^T} \mathcal{C} \times (X - Q)^{S'}; \mathbf{Z}/\ell\mathbf{Z}) \to C_*(U \times_{X^T} \mathcal{C}; \mathbf{Z}/\ell\mathbf{Z})$$

is already an equivalence. Using Proposition 2.3.40, we are reduced to proving that the map

$$\varinjlim_{S' \in \mathrm{Fin}^{\mathrm{s}}} C_*((X - Q)^{S'}; \mathbf{Z}/\ell\mathbf{Z}) \to C_*(\operatorname{Spec} k; \mathbf{Z}/\ell\mathbf{Z})$$

is an equivalence, which follows from the acyclicity of $\mathrm{Ran}(X - Q)$ (Corollary 2.4.13). $\square$

Proof of Lemma 9.6.11. Fix an object $(S, S_0, S_1) \in \Theta$, let s be an element of $S - S_1$, and set $S_0' = S_0 \cup \{s\}$, $S_1' = S_1 \cup \{s\}$. We wish to show that the diagram σ :

$$\begin{array}{ccc} W(S_0' \subseteq S_1' \subseteq S) & \longrightarrow & W(S_0 \subseteq S_1 \subseteq S) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} W(S_0'/E \subseteq S_1'/E \subseteq S/E) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} W(S_0/E \subseteq S_1/E \subseteq S/E) \end{array}$$

is a pullback square in $\mathrm{Shv}_\ell(X^T)$.

For every finite set J_1 equipped with a subset $J_0 \subseteq J_1$, let $P(J_1)$ denote the partially ordered set of all subset $K \subseteq J_1$ and let $\mathcal{I}(J_0 \subseteq J_1)$ denote the subset of $P(J_1)^{\mathrm{op}} \times P(J_1)$ given by those pairs (K_-, K_+) with $K_- \subseteq K_+ \cup J_0$. Note that for every equivalence relation E on S, the quotient map $\beta : S \to S/E$ induces a map of partially ordered sets $\phi_E : \mathcal{I}(S_0 \subseteq S_1) \to \mathcal{I}(S_0/E \subseteq S_1/E)$, given by the formula $\phi_E(K_-, K_+) = (K_-/E, K_+/E)$. We will need the following:

($*$) For each equivalence relation E on S, the map ϕ_E is right cofinal.

To prove ($*$), we must show that for every pair of subsets $I_-, I_+ \subseteq S_1/E$ with $I_- \subseteq I_+ \cup S_0/E$, the simplicial set $\mathrm{N}(B)$ is weakly contractible, where B denotes the partially ordered subset of $\mathcal{I}(S_0 \subseteq S_1)$ consisting of those pairs (K_-, K_+) with $I_- \subseteq K_-/E$ and $K_+/E \subseteq I_+$. To prove this, let B_0 denote the subset of B consisting of those pairs (K_-, K_+) where $K_+ = \beta^{-1} I_+ \cap S_1$. The inclusion $B_0 \hookrightarrow B$ admits a left adjoint (given by $(K_-, K_+) \mapsto (K_-, \beta^{-1} I_+ \cap S_1)$) and therefore induces a weak homotopy equivalence $\mathrm{N}(B_0) \hookrightarrow \mathrm{N}(B)$. We are therefore reduced to proving that the simplicial set $\mathrm{N}(B_0)$ is weakly contractible. This follows from the observation that B_0 has a smallest element, given by the pair $((\beta^{-1} I_+ \cup S_0) \cap S_1, \beta^{-1} I_+ \cap S_1)$.

The proof of ($*$) shows also that each of the maps $\mathcal{I}(S_0' \subseteq S_1') \to \mathcal{I}(S_0'/E, S_1'/E)$ is right cofinal. We will also need the following:

($*'$) The construction $(K_-, K_+) \mapsto (K_- \cup \{s\}, K_+)$ determines a right cofinal map from $\mathcal{I}(S_0 \subseteq S_1)$ to $\mathcal{I}(S_0' \subseteq S_1')$.

To prove ($*'$), fix an object $(K_-', K_+') \in \mathcal{I}(S_0' \subseteq S_1')$. We wish to show that $\mathrm{N}(B')$ is weakly contractible, where B' denotes the partially ordered subset of $\mathcal{I}(S_0 \subseteq S_1)$ spanned by those pairs (K_-, K_+) such that $K_-' \subseteq K_- \cup \{s\}$ and $K_+ \subseteq K_+'$. Let B_0' denote the partially ordered subset of B' spanned by those pairs (K_-, K_+) where $K_+ = K_+'$. Then the inclusion $B_0' \hookrightarrow B'$ admits a left adjoint (given by $(K_-, K_+) \mapsto (K_-, K_+')$) and therefore induces a weak homotopy equivalence $\mathrm{N}(B'_0) \hookrightarrow \mathrm{N}(B')$. We are therefore reduced to showing that $\mathrm{N}(B'_0)$ is weakly contractible. This is clear, since B'_0 has a least element, given by $(S_0 \cup K'_+, K'_+)$.

For every object $\vec{J} = (J_0 \subseteq J_1 \subseteq J)$ in Θ, let $\mathcal{C}(\vec{J})$ denote the fiber product

$$\mathrm{Ran}^{\dagger}_G(X-Q)^{T_0 \subseteq T}_{J_0 \subseteq J_1, \mathrm{aug}} \times_{(X-Q)^{J_1}} (X-Q)^J,$$

whose objects can be identified with tuples $(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ (see Construction 9.6.1). The construction

$$(R, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma) \mapsto (K_-, K_+)$$

determines a Cartesian fibration $\mathcal{C}(\vec{J}) \to \mathcal{I}(J_0 \subseteq J_1)$. We will denote the fiber of this fibration over an object $(K_-, K_+) \in \mathcal{I}(J_0 \subseteq J_1)$ by $\mathcal{C}(\vec{J})_{K_-,K_+}$, so that we have a canonical equivalence

$$W(J_0 \subseteq J_1 \subseteq J) \simeq \varprojlim_{(K_-,K_+) \in \mathcal{I}(J_0 \subseteq J_1)} [\mathcal{C}(\vec{J})_{K_-,K_+}]_{X^J}.$$

Set $\mathcal{J} = \mathcal{I}(S_0 \subseteq S_1)$, so that $(*)$ and $(*')$ supply equivalences

$$W(S_0/E \subseteq S_1/E \subseteq S/E) \simeq \varprojlim_{(K_-,K_+) \in \mathcal{J}} [\mathcal{C}(S_0/E \subseteq S_1/E \subseteq S/E)_{K_-/E, K_+/E}]_{X^T}$$

$$W(S'_0/E \subseteq S'_1/E \subseteq S/E) \simeq \varprojlim_{(K_-,K_+) \in \mathcal{J}} [\mathcal{C}(S'_0/E \subseteq S'_1/E \subseteq S/E)_{(K_- \cup \{s\})/E, K_+/E}]_{X^T}.$$

Fix an object $(K_-, K_+) \in \mathcal{J}$. For each equivalence relation $E \in \mathrm{Equiv}(S)$, set

$$Z(E) = [\mathcal{C}(S_0/E \subseteq S_1/E \subseteq S/E)_{K_-/E, K_+/E}]_{X^T}$$

$$Z'(E) = [\mathcal{C}(S'_0/E \subseteq S'_1/E \subseteq S/E)_{(K_- \cup \{s\})/E, K_+/E}]_{X^T}.$$

Then we can identify σ with a finite limit of diagrams σ_{K_-,K_+} :

$$\begin{array}{ccc} Z'(E_0) & \longrightarrow & Z(E_0) \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} Z'(E) & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} Z(E), \end{array}$$

where E_0 denotes the trivial equivalence relation on S. We will complete the proof by showing that each σ_{K_-,K_+} is a pullback square in $\mathrm{Shv}_\ell(X^T)$. Let τ_{K_-,K_+} denote the commutative diagram in $\mathrm{Shv}(X^T; \mathbf{Z}/\ell\mathbf{Z})$ obtained from σ_{K_-,K_+} by tensoring each entry with $\omega^{-1}_{X^T}$ and reducing modulo ℓ. Since each entry in σ_{K_-,K_+} is ℓ-complete, it will suffice to show that τ_{K_-,K_+} is a pullback square in $\mathrm{Shv}(X^T; \mathbf{Z}/\ell\mathbf{Z})$.

Let U denote the open subset of $(X-Q)^S \times_{\mathrm{Spec}\, k} X^T$ whose k-valued points are given by pairs of maps $\mu : S \to (X-Q)(k)$, $\nu : T \to X(k)$ satisfying $\mu(K_+) \cap \nu(T) = \emptyset = \mu(S_1) \cap \nu(T_0)$ and consider the projection maps

$$(X-Q)^S \xleftarrow{q} U \xrightarrow{q'} X^T.$$

For each equivalence relation $E \in \mathrm{Equiv}(S)$, define $\mathcal{F}_E, \mathcal{F}'_E \in \mathrm{Shv}((X-Q)^S; \mathbf{Z}/\ell\mathbf{Z})$ as in the proof of Proposition 8.5.4. Using Theorem 4.5.4, we can identify τ_{K_-,K_+} with the image of the diagram τ :

$$\begin{array}{ccc} \mathcal{F}'_{E_0} & \longrightarrow & \mathcal{F}_{E_0} \\ \downarrow & & \downarrow \\ \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} \mathcal{F}'_E & \longrightarrow & \varprojlim_{E \in \mathrm{Equiv}^{\circ}(S)} \mathcal{F}_E \end{array}$$

under the exact functor $q'_* q^* : \mathrm{Shv}((X-Q)^S; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Shv}(X^T; \mathbf{Z}/\ell\mathbf{Z})$. It will therefore suffice to show that τ is a pullback diagram, which was established in the proof of Proposition 8.5.4. $\square$

9.7. **Construction of a Pullback Square.** Throughout this section, we fix an algebraically closed field k, an algebraic curve X over k, a smooth affine group scheme G over X, and a finite subset $Q \subseteq X$ for which G is Q-adapted. Our goal in this section is to verify Proposition 9.6.7 by constructing a family of pullback diagrams

$$\begin{array}{ccc} \mathcal{B}_{S_0 \subseteq S, \mathrm{red}} & \longrightarrow & \mathcal{B}^{\circ}_{S_0 \subseteq S} \\ \downarrow & & \downarrow \\ \mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{red}} & \longrightarrow & \mathcal{B}^{\circ, \mathrm{off}}_{S_0 \subseteq S}, \end{array}$$

This will require several auxiliary constructions.

Notation 9.7.1. Let S and T be finite sets. We let $U(S,T)$ denote the open subscheme of $(X-Q)^S \times X^T$ whose k-valued points are pairs of maps $\mu : S \to (X-Q)(k)$, $\nu : T \to X(k)$ such that $\mu(S) \neq \nu(T)$ (as subsets of $X(k)$). We let $U'(S,T)$ denote the open subscheme of $U(S,T)$ whose k-valued points are pairs (μ, ν) where $\mu(S) \nsubseteq \nu(T)$.

Suppose that $\pi : \mathcal{C} \to \mathrm{Ran}_{\mathrm{aug}}(X)$ is an augmented $\mathrm{Ran}(X)$-prestack equipped with a map $\phi : \mathcal{C} \to (X-Q)^S$. For each object $C \in \mathcal{C}$, the image $\pi(C)$ can be identified with a triple $(R, T_0 \subseteq T, \nu)$ where R is a finitely generated k-algebra, T is a nonempty finite set, T_0 is a subset of T, and ν is a map from $\mathrm{Spec}\, R$ into X^T. Then $\phi(C)$ can be identified with a map $\mu : \mathrm{Spec}\, R \to (X-Q)^S$. We let $\mathcal{C}^{\mathrm{off}}$ denote the full subcategory of $\mathcal{C}$ spanned by those objects C for which the product map

$$(\mu, \nu) : \mathrm{Spec}\, R \to (X-Q)^S \times X^T$$

factors through the open subscheme $U(S,T) \subseteq (X-Q)^S \times X^T$, and we let $\mathcal{C}^{\mathrm{skw}}$ denote the full subcategory spanned by those objects for which the product map (μ,ν) factors through $U'(S,T)$. Note that $\mathcal{C}^{\mathrm{skw}}$ and $\mathcal{C}^{\mathrm{off}}$ are augmented $\mathrm{Ran}(X)$-prestacks, and that the inclusions $\mathcal{C}^{\mathrm{skw}} \subseteq \mathcal{C}^{\mathrm{off}} \subseteq \mathcal{C}$ determine maps

$$[\mathcal{C}]_{\mathrm{Ran}_{\mathrm{aug}}(X)} \to [\mathcal{C}^{\mathrm{off}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)} \to [\mathcal{C}^{\mathrm{skw}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}$$

of augmented !-sheaves on $\mathrm{Ran}(X)$.

Notation 9.7.2. Let S be a finite set and let $S_0 \subseteq S$ be a subset. We define augmented $\mathrm{Ran}(X)$-presheaves $\mathcal{B}^{\mathrm{off}}_{S_0 \subseteq S, \mathrm{aug}}$, $\mathcal{B}^{\circ, \mathrm{off}}_{S_0 \subseteq S, \mathrm{aug}}$, $\mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}}$, and $\mathcal{B}^{\circ, \mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}}$ by the formulae

$$\mathcal{B}^{\mathrm{off}}_{S_0 \subseteq S, \mathrm{aug}} = [\mathrm{Ran}^{\dagger}_G(X-Q)^{\mathrm{off}}_{S_0 \subseteq S, \mathrm{aug}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}$$

$$\mathcal{B}^{\circ, \mathrm{off}}_{S_0 \subseteq S, \mathrm{aug}} = [\mathrm{Ran}^{\dagger}_G(X-Q)^{\mathrm{off}}_{S_0 \subseteq S, \mathrm{aug}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}.$$

$$\mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}} = [\mathrm{Ran}^{\dagger}_G(X-Q)^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}$$

$$\mathcal{B}^{\circ, \mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}} = [\mathrm{Ran}^{\dagger}_G(X-Q)^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}}]_{\mathrm{Ran}_{\mathrm{aug}}(X)}.$$

We will denote the underlying lax !-sheaf of $\mathcal{B}^{\mathrm{off}}_{S_0 \subseteq S, \mathrm{aug}}$ by $\mathcal{B}^{\mathrm{off}}_{S_0 \subseteq S}$ and its reduced part by $\mathcal{B}^{\mathrm{off}}_{S_0 \subseteq S, \mathrm{red}}$, and employ analogous notation in the other cases.

We have a commutative diagram

$$\begin{array}{ccccc}
\varprojlim_{\mathrm{prim}} \mathcal{B}_{S_0 \subseteq S, \mathrm{red}} & \xrightarrow{\xi_1} & \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{red}} & \xrightarrow{\xi_2} & \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ}_{S_0 \subseteq S} \\
\downarrow & & \downarrow & & \downarrow \\
 & & \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S, \mathrm{red}} & \longrightarrow & \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S} \\
\downarrow & & \downarrow{\scriptstyle \rho} & & \\
\varprojlim_{\mathrm{prim}} \mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{red}} & \longrightarrow & \varprojlim_{\mathrm{prim}} \mathcal{B}^{\circ,\mathrm{skw}}_{S_0 \subseteq S, \mathrm{red}} . & &
\end{array}$$

We can now formulate the main results of this section:

Proposition 9.7.3. *Let S be a nonempty finite set and let $S_0 \subseteq S$ be a subset. Then the restriction map $\rho : \mathcal{B}^{\mathrm{off}}_{S_0 \subseteq S, red} \to \mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, red}$ is an equivalence in* $\mathrm{Shv}^{\mathrm{lax}}_{\ell}(\mathrm{Ran}(X))$.

Proposition 9.7.4. *Let S be a nonempty finite set and let $S_0 \subseteq S$ be a subset. Then the diagram*

$$\begin{array}{ccc}
\mathcal{B}_{S_0 \subseteq S, \mathrm{aug}} & \longrightarrow & \mathcal{B}^{\circ}_{S_0 \subseteq S, \mathrm{aug}} \\
\downarrow & & \downarrow \\
\mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}} & \longrightarrow & \mathcal{B}^{\circ,\mathrm{skw}}_{S_0 \subseteq S, \mathrm{aug}}
\end{array}$$

is a pullback square of augmented lax !-sheaves on $\mathrm{Ran}(X)$*. In particular, the diagram*

$$\begin{array}{ccc}
\mathcal{B}_{S_0 \subseteq S, red} & \longrightarrow & \mathcal{B}^{\circ}_{S_0 \subseteq S, red} \\
\downarrow & & \downarrow \\
\mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, red} & \longrightarrow & \mathcal{B}^{\circ,\mathrm{skw}}_{S_0 \subseteq S, red}
\end{array}$$

is a pullback square in $\mathrm{Shv}^{\mathrm{lax}}_{\ell}(\mathrm{Ran}(X))$.

Proposition 9.7.5. *Let S be a nonempty finite set and let $S_0 \subseteq S$. Then the diagram*

$$\begin{array}{ccc}
\mathcal{B}^{\circ}_{S_0 \subseteq S, red} & \longrightarrow & \mathcal{B}^{\circ}_{S_0 \subseteq S} \\
\downarrow & & \downarrow \\
\mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S, red} & \longrightarrow & \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}
\end{array}$$

is a pullback square in $\mathrm{Shv}^{\mathrm{lax}}_{\ell}(\mathrm{Ran}(X))$.

Combining Propositions 9.7.3, 9.7.4, and 9.7.5, we obtain a pullback square

$$\begin{array}{ccc}
\mathcal{B}_{S_0 \subseteq S, \mathrm{red}} & \longrightarrow & \mathcal{B}^{\circ}_{S_0 \subseteq S} \\
\downarrow & & \downarrow \\
\mathcal{B}^{\mathrm{skw}}_{S_0 \subseteq S, \mathrm{red}} & \longrightarrow & \mathcal{B}^{\circ,\mathrm{off}}_{S_0 \subseteq S}
\end{array}$$

in $\mathrm{Shv}^{\mathrm{lax}}_{\ell}(\mathrm{Ran}(X))$, which depends functorially on the pair $(S_0 \subseteq S)$. To complete the proof of Proposition 9.6.7, it will suffice to also prove the following:

Proposition 9.7.6. *The primitive limit*

$$\varprojlim_{\text{prim}} \mathcal{B}^{\text{skw}}_{S_0 \subseteq S, \text{aug}}$$

vanishes (in the ∞-category $\text{Shv}^{\text{aug}}_{\ell}(\text{Ran}(X))$ of augmented !-sheaves on $\text{Ran}(X)$). In particular, we have

$$\varprojlim_{\text{prim}} \mathcal{B}^{\text{skw}}_{S_0 \subseteq S, red} \simeq 0 \in \text{Shv}^{\text{lax}}_{\ell}(\text{Ran}(X)).$$

We now turn to the proofs.

Proof of Proposition 9.7.3. Let T be a nonempty finite set; we wish to show that the restriction map

$$\phi : \mathcal{B}^{\text{off}(T)}_{S_0 \subset S, \text{red}} \to \mathcal{B}^{\text{skw}(T)}_{S_0 \subseteq S, \text{red}}$$

is an equivalence in $\text{Shv}_{\ell}(X^T)$. For every element $t \in T$, let $V_t \subseteq (X - Q)^S \times_{\text{Spec}\, k} X^T$ denote the open subset whose k-valued points are pairs (μ, ν) for which $\nu(t) \notin \mu(S)$. Then we have $U(S,T) = U'(S,T) \cup \bigcup_{t \in T} V_t$. For each nonempty subset $T_0 \subseteq T$, we set $V_{T_0} = \bigcap_{t \in T_0} V_t$ and $V'_{T_0} = U'(S,T) \cap \bigcap_{t \in T_0} V_t$. Using Zariski descent, we obtain a pullback diagram

$$\begin{array}{ccc} [\text{Ran}_G(X-Q)_S \times_{X^S} U(S,T)]_{X^T} & \longrightarrow & [\text{Ran}_G(X-Q)_S \times_{X^S} U'(S,T)]_{X^T} \\ \downarrow & & \downarrow \\ \varprojlim_{\emptyset \neq T_0} [\text{Ran}_G(X)_S \times_{X^S} V_{T_0}]_{X^T} & \longrightarrow & \varprojlim_{\emptyset \neq T_0} [\text{Ran}_G(X)_S \times_{X^S} V'_{T_0}]_{X^T} \end{array}$$

is a pulback square in $\text{Shv}_{\ell}(X^T)$. The desired result now follows from passing to fibers in the vertical direction. ☐

Proof of Proposition 9.7.4. Fix a nonempty finite set T and a subset $T_0 \subseteq T$; we will show that the diagram σ :

$$\begin{array}{ccc} \mathcal{B}^{(T_0 \subseteq T)}_{S_0 \subseteq S, \text{aug}} & \longrightarrow & \mathcal{B}^{\circ(T_0 \subseteq T)}_{S_0 \subseteq S, \text{aug}} \\ \downarrow & & \downarrow \\ \mathcal{B}^{\text{skw}(T_0 \subseteq T)}_{S_0 \subseteq S, \text{aug}} & \longrightarrow & \mathcal{B}^{\circ, \text{skw}(T_0 \subseteq T)}_{S_0 \subseteq S, \text{aug}} \end{array}$$

is a pullback square in $\text{Shv}_{\ell}(X^T)$.

For each subset $S_- \subseteq S$, let us identify objects of the prestack $\text{Ran}_G(X-Q)_{S_- \subseteq S} \times_{\text{Spec}\, k} X^T$ with quadruples $(R, \mu, \nu, \mathcal{P}, \alpha)$ where R is a finitely generated k-algebra, $\mu : S \to (X - Q)(R)$ and $\nu : T \to X(R)$ are maps of sets, $\mathcal{P}$ is a G-bundle on X_R, and α is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu(S)|$. Given another subset $S_+ \subseteq S$, we let $\mathcal{C}_{S_-, S_+}$ denote the full subcategory of $\text{Ran}_G(X)_{S_- \subseteq S} \times_{\text{Spec}\, k} X^T$ spanned by those tuples $(R, \mu, \nu, \mathcal{P}, \alpha)$ where $|\mu(S_+)| \cap |\nu(T)| = \emptyset = |\mu(S)| \cap |\nu(T_0)|$, and let $\mathcal{C}^0_{S_-, S_+}$ denote the fiber product $\mathcal{C}_{S_-, S_+} \times_{X^{S \amalg T}} U'(S,T)$. Let $P(S)$ denote the collection of all subsets of S, partially ordered by inclusion, and let $J \subseteq P(S)^{\text{op}} \times P(S)$ denote the set of pairs (S_-, S_+) such that $S_- \subseteq S_+ \cup S_0$. Unwinding the definitions, we see that σ can be identified with the diagram

$$\begin{array}{ccc} \varprojlim_{(S_-, S_+) \in J} [\mathcal{C}_{S_-, S_+}]_{X^T} & \longrightarrow & [\mathcal{C}_{S_0, \emptyset}]_{X^T} \\ \downarrow & & \downarrow \\ \varprojlim_{(S_-, S_+) \in J} [\mathcal{C}^0_{S_-, S_+}]_{X^T} & \longrightarrow & [\mathcal{C}^0_{S_0, \emptyset}]_{X^T}. \end{array}$$

For each pair $(S_-, S_+) \in J$, let $F(S_-, S_+)$ denote the fiber of the canonical map

$$[\mathcal{C}_{S_-,S_+}]_{X^T} \to [\mathcal{C}^0_{S_-,S_+}]_{X^T}.$$

To show that σ is a pullback diagram, it will suffice to show that the canonical map

$$\varprojlim_{(S_-,S_+)\in J} F(S_-, S_+) \to F(S_0, \emptyset)$$

is an equivalence. Note that if $S_+ \neq \emptyset$, then the inclusion $\mathcal{C}^0_{S_-,S_+} \hookrightarrow \mathcal{C}_{S_-,S_+}$ is an equality, so that $F(S_-, S_+) = 0$. It follows that the functor F is a right Kan extension of its restriction to $J' = \{(S_-, S_+) \in J : S_+ = \emptyset\}$. It will therefore suffice to show that the restriction map $\varprojlim_{(S_-,S_+)\in J'} F(S_-, S_+) \to F(S_0, \emptyset)$ is an equivalence. This is clear, since $(S_0, \emptyset)$ is a least element of J'. □

Proof of Proposition 9.7.5. Fix a nonempty finite set T; we wish to show that the diagram

$$\begin{array}{ccc} (\mathcal{B}^{\circ}_{S_0\subseteq S,\mathrm{red}})^{(T)} & \longrightarrow & (\mathcal{B}^{\circ}_{S_0\subseteq S})^{(T)} \\ \downarrow & & \downarrow \\ (\mathcal{B}^{\circ,\mathrm{off}}_{S_0\subseteq S,\mathrm{red}})^{(T)} & \longrightarrow & (\mathcal{B}^{\circ,\mathrm{off}}_{S_0\subseteq S})^{(T)} \end{array}$$

is a pullback square of ℓ-adic sheaves on X^T. Equivalently, we wish to show that the map

$$\varprojlim_{\emptyset\neq T_0\subseteq T} \mathcal{B}^{\circ(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}} \to \varprojlim_{\emptyset\neq T_0\subseteq T} \mathcal{B}^{\circ,\mathrm{off}(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}}.$$

In fact, something stronger is true: for every nonempty subset $T_0 \subseteq T$, the restriction map

$$\mathcal{B}^{\circ(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}} \to \mathcal{B}^{\circ,\mathrm{off}(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}}.$$

This follows immediately from the observation that we have an equality of prestacks

$$(\mathrm{Ran}^{\circ}_G(X-Q)_{S_0\subseteq S,\mathrm{aug}})^{T_0\subseteq T} = (\mathrm{Ran}^{\circ}_G(X-Q)_{S_0\subseteq S,\mathrm{aug}})^{\mathrm{off},T_0\subseteq T},$$

since a pair of maps $\mu : S \to (X-Q)(k)$ and $\nu : T \to X(k)$ cannot simultaneously satisfy $\mu(S) = \nu(T)$ and $\mu(S) \cap \nu(T_0) = \emptyset$. □

Proof of Proposition 9.7.6. Fix a nonempty set T and a subset $T_0 \subseteq T$; we wish to prove that the sheaf

$$\varprojlim_{\mathrm{prim}} \mathcal{B}^{\mathrm{skw}(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}}$$

vanishes in $\mathrm{Shv}_\ell(X^T)$. We can write this sheaf as the cofiber of a map

$$\rho : \varprojlim_{S\in\mathrm{Fin}^s} \varinjlim_{\emptyset\neq S_0\subseteq S} \mathcal{B}^{\mathrm{skw}(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}} \to \varprojlim_{S\in\mathrm{Fin}^s} \mathcal{B}^{\mathrm{skw}(T_0\subseteq T)}_{\emptyset\subseteq S,\mathrm{aug}},$$

which is itself given as a limit of maps

$$\rho_S : \varinjlim_{\emptyset\neq S_0\subseteq S} \mathcal{B}^{\mathrm{skw}(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}} \to \mathcal{B}^{\mathrm{skw}(T_0\subseteq T)}_{\emptyset\subseteq S,\mathrm{aug}}.$$

We will complete the proof by showing that each ρ_S is an equivalence in $\mathrm{Shv}_\ell(X^T)$.

Let $P(S)$ denote the partially ordered set of subsets of S. The construction $S_0 \mapsto \mathcal{B}^{\mathrm{skw}(T_0\subseteq T)}_{S_0\subseteq S,\mathrm{aug}}$ determines a functor $f : P(S)^{\mathrm{op}} \to \mathrm{Shv}_\ell(X^T)$, and we wish to show that f is a colimit diagram. Since the ∞-category $\mathrm{Shv}_\ell(X^T)$ is stable and the domain of f is a cube, it will suffice to show that f is a limit diagram (Proposition HA.1.2.4.13). We are therefore reduced to the problem of showing that the map

$$f(S) \to \varprojlim_{S_0 \subsetneq S} f(S_0)$$

is an equivalence.

For each subset $S_- \subseteq S$, let us identify objects of the prestack $\mathrm{Ran}_G(X)_{S_- \subseteq S} \times_{\mathrm{Spec}\, k} X^T$ with quadruples $(R, \mu, \nu, \mathcal{P}, \alpha)$ where R is a finitely generated k-algebra, $\mu : S \to (X - Q)(R)$ and $\nu : T \to X(R)$ are maps of sets, $\mathcal{P}$ is a G-bundle on X_R, and α is a trivialization of $\mathcal{P}$ over the open set $X_R - |\mu|$. Given another subset $S_+ \subseteq S$, we let $\mathcal{C}_{S_-,S_+}$ denote the full subcategory of $\mathrm{Ran}_G(X)_{S_- \subseteq S} \times_{X^S} U'(S,T)$ spanned by those tuples $(R, \mu, \nu, \mathcal{P}, \alpha)$ where $|\mu(S_+)| \cap |\nu(T)| = \emptyset = |\mu(S)| \cap |\nu(T_0)|$. Unwinding the definitions, we see that the functor f is given by

$$f(S_0) = \varprojlim_{(S_-,S_+)} [\mathcal{C}_{S_-,S_+}]_{X^T},$$

where the limit is taken over all pairs $(S_-, S_+) \in P(S)^{\mathrm{op}} \times P(S)$ satisfying $S_- \subseteq S_+ \cup S_0$. We may therefore identify $\varprojlim_{S_0 \subsetneq S} f(S_0)$ with the limit

$$\varprojlim_{(S_-,S_+) \neq (S,\emptyset)} [\mathcal{C}_{S_-,S_+}]_{X^T}.$$

We are therefore reduced to proving that the canonical map

$$\varprojlim_{(S_-,S_+)} [\mathcal{C}_{S_-,S_+}]_{X^T} \to \varprojlim_{(S_-,S_+) \neq (S,\emptyset)} [\mathcal{C}_{S_-,S_+}]_{X^T}$$

is an equivalence. To prove this, it will suffice to show that the functor $(S_-, S_+) \mapsto [\mathcal{C}_{S_-,S_+}]_{X^T}$ is a right Kan extension of its restriction to $P(S)^{\mathrm{op}} \times (P(S) - \{\emptyset\})$. That is, we will show that for each subset $S_- \subseteq S$, the canonical map

$$\psi : [\mathcal{C}_{S_-,\emptyset}]_{X^T} \to \varprojlim_{\emptyset \neq S_+ \subseteq S} [\mathcal{C}_{S_-,S_+}]_{X^T}$$

is an equivalence in $\mathrm{Shv}_\ell(X^T)$. For each $s \in S$, let V_s denote the open subset of $(X - Q)^S \times_{\mathrm{Spec}\, k} X^T$ whose k-valued points are pairs of maps $\mu : S \to (X - Q)(k)$, $\nu : T \to X(k)$ satisfying $\mu(s) \notin \nu(T)$. Unwinding the definitions, we see that $\mathcal{C}_{S_-,S_+}$ can be identified with the fiber product $\mathcal{C}_{S_-,\emptyset} \times_{U'(S,T)} \bigcap_{s \in S_+} V_s$. The assertion that ψ is an equivalence now follows by Zariski descent, since the open sets $\{V_s\}_{s \in S}$ form a covering of $U'(S,T)$. □

10. The Grothendieck-Lefschetz Trace Formula for $\mathrm{Bun}_G(X)$

Let $\mathbf{F}_q$ denote a finite field q elements, let $\overline{\mathbf{F}}_q$ denote an algebraic closure of $\mathbf{F}_q$, let ℓ be a prime number which is relatively prime to q, and fix an embedding $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$.

Notation 10.0.1. Suppose that $\mathcal{X}$ is a smooth Artin stack over $\mathbf{F}_q$. Let $\mathcal{X}(\mathbf{F}_q)$ denote the groupoid of $\mathbf{F}_q$-valued points of $\mathcal{X}$. For each object $\eta \in \mathcal{X}(\mathbf{F}_q)$, we let $\mathrm{Aut}(\eta)$ denote the automorphism group of η (as an object of the groupoid $\mathcal{X}(\mathbf{F}_q)$), and $|\mathrm{Aut}(\eta)|$ the cardinality of the group $\mathrm{Aut}(\eta)$. Let us assume that each of the groups $\mathrm{Aut}(\eta)$ is finite (this is automatic, for example, if the stack $\mathcal{X}$ has affine diagonal). We let $|\mathcal{X}(\mathbf{F}_q)|$ denote the sum

$$\sum_\eta \frac{1}{|\mathrm{Aut}(\eta)|},$$

where η ranges over a set of representatives for the isomorphism classes of objects of $\mathcal{X}(\mathbf{F}_q)$. We will refer to $|\mathcal{X}(\mathbf{F}_q)|$ as the *mass* of $\mathcal{X}(\mathbf{F}_q)$.

Warning 10.0.2. In the situation of Notation 10.0.1, the groupoid $\mathcal{X}(\mathbf{F}_q)$ might contain infinitely many isomorphism classes of objects. In this case, the sum $\sum_\eta \frac{1}{|\operatorname{Aut}(\eta)|}$ has infinitely many terms. However, since each term is positive, the sum $\sum_\eta \frac{1}{|\operatorname{Aut}(\eta)|}$ converges to a unique element of the set $\mathbf{R}_{\geq 0} \cup \{\infty\}$.

Definition 10.0.3. Let $\mathcal{X}$ be a smooth Artin stack over $\mathbf{F}_q$ of dimension d, let

$$\mathrm{H}^*_{\mathrm{geom}}(\mathcal{X}) = \mathrm{H}^*(\mathcal{X} \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q; \mathbf{Z}_\ell)[\ell^{-1}]$$

denote the geometric cohomology of $\mathcal{X}$ (Notation 6.4.1), and let Frob denote the geometric Frobenius automorphism of $\mathrm{H}^*_{\mathrm{geom}}(\mathcal{X})$. We will say that $\mathcal{X}$ *satisfies the Grothendieck-Lefschetz trace formula* if the pair $(\mathrm{H}^*_{\mathrm{geom}}(\mathcal{X}), \mathrm{Frob}^{-1})$ is summable (Definition 6.3.1) and we have

$$\operatorname{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(\mathcal{X})) = \frac{|\,\mathcal{X}(\mathbf{F}_q)|}{q^d}.$$

Remark 10.0.4. If $\mathcal{X}$ is a smooth Artin stack over $\mathbf{F}_q$ which satisfies the Grothendieck-Lefschetz trace formula, then the mass $|\,\mathcal{X}(\mathbf{F}_q)|$ is finite.

Remark 10.0.5. One can introduce a theory of *compactly supported* cohomology for Artin stacks over $\overline{F}_q$. If $\overline{\mathcal{X}}$ is a smooth Artin stack of dimension d over $\mathbf{F}_q$ for which the cohomology

$$\mathrm{H}^*_c(\mathcal{X} \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q; \mathbf{Z}_\ell)$$

is a finitely generated $\mathbf{Z}_\ell$-module in each degree, then it follows from Poincare duality that $(\mathrm{H}^*_{\mathrm{geom}}(\mathcal{X}); \mathrm{Frob}^{-1})$ is summable if and only if

$$(\mathrm{H}^*_c(\mathcal{X} \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q; \mathbf{Z}_\ell)[\ell^{-1}], \mathrm{Frob})$$

is summable, and in this event we have

$$\operatorname{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(\mathcal{X})) = \frac{\operatorname{Tr}(\mathrm{Frob} \mid \mathrm{H}^*_c(\mathcal{X} \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q; \mathbf{Z}_\ell)[\ell^{-1}])}{q^d}.$$

In this case, $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula if and only if

$$\operatorname{Tr}(\mathrm{Frob} \mid \mathrm{H}^*_c(\mathcal{X} \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q; \mathbf{Z}_\ell)[\ell^{-1}]) = |\,\mathcal{X}(\mathbf{F}_q)|.$$

Note that this condition makes sense even when $\mathcal{X}$ is not smooth. However, we will confine our attention to the case where $\mathcal{X}$ is smooth (which will be sufficient for our applications in this paper).

Our goal in this section is to prove Theorem 1.3.5, which we formulate as follows:

Theorem 10.0.6. *Let X be a smooth complete geometrically connected algebraic curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X. Suppose that the fibers of G are connected and that the generic fiber of G is semisimple. Then the moduli stack $\operatorname{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula.*

In the special case where G is a semisimple group scheme over X, Theorem 10.0.6 was proven by Behrend in [5]. Let us now give an outline of our proof, which will closely follow the methods used in [5].

For algebraic stacks of finite type over $\mathbf{F}_q$, the Grothendieck-Lefschetz trace formula was verified by Behrend in [5]. In §10.1, we prove a weaker version of this result: any (smooth) global quotient stack Y/H (where H is affine) satisfies the Grothendieck-Lefschetz trace formula (Corollary 10.1.4). This is quite relevant to the proof of Theorem 10.0.6, since any quasi-compact open substack $\operatorname{Bun}_G(X)$ can be presented as a global quotient stack (see Corollary 10.4.2).

Unfortunately, we cannot deduce Theorem 10.0.6 directly from the Grothendieck-Lefschetz trace formula for global quotient stacks, because the moduli stack $\mathrm{Bun}_G(X)$ is generally not quasi-compact. Our strategy instead will be to decompose $\mathrm{Bun}_G(X)$ into locally closed substacks $\mathrm{Bun}_G(X)_{[P,\nu]}$ which are more directly amenable to analysis. In §10.2, we lay the foundations by reviewing the notion of a *stratification* of an algebraic stack $\mathcal{X}$. Our main result is that if $\mathcal{X}$ is a smooth Artin stack over $\mathbf{F}_q$ which admits a stratification by locally closed substacks $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ which satisfy the Grothendieck-Lefschetz trace formula, then $\mathcal{X}$ also satisfies the Grothendieck-Lefschetz trace formula provided that a certain convergence condition is satisfied (Proposition 10.2.13; see also Proposition 10.2.11).

To apply the results of §10.2 to our situation, we need to choose a useful stratification of $\mathrm{Bun}_G(X)$. In §10.3, we specialize to the case where G is a split group and review the theory of the *Harder-Narasimhan* stratification, which supplies a decomposition of $\mathrm{Bun}_G(X)$ into locally closed substacks $\mathrm{Bun}_G(X)_{P,\nu}$ where P ranges over standard parabolic subgroups of G and ν ranges over ν ranges over dominant regular cocharacters of the center of the Levi factor $P/\operatorname{rad}_u P$. At the present level of generality, this theory was developed by Behrend and was the main tool used in his proof of Theorem 10.0.6 in the case where G is everywhere semisimple.

In order for a stratification of $\mathrm{Bun}_G(X)$ to be useful to us, we will need to know that the individual strata are more tractable than the entire moduli stack $\mathrm{Bun}_G(X)$ itself: for example, we would like to know that they are quasi-compact. In §10.4, we recall the proof that the Harder-Narasimhan strata $\mathrm{Bun}_G(X)_{P,\nu}$ are quasi-compact (Proposition 10.4.6) in the case of a split group G, and provide a number of tools for establishing related results (by studying the compactness properties of *morphisms* between moduli stacks of the form $\mathrm{Bun}_G(X)$ as G and X vary).

In §10.5, we discuss the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$ in the case where G is semisimple and *generically* split (or, more generally, when the generic fiber is an inner form of a split semisimple group). This is essentially a formal exercise: the stratification of $\mathrm{Bun}_G(X)$ is pulled back from a stratification of $\mathrm{Bun}_{G_{\mathrm{ad}}}(X)$, where G_{ad} denotes the adjoint quotient of G, and the moduli stack $\mathrm{Bun}_{G_{\mathrm{ad}}}(X)$ can be identified with the moduli stack of bundles for the split form of G_{ad}.

In order to prove that the moduli stack $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula, it is not enough to know that $\mathrm{Bun}_G(X)$ can be decomposed into locally closed substacks $\mathrm{Bun}_G(X)_\alpha$ which satisfy the Grothendieck-Lefschetz trace formula: for example, we need to know that sum

$$|\mathrm{Bun}_G(X)(\mathbf{F}_q)| = \sum_\alpha |\mathrm{Bun}_G(X)_\alpha(\mathbf{F}_q)|$$

converges. In the case of the Harder-Narasimhan stratification, the key observation is that the infinite collection of Harder-Narasimhan strata $\{\mathrm{Bun}_G(X)_{P,\nu}\}$ can be decomposed into finitely many families whose members "look alike" (for example, members of the same family have the same ℓ-adic cohomology). In §10.6, we discuss a mechanism which guarantees this behavior: given a G-bundle $\mathcal{P}$ equipped with a reduction to a parabolic subgroup $P \subseteq G$, there is a "twisting" procedure (depending on a few auxiliary choices) for producing a *new* G-bundle $\mathrm{Tw}_{\lambda,D}(\mathcal{P})$. Roughly speaking, this twisting procedure supplies maps

$$\mathrm{Bun}_G(X)_{P,\nu} \to \mathrm{Bun}_G(X)_{P,\nu+\lambda}$$

which exhibit the left hand side as a fiber bundle over the right hand side, whose fibers are affine spaces (strictly speaking, this is only true if we assume that $\mathbf{F}_q$ is a field of sufficiently large characteristic; in general, the twisting construction is only defined "up to" a finite radicial map); see Proposition 10.6.33.

The Harder-Narasimhan stratification of §10.5 is defined in the special case where the group scheme G is everywhere semisimple and the generic fiber of G is split. To treat the general case, we note the generic fiber of G is a semisimple algebraic group over K_X, and therefore splits after passing to some finite Galois extension L of K_X. The field L is then the function field of an algebraic curve $\widetilde{X}$ which is generically étale over X (though not necessarily geometrically connected as an $\mathbf{F}_q$-scheme), and the generic fiber of $G \times_X \widetilde{X}$ is split. In particular, there exists a semisimple group scheme $\widetilde{G}$ over $\widetilde{X}$ and an isomorphism β between $\widetilde{G}$ and $G \times_X \widetilde{X}$ over a dense open subset $U \subseteq \widetilde{X}$. In §10.7, we will show that the group scheme $\widetilde{G}$ can be chosen to have an action of $\mathrm{Gal}(L/K_X)$ (compatible with the action of $\mathrm{Gal}(L/K_X)$ on $\widetilde{X}$) and that the isomorphism β can be chosen to be Γ-equivariant (Proposition 10.7.1). There is then a close relationship between G-bundles on X and Γ-equivariant $\widetilde{G}$-bundles on $\widetilde{X}$, which we will use to "descend" the Harder-Narasimhan stratification of $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ to a stratification of $\mathrm{Bun}_G(X)$ (at least after replacing G by a suitable dilitation). In §10.8, we will show that the latter stratification satisfies the axiomatics developed in §10.2, and thereby obtain a proof of Theorem 10.0.6.

10.1. **The Trace Formula for a Quotient Stack.** Thoughout this section, we fix a finite field $\mathbf{F}_q$ with q elements, an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$, a prime number ℓ which is relatively prime to q, and an embedding $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$. Our goal is to prove the following result:

Proposition 10.1.1. *Let $\mathcal{X}$ be a smooth Artin stack over $\mathbf{F}_q$, let G be a connected linear algebraic group over $\mathbf{F}_q$, and suppose that G acts on $\mathcal{X}$. If $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula, then so does the stack-theoretic quotient $\mathcal{X}/G$.*

Remark 10.1.2. In the statement of Proposition 10.1.1, the assumption that G is affine is not really needed. However, the affine case will be sufficient for our applications.

Example 10.1.3. In the special case where $\mathcal{X} = \operatorname{Spec} \mathbf{F}_q$, we deduce that the classifying stack BG satisfies the Grothendieck-Lefschetz trace formula, which recovers the first assertion of Proposition 6.4.12. However, the proof of Proposition 10.1.1 we give below will use Proposition 6.4.12.

Corollary 10.1.4. *Let Y be a smooth $\mathbf{F}_q$-scheme of finite type and let G be a linear algebraic group over $\mathbf{F}_q$ which acts on $\mathbf{F}_q$. Then the stack-theoretic quotient Y/G satisfies the Lefschetz trace formula.*

Proof. Since G is affine, there exists an embedding of algebraic groups $G \hookrightarrow \mathrm{GL}_n$. Replacing Y by $(Y \times \mathrm{GL}_n)/G$ and G by GL_n, we can reduce to the case where $G = \mathrm{GL}_n$ and in particular where G is connected. In this case, the desired result follows immediately from Proposition 10.1.1 (together with the classical Grothendieck-Lefschetz trace formula). □

Corollary 10.1.5. *Let X be a smooth complete geometrically connected curve over $\mathbf{F}_q$, let G be a smooth affine group scheme over X with connected fibers. Let G' be the smooth affine group scheme over X obtained from G by dilitation along its identity section at an effective divisor $D \subseteq X$ (see Variant A.3.9). If the algebraic stack $\mathrm{Bun}_{G'}(X)$ satisfies the Grothendieck-Lefschetz trace formula, then so does $\mathrm{Bun}_G(X)$.*

Proof. Let H be the connected algebraic group obtained from $G \times_X D$ by Weil restriction along the finite flat map $D \to \operatorname{Spec} \mathbf{F}_q$. Note that $\mathrm{Bun}_{G'}(X)$ can be identified with the moduli stack whose R-valued points are pairs $(\mathcal{P}, \gamma)$, where $\mathcal{P}$ is a G-bundle on X_R and γ is a trivialization of $\mathcal{P}$ on the divisor $D_R = D \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} R$. The algebraic group H acts on $\mathrm{Bun}_{G'}(X)$ by changing trivializations and we can identify $\mathrm{Bun}_G(X)$ with the stack-theoretic quotient $\mathrm{Bun}_{G'}(X)/H$. The desired result now follows from Proposition 10.1.1. □

We now turn to the proof of Proposition 10.1.1. The main ingredient is the following technical convergence result, which will also be useful in §10.2:

Lemma 10.1.6. *Let V be an object of $\mathrm{Mod}_{\mathbf{Q}_\ell}$ which is given as the inverse limit of a tower*

$$\cdots \to V(n+1) \to V(n) \to V(n-1) \to \cdots \to V(0) \to V(-1) \simeq 0$$

Let F be an automorphism of the tower $\{V(n)\}_{n\geq 0}$, and denote also by F the induced automorphism of V. For each $n \geq 0$, let $W(n)$ denote the fiber of the map $V(n) \to V(n-1)$. Suppose that the following conditions are satisfied:

(a) *Each of the pairs $(\mathrm{H}^* W(n), F)$ is summable, in the sense of Definition 6.3.1.*
(b) *The sum $\sum_{n\geq 0} |\,\mathrm{H}^*(W(n))|_F$ converges absolutely.*
(c) *For each d, there exists an integer n_0 such that $W(n) \in (\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -d}$ for $n \geq n_0$.*

Then the pair $(\mathrm{H}^*(V), F)$ is summable. Moreover, we have

$$|\,\mathrm{H}^*(V)|_F \leq \sum_{n\geq 0} |\,\mathrm{H}^*(W(n))|_F$$

$$\mathrm{Tr}(F|\,\mathrm{H}^*(V)) = \sum_{n\geq 0} \mathrm{Tr}(F|\,\mathrm{H}^*(W(n))).$$

Proof. Since each pair $(\mathrm{H}^*(W(n)), F)$ is summable, the graded vector spaces $\mathrm{H}^*(W(n))$ are finite-dimensional in each degree. It follows by induction on n that the graded vector spaces $\mathrm{H}^*(V(n))$ are also finite-dimensional in each degree. Assumption (c) implies that for any fixed d, we have $\mathrm{H}^d(V) \simeq \mathrm{H}^d(V(n))$ for $n \gg 0$, so that the graded vector space $\mathrm{H}^*(V)$ is also finite-dimensional in each degree.

For each integer d, let $|\,\mathrm{H}^d(V)|_F$ denote the sum of the absolute values of the eigenvalues of F (counted with multiplicity) on the complex vector space $\mathrm{H}^d(V) \otimes_{\mathbf{Q}_\ell} \mathbf{C}$. Let $C = \sum_{n\geq 0} |\,\mathrm{H}^*(W(n))|_F$; we wish to show that the sum $\sum_{d\in\mathbf{Z}} |\,\mathrm{H}^d(V)|_F$ is bounded by C. To prove this, it suffices to show that for each integer d_0, the partial sum $\sum_{d\leq d_0} |\,\mathrm{H}^d(V)|_F$ is bounded above by C. Using assumption (c), we deduce that there exists an integer n such that $\mathrm{H}^d(V) \simeq \mathrm{H}^d(V(n))$ for $d \leq d_0$. It will therefore suffice to show that $\sum_{d\leq d_0} |\,\mathrm{H}^d(V(n))|_F \leq C$. This is clear: we have

$$\begin{aligned}
\sum_{d\leq d_0} |\,\mathrm{H}^d(V(n))|_F &\leq \sum_{d\in\mathbf{Z}} |\,\mathrm{H}^d(V(n))|_F \\
&= |\,\mathrm{H}^*(V(n))|_F \\
&\leq \sum_{0\leq m\leq n} |\,\mathrm{H}^*(W(m))|_F \\
&\leq \sum_{0\leq m} |\,\mathrm{H}^*(W(m))|_F \\
&= C,
\end{aligned}$$

where the second inequality follows from iterated application of Remark 6.3.3.

We now complete the proof by verifying the identity

$$\mathrm{Tr}(F|\,\mathrm{H}^*(V)) = \sum_{n\geq 0} \mathrm{Tr}(F|\,\mathrm{H}^*(W(n))).$$

Fix a real number $\epsilon > 0$; we will show that the difference

$$|\,\mathrm{Tr}(F|\,\mathrm{H}^*(V)) - \sum_{n\geq 0} \mathrm{Tr}(F|\,\mathrm{H}^*(W(n)))|$$

is bounded by ϵ. Using assumption (b), we deduce that there exists an integer $n_0 \geq 0$ for which the sum $\sum_{n > n_0} |\operatorname{H}^*(W(n))|_F$ is bounded above by $\frac{\epsilon}{2}$. Form a fiber sequence

$$U \to V \to V(n_0).$$

Applying the first part of the proof to U, we deduce that $(\operatorname{H}^*(U), F)$ is summable with

$$|\operatorname{H}^*(U)|_F \leq \sum_{n > n_0} |\operatorname{H}^*(W(n))|_F \leq \frac{\epsilon}{2}.$$

Using Remark 6.3.3, we obtain

$$\begin{aligned} \operatorname{Tr}(F| \operatorname{H}^*(V)) &= \operatorname{Tr}(F| \operatorname{H}^*(U)) + \operatorname{Tr}(F| \operatorname{H}^*(V(n_0))) \\ &= \operatorname{Tr}(F|H^*(U)) + \sum_{0 \leq n \leq n_0} \operatorname{Tr}(F| \operatorname{H}^*(W(n)). \end{aligned}$$

Subtracting $\sum_{n \geq 0} \operatorname{Tr}(F| \operatorname{H}^*(W(n)))$ from both sides and taking absolute values, we obtain

$$\begin{aligned} |\operatorname{Tr}(F| \operatorname{H}^*(V)) - \sum_{n \geq 0} \operatorname{Tr}(F| \operatorname{H}^* W(n))| &= |\operatorname{Tr}(F| \operatorname{H}^*(U)) - \sum_{n > n_0} \operatorname{Tr}(F| \operatorname{H}^* W(n))| \\ &\leq |\operatorname{Tr}(F| \operatorname{H}^*(U))| + \sum_{n > n_0} |\operatorname{Tr}(F| \operatorname{H}^* W(n)) \\ &\leq |\operatorname{H}^*(U)|_F + \sum_{n > n_0} |\operatorname{H}^*(W(n))|_F \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

as desired. $\square$

Proof of Proposition 10.1.1. Let BG denote the classifying stack of G, so that we have a pullback diagram

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{X}/G \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{F}_q & \longrightarrow & \mathrm{BG}. \end{array}$$

Applying Lemma 7.1.7, we deduce that the induced diagram

$$\begin{array}{ccc} C^*_{\mathrm{geom}}(\mathfrak{X}) & \longleftarrow & C^*_{\mathrm{geom}}(\mathfrak{X}/G) \\ \uparrow & & \uparrow \\ C^*_{\mathrm{geom}}(\operatorname{Spec} \mathbf{F}_q) & \longleftarrow & C^*_{\mathrm{geom}}(\mathrm{BG}) \end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Q}_\ell}$. In other words, we have a canonical equivalence

$$C^*_{\mathrm{geom}}(\mathfrak{X}) \simeq C^*_{\mathrm{geom}}(\mathfrak{X}/G) \otimes_{C^*_{\mathrm{geom}}(\mathrm{BG})} \mathbf{Q}_\ell .$$

Let us regard $C^*_{\mathrm{geom}}(\mathrm{BG})$ as an augmented commutative algebra object of $\mathrm{Mod}_{\mathbf{Q}_\ell}$. Let $\mathfrak{m}$ denote its augmentation ideal, and consider the filtration

$$\cdots \to \mathfrak{m}^{(3)} \to \mathfrak{m}^{(2)} \to \mathfrak{m}^{(1)} \to C^*_{\mathrm{geom}}(\mathrm{BG})$$

introduced in §6.1. We claim that this tower can be regarded as a diagram of $C^*_{\mathrm{geom}}(\mathrm{BG})$-modules, and that the induced action of $C^*_{\mathrm{geom}}(\mathrm{BG})$ on each cofiber

$$\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)} = \operatorname{cofib}(\mathfrak{m}^{(n+1)} \to \mathfrak{m}^{(n)})$$

factors through the augmentation $C^*_{\mathrm{geom}}(\mathrm{BG}) \to \mathbf{Q}_\ell$. This is a general feature of the constructions described in §6.1, but can easily be deduced in this special case from the observation that $\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG})$ is a polynomial ring on generators of even degrees so that $C^*_{\mathrm{geom}}(\mathrm{BG})$ is equivalent to a symmetric algebra $\mathrm{Sym}^*(V)$ for some chain complex V concentrated in even degrees (see the proof of Proposition 6.1.12), together with the identifications $\mathfrak{m}^{(n)} \simeq \mathrm{Sym}^{\geq n} V$ supplied by Example 6.1.8.

It follows from Proposition 6.1.18 that each $\mathfrak{m}^{(n)}$ is belongs to $(\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -n}$ (in fact, the preceding argument even shows that $\mathfrak{m}^{(n)} \in (\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -n}$). Let $\overline{\mathrm{BG}} = \mathrm{BG} \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\overline{\mathbf{F}}_q$. Then $C^*_{\mathrm{geom}}(\mathrm{BG}) = C^*(\overline{\mathrm{BG}}; \mathbf{Z}_\ell)[\ell^{-1}]$. We therefore have equivalences

$$\mathfrak{m}^{(n)} \otimes_{C^*_{\mathrm{geom}}(\mathrm{BG})} C^*_{\mathrm{geom}}(\mathfrak{X}/G) \simeq \mathfrak{m}^{(n)} \otimes_{C^*(\overline{\mathrm{BG}};\mathbf{Z}_\ell)} C^*_{\mathrm{geom}}(\mathfrak{X}/G).$$

Applying Lemma 7.1.6, we conclude that each tensor product $\mathfrak{m}^{(n)} \otimes_{C^*_{\mathrm{geom}}(\mathrm{BG})} C^*_{\mathrm{geom}}(\mathfrak{X}/G)$ belongs to $(\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -n}$, so that the inverse limit

$$\varprojlim_n \mathfrak{m}^{(n)} \otimes_{C^*_{\mathrm{geom}}(\mathrm{BG})} C^*_{\mathrm{geom}}(\mathfrak{X}/G)$$

vanishes. It follows that we can write $C^*_{\mathrm{geom}}(\mathfrak{X}/G)$ as the limit of the tower

$$\{(C^*_{\mathrm{geom}}(\mathrm{BG})/\mathfrak{m}^{(n)}) \otimes_{C^*(\mathrm{BG})} C^*(\mathfrak{X}/G)\}_{n\geq 0}$$

whose successive quotients are given by

$$\begin{aligned} W(n) &= (\mathfrak{m}^{(n)}/\mathfrak{m}^{n+1}) \otimes_{C^*_{\mathrm{geom}}(\mathrm{BG})} C^*_{\mathrm{geom}}(\mathfrak{X}/G) \\ &\simeq (\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}) \otimes_{\mathbf{Q}_\ell} (\mathbf{Q}_\ell \otimes_{C^*_{\mathrm{geom}}(\mathrm{BG})} C^*_{\mathrm{geom}}(\mathfrak{X}/G) \\ &\simeq (\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}) \otimes_{\mathbf{Q}_\ell} C^*_{\mathrm{geom}}(\mathfrak{X}). \end{aligned}$$

Since $\mathfrak{X}$ satisfies the Grothendieck-Lefschetz trace formula, the pair $(\mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X}), \mathrm{Frob}^{-1})$ is summable. It follows that each of the pairs $(\mathrm{H}^*(W(n)), \mathrm{Frob}^{-1})$ is summable with

$$|\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} = |\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}|_{\mathrm{Frob}^{-1}} |\,\mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X})|_{\mathrm{Frob}^{-1}}.$$

Since $\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG})$ can be identified with the direct sum of the cohomologies of the quotients $\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}$, we get

$$\sum_{n\geq 0} |\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} = |\,\mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG})|_{\mathrm{Frob}^{-1}} |\,\mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X})|_{\mathrm{Frob}^{-1}} < \infty.$$

Using Lemma 10.1.6, we conclude that the pair $(\mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X}/G), \mathrm{Frob}^{-1})$ is summable and we obtain the identity

$$\begin{aligned} \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X}/G)) &= \sum_{n\geq 0} \mathrm{Tr}(\mathrm{Frob}^{-1} \,|W(n)) \\ &= \sum_{n\geq 0} \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}))\, \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X})) \\ &= \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{geom}}(\mathrm{BG}))\, \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X})). \end{aligned}$$

Using Proposition 6.4.12 and the fact that $\mathfrak{X}$ satisfies the Grothendieck-Lefschetz trace formula, we obtain

$$\begin{aligned} \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{geom}}(\mathfrak{X}/G)) &= \frac{q^{\dim(G)}}{|G(\mathbf{F}_q)|} \frac{|\,\mathfrak{X}(\mathbf{F}_q)|}{q^{\dim(\mathfrak{X})}} \\ &= q^{-\dim(\mathfrak{X}/G)} \frac{|\,\mathfrak{X}(\mathbf{F}_q)|}{|G(\mathbf{F}_q)|}. \end{aligned}$$

To complete the proof, it will suffice to verify the identity

$$|(\mathfrak{X}/G)(\mathbf{F}_q)| = \frac{|\mathfrak{X}(\mathbf{F}_q)|}{|G(\mathbf{F}_q)|}. \tag{18}$$

For each object $\eta \in (\mathfrak{X}/G)(\mathbf{F}_q)$, let $\mathcal{C}_\eta$ denote the full subcategory of $\mathfrak{X}(\mathbf{F}_q)$ spanned by those objects C whose image in $(\mathfrak{X}/G)(\mathbf{F}_q)$ is isomorphic to η (where the isomorphism is *not* specified), so that we can write $\mathfrak{X}(\mathbf{F}_q)$ as a disjoint union of the groupoids $\mathcal{C}_\eta$ where η ranges over all isomorphism classes of objects of $(\mathfrak{X}/G)(\mathbf{F}_q)$. To prove (18), it will suffice to show that for each $\eta \in (\mathfrak{X}/G)(\mathbf{F}_q)$, we have an equality

$$\frac{1}{|\operatorname{Aut}(\eta)|} = \frac{1}{|G(\mathbf{F}_q)|} \sum_{C \in \mathcal{C}_\eta} \frac{1}{|\operatorname{Aut}(C)|},$$

where the sum is taken over all isomorphism classes of objects of $\mathcal{C}_\eta$.

The object η can be regarded as a map $\operatorname{Spec} \mathbf{F}_q \to (\mathfrak{X}/G)$, so we can consider the fiber product $Y = \mathfrak{X} \times_{\mathfrak{X}/G} \operatorname{Spec} \mathbf{F}_q$, which is a torsor for the algebraic group G. The finite group $\operatorname{Aut}(\eta)$ acts on Y, and therefore acts on the finite set $Y(\mathbf{F}_q)$. Unwinding the definitions, we can identify $\mathcal{C}_\eta$ with the groupoid-theoretic quotient of $Y(\mathbf{F}_q)$ by the action of $\operatorname{Aut}(\eta)$. We may therefore identify the set of isomorphism classes of objects of $\mathcal{C}_\eta$ with the set of orbits of $\operatorname{Aut}(\eta)$ acting on $Y(\mathbf{F}_q)$. For each $y \in Y(\mathbf{F}_q)$, the automorphism group of the corresponding object $C \in \mathcal{C}_\eta$ can be identified with the stabilizer $\operatorname{Aut}(\eta)_y = \{\phi \in \operatorname{Aut}(\eta) : \phi(y) = y\}$. We therefore have

$$\begin{aligned} \sum_{C \in \mathcal{C}_\eta} \frac{1}{|\operatorname{Aut}(C)|} &= \sum_{y \in Y(\mathbf{F}_q)} \frac{1}{|\operatorname{Aut}(\eta)/\operatorname{Aut}(\eta)_y|} \frac{1}{|\operatorname{Aut}(\eta)_y|} \\ &= \sum_{y \in Y(\mathbf{F}_q)} \frac{1}{|\operatorname{Aut}(\eta)|} \\ &= \frac{|Y(\mathbf{F}_q)|}{|\operatorname{Aut}(\eta)|}. \end{aligned}$$

To complete the proof, it will suffice to show that the finite sets $Y(\mathbf{F}_q)$ and $G(\mathbf{F}_q)$ have the same size. This follows from Lang's theorem, the $\mathbf{F}_q$-scheme Y is a G-torsor and is therefore G-equivariantly isomorphic to G (by virtue of our assumption that G is connected). □

10.2. **Stratifications.** Throughout this section, we let $\mathbf{F}_q$ denote a finite field with q elements, $\overline{\mathbf{F}}_q$ an algebraic closure of $\mathbf{F}_q$, ℓ a prime number which is relatively prime to q, and $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$ an embedding of fields.

Let $\mathfrak{X}$ be a smooth Artin stack over $\mathbf{F}_q$. Our goal is to give a concrete geometric criterion which can be used to prove that $\mathfrak{X}$ satisfies the Grothendieck-Lefschetz trace formula (Definition 10.0.3). According to Corollary 10.1.4, this is true whenever $\mathfrak{X}$ can be written as a quotient of a quasi-projective variety by the action of a linear algebraic group. Unfortunately, this is not good enough for our purposes: moduli stacks of the form $\mathfrak{X} = \operatorname{Bun}_G(X)$ cannot be written as global quotients (except in trivial cases) because they generally fail to be quasi-compact. We will address this issue by breaking $\mathfrak{X}$ up into pieces, each of which is quasi-compact.

Definition 10.2.1. Let $\mathfrak{X}$ be an Artin stack. A *stratification* of $\mathfrak{X}$ consists of the following data:

- (a) A partially ordered set A.
- (b) A collection of open substacks $\{\mathcal{U}_\alpha \subseteq \mathfrak{X}\}_{\alpha \in A}$ satisfying $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ when $\alpha \leq \beta$.

This data is required to satisfy the following conditions:

- For each index $\alpha \in A$, the set $\{\beta \in A : \beta \leq \alpha\}$ is finite.
- For every field k and every map $\eta : \operatorname{Spec} k \to \mathcal{X}$, the set $\{\alpha \in A : \ \eta$ factors through $\mathcal{U}_\alpha\ \}$ has a smallest element.

Notation 10.2.2. Let $\mathcal{X}$ be an Artin stack equipped with a stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$. For each $\alpha \in A$, we let $\mathcal{X}_\alpha$ denote the reduced closed substack of $\mathcal{U}_\alpha$ given by the complement of $\bigcup_{\beta < \alpha} \mathcal{U}_\beta$. Each $\mathcal{X}_\alpha$ is a locally closed substack of $\mathcal{X}$; we will refer to these locally closed substacks as the *strata* of $\mathcal{X}$.

Remark 10.2.3. Let $\mathcal{X}$ be an Artin stack. A stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is determined by the partially ordered set A together with the collection of locally closed substacks $\{\mathcal{X}_\alpha\}_{\alpha \in A}$: each $\mathcal{U}_\alpha$ can be characterized by the fact that it is an open substack of $\mathcal{X}$ and that, if k is a field, then a map $\eta : \operatorname{Spec} k \to \mathcal{X}$ factors through $\mathcal{U}_\alpha$ if and only if it factors through $\mathcal{X}_\beta$ for some $\beta \leq \alpha$. Because of this, we will generally identify stratification of $\mathcal{X}$ with the collection of locally closed substacks $\{\mathcal{X}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ (where the partial ordering of A is understood to be implicitly specified).

Remark 10.2.4. Let $\mathcal{X}$ be an Artin stack equipped with a stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$. If k is a field, then for any map $\eta : \operatorname{Spec} k \to \mathcal{X}$ there is a *unique* index $\alpha \in A$ such that η factors through $\mathcal{X}_\alpha$. In other words, $\mathcal{X}$ is a *set-theoretic* union of the locally closed substacks $\mathcal{X}_\alpha$.

Remark 10.2.5 (Functoriality)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a map of Artin stacks. Suppose that $\mathcal{Y}$ is equipped with a stratification $\{\mathcal{U}_\alpha \subseteq \mathcal{Y}\}_{\alpha \in A}$. Then $\{\mathcal{U}_\alpha \times_{\mathcal{Y}} \mathcal{X} \subseteq \mathcal{X}\}_{\alpha \in A}$ is a stratification of $\mathcal{X}$ (indexed by the same partially ordered set A). The corresponding strata of $\mathcal{X}$ are given by the reduced locally closed substacks

$$\mathcal{X}_\alpha = (\mathcal{Y}_\alpha \times_{\mathcal{Y}} \mathcal{X})_{\mathrm{red}}.$$

We now make some elementary observations about the behavior of stratifications with respect to the actions of finite groups, which will be useful in §10.8.

Remark 10.2.6 (Stratification of Fixed Point Stacks)**.** Let $\mathcal{X}$ be an Artin stack equipped with an action of a finite group Γ. Suppose that $\mathcal{X}$ is equipped with a stratification $\{\mathcal{U}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ which is Γ-*equivariant* in the following sense: the group Γ acts on A (by monotone maps) and for each $\alpha \in A$, $\gamma \in \Gamma$ the open substack $\mathcal{U}_{\gamma(\alpha)}$ is the image of $\mathcal{U}_\alpha$ under the automorphism of $\mathcal{X}$ determined by γ.

Let $\mathcal{X}^\Gamma$ denote the (homotopy) fixed point stack for the action of Γ on $\mathcal{X}$, and let A^Γ denote the set of fixed points for the action of Γ on A. For each $\alpha \in A$, the open substack $\mathcal{U}_\alpha \subseteq \mathcal{X}$ inherits an action of Γ, and the fixed point stack $\mathcal{U}_\alpha^\Gamma$ can be regarded as an open substack of $\mathcal{X}^\Gamma$. Moreover, the collection $\{\mathcal{U}_\alpha^\Gamma \subseteq \mathcal{X}^\Gamma\}_{\alpha \in A^\Gamma}$ is a stratification of $\mathcal{X}^\Gamma$. For each $\alpha \in A^\Gamma$, the corresponding locally closed substack of $\mathcal{X}^\Gamma$ can be identified with the reduced stack $((\mathcal{X}_\alpha)^\Gamma)_{\mathrm{red}}$.

Remark 10.2.7. Let $\mathcal{X}$ be an Artin stack equipped with an action of a finite group Γ, and suppose we are given a stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ which is Γ-equivariant (as in Remark 10.2.6). Let A/Γ denote the quotient of A by the action of Γ, and for each $\alpha \in A$ let $[\alpha]$ denote its image in A/Γ. We can endow A/Γ with the structure of a partially ordered set by writing $[\alpha] \leq [\alpha']$ if there exists an element $\gamma \in \Gamma$ such that $\alpha \leq \gamma(\alpha')$. For each $[\alpha] \in A/\Gamma$, let $\mathcal{U}_{[\alpha]}$ denote the open substack of $\mathcal{X}$ given by the union $\bigcup_{\gamma \in \Gamma} \mathcal{U}_{\gamma(\alpha)}$. Then $\{\mathcal{U}_{[\alpha]}\}_{[\alpha] \in A/\Gamma}$ is a stratification of $\mathcal{X}$ indexed by the partially ordered set A/Γ. For each $[\alpha] \in A/\Gamma$, the corresponding stratum $\mathcal{X}_{[\alpha]}$ can be identified with the disjoint union $\amalg_{\alpha'} \mathcal{X}_{\alpha'}$, where α' ranges over those elements of A having the form $\gamma(\alpha)$ for some $\gamma \in \Gamma$.

Remark 10.2.8. Let $\mathcal{X}$ be an Artin stack equipped with an action of a finite group Γ. Suppose we are given a stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$, where each $\mathcal{U}_\alpha$ is Γ-invariant. Then each quotient $\mathcal{U}_\alpha / \Gamma$

can be regarded as an open substack of $\mathcal{X}/\Gamma$, and the collection of open substacks $\{\mathcal{U}_\alpha/\Gamma\}_{\alpha\in A}$ determines a stratification of $\mathcal{X}/\Gamma$ whose strata can be identified with the quotients $\mathcal{X}_\alpha/\Gamma$.

More generally, suppose that the stratification $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ is merely Γ-equivariant in the sense of Remark 10.2.6. We can then apply the preceding remark to the induced stratification $\{\mathcal{U}_{[\alpha]}\}_{\alpha\in A/\Gamma}$ by Γ-invariant open substacks. This yields a stratification of $\mathcal{X}/\Gamma$ by open substacks $\{\mathcal{U}_{[\alpha]}/\Gamma\}_{\alpha\in A/\Gamma}$, where each stratum $(\mathcal{X}/\Gamma)_{[\alpha]}$ can be identified with the quotient $\mathcal{X}_\alpha/\Gamma_\alpha$, where Γ_α denotes the subgroup of Γ which stabilizes α.

Definition 10.2.9. Let $\mathcal{X}$ be an Artin stack of finite type over $\operatorname{Spec}\mathbf{F}_q$. We will say that a stratification $\{\mathcal{X}_\alpha\}_{\alpha\in A}$ of $\mathcal{X}$ is *convergent* if there exists a finite collection of Artin stacks $\mathcal{T}_1,\ldots,\mathcal{T}_n$ over $\operatorname{Spec}\mathbf{F}_q$ with the following properties:

(1) For each $\alpha\in A$, there exists $i\in\{1,2,\ldots,n\}$ and a diagram of Artin stacks
$$\mathcal{T}_i \xrightarrow{f} \widetilde{\mathcal{X}}_\alpha \xrightarrow{g} \mathcal{X}_\alpha$$
where the map f is a fiber bundle (locally trivial with respect to the étale topology) whose fibers are affine spaces of some fixed dimension d_α and the map g is surjective, finite, and radicial.

(2) The nonnegative integers d_α appearing in (2) satisfy $\sum_{\alpha\in A} q^{-d_\alpha}<\infty$.

(3) For $1\le i\le n$, the Artin stack $\mathcal{T}_i$ can be written as a stack-theoretic quotient Y/G, where Y is an algebraic space of finite type over $\mathbf{F}_q$ and G is a linear algebraic group over $\mathbf{F}_q$ which acts on Y.

Remark 10.2.10. In the situation of Definition 10.2.9, hypothesis (2) guarantees that the set A is at most countable.

We can now state the main result of this section:

Proposition 10.2.11. *Let $\mathcal{X}$ be a smooth Artin stack of dimension d over* $\operatorname{Spec}\mathbf{F}_q$*. If $\mathcal{X}$ admits a convergent stratification, then $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula.*

Remark 10.2.12. In the statement of Proposition 10.2.11, the hypothesis that $\mathcal{X}$ be smooth is not really important (see Remark 10.0.5).

We will deduce Proposition 10.2.11 from the following variant:

Proposition 10.2.13. *Let $\mathcal{X}$ be a smooth Artin stack of dimension d over* $\operatorname{Spec}\mathbf{F}_q$*. Suppose that there exists a stratification $\{\mathcal{X}_\alpha\}_{\alpha\in A}$ of $\mathcal{X}$ and a finite collection of Artin stacks $\{\mathcal{T}_i\}_{1\le i\le n}$ over* $\operatorname{Spec}\mathbf{F}_q$ *which satisfy conditions* (1) *and* (2) *of Definition 10.2.9, together with the following variant of* (3)*:*

(3′) *Each $\mathcal{T}_i$ is smooth of constant dimension over* $\operatorname{Spec}\mathbf{F}_q$ *and satisfies the Grothendieck-Lefschetz trace formula.*

Then $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula.

Corollary 10.2.14. *Let X be an algebraic space which is smooth (of constant dimension) and of finite type over* $\operatorname{Spec}\mathbf{F}_q$*. Then X satisfies the Grothendieck-Lefschetz trace formula.*

Proof. Every reduced closed $Y\subseteq X$ is a quasi-compact, quasi-separated algebraic space of finite type over $\operatorname{Spec}\mathbf{F}_q$, and therefore contains a nonempty affine open subset $U\subseteq Y$. Since the field $\mathbf{F}_q$ is perfect, we may assume (shrinking U if necessary) that U is smooth of constant dimension over $\mathbf{F}_q$. It follows by Noetherian induction that X admits a *finite* stratification $\{X_\alpha\}_{\alpha\in A}$ where each stratum X_α is an affine scheme which is smooth over $\operatorname{Spec}\mathbf{F}_q$. The desired result now follows from Proposition 10.2.13 (taking the algebraic stacks $\mathcal{T}_i$ to be the strata X_α). □

We begin by showing that Proposition 10.2.13 implies Proposition 10.2.11:

Proof of Proposition 10.2.11. Let $\mathcal{X}$ be a smooth Artin stack equipped with a convergent stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$. To prove that $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula, it will suffice (by virtue of Proposition 10.2.13) to show that $\mathcal{X}$ admits another stratification $\{\mathcal{Y}_\beta\}_{\beta \in B}$ which satisfies the hypotheses of Proposition 10.2.13.

Since the stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ is convergent, there exists a finite collection $\{\mathcal{T}_i\}_{1 \leq i \leq m}$ of Artin stacks of finite type over $\operatorname{Spec} \mathbf{F}_q$ which satisfies conditions (1), (2), and (3) of Definition 10.2.9. In particular, condition (3) implies that we can write each $\mathcal{T}_i$ as a stack-theoretic quotient Y_i / G_i, where Y_i is an algebraic space of finite type over $\mathbf{F}_q$ and each G_i is a linear algebraic group over $\mathbf{F}_q$. Choose an integer $n \geq 0$ such that each of the algebraic spaces Y_i has dimension $\leq n$. We define a sequence of locally closed substacks

$$Z_{i,n}, Z_{i,n-1}, Z_{i,n-2}, \ldots, Z_{i,0} \subseteq Y_i$$

by descending induction as follows: for $0 \leq j \leq n$, let $Z_{i,j}$ denote the largest open subset of $(Y_i - \bigcup_{j' > j} Z_{i,j'})$ which is smooth of dimension j over $\operatorname{Spec} \mathbf{F}_q$ (where we regard $Y_i - \bigcup_{j' > j} Z_{i,j'}$ as a reduced closed subscheme of Y). Note that the action of G_i on Y_i preserves each $Z_{i,j}$, so that we can regard the quotient $Y_{i,j}/G_i$ as a locally closed substack $\mathcal{T}_{i,j} \subseteq \mathcal{T}_i$ which is smooth of dimension $j - \dim(G_i)$ over $\operatorname{Spec} \mathbf{F}_q$.

Condition (1) of Definition 10.2.9 implies that for each $\alpha \in A$, there exists an integer $i(\alpha) \in \{1, \ldots, m\}$ and a pair of maps

$$\mathcal{T}_{i(\alpha)} \xrightarrow{f_\alpha} \widetilde{\mathcal{X}}_\alpha \xrightarrow{g_\alpha} \mathcal{X}_\alpha,$$

where f_α is an étale fiber bundle whose fibers are affine spaces of some dimension d_α and the map g_α is surjective, finite, and radicial. In particular, the morphism f_α is smooth of constant dimension; it follows that each of the closed substacks $\mathcal{T}_{i(\alpha),j} \subseteq \mathcal{T}_{i(\alpha)}$ can be realized as a fiber product

$$\mathcal{T}_{i(\alpha)} \times_{\widetilde{\mathcal{X}}_\alpha} \widetilde{\mathcal{X}}_{\alpha,j},$$

where $\{\widetilde{\mathcal{X}}_{\alpha,j}\}_{0 \leq j \leq n}$ is the collection of locally closed substacks of $\widetilde{\mathcal{X}}_\alpha$ defined inductively by taking $\widetilde{\mathcal{X}}_{\alpha,j}$ to be the largest open substack of $(\widetilde{\mathcal{X}}_\alpha - \bigcup_{j' > j} \widetilde{\mathcal{X}}_{\alpha,j'})$ which is smooth of dimension $j - d_\alpha - \dim(G_{i(\alpha)})$ over $\mathbf{F}_q$. Since g_α is a universal homeomorphism, each of the (reduced) locally closed substacks $\widetilde{\mathcal{X}}_{\alpha,j}$ is given set-theoretically as the inverse image of a reduced locally closed substack $\mathcal{X}_{\alpha,j} \subseteq \mathcal{X}_\alpha$, and the projection map $\widetilde{\mathcal{X}}_{\alpha,j} \to \mathcal{X}_{\alpha,j}$ is surjective, finite, and radicial.

Let $B = A \times \{0, \ldots, n\}$. We will regard B as equipped with the lexicographical ordering (so that $(\alpha, j) \leq (\alpha', j')$ if either $\alpha < \alpha'$ or $\alpha = \alpha'$ and $j \leq j'$). Then $\{\mathcal{X}_{\alpha,j}\}_{(\alpha,j) \in B}$ is a stratification of $\mathcal{X}$. We claim that this stratification satisfies the hypotheses of Proposition 10.2.13. By construction, for each $(\alpha, j) \in B$, we have a diagram

$$\mathcal{T}_{\alpha(i),j} \to \widetilde{\mathcal{X}}_{\alpha,j} \to \mathcal{X}_{\alpha,j}$$

where the first map is an étale fiber bundle whose fibers are affine spaces of dimension d_α, and the second map is surjective, finite, and radicial. Moreover, we have

$$\sum_{(\alpha,j) \in B} q^{-d_\alpha} = (n+1) \sum_{\alpha \in A} q^{-d_\alpha} < \infty.$$

To complete the proof, it will suffice to verify condition $(3')$: each of the smooth Artin stacks $\mathcal{T}_{i,j}$ satisfies the Grothendieck-Lefschetz trace formula. Choose an embedding $G_i \hookrightarrow \mathrm{GL}_d$, so that we can describe $\mathcal{T}_{i,j}$ as the stack-theoretic quotient of $(Y_{i,j} \times \mathrm{GL}_d)/G$ by the action of

GL_d. Using Proposition 10.1.1, we are reduced to showing that $(Y_{i,j} \times \mathrm{GL}_d)/G$ satisfies the Grothendieck-Lefschetz trace formula, which follows from Corollary 10.2.14. □

The proof of Proposition 10.2.13 will require some preliminaries.

Lemma 10.2.15 (Gysin Sequence)**.** *Let X and Y be smooth quasi-projective varieties over an algebraically closed field k, let $g : Y \to X$ be a finite radicial morphism, and let $U \subseteq X$ be the complement of the image of g. Then there is a canonical fiber sequence*

$$C^{*-2d}(Y; \mathbf{Z}_\ell(-d)) \to C^*(X; \mathbf{Z}_\ell) \to C^*(U; \mathbf{Z}_\ell),$$

where d denotes the relative dimension $\dim(X) - \dim(Y)$.

Proof. If $f : Z' \to Z$ is a proper morphism of quasi-projective k-schemes, let $\omega_{Z'/Z} = f^! \underline{\mathbf{Z}}_{\ell Z}$ denote the relative dualizing complex of f. Note that if Z and Z' are smooth of constant dimension, we have

$$\begin{aligned} \omega_{Z'/Z} &= f^! \underline{\mathbf{Z}}_{\ell Z} \\ &\simeq f^!(\omega_Z^{-1} \otimes \omega_Z) \\ &\simeq f^* \omega_Z^{-1} \otimes f^! \omega_Z \\ &\simeq f^* \omega_Z^{-1} \otimes \omega_{Z'} \\ &\simeq \Sigma^{-2\dim Z} \underline{\mathbf{Z}}_{\ell Z'}(-\dim Z) \otimes \Sigma^{2\dim(Z')} \underline{\mathbf{Z}}_{\ell Z'}(\dim Z') \\ &\simeq \Sigma^{2(\dim Z' - \dim Z)} \underline{\mathbf{Z}}_{\ell Z'}(\dim Z' - \dim Z). \end{aligned}$$

In particular, we have $\omega_{Y/X} \simeq \Sigma^{-2d} \underline{\mathbf{Z}}_{\ell Y}(-d)$.

Let $Y_0 \subseteq X$ denote the image of g, regarded as a reduced closed subscheme of Y. Then g restricts to finite radicial surjection $g_0 : Y \to Y_0$, and we have $\omega_{Y/X} \simeq g_0^! \omega_{Y_0/X}$. Let $j : U \hookrightarrow X$ and $i : Y_0 \hookrightarrow X$ denote the inclusion maps, so that we have a fiber sequence of sheaves

$$i_* i^! \underline{\mathbf{Z}}_{\ell X} \to \underline{\mathbf{Z}}_{\ell X} \to j_* j^* \underline{\mathbf{Z}}_{\ell X}.$$

Passing to global sections, we obtain a fiber sequence

$$C^*(Y_0; \omega_{Y_0/X}) \to C^*(X; \mathbf{Z}_\ell) \to C^*(U; \mathbf{Z}_\ell).$$

The map g_0 is a finite radicial surjection, and therefore induces an equivalences between the étale sites of Y and Y_0. It follows that the counit map

$$g_{0*} g_0^! \omega_{Y_0/X} \to \omega_{Y_0/X}$$

is an equivalence, so we have equivalences

$$\begin{aligned} C^*(Y_0; \omega_{Y_0/X}) &\simeq C^*(Y_0; g_{0*} g_0^! \omega_{Y_0/X}) \\ &\simeq C^*(Y; g_0^! \omega_{Y_0/X}) \\ &\simeq C^*(Y; \omega_{Y/X}) \\ &\simeq C^{*-2d}(Y; \mathbf{Z}_\ell(-d)). \end{aligned}$$

□

Lemma 10.2.15 immediately implies a corresponding result for algebraic stacks:

Lemma 10.2.16. *Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth Artin stacks of constant dimension over an algebraically closed field k, let $g : \mathcal{Y} \to \mathcal{X}$ be a finite radicial morphism, and let $\mathcal{U} \subseteq \mathcal{X}$ be the open substack of $\mathcal{X}$ complementary to the image of g. Then there is a canonical fiber sequence*

$$C^{*-2d}(\mathcal{Y}; \mathbf{Z}_\ell(-d)) \to C^*(\mathcal{X}; \mathbf{Z}_\ell) \to C^*_{\mathrm{geom}}(\mathcal{Y}; \mathbf{Z}_\ell),$$

where d denotes the relative dimension $\dim(\mathcal{X}) - \dim(\mathcal{Y})$.

Proof. Let $\mathcal{C}$ denote the category whose objects are affine k-schemes X equipped with a smooth morphism $X \to \mathcal{X}$. For each object $X \in \mathcal{C}$, let $Y_X = \mathcal{Y} \times_{\mathcal{X}} X$ and let $U_X = \mathcal{U} \times_{\mathcal{X}} X$. Lemma 10.2.15 then supplies a fiber sequence

$$C^{*-2d}(Y_X; \mathbf{Z}_\ell(-d)) \to C^*(X; \mathbf{Z}_\ell) \to C^*(U_X; \mathbf{Z}_\ell).$$

The construction of this fiber sequence depends functorially on X. We may therefore pass to the limit to obtain a fiber sequence

$$\varprojlim_{X \in \mathcal{C}} C^{*-2d}(Y_X; \mathbf{Z}_\ell(-d)) \to \varprojlim_{X \in \mathcal{C}} C^*(X; \mathbf{Z}_\ell) \to \varprojlim_{X \in \mathcal{C}} C^*(U_X; \mathbf{Z}_\ell).$$

The desired result now follows from the identifications

$$C^{*-2d}(\mathcal{Y}; \mathbf{Z}_\ell(-d)) \simeq \varprojlim_{X \in \mathcal{C}} C^{*-2d}(Y_X; \mathbf{Z}_\ell(-d))$$

$$C^*(\mathcal{X}; \mathbf{Z}_\ell) \simeq \varprojlim_{X \in \mathcal{C}} C^*(X; \mathbf{Z}_\ell)$$

$$C^*(\mathcal{U}; \mathbf{Z}_\ell) \simeq \varprojlim_{X \in \mathcal{C}} C^*(U_X; \mathbf{Z}_\ell).$$

□

Lemma 10.2.17. *Let X be an affine $\mathbf{F}_q$-scheme of finite type which becomes isomorphic to an affine space $\mathbf{A}^e$ after passing to some finite extension of $\mathbf{F}_q$. Then the set $X(\mathbf{F}_q)$ has q^e elements.*

Proof. By virtue of the Grothendieck-Lefschetz trace formula, it will suffice to show that $\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{geom}}(X))$ is equal to 1. Equivalently, we must show that the trace of Frob^{-1} on the reduced cohomology $\mathrm{H}^*_{\mathrm{red}}(X \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\overline{\mathbf{F}}_q; \mathbf{Q}_\ell)$ vanishes. But this reduced cohomology itself vanishes, since $X \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\overline{\mathbf{F}}_q$ is isomorphic to an affine space over $\mathrm{Spec}\,\overline{\mathbf{F}}_q$. □

Proof of Proposition 10.2.13. Let $d = \dim(\mathcal{X})$ and let $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ be the given stratification of $\mathcal{X}$. The set A is at most countable (Remark 10.2.10). By adding additional elements to A and assigning to those additional elements the empty substack of $\mathcal{X}$, we may assume that A is infinite. Using our assumption that $\{\beta \in A : \beta \leq \alpha\}$ is finite for each $\alpha \in A$, it follows that we can choose an enumeration

$$A = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$$

where each initial segment $\{\alpha_0, \ldots, \alpha_n\}$ is a downward-closed subset of A. We can then write $\mathcal{X}$ as the union of an increasing sequence of open substacks

$$\mathcal{U}_0 \hookrightarrow \mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \hookrightarrow \cdots$$

where $\mathcal{U}_n$ is characterized by the requirement that if k is a field, then a map $\eta : \mathrm{Spec}\,k \to \mathcal{X}$ factors through $\mathcal{U}_n$ if and only if it factors through one of the substacks $\mathcal{X}_{\alpha_0}, \mathcal{X}_{\alpha_1}, \ldots, \mathcal{X}_{\alpha_n}$.

By hypothesis, there exists a finite collection $\{\mathcal{T}_i\}_{1 \leq i \leq m}$ of smooth Artin stacks over $\mathrm{Spec}\,\mathbf{F}_q$, where each $\mathcal{T}_i$ has some fixed dimension d_i and satisfies the Grothendieck-Lefschetz trace formula, and for each $n \geq 0$ there exists an index $i(n) \in \{1, \ldots, m\}$ and a diagram

$$\mathcal{T}_{i(n)} \xrightarrow{f_n} \widetilde{\mathcal{X}}_{\alpha_n} \xrightarrow{g_n} \mathcal{X}_{\alpha_n},$$

where f_n is an étale fiber bundle whose fibers are affine spaces of some fied dimension $e(n)$ and g_n is a finite radicial surjection. Set

$$\overline{\mathcal{X}} = \mathcal{X} \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\overline{\mathbf{F}}_q$$

$$\overline{\mathcal{U}}_n = \mathcal{U}_n \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\overline{\mathbf{F}}_q$$

$$\overline{\mathcal{T}}_i = \mathcal{T}_i \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q.$$

The map f_n induces an isomorphism on ℓ-adic cohomology. Applying Lemma 10.2.16 to the finite radicial map

$$g_n : \widetilde{\mathcal{X}}_{\alpha_n} \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q \to \overline{\mathcal{U}}_n,$$

we obtain fiber sequences

$$C^{*-2e'_n}(\widetilde{\mathcal{T}}_{i(n)}; \mathbf{Z}_\ell(-e'_n)) \to C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell) \to C^*(\overline{\mathcal{U}}_{n-1}; \mathbf{Z}_\ell)$$

where $e'_n = e_n + d - d_{i(n)}$ denotes the relative dimension of the map $\widetilde{\mathcal{X}}_{\alpha_n} \to \mathcal{X}$.

We have a canonical equivalence

$$\theta : C^*(\overline{\mathcal{X}}; \mathbf{Z}_\ell) \simeq \varprojlim_n C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell).$$

Our convergence assumption

$$\sum_{n \geq 0} q^{-e_n} < \infty$$

guarantees that the sequence of integers $\{e_n\}_{n \geq 0}$ tends to infinity and therefore the sequence $\{e'_n\}_{n \geq 0}$ also tends to infinity. It follows that the restriction maps

$$\mathrm{H}^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell) \to \mathrm{H}^*(\overline{\mathcal{U}}_{n-1}; \mathbf{Z}_\ell)$$

are isomorphisms for $n \gg *$, so that θ also induces an equivalence

$$C^*(\overline{\mathcal{X}}; \mathbf{Z}_\ell)[\ell^{-1}] \simeq \varprojlim_n C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell)[\ell^{-1}].$$

Set $V(n) = C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell)[\ell^{-1}] = C^*_{\mathrm{geom}}(\mathcal{U}_n)$ and let $W(n)$ denote the fiber of the restriction map $V(n) \to V(n-1)$ (with the convention that $W(0) = V(0)$). The above calculation gives

$$W(n) = C^{*-2e'_n}_{\mathrm{geom}}(\mathcal{T}_{i(n)})(-e'_n).$$

Since each $\mathcal{T}_i$ satisfies the Grothendieck-Lefschetz trace formula, the cohomologies of $W(n)$ are finite-dimensional in each degree and we have

$$|\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} = q^{-e'_n} |\,\mathrm{H}^*_{\mathrm{geom}}(\mathcal{T}_{i(n)}\,|_{\mathrm{Frob}^{-1}}|$$

$$\mathrm{Tr}(\mathrm{Frob}^{-1}\,|\,\mathrm{H}^*(W(n))) = q^{-e'_n}\,\mathrm{Tr}(\mathrm{Frob}^{-1}\,|\,\mathrm{H}^*_{\mathrm{geom}}(\mathcal{T}_{i(n)})) = q^{-e_n-d} |\,\mathcal{T}_{i(n)}(\mathbf{F}_q)|.$$

In particular, we have

$$\begin{aligned} \sum_{n \geq 0} |\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} &= \sum_{n \geq 0} q^{-e'_n} |\,\mathrm{H}^*_{\mathrm{geom}}(\mathcal{T}_{i(n)}\,|_{\mathrm{Frob}^{-1}} \\ &\leq \sum_{n \geq 0} q^{-e_n} \sum_{1 \leq i \leq m} q^{d_i - d} |\,\mathrm{H}^*_{\mathrm{geom}}(\mathcal{T}_i)|_{\mathrm{Frob}^{-1}} \\ &< \infty. \end{aligned}$$

Invoking Lemma 10.1.6, we conclude that $(\mathrm{H}^*_{\mathrm{geom}}(\mathcal{X}), \mathrm{Frob}^{-1})$ is summable, with

$$\begin{aligned} \mathrm{Tr}(\mathrm{Frob}^{-1}\,|\,\mathrm{H}^*_{\mathrm{geom}}(\mathcal{X})) &= \sum_{n \geq 0} \mathrm{Tr}(\mathrm{Frob}^{-1}\,|\,\mathrm{H}^*(W(n)) \\ &= \sum_{n \geq 0} q^{-e_n-d} |\,\mathcal{T}_{i(n)}(\mathbf{F}_q)|. \end{aligned}$$

On the other hand, the stratification $\{\mathcal{X}_{\alpha_n}\}_{n \geq 0}$ of $\mathcal{X}$ gives the identity

$$\frac{|\,\mathcal{X}(\mathbf{F}_q)|}{q^d} = \sum_{n \geq 0} q^{-d} |\,\mathcal{X}_{\alpha_n}(\mathbf{F}_q)|.$$

It will therefore suffice to prove that for each $n \geq 0$, we have

$$|\mathcal{T}_{i(n)}(\mathbf{F}_q)| = q^{e_n} |\mathcal{X}_{\alpha_n}(\mathbf{F}_q)|.$$

Since g_n is a finite radicial surjection, it induces an equivalence of categories $\widetilde{\mathcal{X}}_{\alpha_n}(\mathbf{F}_q) \simeq \mathcal{X}_{\alpha_n}(\mathbf{F}_q)$. It will therefore suffice to show that each object of the groupoid $\widetilde{\mathcal{X}}_{\alpha_n}(\mathbf{F}_q)$ can be lifted in exactly q^{e_n} ways to an object of the groupoid $\mathcal{T}_{i(n)}(\mathbf{F}_q)$ via the map f_n, which is an immediate consequence of Lemma 10.2.17. □

10.3. **The Harder-Narasimhan Stratification (Split Case).** Throughout this section, we fix an algebraically closed field k, an algebraic curve X over k, and a reductive algebraic group G over k. Let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles on X. Our goal in this section is to review the theory of the *Harder-Narasimhan* stratification of $\mathrm{Bun}_G(X)$. We will merely give an expository account here, referring the reader to [5] or [47] for more details.

For the remainder of this section, we fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We will say that a parabolic subgroup $P \subseteq G$ is *standard* if it contains B.

Notation 10.3.1. For every linear algebraic group H over k, we let $\mathrm{Hom}(H, \mathbf{G}_m)$ denote the character group of H (a finitely generated abelian group). We let $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$ denote the abelian group of homomorphisms from $\mathrm{Hom}(H, \mathbf{G}_m)$ to $\mathbf{Z}$. Given elements $\mu \in \mathrm{Hom}(H, \mathbf{G}_m)$ and $\nu \in \mathrm{Hom}(H, \mathbf{G}_m)^\vee$, we let $\langle \mu, \nu \rangle \in \mathbf{Z}$ denote the integer given by evaluating ν on μ.

Definition 10.3.2. Let H be a linear algebraic group over k and let $\mathcal{P}$ be an H-bundle on X. For every character $\mu : H \to \mathbf{G}_m$, the H-bundle $\mathcal{P}$ determines a $\mathbf{G}_m$-bundle $\mathcal{P}_\mu$ on X, which we will identify with the corresponding line bundle. We let $\deg(\mathcal{P})$ denote the element of $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$ given by $\mu \mapsto \deg(\mathcal{L}_\mu)$. We will refer to $\deg(\mathcal{P})$ as the *degree of* $\mathcal{P}$.

Let ν be an element of $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$. We let $\mathrm{Bun}_H^\nu(X)$ denote the full subategory of $\mathrm{Bun}_H(X)$ spanned by those pairs $(R, \mathcal{P})$ where R is a finitely generated k-algebra and $\mathcal{P}$ is an H-bundle on the relative curve X_R having the property that for each k-valued point $\eta : \operatorname{Spec} k \to \operatorname{Spec} R$, the fiber $\mathcal{P}_\eta = \mathcal{P} \times_{\operatorname{Spec} R} \operatorname{Spec} k$ has degree ν, when regarded as an H-bundle on X. We will refer to $\mathrm{Bun}_H^\nu(X)$ as the *moduli stack of H-bundles of degree ν on X*.

Remark 10.3.3. In the situation of Definition 10.3.2, let R be a finitely generated k-algebra and let $\mathcal{P}$ be an H-bundle on X_R. The construction

$$\eta \mapsto \deg(\mathcal{P}_\eta)$$

determines a map from the closed points of $\operatorname{Spec} R$ to $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$ which is locally constant for the Zariski topology. It follows that each $\mathrm{Bun}_H^\nu(X)$ is a closed and open substack of $\mathrm{Bun}_H(X)$; in particular, it is a smooth Artin stack over k. Moreover, we can identify $\mathrm{Bun}_H(X)$ with the disjoint union

$$\coprod_{\nu \in \mathrm{Hom}(H, \mathbf{G}_m)^\vee} \mathrm{Bun}_H^\nu(X)$$

(taken in the 2-category of Artin stacks over k).

Notation 10.3.4. Let H be a linear algebraic group over k and let $\mathfrak{h}$ denotes its Lie algebra. The adjoint action of H on $\mathfrak{h}$ determines a character

$$H \to \mathrm{GL}(\mathfrak{h}) \xrightarrow{\det} \mathbf{G}_m,$$

which we will denote by $2\rho_H$ and regard as an element of $\mathrm{Hom}(H, \mathbf{G}_m)$.

Remark 10.3.5. Specializing to the case where H is the standard Borel subgroup $B \subseteq G$, we can identify $\mathrm{Hom}(B, \mathbf{G}_m)$ with the character lattice of G. In this case, the element $2\rho_B \in \mathrm{Hom}(G, \mathbf{G}_m)$ is the sum of the positive roots of G. Beware that $2\rho_B$ is generally not divisible by 2 in $\mathrm{Hom}(B, \mathbf{G}_m)$ (however, it is divisible by 2 when G is semisimple and simply connected: in this case, ρ_B can be identified with the sum of the fundamental weights of G).

Definition 10.3.6. Let $\mathcal{P}$ be a G-bundle on X. We will say that $\mathcal{P}$ is *semistable* if, for every standard parabolic subgroup $P \subseteq G$ and every reduction of $\mathcal{P}$ to to a P-bundle $\mathcal{Q}$, we have $\langle 2\rho_P, \deg(\mathfrak{p}_{\mathcal{Q}}) \leq 0$.

More generally, if R is a finitely generated k-algebra and $\mathcal{P}$ is a G-bundle on the relative curve X_R, we say that $\mathcal{P}$ is *semistable* if, for every k-valued point $\eta : \mathrm{Spec}\, k \to \mathrm{Spec}\, R$, the fiber $\mathcal{P}_\eta = \mathcal{P} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, k$ is semistable (when viewed as a G-bundle on X). We let $\mathrm{Bun}_G(X)^{\mathrm{ss}}$ denote the prestack given by the full subcategory of $\mathrm{Bun}_G(X)$ spanned by those pairs $(R, \mathcal{P})$, where R is a finitely generated k-algebra and $\mathcal{P}$ is a semistable G-bundle on X_R. We will refer to $\mathrm{Bun}_G(X)$ as the *moduli stack of semistable G-bundles*.

Remark 10.3.7. Let $P \subseteq G$ be a standard parabolic subgroup and let $\mathcal{Q}$ be a G-bundle on X. Let U denote the unipotent radical of P and let $\mathfrak{u}$ denote its Lie algebra. We then have an exact sequence

$$0 \to \mathfrak{u} \to \mathfrak{p} \to \mathfrak{p}/\mathfrak{u} \to 0$$

of representations of P. Note that the action of P on $\mathfrak{p}/\mathfrak{u}$ factors through the adjoint quotient of P/U, and is therefore given by a map $P/U \to \mathrm{SL}(\mathfrak{p}/\mathfrak{u})$. It follows that the character $2\rho_P \in \mathrm{Hom}(P, \mathbf{G}_m)$ can be identified with the character

$$P \to \mathrm{GL}(\mathfrak{u}) \xrightarrow{\det} \mathbf{G}_m.$$

Remark 10.3.8. Let G_{ad} denote the adjoint quotient of G. For every standard parabolic subgroup $P \subseteq G$, we let P_{ad} denote the image of P in G_{ad}. If $\mathcal{Q}$ is a P-bundle, we let $\mathcal{Q}_{\mathrm{ad}}$ denote the associated P_{ad}-bundle. Note that the natural map $P \to P_{\mathrm{ad}}$ induces an isomorphism from the unipotent radical of P to the unipotent radical of P_{ad}. It follows from Remark 10.3.7 the induced map

$$\mathrm{Hom}(P_{\mathrm{ad}}, \mathbf{G}_m) \to \mathrm{Hom}(P, \mathbf{G}_m)$$

carries $2\rho_{P_{\mathrm{ad}}}$ to $2\rho_P$.

Remark 10.3.9. Let $\mathcal{P}$ be a G-bundle on X. For every standard parabolic subgroup $P \subseteq G$, there is a canonical bijection between the set of P-reductions of $\mathcal{P}$ to the set of P_{ad}-reductions of $\mathcal{P}_{\mathrm{ad}}$, given (at the level of bundles) by the construction $\mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{ad}}$. It follows from Remark 10.3.8 that we have

$$\langle 2\rho_P, \deg(\mathcal{Q}) \rangle = \langle 2\rho_{P_{\mathrm{ad}}}, \deg(\mathcal{Q})_{\mathrm{ad}} \rangle$$

$\mathcal{P}$ is semistable if and only if $\mathcal{P}_{\mathrm{ad}}$ is semistable. Consequently, we have a pullback diagram of prestacks

$$\begin{array}{ccc} \mathrm{Bun}_G(X)^{\mathrm{ss}} & \longrightarrow & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G_{\mathrm{ad}}}(X)^{\mathrm{ss}} & \longrightarrow & \mathrm{Bun}_{G_{\mathrm{ad}}}(X). \end{array}$$

Variant 10.3.10. Let $P \subseteq G$ be a standard parabolic subgroup and let $U \subseteq P$ be its unipotent radical. Then P/U is a reductive algebraic group over X. We say that a P-bundle $\mathcal{Q}$ on X is *semistable* if the associated (P/U)-bundle on X is semistable. We let $\mathrm{Bun}_P(X)^{\mathrm{ss}}$ denote the fiber product

$$\mathrm{Bun}_P(X) \times_{\mathrm{Bun}_{P/U}(X)} \mathrm{Bun}_{P/U}(X)^{\mathrm{ss}};$$

we will refer to $\mathrm{Bun}_P(X)^{\mathrm{ss}}$ as the *moduli stack of semistable P-bundles*. For each element $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee$, we let $\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}}$ denote the intersection $\mathrm{Bun}_P^\nu(X) \cap \mathrm{Bun}_P(X)^{\mathrm{ss}}$, which we will refer to as the *moduli stack of semistable P-bundles of degree ν*.

Remark 10.3.11. Let $G \to G'$ be a central isogeny of reductive algebraic groups over k. We let B' and T' denote the images of B and T in G', so that B' is a Borel subgroup of G' and T' is a maximal torus in B'. For every standard parabolic subgroup $P \subseteq G$, let P' denote the image of P in G', so that P' is a standard parabolic subgroup of G'. The natural map $P \to P'$ induces an injection of finitely generated free abelian groups

$$\mathrm{Hom}(P, \mathbf{G}_m)^\vee \to \mathrm{Hom}(P', \mathbf{G}_m)^\vee,$$

$$\nu \mapsto \nu'.$$

For each $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee$ we have $\mathrm{Bun}_P^\nu(X) \simeq \mathrm{Bun}_P(X) \times_{\mathrm{Bun}_{P'}(X)} \mathrm{Bun}_{P'}^{\nu'}(X)$, and Remark 10.3.9 gives $\mathrm{Bun}_P(X)^{\mathrm{ss}} \simeq \mathrm{Bun}_P(X) \times_{\mathrm{Bun}_{P'}(X)} \mathrm{Bun}_{P'}(X)^{\mathrm{ss}}$.

Construction 10.3.12. Let $P \subseteq G$ be a standard parabolic subgroup and let $U \subseteq P$ be its unipotent radical. Then there is a unique subgroup $H \subseteq P$ which contains T for which the composite map

$$H \hookrightarrow P \to P/U$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc} & \mathrm{Hom}(P, \mathbf{G}_m) & \\ \nearrow & & \searrow \\ \mathrm{Hom}(P/U, \mathbf{G}_m) & \longrightarrow & \mathrm{Hom}(H, \mathbf{G}_m) \end{array}$$

where the bottom map and the left diagonal map are isomorphisms, so the right diagonal map is an isomorphism as well.

Let $\mathfrak{Z}(H)$ denote the center of H, which we regard as a subgroup of T. Since H is a reductive group, the canonical map

$$\mathrm{Hom}(H, \mathbf{G}_m) \to \mathrm{Hom}(\mathfrak{Z}(H), \mathbf{G}_m)$$

is a rational isomorphism. In particular, we have a canonical map

$$\begin{array}{rcl} \mathrm{Hom}(T, \mathbf{G}_m) & \to & \mathrm{Hom}(\mathfrak{Z}(H), \mathbf{G}_m) \\ & \to & \mathrm{Hom}(\mathfrak{Z}(H), \mathbf{G}_m) \otimes \mathbf{Q} \\ & \xleftarrow{\sim} & \mathrm{Hom}(H, \mathbf{G}_m) \otimes \mathbf{Q} \\ & \xleftarrow{\sim} & \mathrm{Hom}(P, \mathbf{G}_m) \otimes \mathbf{Q}\,. \end{array}$$

In particular, every character $\alpha \in \mathrm{Hom}(T, \mathbf{G}_m)$ determines a map $\mathrm{Hom}(P, \mathbf{G}_m)^\vee \to \mathbf{Q}$, which we will denote by $\nu \mapsto \langle \alpha, \nu \rangle$.

Let $\mathbf{\Delta} \subseteq \mathrm{Hom}(T, \mathbf{G}_m)$ denote the set of simple roots of G and let $\mathbf{\Delta}_P \subseteq \mathbf{\Delta}$ denote the subset consisting of those roots α such that $-\alpha$ is not a root of P. We will say that an element $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee$ is *dominant* if $\langle \alpha, \nu \rangle \geq 0$ for each $\alpha \in \mathbf{\Delta}_P$, and we will say that ν is *dominant regular* if $\langle \alpha, \nu \rangle > 0$ for each $\alpha \in \mathbf{\Delta}_P$. We let $\mathrm{Hom}(P, \mathbf{G}_m)^\vee_{\geq 0}$ denote the subset of $\mathrm{Hom}(P, \mathbf{G}_m)^\vee$ spanned by the dominant elements and $\mathrm{Hom}(P, \mathbf{G}_m)^\vee_{>0}$ the subset consisting of dominant regular elements.

We can now state the main result that we will need. For a proof, we refer the reader to [5] or [47].

Theorem 10.3.13. (*a*) *For each standard parabolic subgroup $P \subseteq G$, the inclusion*

$$\mathrm{Bun}_P(X)^{\mathrm{ss}} \hookrightarrow \mathrm{Bun}_P(X)$$

is an open immersion. In particular, $\mathrm{Bun}_P(X)^{\mathrm{ss}}$ *is a smooth Artin stack over* $\operatorname{Spec} k$, *which can be written as a disjoint union*

$$\coprod_{\nu \in \mathrm{Hom}(P,\mathbf{G}_m)^\vee} \mathrm{Bun}_P^\nu(X)^{\mathrm{ss}}.$$

(*b*) *For each standard parabolic subgroup $P \subseteq G$ and each $\nu \in \mathrm{Hom}(P,\mathbf{G}_m)^\vee_+$, there exists a locally closed substack $\mathrm{Bun}_G(X)_{P,\nu} \subseteq \mathrm{Bun}_G(X)$ which is characterized by the following property: the natural map $\mathrm{Bun}_P(X) \to \mathrm{Bun}_G(X)$ restricts to a surjective finite radicial map*

$$\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)_{P,\nu}.$$

(*c*) *Let A be the collection of all pairs (P,ν), where P is a standard parabolic subgroup of G and ν is an element of $\mathrm{Hom}(P,\mathbf{G}_m)^\vee_{>0}$. Then the collection of locally closed substacks $\{\mathrm{Bun}_G(X)_{P,\nu}\}_{(P,\nu)\in A}$ determines a stratification of $\mathrm{Bun}_G(X)$ (for a suitably chosen partial ordering of A; see Remark 10.2.3).*

We will refer to the stratification of $\mathrm{Bun}_G(X)$ whose existence is guaranteed by Theorem 10.3.13 as the *Harder-Narasimhan stratification*.

Remark 10.3.14. If the field k has characteristic zero, or if $G = \mathrm{GL}_n$, or more generally if if the characteristic of k does not belong to a finite set of "bad primes" which may depend on G, then assertion (*b*) can be strengthened: the maps $\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)_{P,\nu}$ are equivalences. However, this is not true in general; see [25] for a more thorough discussion.

Remark 10.3.15. Let $G \to G'$ be a central isogeny of reductive algebraic groups over k. It follows from Remark 10.3.11 that the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$ is the pullback of the Harder-Narasimhan stratification of $\mathrm{Bun}_{G'}(X)$.

We conclude this section with a few remarks about the naturality of the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$.

Remark 10.3.16 (Functoriality in X)**.** Let ψ be an automorphism of X *as an abstract scheme*, so that ψ induces an automorphism ψ_0 of the field $k = \mathrm{H}^0(X;\mathcal{O}_X)$ which we do not assume to be the identity. Then σ fits into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\psi_0} & \operatorname{Spec} k. \end{array}$$

We can regard the reductive algebraic group G, the Borel subgroup $B \subseteq G$, and the maximal torus $T \subseteq B$ as defined over any subfield $k_0 \subseteq k$ (by taking the split form of G over k_0), so that ψ induces an automorphism ϕ of the algebraic stack $\mathrm{Bun}_G(X)$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_G(X) & \xrightarrow{\phi} & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\psi_0} & \operatorname{Spec} k. \end{array}$$

All of the constructions appearing this section are natural in X: in particular, the automorphism ϕ carries each Harder-Narasimhan stratum $\mathrm{Bun}_G(X)_{P,\nu}$ into itself.

Remark 10.3.17. Suppose that k is an algebraic closure of a perfect subfield $k_0 \subseteq k$, and that X is defined over k_0: that is, we can write $X = X_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec} k$, where X_0 is an algebraic curve over k_0. Let $\operatorname{Bun}_G(X_0)$ denote the moduli stack of G-bundles on X_0, which we regard as a smooth Artin stack over k_0. We have $\operatorname{Bun}_G(X) \simeq \operatorname{Bun}_G(X_0) \times_{\operatorname{Spec} k_0} \operatorname{Spec} k$. It follows that there is a bijective correspondence between open substacks of $\operatorname{Bun}_G(X_0)$ and $\operatorname{Gal}(k/k_0)$-equivariant open substacks of $\operatorname{Bun}_G(X)$. Invoking Remark 10.3.16, we see that the Harder-Narasimhan stratification of $\operatorname{Bun}_G(X)$ is defined over k_0: that is, there is a stratification of $\operatorname{Bun}_G(X_0)$ by locally closed substacks $\{\operatorname{Bun}_G(X_0)_{P,\nu}\}$ satisfying

$$\operatorname{Bun}_G(X)_{P,\nu} = \operatorname{Bun}_G(X_0)_{P,\nu} \times_{\operatorname{Spec} k_0} \operatorname{Spec} k.$$

Remark 10.3.18 (Functoriality in G)**.** Choose a pinning $(B, T, \{\phi_\alpha : \mathbf{G}_a \to B\})$ of the algebraic group G, so that the outer automorphism group $\operatorname{Out}(G)$ acts on G by pinned automorphisms. This determines an action of the outer automorphism group $\operatorname{Out}(G)$ on the moduli stack $\operatorname{Bun}_G(X)$. This action preserves the Harder-Narasimhan stratification of $\operatorname{Bun}_G(X)$, but permutes the strata. More precisely, let $A = \{(P, \nu)\}$ be as in the statement of Theorem 10.3.13. The automorphism group $\operatorname{Out}(G)$ acts on A by the construction

$$(\sigma \in \operatorname{Out}(G), (P, \nu) \in A) \mapsto (\sigma(P), \nu_\sigma)$$

where $\langle \mu, \nu_\sigma \rangle = \langle \mu \circ \sigma, \nu \rangle$ for $\mu \in \operatorname{Hom}(\sigma(P), \mathbf{G}_m)$. For each $\sigma \in \operatorname{Out}(G)$, the associated automorphism $\psi_\sigma : \operatorname{Bun}_G(X) \simeq \operatorname{Bun}_G(X)$ restricts to equivalences

$$\operatorname{Bun}_G(X)_{P,\nu} \simeq \operatorname{Bun}_G(X)_{\sigma(P),\nu_\sigma}.$$

10.4. **Quasi-Compactness of Moduli Spaces of Bundles.** Let X be an algebraic curve over a finite field $\mathbf{F}_q$ and let G be a smooth affine group scheme over X with connected fibers and semisimple generic fiber. In §10.8, we will prove that the moduli stack $\operatorname{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula (Theorem 10.0.6). The main obstacle to overcome is that the moduli stack $\operatorname{Bun}_G(X)$ is not quasi-compact. Our strategy for addressing this problem is to use the Harder-Narasimhan stratification of §10.3 to decompose $\operatorname{Bun}_G(X)$ into pieces which are easier to analyze. Our goal in this section is to supply some tools which can be used to verify that these individual pieces are amenable to analysis. We begin with the following:

Proposition 10.4.1. *Let X be an algebraic curve over a field k and let G be a smooth affine group over X. Let $\mathcal{U}$ be a quasi-compact open substack of $\operatorname{Bun}_G(X)$. Then there exists an effective divisor $D \subseteq X$ such that the fiber product $\operatorname{Bun}_G(X, D) \times_{\operatorname{Bun}_G(X)} \mathcal{U}$ is an algebraic space (here $\operatorname{Bun}_G(X, D)$ denotes the moduli stack of G-bundles on X equipped with a trivialization on D; see Definition 3.2.1).*

Proof. Since $\mathcal{U}$ is quasi-compact, we can choose a smooth surjection $\operatorname{Spec} R \to \mathcal{U}$, corresponding to a G-bundle $\mathcal{P}$ on the relative curve X_R. Since the diagonal of $\operatorname{Bun}_G(X)$ is affine, automorphisms of the G-bundle $\mathcal{P}$ are parametrized by an affine R-scheme of finite type Y. The identity automorphism of $\mathcal{P}$ determines a closed immersion R-schemes $s : \operatorname{Spec} R \to Y$; let us denote the image of this map by $Y' \subseteq Y$.

Fix a closed point $x \in X$. For each $n \geq 0$, let $D_n \subseteq X$ denote the divisor given by the nth multiple of X, and let Y_n denote the closed subscheme of Y classifying automorphisms of $\mathcal{P}$ which restrict to the identity over the divisor D_n.

Let $\mathcal{O}_x$ denote the complete local ring of x at X, which we can identify with a formal power series ring $k'[[t]]$ for some finite extension k' of k. For any Noetherian R-algebra A, we can identify the formal completion of X_A along the closed subscheme $\{x\} \times_{\operatorname{Spec} k} \operatorname{Spec} A$ with the formal spectrum of the power series ring $A'[[t]]$, where $A' = A \otimes_k k'$. The map $\operatorname{Spec} A'[[t]] \to X_A$ is schematically dense, so any automorphism of $\mathcal{P} \times_{X_R} X_A$ which restricts to the identity on the

$\operatorname{Spec} A'[[t]]$ must coincide with the identity. In other words, we have $\bigcap_{n \geq 0} Y_n = Y'$ (as closed subschemes of Y). Since Y is a Noetherian scheme, we must have $Y' = Y_n$ for $n \gg 0$. It then follows that $\operatorname{Bun}_G(X, D_n) \times_{\operatorname{Bun}_G(X)} \mathcal{U}$ is an algebraic space. □

Corollary 10.4.2. *Let X be an algebraic curve over a field k and let G be a smooth affine group scheme over X. Suppose we are given a quasi-compact Artin stack $\mathcal{Y}$ over k equipped with a map $f : \mathcal{Y} \to \operatorname{Bun}_G(X)$. Assume that f is representable by quasi-compact, quasi-separated algebraic spaces (in other words, for every map $\operatorname{Spec} R \to \operatorname{Bun}_G(X)$, the fiber product $\mathcal{Y} \times_{\operatorname{Bun}_G(X)} \operatorname{Spec} R$ is a quasi-compact, quasi-separated algebraic space). Then $\mathcal{Y}$ can be written as a quotient Y/H, where Y is a quasi-compact, quasi-separated algebraic space over k and H is a linear algebraic group over k.*

Proof. Since $\mathcal{Y}$ is quasi-compact, the map f factors through a quasi-compact open substack $\mathcal{U} \subseteq \operatorname{Bun}_G(X)$. Using Proposition 10.4.1, we can choose an effective divisor $D \subseteq X$ such that the fiber product

$$Z = \mathcal{U} \times_{\operatorname{Bun}_G(X)} \operatorname{Bun}_G(X, D)$$

is an algebraic space. Since Z is affine over $\mathcal{U}$, it is quasi-compact and quasi-separated. Set

$$Y = \mathcal{Y} \times_{\operatorname{Bun}_G(X)} \operatorname{Bun}_G(X, D) \simeq \mathcal{Y} \times_{\operatorname{Bun}_G(X)} Z$$

Since f is representable by quasi-compact quasi-separated algebraic spaces, it follows that Y is a quasi-compact quasi-separated algebraic space. Let H denote the Weil restriction of $G \times_X D$ along the finite flat map $D \to \operatorname{Spec} k$. Then H is a linear algebraic group acting on $\operatorname{Bun}_G(X, D)$, and we can identify $\operatorname{Bun}_G(X)$ with the (stack-theoretic) quotient $\operatorname{Bun}_G(X, D)/H$. It follows that H acts on $Y = \mathcal{Y} \times_{\operatorname{Bun}_G(X)} \operatorname{Bun}_G(X, D)$ (via its action on the second factor) with quotient

$$Y/H \simeq \mathcal{Y} \times_{\operatorname{Bun} G(X)} \operatorname{Bun}_G(X, D)/H \simeq \mathcal{Y}.$$

□

Our next few results concern quasi-compactness properties of $\operatorname{Bun}_G(X)$ as G and X vary.

Proposition 10.4.3. *Let X be an algebraic curve over a field k and let $f : G \to G'$ be a morphism of smooth affine group schemes over X. Suppose that f is an isomorphism at the generic point of X. Then the induced map $\operatorname{Bun}_G(X) \to \operatorname{Bun}_{G'}(X)$ is quasi-compact.*

Proof. Without loss of generality, we may assume that k is algebraically closed. Let $U \subseteq X$ be an open set over which f is an isomorphism, so that the inverse of f defines a map

$$g : G' \times_X U \to G.$$

Using Proposition A.3.11, we deduce that there is an effective divisor $D' \subseteq X$ (disjoint from U) such that, if $\overline{G}'$ is the group scheme over X obtained from G' by dilitation at D' along its identity section (see Variant A.3.9), then g extends to a map $\overline{g} : \overline{G}' \to G$ of group schemes over X. Applying the same argument to the map $f_0 : G \times_X U \to \overline{G}'$ determined by f, we conclude that there is an effective divisor $D \subseteq X$ (again disjoint from U) such that, if $\overline{G}$ denotes the group scheme obtained from G by dilitation at the divisor D along its identity section, then f_0 extends to a map $\overline{f} : \overline{G} \to \overline{G}'$. We have canonical equivalences

$$\operatorname{Bun}_{\overline{G}}(X) \simeq \operatorname{Bun}_G(X, D) \qquad \operatorname{Bun}_{\overline{G}'}(X) \simeq \operatorname{Bun}_{G'}(X, D'),$$

so that the maps f, $\overline{g}$, and $\overline{f}$ give a diagram of algebraic stacks

$$\operatorname{Bun}_G(X, D) \to \operatorname{Bun}_{G'}(X, D') \to \operatorname{Bun}_G(X) \to \operatorname{Bun}_{G'}(X).$$

Note that the composite map $\operatorname{Bun}_G(X, D) \to \operatorname{Bun}_G(X)$ is surjective (since any G-bundle on the a relative divisor $D \times_{\operatorname{Spec} k} \operatorname{Spec} R$ can be trivialized étale locally on $\operatorname{Spec} R$), so the map

$\mathrm{Bun}_{G'}(X, D') \to \mathrm{Bun}_G(X)$ is also surjective. Consequently, to prove that the map $\mathrm{Bun}_G(X) \to \mathrm{Bun}_{G'}(X)$ is quasi-compact, it will suffice to show that the composite map $\phi : \mathrm{Bun}_{G'}(X, D') \to \mathrm{Bun}_{G'}(X)$ is quasi-compact. This is clear, because ϕ is an affine morphism (it is a torsor for the affine group scheme over k given by the Weil restriction of $G' \times_X D'$ along the finite flat map $D' \to \operatorname{Spec} k$). □

Proposition 10.4.4. *Let k be a field, let $f : \widetilde{X} \to X$ be a non-constant morphism of algebraic curves over k, and let G be a smooth affine group scheme over X. Then the canonical map of algebraic stacks $\mathrm{Bun}_G(X) \to \mathrm{Bun}_G(\widetilde{X})$ (given by pullback along f) is an affine morphism.*

Proof. Fix a map $\operatorname{Spec} R \to \mathrm{Bun}_G(\widetilde{X})$, corresponding to a G-bundle $\mathcal{P}$ on the relative curve $\widetilde{X}_R$. We wish to show the fiber product $\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G(\widetilde{X})} \operatorname{Spec} R$ is representable by an affine R-scheme. Let Y denote the fiber product $\widetilde{X}_R \times_{X_R} \widetilde{X}_R$ and let

$$\pi_1, \pi_2 : Y \to \widetilde{X}_R$$

denote the two projection maps. Let $\mathrm{Iso}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P})$ denote the affine Y-scheme whose A-valued points are G-bundle isomorphisms of $(\pi_1^* \mathcal{P}) \times_Y \operatorname{Spec} A$ with $(\pi_2^* \mathcal{P}) \times_Y \operatorname{Spec} A$. Let Z denote the affine R-scheme obtained by Weil restriction of $\mathrm{Iso}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P})$ along the proper flat morphism $Y \to \operatorname{Spec} R$. It now suffices to observe that $\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G(\widetilde{X})} \operatorname{Spec} R$ can be identified with a closed subscheme of Z: the A-valued point of Z correspond to G-bundle isomorphisms

$$\gamma : (\pi_1^* \mathcal{P} \times_{\operatorname{Spec} R} \operatorname{Spec} A) \simeq (\pi_2^* \mathcal{P} \times_{\operatorname{Spec} R} \operatorname{Spec} A),$$

while the A-valued points of $\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G(\widetilde{X})} \operatorname{Spec} R$ correspond to such G-bundle isomorphisms which satisfy a cocycle condition (since the finite flat morphism $\widetilde{X}_A \to X_A$ is of effective descent for G-bundles). □

Proposition 10.4.5. *Let X be an algebraic curve over a field k, let G be a semisimple group scheme over X, and let G_{ad} denote the adjoint quotient of G. Assume that the generic fiber of G is split. Then the natural map*

$$\mathrm{Bun}_G(X) \to \mathrm{Bun}_{G_{\mathrm{ad}}}(X)$$

is quasi-compact.

Proof. Without loss of generality, we may assume that k is algebraically closed. Let G_0 denote the generic fiber of G. Since G_0 is split, we can choose a Borel subgroup $B_0 \subseteq G_0$ and a split maximal torus $T_0 \subseteq B_0$. Since G is semisimple, the X-scheme parametrizing Borel subgroups of G is proper over X; it follows from the valuative criterion of properness that B_0 extends uniquely to a Borel subgroup $B \subseteq G$ (given by the scheme-theoretic closure of B_0 in G). Let U denote the unipotent radical of B, and let $T = B/U$. Then T is an algebraic torus over X whose generic fiber is split (since it is isomorphic to T_0); it follows that T itself is a split torus. Let B_{ad} denote the image of B in the adjoint quotient G_{ad}, and let T_{ad} denote the quotient of B_{ad} by its unipotent radical.

Let R be a finitely generated k-algebra and suppose we are given a map $f : \operatorname{Spec} R \to \mathrm{Bun}_{G_{\mathrm{ad}}}(X)$; we wish to prove that the fiber product $\operatorname{Spec} R \times_{\mathrm{Bun}_{G_{\mathrm{ad}}}(X)} \mathrm{Bun}_G(X)$ is quasi-compact. This assertion can be tested locally with respect to the étale topology on $\operatorname{Spec} R$; we may therefore assume without loss of generality that f factors through $\mathrm{Bun}_{B_{\mathrm{ad}}}(X)$ (Theorem

3.7.1). Since the diagram of algebraic stacks

$$\begin{array}{ccccc} \mathrm{Bun}_G(X) & \longleftarrow & \mathrm{Bun}_B(X) & \longrightarrow & \mathrm{Bun}_T(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bun}_{G_{\mathrm{ad}}}(X) & \longleftarrow & \mathrm{Bun}_{P_{\mathrm{ad}}}(X) & \longrightarrow & \mathrm{Bun}_{T_{\mathrm{ad}}}(X) \end{array}$$

consists of pullback squares, it will suffice to show that the fiber product $\mathrm{Spec}\, R \times_{\mathrm{Bun}_{T_{\mathrm{ad}}}(X)} \mathrm{Bun}_T(X)$ is quasi-compact. We are therefore reduced to proving that the map $\mathrm{Bun}_T(X) \to \mathrm{Bun}_{T_{\mathrm{ad}}}(X)$ is quasi-compact.

Let $\mathrm{Pic}(X) = \mathrm{Bun}_{\mathbf{G}_m}(X)$ denote the Picard stack of X; a choice of k-rational point $x \in X$ determines a splitting

$$\mathrm{Pic}(X) = \mathbf{Z} \times J(X) \times \mathrm{B}\,\mathbf{G}_m$$

where $J(X)$ is the Jacobian variety of X. Let $\Lambda = \mathrm{Hom}(\mathbf{G}_m, T)$ denote the cocharacter lattice of T and let $\Lambda_{\mathrm{ad}} = \mathrm{Hom}(\mathbf{G}_m, T_{\mathrm{ad}})$ denote the cocharacter lattice of T_{ad}. We wish to show that the natural map

$$\mathrm{Bun}_T(X) \simeq \Lambda \otimes_{\mathbf{Z}} \mathrm{Pic}(X) \to \Lambda_{\mathrm{ad}} \otimes_{\mathbf{Z}} \mathrm{Pic}(X) \simeq \mathrm{Bun}_{T_{\mathrm{ad}}}(X)$$

is quasi-compact. This is clear: the preimage of each connected component of $\mathrm{Bun}_{T_{\mathrm{ad}}}(X)$ is either empty or isomorphic to a product of finitely many copies of $J(X) \times \mathrm{B}\,\mathbf{G}_m$. □

Proposition 10.4.6. *Let X be an algebraic curve over a field k and let G be a split reductive group over k. Fix a Borel subgroup $B \subseteq G$ containing a split maximal torus $T \subseteq B$. For every standard parabolic subgroup $P \subseteq G$ and every element $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee$, the algebraic stack $\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}}$ is quasi-compact.*

Proof of Proposition 10.4.6. Without loss of generality, we may assume that k is algebraically closed. We proceed in several steps.

(*a*) Suppose first that G is a torus, and let $\Lambda = \mathrm{Hom}(\mathbf{G}_m, G)$ denote the cocharacter lattice of G. In this case, the only parabolic subgroup $P \subseteq G$ is the group G itself, and we have $\mathrm{Hom}(P, \mathbf{G}_m)^\vee \simeq \Lambda$. For each $\nu \in \Lambda$, the moduli stack $\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}} = \mathrm{Bun}_P^\nu(X)$ can be identified (after choosing a k-rational point $x \in X$) with a product of finitely many copies of $\mathrm{Pic}^0(X) \simeq J(X) \times \mathrm{B}\,\mathbf{G}_m$ (as in the proof of Proposition 10.4.5), and is therefore quasi-compact.

(*b*) We claim that if Proposition 10.4.6 is valid for the quotient $G' = P/\mathrm{rad}_u(P)$ (regarded as a parabolic subgroup of itself), then it is valid for the parabolic subgroup P. To prove this, it suffices to show that the map $\mathrm{Bun}_P(X) \to \mathrm{Bun}_{G'}(X)$ is quasi-compact. Note that the unipotent radical $\mathrm{rad}_u(P)$ is equipped with a finite filtration by normal subgroups

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_m = \mathrm{rad}_u(P)$$

where each quotient U_i/U_{i-1} is isomorphic to a vector group equipped with a linear action of P (which necessarily factors through the quotient P/U_i) We claim that each of the maps

$$\mathrm{Bun}_{P/U_{i-1}}(X) \to \mathrm{Bun}_{P/U_i}(X)$$

is quasi-compact. To prove this, fix a map $\mathrm{Spec}\, R \to \mathrm{Bun}_{P/U_i}(X)$, given by a P/U_i-torsor $\mathcal{P}$ on the relative curve X_R. Via the linear action of P/U_i on U_i/U_{i-1}, we obtain a vector bundle $\mathcal{E}_i$ on X_R. The obstruction to lifting $\mathcal{P}$ to a P/U_{i-1}-bundle is measured by a cohomology class $\eta \in \mathrm{H}^2(X_R; \mathcal{E}_i)$, which automatically vanishes since X_R is a curve over an affine scheme. Choose a lifting of $\mathcal{P}$ to a (P/U_{i-1})-torsor on X_R. Then the fiber product

$$\mathcal{Y} = \operatorname{Spec} R \times_{\operatorname{Bun}_{P/U_i}(X)} \operatorname{Bun}_{P/U_{i-1}}(X)$$

can be identified with the stack whose A-valued points (where A is an R-algebra) correspond to $\mathcal{E}_i$-torsors on the relative curve X_A. We wish to prove that $\mathcal{Y}$ is quasi-compact. If $D \subseteq X$ is an effective divisor, let $\mathcal{Y}_D$ denote the algebraic stack whose A-valued points are $\mathcal{E}_i$-torsors on X_A which are equipped with a trivialization along the relative divisor $D \times_{\operatorname{Spec} k} \operatorname{Spec} A$. The evident forgetful functor $\mathcal{Y}_D \to \mathcal{Y}$ is surjective, so it will suffice to prove that we can choose D such that $\mathcal{Y}_D$ is quasi-compact. Note that if $\deg(D) \gg 0$, then $\mathrm{H}^0(X_R; \mathcal{E}_i(-D)) \simeq 0$ and $\mathrm{H}^1(X_R; \mathcal{E}_i(-D))$ will be a projective R-module M of finite rank; in this case, we can identify $\mathcal{Y}_D$ with the affine scheme $\operatorname{Spec} \operatorname{Sym}^*_R(M^\vee)$.

(*c*) We now prove Proposition 10.4.6 in general. By virtue of (*b*), it will suffice to treat the case where $P = G$. Set $\Lambda_0 = \operatorname{Hom}(G, \mathbf{G}_m)^\vee$ and $\Lambda = \operatorname{Hom}(B, \mathbf{G}_m)^\vee \simeq \operatorname{Hom}(\mathbf{G}_m, T)$. The inclusion $B \hookrightarrow G$ induces a surjective map of lattices $\chi : \Lambda \to \Lambda_0$. It follows from steps (*a*) and (*b*) that for each $\lambda \in \Lambda$, the moduli stack $\operatorname{Bun}_B^\lambda(X)^{\mathrm{ss}} = \operatorname{Bun}_B^\lambda(X)$ is quasi-compact. To complete the proof, it will suffice to show that for each $\nu \in \Lambda_0$, we can find a finite subset $S \subseteq \chi^{-1}\{\nu\}$ for which the map

$$\amalg_{\lambda \in S} \operatorname{Bun}_B^\lambda(X) \to \operatorname{Bun}_G^\nu(X)^{\mathrm{ss}}$$

is surjective.

Let g denote the genus of the algebraic curve X, let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots of G (which we identify with elements of $\Lambda^\vee$), and let 2ρ denote the sum of the positive roots of G. We will show that the set

$$S = \{\lambda \in \Lambda : \chi(\lambda) = \nu, \langle 2\rho, \lambda \rangle \leq 0, \langle \alpha_i, \lambda \rangle \geq \min\{1 - g, 0\}\}$$

has the desired property. Note that S can be identified with the set of lattice points belonging to the locus

$$S_{\mathbf{R}} = \{\lambda \in \Lambda \otimes \mathbf{R} : \chi(\lambda) = \nu, \langle 2\rho, \lambda \rangle \leq 0, \langle \alpha_i, \lambda \rangle \geq \min\{1 - g, 0\}\}$$

which is a simplex in the real vector space $\Lambda \otimes \mathbf{R}$; this proves that S is finite. We will complete the proof by showing that if $\mathcal{P}$ is a semistable G-bundle of degree ν, then there exists $\lambda \in S$ such that $\mathcal{P}$ can be reduced to a B-bundle of degree λ. Note that in this case the conditions $\chi(\lambda) = \nu$ and $\langle 2\rho, \lambda \rangle \leq 0$ are automatic (if the second condition were violated, then $\mathcal{P}$ would not be semistable). It will therefore suffice to prove the following:

(∗) Let $\mathcal{P}$ be a G-bundle on X. Then $\mathcal{P}$ can be reduced to a B-bundle $\mathcal{Q}$ satisfying

$$\langle \deg(\mathcal{Q}), \alpha_i \rangle \geq \min\{1 - g, 0\}$$

for $1 \leq i \leq r$.

Let $C = \{\lambda \in \Lambda : \langle \alpha_i, \lambda \rangle \geq 0 \text{ for } 1 \leq i \leq r\}$ be the dominant Weyl chamber in Λ, and let W denote the Weyl group of G. Then W acts on Λ, and every W-orbit in Λ contains an element of C. Let $\mathcal{P}$ be as in (∗). Then $\mathcal{P}$ admits a B-reduction $\mathcal{Q}$ (to see this, it suffices to show that the associated bundle over the adjoint quotient G_{ad} admits a B_{ad}-reduction, where B_{ad} denotes the image of B in G_{ad}; this is a special case of Theorem 3.7.1). Write $\deg(\mathcal{Q}) = w\lambda$, where $\lambda \in C$ and $w \in W$. Let us assume that $\mathcal{Q}$ and w have been chosen so that the w has minimal length. We will prove (∗) by showing that $\langle \alpha_i, \deg(\mathcal{Q}) \rangle \geq \min\{1 - g, 0\}$ for $1 \leq i \leq r$. Suppose otherwise: then there exists a simple root α such that $\langle \deg(\mathcal{Q}), \alpha_i \rangle < 0$ and $\langle \alpha_i, \deg(\mathcal{Q}) \rangle < 1 - g$. Let $w_i \in W$ denote the simple reflection corresponding to the root α_i. The condition $\langle \deg(\mathcal{Q}), \alpha_i \rangle < 0$ implies that $w_i w$ has smaller length than w. We will obtain a contradiction by showing that $\mathcal{P}$ admits a reduction to a B-bundle having degree $w_i \deg(\mathcal{Q}) = (w_i w)\lambda$. For this, it suffices to establish the following:

$(*')$ Let $\mathcal{Q}$ be a B-bundle on X and let α_i be a simple root of G satisfying $\langle \alpha_i, \deg(\mathcal{Q}) \rangle < 1-g$. Then there exists another B-bundle $\mathcal{Q}'$ on X such that $\deg(\mathcal{Q}') = w_i \deg(\mathcal{Q})$, and $\mathcal{Q}$ and $\mathcal{Q}'$ determine isomorphic G-bundles on X.

To prove $(*)$, let $P \subseteq G$ denote the parabolic subgroup generated by B together with the root subgroup corresponding to $-\alpha_i$, and let $\mathcal{Q}_P$ denote the P-bundle determined by $\mathcal{Q}$. We will show that $\mathcal{Q}_P$ admits a B-reduction $\mathcal{Q}'$ satisfying $\deg(\mathcal{Q}') = w_i \deg(\mathcal{Q})$. Note that there is a bijective correspondence between B-reductions of $\mathcal{Q}_P$ and $(B/\operatorname{rad}_u(P))$-reductions of the induced $(P/\operatorname{rad}_u P)$-bundle $\mathcal{Q}_{P/\operatorname{rad}_u P}$. Replacing G by $P/\operatorname{rad}_u P$, we are reduced to the problem of proving $(*')$ in the special case where G has semisimple rank 1 (that is, where α_i is the *only* root of G). In this case, we will prove that $\mathcal{Q}$ can be reduced to a T-bundle $\mathcal{Q}_0$. The element $w_i \in W$ determines an automorphism of T which becomes inner in G, and therefore induces an automorphism of the set of isomorphism classes of T-bundles with itself which does not change the isomorphism class of the associated G-bundle. This automorphism carries $\mathcal{Q}_0$ to the isomorphism class of another T-bundle $\mathcal{Q}_0^{w_i}$, and we can complete the proof of $(*')$ by taking $\mathcal{Q}'$ to be the B-bundle determined by $\mathcal{Q}_0^{w_i}$. We conclude by observing that the obstruction to choosing the reduction $\mathcal{Q}_0$ is given by an element of $\mathrm{H}^1(X; \mathcal{L})$, where $\mathcal{L}$ is the line bundle on X obtained from $\mathcal{Q}$ via the (linear) action of B on $\operatorname{rad}_u(B) \simeq \mathbf{G}_a$. An elementary calculation shows that the degree of $\mathcal{L}$ is given by $-\langle \alpha_i, \deg(\mathcal{Q}) \rangle > g-1$, so that $\mathrm{H}^1(X; \mathcal{L})$ vanishes by the Riemann-Roch theorem.

□

10.5. **The Harder-Narasimhan Stratification (Generically Split Case).** Throughout this section, we fix a perfect field k and an algebraic curve X over k. If G is a split reductive group scheme over X, then the moduli stack $\operatorname{Bun}_G(X)$ can be equipped with the Harder-Narasimhan stratification introduced in §10.3 (see Remark 10.3.17). In this section, we discuss a version which can be defined using assuming only that the generic fiber of G is split (or, more generally, that the generic fiber of G is an inner form).

We begin by introducing some terminology. For the remainder of this section, we fix a split reductive algebraic group G_0 over k, a Borel subgroup $B_0 \subseteq G_0$, and a split maximal torus $T_0 \subseteq B_0$. We will say that a parabolic subgroup $P_0 \subseteq G_0$ is *standard* if it contains B_0. We let $G_{0\,\mathrm{ad}}$ denote the adjoint quotient of G_0; for any subgroup $H_0 \subseteq G_0$, we let $H_{0\,\mathrm{ad}}$ denote the image of H_0 in G_{ad}.

If Y is a k-scheme and G is a group scheme over Y, we will say that G is a *form of* G_0 *over* Y if there exists an étale surjection $\widetilde{Y} \to Y$ such that $G \times_Y \widetilde{Y}$ is isomorphic to $G_0 \times_{\operatorname{Spec} k} \widetilde{Y}$ as a group scheme over Y. Note that this condition implies that G is a reductive group scheme over Y, and that the adjoint quotient G_{ad} of G is a form of $G_{0\,\mathrm{ad}}$ over Y.

Notation 10.5.1. Let Y be a k-scheme and let G be a form of G_0 over Y. We let $\operatorname{Iso}(G, G_0)$ denote the Y-scheme parametrizing isomorphisms of G with G_0 (so that the R-valued points of $\operatorname{Iso}(G, G_0)$ are isomorphisms of $G \times_Y \operatorname{Spec} R$ with $G_0 \times_{\operatorname{Spec} k} \operatorname{Spec} R$ as group schemes over R). Then $\operatorname{Iso}(G, G_0)$ is an $\operatorname{Aut}(G_0)$-torsor over Y, where $\operatorname{Aut}(G_0)$ denotes the automorphism group of G_0. The automorphism group $\operatorname{Aut}(G_0)$ fits into an exact sequence

$$0 \to G_{0\,\mathrm{ad}} \to \operatorname{Aut}(G_0) \to \operatorname{Out}(G_0) \to 0,$$

where $\mathrm{Out}(G_0)$ denotes the (constant) group of outer automorphisms of G_0 (if G_0 is semisimple, then $\mathrm{Out}(G_0)$ is finite). Let $\mathrm{Out}(G, G_0)$ denote the quotient $G_0 \backslash \mathrm{Out}(G, G_0)$, which we regard as an $\mathrm{Out}(G_0)$-torsor over Y. In particular, $\mathrm{Out}(G, G_0)$ is a scheme equipped with an étale surjection $\mathrm{Out}(G, G_0) \to Y$ (which is finite étale in the case where G_0 is semisimple).

Definition 10.5.2. Let Y be a k-scheme and let G be a form of G_0 over Y. An *inner structure* on G is a section of the projection map $\mathrm{Out}(G, G_0) \to Y$. An *inner form of G_0 over Y* is a pair (G, σ), where G is a form of G_0 over Y and σ is an inner structure on G.

Example 10.5.3. Let G be a form of G_0 over a k-scheme Y. Any isomorphism $\beta : G \simeq G_0 \times_{\mathrm{Spec}\, k} Y$ determines an inner structure on G (in particular, the split form of G_0 over Y admits an inner structure), and every inner structure on G arises in this way étale locally on Y. Moreover, if β' is another such isomorphism, then β' determines the same inner structure on G if and only if the isomorphism

$$\beta'^{-1} \circ \beta : G \to G$$

is given by conjugation by a Y-valued point of G_{ad}.

Example 10.5.4. Let Y be a connected normal k-scheme with fraction field K_Y and let G be a form of G_0 over Y. Then every inner structure on the algebraic group $G \times_Y \mathrm{Spec}\, K_Y$ extends uniquely to an inner structure on G. inner structures on G to inner structures on $G \times_Y U$ is bijective. In particular, the group scheme G admits an inner structure whenever the generic fiber of G is split.

In the special case where k is algebraically closed and Y is an algebraic curve over k, the converse holds: since the fraction field K_Y has dimension ≤ 1, the generic fiber G is automatically quasi-split, so that G admits an inner structure if and only if the generic fiber of G is split.

Remark 10.5.5. Let G be a form of G_0 over a k-scheme Y. Then the group $\mathrm{Out}(G_0)$ acts on the collection of inner structures on G. If G admits an inner structure and Y is connected, then this action is simply transitive.

Construction 10.5.6. Let (G, σ) be an inner form of G_0 over a k-scheme Y. We let $\mathrm{Iso}^{\sigma}(G, G_0)$ denote the fiber product $Y \times_{\mathrm{Out}(G,G_0)} \mathrm{Iso}(G, G_0)$. Then $\mathrm{Iso}(G, G_0)$ is a *bitorsor* for the groups G_{ad} and $G_{0\,\mathrm{ad}}$: that is, it is equipped with commuting actions of the Y-schemes G_{ad} (on the right) and $G_{0\,\mathrm{ad}} \times_{\mathrm{Spec}\, k} Y$ (on the left), each of which is simply transitive locally for the étale topology. It follows that the construction

$$\mathcal{P} \mapsto \mathrm{Iso}(G, G_0) \otimes_{G_{\mathrm{ad}}} \mathcal{P} = (\mathrm{Iso}(G, G_0) \times_Y \mathcal{P})/G_{\mathrm{ad}}$$

induces an equivalence from the category $\mathrm{Tors}_{G_{\mathrm{ad}}}(Y)$ of G_{ad}-torsors on Y to the category $\mathrm{Tors}_{G_{0\,\mathrm{ad}}}(Y)$ of $G_{0\,\mathrm{ad}}$-torsors on Y.

Suppose that (G, σ) is an inner form of G_0 over the algebraic curve X. Applying the above reasoning to k-schemes of the form X_R where R is a k-algebra, we obtain an equivalence of algebraic stacks

$$\epsilon_\sigma : \mathrm{Bun}_{G_{\mathrm{ad}}}(X) \simeq \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X).$$

Warning 10.5.7. In the situation of Construction 10.5.6, the equivalence

$$\epsilon_\sigma : \mathrm{Bun}_{G_{\mathrm{ad}}}(X) \simeq \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)$$

depends on the choice of inner structure σ. Note that the group $\mathrm{Out}(G_0)$ acts simply transitively on the set of inner structures on G; in particular, any other inner structure on G can be written as $g(\sigma)$ where $g \in \mathrm{Out}(G_0)$. In this case, we have a commutative diagram

$$\begin{array}{ccc} & \mathrm{Bun}_{G_{\mathrm{ad}}}(X) & \\ {}^{\epsilon_\sigma}\swarrow & & \searrow^{\epsilon_{g(\sigma)}} \\ \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X) & \longrightarrow & \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X) \end{array}$$

where the lower horizontal map is the automorphism induced by g (which we can identify with a pinned automorphism of the algebraic $G_{0\,\mathrm{ad}}$).

Construction 10.5.8 (The Harder-Narasimhan Stratification)**.** Let (G, σ) be an inner form of G_0 over X, so that σ determines a map

$$\mathrm{Bun}_G(X) \to \mathrm{Bun}_{G_{\mathrm{ad}}}(X) \xrightarrow{\epsilon_\sigma} \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X).$$

Let A denote the set of all pairs (P_0, ν), where $P_0 \subseteq G_0$ is a standard parabolic subgroup and $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}$. For each element $(P_0, \nu) \in A$, let $\mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)_{P_{0\,\mathrm{ad}},\nu}$ denote the corresponding stratum of the Harder-Narasimhan stratification of $\mathrm{Bun}_{G_0}(X)$ (see Remark 10.3.17). We let $\mathrm{Bun}_G(X)^\sigma_{P_0,\nu}$ denote the reduced locally closed substack of $\mathrm{Bun}_G(X)$ given by

$$(\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)} \mathrm{Bun}_{G_0}(X)_{P_{0\,\mathrm{ad}},\nu})_{\mathrm{red}}.$$

Then $\{\mathrm{Bun}_G(X)^\sigma_{P_0,\nu}\}_{(P_0,\nu)\in A}$ is a stratification of $\mathrm{Bun}_G(X)$, which we will refer to as the *Harder-Narasimhan stratification*.

Warning 10.5.9. In the special case where the reductive group scheme G is split, the Harder-Narasimhan stratification of Construction 10.5.8 is not quite the same as the Harder-Narasimhan stratification of Theorem 10.3.13. The former stratification is indexed by the set

$$A = \{(P_0, \nu) : \ P_0 \subseteq G_0 \text{ is a standard parabolic subgroup and } \nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}\ \},$$

while the second stratification is indexed by the set

$$B = \{P_0, \overline{\nu} : \ P_0 \subseteq G_0 \text{ is a standard parabolic subgroup and } \overline{\nu} \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee_{>0}\}$$

For every standard parabolic subgroup $P_0 \subseteq G_0$, there is a canonical lattices

$$\rho_{P_0} : \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee \to \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee,$$

and for each $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee$ we have

$$\mathrm{Bun}_G(X)^\sigma_{P_0,\nu} = \amalg_{\rho_{P_0}(\overline{\nu})=\nu} \mathrm{Bun}_G(X)_{P_0,\overline{\nu}},$$

where the left hand side refers to the stratification of Construction 10.5.8 (where σ denotes the inner structure determined by a splitting of G) and the right hand side refers to the stratification of Theorem 10.3.13.

If the group G_0 is semisimple, then the map ρ_{P_0} is injective for every standard parabolic $P_0 \subseteq G_0$. In this case, we can regard B as a subset of A, and we have

$$\mathrm{Bun}_G(X)^\sigma_{P_0,\nu} = \begin{cases} \mathrm{Bun}_G(X)_{P_0,\nu} & \text{if } (P_0,\nu) \in B \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, the only difference between the stratifications of Construction 10.5.8 and Theorem 10.3.13 is that the former includes some "superfluous" empty strata (indexed by elements of A that do not belong to B).

If the group G_0 is not semisimple, then the maps ρ_{P_0} fail to be injective. In this case, the stratification of Theorem 10.3.13 is much finer than the stratification of Construction 10.5.8. For example, if $G_0 = \mathbf{G}_m$, then we can identify $\mathrm{Bun}_G(X) = \mathrm{Bun}_{G_0}(X)$ with the Picard stack $\mathrm{Pic}(X)$ of line bundles on X. The stratification of Construction 10.5.8 is trivial (there is only one stratum, consisting of the entire moduli stack $\mathrm{Pic}(X)$), but the stratification of Theorem 10.3.13 reproduces the decomposition of $\mathrm{Pic}(X)$ as a disjoint union $\amalg_{n \in \mathbf{Z}} \mathrm{Pic}^n(X)$, where $\mathrm{Pic}^n(X)$ denotes the moduli stack of line bundles of degree n on X.

Warning 10.5.10. Let (G, σ) be an inner form of G_0 over X. If the group scheme G is split, then the strata $\mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu}$ are empty when ν does not belong to image of the restriction map

$$\rho_{P_0} : \mathrm{Hom}(P_0, \mathbf{G}_m)^{\vee} \to \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}.$$

However, this is generally not true if G is not split.

Warning 10.5.11. Let G be a form of G_0 over X. Suppose that G_{ad} admits an inner structure σ, and let $\{\mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu}\}_{(P_0,\nu)\in A}$ be the stratification of Construction 10.5.8. The collection of locally closed substacks $\{\mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu} \subseteq \mathrm{Bun}_G(X)\}$ does not depend on the choice of σ. However, the indexing of this collection of locally closed substacks by the set A *does* depend on σ. More precisely, for each element $g \in \mathrm{Out}(G_0)$, we have

$$\mathrm{Bun}_G(X)^{g(\sigma)}_{g(P_0),\nu_g} = \mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu}$$

(as locally closed substacks of $\mathrm{Bun}_G(X)$), where the left hand side as defined as in Remark 10.3.18. This equality follows immediately from Remark 10.3.18 together with Warning 10.5.7.

Remark 10.5.12 (Functoriality in X and G)**.** Let ψ be an automorphism of X *as an abstract scheme*, so that ψ induces an automorphism ψ_0 of the field $k = \mathrm{H}^0(X; \mathcal{O}_X)$ which we do not assume to be the identity. Let G be form of G_0 over X and let $\overline{\psi}$ be an automorphism of G which covers the automorphism ψ of X. Then the pair $(\psi, \overline{\psi})$ determines an automorphism ϕ of the algebraic stack $\mathrm{Bun}_G(X)$, which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_G(X) & \xrightarrow{\phi} & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \xrightarrow{\psi_0} & \mathrm{Spec}\, k. \end{array}$$

Suppose that G admits an inner structure σ. The image of σ under $\overline{\psi}$ determines another inner structure σ' on G. Using Remarks 10.3.16 and 10.3.18, we see that ϕ restricts to give equivalences of Harder-Narasimhan strata

$$\mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu} \simeq \mathrm{Bun}_G(X)^{\sigma'}_{P_0,\nu}.$$

In other words, the automorphism ϕ of $\mathrm{Bun}_G(X)$ preserves the decomposition of $\mathrm{Bun}_G(X)$ into locally closed substacks $\{\mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu}\}_{(P_0,\nu)\in A}$, and permutes the strata by means of the action of the group $\mathrm{Out}(G_0)$ on A.

10.6. **Comparing Harder-Narasimhan Strata.** Throughout this section, we fix a field k, an algebraic curve X over k, and a split reductive algebraic group G_0 over k. Fix a Borel subgroup $B_0 \subseteq G_0$, a split maximal torus $T_0 \subseteq B_0$, and a parabolic subgroup $P_0 \subseteq G_0$ which contains B_0. If (G, σ) is an inner form of G_0 over X, then we can regard the moduli stack $\mathrm{Bun}_G(X)$ as equipped with the Harder-Narasimhan stratification of Construction 10.5.8. Our goal in this section is to study the relationships between the strata $\mathrm{Bun}^{\sigma}_G(X)_{P_0,\nu}$ as ν varies.

We begin by introducing some terminology.

Definition 10.6.1. Let (G, σ) be an inner form of G_0 over a k-scheme Y. We will say that a parabolic subgroup $P \subseteq G$ is *of type* P_0 if there exists an étale surjection $\widetilde{Y}$ and an isomorphism

$$\beta : G \times_Y \widetilde{Y} \to G_0 \times_{\operatorname{Spec} k} \widetilde{Y}$$

which is compatible with σ and which restricts to an isomorphism of $P \times_Y \widetilde{Y}$ with $P_0 \times_{\operatorname{Spec} k} \widetilde{Y}$.

If $\mathcal{P}$ is a G-torsor on Y, we let $G_{\mathcal{P}}$ denote group scheme over Y whose R-valued points are G-bundle automorphisms of $\mathcal{P} \times_Y \operatorname{Spec} R$. The inner structure σ on G determines an inner structure $\sigma_{\mathcal{P}}$ on $G_{\mathcal{P}}$. We will say that a subgroup $P \subseteq G_{\mathcal{P}}$ is a P_0*-structure* on $\mathcal{P}$ if it a parabolic subgroup of type P_0.

Example 10.6.2. Let (G, σ) be an inner form of G_0 over a k-scheme Y, and let $P \subseteq G$ be a parabolic subgroup of type P_0. Let $\mathcal{Q}$ be a P-torsor on Y and let $\mathcal{P}$ be the associated G-torsor. Since any automorphism of $\mathcal{Q}$ (as a P-torsor) determines an automorphism of $\mathcal{P}$ (as a G-torsor), there is a canonical map of group schemes

$$P_{\mathcal{Q}} \to G_{\mathcal{P}}$$

which exhibits $P_{\mathcal{Q}}$ as a parabolic subgroup of $G_{\mathcal{P}}$ of type P_0. This construction determines an equivalence of categories

$$\{ \text{ } P\text{-bundles on } Y \text{ } \} \to \{ \text{ } G\text{-bundles } \mathcal{P} \text{ on } Y \text{ equipped with a } P_0\text{-structure}\}.$$

In particular, if $G = G_0 \times_{\operatorname{Spec} k} Y$ is the split form of G_0 over Y, then we obtain an equivalence

$$\{ \text{ } P_0\text{-bundles on } Y \text{ } \} \to \{ \text{ } G\text{-bundles on } Y \text{ equipped with a } P_0\text{-structure}\}.$$

Remark 10.6.3. Suppose that G_0 is an adjoint semisimple group. If (G, σ) is an inner form of G_0 over a k-scheme Y, then Construction 10.5.6 determines a canonical equivalence from the category of G-bundles on Y to the category of G_0-torsors on Y, which we will denote by $\mathcal{P} \mapsto \mathcal{P}_0$. By functoriality, we can identify the automorphism group scheme $G_{\mathcal{P}}$ of $\mathcal{P}$ (as a G-torsor) with the automorphism group scheme $(G_0 \times_{\operatorname{Spec} k} Y)_{\mathcal{P}_0}$ of $\mathcal{P}_0$ (as a G_0-torsor). In particular, there is a canonical bijection between the set of P_0-structures on $\mathcal{P}$ and the set of P_0-structures on $\mathcal{P}_0$. Combining this observation with Example 10.6.2, we obtain an equivalence of categories

$$\{ \text{ } G\text{-torsors on } Y \text{ with a } P_0\text{-structure } \} \simeq \{ \text{ } P_0\text{-torsors on } Y \text{ } \}.$$

Remark 10.6.4. Let (G, σ) be an inner form of G_0 over a k-scheme Y, and let G_{ad} denote the adjoint quotient of G. Then G_{ad} is a form of the adjoint group $G_{0\,\mathrm{ad}}$ over Y, and σ determines an inner structure σ_{ad} on G_{ad}. For any parabolic subgroup $P \subseteq G$, let P_{ad} denote the image of P in G_{ad}. The construction $P \mapsto P_{\mathrm{ad}}$ determines a bijective correspondence between parabolic subgroups of G and parabolic subgroups of G_{ad}; moreover, a parabolic subgroup $P \subseteq G$ has type P_0 if and only if $P_{\mathrm{ad}} \subseteq G_{\mathrm{ad}}$ has type $P_{0\,\mathrm{ad}} \subseteq G_{0\,\mathrm{ad}}$. It follows that if $\mathcal{P}$ is a G-torsor on Y and $\mathcal{P}_{\mathrm{ad}}$ denotes the associated G_{ad}-torsor, then there is a canonical bijection from the set of P_0-structures on $\mathcal{P}$ to the set of $P_{0\,\mathrm{ad}}$ structures on $\mathcal{P}_{\mathrm{ad}}$.

Definition 10.6.5. Let (G, σ) be an inner form of G_0 over X. For every standard parabolic subgroup $P_0 \subseteq G_0$, we let $\operatorname{Bun}_{G,P_0}(X)$ denote the stack whose R-valued points are pairs $(\mathcal{P}, P)$, where $\mathcal{P}$ is a G-bundle on X_R and $P \subseteq G_{\mathcal{P}}$ is a parabolic subgroup of type P_0. We will refer to $\operatorname{Bun}_{G,P_0}(X)$ as the *moduli stack of G-bundles with a P_0-structure.*

Warning 10.6.6. Though it is not apparent from our notation, the moduli stack $\operatorname{Bun}_{G,P_0}(X)$ depends on the choice of inner structure σ on G. Modifying σ by an element $g \in \operatorname{Out}(G_0)$ has the effect of replacing $\operatorname{Bun}_{G,P_0}(X)$ with $\operatorname{Bun}_{G,g(P_0)}(X)$.

Example 10.6.7. Let (G, σ) be an inner form of G_0 over X. If there exists a parabolic subgroup $P \subseteq G$ of type P_0, then Example 10.6.2 furnishes a canonical equivalence $\mathrm{Bun}_P(X) \simeq \mathrm{Bun}_{G,P_0}(X)$.

Example 10.6.8. Suppose that G_0 is an adjoint semisimple algebraic group over k, and let (G, σ) be an inner form of G_0 over X. It follows from Remark 10.6.3 that the bitorsor $\mathrm{Iso}^{\sigma}(G, G_0)$ determines an equivalence

$$\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_{G',P_0}(X),$$

where G' denotes the split form of G_0 over X. Combining this with Example 10.6.7, we obtain a canonical equivalence $\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_{P_0}(X)$.

Example 10.6.9. Let (G, σ) be an inner form of G_0 over X. Then Remark 10.6.4 furnishes a pullback diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \longrightarrow & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G_{\mathrm{ad}},P_{0\,\mathrm{ad}}}(X) & \longrightarrow & \mathrm{Bun}_{G_{\mathrm{ad}}}(X). \end{array}$$

Combining this with Example 10.6.8, we obtain an equivalence

$$\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_G(X) \times_{\mathrm{Bun}_{G_{\mathrm{ad}}}(X)} \mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X).$$

It follows from this that $\mathrm{Bun}_{G,P_0}(X)$ is an Artin stack which is locally of finite type over k, and that the diagonal of $\mathrm{Bun}_{G,P_0}(X)$ is affine.

Remark 10.6.10. If (G, σ) is an inner form of G_0 over X, then the moduli stack $\mathrm{Bun}_{G,P_0}(X)$ is smooth over $\mathrm{Spec}\, k$. We will not need this fact and therefore omit the proof.

Notation 10.6.11. Let (G, σ) be an inner form of G_0 over X. We let $\mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}}$ denote the fiber product

$$\mathrm{Bun}_{G,P_0}(X) \times_{\mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X)} \mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X)^{\mathrm{ss}}.$$

Then $\mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}}$ is an open substack of $\mathrm{Bun}_{G,P_0}(X)$ which we will refer to as the *semistable locus* of $\mathrm{Bun}_{G,P_0}(X)$.

For each element $\mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}$, we let $\mathrm{Bun}^{\nu}_{G,P_0}(X)$ denote the fiber product

$$\mathrm{Bun}_{G,P_0}(X) \times_{\mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X)} \mathrm{Bun}^{\nu}_{P_{0\,\mathrm{ad}}}(X).$$

Then each $\mathrm{Bun}^{\nu}_{G,P_0}(X)$ is an open substack of $\mathrm{Bun}_{G,P_0}(X)$, and we can identify $\mathrm{Bun}_{G,P_0}(X)$ with the disjoint union

$$\amalg_{\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}} \mathrm{Bun}^{\nu}_{G,P_0}(X).$$

We let $\mathrm{Bun}^{\nu}_{G,P_0}(X)^{\mathrm{ss}}$ denote the intersection $\mathrm{Bun}^{\nu}_{G,P_0}(X) \cap \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}}$. It follows from Theorem 10.3.13 and Example 10.6.9 that if $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}_{>0}$, then the canonical map

$$\mathrm{Bun}^{\nu}_{G,P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)$$

restricts to a finite radicial surjection

$$\mathrm{Bun}^{\nu}_{G,P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)_{P_0,\nu}.$$

We now consider the relationship between the moduli stacks $\mathrm{Bun}^{\nu}_{G,P_0}(X)$ as ν varies.

Definition 10.6.12. Let G be a reductive group scheme over a k-scheme Y, and let $P \subseteq G$ be a parabolic subgroup. Let $\operatorname{rad}_u(P)$ denote the unipotent radical of P, so that we have an exact sequence

$$0 \to \operatorname{rad}_u(P) \to P \xrightarrow{\pi} P/\operatorname{rad}_u(P) \to 0$$

A *Levi decomposition of* P is a section of the map π (which then determines a semidirect product decomposition $P \simeq \operatorname{rad}_u(P) \rtimes (P/\operatorname{rad}_u(P))$).

Remark 10.6.13. Suppose that G is a reductive group scheme over Y and that $P \subseteq G$ is a parabolic subgroup. Then we always find a Levi decomposition ψ of P locally for the étale topology: for example, if $T \subseteq P$ is a maximal torus and $Z \subseteq T$ is the preimage in T of the center of $P/\operatorname{rad}_u(P)$, then the centralizer of Z in P is a subgroup H for which the composite map

$$H \hookrightarrow P \to P/\operatorname{rad}_u P$$

is an isomorphism, so the inverse isomorphism $P/\operatorname{rad}_u P \simeq H \hookrightarrow P$ is a Levi decomposition of P. Moreover, if P admits a Levi decomposition $\psi : P/\operatorname{rad}_u P \to P$, then ψ is unique up to conjugation by a Y-valued point of $P/\operatorname{rad}_u P$. More precisely, the collection of Levi decompositions of P can be regarded as a torsor for $\operatorname{rad}_u(P)$ which is locally trivial for the étale topology. Since the unipotent radical $\operatorname{rad}_u(P)$ admits a finite filtration whose successive quotients are vector groups, it follows that this torsor is trivial whenever Y is affine (in particular, it is locally trivial with respect to the Zariski topology).

Notation 10.6.14. Let $\mathfrak{Z}_0$ denote the center of the reductive algebraic $P_0/\operatorname{rad}_u(P_0)$; this is a split diagonalizable group over k, and let $\Lambda = \operatorname{Hom}(\mathbf{G}_m, \mathfrak{Z}_0)$ denote the cocharacter lattice of $\mathfrak{Z}_0$. There is a canonical bilinear map of abelian groups

$$\operatorname{Hom}(P_0, \mathbf{G}_m) \times \Lambda \to \mathbf{Z},$$

which carries a pair (μ, λ) to the composite map

$$\mathbf{G}_m \xrightarrow{\lambda} \mathfrak{Z} \subseteq P_0/\operatorname{rad}_u(P_0) \xrightarrow{\mu} \mathbf{G}_m,$$

regarded as an element of $\operatorname{Hom}(\mathbf{G}_m, \mathbf{G}_m) \simeq \mathbf{Z}$. This bilinear map determines an injective map of lattices $\Lambda \hookrightarrow \operatorname{Hom}(P_0, \mathbf{G}_m)^\vee$. In what follows, we will generally abuse notation by identifying Λ with its image in $\operatorname{Hom}(P_0, \mathbf{G}_m)^\vee$. We let $\Lambda_{\geq 0}$ denote the inverse image of $\operatorname{Hom}(P, \mathbf{G}_m)^\vee_{\geq 0}$ under this map (in other words, the collection of those elements $\lambda \in \Lambda$ having the property that $\langle \alpha, \lambda \rangle \geq 0$ for every simple root of G_0).

Suppose that (G, σ) is an inner form of G_0 over a k-scheme Y, and let $P \subseteq G$ be a parabolic subgroup of type P_0. Then σ determines an isomorphism

$$\mathfrak{Z}(P/\operatorname{rad}_u(P)) \simeq \mathfrak{Z}_0 \times_{\operatorname{Spec} k} Y.$$

If $\psi : P/\operatorname{rad}_u(P) \to P$ is a Levi decomposition of P, then ψ restricts to a map of group schemes $\mathfrak{Z}_0 \times_{\operatorname{Spec} k} Y \to P$. In particular, every element $\lambda \in \Lambda$ determines a map $\mathbf{G}_m \to P$ of group schemes over Y, which we will denote by $\psi(\lambda)$.

Definition 10.6.15. Let Y be a scheme. An *effective Cartier divisor* on Y is a closed subscheme $D \subseteq Y$ for which the corresponding ideal sheaf $\mathcal{I}_D \subseteq \mathcal{O}_Y$ is invertible. A *local parameter* for D is a global section of $\mathcal{I}_D$ which generates $\mathcal{I}_D$ at every point.

Remark 10.6.16. Let Y be a scheme. Then every effective Cartier divisor $D \subseteq Y$ admits a local parameter locally with respect to the Zariski topology on Y.

Definition 10.6.17. Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $\mathcal{P}$ be a G-bundle on X equipped with a P_0-structure $P \subseteq G_{\mathcal{P}}$, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. A *$\lambda$-twist of* $(\mathcal{P}, P)$ *along* D is a pair $(\mathcal{P}', \gamma)$, where $\mathcal{P}'$ is a G-bundle on Y and γ is a G-bundle isomorphism

$$\mathcal{P} \times_Y (Y - D) \simeq \mathcal{P}' \times_Y (Y - D)$$

having the following property:

(∗) Let $U \subseteq Y$ be an open subset having the property that $P \times_Y U$ admits a Levi decomposition $\psi : (P/\operatorname{rad}_u P) \times_Y U \to P \times_Y U$ and the Cartier divisor $(D \cap U) \subseteq U$ admits a local parameter t. Then the G-bundle isomorphism

$$\mathcal{P} \times_Y (U - (D \cap U)) \overset{\psi(\lambda)(t)^{-1}}{\to} \mathcal{P} \times_Y (U - (D \cap U)) \overset{\gamma'}{\mathcal{P}} \times_Y (U - (D \cap U))$$

extends to a G-bundle isomorphism $\mathcal{P} \times_Y U \simeq \mathcal{P}' \times_Y U$.

Remark 10.6.18. In the situation of condition (∗) above, the extension $\overline{\gamma}_U$ is automatically unique (since the inclusion

$$\mathcal{P} \times_Y (U - (D \cap U)) \hookrightarrow \mathcal{P} \times_Y U$$

is complementary to a Cartier divisor, and therefore schematically dense).

Proposition 10.6.19. *Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $\mathcal{P}$ be a G-bundle on X equipped with a P_0-structure $P \subseteq G_{\mathcal{P}}$, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. Then there exists a G-bundle $\mathcal{P}'$ on Y and an isomorphism*

$$\gamma : \mathcal{P} \times_Y (Y - D) \simeq \mathcal{P}' \times_Y (Y - D)$$

for which the pair $(\mathcal{P}', \gamma)$ is a λ-twist of $\mathcal{P}$ along D. Moreover, the pair $(\mathcal{P}', \gamma)$ is unique up to unique isomorphism.

Lemma 10.6.20. *Let (G, σ) be an inner form of G_0 over an affine k-scheme $Y = \operatorname{Spec} R$, let $P \subseteq G$ be a parabolic subgroup of type P_0, and let $\psi : P/\operatorname{rad}_u P \to P$ be a Levi decomposition of P. For any element $\lambda \in \Lambda_{\geq 0}$ and any regular element $t \in R$, the automorphism of $P \times_{\operatorname{Spec} R} \operatorname{Spec} R[t^{-1}]$ given by conjugation by $\psi(\lambda)(t)$ extends to a group scheme homomorphism $P \to P$.*

Proof. The assertion is local with respect to the étale topology on $\operatorname{Spec} R$. We may therefore assume without loss of generality that $G = G_0 \times_{\operatorname{Spec} k} \operatorname{Spec} R$ and $P = P_0 \times_{\operatorname{Spec} k} \operatorname{Spec} R$, and that the image of ψ is $H_0 \times_{\operatorname{Spec} k} \operatorname{Spec} R$, where $H_0 \subseteq P_0$ is the unique Levi factor which contains the chosen maximal torus T_0.

Let $\{\alpha_1, \ldots, \alpha_m\}$ be an enumeration of the roots of P_0 which are not roots of H_0. For $1 \leq i \leq m$, let $f_i : \mathbf{G}_a \to P_0$ be a parametrization of the corresponding root space. It follows from the structure theory of reductive groups (and their parabolic subgroups) that the map

$$H_0 \times \mathbf{G}_a^m \to P_0$$

$$(h, y_1, \ldots, y_m) \mapsto h f_1(a_1) f_2(a_2) \ldots f_n(a_n)$$

is an isomorphism. In particular, for every R-algebra A, the preceding construction gives a bijection $H_0(A) \times A^m \to P(A)$. When A is an $R[t^{-1}]$-algebra, then conjugation by $\psi(\lambda)(t)$ determines an automorphism of $P(A)$ which corresponds (under the preceding bijection) to the bijection of $H_0(A) \times A^m$ with itself given by

$$(h, a_1, \ldots, a_m) \mapsto (h, \alpha_1(\psi(\lambda)(t))a_1, \ldots, \alpha_m(\psi(\lambda)(t))a_m).$$

Our assumption that $\lambda \in \Lambda_{\geq 0}$ guarantees that each $\alpha_i(\psi(\lambda)(t))$ belongs to the subset $R \subseteq R[t^{-1}]$, so that this construction extends to a morphism of R-schemes $P \to P$. This map is automatically a group homomorphism (this follows from the schematic density of $\operatorname{Spec} R[t^{-1}]$ in $\operatorname{Spec} R$). □

Proof of Proposition 10.6.19. The assertion is local on Y with respect to the Zariski topology. We may therefore assume that $Y = \operatorname{Spec} R$ is affine and that $D \subseteq Y$ is the Cartier divisor is the vanishing locus of a regular element $t \in R$. Since Y is affine, it admits a Levi decomposition $\psi : P/\operatorname{rad}_u P \to P$. In this case, we can take $\mathcal{P}' = \mathcal{P}$ and γ to be the automorphism of $\mathcal{P} \times_Y (Y - D)$ determined by the element $\psi(\lambda)(t)$. It follows tautologically that the pair $(\mathcal{P}, \gamma)$ is characterized uniquely up to unique isomorphism by the requirement that the composite map

$$\mathcal{P} \times_Y (Y - D) \xrightarrow{\psi(\lambda)(t)^{-1}} \mathcal{P} \times_Y (Y - D) \xrightarrow{\gamma} \mathcal{P}' \times_Y (Y - D)$$

extends to an isomorphism of $\mathcal{P}$ with $\mathcal{P}'$. To complete the proof, it will suffice to show that the pair $(\mathcal{P}', \gamma)$ is a λ-twist of $\mathcal{P}$ along D: that is, after replacing Y by any open subset $U \subseteq Y$ and choosing a different local parameter t' for D and a different Levi decomposition $\psi' : P/\operatorname{rad}_u P \to P$, the composite map

$$\mathcal{P} \times_Y (Y - D) \xrightarrow{\psi'(\lambda)(t')^{-1}} \mathcal{P} \times_Y (Y - D) \xrightarrow{\gamma} \mathcal{P}' \times_Y (Y - D)$$

also extends to G-bundle isomorphism of $\mathcal{P}$ with $\mathcal{P}'$. In other words, we wish to show that the difference $\psi'(\lambda)(t')^{-1}\psi(\lambda)(t)$ (which we regard as an element of the group $G_{\mathcal{P}}(R[t^{-1}])$) belongs to the subgroup $G_{\mathcal{P}}(R) \subseteq G_{\mathcal{P}}(R[t^{-1}])$. Note that we can write $t' = ut$, where $u \in R$ is a unit, so that

$$\psi'(\lambda)(t')^{-1}\psi(\lambda)(t) = \psi'(\lambda)(u)^{-1}\psi'(\lambda)(t)^{-1}\psi(\lambda)(t)$$

where the first factor belongs to $G_{\mathcal{P}}(R)$. It will therefore suffice to treat the case where $t' = t$.

Since Levi decompositions of P are unique up to the action of $\operatorname{rad}_u(P)$, we can choose an element $g \in \operatorname{rad}_u(P)(R) \subseteq P(R)$ such that $\psi'(\lambda)(t) = g\psi(\lambda)(t)g^{-1}$. We are therefore reduced to proving that $\psi(\lambda)(t)^{-1}g^{-1}\psi(\lambda)(t)$ belongs to $P(R)$, which follows from Lemma 10.6.20. □

Notation 10.6.21. Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. Suppose we are given a G-torsor $\mathcal{P}$ on Y and a P_0-structure $P \subseteq G_{\mathcal{P}}$. Proposition 10.6.19 implies that there exists an (essentially unique) λ-twist of $\mathcal{P}$ along D; we will denote the underlying G-bundle of this twist by $\operatorname{Tw}_{\lambda,D}(\mathcal{P}, P)$.

Example 10.6.22. Let $G_0 = P_0 = \mathbf{G}_m$ and let $\lambda \in \Lambda = \operatorname{Hom}(\mathbf{G}_m, \mathbf{G}_m)$ be the identity map. Then G_0 has a unique inner form G over any k-scheme Y (given by the multiplicative group over Y), and we can identify G-torsors with line bundles on Y. Any such torsor admits a unique P_0-structure. If $D \subseteq Y$ is an effective Cartier divisor and $\mathcal{L}$ is a line bundle on Y, then we have

$$\operatorname{Tw}_{\lambda,D}(\mathcal{L}) = \mathcal{L}(D) = \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{I}_D^{-1}.$$

In the situation of Notation 10.6.21, the twist $\operatorname{Tw}_{\lambda,D}(\mathcal{P}, P)$ comes equipped with a tautological isomorphism

$$\gamma : \operatorname{Tw}_{\lambda,D}(\mathcal{P}, P) \times_Y (Y - D) \simeq \mathcal{P} \times_Y (Y - D).$$

In particular, we obtain an isomorphism of group schemes

$$G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D) \simeq G_{\mathcal{P}} \times_Y (Y - D).$$

Under this isomorphism, the parabolic subgroup $P \subseteq \mathcal{G}_{\mathcal{P}}$ determines a parabolic subgroup $P_\gamma^\circ \subseteq G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D)$.

Proposition 10.6.23. *Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $\mathcal{P}$ be a G-bundle on X equipped with a P_0-structure $P \subseteq G_{\mathcal{P}}$, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. Then the subgroup $P_\gamma^\circ \subseteq G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D)$ can be extended uniquely to a parabolic subgroup $P_\gamma \subseteq G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)}$ of type P_0.*

Remark 10.6.24. The uniqueness assertion of Proposition 10.6.23 is immediate: if P°_γ can be extended to a parabolic subgroup $P_\gamma \subseteq G_{\mathrm{Tw}_{\lambda,D}(\mathcal{P},P)}$, then P_γ can be characterized as the scheme-theoretic closure of P°_γ in $G_{\mathrm{Tw}_{\lambda,D}(\mathcal{P},P)}$.

Proof of Proposition 10.6.23. By virtue of the uniqueness supplied by Remark 10.6.24, the assertion of Proposition 10.6.23 is local with respect to the étale topology on Y. We may therefore assume without loss of generality that the torsor $\mathcal{P}$ is trivial (so that $G_{\mathcal{P}} \simeq G$), that $P \subseteq G$ admits a Levi decomposition ψ, that $Y = \operatorname{Spec} R$ is affine, and that D is the vanishing locus of a regular element $t \in R$. In this case, the proof of Proposition 10.6.19 shows that we can take the twist $\mathrm{Tw}_{\lambda,D}(\mathcal{P}, P)$ to be the trivial G-torsor and γ to be the map given by right multiplication by $\psi(\lambda)(t)$. It follows that the isomorphism

$$G_{\mathrm{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D) \simeq G_{\mathcal{P}} \times_Y (Y - D).$$

corresponds to the automorphism of G given by conjugation by $\psi(\lambda)(t)$. Since $\psi(\lambda)(t)$ belongs to $P(R[t^{-1}])$, conjugation by $\psi(\lambda)(t)$ carries $P \times_Y (Y - D)$ into itself; we can therefore identify P°_γ with the subgroup $P \times_Y (Y - D) \subseteq G \times_Y (Y - D)$, which extends to the parabolic subgroup $P \subseteq G$. □

Construction 10.6.25. Let (G, σ) be an inner form of G_0 over the curve X, let $D \subseteq X$ be an effective divisor, and let λ be an element of $\Lambda_{\geq 0}$. If R is a finitely generated k-algebra, $\mathcal{P}$ is a G-bundle on X_R, and $P \subseteq G_{\mathcal{P}}$ is a P_0-structure on $\mathcal{P}$, then we can regard $\mathrm{Tw}_{\lambda,D_R}(\mathcal{P}, P)$ as another G-bundle on X_R, equipped with the P_0-structure P' supplied by Proposition 10.6.23. The construction

$$(\mathcal{P}, P) \mapsto (\mathrm{Tw}_{\lambda,D_R} \mathcal{P}, P')$$

depends functorially on R and therefore determines a map of algebraic stacks

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{G,P_0}(X) \to \mathrm{Bun}_{G,P_0}(X),$$

which we will refer to as *twisting by λ along D*.

Example 10.6.26. In the situation of Construction 10.6.25, suppose that $P_0 = G_0$, so that $\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_G(X)$. In this case, the element $\lambda \in \Lambda_{\geq 0} = \Lambda$ can be regarded as a cocharacter of the center $\mathfrak{Z}(G)$, which determines an action

$$m_\lambda : \mathrm{Bun}_{\mathbf{G}_m}(X) \times_{\operatorname{Spec} k} \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X).$$

Unwinding the definitions, we see that if $D \subseteq X$ is an effective divisor, then the map $\mathrm{Tw}_{\lambda,D}$: $\mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X)$ is given by $\mathcal{P} \mapsto m_\lambda(\mathcal{O}_X(D), \mathcal{P})$. In particular, $\mathrm{Tw}_{\lambda,D}$ is an automorphism of $\mathrm{Bun}_G(X)$ which preserves the semistable locus $\mathrm{Bun}_G(X)^{\mathrm{ss}}$ and restricts to equivalences

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}^{\nu}_G(X) \simeq \mathrm{Bun}^{\nu+\deg(D)\lambda}_G(X);$$

here we identify Λ with a sublattice of $\mathrm{Hom}(G_0, \mathbf{G}_m)^\vee$ as in Notation 10.6.14.

Example 10.6.27. In the special case where $G = G_0 \times_{\operatorname{Spec} k} X$ is the split form of G_0, we can regard Construction 10.6.25 as giving a map

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{P_0}(X) \to \mathrm{Bun}_{P_0}(X);$$

see Example 10.6.7.

Remark 10.6.28. In the situation of Example 10.6.27, let us abuse notation by identifying λ with an element of $\mathrm{Hom}(P_0/\mathrm{rad}_u P_0, \mathbf{G}_m)^\vee$. Then the diagram

$$\begin{CD} \mathrm{Bun}_{P_0}(X) @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}_{P_0}(X) \\ @VVV @VVV \\ \mathrm{Bun}_{P_0/\mathrm{rad}_u(P_0)}(X) @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}_{P_0/\mathrm{rad}_u(P_0)}(X) \end{CD}$$

commutes up to canonical isomorphism. Combining this observation with Example 10.6.26, we deduce that $\mathrm{Tw}_{\lambda,D}$ restricts to give maps

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}}$$

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}^{\nu}_{P_0}(X) \to \mathrm{Bun}^{\nu+\deg(D)\lambda}_{P_0}(X)$$

which fit into pullback squares

$$\begin{CD} \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}} @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}} \\ @VVV @VVV \\ \mathrm{Bun}_{P_0}(X) @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}_{P_0}(X) \end{CD}$$

$$\begin{CD} \mathrm{Bun}^{\nu}_{P_0}(X) @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}^{\nu}_{P_0}(X) \\ @VVV @VVV \\ \mathrm{Bun}_{P_0}(X) @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}_{P_0}(X). \end{CD}$$

Remark 10.6.29. In the situation of Construction 10.6.25, suppose that the algebraic group G_0 is semisimple and adjoint. Then the diagram

$$\begin{CD} \mathrm{Bun}_{G,P_0}(X) @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}_{G,P_0}(X) \\ @VVV @VVV \\ \mathrm{Bun}_{P_0}(X) @>{\mathrm{Tw}_{\lambda_{\mathrm{ad}},D}}>> \mathrm{Bun}_{P_0}(X) \end{CD}$$

commutes up to canonical isomorphism, where the vertical maps are the equivalences of Example 10.6.8.

Remark 10.6.30. Let (G, σ) be an inner form of G_0 over X, let $D \subseteq X$ be an effective divisor, and let $\lambda \in \Lambda_{\geq 0}$. Let λ_{ad} denote the image of λ in the lattice

$$\Lambda_{\mathrm{ad}} = \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(P_{0\,\mathrm{ad}}/\mathrm{rad}_u P_{0\,\mathrm{ad}})).$$

Then the diagram

$$\begin{CD} \mathrm{Bun}_{G,P_0}(X) @>{\mathrm{Tw}_{\lambda,D}}>> \mathrm{Bun}_{G,P_0}(X) \\ @VVV @VVV \\ \mathrm{Bun}_{G_{\mathrm{ad}},P_{0\,\mathrm{ad}}}(X) @>{\mathrm{Tw}_{\lambda_{\mathrm{ad}},D}}>> \mathrm{Bun}_{G_{\mathrm{ad}},P_{0\,\mathrm{ad}}}(X) \end{CD}$$

commutes up to canonical isomorphism. Combining this observation with Remarks 10.6.28 and 10.6.29, we conclude that the $\mathrm{Tw}_{\lambda,D}$ restricts to give maps

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}}$$

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}^{\nu}_{G,P_0}(X) \to \mathrm{Bun}^{\nu+\deg(D)\lambda}_{G,P_0}(X)$$

which fit into pullback squares

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}} & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \end{array}$$

$$\begin{array}{ccc} \mathrm{Bun}^{\nu}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}^{\nu}_{G,P_0}(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X). \end{array}$$

Remark 10.6.31 (Functoriality)**.** Let ψ be an automorphism of X as an abstract scheme, so that ψ determines an automorphism ψ_0 of the field $k = \mathrm{H}^0(X; \mathcal{O}_X)$ fitting into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\psi_0} & \operatorname{Spec} k. \end{array}$$

Let G be a form of G_0 over X equipped with an automorphism $\overline{\psi}$ compatible with the automorphism ψ of X. Then $\overline{\psi}$ determines an automorphism of the set of inner structures on G. In particular, if σ is an inner structure on G, then we can form a new inner structure $\overline{\psi}(\sigma)$ which can be written as $g\sigma$ for some unique element $g \in \mathrm{Out}(G_0)$. The pair $(\psi, \overline{\psi})$ determines an automorphism ϕ of $\mathrm{Bun}_G(X)$ which we can lift to an equivalence $\overline{\phi} : \mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_{G,g(P_0)}(X)$ (see Warning 10.6.6). Each element $\lambda \in \Lambda = \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(P_0/\operatorname{rad}_u(P_0))$ determines an element $g(\lambda) \in \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(g(P_0)/\operatorname{rad}_u g(P_0)))$, and each effective divisor $D \subseteq X$ determines a divisor $\phi(D) \subseteq X$. It follows immediately from our constructions that the diagram of algebraic stacks

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \\ \downarrow{\scriptstyle \overline{\phi}} & & \downarrow{\scriptstyle \overline{\phi}} \\ \mathrm{Bun}_{G,g(P_0)}(X) & \xrightarrow{\mathrm{Tw}_{g(\lambda),\phi(D)}} & \mathrm{Bun}_{G,g(P_0)}(X) \end{array}$$

commutes up to canonical isomorphism.

Remark 10.6.32 (Group Actions)**.** Let Γ be a finite group which acts on X as an abstract scheme, let G be a form of G_0 over X equipped with a compatible action of Γ, so that the group Γ acts on the moduli stack $\mathrm{Bun}_G(X)$.

Fix an inner structure σ on X. The collection of all inner structures on G forms a torsor Σ for the group $\mathrm{Out}(G_0)$, and the group Γ acts on Σ by $\mathrm{Out}(G_0)$-torsor automorphisms. The choice of element $\sigma \in \Sigma$ determines an isomorphism $\mathrm{Out}(G_0) \simeq \Sigma$ of $\mathrm{Out}(G_0)$-torsors, so that the action of Γ on Σ is via a group homomorphism $\Gamma \to \mathrm{Out}(G_0)$. Let P_0 be a standard parabolic subgroup of G_0 which is invariant under the action of Γ (by pinned automorphisms of G_0). Then the action of Γ on $\mathrm{Bun}_G(X)$ lifts canonically to an action of Γ on $\mathrm{Bun}_{G,P_0}(X)$.

The group Γ acts on the lattice $\Lambda = \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(P_0/\operatorname{rad}_u(P_0))$. Suppose that λ is a Γ-invariant element of $\lambda_{\geq 0}$, and let $D \subseteq X$ be an effective divisor which is Γ-invariant. Using the canonical isomorphisms of Remark 10.6.31, we can promote the map

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{G,P_0}(X) \to \mathrm{Bun}_{G,P_0}(X)$$

of Construction 10.6.25 to a Γ-equivariant morphism of algebraic stacks.

The main property of Construction 10.6.25 that we will need is the following:

Proposition 10.6.33. *Let Γ be a finite group acting on X via k-scheme automorphisms, let G be a Γ-equivariant group scheme over X, and let σ be an inner structure on G (so that the choice of σ determines a group homomorphism $\Gamma \to \mathrm{Out}(G_0)$). Let $P_0 \subseteq G_0$ be a standard parabolic which is Γ-invariant, let $\lambda \in \Lambda_{\geq 0}$ be Γ-invariant, and let $D \subseteq X$ be a Γ-invariant effective divisor, so that $\mathrm{Tw}_{\lambda,D}$ induces a map of (homotopy) fixed point stacks*

$$\phi : \mathrm{Bun}_{G,P_0}(X)^{\Gamma} \to \mathrm{Bun}_{G,P_0}(X)^{\Gamma}.$$

If D is étale over $\mathrm{Spec}\, k$ and the action of Γ on D is free, then ϕ is a fiber bundle (locally trivial in the étale topology) whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_P, \lambda\rangle$.

To prove Proposition 10.6.33, we may assume without loss of generality that the field k is separably closed. It will be convenient to introduce a local variant of Construction 10.6.25. For each point $x \in D$, choose a local coordinate t_x for X at the point x, so that the complete local ring $\mathcal{O}_x$ can be identified with the power series ring $k[[t_x]]$.

If R is a finitely generated k-algebra, we let $X^{\wedge}_{R,x}$ denote the formal completion of X_R along the closed subscheme $\{x\} \times_X \mathrm{Spec}\, R$, which we can identify with the formal spectrum $\mathrm{Spf}\, R[[t_x]]$. Let $\mathrm{Bun}_{G,P_0}(X^{\wedge}_x)$ denote the (non-algebraic) moduli stack of G-bundles on $X^{\wedge}_x$ equipped with a P_0-structure. More precisely, $\mathrm{Bun}_{G,P_0}(X^{\wedge}_x)$ denotes the stack whose R-valued points are pairs $(\mathcal{P}, P)$ where $\mathcal{P}$ is a G-bundle $\mathcal{P}$ on $\mathrm{Spec}\, R[[t_x]]$ (or equivalently on the formal scheme $\mathrm{Spf}\, R[[t_x]]$) equipped with a parabolic subgroup $P \subseteq G_{\mathcal{P}}$ of type P_0. If we let $D_{xR} \subseteq \mathrm{Spec}\, R[[t_x]]$ denote the Cartier divisor given by the vanishing locus of t_x, then a variant of Construction 10.6.25 determines a morphism of stacks

$$\mathrm{Tw}_{\lambda,\{x\}} : \mathrm{Bun}_{G,P_0}(X^{\wedge}_{\{x\}}) \to \mathrm{Bun}_{G,P_0}(X^{\wedge}_{\{x\}}).$$

We have a commutative diagram of stacks

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \\ \downarrow & & \downarrow \\ \prod_{x\in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x) & \xrightarrow{\mathrm{Tw}_{\lambda,\{x\}}} & \prod_{x\in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x). \end{array}$$

which is easily seen to be a pullback square (since the operation of twisting a G-bundle $\mathcal{P}$ by λ along D does not change $\mathcal{P}$ over the open set $X - D$). Passing to Γ-invariants, we obtain another pullback square

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X)^{\Gamma} & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X)^{\Gamma} \\ \downarrow & & \downarrow \\ (\prod_{x\in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x))^{\Gamma} & \xrightarrow{\mathrm{Tw}_{\lambda,\{x\}}} & (\prod_{x\in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x))^{\Gamma}. \end{array}$$

Let $D_0 \subseteq D$ denote a subset consisting of one element from each Γ orbit. Since the action of Γ on D is free, we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X)^{\Gamma} & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X)^{\Gamma} \\ \downarrow & & \downarrow \\ \prod_{x\in D_0} \mathrm{Bun}_{G,P_0}(X_x^{\wedge}) & \xrightarrow{\mathrm{Tw}_{\lambda,\{x\}}} & \prod_{x \in D_0} \mathrm{Bun}_{G,P_0}(X_x^{\wedge}). \end{array}$$

Consequently, Proposition 10.6.33 reduces to the following local assertion (which makes no reference to the group Γ):

Proposition 10.6.34. *In the situation above, each of the maps*

$$\mathrm{Tw}_{\lambda,\{x\}} : \mathrm{Bun}_{G,P_0}(X_x^{\wedge}) \to \mathrm{Bun}_{G,P_0}(X_x^{\wedge})$$

is a fiber bundle (locally trivial for the étale topology) whose fibers are affine spaces of dimension $\langle 2\rho_P, \lambda\rangle$.

Proof. Let t be a generator of the maximal ideal in the complete local ring $\mathcal{O}_x$. Since the power series ring $\mathcal{O}_x \simeq k[[t_x]]$ is strictly Henselian, the group scheme G splits over $\mathcal{O}_x$. Using Example 10.6.7, we obtain an identification $\mathrm{Bun}_{G,P_0}(X_x^{\wedge}) \simeq \mathrm{Bun}_{P_0}(X_x^{\wedge})$, where $\mathrm{Bun}_{P_0}(X_x^{\wedge})$ denotes the stack whose R-valued points are P_0-bundles on $X_{R,x}^{\wedge} \simeq \mathrm{Spf}\, R[[t]]$. Let $H_0 \subseteq P_0$ be the unique subgroup which contains the maximal torus T_0 and which maps isomorphically onto the reductive quotient $P_0/\mathrm{rad}_u(P_0)$. We will identify λ with a cocharacter of the center of H_0, so that we can identify $\lambda(t)$ with an element of the group $H_0(K_x)$.

Since P_0 is smooth over k, a P_0-bundle on $X_{R,x}^{\wedge}$ is trivial if and only if its restriction to the subscheme $\{x\} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R \subseteq X_R$ is trivial. In particular, any P_0-bundle on $X_{R,x}^{\wedge}$ can be trivialized locally with respect to the étale topology on $\mathrm{Spec}\, R$. Moreover, the automorphism group of the trivial P_0-bundle on $X_{R,x}^{\wedge} \simeq \mathrm{Spf}\, R[[t]]$ can be identified with the group $P_0(R[[t]])$. It follows that $\mathrm{Bun}_P(X_x^{\wedge})$ can be identified with the classifying stack (taken with respect to the étale topology) of the group-valued functor $R \mapsto P_0(R[[t]])$.

For every k-algebra R, let us view $P_0(R[[t]])$ as a subgroup of the larger group $P_0(R[[t]][t^{-1}])$. It follows from Lemma 10.6.20 that conjugation by $\lambda(t)$ determines a group homomorphism from $P_0(R[[t]])$ to itself; let us denote the image of this homomorphism by $P_0'(R[[t]])$.

Fix a map $\eta : \mathrm{Spec}\, R \to \mathrm{Bun}_{P_0}(X_x^{\wedge})$; we wish to show that the fiber product

$$Y = \mathrm{Bun}_{P_0}(X_x^{\wedge}) \times_{\mathrm{Bun}_{P_0}(X_x^{\wedge})} \mathrm{Spec}\, R$$

is representable by an affine R-scheme which is locally (with respect to the étale topology on $\mathrm{Spec}\, R$) isomorphic to $\mathbf{A}^{\langle 2\rho_{P_0}, \lambda\rangle}$. The map η classifies some P-bundle on $X_{Rx}^{\wedge}$, which we may assume to be trivial (after passing to an étale cover of $\mathrm{Spec}\, R$). Unwinding the definitions, we see that Y can be identified with the sheafification (with respect to the étale topology) of the functor

$$F : \mathrm{Ring}_R \to \mathcal{S}\mathrm{et}$$
$$F(A) = P_0(A[[t]])/P_0'(A[[t]]).$$

We will compete the proof by showing that that the functor F is representable by an affine space of dimension $\langle 2\rho_{P_0}, \lambda\rangle$ over $\mathrm{Spec}\, R$ (and is therefore already a sheaf with respect to the étale topology).

Let U denote the unipotent radical of P_0, so that $P_0(A[[t]])$ factors as a semidirect product $U(A[[t]]) \rtimes H_0(A[[t]])$. This decomposition is invariant under conjugation by $\lambda(t)$ and therefore determines an analogous decomposition $P_0'(A[[t]]) \simeq U'(A[[t]]) \rtimes H_0'(A[[t]])$. Since $\lambda(t)$ is central in $H_0(A[[t]][t_x^{-1}])$, we have $H_0'(A[[t]]) = H_0(A[[t]])$. It follows that the functor $F : \mathrm{Ring}_R \to \mathcal{S}\mathrm{et}$ above can be described by the formula $F(A) = U(A[[t]])/U'(A[[t]])$.

Let $\{\alpha_1, \ldots, \alpha_m\} \subseteq \mathrm{Hom}(T_0, \mathbf{G}_m)$ be the collection of roots of P_0 which are not roots of H_0. For $1 \leq i \leq m$, let $f_i : \mathbf{G}_a \to U$ be a parametrization of the root subgroup corresponding to α_i. For $A \in \mathrm{Ring}_R$, every element of the group $U(A[[t]])$ has a unique representation as a product

$$f_1(a_1(t))f_2(a_2(t)) \cdots f_m(a_m(t))$$

where $a_i(t) \in A[[t]]$. As in the proof of Lemma 10.6.20, we can identify $U'(A[[t]])$ with the subgroup of $U(A[[t]])$ spanned by those products where each $a_i(t)$ is divisible by $t^{\langle \alpha_i, \lambda \rangle}$.

Reordering the roots $\{\alpha_1, \ldots, \alpha_m\}$ if necessary, we may assume that for $0 \leq i \leq m$, the image of the map

$$\prod_{1 \leq i' \leq i} f_i : \mathbf{A}^i \to U_0$$

is a normal subgroup $U_i \subseteq U$. Then every $A[[t]]$-valued point of the quotient U/U_i has a unique representation as a product

$$f_{i+1}(a_{i+1}(t))f_{i+2}(a_{i+2}(t)) \cdots f_m(a_m(t))$$

where $a_j(t) \in A[[t]]$. Let $V_i(A)$ denote the subgroup of $(U/U_i)(A[[t]])$ consisting of those products where each $a_j(t)$ is divisible by $t^{\langle \alpha_j, \lambda \rangle}$, and let $F_i : \mathrm{Ring}_R \to \mathcal{S}\mathrm{et}$ be the functor given by $F_i(A) = (U/U_i)(A[[t]])/V_i(A)$. We will prove the following:

($*$) For $0 \leq i \leq n$, the functor F_i is representable by an affine space of dimension

$$\sum_{i < j \leq m} \langle \alpha_j, \lambda \rangle$$

over $\mathrm{Spec}\, R$.

Note that $F_0 = F$ and that

$$\sum_{1 \leq j \leq m} \langle \alpha_j, \lambda \rangle = \langle 2\rho_P, \lambda \rangle,$$

so that when $i = 0$ assertion ($*$) asserts that F is representable by an affine space of dimension $\langle 2\rho_P, \lambda \rangle$ over $\mathrm{Spec}\, R$. We will prove ($*$) by descending induction on i, the case $i = m$ being trivial. To carry out the inductive step, we note that for $1 \leq i \leq m$ we have natural exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t^{\langle \alpha_i, \lambda \rangle} A[[t]] & \longrightarrow & V_{i-1}(A) & \longrightarrow & V_i(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A[[u]] & \longrightarrow & (U/U_{i-1})(A[[u]]) & \longrightarrow & (U/U_i)(A[[u]]) & \longrightarrow & 0 \end{array}$$

where the vertical maps are injective and each of the exact sequences is a central extension. It follows from a diagram chase that we can identify $F_i(A)$ with the quotient of $F_{i-1}(A)$ by a free action of the quotient $A[[t]]/t^{\langle \alpha_i, \lambda \rangle} A[[t]]$, and that this identification depends functorially on A. In other words, the functor F_{i-1} can be identified with a $\mathbf{G}_a^{\langle \alpha_i, \lambda \rangle}$-torsor over F_i. By the inductive hypothesis, the functor F_i is representable by an affine scheme, so that any $\mathbf{G}_a$-torsor over F_i is trivial. We therefore obtain

$$\begin{aligned} F_{i-1} &\simeq \mathbf{G}_a^{\langle \alpha_i, \lambda \rangle} \times F_i \\ &\simeq \mathbf{A}^{\langle \alpha_i, \lambda} \times (\mathbf{A}^{\sum_{i < j \leq m} \langle \alpha_j, \lambda} \times \mathrm{Spec}\, R) \\ &\simeq \mathbf{A}^{\sum_{i \leq j \leq m} \langle \alpha_j, \lambda \rangle} \times \mathrm{Spec}\, R, \end{aligned}$$

as desired. □

10.7. **Reductive Models.** Throughout this section, we fix a field k and an algebraic curve X over k. Let K_X be the fraction field of X and let G_0 be a reductive algebraic group over K_X. If the field k is algebraically closed, then the function field K_X has dimension ≤ 1, so the algebraic group G_0 is quasi-split. It follows that there is a finite Galois extension L of K_X, a split reductive group G over k on which $\mathrm{Gal}(L/K_X)$ acts by pinned automorphisms, and a $\mathrm{Gal}(L/K_X)$-equivariant isomorphism

$$G_0 \times_{\operatorname{Spec} K_X} \operatorname{Spec} L \simeq G \times_{\operatorname{Spec} k} \operatorname{Spec} L.$$

In particular, there is $\mathrm{Gal}(L/K_X)$-equivariant isomorphism of $G_0 \times_{\operatorname{Spec} K_X} \operatorname{Spec} L$ with the generic fiber of the split reductive group scheme $G \times_{\operatorname{Spec} k} \widetilde{X}$, where $\widetilde{X}$ denotes the algebraic curve with function field L. Our goal in this section is to establish an analogous (but weaker) result which does not require the assumption that k is algebraically closed:

Proposition 10.7.1. *Let G_0 be a reductive algebraic group over the fraction field K_X. Then there exists a finite Galois extension L of K_X, a reductive group scheme G over the curve $\widetilde{X}$ with function field L, an action of $\mathrm{Gal}(L/K_X)$ on G (compatible with the tautological action of $\mathrm{Gal}(L/K_X)$ on $\widetilde{X}$), and a $\mathrm{Gal}(L/K_X)$-equivariant isomorphism*

$$G_0 \times_{\operatorname{Spec} K_X} \operatorname{Spec} L \simeq G \times_{\widetilde{X}} \operatorname{Spec} L.$$

Remark 10.7.2. In the situation of Proposition 10.7.1, if L is a Galois extension of K_X for which there exists a $\mathrm{Gal}(L/K_X)$-equivariant group scheme on the associated algebraic curve $\widetilde{X}$ whose generic fiber is $\mathrm{Gal}(L/K_X)$-equivariantly isomorphic to $G_0 \times_{\operatorname{Spec} K_X} \operatorname{Spec} L$, then any larger Galois extension L' has the same property (the inclusion $L \hookrightarrow L'$ induces a map of algebraic curves $\widetilde{X}' \to \widetilde{X}$, and the pullback $G \times_{\widetilde{X}} \widetilde{X}'$ is a reductive group scheme over $\widetilde{X}'$ having the desired properties). We are therefore free to assume that the Galois extension L appearing in Proposition 10.7.1 is as large as we like: in particular, we may assume that the algebraic group G_0 splits over L.

Warning 10.7.3. In the situation of Proposition 10.7.1, the algebraic curve $\widetilde{X}$ is connected, but need not be geometrically connected (when regarded as a k-scheme). If we want to guarantee that the generic fiber of G is split reductive (as in Remark 10.7.2), this is unavoidable: if k' is a Galois extension of k and the group scheme G_0 is obtained by Weil restriction along the field extension $K_X \hookrightarrow K_X \otimes_k k'$, then any Galois extension L of K_X which splits G_0 must contain k'.

Remark 10.7.4. There are two main differences between Proposition 10.7.1 (which applies over any ground field k) and the discussion which precedes it (which applies when the ground field k is algebraically closed):

- Proposition 10.7.1 guarantees the existence of a reductive group scheme G over $\widetilde{X}$, but does not guarantee that this reductive group scheme is constant (though we can arrange that it is split at the generic point of $\widetilde{X}$, by virtue of Remark 10.7.2).
- Proposition 10.7.1 gives no information about the action of $\mathrm{Gal}(L/K_X)$ on the group scheme G: in particular, this action need not preserve a pinning of G, even at the generic point of $\widetilde{X}$.

We will deduce Proposition 10.7.1 from the following local assertion:

Lemma 10.7.5. *Let K be the fraction field of a complete discrete valuation ring $\mathcal{O}_K$ and let G_0 be a reductive algebraic group over K. Then there exists a finite Galois extension L of K, a* *reductive group scheme G over the ring of integers $\mathcal{O}_L \subseteq L$ equipped with an action of* $\mathrm{Gal}(L/K)$ *(compatible with the tautological action of* $\mathrm{Gal}(L/K)$ *over* $\mathcal{O}_L$*), and a* $\mathrm{Gal}(L/K)$*-equivariant isomorphism*

$$G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \simeq G \times_{\mathrm{Spec}\, \mathcal{O}_L} \mathrm{Spec}\, L.$$

Proof. Choose a maximal torus $T \subseteq G_0$ which is defined over K. Let L_0 be a finite Galois extension of K for which the torus T splits over L_0. We will show that if L is a Galois extension of K which contains L_0 and whose ramification degree over L_0 is divisible by $\deg(L_0/K)$, then L has the desired property.

For every Galois extension L of K which contains L_0, let $M(L)$ denote the set of isomorphism classes of pairs (G, α), where G is a reductive group scheme over $\mathcal{O}_L$ and α is an isomorphism (not assumed to be $\mathrm{Gal}(L/K)$-equivariant)

$$G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \simeq G \times_{\mathrm{Spec}\, \mathcal{O}_L} \mathrm{Spec}\, L$$

of reductive algebraic groups over L which has the following additional property: the scheme-theoretic image of composite map

$$T \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \hookrightarrow G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \xrightarrow{\alpha} G$$

is a torus $\overline{T}$ (automatically split) over $\mathrm{Spec}\, \mathcal{O}_L$. The group $\mathrm{Gal}(L/K)$ acts on the set $M(L)$; to prove Lemma 10.7.5, it will suffice to show that there is an element of $M(L)$ which is fixed by $\mathrm{Gal}(L/K)$ (provided that L is sufficiently large).

Let G_{ad} denote the quotient of G_0 by its center, and let T_{ad} denote the image of T in G_{ad}. For every element $g \in T_{\mathrm{ad}}(L)$, conjugation by g determines an automorphism c_g of $G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ which acts trivially on T. The construction $(G, \alpha) \mapsto (G, \alpha \circ c_g)$ determines an action of $T_{\mathrm{ad}}(L)$ on the set $M(L)$. We claim that this action is transitive: in other words, given any pair of elements $(G, \alpha), (G', \alpha') \in M(L)$, there exists an isomorphism $\beta : G \simeq G'$ such that the composite map

$$G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \xrightarrow{\alpha} G \times_{\mathrm{Spec}\, \mathcal{O}_L} \mathrm{Spec}\, L \xrightarrow{\beta} G' \xrightarrow{\alpha'^{-1}} G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$$

is given by conjugation by some $g \in T_{\mathrm{ad}}(L)$. To prove this, let $\overline{T} \subseteq G$ and $\overline{T}' \subseteq G'$ denote the scheme-theoretic images of $T \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ under α and α', respectively. Then $\overline{T}$ and $\overline{T}'$ are split tori over $\mathcal{O}_L$, so that the identification between their generic fibers (supplied by α and α') extends uniquely to an isomorphism $\beta_0 : \overline{T} \simeq \overline{T}'$. Let B be a Borel subgroup of $G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ containing $T \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$. Since $\overline{T}$ is a maximal split torus in G, there is a unique Borel subgroup $\overline{B} \subseteq G$ containing $\overline{T}$ with $\alpha^{-1}\overline{B} = B$. Similarly, there is a unique Borel subgroup $\overline{B}' \subseteq G'$ which contains $\overline{T}'$ satisfying $\alpha'^{-1}\overline{B}' = B$. Since the ring of integers $\mathcal{O}_L$ is a discrete valuation ring, the pairs $(\overline{T}, \overline{B})$ and $(\overline{T}', \overline{B}')$ can be extended to pinnings of the group schemes G and G', respectively. It follows that there is a unique pinned isomorphism $\beta : G \to G'$ which restricts to the identity on the Dynkin diagram of their common generic fiber $G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$. By construction, the composition

$$G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L \xrightarrow{\alpha} G \times_{\mathrm{Spec}\, \mathcal{O}_L} \mathrm{Spec}\, L \xrightarrow{\beta} G' \xrightarrow{\alpha'^{-1}} G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$$

is an automorphism of $G_0 \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ which restricts to the identity on $T \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$, and is therefore given by conjugation by some element $g \in G_{\mathrm{ad}}(L)$. Since g centralizes $T \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$, it belongs to the subgroup $T_{\mathrm{ad}}(L) \subseteq G_{\mathrm{ad}}(L)$.

Let $\Lambda = \mathrm{Hom}(\mathbf{G}_m, T_{\mathrm{ad}} \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L)$ denote the cocharacter lattice of the split torus $T_{\mathrm{ad}} \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$, so that we can identify $T_{\mathrm{ad}}(L)$ with the tensor product $\Lambda \otimes L^{\times}$. Note that if (G, α) is any element of $M(L)$, then (G, α) is isomorphic to $(G, \alpha \circ c_g)$ if and only if conjugation by the element $g \in T_{\mathrm{ad}}(L)$ extends to an automorphism of the group scheme G: this is equivalent to the assertion that for each root α of the split group $G_0 \times_{\operatorname{Spec} K} \operatorname{Spec} L$, the induced map

$$\Lambda \otimes L^{\times} \xrightarrow{\alpha} L^{\times}$$

belongs to $\mathcal{O}_L^{\times}$, which is equivalent to the assertion that g belongs to the subgroup

$$\Lambda \otimes \mathcal{O}_L^{\times} \subseteq \Lambda \otimes L^{\times} \simeq T_{\mathrm{ad}}(L).$$

It follows that we can regard $M(L)$ as a torsor for the quotient group

$$(\Lambda \otimes L^{\times})/(\Lambda \otimes \mathcal{O}_L^{\times}) \simeq \Lambda \otimes \mathbf{Z}_L,$$

where $\mathbf{Z}_L = L^{\times}/\mathcal{O}_L^{\times}$ denotes the value group of L. Note that the group $\mathbf{Z}_L$ is canonically isomorphic to $\mathbf{Z}$ (so $\Lambda \otimes \mathbf{Z}_L$ is canonically isomorphic to Λ); however, in the arguments which follow, it will be convenient not to make use of this.

Let us fix an element $x_0 \in M(L_0)$, which determines an element $x_L \in M(L)$ for every finite extension L of L_0. The action of $\Lambda \otimes \mathbf{Z}_L$ on $M(L)$ determines a bijective map

$$\gamma_L : \Lambda \otimes \mathbf{Z}_L \to M(L)$$

$$\gamma_L(0) = x_L.$$

The set $M(L)$ admits a unique abelian group structure for which γ_L is an isomorphism of abelian groups. Note that if L is an extension of L_0 having ramification degree d, then we can identify $\mathbf{Z}_L$ with $\frac{1}{d}\mathbf{Z}_{L_0}$. It follows that for any element $y \in M(L_0)$, the image of y in $M(L)$ is divisible by d.

The action of the Galois group $\operatorname{Gal}(L/K)$ on $M(L)$ does not preserve the group structure on $M(L)$ (because the element x_L is not necessarily $\operatorname{Gal}(L/K)$-invariant). However, the action of $\operatorname{Gal}(L/K)$ is affine-linear: that is, for each $g \in \operatorname{Gal}(L/K)$ we have the identity $g(y+z) = g(y) + g(z) - g(0)$ in $M(L)$. Suppose that L is a Galois extension of K having ramification degree divisible by $\deg(L_0/K)$. Then for each $g \in \operatorname{Gal}(L_0/K)$, the image of $g(x_0)$ in $M(L)$ is divisible by $\deg(L_0/K)$. It follows that the average

$$\sum_{g \in \operatorname{Gal}(L_0/K)} \frac{g(x_0)}{\deg(L_0/K)}|$$

is a well-defined element of $M(L)$, and this element is clearly fixed under the action of $\operatorname{Gal}(L/K)$. □

Proof of Proposition 10.7.1. Let G_0 be a reductive algebraic group over K_X. Then we can choose a dense open subset $U \subseteq X$ such that G_0 extends to a reductive group scheme G_U over U. Let S denote the finite set of closed points of X which do not belong to U. For each $x \in S$, let $\mathcal{O}_x$ denote the complete local ring of X at the point x, let K_x denote its fraction field, and let $G_{0x} = G_0 \times_{\operatorname{Spec} K_X} \operatorname{Spec} K_x$ be the associated reductive algebraic group over K_x. It follows from Lemma 10.7.5 that for each point $x \in X$, there exists a finite Galois extension L_x of K_x, a reductive algebraic group G_x over the ring of integers $\mathcal{O}_{L_x}$, an action of $\operatorname{Gal}(L_x/K_x)$ on G_x (compatible with its action on $\mathcal{O}_{L_x}$), and a $\operatorname{Gal}(L_x/K_x)$-equivariant isomorphism

$$G_{0x} \times_{\operatorname{Spec} K_x} \operatorname{Spec} L_x \simeq G_x \times_{\operatorname{Spec} \mathcal{O}_{L_x}} \operatorname{Spec} L_x.$$

Let L be a Galois extension of K_X which is large enough that for each $x \in S$, the tensor product $L \otimes_{K_X} K_x$ contains an isomorphic copy of L_x. Enlarging the fields L_x if necessary (see Remark 10.7.2), we may assume that each L_x appears as a direct factor in the tensor product $L \otimes_{K_X} K_x$. Then L is the fraction field of an algebraic curve $\widetilde{X}$ (which is not necessarily geometrically connected over k). Let $\widetilde{U}$ denote the inverse image of U in $\widetilde{K}$, so that the pullback of G_U determines a $\mathrm{Gal}(L/K_X)$-equivariant reductive group scheme $G_{\widetilde{U}}$ over $\widetilde{U}$. To complete the proof, it will suffice to show that $G_{\widetilde{U}}$ admits a $\mathrm{Gal}(L/K_X)$-equivariant extension to a reductive group scheme over $\widetilde{X}$. To construct such an extension, it suffices to show that we can solve the analogous problem after replacing X by $\operatorname{Spec} \mathcal{O}_x$ for $x \in S$, which is precisely the content of Lemma 10.7.5. □

10.8. **Proof of the Trace Formula.** Throughout this section, we fix a finite field $\mathbf{F}_q$ with q elements, an algebraic closure $\overline{\mathbf{F}}_q$ of q, a prime number ℓ which is relatively prime to q, an embedding $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$, an algebraic curve X over $\mathbf{F}_q$, and a smooth affine group scheme G over X. Assume that G has connected fibers and that the generic fiber of G is semisimple. Our goal is to prove Theorem 10.0.6, which asserts that the moduli stack $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula. Our strategy will be to show that after suitably modifying G, the moduli stack $\mathrm{Bun}_G(X)$ admits a convergent stratification (Definition 10.2.9).

Let K_X denote the fraction field of X. According to Proposition 10.7.1, there is a finite Galois extension L of K_X, where L is the function field of an algebraic curve $\widetilde{X}$, a semisimple group scheme $\widetilde{G}$ over $\widetilde{X}$ equipped with a compatible action of $\mathrm{Gal}(L/K_X)$, and a $\mathrm{Gal}(L/K_X)$-equivariant isomorphism

$$G \times_X \operatorname{Spec} L \simeq \widetilde{G} \times_{\widetilde{X}} \operatorname{Spec} L.$$

Moreover, we may further assume that the generic fiber of $\widetilde{G}$ is split (Remark 10.7.2). The algebraic curve $\widetilde{X}$ is not necessarily geometrically connected when regarded as an $\mathbf{F}_q$-scheme. The algebraic closure of $\mathbf{F}_q$ in L is a finite field $\mathbf{F}_{q^d}$ with q^d elements for some $d \geq 0$; let us fix an embedding of this field into $\overline{\mathbf{F}}_q$.

Let $X' = X \times_{\operatorname{Spec} \mathbf{F}_q} \operatorname{Spec} \overline{\mathbf{F}}_q$ and let $G' = G \times_X \overline{X}$; similarly we define $\widetilde{X}' = \widetilde{X} \times_{\operatorname{Spec} \mathbf{F}_{q^d}} \operatorname{Spec} \overline{\mathbf{F}}_q$ and $\widetilde{G}' = \widetilde{G} \times_{\widetilde{X}} \widetilde{X}'$. For each effective divisor $Q \subseteq \overline{X}$, let $D^Q(G')$ denote the group scheme over $\overline{X}$ obtained by dilitation of G' along its identity section (Variant A.3.9). Using Proposition A.3.11, we see that if Q is large enough, then the equivalence $G \times_X \operatorname{Spec} L \simeq \widetilde{G} \times_{\widetilde{X}} \operatorname{Spec} L$ extends to a homomorphism

$$\overline{\beta} : D^Q(G') \times_{X'} \widetilde{X}' \to \widetilde{G}'.$$

Enlarging Q if necessary, we may assume that Q is invariant under the action of the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$, so that the group scheme $D^Q(G')$ and the map β are defined over $\mathbf{F}_q$: that is, we have a dilitation $D^Q(G) \to G$ and a map of group schemes

$$D^Q(G) \times_X \widetilde{X} \to \widetilde{G}$$

which is an isomorphism at the generic point of $\widetilde{X}$. According to Corollary 10.1.5, to prove that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula, it will suffice to show that $\mathrm{Bun}_{D^Q(G)}(X)$ satisfies the Grothendieck-Lefschetz trace formula. We may therefore replace G by $D^Q(G)$ and thereby reduce to the case where there exists a homomorphism of group schemes

$$\beta : G \times_X \widetilde{X} \to \widetilde{G}$$

which is an isomorphism at the generic point of $\widetilde{X}$. Note that β is automatically $\mathrm{Gal}(L/K_X)$-equivariant (since this can be tested at the generic point of X).

Let G_0 denote the split form of $\widetilde{G}$, which we regard as a semisimple algebraic group over $\mathbf{F}_q$. Fix a Borel subgroup $B_0 \subseteq G_0$ and a split maximal torus $T_0 \subseteq B_0$. Let Σ denote the set of inner structures on the group scheme $\widetilde{G}$ (Definition 10.5.2). Since the generic fiber of $\widetilde{G}$ is split, the set Σ is nonempty (Example 10.5.4), and is therefore a torsor for the outer automorphism group $\mathrm{Out}(G_0)$ (Remark 10.5.5). Fix an element $\sigma \in \Sigma$, which supplies an isomorphism $\mathrm{Out}(G_0) \simeq \Sigma$

of $\mathrm{Out}(G_0)$-torsors. The group $\mathrm{Gal}(L/K_X)$ acts on the pair $(\widetilde{X}, \widetilde{G})$ and therefore acts on the set Σ by $\mathrm{Out}(G_0)$-torsor automorphisms; let us identify this action with a group homomorphism $\rho : \mathrm{Gal}(L/K_X) \to \mathrm{Out}(G_0)$.

Let $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ denote the moduli stack of $\widetilde{G}$-bundles on $\widetilde{X}$, where we regard $\widetilde{X}$ as a geometrically connected algebraic curve over $\mathrm{Spec}\,\mathbf{F}_{q^d}$ (so that $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ is a smooth Artin stack over $\mathbf{F}_{q^d}$). Let A denote the set of all pairs (P_0, ν), where P_0 is a parabolic subgroup of G_0 which contains B_0 and $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}_{>0}$. We let $\{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}\}_{(P_0,\nu)\in A}$ denote the Harder-Narasimhan stratification of $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ determined by the choice of inner structure $\sigma \in \Sigma$ (Construction 10.5.8). The Galois group $\mathrm{Gal}(L/K_X)$ acts on the moduli stack $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$. According to Remark 10.5.12, this action permutes the Harder-Narasimhan strata (via the action of $\mathrm{Gal}(L/K_X)$ on A determined by the homomorphism ρ).

The Galois group $\mathrm{Gal}(\mathbf{F}_{q^d}/\mathbf{F}_q)$ is canonically isomorphic to the cyclic group $\mathbf{Z}/d\mathbf{Z}$ generated by the Frobenius map $t \mapsto t^q$, and the Galois group $\mathrm{Gal}(L/K_X)$ fits into a short exact sequence

$$0 \to \Gamma \to \mathrm{Gal}(L/K_X) \to \mathbf{Z}/d\mathbf{Z} \to 0$$

where Γ denotes the Galois group of L over $K_X \otimes_{\mathbf{F}_q} \mathbf{F}_{q^d}$. Let A^Γ denote the set of fixed points for the action of Γ on A. According to Remark 10.2.6, the homotopy fixed point stack $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma$ inherits a stratification by locally closed substacks

$$\{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma_{P_0,\nu}\}_{(P_0,\nu)\in A^\Gamma},$$

where each stratum $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma_{P_0,\nu}$ is given by

$$((\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^\Gamma)_{\mathrm{red}}.$$

We will regard $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma$ as an algebraic stack (not necessarily smooth) over $\mathrm{Spec}\,\mathbf{F}_{q^d}$. This algebraic stack inherits a residual action of the group $\mathbf{Z}/d\mathbf{Z}$ (compatible with the action of $\mathbf{Z}/d\mathbf{Z} \simeq \mathrm{Gal}(\mathbf{F}_{q^d}/\mathbf{F}_q)$ on $\mathrm{Spec}\,\mathbf{F}_{q^d}$), so we can consider the stack-theoretic quotient

$$\mathcal{X} = \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma/(\mathbf{Z}/d\mathbf{Z})$$

as an algebraic stack over $\mathbf{F}_q$. Moreover, we have

$$\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma \simeq \mathcal{X} \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\mathbf{F}_{q^d}.$$

Remark 10.8.1. Let $\overline{X}$ denote the stack-theoretic quotient of $\widetilde{X}$ by the action of $\mathrm{Gal}(L/K_X)$. Then $\overline{X}$ is a "stacky curve" over $\mathrm{Spec}\,\mathbf{F}_q$, and there is a natural map $\pi : \overline{X} \to X$ which exhibits X as the coarse moduli space of X'. The action of $\mathrm{Gal}(L/K_X)$ on $\widetilde{G}$ allows us to descend $\widetilde{G}$ to an affine group scheme $\overline{G} = \widetilde{G}/\mathrm{Gal}(L/K_X)$ over $\overline{X}$, and we can think of $\mathcal{X}$ as the moduli stack (defined over $\mathbf{F}_q$) of $\overline{G}$-bundles on $\overline{X}$.

Let $A^\Gamma/(\mathbf{Z}/d\mathbf{Z})$ be the quotient of A^Γ by the action of $\mathbf{Z}/d\mathbf{Z}$; for each object $(P_0,\nu) \in A^\Gamma$, we let $[P_0,\nu]$ denote the image of (P_0,ν) in the quotient $A^\Gamma/(\mathbf{Z}/d\mathbf{Z})$. It follows from Remark 10.2.8 that $\mathcal{X}$ inherits a stratification by locally closed substacks $\{\mathcal{X}_{[P_0,\nu]}\}_{[P_0,\nu]\in A^\Gamma/(\mathbf{Z}/d\mathbf{Z})}$, where each $\mathcal{X}_{[P_0,\nu]}$ can be identified with the quotient

$$\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma_{P_0,\nu}/H$$

where H denotes the subgroup of $\mathbf{Z}/d\mathbf{Z}$ which stabilizes the element $(P_0,\nu) \in A^\Gamma$.

Let $\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}}$ denote the fiber product $\mathrm{Bun}_G(X) \times_{\mathrm{Spec}\,\mathbf{F}_q} \mathrm{Spec}\,\mathbf{F}_{q^d}$. The $\mathrm{Gal}(L/K_X)$-equivariant map $\beta : G \times_X \widetilde{X} \to \widetilde{G}$ induces a morphism of algebraic stacks

$$\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma$$

over $\mathbf{F}_{q^d}$ which descends to a morphism $\mathrm{Bun}_G(X) \to \mathcal{X}$ of algebraic stacks over $\mathbf{F}_q$. Applying Remark 10.2.5, we obtain a stratification of $\mathrm{Bun}_G(X)$ by locally closed substacks

$$\{\mathrm{Bun}_G(X)_{[P_0,\nu]}\}_{[P_0,\nu]\in A^\Gamma/(\mathbf{Z}/d\mathbf{Z})},$$

where

$$\mathrm{Bun}_G(X)_{[P_0,\nu]} = (\mathrm{Bun}_G(X) \times_{\mathcal{X}} \mathcal{X}_{[P_0,\nu]})_{\mathrm{red}}.$$

By virtue of Proposition 10.2.11, in order to prove that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula, it will suffice to verify the following:

Proposition 10.8.2. *The stratification* $\{\mathrm{Bun}_G(X)_{[P_0,\nu]}\}_{[P_0,\nu]\in A^\Gamma/(\mathbf{Z}/d\mathbf{Z})}$ *is convergent, in the sense of Definition 10.2.9.*

The remainder of this section is devoted to the proof of Proposition 10.8.2. We begin by analyzing the Harder-Narasimhan stratification of $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$. Recall that for each $(P_0,\nu) \in A$, the Harder-Narasimhan stratum $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}$ is equipped with a finite surjective radicial map

$$\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}} \to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}$$

(see Notation 10.6.11). If $(P_0,\nu) \in A^\Gamma$, then the group Γ acts on both $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}}$ and $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}$ (via its action on $\widetilde{X}$ as an algebraic curve over $\operatorname{Spec}\mathbf{F}_{q^d}$ together with its action on the group G_0 via the homomorphism ρ), and therefore determines a map of fixed point stacks

$$\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \to (\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^\Gamma.$$

Lemma 10.8.3. *For each* $(P_0,\nu) \in A^\Gamma$*, the map*

$$\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \to (\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^\Gamma.$$

is a finite radicial surjection.

Proof. Choose a map $\operatorname{Spec} R \to (\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^\Gamma$ set $Y = \operatorname{Spec} R \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}} \mathrm{Bun}^{\nu}_{G,P_0}(\widetilde{X})^{\mathrm{ss}}$. Theorem 10.3.13 implies that Y is a scheme and the map $Y \to \operatorname{Spec} R$ is surjective, finite, and radicial (see Notation 10.6.11). The group Γ acts on Y, and we have

$$\operatorname{Spec} R \times_{(\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^\Gamma} \mathrm{Bun}^{\nu}_{G,P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \simeq Y^\Gamma.$$

To prove Lemma 10.8.3, we must show that the map $Y^\Gamma \to \operatorname{Spec} R$ is also surjective, finite, and radicial. Since Y^Γ can be identified with a closed subscheme of Y, the only nontrivial point is to prove surjectivity. Fix an algebraically closed field κ and a map $\eta : \operatorname{Spec}\kappa \to \operatorname{Spec} R$. Since the map $Y \to \operatorname{Spec} R$ is a radicial surjection, the map η lifts *uniquely* to a map $\overline{\eta} : \operatorname{Spec}\kappa \to Y$. It follows from the uniqueness that $\overline{\eta}$ is invariant under the action of Γ, and therefore factors through the closed subscheme $Y^\Gamma \subseteq Y$. □

Construction 10.8.4. The stratification of $\mathrm{Bun}_G(X)$ that we have defined above is obtained by pulling back the Harder-Narasimhan stratification of $\mathrm{Bun}_{G_0}(\widetilde{X})$ along a certain $(\mathbf{Z}/d\mathbf{Z})$-equivariant map

$$\mathrm{Bun}_G(X) \times_{\operatorname{Spec}\mathbf{F}_q} \operatorname{Spec}\mathbf{F}_{q^d} \to \mathrm{Bun}_{G_0}(\widetilde{X})^\Gamma,$$

and then taking quotients by the action of $\mathbf{Z}/d\mathbf{Z}$.

Let (P_0,ν) be an element of A^Γ. We let $\mathcal{Y}_{[P_0,\nu]}$ denote the reduced algebraic stack

$$(\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma} \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma})_{\mathrm{red}},$$

which carries an action of the subgroup $d'\mathbf{Z}/d\mathbf{Z} \subseteq \mathbf{Z}/d\mathbf{Z}$ which stabilizes $(P_0,\nu) \in A^\Gamma$.

Proposition 10.8.5. *Let $(P_0,\nu)\in A^\Gamma$ and let $C\subseteq \mathbf{Z}/d\mathbf{Z}$ be a subgroup which stabilizes (P_0,ν). Then, as an algebraic stack over $\mathbf{F}_q$, the quotient $\mathcal{Y}_{[P_0,\nu]}/C$ can be written as a stack-theoretic quotient Y/H, where Y is a quasi-compact quasi-separated algebraic space over $\mathbf{F}_q$ and H is a linear algebraic group over $\mathbf{F}_q$.*

Proof. Let $C'\subseteq \mathbf{Z}/d\mathbf{Z}$ be the stabilizer of (P_0,ν) in A^Γ. It follows from Lemma 10.8.3 that $\mathcal{Y}_{[P_0,\nu]}/C'$ admits a surjective finite radicial map $\mathcal{Y}_{[P_0,\nu]}\to \mathrm{Bun}_G(X)_{P_0,\nu}$. The projection map $\mathcal{Y}_{[P_0,\nu]}/C\to \mathcal{Y}_{[P_0,\nu]}/C'$ is finite étale and the inclusion $\mathrm{Bun}_G(X)_{P_0,\nu}\hookrightarrow \mathrm{Bun}_G(X)$ is a locally closed immersion. It follows that the composite map

$$\mathcal{Y}_{[P_0,\nu]}/C\to \mathcal{Y}_{[P_0,\nu]}/C'\to \mathrm{Bun}_G(X)_{P_0,\nu}\hookrightarrow \mathrm{Bun}_G(X)$$

is quasi-finite. By virtue of Corollary 10.4.2, Proposition 10.8.5 is equivalent to the statement that the algebraic stack $\mathcal{Y}_{[P_0,\nu]}/C$ is quasi-compact. Since the quotient map $\mathcal{Y}_{[P_0,\nu]}\to \mathcal{Y}_{[P_0,\nu]}/C$ is surjective, it will suffice to show that $\mathcal{Y}_{[P_0,\nu]}$ is quasi-compact.

Using Propositions 10.4.4 and 10.4.3, we deduce that the composite map

$$\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}}\xrightarrow{g}\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma\to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$$

is quasi-compact. Since the algebraic stack $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ has affine diagonal, the map $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma\to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ is affine and, in particular, quasi-separated. It follows that g is quasi-compact. Consequently, to prove that

$$\mathcal{Y}_{[P_0,\nu]}=(\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}}\times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma}\mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma})_{\mathrm{red}},$$

is quasi-compact, it will suffice to show that $\mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}$ is quasi-compact. Using the affine morphism $\mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}\to \mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}}$, we are reduced to proving that $\mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}}$ is quasi-compact. It now suffices to observe that we have a pullback diagram

$$\begin{array}{ccc}
\mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}} & \longrightarrow & \mathrm{Bun}^\nu_{P_{0\,\mathrm{ad}}}(\widetilde{X})^{\mathrm{ss}} \\
\downarrow & & \downarrow \\
\mathrm{Bun}_{\widetilde{G}}(X) & \longrightarrow & \mathrm{Bun}_{\widetilde{G}_{\mathrm{ad}}}(X),
\end{array}$$

where the lower horizontal map is quasi-compact (Proposition 10.4.5) and the upper right hand corner is quasi-compact (Proposition 10.4.6). □

Note that every stratum of $\mathrm{Bun}_G(X)$ admits a surjective finite radicial morphism from a quotient stack $\mathcal{Y}_{[P_0,\nu]}/C$, where $C\subseteq \mathbf{Z}/d\mathbf{Z}$ (and no two strata correspond to the same quotient). Here there are only finitely many choices for the parabolic subgroup P_0 and for the subgroup C. Consequently, Proposition 10.8.2 is a consequence of Proposition 10.8.5 together with the following:

Proposition 10.8.6. *Let d' be a divisor of d, let $P_0\subseteq G_0$ be a standard parabolic subgroup which is fixed under the action of the subgroup*

$$\Gamma'=\mathrm{Gal}(L/K_X)\times_{\mathbf{Z}/d\mathbf{Z}}(d'\mathbf{Z}/d\mathbf{Z})\subseteq \mathrm{Gal}(L/K_X),$$

and let $B\subseteq \mathrm{Hom}(P_{0\,\mathrm{ad}},\mathbf{G}_m)^\vee_{>0}$ be the subset consisting of those elements ν which are fixed by Γ'. Then there exists a finite subset $B_0\subseteq B$ with the following properties:

(1) *For each $\nu\in B$, there exists $\nu_0\in B_0$ and a $(d'\mathbf{Z}/d\mathbf{Z})$-equivariant map of algebraic stacks*

$$\mathcal{Y}_{[P_0,\nu_0]}\to \mathcal{Y}_{[P_0,\nu]}$$

which exhibits $\mathcal{Y}_{[P_0,\nu_0]}$ as a fiber bundle (locally trivial with respect to the étale topology) of some rank e_ν over $\mathcal{Y}_{[P_0,\nu]}$.

(2) *For every real number $r > 1$, the infinite sum $\sum_{\nu \in B} r^{-e_\nu}$ converges.*

To prove Proposition 10.8.6, we are free to replace the ground field $\mathbf{F}_q$ by $\mathbf{F}_{q^{d'}}$ and thereby reduce to the case where $d' = 1$. For the remainder of this section, we will fix a standard parabolic subgroup $P_0 \subseteq G_0$ which is invariant under the action of the Galois group $\mathrm{Gal}(L/K_X)$; we will prove that Proposition 10.8.6 is valid for P_0 (in the case $d' = 1$). To simplify our notation, for $\nu \in B$ we will denote the algebraic stack $\mathcal{Y}_{[P_0,\nu]}$ simply by $\mathcal{Y}_\nu$.

Let $\Lambda \subseteq \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee$ be as in Notation 10.6.14. Let $\mathbf{\Delta}_{P_0} = \{\alpha_1, \ldots, \alpha_m\}$ be the collection of simple roots α of G_0 such that $-\alpha$ is not a root of P_0. The construction

$$\lambda \mapsto \{\langle \alpha_i, \lambda \rangle\}_{1 \leq i \leq m}$$

determines an injective map $\Lambda \hookrightarrow \mathbf{Z}^m$ between finitely generated abelian groups of the same rank. It follows that we can choose an integer $N > 0$ such that the image of Λ contains $N\mathbf{Z}^m$: in other words, we can find elements $\lambda_1, \ldots, \lambda_m \in \Lambda$ satisfying

$$\langle \alpha_j, \lambda_i \rangle = \begin{cases} N & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Every element $\nu \,\mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee$ can be written uniquely in the form $\sum c_i \lambda_i$, where the elements c_i are rational numbers. We observe that ν belongs to $\mathrm{Hom}(P_0, \mathbf{G}_m)^\vee_{>0}$ if and only if each of the rational numbers c_i is positive, and that ν is fixed by the action of $\mathrm{Gal}(L/K_0)$ if and only if we have $c_i = c_j$ whenever the roots α_i and α_j are conjugate by the action of $\mathrm{Gal}(L/K_X)$ (which acts on the set $\mathbf{\Delta}_{P_0}$ via the group homomorphism $\rho : \mathrm{Gal}(L/K_X) \to \mathrm{Out}(G_0)$).

Since L is a Galois extension of K_X, the map $\pi : \widetilde{X} \to X$ is generically étale. Choose a closed point $x \in X$ such that π is étale over the point x and the map $\beta : G \times_X \widetilde{X} \to \widetilde{G}$ is an isomorphism when restricted to the inverse image of x. Let $D \subseteq \widetilde{X}$ be the effective divisor given by the inverse image of x, and let $\deg(D)$ denote the degree of D over $\mathbf{F}_{q^d}$. Let us say that an element $\nu = \sum c_i \lambda_i \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$ is *minimal* if each of the coefficients c_i satisfy $0 < c_i \leq \deg(D)$. Note that every element $\nu \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$ can be written uniquely in the form $\nu_0 + \sum c_i \deg(D) \lambda_i$, where ν_0 is minimal and each c_i is an integer. Moreover, ν belongs to $\mathrm{Hom}(P_0, \mathbf{G}_m)^\vee_{>0}$ if and only if each of the integers c_i is nonnegative, and ν is fixed by $\mathrm{Gal}(L/K_X)$ if and only if ν_0 and $\sum c_i \lambda_i$ are both fixed by $\mathrm{Gal}(L/K_X)$. We will deduce Proposition 10.8.6 from the following more precise result:

Proposition 10.8.7. *Let $\nu \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$ be an element which is minimal and fixed by the action of $\mathrm{Gal}(L/K_X)$. For every element $\lambda \in \Lambda_{\geq 0}$ which is fixed by the action of $\mathrm{Gal}(L/K_X)$, there exists a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map $\mathcal{Y}_\nu \to \mathcal{Y}_{\nu + \deg(D)\lambda}$ which exhibits $\mathcal{Y}_\nu$ as a fiber bundle (locally trivial with respect to the étale topology) whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|} \langle 2\rho_{P_0}, \lambda \rangle$.*

Proof. For each $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}$, let $\mathcal{Z}_\nu$ denote the fiber product

$$\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma} \mathrm{Bun}^\nu_{\widetilde{G}, P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}.$$

Let $U = X - \{x\}$, which we regard as an open subset of X, and let $\widetilde{U}$ denote the inverse image of U in $\widetilde{X}$. Let $\mathrm{Bun}_G(U)$ denote the moduli stack of G-bundles on U: that is, the (non-algebraic) stack over $\mathbf{F}_q$ whose R-valued points are given by G-torsors on the open curve $U_R = U \times_{\mathrm{Spec}\, \mathbf{F}_q} \mathrm{Spec}\, R$, and define $\mathrm{Bun}_{\widetilde{G}}(\widetilde{U})$ similarly. Since the map $\widetilde{X} \to X$ is étale over the point x and the map $G \times_X \widetilde{X} \to \widetilde{G}$ is an isomorphism over $\{x\}$, the diagram

$$\begin{array}{ccc} \mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G(U)_{\mathbf{F}_{q^d}} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(\widetilde{U})^{\Gamma} \end{array}$$

is a pullback square. It follows that we can identify $\mathcal{Z}_\nu$ with the fiber product

$$\mathrm{Bun}_G(U)_{\mathbf{F}_{q^d}} \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{U})^{\Gamma}} \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}.$$

For each $\lambda \in \Lambda_{\geq 0}$, the twisting map

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X}) \to \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X})$$

is $\mathrm{Gal}(L/K_X)$-equivariant (Remark 10.6.32) and therefore induces a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map

$$u : \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X})^{\Gamma} \to \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X})^{\Gamma}.$$

It follows from Remark 10.6.30 that u restricts to a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map

$$u_0 : \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \to \mathrm{Bun}^{\nu+\deg(D)\lambda}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}.$$

The map u_0 is a pullback of u, and therefore (by virtue of Proposition 10.6.33) is an étale fiber bundle whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle$. Note that the twisting construction does not modify bundles over the open set $\widetilde{U}$: in other words, the diagram

$$\begin{array}{ccc} \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(X)^{\Gamma} \\ \downarrow u_0 & & \downarrow \mathrm{id} \\ \mathrm{Bun}^{\nu+\deg(D)\lambda}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(X)^{\Gamma} \end{array}$$

commutes up to canonical isomorphism. This isomorphism allows us to lift u_0 to a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map

$$\overline{u}_0 : \mathcal{Z}_\nu \to \mathcal{Z}_{\nu+\deg(D)\lambda}$$

which is a pullback of u_0 and therefore also an étale fiber bundle whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle$.

By definition, we have $\mathcal{Y}_\nu = (\mathcal{Z}_\nu)_{\mathrm{red}}$ and $\mathcal{Y}_{\nu+\deg(D)\lambda} = (\mathcal{Z}_{\nu+\deg(D)})_{\mathrm{red}}$. Consequently, $\overline{u}_0$ induces a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map $v : \mathcal{Y}_\nu \to \mathcal{Y}_{\nu+\deg(D)\lambda}$ which factors as a composition

$$\mathcal{Y}_\nu \xrightarrow{v'} \mathcal{Y}_{\nu+\deg(D)\lambda} \times_{\mathcal{Z}_{\nu+\deg(D)\lambda}} \mathcal{Z}_\nu \xrightarrow{v''} \mathcal{Y}_{\nu+\deg(D)\lambda}.$$

The map v'' is a pullback of $\overline{u}_0$ and is therefore an étale fiber bundle whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle$. To complete the proof, it will suffice to show that v' is an equivalence. It is clear that v' induces an equivalence of the underlying reduced substacks. Since $\mathcal{Y}_\nu$ is reduced, we only need to show that the fiber product $\mathcal{Y}_{\nu+\deg(D)\lambda} \times_{\mathcal{Z}_{\nu+\deg(D)\lambda}} \mathcal{Z}_\nu$ is also reduced. This follows from the fact that $\mathcal{Y}_{\nu+\deg(D)\lambda}$ is reduced, since the morphism v'' is smooth. □

Proof of Proposition 10.8.6. We will show that the subset

$$B_0 = \{\nu \in B : \ \nu \text{ is minimal}\} \subseteq B$$

satisfies the requirements of Proposition 10.8.6. The only nontrivial point is to prove the convergence of the infinite sum $\sum_{\nu \in B} r^{-e_\nu}$ for $r > 1$. By virtue of Proposition 10.8.7, we can write this sum as

$$|B_0| \sum_{\lambda \in Z} r^{-\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle},$$

where the sum is taken over the set of all $\mathrm{Gal}(L/K_X)$-invariant elements $\lambda \in \Lambda$ which can be written in the form $\sum_{1 \leq i \leq m} c_i \lambda_i$ where each c_i is a nonnegative integer. Up to a constant factor of $|B_0|$, this sum is dominated by the larger infinite sum

$$\begin{aligned}
\sum_{c_1,\ldots,c_m \geq 0} r^{-\frac{N}{|\Gamma|}\langle 2\rho_{P_0}, \sum c_i \lambda_i\rangle} &= \sum_{c_1,\ldots,c_m \in \mathbf{Z}_{\geq 0}} (r^{-\frac{\deg(D)}{|\Gamma|}})^{\sum c_i \langle 2\rho_{P_0}, \lambda_i\rangle} \\
&= \prod_{1 \leq i \leq m} (\frac{r^{a_i}}{r^{a_i} - 1}) \\
&< \infty,
\end{aligned}$$

where $a_i = \frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda_i\rangle$. □

Appendix A

In this appendix, we review some general facts about group schemes, principal bundles, and algebraic curves which will be needed throughout this paper. We recommend that the reader refer to it only as needed.

A.1. G-Bundles.

Definition A.1.1. Let X be a scheme, and let G be a group scheme over X. For every X-scheme Y, let $G_Y = G \times_X Y$ denote the associated group scheme over Y. By a *G-bundle on Y*, we will mean a Y-scheme $\mathcal{P}$ equipped with an action

$$G_Y \times_Y \mathcal{P} \simeq G \times_X \mathcal{P} \to \mathcal{P}$$

of G_Y (in the category of Y-schemes) which is locally trivial in the following sense: there exists a faithfully flat map $U \to Y$ and a G_Y-equivariant isomorphism $U \times_Y \mathcal{P} \simeq U \times_Y G_Y \simeq U \times_X G$.

Remark A.1.2. Let G be an group scheme over X. Then G represents a functor h_G from the category of X-schemes to the category of groups, and the functor h_G is a sheaf for the fpqc topology. Every G-bundle $\mathcal{P}$ on X represents a functor $h_{\mathcal{P}}$ which can be regarded as an h_G-torsor, locally trivial for the fpqc topology. If G is affine, then every h_G-torsor arises in this way (since affine morphisms satisfy effective descent for the fpqc topology). For this reason, we will generally use the terminology *G-bundle* and *G-torsor* interchangeably when G is affine (which will be satisfied for all of our applications).

Notation A.1.3. Let X be a scheme and let G be an affine group scheme over X. For every X-scheme Y, we let $\mathrm{Tors}_G(Y)$ denote the category whose objects are G-bundles on Y, and whose morphisms are isomorphisms of G-bundles.

Remark A.1.4. In the special case where G is a smooth over X, any G-bundle $\mathcal{P}$ on an X-scheme Y is smooth as a Y-scheme. It follows that $\mathcal{P}$ can be trivialized over an étale covering $U \to Y$.

Notation A.1.5. Let R be a commutative ring and let G be an affine group scheme over $\operatorname{Spec} R$. Then we can write $G = \operatorname{Spec} \mathcal{O}_G$, where $\mathcal{O}_G$ is a commutative Hopf algebra over R. We let $\operatorname{Rep}(G)$ denote the category of left $\mathcal{O}_G$-comodules which are projective R-modules of finite rank.

Let $V \in \operatorname{Rep}(G)$, let X be an R-scheme, and let $\pi : \mathcal{P} \to X$ be a G-bundle on X. The left $\mathcal{O}_G$-coaction on V determines descent data on the locally free sheaf $V \otimes_R \mathcal{O}_{\mathcal{P}}$, so that we can write

$$V \otimes_R \mathcal{O}_{\mathcal{P}} = \pi^* V_{\mathcal{P}}$$

for some locally free $\mathcal{O}_X$-module $V_{\mathcal{P}}$. We will refer to $V_{\mathcal{P}}$ as the *vector bundle associated to $\mathcal{P}$ by the representation V*.

Proposition A.1.6. *Let R be a Dedekind ring and let G be a flat affine group scheme over R, let S be an arbitrary scheme, and let $\operatorname{Vect}(X)$ be the category of locally free $\mathcal{O}_S$-modules. Let $\mathcal{C}$ denote the category whose objects are pairs $(f, \mathcal{P})$, where $f : S \to \operatorname{Spec} R$ is a morphism of schemes and $\mathcal{P}$ is a G-bundle on S (morphisms in the category $\mathcal{C}$ are given by isomorphisms of G-bundles). For every such pair $(f, \mathcal{P})$, let $f^*_{\mathcal{P}} : \operatorname{Rep}(G) \to \operatorname{Vect}(X)$ be the functor given by $f^*_{\mathcal{P}}(V) = V_{\mathcal{P}}$. Then the construction*

$$(f, \mathcal{P}) \mapsto (f^*_{\mathcal{P}} : \operatorname{Rep}(G) \to \operatorname{Vect}(S))$$

induces an equivalence from $\mathcal{C}$ to the category of symmetric monoidal functors from $\operatorname{Rep}(G)$ to $\operatorname{Vect}(S)$ which preserve zero objects and exact sequences (with morphisms given by symmetric monoidal natural transformations).

Proof. See Corollary SAG.2.3.6.15. □

Corollary A.1.7. *Let R be a Dedekind ring and let G be a flat affine group scheme over R. If A is a Noetherian R-algebra which is complete with respect to an ideal $I \subseteq A$, then the groupoid $\operatorname{Tors}_G(A)$ is equivalent to the homotopy inverse limit of the groupoids $\operatorname{Tors}_G(A/I^n)$.*

In other words, giving a G-bundle on the affine scheme $\operatorname{Spec} A$ is equivalent to giving a G-bundle on the affine formal scheme $\operatorname{Spf} A$.

Proof. This follows from Proposition A.1.6, since $\operatorname{Vect}(\operatorname{Spec} A)$ is equivalent to the homotopy inverse limit of the groupoids $\operatorname{Vect}(\operatorname{Spec} A/I^n)$. □

Proposition A.1.8. *Let R be a discrete valuation ring with fraction field K and let G be a flat affine group scheme over R with generic fiber G_K. Then any map $G_K \to \mathbf{G}_m$ of group schemes over K extends to a map $G \to \mathbf{G}_m$ of group schemes over R.*

Proof. Let t denote a uniformizer of R and let $\mathcal{O}_G$ be the ring of functions on G, so that $\mathcal{O}_G[t^{-1}]$ is the ring of functions on the group scheme G_K. Let $\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes_R \mathcal{O}_G$ be the comultiplication associated to the group structure on G. Then a group homomorphism $G_K \to \mathbf{G}_m$ is classified by an invertible element $x \in \mathcal{O}_G[t^{-1}]$ satisfying $\Delta(x) = x \otimes x$. We will show that x automatically belongs to $\mathcal{O}_G$. Applying the same argument to x^{-1}, we deduce that x is an invertible element of $\mathcal{O}_G$ and therefore defines a group homomorphism $G \to \mathbf{G}_m$.

Assume that $x \notin \mathcal{O}_G$. Then there exists some least integer n such that $t^n x \in \mathcal{O}_G$. Let M be the quotient of $\mathcal{O}_G$ by the R-submodule generated by $t^n x$. We claim that M is torsion-free as an R-module: that is, that the map $t : M \to M$ is injective. To prove this, we note that if $y \in \mathcal{O}_G$ and its image $\overline{y} \in M$ satisfies $t\overline{y} = 0$, then $ty = at^n x$ for some $a \in R$, so that (using the flatness of $\mathcal{O}_G$ over R) we have $y = at^{n-1}x$. Since $t^{n-1}x \notin \mathcal{O}_G$, the element $a \in R$ cannot be invertible; it follows that y is a multiple of $t^n x$ and therefore $\overline{y} = 0$.

Since R is a discrete valuation ring, M is a flat R-module. Let ϕ denote the unit map $\mathcal{O}_G[t^{-1}]/\mathcal{O}_G \to \mathcal{O}_G \otimes_R (\mathcal{O}_G[t^{-1}]/\mathcal{O}_G)$, so that we have an exact sequence

$$0 \to R \otimes_R \mathcal{O}_G[t^{-1}]/\mathcal{O}_G \xrightarrow{\phi} \mathcal{O}_G \otimes_R \mathcal{O}_G[t^{-1}]/\mathcal{O}_G \to M \otimes_R \mathcal{O}_G[t^{-1}]/\mathcal{O}_G \to 0$$

Since $\Delta(x) = x \otimes x$, we have $\Delta(t^n x) = t^n x \otimes x \in \mathcal{O}_G \otimes \mathcal{O}_G$. If $\overline{x}$ denotes the image of x in $\mathcal{O}_G[t^{-1}]/\mathcal{O}_G$, we conclude that $\phi(1 \otimes \overline{x}) = 0$, so that the injectivity of ϕ guarantees that $\overline{x} = 0$. This proves that $x \in \mathcal{O}_G$, as desired. □

A.2. **Curves and Divisors.** In this section, we review some elementary facts about divisors on algebraic curves which will be needed in the body of the paper.

Definition A.2.1. Let S be a scheme. A *relative curve* over S is a smooth, proper, geometrically connected morphism $\pi : X \to S$ of relative dimension 1.

Let $\pi : X \to S$ be a relative curve. An *effective divisor* in X is a closed subscheme $D \subseteq X$ such that the composite map $D \hookrightarrow X \xrightarrow{f} S$ is finite and flat. If D is finite and flat of degree d over S, then we will say that D is an *effective divisor of degree d*.

Example A.2.2. Let $\pi : X \to S$ be a relative curve, and let $\alpha : \mathcal{L} \to \mathcal{L}'$ be a morphism between line bundles on X. Suppose that, for each point $s \in S$, the induced map $\mathcal{L}_s \to \mathcal{L}'_s$ of line bundles on $X_s = X \times_S \{s\}$ is nonzero. Then the vanishing locus of α is an effective divisor in X.

Notation A.2.3. Let $\pi : X \to S$ be a relative curve, and suppose we are given effective divisors $D, D' \subseteq X$, defined by invertible ideal sheaves $\mathcal{O}_X(-D), \mathcal{O}_X(-D') \subseteq \mathcal{O}_X$. We let $D + D'$ denote the closed subscheme of X defined by product $\mathcal{O}_X(-D)\,\mathcal{O}_X(-D') \subseteq \mathcal{O}_X$. For each integer $n \geq 0$, we let nD denote the sum of n copies of D.

Let J be a finite set, let S' be an S-scheme, and suppose we are given a map $\beta : J \to \mathrm{Hom}_S(S', X)$. Let $X_{S'}$ denote the fiber product $S' \times_S X$ (viewed as a relative curve over S'), and for each $j \in J$ we let $\Gamma_j \subseteq X_{S'}$ denote the graph of the morphism $\beta(j) \in \mathrm{Hom}_S(S', X)$: this is a closed subscheme of $X_{S'}$ which we regard as an effective divisor of degree 1. We let $|\beta|$ denote the sum $\sum_{j \in J} \Gamma_j \subseteq X_{S'}$, so that $|\beta|$ is an effective divisor of degree equal to the cardinality of the set J.

Remark A.2.4. In the situation of Notation A.2.3, we will often abuse notation by denoting the effective divisor $|\beta|$ by $|J|$ (if the map β is clear from context).

If J is given as a subset of $\mathrm{Hom}_S(S', X)$, we let $|J|$ denote the effective divisor $|\iota|$, where $\iota : J \hookrightarrow \mathrm{Hom}_S(S', X)$ denotes the inclusion map. Note that for an arbitrary map $\beta : J \to \mathrm{Hom}_S(S', X)$, the divisors $|\beta|$ and $|\beta(J)|$ have the same underlying topological space, but generally have different scheme structures (unless β is injective).

Remark A.2.5. Let $\pi : X \to S$ be a relative curve, and let $D, D' \subseteq X$ be effective divisors with $D' \subseteq D$. Then we can write $D = D' + D''$ for a unique effective divisor $D'' \subseteq X$. To prove this, let $\mathcal{O}_X(-D)$ and $\mathcal{O}_X(-D')$ be the ideal sheaves of D and D', respectively. Then $\mathcal{O}_X(-D')$ is an invertible sheaf with inverse $\mathcal{O}_X(D')$. We can then take D'' to be the effective divisor defined by the invertible ideal sheaf given by the image of the natural map

$$\mathcal{O}_X(-D) \otimes \mathcal{O}_X(D') \hookrightarrow \mathcal{O}_X(-D') \otimes \mathcal{O}_X(D') \simeq \mathcal{O}_X\,.$$

Proposition A.2.6. *Let S be a scheme, let $f : X \to S$ be a relative curve and suppose we are given closed subschemes $Y, Z \subseteq X$ such that $Y \cap Z = \emptyset$, and neither Y nor Z contains a fiber of f. Then there exists a surjective étale morphism $S' \to S$ and an effective divisor $D \subseteq X \times_S S'$ of constant degree $d \geq 0$ which contains $Y \times_S S'$ and is disjoint from $Z \times_S S'$.*

Lemma A.2.7. *Let R be a commutative ring, let $f : X \to \operatorname{Spec} R$ be a relative curve, and suppose we are given closed subschemes $Y, Z \subseteq X$ such that $Y \cap Z = \emptyset$. Suppose further that there exist effective divisors $D, D' \subseteq X$ (of degrees $d, d' > 0$) such that $D \cap Z = D' \cap Y = \emptyset$. Then there exists an effective divisor $D'' \subseteq X$ (of constant degree $d'' > 0$) which contains Y and is disjoint from Z.*

Proof. Replacing Z by $Z \cup D'$ if necessary, we may assume that the projection map $Z \to \operatorname{Spec} R$ is surjective. Enlarging Y and Z if necessary, we may suppose that Y and Z are of finite presentation over X. Then X, D, D', Y, and Z are defined over some finitely generated subring $R_0 \subseteq R$; replacing R by R_0, we may suppose that R is Noetherian.

The inclusion $D \hookrightarrow X$ is a closed immersion, defined by an invertible ideal sheaf $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$. For every coherent sheaf $\mathcal{F}$ on X, we let $\mathcal{F}(nD)$ denote the tensor product of $\mathcal{F}$ with $\mathcal{O}(-D)^{\otimes -n}$. Let $\mathcal{I}_Y$ denote the ideal sheaf defining the closed subscheme $Y \subseteq X$, and let $i : Z \to X$ denote the inclusion map. Since Y and Z do not intersect, the composite map

$$\mathcal{I}_Y \hookrightarrow \mathcal{O}_X \to i_* \mathcal{O}_Z$$

is an epimorphism. Let $\mathcal{F}$ denote the kernel of this epimorphism. For every integer n, we have an exact sequence of sheaves

$$0 \to \mathcal{F}(nD) \to \mathcal{I}_Y(nD) \to (i_* \mathcal{O}_Z)(nD) \to 0.$$

The line bundle $\mathcal{O}_X(D)$ is ample, so that $\mathrm{H}^1(X; \mathcal{F}(nD)) \simeq 0$ for $n \gg 0$. In particular, we can choose $n > 0$ such that the map

$$\mathrm{H}^0(X; \mathcal{I}_Y(nD)) \to \mathrm{H}^0(X; (i_* \mathcal{O}_Z)(nD)) \simeq \mathrm{H}^0(Z; \mathcal{O}_Z)$$

is surjective. It follows that there exists a section f of $\mathcal{I}_Y(nD) \subseteq \mathcal{O}_X(nD)$ which takes the value 1 on Z. Since $Z \to \operatorname{Spec} R$ is surjective, the section f does not vanish on any fiber of the map f. The vanishing locus of f is an effective divisor of degree nd (Example A.2.2) which contains Y and is disjoint from Z. □

Proof of Proposition A.2.6. The assertion is local on S with respect to the étale topology. We may therefore assume without loss of generality that $S = \operatorname{Spec} R$ is affine. Since Y and Z do not contain any fibers of f, the maps

$$X - Y \to \operatorname{Spec} R \leftarrow X - Z$$

are smooth surjections. Passing to an étale cover of $\operatorname{Spec} R$, we may assume that each of these maps admits a section. In this case, the desired result follows from Lemma A.2.7. □

The following statement is closely related to Proposition A.2.6:

Proposition A.2.8. *Let S be a Noetherian scheme, let $\pi : X \to S$ be a relative curve and let $K \subseteq X$ be a closed subset. Suppose that $s \in S$ is a point such that K does not contain the fiber $\pi^{-1}\{s\}$. Then there exists an étale map $U \to S$ whose image contains s and an effective divisor $D \subseteq X \times_S U$ of constant degree whose underlying topological space contains $K \times_S U$.*

Proof. The assertion is local on S; we may therefore assume without loss of generality that $S = \operatorname{Spec} R$ is affine. Replacing R by an étale R-algebra, we may assume that the map π admits a section. It follows that X admits an ample line bundle $\mathcal{L}$ of degree one. Let us regard K as a closed subscheme of X (by giving it the reduced structure), and let $\mathcal{I} \subseteq \mathcal{O}_X$ be its ideal sheaf.

For $n \gg 0$, the coherent sheaf $\mathcal{I} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. Since $\pi^{-1}\{s\}$ is not contained in K, there exists a global section of $\mathcal{I} \otimes \mathcal{L}^{\otimes n}$ which does not vanish at the generic point of $\pi^{-1}\{s\}$. We can identify this section with a map of coherent sheaves

$$\alpha : \mathcal{L}^{-n} \to \mathcal{I} \subseteq \mathcal{O}_X .$$

Then the vanishing locus of α is a closed subscheme $D \subseteq X$ containing K. Since X is flat over S, the map $X - D \to S$ has open image. Replacing S by an open subset if necessary, we may assume that the map $X - D \to S$ is surjective: that is, g does not vanish on any fiber of π. It follows that D is an effective divisor of constant degree n which contains K (Example A.2.2). □

Proposition A.2.9. *Let $\pi : X \to S$ be a relative curve, and let $D \subseteq X$ be an effective divisor of degree n. Then there exists a finite flat surjective map $S' \to S$ and a map $\beta : \{1, \ldots, n\} \to \mathrm{Hom}_S(S', X)$ such that $D \times_S S'$ coincides with $|\beta|$ (as effective divisors in $X_{S'}$).*

Proof. We proceed by induction on n. If $n = 0$, then $D = \emptyset$ and there is nothing to prove. Otherwise, the map $D \to S$ finite, flat, and surjective. Replacing S by D, we may reduce to the case where $\pi|_D$ admits a section s. Let $D' \subseteq X$ denote the scheme-theoretic image of s. Then $D' \subseteq D$, so we can write $D = D' + D''$ where D'' is an effective divisor of degree $n - 1$ (Remark A.2.5).

The inductive hypothesis implies that there exists a finite, flat, surjective map $S' \to S$ such that $S' \times_S D'' = |\beta_0|$, for some map $\beta_0 : \{1, \ldots, n-1\} \to \mathrm{Hom}_S(S', X)$. It now suffices to take $\beta : \{1, \ldots, n\} \to \mathrm{Hom}_S(S', X)$ to be the unique map extending β_0 for which $\beta(n)$ is the composition $S' \to S \xrightarrow{s} X$. □

Corollary A.2.10. *Let S be a scheme, let $\pi : X \to S$ be a relative curve and suppose we are given closed subschemes $Y, Z \subseteq X$ such that $Y \cap Z = \emptyset$, and that neither Y nor Z contains a fiber of π. Then there exists a quasi-finite flat surjection $S' \to S$ and a map $\beta : \{1, \ldots, n\} \to \mathrm{Hom}_S(S', X)$ such that $|\beta| \subseteq X_{S'}$ contains $S' \times_S Y$ and is disjoint from $S' \times_S Z$.*

Proof. Combine Propositions A.2.6 and A.2.9. □

A.3. **Dilitations.** Throughout this section, we fix a smooth algebraic curve X over an algebraically closed field k (not necessarily complete). We give a brief review of the theory of *dilitations* (or *affine blow-ups*) for schemes over X, which will be useful at several points in the body of this paper.

Construction A.3.1. Let $\phi : Y \to X$ be a map of separated k-schemes, and let $y \in Y(k)$ be a k-valued point of Y. Since Y is separated, y is a closed immersion (when regarded as a map of schemes from $\mathrm{Spec}\, k$ into Y). We let $\mathcal{I}_y \subseteq \mathcal{O}_Y$ denote the associated quasi-coherent ideal sheaf, and $\mathcal{O}_X(-\phi(y)) \subseteq \mathcal{O}_X$ the ideal sheaf associated to the composite map $\mathrm{Spec}\, k \xrightarrow{y} Y \xrightarrow{\phi} X$. Since X is a smooth curve, $\mathcal{O}_X(-\phi(y))$ is an invertible sheaf on X. If $\mathcal{F}$ is a quasi-coherent sheaf on Y and n is an integer, we let $\mathcal{F}(n)$ denote the tensor product $\mathcal{F} \otimes \phi^* \mathcal{O}_X(-\phi(y))^{\otimes -n}$. We have a canonical map $\phi^* \mathcal{O}_X(-\phi(y)) \to \mathcal{I}_y$, which induces maps $\mathcal{I}_y^m(m) \to \mathcal{I}_y^{m+1}(m+1)$ for $m \geq 0$. Let $\mathcal{A}_y = \varinjlim \mathcal{I}_y^m(m)$, so that $\mathcal{A}_y$ is a quasi-coherent sheaf of commutative algebras on Y. We let $D_y(Y)$ denote the relative spectrum $\mathrm{Spec}\, \mathcal{A}_y$, so that we have an affine map of schemes $D_y(Y) \to Y$. We will refer to $D_y(Y)$ as the *dilitation of Y at the point y*.

Remark A.3.2. In the situation of Construction A.3.1, suppose that the map $\phi : Y \to X$ is smooth. Then the composite map $D_y(Y) \xrightarrow{\psi} Y \to X$ is smooth. In particular, $D_y(Y)$ is smooth over k. We can describe $\mathcal{A}_y$ as a subsheaf of the sheaf of rational functions on Y, generated over $\mathcal{O}_Y$ by local sections of the form $\frac{f}{\phi^* g}$, where f is a section of $\mathcal{O}_Y$ which vanishes at the point y, and g is a section of $\mathcal{O}_X$ which has a simple pole at $\phi(y)$. Note that the map of tangent bundles $T_{D_y(Y)/X} \to \psi^* T_{Y/X}$ (obtained by differentiating ψ) induces an isomorphism $T_{D_y(Y)/X} \simeq \psi^*(T_{Y/X}(-\phi(y)))$.

Let U be the complement of $\phi(y)$ in X. Then the map $D_y(Y) \to Y$ induces an isomorphism $D_y(Y) \times_X U \to Y \times_X U$. Moreover, the fiber $D_y(Y) \times_X \operatorname{Spec} k$ (taken over the point $\phi(y)$) is canonically isomorphic to the tangent space to the fiber $Y \times_X \{\phi(y)\}$ at the point y.

Remark A.3.3. In the situation of Construction A.3.1, we can identify $D_y(Y)$ with an open subscheme of the scheme obtained from Y by blowing up the point y. More precisely, it is the open subscheme complementary to the proper transform of the fiber $Y_{\phi(y)} = Y \times_X \operatorname{Spec} k$.

Example A.3.4. In the situation of Construction A.3.1, suppose that the map $\phi : Y \to X$ is an isomorphism. Then the projection map $D_y(Y) \to Y$ is also an isomorphism.

Remark A.3.5. Construction A.3.1 is functorial in the pair (Y, y). That is, suppose that we are given a commutative diagram of separated schemes

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & Y' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & X. & \end{array}$$

Let y be a k-point of Y and let y' be its image in Y'. Then ψ induces a map of schemes $D_y(Y) \to D_{y'}(Y')$. In particular, every section $s : X \to Y$ of the map $\phi : Y \to X$ determines a section $\widetilde{s} : X \to D_y(Y)$ of the induced map $D_y(Y) \to X$.

Definition A.3.6. Let $\phi : Y \to X$ be a map of k-schemes. Suppose we are given a pair of sections $s, s' : X \to Y$ of the map ϕ, and let $x \in X(k)$ be a k-point of X. We will say that s and s' *agree to order* n at x if they coincide on the closed subscheme of X defined by the ideal sheaf $\mathcal{O}_X(-(n+1)x)$.

Proposition A.3.7. *Let $\phi : Y \to X$ be a map of separated k-schemes and let $s, s' : X \to Y$ be sections of ϕ. Suppose that s and s' agree to order $n \geq 1$ at some k-point $x \in X(k)$, and let $y = s(x) = s'(x) \in Y(k)$. Let $\widetilde{s}, \widetilde{s}' : X \to D_y(Y)$ be the sections determined by s and s', respectively (see Remark A.3.5). Then $\widetilde{s}$ and $\widetilde{s}'$ agree to order $(n-1)$.*

Proof. Let $\mathcal{O}_{X,x}$ denote the local ring of X at the point x, let $\mathfrak{m}_x$ denote its maximal ideal, and choose a uniformizer t of $\mathfrak{m}_x$. We let $\mathcal{O}_{Y,y}$ denote the local ring of Y at y and $\mathfrak{m}_y$ its maximal ideal. Then s and s' induce ring homomorphisms $\rho, \rho' : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. We can write $D_y(Y) \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}$ as $\operatorname{Spec} A$, where A is given by the direct limit $\varinjlim \mathfrak{m}_y^m$, with transition maps given by multiplication by t. The sections $\widetilde{s}, \widetilde{s}'$ are classified by ring homomorphisms $\widetilde{\rho}, \widetilde{\rho}' : A \to \mathcal{O}_{X,x}$, whose restrictions to $\mathfrak{m}_y^m$ are given by the compositions

$$\mathfrak{m}_y^m \xrightarrow{\rho} \mathfrak{m}_x^m \xrightarrow{t^{-m}} \mathcal{O}_{X,x} \qquad \mathfrak{m}_y^m \xrightarrow{\rho'} \mathfrak{m}_x^m \xrightarrow{t^{-m}} \mathcal{O}_{X,x}.$$

We wish to show that these maps agree modulo the ideal $\mathfrak{m}_x^n$. Since A is generated as a $\mathcal{O}_{X,x}$-algebra by $\mathfrak{m}_y$, we are reduced to proving this in the case $m = 1$. In this case, the desired result follows immediately from our assumption that ρ and ρ' agree modulo $\mathfrak{m}_x^{n+1}$. □

Construction A.3.8. Let $\phi : Y \to X$ be a map of separated k-schemes, and let $s : X \to Y$ be a section of Y. Let $x \in X(k)$ be a k-valued point. We define a tower of k-schemes

$$\cdots \to D^2_{s,x}(Y) \to D^1_{s,x}(Y) \to D^0_{s,x}(Y) \to X,$$

equipped with a compatible family sections $s_n : X \to D^n_{s,x}(Y)$ by induction as follows. Set $D^0_{s,x}(Y) = Y$, and $s_0 = s$. Assuming that $D^n_{s,x}(Y)$ and the map $s_n : X \to D^n_{s,x}(Y)$ have been constructed, we set $y_n = s_n(x)$, define $D^{n+1}_{s,x}(Y)$ to be the dilitation of $D^n_{s,x}(Y)$ at the point y_n, and define $s_{n+1} = \widetilde{s}_n$ to be the section of the map $D^{n+1}_{s,x}(Y) \to X$ determined by the map s_n, as in Remark A.3.5. We will refer to $D^n_{s,x}(Y)$ as the *nth order dilitation of Y along s at the point x*.

Variant A.3.9. Let $\phi : Y \to X$ be a morphism of separated k-schemes equipped with a section s, and let $Q \subseteq X$ be an effective divisor which we can write as $\sum n_i x_i$. We let $D^Q_s(Y)$ denote the scheme obtained from Y by applying an (n_i)th order dilitation of Y along s at the point x_i for each i (it is easy to see that this is independent of the ordering in which we perform the dilitations, since the dilitation of Y at a point $x \in X$ does not change Y on the open set $Y \times_X (X - \{x\})$.

Suppose that k is a separable algebraic extension of some subfield $k_0 \subseteq k$ and that the k-schemes X and Y, the map ϕ, the section s, and the divisor D are all defined over k_0. Then the dilitation $D^Q_s(Y)$ is also defined over k_0 (this is an easy exercise in Galois descent).

Remark A.3.10. In the situation of Construction A.3.8, the nth order dilitation of Y along s is characterized by the following universal mapping property: for any *flat* X-scheme Z, composition with the map $D^n_{s,x}(Y) \to Y$ induces an injection

$$\mathrm{Hom}_X(Z, D^n_{s,x}(Y)) \to \mathrm{Hom}_X(Z, Y)$$

(here $\mathrm{Hom}_X(Z,Y)$ denotes the set of X-scheme morphisms from Z to Y and $\mathrm{Hom}_X(Z, D^n_{s,x}(Y))$ is defined similarly), whose image is the collection of those morphisms $Z \to Y$ for which the composition

$$Z \times_X E \to Z \to Y$$

is equal to the composition

$$Z \times_X E \to E \hookrightarrow X \xrightarrow{s} Y,$$

where $E \subseteq X$ is the effective divisor given by nx (this follows by induction on n; in the case $n = 1$, it is immediate from the definition of the dilitation).

Proposition A.3.11. *Let $x \in X(k)$ be a k-valued point, and let $\phi : Y \to X$ be a separated map of integral k-schemes equipped with a section $s : X \to Y$. Let U be the open subscheme of X obtained by removing the point x, and suppose we are given a map of k-schemes $\psi : Y \times_X U \to Z$, where Z is a separated scheme of finite type over k. Suppose further that the composition*

$$U \xrightarrow{s} Y \times_X U \xrightarrow{\psi} Z$$

extends to a map $h : X \to Z$. Then there exists an integer $n \geq 0$ and a map $\overline{\psi} : D^n_{s,x}(Y) \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} D^n_{s,x}(Y) \times_X U & \longrightarrow & D^n_{s,x}(Y) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \overline{\psi}} \\ Y \times_X U & \xrightarrow{\psi} & Z. \end{array}$$

Proof. The assertion is local on X. We may therefore assume without loss of generality that $X = \operatorname{Spec} R$ is affine, and that the maximal ideal $\mathfrak{m}_x \subseteq R$ defining the point x is generated by an element $t \in R$. Choose an affine open subscheme $Y_0 \subseteq Y$ containing the point $y = s(x)$. Replacing X by $s^{-1}Y_0$, we may assume that h factors through Y_0. Note that for each $n \geq 0$, $D^n_{s,x}(Y)$ is covered by the open sets $D^n_{s,x}(Y_0)$ and $D^n_{s,x}(Y) \times_X U \simeq Y \times_X U$, which intersect in $D^n_{s,x}(Y_0) \times_X U \simeq Y_0 \times_X U$. Consequently, to prove the existence of $\overline{\psi}$, we may replace Y by Y_0 and thereby reduce to the case where $Y = \operatorname{Spec} A$ is affine. We will abuse notation by identifying $t \in R$ with its image in A. Moreover, the section s determines an R-algebra homomorphism $\rho : A \to R$.

Choose an affine open subscheme $Z_0 \subseteq Z$ containing the point $z = h(x)$. Replacing X by $h^{-1}Z_0$, we may assume that h factors through Z_0. The fiber product

$$Z_0 \times_Z (Y_0 \times_X U)$$

is a quasi-compact open subscheme of $Y_0 \times_X U$, and is therefore defined by a finitely generated ideal $I \subseteq A[t^{-1}]$. Choose a finite sequence of elements $f_1, \ldots, f_m \in A$ whose images in $A[t^{-1}]$ generate the ideal I. Since $h(X) \subseteq Z_0$, $s(U)$ is contained in $Z_0 \times_Z (Y_0 \times_X U)$, so that $\rho(f_i) \neq 0$ for $1 \leq i \leq m$. Since $\mathfrak{m}_x \subseteq R$ is a principal ideal, we can write $\rho(f_1) = t^d g$, where $d \geq 0$ is an integer and $g \in R - \mathfrak{m}_x$. If $d > 0$, then f_1 belongs to the maximal ideal $\mathfrak{m}_y \subseteq A$ defining the point $y = s(x)$. Write $D_y(Y) = \operatorname{Spec} A'$. By construction, every element of the image of $\mathfrak{m}_y$ in A' is divisible by t. Replacing Y by $D_y(Y)$, we may assume that $f_1 = t f'_1$ for some $f'_1 \in A$. Note that $\rho(f'_1) = t^{d-1} g$. Repeating this argument, we may reduce to the case where $\rho(f_1) \notin \mathfrak{m}_x$. Similarly we may assume that $\rho(f_i) \notin \mathfrak{m}_x$ for $1 \leq i \leq m$. Consequently, none of the functions $f_1, \ldots, f_m$ vanishes at the point y. We may therefore choose an affine open subscheme $Y_1 \subseteq Y$, containing y, on which each of the functions f_i is invertible. Replacing X by $s^{-1}Y_1$, we may assume that s factors through Y_1. Replacing Y by Y_1, we can assume that Y is affine and that the map ψ factors through Z_0. We may therefore replace Z by Z_0 and thereby reduce to the case where $Z \simeq \operatorname{Spec} B$ is affine.

For each $n \geq 0$, write $D^n_{s,x}(Y) = \operatorname{Spec} A^{(n)}$ for some A-algebra $A^{(n)}$. Since Y is integral, each $A^{(n)}$ can be identified with a subalgebra of $A[t^{-1}]$. The map ψ induces a ring homomorphism $\nu : B \to A[t^{-1}]$. We wish to prove that ν factors through $A^{(n)}$ for $n \gg 0$. Since Z is of finite type over k, B is finitely generated as a k-algebra. It will therefore suffice to show that for every element $b \in B$, we have $\nu(b) \in A^{(n)}$ for some $n \geq 0$. Write $\nu(b) = t^{-n} a$ for some $a \in A$; we will show that $\nu(b) \in A^{(n)}$. The proof proceeds by induction on n, the case $n = 0$ being trivial. If $n > 0$, then $\nu(t^n b) = a$, so that $\rho(a) \in R$ belongs to the image of $t^n B$ in R (under the ring homomorphism classifying the map h) and therefore $a \in \mathfrak{m}_y$. It follows that we can write $a = t a'$ where $a' \in A^{(1)}$. Then $\nu(b) = t^{1-n} a'$, and the desired result follows by applying the inductive hypothesis (after replacing Y by $D_y(Y)$). □

A.4. **Automorphisms of Semisimple Algebraic Groups.** Throughout this appendix, we fix an algebraically closed field k and a simply connected semisimple algebraic group G over k. Our goal is to establish some elementary facts about automorphisms of G which will be needed in the body of the paper.

Recall that a *pinning* of G is a triple $(B, T, \{\phi_\alpha : \mathbf{G}_a \to B\})$ where B is a Borel subgroup of G, T is a maximal torus contained in B, and $\{\phi_\alpha : \mathbf{G}_a \to B\}$ is a collection of maps which restrict to isomorphisms from $\mathbf{G}_a$ to the root subgroups of B corresponding to the simple roots (with respect to the maximal torus T). If σ is an automorphism of G, we say that σ *respects the pinning* $(B, T, \{\phi_\alpha\})$ if it carries B, T, and $\{\phi_\alpha\}$ into themselves. Let $\operatorname{Aut}_{(B,T,\{\phi_\alpha\})}(G)$ denote the group of all automorphisms of G which respect a pinning $(B, T, \{\phi_\alpha\})$. Then the group $\operatorname{Aut}_{(B,T,\{\phi_\alpha\})}(G)$ is isomorphic to the group of automorphisms of the Dynkin diagram of G. Consequently, if Γ is any group acting on the Dynkin diagram of G, then we can regard Γ as a group which acts on G preserving a pinning $(B, T, \{\phi_\alpha\})$. In this case, we will say that Γ acts on G *via automorphisms of its Dynkin diagram.*

Proposition A.4.1. *Let Γ be a finite group acting on G which preserves a pinning $(B, T, \{\phi_\alpha\})$. Let $N(T)$ denote the normalizer of T, and let G_0 denote the identity component of the fixed* *point locus $G^\Gamma \subseteq G$. Then there exists a k-valued point g of the intersection $N(T) \cap G_0$ which represents the longest element of the Weyl group $W = N(T)/T$.*

Proof. Let Φ denote the set of roots of G (which we regard as a subset of the character lattice $\mathrm{Hom}(T, \mathbf{G}_m)$), so that we can write $\Phi = \Phi_+ \amalg \Phi_-$ where Φ_+ is the set of roots of B, and $\Phi_- = \{-\alpha : \alpha \in \Phi_+\}$. Let W_0 denote the image of $N(T) \cap G_0$ in W, and choose an element $w \in W_0$ for which the set $\Phi_+ \cap w(\Phi_+)$ is as small as possible. We will complete the proof by showing that $\Phi_+ \cap w(\Phi_+) = \emptyset$ (so that w is the longest element of W). Suppose otherwise; then there exists a positive root $\alpha \in \Phi_+$ such that $w(\alpha)$ also belongs to Φ_+. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots, so that we can write $\alpha = \sum c_i \alpha_i$ where the coefficients c_i are nonnegative. Since $w(\alpha)$ is a positive root, at least one of the roots $w(\alpha_i)$ must also be positive. Replacing α by α_i, we may suppose that α is a simple root. Let Δ_0 denote the orbit of α under the action of the group Γ.

For every simple root $\beta \in \Delta$, let $\beta^\vee$ denote the corresponding coroot and let $s_\beta \in W$ be the corresponding simple reflection. Let us identify Δ with a set of vertices of the Dynkin diagram of G, and let Y denote the set of connected components of Δ_0 (regarded as a subgraph of the Dynkin diagram of G). For each $y \in Y$, let $G_y \subseteq G$ denote the Levi subgroup determined by the corresponding subset of the Dynkin diagram of G, let $\Phi_{+,y}$ denote the subset of Φ_+ consisting of positive roots of G_y, and let s_y denote the image in W of the longest element of the Weyl group of G_y.

Let $\Phi_{+,Y} = \bigcup_{y \in Y} \Phi_{+,y}$. Since w belongs to W_0, the set $\{\beta \in \Phi_+ : w(\beta) \in \Phi_+\}$ is Γ-invariant, and therefore contains $\Phi_{+,Y}$. Note that the elements $\{s_y\}_{y \in Y}$ commute; let s denote their product in W. Then the action of s carries $\Phi_{+,Y}$ to $-\Phi_{+,Y}$, and carries the set $\Phi_+ \setminus \Phi_{+,Y}$ to itself. Consequently, we have

$$\Phi_+ \cap ws(\Phi_+) = (\Phi_+ \cap w(\Phi_+)) \setminus w(\Phi_{+,Y}) \subsetneq \Phi_+ \cap w(\Phi_+).$$

We will complete the proof by showing that $s \in W_0$ (so that $ws \in W_0$, contradicting our choice of w). To prove this, we are free to replace G by the subgroup determined by the Dynkin diagram Δ_0. Replacing G by its simply connected cover, we may suppose that G is simply connected, so that G factors as a product $\prod_{y \in Y} G_y$. This determines a factorization of the Weyl group W as a product $\prod_{y \in Y} W_y$, under which the element s corresponds to the tuple $\{s_y\}_{y \in Y}$. Fix an element $y \in Y$, and let Γ_y denote the stabilizer of y in Γ. Then we have a canonical isomorphism $G^\Gamma \simeq G_y^{\Gamma_y}$. We may therefore replace G by G_y and Γ by Γ_y, and thereby reduce to the case where G is a simple group and Γ acts transitively on the set of vertices of the Dynkin diagram of G. Invoking the classification of semisimple algebraic groups, we see that there are only two possibilities for the structure of G:

(*a*) We have $G = \mathrm{SL}_2$, equipped with the trivial action of Γ. In this case, there is nothing to prove.

(*b*) We have $G = \mathrm{SL}_3$, where Γ contains an element γ which nontrivially permutes the two vertices of the Dynkin diagram of G. Let us identify the Borel subgroup B and its opposite with the subset of G consisting of upper and lower triangular matrices, respectively. A simple calculation shows that the action of γ is given by the formulae

$$\sigma \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y & xy - z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ xy - z & x & 1 \end{pmatrix}.$$

Choose an element c of k satisfying $c^2 = 2$. Then we obtain algebraic group homomorphisms $u, v : \mathbf{G}_a \to G_0$ given by the formulae

$$u(t) = \begin{pmatrix} 1 & ct & t^2 \\ 0 & 1 & ct \\ 0 & 0 & 1 \end{pmatrix} \qquad v(t) = \begin{pmatrix} 1 & 0 & 0 \\ ct & 1 & 0 \\ t^2 & ct & 1 \end{pmatrix}.$$

A simple calculation gives

$$u(1)v(-1)u(1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which is an element of $N(T)$ representing the longest element of the Weyl group W.

□

Fix a Borel subgroup $B \subseteq G$. Recall that a pair of Borel subgroups $B', B'' \subseteq G$ are said to be *in general position* if the intersection $B' \cap B''$ is a maximal torus of G. We will need the following fact:

Proposition A.4.2. *Let Γ be a finite group acting faithfully on G which respects a pinning $(B, T, \{u_\alpha\})$. Suppose that G does not have a simple factor of type A_{2m} which is acted on nontrivially by its stabilizer in Γ. Then there exists a finite collection of closed points $g_1, \ldots, g_n$ belonging to the identity component of G^Γ with the following property: every Borel subgroup of G is in general position with respect to some conjugate $g_i B g_i^{-1}$.*

Remark A.4.3. The hypothesis that G does not have a simple factor of type A_{2m} with a nontrivial action of its stabilizer is equivalent to the requirement that the Dynkin diagram of G does not contain any edge connecting two vertices belonging to the same orbit of Γ.

Proof of Proposition A.4.2. For each point $g \in G(k)$, let $W_g \subseteq G/B$ be the open set whose points are given by cosets hB such that hBh^{-1} is in general position with respect to gBg^{-1}. Let W be the union of the open sets W_g as g ranges over all k-valued points belonging to the identity component of G^Γ. Since G/B is quasi-compact, it will suffice to show that $W = G/B$. Suppose otherwise. Then $Y = G/B - W$ is a closed subset of G/B which is invariant under the action of the identity component of G^Γ.

Let Δ be the set of simple roots of G and let $\pi : \Delta \to \Delta/\Gamma$ denote the projection map. For each element $\overline{\alpha} \in \Delta/\Gamma$, we define a map $f_{\overline{\alpha}} : \mathbf{A}^1 \to G$ by the formula

$$f_{\overline{\alpha}}(t) = \prod_{\pi(\alpha) = \overline{\alpha}} u_\alpha(t).$$

Note that our hypothesis on G guarantees that the elements $\{u_\alpha(t)\}_{\pi(\alpha)=\overline{\alpha}}$ commute with one another, so that this product is independent of the ordering of the set $\pi^{-1}\{\overline{\alpha}\} \subseteq \Delta$. Let $f : \mathbf{A}^1 \to G$ denote the product of the maps $\{f_{\overline{\alpha}}\}_{\overline{\alpha} \in \Delta/\Gamma}$ (with respect to any ordering of the set Δ/Γ). Let $U \subseteq G$ denote the closed subgroup generated by the image of f. Because U is contained in G^Γ, it acts on the set Y. The algebraic group U is connected (since $\mathbf{A}^1$ is connected) and solvable (since it is contained in B), so that the action of U on Y has a fixed point hB. Then U is contained in the Borel subgroup hBh^{-1}. Note that $f(1)$ is a regular unipotent element of G (Lemma 3.2 of [52]), and is therefore contained in a *unique* Borel subgroup of G. We therefore have $B = hBh^{-1}$. Since $hB \in Y$, we conclude that B cannot be in general position with respect to gBg^{-1} for any point g belonging to the identity component of G^Γ, contradicting Proposition A.4.1. □

Proposition A.4.2 fails for groups of type A_{2n} if the field k has characteristic 2. However, we have the following:

Proposition A.4.4. *Let G be a simple algebraic group of type A_{2n} and let $\Gamma = \mathbf{Z}/2\mathbf{Z}$ be group of automorphisms of G which preserve a pinning $(B, T, \{u_\alpha\})$. Let $W \subseteq G/B$ be the open set defined in the proof of Proposition A.4.2. Then one of the following assertions holds:*

(1) *The field k has characteristic $\neq 2$ and $W = G/B$.*
(2) *The field k has characteristic 2 and the closed subset $Y = G/B - W$ can be written as a disjoint union $K_- \amalg K_+$, where the components K_- and K_+ are exchanged by the action of Γ.*

The proof of Proposition A.4.4 depends on the following elementary linear-algebraic fact:

Lemma A.4.5. *Let V be a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form $b : V \times V \to V$, and suppose that b satisfies $b(v, v) = 0$ if k is of characteristic 2. Let $V' \subseteq V$ be a subspace such that $\dim(V) \leq 2\dim(V')$. Then there exists a subspace $V'' \subseteq V$ such that $V' \cap V'' = \{0\}$, $V' + V'' = V$, and b vanishes on $V'' \times V''$.*

Proof. Let $V'' \subseteq V$ be maximal among those isotropic subspaces of V for which $V' \cap V'' = \{0\} = V'^{\perp} \cap V''$. Then the intersections $V' \cap V''^{\perp}$ and $V'^{\perp} \cap V''^{\perp}$ have codimension $\dim(V'')$ in V' and $V'^{\perp}$, respectively. The quotient $W = V''^{\perp}/V''$ inherits a nondegenerate symmetric bilinear form, and we have injective maps

$$V' \cap V''^{\perp} \to W \qquad V'^{\perp} \cap V''^{\perp} \to W$$

whose images are orthogonal subspaces S and $S^{\perp}$ of dimension $\dim(V') - \dim(V'')$ and $\dim(V) - \dim(V') - \dim(V'')$, respectively. The maximality of V' implies that every nonzero isotropic vector of W is contained in either S or $S^{\perp}$, and is therefore contained in $S \cap S^{\perp}$. Since V is generated by isotropic vectors, it follows that

$$\dim(V) - 2\dim(V'') = \dim(W) \leq \dim(S) = \dim(V') - \dim(V''),$$

so that $\dim(V) \leq \dim(V') + \dim(V'')$ and therefore $V' + V'' = V$. □

Proof of Proposition A.4.4. Without loss of generality we may assume that G is simply connected so that there is an isomorphism $G \simeq \mathrm{SL}(V)$, where V is a vector space of dimension $2n + 1$ over k. Then we can identify k-valued points of the quotient G/B with complete flags

$$(0) = F_0 \subset F_1 \subset \cdots \subset F_{2n} \subset F_{2n+1} = V$$

in the vector space V, where two flags $F_\bullet$ and $F'_\bullet$ are in general position if $F_i \cap F'_j = \{0\}$ for $i + j \leq 2n + 1$. Let σ denote the nontrivial element Γ. Then there is a nondegenerate symmetric bilinear form $b : V \times V \to k$ (unique up to scalar multiplication) for which σ is characterized by the formula

$$b(g^\sigma v, v') = b(v, g^{-1} v').$$

The Borel subgroup B can be identified with the stabilizer of a particular flag

$$(0) = V_0 \subset V_1 \subset \cdots \subset V_{2n} \subset V_{2n+1} = V$$

for which each V_i is the orthogonal complement (with respect to the bilinear form b) of V_{2n+1-i}.

Let G_0 denote the identity component of G^Γ. For each complete flag $F_\bullet$ in V, let $G_{0,F_\bullet}$ denote the closed subset of G_0 whose k-valued points $g \in G_0(k)$ such that $F_\bullet$ is not in general position with respect to $gV_\bullet$. Note that $G_{0,F_\bullet}$ can be written as a union of closed subsets

$$G^1_{0,F_\bullet} \cup \cdots \cup G^{2n}_{0,F_\bullet},$$

where $G^d_{0,F_\bullet}$ consists of those points g for which $gV_d \cap F_{2n+1-d} \neq \{0\}$. Since G_0 is irreducible, it follows that $G_0 = G_{0,F_\bullet}$ if and only if $G_0 = G^d_{0,F_\bullet}$ for some $1 \leq d \leq 2n$. We may therefore write Y as a union $\bigcup_{1 \leq d \leq 2n} Y_d$, where Y_d consists of those flags $F_\bullet$ such that $gV_d \cap F_{2n+1-d} \neq \{0\}$ for all $g \in G_0(k)$. Note that the involution σ carries Y_d to Y_{2n+1-d}. We will prove the following:

($*$) Let $1 \leq d \leq n$. Then the set $Y_d \subseteq G/B$ contains no fixed points for σ, and is empty unless the characteristic of k is equal to 2.

Proposition A.4.4 follows immediately from ($*$) (if the characteristic of k is equal to 2, we can take $K_- = Y_1 \cup \cdots \cup Y_n$ and $K_+ = Y_{n+1} \cup \cdots \cup Y_{2n}$).

Note that the field k has characteristic different from 2, then we can identify the fixed point subgroup G^Γ with the orthogonal group $\mathrm{O}(V)$, so that the identity component of G^Γ is given by $\mathrm{SO}(V)$. When k has characteristic 2, the bilinear form b restricts to a symplectic form on the subspace $V' = \{v \in V : b(v,v) = 0\}$, and the fixed point locus G^Γ (with its reduced scheme structure) is isomorphic to the symplectic group $\mathrm{Sp}(V')$. In either case, the identity component of G^Γ acts transitively on the collection of all d-dimensional isotropic subspaces of V for $1 \leq d \leq n$. Consequently, a flag $F_\bullet$ belongs to Y_d if and only if $F_{2n+1-d} \cap F' \neq \{0\}$ for every isotopic subspace F' of dimension d. We conclude that ($*$) follows immediately from Lemma A.4.5 when the characteristic of k is different from 2.

We will henceforth assume that the characteristic of k is equal to 2. In this case, we will deduce ($*$) from the following:

($*'$) Let $1 \leq d \leq n$. Then a complete flag $F_\bullet$ belongs to Y_d if and only if every vector belonging to F_{2n+1-d} is isotropic.

Assume ($*'$) for the moment. The subset $V' = \{v \in V : b(v,v) = 0\}$ is a linear subspace of codimension 1 in V, so we can choose a vector $v_0 \in V$ such that $V' = \{v \in V : b(v, v_0) = 0\}$. Assertion ($*'$) implies that a flag $F_\bullet$ is contained in Y_d if and only if $v_0 \in F^\perp_{2n+1-d}$. If, in addition, $F_\bullet$ is fixed by the involution σ, then we deduce that $F^\perp_{2n+1-d} = F_d$, so that $v_0 \in F_d \subseteq F_{2n+1-d}$. It follows that $b(v_0, v_0) = 0$, so that $v_0 \in V'$. Then b descends to a nondegenerate symplectic form on the vector space V'/kv_0. Since a vector space of odd dimension cannot support a nondegenerate symplectic form, we conclude that σ has no fixed points on Y_d.

We now prove ($*'$). Suppose first that every vector in F_{2n+1-d} is isotropic. Then the characteristic of k is equal to 2 (by the argument given above) and we have $F_{2n+1-d} \subseteq V'$. For any k-valued point g of G^Γ, the subspace $g(V_d)$ is isotopic and therefore contained in V', so has nontrivial intersection with F_{2n+1-d} in V'.

For the converse, suppose that F_{2n+1-d} contains a non-isotropic vector. Then $F_{2n+1-d} \cap V'$ has codimension $d \leq n$ in V'. Applying Lemma A.4.5, we deduce that there is an isotropic subspace $F' \subseteq V'$ of dimension d such that $F' \cap F_{2n+1-d} = \{0\}$, so that $F_\bullet \notin Y_d$. □

Corollary A.4.6. *Let Γ be a finite group acting on G which respects a pinning $(B, T, \{u_\alpha\})$. Then the canonical map $\theta : G^\Gamma \to (G/B)^\Gamma$ is surjective.*

Proof. Since G is simply connected, it factors as a product of simple factors $\prod_{i \in I} G_i$. Let $I_0 \subseteq I$ be a set of representatives for the orbits of Γ on I, and for each $i \in I_0$ let Γ_i denote the stabilizer of $i \in \Gamma$. Then θ can be identified with the product

$$\prod_{i \in I_0} G_i^{\Gamma_i} \to \prod_{i \in I_0} (G_i/B_i)^{\Gamma_i},$$

where B_i denotes the intersection of B with G_i. We may therefore assume without loss of generality that G is simple.

We first treat the special case where $G = \mathrm{SL}(V)$ for a vector space V of odd dimension $2n+1$, the field k has characteristic 2, and the action of Γ is nontrivial. Let $b : V \times V \to k$

be as in the proof of Proposition A.4.4 and identify the k-valued points of G/B with complete flags

$$(0) = F_0 \subset F_1 \subset \cdots \subset F_{2n} \subset F_{2n+1} = V$$

so that a flag $F_\bullet$ is Γ-invariant if and only if $F_d^\perp = F_{2n+1-d}$ for $1 \leq d \leq 2n$. Let $F_\bullet$ and $F'_\bullet$ be two such flags; we wish to show that there exists an automorphism of V which preserves the bilinear form b and carries $F_\bullet$ to $F'_\bullet$. For $1 \leq i \leq n$, let v_i be an element of $F_i \setminus F_{i-1}$, so that F_i is the linear span of the set of vectors $\{v_1, \ldots, v_i\}$, and define $v'_1, \ldots, v'_n$ similarly. Note that $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_n\}$ are bases for Lagrangian subspaces of the symplectic vector space $V' = \{v \in V : b(v,v) = 0\}$, so there exists a symplectic automorphism of V' which carries each v_i to v'_i, and this symplectic automorphism extends uniquely to an automorphism of V which preserves the symmetric bilinear form b.

We now treat the non-exceptional cases (that is, the cases where $G \neq \mathrm{SL}_{2n+1}$, Γ is trivial, or k does not have characteristic 2). Let B_- be the unique Borel subgroup of G which contains T and is in general position with respect to B and let U_- denote the unipotent radical of B_-, so that evaluation on the identity coset determines an open immersion $U_- \hookrightarrow G/B$ whose image is the open subset $V \subseteq G/B$ consisting of those cosets gB for which gBg^{-1} is in general position with respect to B_-. Propositions A.4.2 and A.4.4 imply that G/B can be written as a union $\bigcup_{1 \leq i \leq n} g_i V$ for some finite collection of elements $g_1, \ldots, g_n \in G(k)^\Gamma$. It will therefore suffice to show that every Γ-invariant element of $g_i V$ can be lifted to a Γ-invariant element of G. Since g_i is Γ-invariant, we are reduced to proving that the map $(G \times_{G/B} V)^\Gamma \to V^\Gamma$ is surjective. This is clear, since the composite map

$$U_-^\Gamma \to (G \times_{G/B} V)^\Gamma \to V^\Gamma$$

is an isomorphism. □

Corollary A.4.7. *Let Γ be a finite group acting on G which respects a pinning $(B, T, \{u_\alpha\})$, and let G_0 be the identity component of G^Γ. Then $B \cap G_0$ is a Borel subgroup of G_0 (when endowed with the reduced scheme structure).*

Proof. Let us regard G^Γ, B^Γ, and $(G/B)^\Gamma$ as closed subschemes of G, B, and (G/B), respectively (endowed with the reduced scheme structure). Then G^Γ and B^Γ are reduced group schemes over k, and therefore smooth. Since k is algebraically closed, the smooth locus of $(G/B)^\Gamma$ is dense in $(G/B)^\Gamma$. It follows from Corollary A.4.6 that the group G^Γ acts transitively on $(G/B)^\Gamma$, from which we deduce that $(G/B)^\Gamma$ is smooth. We have an evident map of smooth varieties $\phi : G^\Gamma/B^\Gamma \to (G/B)^\Gamma$, and Corollary A.4.6 guarantees that this map is bijective on k-points.

We now claim that ϕ is finite. To prove this, it will suffice to show that for every closed point x of $(G/B)^\Gamma$, there exists an étale neighborhood U of x such that the induced map $\phi_U : (G^\Gamma/B^\Gamma) \times_{(G/B)^\Gamma} U \to U$ is finite. Let $\widetilde{x}$ denote a k-valued point of U lying over x. Choosing U sufficiently fine, we can arrange that the domain of ϕ_U decomposes as a disjoint union $V \amalg W$, where $\phi_U|_V$ is finite and the image of $\phi_U|_W$ does not contain the point $\widetilde{x}$. Since ϕ_U is bijective on k-valued points, it follows that $\widetilde{x}$ belongs to the image of $\phi_U|_V$. Since ϕ_U is a quasi-finite map between smooth varieties, it is flat, so that the image of $\phi_U|_V$ is an open subset $U_0 \subseteq U$ containing $\widetilde{x}$. Replacing U by U_0, we may assume that $\phi_U|_V$ is surjective. The bijectivity of ϕ_U then implies that $W = \emptyset$, so that ϕ_U is finite as desired.

Because ϕ is finite, the quotient G^Γ/B^Γ is proper over $\operatorname{Spec} k$. Since $G_0/(B \cap G_0)$ can be identified with a connected component of G^Γ/B^Γ, we deduce that $G_0/(B \cap G_0)$ is also proper over $\operatorname{Spec} k$: that is, $B \cap G_0$ is a parabolic subgroup of G_0. Since $B \cap G_0$ is a subgroup of B, it is solvable, and therefore a Borel subgroup of G_0. □

Remark A.4.8. In the situation of Corollary A.4.7, let B_- denote the unique Borel subgroup containing T which is in general position with respect to B. Then Corollary A.4.7 implies that $B_- \cap G_0$ is a Borel subgroup of G_0, and the intersection $(B_- \cap G_0) \cap (B \cap G_0) = T \cap G_0$ is diagonalizable. It follows that $B_- \cap G_0$ and $B \cap G_0$ are in general position with respect to one another, and that the intersection $T \cap G_0$ is a maximal torus of G_0. In particular, the algebraic group G_0 is reductive.

Definition A.4.9. Let G be a semisimple group equipped with a pinning $(B, T, \{u_\alpha\})$, and let $\mu \in \mathrm{Hom}(\mathbf{G}_m, T)$ be a cocharacter of T. We say that μ is *dominant* if, for every character $\alpha \in \mathrm{Hom}(T, \mathbf{G}_m)$ which is a root of B (that is, a positive root of G), we have $\langle \mu, \alpha \rangle \geq 0$. We say that μ is *strictly dominant* if $\langle \mu, \alpha \rangle > 0$ for every root α of B. Here $\langle , \rangle$ denotes the canonical pairing

$$\mathrm{Hom}(\mathbf{G}_m, T) \times \mathrm{Hom}(T, \mathbf{G}_m) \to \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) \simeq \mathbf{Z}.$$

Proposition A.4.10. *Let Γ be a finite group acting on G which respects a pinning $(B, T, \{u_\alpha\})$, let G_0 be the identity component of G^Γ, and let $T_0 = T \cap G_0$. Then the inclusion $i : T_0 \hookrightarrow T$ carries strictly dominant cocharacters of T_0 to strictly dominant cocharacters of T.*

Proof. The inclusion i determines group homomorphisms

$$i_* : \mathrm{Hom}(\mathbf{G}_m, T_0) \to \mathrm{Hom}(\mathbf{G}_m, T) \qquad i^* : \mathrm{Hom}(T, \mathbf{G}_m) \to \mathrm{Hom}(T_0, \mathbf{G}_m).$$

Let μ be a strictly dominant cocharacter of T_0; we wish to prove that $i_*\mu$ is a strictly dominant cocharacter of T. Equivalently, we wish to prove that the integer

$$\langle i_*\mu, \alpha \rangle = \langle \mu, i^*\alpha \rangle$$

is positive for every positive root α of G. Without loss of generality, we may suppose that α is a simple root, which determines a vertex v of the Dynkin diagram of G. Arguing as in the proof of Proposition A.4.1, we can replace G by a subgroup whose Dynkin diagram coincides with the Γ-orbit of v, and thereby reduce to the case where Γ acts transitively on the simple roots of G. Replacing G by its simply connected cover, we may assume that G factors as a product of simple factors $\prod G_i$, which are permuted transitively by Γ. Replacing G by one of the factors G_i and Γ by the stabilizer of G_i, we may assume either that $G = \mathrm{SL}_2$ (with a trivial action of Γ) or that $G = \mathrm{SL}_3$ (with the nontrivial action of Γ described in the proof of Proposition A.4.1). In the first case, the result is obvious. In the second, we note that the homomorphism $u : \mathbf{G}_a \to G_0$ appearing in the proof of Proposition A.4.1 is is T_0-equivariant, where T_0 acts on $\mathbf{G}_a$ by the character $i^*\alpha$. Let d denote the degree of u, regarded as an isogeny from $\mathbf{G}_a$ to the unipotent radical of $G_0 \cap B$ (so that $d = 1$ unless the ground field k has characteristic 2, in which case we have $d = 2$). Then $di^*(\alpha)$ is a positive root of G_0, so that $\langle \mu, i^*\alpha \rangle = \frac{1}{d} \langle \mu, di^*\alpha \rangle > 0$ by virtue of our assumption that μ is strictly dominant. □

A.5. **A Relative Künneth Formula.** Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. In §2.3, we showed that if $\mathcal{C}$ and $\mathcal{C}'$ are prestacks, then there is a canonical equivalence

$$C_*(\mathcal{C}; \mathbf{Z}_\ell) \otimes C_*(\mathcal{C}'; \mathbf{Z}_\ell) \simeq C_*(\mathcal{C} \times_{\mathrm{Spec}\, k} \mathcal{C}'; \mathbf{Z}_\ell)$$

(see Proposition 2.3.40). In this section, we will discuss analogous results where the fiber product is taken not over $\mathrm{Spec}\, k$, but over an arbitrary quasi-projective k-scheme X. For example, we will show that given a suitable pair of maps $\mathcal{C} \to X \leftarrow \mathcal{C}'$, there is a canonical equivalence

$$[\mathcal{C}]_X \otimes^! [\mathcal{C}']_X \simeq [\mathcal{C} \times_X \mathcal{C}']_X$$

in the ∞-category of ℓ-adic sheaves on X, where $\otimes^!$ denotes the !-tensor product introduced in §4.6 and $[\mathcal{C}]_X$ denotes the cohomology sheaf of the projection map $\mathcal{C} \to X$ (see §5.1). However, this statement is valid in somewhat less generality than Proposition 2.3.40:

- Since we are working in cohomology rather than homology, we generally only expect a Künneth formula under the assumption that $\mathcal{C}$ and $\mathcal{C}'$ satisfy some reasonable finiteness conditions. We will therefore restrict our attention to the case where $\mathcal{C}$ and $\mathcal{C}'$ are quasi-compact Artin stacks with affine diagonal (the condition that the diagonal be affine can be relaxed, but is satisfied for all the Artin stacks of interest to us in this paper).
- The formation of fiber products over X is generally a rather severe operation from a topological point of view. We will therefore restrict our attention to the case where the projection maps $\mathcal{C} \to X \leftarrow \mathcal{C}'$ are smooth (in which case the formation of the cohomology sheaves $[\mathcal{C}]_X$ and $[\mathcal{C}']_X$ is compatible with base change; see Proposition 5.1.9).

Let us now outline the contents of this section. Our main objective will be to give careful constructions of the ∞-category $\mathrm{Shv}_\ell^!$ and the 2-category $\mathrm{AlgStack}^!$ of Definition 5.1.18 and 5.1.19, and of the functor $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ described in Proposition 5.1.20. The essential properties of our constructions can be summarized as follows:

(a) The ∞-category $\mathrm{Shv}_\ell^!$ is comprised of pairs $(X, \mathcal{F})$, where X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$. This ∞-category is equipped with a forgetful functor $\rho : \mathrm{Shv}_\ell^! \to \mathrm{Sch}_k^{\mathrm{pr}}$, where $\mathrm{Sch}_k^{\mathrm{pr}}$ denotes the category whose objects are quasi-projective k-schemes and whose morphisms are proper maps. This forgetful functor is both a Cartesian fibration and a coCartesian fibration; to every proper morphism $f : X \to Y$ between quasi-projective k-schemes, it determines a pair of adjoint functors

$$\mathrm{Shv}_\ell(X) \underset{f^!}{\overset{f_*}{\rightleftarrows}} \mathrm{Shv}_\ell(Y).$$

(b) The ∞-category $\mathrm{Shv}_\ell^!$ is equipped with a symmetric monoidal structure, whose underlying tensor product is given by

$$(X, \mathcal{F}) \otimes (X', \mathcal{F}') = (X \times X', \mathcal{F} \boxtimes \mathcal{F}').$$

The forgetful functor $\rho : \mathrm{Shv}_\ell^! \to \mathrm{Sch}_k^{\mathrm{pr}}$ is a symmetric monoidal functor, and the collection of ρ-Cartesian and ρ-coCartesian morphisms are stable under the formation of tensor products (this is essentially a restatement of Propositions 4.6.2 and 4.6.7).

(c) The 2-category $\mathrm{AlgStack}^!$ is comprised of pairs $(X, \mathcal{C})$, where X is a quasi-projective k-scheme and $\mathcal{C}$ is a quasi-compact Artin stack with affine diagonal equipped with a smooth map $\mathcal{C} \to X$. The construction $(X, \mathcal{C}) \mapsto X$ determines a forgetful functor $\rho' : \mathrm{AlgStack}^! \to \mathrm{Sch}_k^{\mathrm{pr}}$; let us denote the fiber of ρ' over a quasi-projective k-scheme X by $\mathrm{AlgStack}^!_X$ (by construction, this will be the *opposite* of the 2-category of quasi-compact Artin stacks with affine diagonal which are smooth over X). The functor ρ' is a Cartesian fibration: it associates to each proper morphism $f : X \to Y$ a pullback functor $\mathrm{AlgStack}^!_Y \to \mathrm{AlgStack}^!_X$, given at the level of objects by $\mathcal{C} \mapsto \mathcal{C} \times_Y X$.

(d) The 2-category $\mathrm{AlgStack}^!$ admits a symmetric monoidal structure whose underlying tensor product is given by

$$(X, \mathcal{C}) \otimes (X', \mathcal{C}') = (X \times X', \mathcal{C} \times_{\mathrm{Spec}\, k} \mathcal{C}').$$

The forgetful functor ρ' is symmetric monoidal, and the collection of ρ'-Cartesian morphisms is closed under tensor products.

(e) The functor $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ is given on objects by the formula $\Phi(X, \mathcal{C}) = (X, [\mathcal{C}]_X)$. It is a symmetric monoidal functor: that is, for every pair of objects $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ of $\mathrm{AlgStack}^!$, we have a canonical equivalence of ℓ-adic sheaves

$$[\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}']_{X \times X'} \simeq [\mathcal{C}]_X \boxtimes [\mathcal{C}']_{X'}.$$

(f) We have a commutative diagram of symmetric monoidal functors

$$\begin{array}{ccc} \mathrm{AlgStack}^! & \xrightarrow{\Phi} & \mathrm{Shv}_\ell^! \\ & {\scriptstyle \rho'} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathrm{Sch}_k^{\mathrm{pr}} . & \end{array}$$

Moreover, the functor Φ carries ρ'-Cartesian morphisms to ρ-Cartesian morphisms. More concretely, for every object $(X, \mathcal{C})$ of $\mathrm{AlgStack}^!$ and every proper morphism $f : Y \to X$, we have a canonical equivalence $[\mathcal{C} \times_X Y]_Y \simeq f^! [\mathcal{C}]_X$ (see Proposition 5.1.9).

Example A.5.1. The relative Künneth formula

$$[\mathcal{C}]_X \otimes^! [\mathcal{C}']_X \simeq [\mathcal{C} \times_X \mathcal{C}']_X$$

for $\mathcal{C}, \mathcal{C}' \in \mathrm{AlgStack}_X^!$ is an immediate consequence of (e) and (f) above.

Remark A.5.2. Example A.5.1 is somewhat misleading: the Künneth formula

$$[\mathcal{C}]_X \otimes^! [\mathcal{C}']_X \simeq [\mathcal{C} \times_X \mathcal{C}']_X$$

can be deduced easily from the results of §5.1 together with Proposition A.5.19 below: it does not rely on the careful treatment of homotopy coherence which is the main objective of this section. Issues of homotopy coherence arise when we want to discuss commutative algebra objects of $\mathrm{Shv}_\ell^!$, which play an essential role in §5.5, §5.6, and §5.7.

Warning A.5.3. Our definitions (and the verification of properties (a) through (f)) will require a somewhat elaborate series of categorical constructions. These constructions are entirely formal: most of the essential geometric input comes from the results of §4.6 and §5.1. Consequently, this section can be safely skipped by a reader who is willing to accept the existence of the functor $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ satisfying (a) through (f).

For the remainder of this section, we will assume that the reader is familiar with the language of Cartesian and coCartesian fibrations of ∞-categories (see [34]) and with the language of symmetric monoidal ∞-categories (see [35]).

Notation A.5.4. Let $\mathcal{C}\mathrm{at}_\infty$ denote the ∞-category of ∞-categories (in the ensuing discussion, we will ignore issues of size), and $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ the ∞-category whose objects are functors $f : \mathcal{C} \to \mathcal{D}$. Let $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ denote the subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ whose objects are Cartesian fibrations $f : \mathcal{C} \to \mathcal{D}$ and whose morphisms are diagrams

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{g} & \mathcal{C}' \\ \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f'} \\ \mathcal{D} & \longrightarrow & \mathcal{D}' \end{array}$$

where g carries f-Cartesian morphisms of $\mathcal{C}$ to f'-Cartesian morphisms of $\mathcal{C}'$. Similarly, we let $\mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ denote the subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ whose objects are coCartesian fibrations $f : \mathcal{C} \to \mathcal{D}$ and whose morphisms are diagrams

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{g} & \mathcal{C}' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathcal{D} & \longrightarrow & \mathcal{D}' \end{array}$$

where g carries f-coCartesian morphisms of $\mathcal{C}$ to f'-coCartesian morphisms of $\mathcal{C}'$

The construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ determines an equivalence σ of $\mathcal{C}\mathrm{at}_\infty$ with itself. Suppose that $q : \mathcal{C} \to \mathcal{D}$ is a Cartesian fibration of ∞-categories. Then q is classified by a functor $\chi : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, given informally by the formula $\chi(D) = \mathcal{C} \times_{\mathcal{D}} \{D\}$. The composition $\sigma \circ \chi : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ classifies another Cartesian fibration $q' : \mathcal{C}' \to \mathcal{D}$, whose fibers are equivalent to the opposites of the fibers of q. We will refer to q' as the *Cartesian dual* of the Cartesian fibration q. The formation of Cartesian duals determines an equivalence from the ∞-category $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ to itself (for an explicit construction of this equivalence, we refer the reader to §SAG.4.3.4).

If $q : \mathcal{C} \to \mathcal{D}$ is a coCartesian fibration of ∞-categories, then q is classified by a functor $\chi : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$. The composition $\sigma \circ \chi$ classifies another coCartesian fibration $q'' : \mathcal{C}'' \to \mathcal{D}$, which we will refer to as the *coCartesian dual* of q. The formation of coCartesian duals determines an equivalence of $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ with itself.

Warning A.5.5. Let $q : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories which is both a Cartesian fibration and a coCartesian fibration. Then we can consider *either* the coCartesian dual $q' : \mathcal{C}' \to \mathcal{D}$ or the Cartesian dual $q'' : \mathcal{C}'' \to \mathcal{D}$ of q. For each object $D \in \mathcal{D}$, we have a canonical equivalence

$$\mathcal{C}' \times_{\mathcal{D}} \{D\} \simeq (\mathcal{C} \times_{\mathcal{D}} \{D\})^{\mathrm{op}} \simeq \mathcal{C}'' \times_{\mathcal{D}} \{D\}.$$

However, the ∞-categories $\mathcal{C}'$ and $\mathcal{C}''$ are usually *not* equivalent to one another.

Remark A.5.6. We regard $\mathcal{C}\mathrm{at}_\infty$ as a symmetric monoidal ∞-category via the Cartesian product, so that the ∞-category $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ of commutative algebra objects of $\mathcal{C}$ can be identified with the ∞-category of symmetric monoidal ∞-categories. Using the canonical isomorphism $\mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)) \simeq \mathrm{CAlg}(\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty))$, we can identify the objects of $\mathrm{CAlg}(\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty))$ with symmetric monoidal functors $f : \mathcal{C} \to \mathcal{D}$. We have fully faithful embeddings

$$\mathrm{CAlg}(\mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)) \xhookrightarrow{\theta} \mathrm{CAlg}(\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)) \xhookleftarrow{\theta'} \mathrm{CAlg}(\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)).$$

The essential image of θ can be identified with the collection of symmetric monoidal functors $f : \mathcal{C} \to \mathcal{D}$ which are equivalent to coCartesian fibrations, which have the additional property that the collection of f-coCartesian morphisms in $\mathcal{C}$ is closed under tensor products by objects of $\mathcal{C}$. Similarly, we can identify the essential image of the functor θ' with those symmetric monoidal functors $f : \mathcal{C} \to \mathcal{D}$ which are equivalent to Cartesian fibrations having the property that the collection of f-Cartesian morphisms in $\mathcal{C}$ is closed under tensor products by objects of $\mathcal{C}$. We will refer to these types of symmetric monoidal functors as *symmetric monoidal coCartesian fibrations* and *symmetric monoidal Cartesian fibrations*, respectively.

Remark A.5.7. Let $\mathcal{D}$ be a symmetric monoidal ∞-category. Using Proposition HA.2.4.3.16, we see that the following types of data are equivalent:

- Symmetric monoidal coCartesian fibrations $q : \mathcal{C} \to \mathcal{D}$.
- Lax symmetric monoidal functors $\chi : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ (that is, functors equipped with multiplication maps $\chi(D) \times \chi(D') \to \chi(D \otimes D')$ which are coherently unital, commutative, and associative, but need not be equivalences).

Concretely, this equivalence is implemented by the construction which assigns to each coCartesian fibration $q : \mathcal{C} \to \mathcal{D}$ the functor χ given by $\chi(D) = \mathcal{C} \times_{\mathcal{D}} \{D\}$.

Example A.5.8. In the situation of Remark A.5.7, suppose that $\mathcal{D}$ admits finite coproducts and that the symmetric monoidal structure on $\mathcal{D}$ is given by the formation of coproducts. Using Theorem HA.2.4.3.18, we see that the data of a lax symmetric monoidal functor from $\mathcal{D}$ to $\mathcal{C}\mathrm{at}_\infty$ is equivalent to the data of an arbitrary functor from $\mathcal{D}$ to the ∞-category $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ of symmetric monoidal ∞-categories. Concretely, if $\chi : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ is a lax monoidal functor, then for each object $D \in \mathcal{D}$ the "fold map" $e : D \amalg D \to D$ induces a multiplication map

$$\chi(D) \times \chi(D) \to \chi(D \amalg D) \xrightarrow{\chi(e)} \chi(D)$$

which endows $\chi(D)$ with the structure of a symmetric monoidal ∞-category. Conversely, for any functor $\chi : \mathcal{D} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$, the tensor product $\otimes$ on the ∞-categories $\chi(D \amalg D')$ determines maps

$$\chi(D) \times \chi(D') \to \chi(D \amalg D') \times \chi(D \amalg D') \xrightarrow{\otimes} \chi(D \amalg D')$$

which exhibit χ as a lax symmetric monoidal functor.

Construction A.5.9. For each quasi-projective k-scheme X, we regard $\mathrm{Shv}_\ell(X)$ as a symmetric monoidal ∞-category with respect to the usual tensor product of sheaves (see Remark 4.3.40). The construction $X \mapsto \mathrm{Shv}_\ell(X)$ determines a functor

$$\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty).$$

Applying the construction of Example A.5.8, we see that this functor classifies a symmetric monoidal coCartesian fibration $\mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$ (here the symmetric monoidal structure on Sch_k is given by the formation of Cartesian products). We can describe the symmetric monoidal ∞-category $\mathrm{Shv}_\ell^\star$ more informally as follows:

- (a) The objects of the ∞-category $\mathrm{Shv}_\ell^\star$ are pairs $(X, \mathcal{F})$, where X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$.
- (b) Given a pair of objects $(X, \mathcal{F}), (X', \mathcal{F}') \in \mathrm{Shv}_\ell^\star$, a morphism from $(X, \mathcal{F})$ to $(X', \mathcal{F}')$ is given by a pair (f, α), where $f : X' \to X$ is a morphism of k-schemes, and $\alpha : f^* \mathcal{F} \to \mathcal{F}'$ is a morphism in $\mathrm{Shv}_\ell(X')$.
- (c) The tensor product on $\mathrm{Shv}_\ell^\star$ is given by the formula

$$(X, \mathcal{F}) \otimes (X', \mathcal{F}') = (X \times X', \mathcal{F} \boxtimes \mathcal{F}').$$

Proposition A.5.10. *The forgetful functor $q : \mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$ is a symmetric monoidal Cartesian fibration.*

Proof. The assertion that q is a Cartesian fibration follows from the fact that for every map $f : X \to X'$ in Sch_k, the pullback functor $f^* : \mathrm{Shv}_\ell(X') \to \mathrm{Shv}_\ell(X)$ admits a right adjoint. The fact that the collection of q-Cartesian morphisms in $\mathrm{Shv}_\ell^\star$ is closed under tensor products by objects of $\mathrm{Shv}_\ell^\star$ is a reformation of Proposition 4.6.2. □

Construction A.5.11. Let us regard the symmetric monoidal Cartesian fibration q of Proposition A.5.10 as a commutative algebra object of the ∞-category $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$. Applying the Cartesian duality functor $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty) \simeq \mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ to $\mathrm{Shv}^\star$, we obtain a new symmetric monoidal Cartesian fibration which we will denote by $\mathrm{Shv}_\ell^{\diamond, \mathrm{op}} \to \mathrm{Sch}_k^{\mathrm{op}}$. We can describe the symmetric monoidal ∞-category $\mathrm{Shv}_\ell^\diamond$ more informally as follows:

- (a) The objects of $\mathrm{Shv}_\ell^\diamond$ are pairs $(X, \mathcal{F})$, where $X \in \mathrm{Sch}_k$ and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$.
- (b) A morphism from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ in $\mathrm{Shv}_\ell^\diamond$ consists of a map of k-schemes $f : X \to Y$ together with a morphism $f_* \mathcal{F} \to \mathcal{G}$ in the ∞-category $\mathrm{Shv}_\ell(Y)$.

(*c*) The tensor product on $\mathrm{Shv}_\ell^\diamond$ is given by $(X, \mathcal{F}) \otimes (Y, \mathcal{G}) = (X \times Y, \mathcal{F} \boxtimes \mathcal{G})$.

We let $\mathrm{Shv}_\ell^!$ denote the fiber product $\mathrm{Shv}_\ell^\diamond \times_{\mathrm{Sch}_k} \mathrm{Sch}_k^{\mathrm{pr}}$.

Proposition A.5.12. *The projection map*

$$r : \mathrm{Shv}_\ell^! \to \mathrm{Sch}_k^{\mathrm{pr}}$$

is a symmetric monoidal Cartesian fibration.

Proof. The assertion that r is a Cartesian fibration follows from the fact that for every proper morphism $f : X \to X'$ in Sch_k, the functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X')$ admits a right adjoint $f^!$. The fact that the collection of r-coCartesian morphisms in $\mathrm{Shv}_\ell^{!\,\mathrm{op}}$ is closed under tensor product by objects of $\mathrm{Shv}_\ell^{!\,\mathrm{op}}$ is a reformation of Corollary 4.6.8. □

Notation A.5.13. We define a 2-category RelStack as follows:

- An object of RelStack consists of a quasi-projective k-scheme X together with a morphism of prestacks $\pi : \mathcal{C} \to X$ (here we abuse notation by identifying X with the corresponding prestack).
- A morphism from $\pi : \mathcal{C} \to X$ to $\pi' : \mathcal{C}' \to X'$ in the category RelStack is a commutative diagram of prestacks

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & \mathcal{C}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{f} & X' \end{array}$$

  where f is proper. We regard the collection of morphisms from π to π' as a category, where

$$\mathrm{Hom}((\phi, f), (\phi', f'))$$

  is empty unless $f = f'$, in which case it is the set of all isomorphisms of ϕ with ϕ' which are compatible with π.

We will abuse notation by identifying RelStack with the its associated ∞-category. We let AlgStack denote the full subcategory of RelStack spanned by those maps $\pi : \mathcal{C} \to X$ where where $\mathcal{C}$ is a quasi-compact Artin stack with affine diagonal and the map $\mathcal{C} \to X$ is smooth. We regard RelStack and AlgStack as symmetric monoidal ∞-categories, where the symmetric monoidal structure is given by the Cartesian product.

Construction A.5.14. The construction $(\pi : \mathcal{C} \to X) \mapsto X$ determines symmetric monoidal Cartesian fibrations

$$\mathrm{RelStack} \to \mathrm{Sch}_k^{\mathrm{pr}} \qquad \mathrm{AlgStack} \to \mathrm{Sch}_k^{\mathrm{pr}}.$$

We will denote the Cartesian duals of these fibrations by $\mathrm{RelStack}^!$ and $\mathrm{AlgStack}^!$, respectively.

Remark A.5.15. More informally, we can identify $\mathrm{RelStack}^!$ with an ∞-category whose objects are pairs $(X, \mathcal{C})$, where X is a quasi-projective k-scheme and $\mathcal{C}$ is a prestack equipped with a map $\mathcal{C} \to X$; a morphism from $(X, \mathcal{C})$ to $(X', \mathcal{C}')$ in $\mathrm{RelStack}^!$ consists of a proper morphism of k-schemes $f : X \to X'$ together with a map of prestacks $X \times_{X'} \mathcal{C}' \to \mathcal{C}$. We can identify $\mathrm{AlgStack}^!$ with the full subcategory of $\mathrm{RelStack}^!$ spanned by those pairs $(X, \mathcal{C})$ where $\mathcal{C}$ is a quasi-compact Artin stack with affine diagonal and the projection $\mathcal{C} \to X$ is smooth.

The definition of the functor $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ will require a few auxiliary constructions.

Notation A.5.16. Let $\mathrm{RelStack}^{\star}$ denote the fiber product $\mathrm{RelStack} \times_{\mathrm{Sch}_k} (\mathrm{Shv}_\ell^{\star})^{\mathrm{op}}$, where $\mathrm{Shv}_\ell^{\star}$ is defined as in Construction A.5.9. More informally, $\mathrm{RelStack}^{\star}$ is the ∞-category whose objects are triples $(X, \mathcal{C}, \mathcal{F})$ where X is a quasi-projective k-scheme, $\mathcal{C}$ is a prestack equipped with a map $\pi : \mathcal{C} \to X$, and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$. We regard $\mathrm{RelStack}^{\star}$ as a symmetric monoidal ∞-category, with tensor product given on objects by the formula

$$(X, \mathcal{C}, \mathcal{F}) \otimes (X', \mathcal{C}', \mathcal{F}') = (X \times_{\mathrm{Spec}\, k} X', \mathcal{C} \times_{\mathrm{Spec}\, k} \mathcal{C}', \mathcal{F} \boxtimes \mathcal{F}').$$

Construction A.5.17. The construction $(\pi : \mathcal{C} \to X) \mapsto \mathcal{C}^{\mathrm{op}}$ determines a functor from the 2-category RelStack to the 2-category of categories. This functor classifies a coCartesian fibration of ∞-categories $\overline{\mathrm{RelStack}} \to \mathrm{RelStack}$. We will identify objects of $\overline{\mathrm{RelStack}}$ with triples $(X, \mathcal{C}, \eta)$, where X is a quasi-projective k-scheme, $\mathcal{C}$ is a prestack equipped with a map $\pi : \mathcal{C} \to X$, and η is an object of $\mathcal{C}$ which we identify with a map $\eta : \mathrm{Spec}\, R_\eta \to \mathcal{C}$. The construction $(X, \mathcal{C}, \eta) \mapsto \mathrm{Spec}\, R_\eta$ determines a forgetful functor $\rho : \overline{\mathrm{RelStack}} \to \mathrm{Sch}_k$. We define an ∞-category $\mathrm{RelStack}^{\star}_{\mathrm{aux}}$ equipped with a forgetful functor $\mathrm{RelStack}^{\star}_{\mathrm{aux}} \to \mathrm{RelStack}$ so that the following universal property is satisfied: for every simplicial set K equipped with a map $K \to \mathrm{RelStack}$, we have an isomorphism

$$\mathrm{Fun}_{\mathrm{RelStack}}(K, \mathrm{RelStack}^{\star}_{\mathrm{aux}}) \simeq \mathrm{Fun}_{\mathrm{Sch}_k}(K \times_{\mathrm{RelStack}} \overline{\mathrm{RelStack}}, (\mathrm{Shv}_\ell^{\star})^{\mathrm{op}}).$$

It follows from Corollary HTT.3.2.2.12 that the projection map $\mathrm{RelStack}^{\star}_{\mathrm{aux}} \to \mathrm{RelStack}$ is a coCartesian fibration. Unwinding the definitions, we can identify the objects of $\mathrm{RelStack}^{\star}_{\mathrm{aux}}$ with triples $(X, \mathcal{C}, \{\mathcal{F}_\eta\}_{\eta \in \mathcal{C}})$, where X is a quasi-projective k-scheme, $\mathcal{C}$ is a prestack equipped with a map $\pi : \mathcal{C} \to X$, and $\{\mathcal{F}_\eta\}_{\eta \in \mathcal{C}}$ is a diagram which assigns to each point $\eta : \mathrm{Spec}\, R_\eta \to \mathcal{C}$ an ℓ-adic sheaf $\mathcal{F}_\eta \in \mathrm{Shv}_\ell(\mathrm{Spec}\, R_\eta; \mathbf{Z}_\ell)$ and to each morphism $f : \eta \to \eta'$ in $\mathcal{C}$ a map $f^* \mathcal{F}_\eta \to \mathcal{F}_{\eta'}$ in $\mathrm{Shv}_\ell(\mathrm{Spec}\, R_{\eta'}; \mathbf{Z}_\ell)$. There is an evident pullback functor $G : \mathrm{RelStack}^{\star} \to \mathrm{RelStack}^{\star}_{\mathrm{aux}}$, given on objects by the formula $F(X, \mathcal{C}, \mathcal{F}) = (X, \mathcal{C}, \{(\pi \circ \eta)^* \mathcal{F}\})$. The functor G admits a left adjoint $F : \mathrm{RelStack}^{\star}_{\mathrm{aux}} \to \mathrm{RelStack}^{\star}$, given on objects by the formula

$$F(X, \mathcal{C}, \{\mathcal{F}_\eta\}) = (X, \varprojlim_{\eta \in \mathcal{C}} (\pi \circ \eta)_* \mathcal{F}_\eta).$$

The composition $F \circ G$ determines a functor from $\mathrm{RelStack}^{\star}$ to itself, which we will denote by $(X, \mathcal{C}, \mathcal{F}) \mapsto (X, \mathcal{C}, [\mathcal{C}]_{\mathcal{F}})$. Here $[\mathcal{C}]_{\mathcal{F}} \in \mathrm{Shv}_\ell(X)$ denotes the sheaf given by the limit

$$\varprojlim (\pi \circ \eta)_* (\pi \circ \eta)^* \mathcal{F},$$

where η ranges over all maps $\mathrm{Spec}\, R_\eta \to \mathcal{C}$.

We regard $\mathrm{RelStack}^{\star}_{\mathrm{aux}}$ as a symmetric monoidal ∞-category, where the tensor product is given on objects by the formula

$$(X', \mathcal{C}', \{\mathcal{F}'_{\eta'}\}) \otimes (X'', \mathcal{C}'', \{\mathcal{F}''_{\eta''}\}) = (X' \times_{\mathrm{Spec}\, k} X'', \mathcal{C}' \times_{\mathrm{Spec}\, k} \mathcal{C}'', \{\mathcal{F}_\eta\}),$$

where $\{\mathcal{F}_\eta\}$ is the diagram which assigns to each point $\eta = (\eta', \eta'') \in \mathcal{C}' \times_{\mathrm{Spec}\, k} \mathcal{C}''$ the tensor product $\mathcal{F}_\eta = \mathcal{F}'_{\eta'} \otimes \mathcal{F}''_{\eta''} \in \mathrm{Shv}_\ell(\mathrm{Spec}\, R_\eta)$. Note that G is a symmetric monoidal functor, so that F is a colax symmetric monoidal functor and therefore the composite functor

$$(X, \mathcal{C}, \mathcal{F}) \mapsto (X, \mathcal{C}, [\mathcal{C}]_{\mathcal{F}})$$

is also colax symmetric monoidal. Taking opposite categories and composing with the projection map $(\mathrm{RelStack}^{\star})^{\mathrm{op}} \to \mathrm{Shv}_\ell^{\star}$, we obtain a lax symmetric monoidal functor

$$\Psi : \mathrm{RelStack}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^{\star} \to \mathrm{Shv}_\ell^{\star},$$

given on objects by the formula

$$(X, \mathcal{C}, \mathcal{F}) \mapsto (X, [\mathcal{C}]_{\mathcal{F}}).$$

In particular, for every pair of objects $(X, \mathcal{C}, \mathcal{F}), (X', \mathcal{C}', \mathcal{F}') \in \mathrm{RelStack}^{\star}$, we have a canonical map

$$[\mathcal{C}]_{\mathcal{F}} \otimes [\mathcal{C}']_{\mathcal{F}'} \to [\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}']_{\mathcal{F} \boxtimes \mathcal{F}'}$$

in the ∞-category $\mathrm{Shv}_{\ell}(X \times X')$.

The functor Ψ of Construction A.5.17 is not symmetric monoidal. However, we will prove that it restricts to a symmetric monoidal functor on a reasonably large subcategory of

$$\mathrm{RelStack}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_{\ell}^{\star}.$$

Notation A.5.18. For every quasi-projective k-scheme X, let $\mathrm{Shv}_{\ell}(X)_{<\infty} = \bigcup_n \mathrm{Shv}_{\ell}(X)_{\leq n}$ denote the full subcategory of $\mathrm{Shv}_{\ell}(X)$ spanned by the truncated ℓ-adic sheaves (see §4.4). Let $\mathrm{Shv}_{<\infty}^{\star}$ denote the full subcategory of $\mathrm{Shv}_{\ell}^{\star}$ spanned by those pairs $(X, \mathcal{F})$ where $\mathcal{F}$ belongs to $\mathrm{Shv}_{\ell}(X)_{<\infty}$.

Proposition A.5.19. *The functor Ψ of Construction A.5.17 restricts to a symmetric monoidal functor*

$$\mathrm{AlgStack}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_{<\infty}^{\star} \to \mathrm{Shv}_{\ell}^{\star}.$$

Proof. It follows immediately from the definitions that Ψ preserves unit objects. Let X and X' be quasi-projective k-schemes, let $\mathcal{C}$ and $\mathcal{C}'$ be quasi-compact Artin stacks with affine diagonals equipped with smooth morphisms $\pi : \mathcal{C} \to X$ and $\pi' : \mathcal{C}' \to X'$. Suppose we are given ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_{\ell}(X)_{<\infty}$ and $\mathcal{F}' \in \mathrm{Shv}_{\ell}(X')_{<\infty}$. We wish to prove that the canonical map

$$\theta : [\mathcal{C}]_{\mathcal{F}} \boxtimes [\mathcal{C}']_{\mathcal{F}'} \to [\mathcal{C} \times_{\operatorname{Spec} k} \mathcal{C}']_{\mathcal{F} \boxtimes \mathcal{F}'}$$

is an equivalence in $\mathrm{Shv}_{\ell}(X \times_{\operatorname{Spec} k} X')$. Shifting $\mathcal{F}$ and $\mathcal{F}'$ if necessary, we may assume that $\mathcal{F} \in \mathrm{Shv}_{\ell}(X)_{\leq 0}$ and $\mathcal{F}' \in \mathrm{Shv}_{\ell}(X')_{\leq 0}$.

Choose affine schemes U_0 and U_0' equipped with smooth surjections $\rho : U_0 \to \mathcal{C}$ and $\rho' : U_0' \to \mathcal{C}'$. Let $U_{\bullet}$ denote the simplicial affine scheme given by the iterated fiber product of U_0 with itself over $\mathcal{C}$, and define $U_{\bullet}'$ similarly, and consider the natural maps $\phi_{\bullet} : U_{\bullet} \to X$ and $\phi_{\bullet}' : U_{\bullet}' \to X'$. Using Notation A.5.18, we can identify θ with the canonical map

$$(\mathrm{Tot}[U_{\bullet}]_{\mathcal{F}}) \boxtimes (\mathrm{Tot}[U_{\bullet}']_{\mathcal{F}'}) \to (\mathrm{Tot}[U_{\bullet} \times_{\operatorname{Spec} k} U_{\bullet}']_{\mathcal{F} \boxtimes \mathcal{F}'}).$$

Using Example 5.1.3 and Proposition 4.6.2, we can identify the codomain of θ with the totalization $\mathrm{Tot}([U_{\bullet}]_{\mathcal{F}} \boxtimes [U_{\bullet}']_{\mathcal{F}'})$. Example 5.1.3 also show that $[U_{\bullet}]_{\mathcal{F}}$ and $[U_{\bullet}']_{\mathcal{F}'}$ can be identified with cosimplicial objects of $\mathrm{Shv}_{\ell}(X)_{\leq 0}$ and $\mathrm{Shv}_{\ell}(X')_{\leq 0}$, respectively. The desired result now follows from Proposition 4.6.17. □

Remark A.5.20. The smoothness of the morphisms $\pi : \mathcal{C} \to X$ and $\pi' : \mathcal{C}' \to X'$ is not needed in the proof of Proposition A.5.19. However, it is important for the applications of Proposition A.5.19 which follow.

Definition A.5.21. Let $q : \mathcal{A} \to \mathcal{C}$ be a Cartesian fibration of ∞-categories, and let $r : \mathcal{B} \to \mathcal{C}$ and $s : \mathcal{D} \to \mathcal{C}$ be coCartesian fibrations. Let $\lambda : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{D}$ be a functor for which the diagram

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \xrightarrow{\lambda} & \mathcal{D} \\ & \searrow \quad \swarrow_{s} & \\ & \mathcal{C} & \end{array}$$

commutes. We will say that λ is *balanced* if, for every morphism (α, β) in the fiber product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$, if α is a q-Cartesian morphism in $\mathcal{A}$ and β is an r-coCartesian morphism in $\mathcal{B}$, then $\lambda(\alpha, \beta)$ is an s-coCartesian morphism in $\mathcal{D}$.

The collection of all commutative diagrams

$$\begin{array}{cccccc} & & \mathcal{A}\times_{\mathcal{C}}\mathcal{B} & \longrightarrow & & \mathcal{D} \\ & \swarrow & & \searrow & & \downarrow{\scriptstyle s} \\ \mathcal{A} & & & & \mathcal{B} & \\ & \searrow{\scriptstyle q} & & \swarrow{\scriptstyle r} & & \downarrow \\ & & \mathcal{C} & \xrightarrow{\mathrm{id}} & & \mathcal{C} \end{array}$$

where q is a Cartesian fibration, r is a coCartesian fibration, s is a coCartesian fibration, and λ is a balanced functor can be organized into an ∞-category which we will denote by $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Bal}}$.

Remark A.5.22. Let $q : \mathcal{A} \to \mathcal{C}$, $r : \mathcal{B} \to \mathcal{C}$, and $s : \mathcal{D} \to \mathcal{C}$ be as in Definition A.5.21, and suppose that q, r, and s are classified by functors $\chi_q : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, $\chi_r, \chi_s : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$. Using Corollary HTT.3.2.2.12, one can show that the data of a balanced functor $\lambda : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{D}$ is equivalent to the data of a natural transformation $\chi_r \to U$, where $U : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ is given by the formula $U(C) = \mathrm{Fun}(\chi_q(C), \chi_s(C))$.

Let $\sigma : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ denote the functor which assigns to each ∞-category its opposite. Any natural transformation from χ_r to U determines a natural transformation from $\sigma \circ \chi_r$ to the functor $\sigma \circ U$ given by

$$(\sigma \circ U)(C) = \mathrm{Fun}(\chi_q(C), \chi_s(C))^{\mathrm{op}} = \mathrm{Fun}(\chi_q(C)^{\mathrm{op}}, \chi_s(C)^{\mathrm{op}}).$$

It follows that the data of a balanced functor $\lambda : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{D}$ is equivalent to the data of a balanced functor $\mathcal{A}' \times_{\mathcal{C}} \mathcal{B}^\circ \to \mathcal{D}'$, where $\mathcal{A}' \to \mathcal{C}$ denotes the Cartesian dual of q, $\mathcal{B}^\circ \to \mathcal{C}$ denotes the coCartesian dual of $\mathcal{B}$, and $\mathcal{D}' \to \mathcal{C}$ denotes the coCartesian dual of $\mathcal{D}$. We will refer to λ' as the *balanced dual* of λ. With more effort, one can show that the construction $\lambda \mapsto \lambda'$ determines an equivalence of the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Bal}}$ with itself.

Proposition A.5.23. *Let* $\mathrm{Shv}_\ell^{\star,\mathrm{pr}}$ *denote the fiber product* $\mathrm{Shv}_\ell^\star \times_{\mathrm{Sch}_k^{\mathrm{op}}} (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$. *Then the functor* Ψ *of Construction A.5.17 induces a balanced map*

$$\Psi^{\mathrm{pr}} : \mathrm{RelStack} \times_{\mathrm{Sch}_k^{\mathrm{pr}}} (\mathrm{Shv}_\ell^{\star,\mathrm{pr}})^{\mathrm{op}} \to (\mathrm{Shv}_\ell^{\star,\mathrm{pr}})^{\mathrm{op}}.$$

Proof. Let $f : X' \to X$ be a map of quasi-projective k-schemes, let $\mathcal{C}$ be a prestack equipped with a map $\pi : \mathcal{C} \to X$, let $\mathcal{C}'$ be the fiber product $\mathcal{C} \times_X X'$, and let $\mathcal{F} \in \mathrm{Shv}_\ell(X')$. We wish to show that if f is proper, then the canonical map $\theta : [\mathcal{C}]_{f_* \mathcal{F}} \to f_*[\mathcal{C}']_{\mathcal{F}}$ is an equivalence in $\mathrm{Shv}_\ell(X)$. Note that since the functor f_* commutes with limits, the codomain of θ is given by

$$\varprojlim_{(\eta,u)} f_* u_* u^* \mathcal{F} \simeq \varprojlim_{(\eta,u)} (\pi \circ \eta)_* u^* \mathcal{F},$$

where the limit is taken over all pairs consisting of a point $\eta : \mathrm{Spec}\, R_\eta \to \mathcal{C}$ together with a map of X-schemes $u : \mathrm{Spec}\, R_\eta \to X'$. For each point η, form a pullback diagram

$$\begin{array}{ccc} Y_\eta & \xrightarrow{g_\eta} & \mathrm{Spec}\, R_\eta \\ \downarrow{\scriptstyle v_\eta} & & \downarrow{\scriptstyle \pi\circ\eta} \\ X' & \xrightarrow{f} & X. \end{array}$$

We can therefore identify θ with the canonical map

$$\varprojlim_{\eta} (\pi \circ \eta)_* (\pi \circ \eta)^* f_* \mathcal{F} \to \varprojlim_{\eta} f_* (\pi \circ \eta)_* (\pi \circ \eta)^* \mathcal{F}.$$

Since f is proper, the desired result follows from the proper base change theorem (Theorem 4.5.4). $\square$

Remark A.5.24. Using Proposition A.5.23, we see that the map Ψ of Construction A.5.17 determines a balanced functor

$$\Psi^! : \mathrm{RelStack}^! \times_{\mathrm{Sch}_k^{\mathrm{pr}}} \mathrm{Shv}_\ell^! \to \mathrm{Shv}_\ell^!,$$

which is again given on objects by the formula

$$(X, \mathcal{C}, \mathcal{F}) \mapsto (X, [\mathcal{C}]_{\mathcal{F}}).$$

Notation A.5.25. Let $\mathrm{Shv}_{<\infty}^{\star,\mathrm{pr}}$ denote the full subcategory of $\mathrm{Shv}_\ell^{\star,\mathrm{pr}}$ spanned by those objects $(X, \mathcal{F})$ where $\mathcal{F} \in \mathrm{Shv}_\ell(X)_{<\infty}$. Then the forgetful functor $\mathrm{Shv}_{<\infty}^{\star,\mathrm{pr}} \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ is a Cartesian fibration; let us denote its Cartesian dual by $\mathrm{Shv}_{<\infty}^! \to \mathrm{Sch}_k^{\mathrm{pr}}$. It follows from Propositions A.5.19 and A.5.23 that the restriction of Ψ determines a balanced functor

$$\mathrm{AlgStack} \times_{\mathrm{Sch}_k^{\mathrm{pr}}} (\mathrm{Shv}_{<\infty}^{\star,\mathrm{pr}})^{\mathrm{op}} \to (\mathrm{Shv}_\ell^{\star,\mathrm{pr}})^{\mathrm{op}}$$

which can be regarded as a commutative algebra object of the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Bal}}$. Consequently, its balanced dual

$$\Psi_{<\infty}^! : \mathrm{AlgStack}^! \times_{\mathrm{Sch}_k^{\mathrm{pr}}} \mathrm{Shv}_{<\infty}^! \to \mathrm{Shv}_\ell^!$$

is also a symmetric monoidal functor.

Construction A.5.26. The construction $X \mapsto (X, \omega_X)$ determines a symmetric monoidal section s of the coCartesian fibration $\mathrm{Shv}_\ell^! \to \mathrm{Sch}_k^{\mathrm{pr}}$. We let $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ denote the functor obtained by composing s with the functor $\Psi_{<\infty}^!$ of Notation A.5.25.

Proposition A.5.27. *The functor $\Phi : \mathrm{AlgStack}^! \to \mathrm{Shv}_\ell^!$ fits into a commutative diagram of symmetric monoidal functors*

$$\begin{array}{ccc} \mathrm{AlgStack}^! & \xrightarrow{\Phi} & \mathrm{Shv}_\ell^! \\ & {\scriptstyle \rho'} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathrm{Sch}_k^{\mathrm{pr}} . & \end{array}$$

Moreover, Φ carries ρ'-Cartesian morphisms to ρ-Cartesian morphisms.

Proof. The first assertion is follows immediately from our construction, and the second is a consequence of Proposition 5.1.9. $\square$

A.6. **Connectivity of the Fat Diagonal.** Our goal in this section is to prove the following result which will be needed in §9.2:

Proposition A.6.1. *Let X be a smooth (not necessarily complete) algebraic curve defined over an algebraically closed field k, let ℓ be a prime number which is invertible in k, let n be a positive integer, and let $\Delta \subseteq X^n$ denote the (reduced) closed subscheme whose k-valued points are n-tuples $(x_1, \ldots, x_n)$ where $x_i = x_j$ for some $i \neq j$. Then the restriction map $\theta : \mathrm{H}^m(X^n; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^m(\Delta; \mathbf{Z}/\ell\mathbf{Z})$ is an injection when $m = n-2$ and an isomorphism for $m < n-2$.*

We begin by proving the analogous result in the setting of ordinary topological spaces.

Proposition A.6.2. *Let K be a connected CW complex, let $n \geq 0$ be an integer, and let $\Delta \subseteq K^n$ be the subset consisting of those n-tuples of points $(x_1, \ldots, x_n)$ such that $x_i = x_j$ for some $i \neq j$. For every field Λ, the map*

$$\mathrm{H}^m(K^n; \Lambda) \to \mathrm{H}^m(\Delta; \Lambda)$$

is an injection when $m = n - 2$ and an isomorphism for $m < n - 2$.

Lemma A.6.3. *Let K be a connected manifold of dimension d, let S be a finite set, and let $U \subseteq K^S$ be the open subset consisting of injective maps $S \to K$. For every local system of abelian groups $\mathcal{A}$ on U, the homology groups $\mathrm{H}_m(U; \mathcal{A})$ vanish for $m > (d-1)|S| + 1$.*

Proof. We proceed by induction on the cardinality of S. If $|S| = 1$, then U is homeomorphic to K, and the desired result follows from the fact that K has dimension d. Suppose that $|S| > 1$, and write $S = S' \cup \{s\}$ where S' nonempty. Let $V \subseteq K^{S'}$ be the open subset consisting of injective maps $S' \to K$. The projection map $\pi : U \to V$ is a fiber bundle, whose fiber over a point $v \in V$ can be identified with the manifold $K - v(S')$. For each integer $i \geq 0$, let $\mathcal{B}_i$ denote the local system on V given by $v \mapsto \mathrm{H}_i(\pi^{-1}\{v\}; \mathcal{A}\,|_{\pi^{-1}\{v\}})$. We then have a Leray-Serre spectral sequence

$$\mathrm{H}_s(V; \mathcal{B}_t) \Rightarrow \mathrm{H}_{s+t}(U; \mathcal{A}).$$

The left hand side vanishes for $s > (d-1)|S'| + 1$ by the inductive hypothesis. It will therefore suffice to show that $\mathcal{B}_t \simeq 0$ for $t \geq d$. This is clear, since each fiber $\pi^{-1}\{v\}$ is a manifold of dimension d which has no compact connected components. □

Proof of Proposition A.6.2. Since both sides are compatible with filtered colimits, we may assume without loss of generality that K is a finite CW complex. In this case, K is homotopy equivalent to a compact oriented manifold with boundary M. Replacing K by M, we may assume that K is a compact oriented manifold with boundary. Replacing K by $K \times [0,1] \times [0,1]$, we may assume that the boundary ∂K is connected.

Let d be the dimension of K. Since K^n is a compact space, we have a long exact sequence

$$\cdots \to \mathrm{H}^m_c(K^n - \Delta; \Lambda) \to \mathrm{H}^m(K^n; \Lambda) \to \mathrm{H}^m(\Delta; \Lambda) \to \mathrm{H}^{m+1}_c(K^n - \Delta; \Lambda) \to \cdots$$

It will therefore suffice to show that the compactly supported cohomology group $\mathrm{H}^m_c(K^n - \Delta; \Lambda)$ vanishes for $m \leq n - 2$.

For every subset $S \subseteq \{1, \ldots, n\}$, let N_S denote the subset of $K^n - \Delta$ consisting of those points $(x_1, \ldots, x_n)$ satisfying

$$x_i \in \begin{cases} \partial K & \text{if } i \in S \\ K - \partial K & \text{if } i \notin S. \end{cases}$$

The space $K^n - \Delta$ admits a stratification whose open strata are the sets N_S. It will therefore suffice to show that $\mathrm{H}^m_c(N_S; \Lambda) \simeq 0$ for $m \leq n - 2$. Note that N_S is a an oriented manifold of dimension $nd - |S|$, so that Poincare duality supplies isomorphisms $\mathrm{H}^m_c(N_S; \Lambda) \simeq \mathrm{H}_{nd-m-|S|}(N_S; \Lambda)$. It will therefore suffice to show that the homology groups $\mathrm{H}_i(N_S; \Lambda)$ vanish for $i > n(d-1) + 2 - |S|$.

Let $S' = \{1, \ldots, n\} - S$. Then N_S is homeomorphic to the product $U \times V$, where U is the subset of $(\partial K)^S$ consisting of injective maps from S into ∂K, and V is the subset of $(K - \partial K)^{S'}$ consisting of injective maps from S' into $(K - \partial K)$. It follows from Lemma A.6.3 that the homology groups $\mathrm{H}_i(U; \Lambda)$ vanish for $i > |S|(d-2) + 1$ and that the homology groups $\mathrm{H}_j(V; \Lambda)$ vanish for $j > |S'|(d-1) + 1$. The desired result now follows from the existence of a Künneth isomorphism

$$\mathrm{H}_*(N_S; \Lambda) \simeq \mathrm{H}_*(U; \Lambda) \otimes_\Lambda \mathrm{H}_*(V; \Lambda).$$

□

Proof of Proposition A.6.1. Suppose first that k has characteristic zero. Choose an algebraically closed subfield $k_0 \subseteq k$ of finite transcendence degree over $\mathbf{Q}$ such that X is defined over k_0. Replacing k by k_0, we may assume that k has finite transcendence degree over $\mathbf{Q}$ so that there exists an embedding $k \hookrightarrow \mathbf{C}$. Replacing X by $\operatorname{Spec} \mathbf{C} \times_{\operatorname{Spec} k} X$, we can reduce to the case where $k = \mathbf{C}$. We may therefore assume that $k = \mathbf{C}$. In this case, we can identify θ with the restriction map $\mathrm{H}^m(X^n(\mathbf{C}); \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{H}^m(\Delta(\mathbf{C}); \mathbf{Z}/\ell\mathbf{Z})$ induced by the inclusion of topological spaces $\Delta(\mathbf{C}) \hookrightarrow X^n(\mathbf{C})$ and the desired result follows from Proposition A.6.2.

Now suppose that k has characteristic $p > 0$. Let Y denote a smooth compactification of X, let g be the genus of Y, and let $\mathcal{M}_{g,0}$ denote the moduli stack of smooth curves of genus g. Then Y is classified by a map $\eta_0 : \operatorname{Spec} k \to \mathcal{M}_{g,0}$. Since $\mathcal{M}_{g,0}$ is smooth over $\operatorname{Spec} \mathbf{Z}$, we can extend η_0 to a map $\eta : \operatorname{Spec} W(k) \to \mathcal{M}_{g,0}$, where $W(k)$ denotes the ring of Witt vectors of k. The map η classifies a smooth projective curve $\pi : \overline{Y} \to \operatorname{Spec} W(k)$. The complement of X in Y is the union of a finite collection of points $y_1, \ldots, y_m \in Y(k)$. Since $\overline{Y}$ is smooth over $\operatorname{Spec} W(k)$, we can extend each y_i to a section $\overline{y}_i : \operatorname{Spec} W(k) \to \overline{Y}$ of the map π. Let $\overline{X}$ denote the complement of the images of the maps $\overline{y}_i$ in $\overline{Y}$, let $\overline{X}^n$ denote the nth power of $\overline{X}$ in the category of $W(k)$-schemes, and define $\overline{\Delta}$ similarly. Let

$$\phi : \overline{X}^n \to \operatorname{Spec} W(k) \qquad \phi_0 : \overline{\Delta} \to \operatorname{Spec} W(k)$$

denote the projection maps. Since $\overline{Y}$ is smooth and proper over $W(k)$, and the complement of $\overline{X}$ in $\overline{Y}$ is smooth and proper over $W(k)$, it follows that the direct image $\phi_* \underline{\Lambda}_{\overline{X}^n}$ is lisse and compatible with base change in $\operatorname{Spec} W(k)$. A similar argument shows that the cohomologies of $\phi_{0*} \underline{\Lambda}_{\overline{\Delta}}$ are lisse and compatible with base change. Let $\mathcal{F}$ denote the fiber of the restriction map

$$\phi_* \underline{\Lambda}_{\overline{X}^n} \to \phi_{0*} \underline{\Lambda}_{\overline{\Delta}};$$

we wish to show that the stalk of $\mathcal{F}$ at the closed point of $\operatorname{Spec} W(k)$ is concentrated in cohomological degrees $> n-2$. Since $\mathcal{F}$ is lisse, it suffices to prove that the stalk of $\mathcal{F}$ at the generic point of $\operatorname{Spec} W(k)$ is concentrated in cohomological degrees $> n-2$, which follows from the first part of the proof. □

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