Designing Quantum Experiments via Graph Theory

Rishikesh Gajjala
NYU Abu Dhabi

Quantum Colloquium
May 12, 2025

Greenberger–Horne–Zeilinger (GHZ) state

- A GHZ state is an entangled quantum state of $n(> 2)$ qubits.

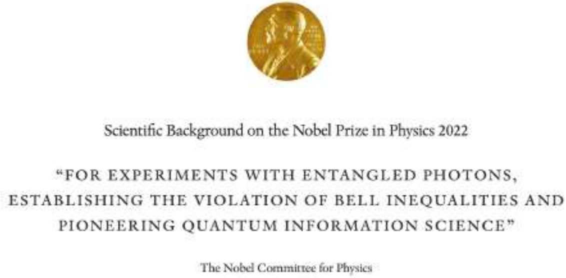
Greenberger–Horne–Zeilinger (GHZ) state

- A GHZ state is an entangled quantum state of $n(> 2)$ qubits.

- In 1999, a three-photon GHZ state was first produced experimentally.
  - $\left|\mathrm{GHZ}_{3,2}\right\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$

Greenberger–Horne–Zeilinger (GHZ) state

- A GHZ state is an entangled quantum state of $n(> 2)$ qubits.
- In 1999, a three-photon GHZ state was first produced experimentally.
  - $|\mathrm{GHZ}_{3,2}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
- They also established the violation of Bell inequalities, as predicted by theory



Generating GHZ states of large dimension

- The entanglement shared among several parties is used in:

Generating GHZ states of large dimension

- The entanglement shared among several parties is used in:
  - Communication protocols (Distributed quantum computing)

Generating GHZ states of large dimension

- The entanglement shared among several parties is used in:
  - Communication protocols (Distributed quantum computing)
  - Cryptographic applications (Quantum Byzantine agreement)

Generating GHZ states of large dimension

- The entanglement shared among several parties is used in:
  - Communication protocols (Distributed quantum computing)
  - Cryptographic applications (Quantum Byzantine agreement)

- Huge effort is beeing made by several experimental groups around the world to push the size of GHZ states

Generating GHZ states of large dimension

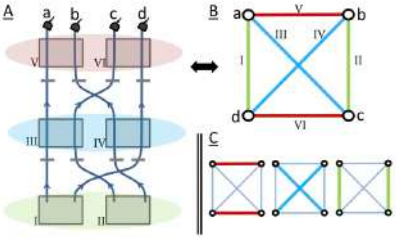
- The entanglement shared among several parties is used in:
  - Communication protocols (Distributed quantum computing)
  - Cryptographic applications (Quantum Byzantine agreement)

- Huge effort is beeing made by several experimental groups around the world to push the size of GHZ states
  - Photonic technology is one of the main players in this game

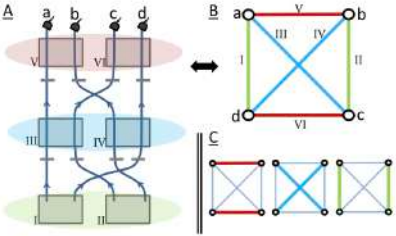
Generating GHZ states of large dimension

- The entanglement shared among several parties is used in:
  - Communication protocols (Distributed quantum computing)
  - Cryptographic applications (Quantum Byzantine agreement)
- Huge effort is beeing made by several experimental groups around the world to push the size of GHZ states
  - Photonic technology is one of the main players in this game
- Can high-dimensional GHZ states be created in quantum optical experiments with probabilistic photon sources and linear optics?
- Part I of the talk: How does this question translate to graph theory?
- Part II of the talk: How to approach this graph theoretic question?

Graph for a experiment for 3-dimensional 4-photon GHZ state

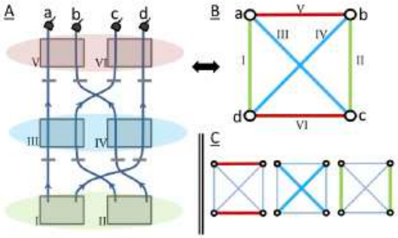


Graph for a experiment for 3-dimensional 4-photon GHZ state



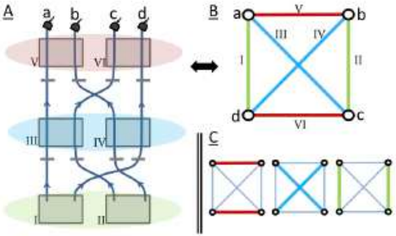
- Undirected coloured graphs correspond to quantum optical experiments, using probabilistic photon-pair sources and linear optics.

Graph for a experiment for 3-dimensional 4-photon GHZ state



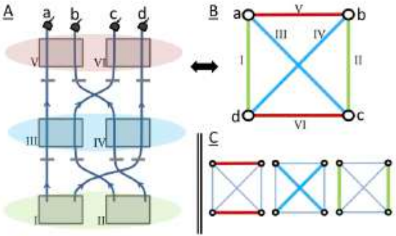
- Undirected coloured graphs correspond to quantum optical experiments, using probabilistic photon-pair sources and linear optics.
- Vertices correspond to photon detectors in the output of some photon path.

Graph for a experiment for 3-dimensional 4-photon GHZ state



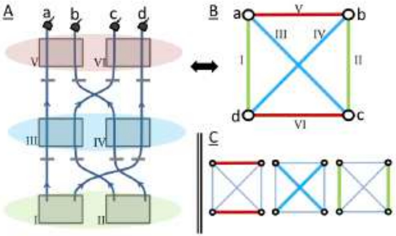
- Undirected coloured graphs correspond to quantum optical experiments, using probabilistic photon-pair sources and linear optics.
- Vertices correspond to photon detectors in the output of some photon path.
- Edges correspond to photon pairs that emerge from two photon paths.

Graph for a experiment for 3-dimensional 4-photon GHZ state

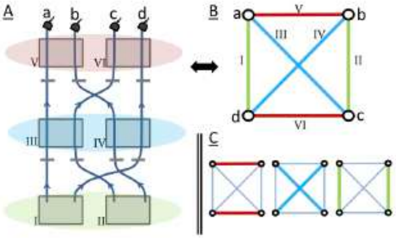


- Undirected coloured graphs correspond to quantum optical experiments, using probabilistic photon-pair sources and linear optics.
- Vertices correspond to photon detectors in the output of some photon path.
- Edges correspond to photon pairs that emerge from two photon paths.
- Perfect matchings correspond to a multi-photon event where each single photon detector detects a photon.

Graph for a experiment for 3-dimensional 4-photon GHZ state

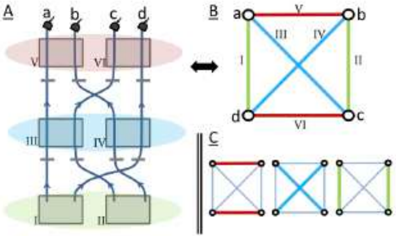


Graph for a experiment for 3-dimensional 4-photon GHZ state



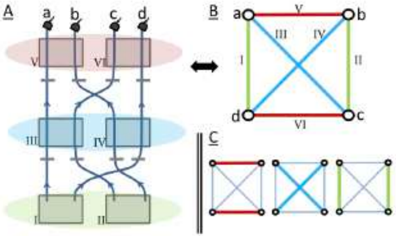
- Edge colors correspond to the mode number of photons in the path.

Graph for a experiment for 3-dimensional 4-photon GHZ state



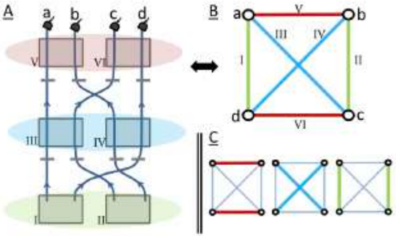
- Edge colors correspond to the mode number of photons in the path.
  - Can be bi-colored, as the two photons can have different mode numbers

Graph for a experiment for 3-dimensional 4-photon GHZ state



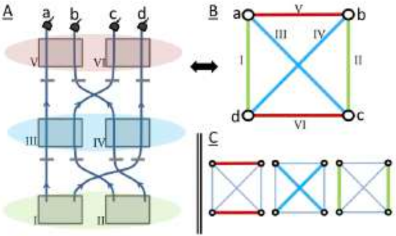
- Edge colors correspond to the mode number of photons in the path.
  - Can be bi-colored, as the two photons can have different mode numbers
- Edge weights correspond to the amplitude of the corresponding photon pair

Graph for a experiment for 3-dimensional 4-photon GHZ state



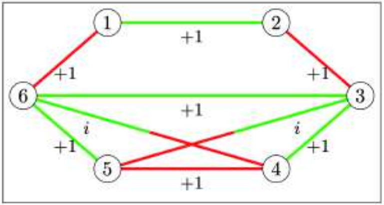
- Edge colors correspond to the mode number of photons in the path.
  - Can be bi-colored, as the two photons can have different mode numbers
- Edge weights correspond to the amplitude of the corresponding photon pair
- Mode number of photons in a path and amplitude of a photon pair can be set by the experimenter

Graph for a experiment for 3-dimensional 4-photon GHZ state

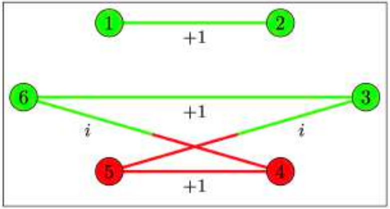


- Edge colors correspond to the mode number of photons in the path.
  - Can be bi-colored, as the two photons can have different mode numbers
- Edge weights correspond to the amplitude of the corresponding photon pair
- Mode number of photons in a path and amplitude of a photon pair can be set by the experimenter
- Therefore, we are interested in finding a colouring and weight assignment for a given graph, satisfying **certain properties**.

The GHZ property

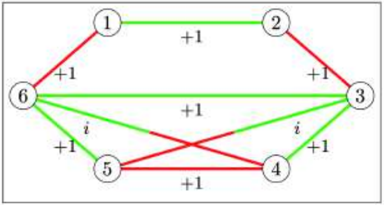


(a) An edge coloured edge-weighted graph G_c^w

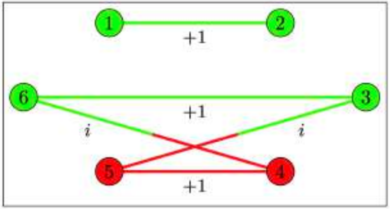


(b) $\mathcal{F}(G_c^w, vc)$, where $vc(i)$ is green for $i \in [1, 2, 3, 6]$ and red for $i \in [4, 5]$

The GHZ property



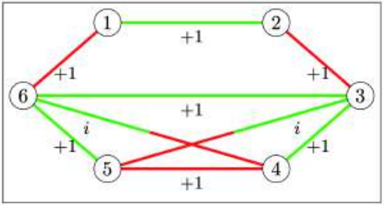
(a) An edge coloured edge-weighted graph G_c^w



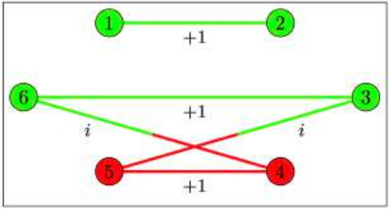
(b) $\mathcal{F}(G_c^w, vc)$, where $vc(i)$ is green for $i \in [1, 2, 3, 6]$ and red for $i \in [4, 5]$

- In a GHZ state, *impure* terms have weight 0.

The GHZ property



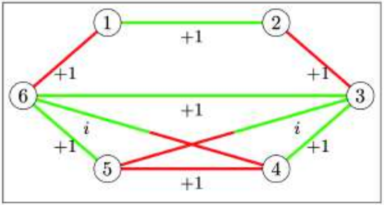
(a) An edge coloured edge-weighted graph G_c^w



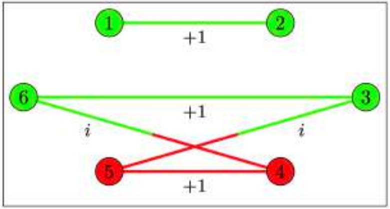
(b) $\mathcal{F}(G_c^w, vc)$, where $vc(i)$ is green for $i \in [1, 2, 3, 6]$ and red for $i \in [4, 5]$

- In a GHZ state, *impure* terms have weight 0.
- Weight of graphs filtered by non-monochromatic vertex colourings should is 0.

The GHZ property



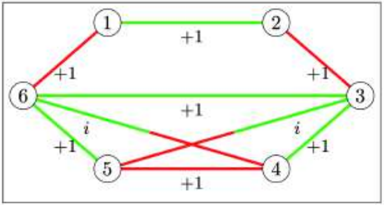
(a) An edge coloured edge-weighted graph G_c^w



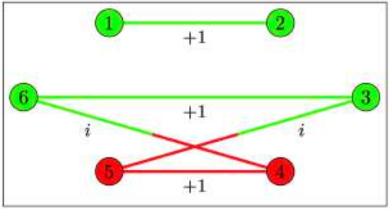
(b) $\mathcal{F}(G_c^w, vc)$, where $vc(i)$ is green for $i \in [1, 2, 3, 6]$ and red for $i \in [4, 5]$

- In a GHZ state, *impure* terms have weight 0.
- Weight of graphs filtered by non-monochromatic vertex colourings should is 0.
  - $w(gggrrg) = i \cdot i \cdot 1 + 1 \cdot 1 \cdot 1 = -1 + 1 = 0$

The GHZ property



(a) An edge coloured edge-weighted graph G_c^w

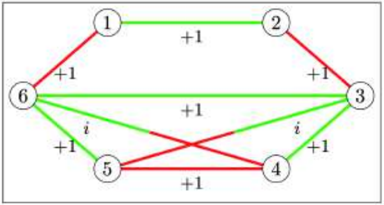


(b) $\mathcal{F}(G_c^w, vc)$, where $vc(i)$ is green for $i \in [1, 2, 3, 6]$ and red for $i \in [4, 5]$

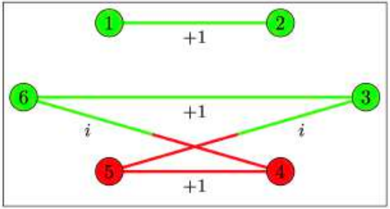
- In a GHZ state, *impure* terms have weight 0.
- Weight of graphs filtered by non-monochromatic vertex colourings should is 0.
  - $w(gggrrg) = i \cdot i \cdot 1 + 1 \cdot 1 \cdot 1 = -1 + 1 = 0$

- In a GHZ state, *pure* terms have weight 1.

The GHZ property



(a) An edge coloured edge-weighted graph G_c^w

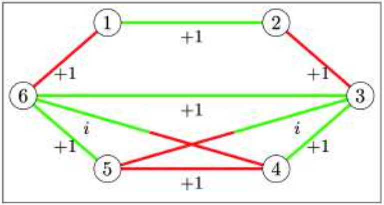


(b) $\mathcal{F}(G_c^w, vc)$, where $vc(i)$ is green for $i \in [1,2,3,6]$ and red for $i \in [4,5]$

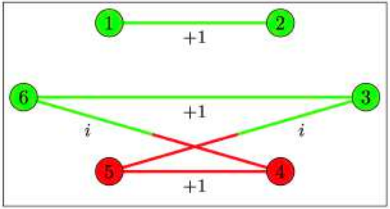
- In a GHZ state, *impure* terms have weight 0.
- Weight of graphs filtered by non-monochromatic vertex colourings should is 0.
  - $w(gggrrg) = i \cdot i \cdot 1 + 1 \cdot 1 \cdot 1 = -1 + 1 = 0$

- In a GHZ state, *pure* terms have weight 1.
- Weight of graphs filtered by monochromatic vertex colourings is 1

The GHZ property



(a) An edge coloured edge-weighted graph G_c^w



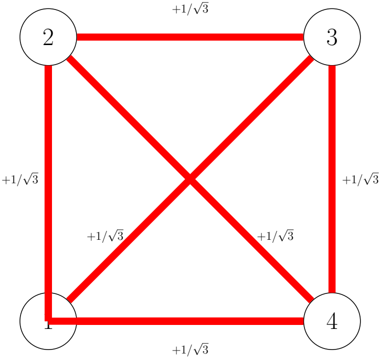
(b) $\mathcal{F}(G_c^w, vc)$, where $vc(i)$ is green for $i \in [1, 2, 3, 6]$ and red for $i \in [4, 5]$

- In a GHZ state, *impure* terms have weight 0.
- Weight of graphs filtered by non-monochromatic vertex colourings should is 0.
  - $w(gggrrg) = i \cdot i \cdot 1 + 1 \cdot 1 \cdot 1 = -1 + 1 = 0$

- In a GHZ state, *pure* terms have weight 1.
- Weight of graphs filtered by monochromatic vertex colourings is 1
  - $w(gggggg) = 1 \cdot 1 \cdot 1 = 1$

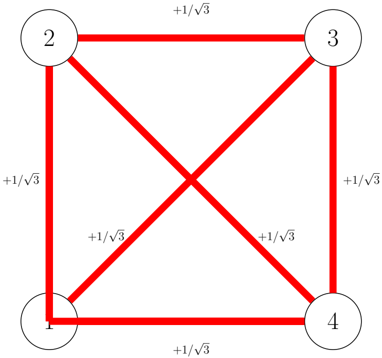
- $\left|\mathrm{GHZ}_{6,2}\right\rangle = \frac{1}{\sqrt{2}}(|000000\rangle + |111111\rangle)$

One dimensional GHZ state



1

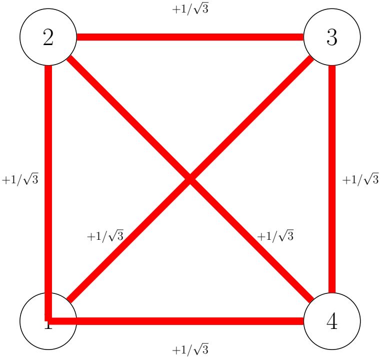
One dimensional GHZ state



- There is one inherited vertex colouring which is monochromatic and its weights sum up to one.

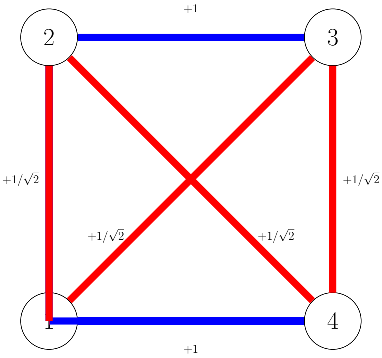
1

One dimensional GHZ state



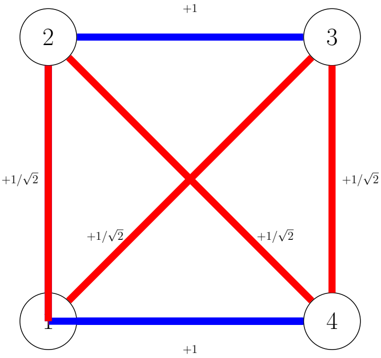
- There is one inherited vertex colouring which is monochromatic and its weights sum up to one.
- $\left|\mathrm{GHZ}_{4,1}\right\rangle = |0000\rangle$

Two dimensional GHZ state



1

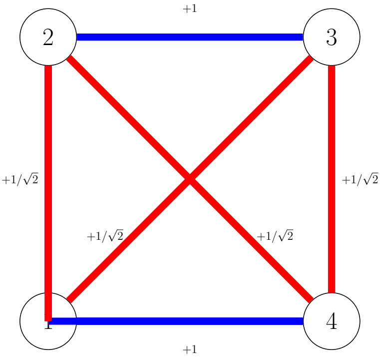
Two dimensional GHZ state



- There are two inherited vertex colourings which are monochromatic and their weights sum up to one.

1

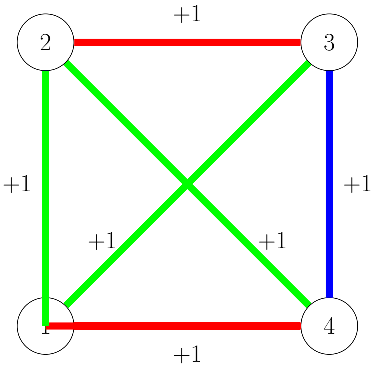
Two dimensional GHZ state



- There are two inherited vertex colourings which are monochromatic and their weights sum up to one.
- $\left|\mathrm{GHZ}_{4,2}\right\rangle = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$

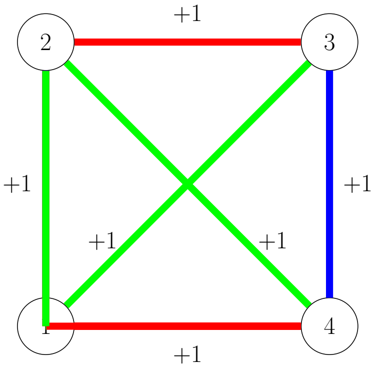
1

Three dimensional GHZ state



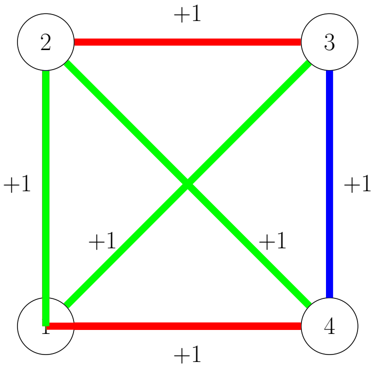
1

Three dimensional GHZ state



- There are three inherited vertex colourings which are monochromatic and their weights sum up to one.

Three dimensional GHZ state



- There are three inherited vertex colourings which are monochromatic and their weights sum up to one.
- $\left|\mathrm{GHZ}_{4,3}\right\rangle = \frac{1}{\sqrt{3}}\left(|0000\rangle + |1111\rangle + |2222\rangle\right)$

1

Dimension of GHZ state and matching index

- The best dimension achievable over all possible colourings and weights which satisfy GHZ property is the **matching index** of a graph, $\bar{\mu}(G)$.

Dimension of GHZ state and matching index

- The best dimension achievable over all possible colourings and weights which satisfy GHZ property is the **matching index** of a graph, $\bar{\mu}(G)$.
- Matching index of K_4 is 3

Dimension of GHZ state and matching index

- The best dimension achievable over all possible colourings and weights which satisfy GHZ property is the **matching index** of a graph, $\bar{\mu}(G)$.
- Matching index of K_4 is 3

Question 1

Are there graphs for with a high matching index?
If yes, we can design experiments to generate a high-dimensional Greenberger-Horne-Zeilinger state!

Dimension of GHZ state and matching index

- The best dimension achievable over all possible colourings and weights which satisfy GHZ property is the **matching index** of a graph, $\bar{\mu}(G)$.
- Matching index of K_4 is 3

Question 1

Are there graphs for with a high matching index?
If yes, we can design experiments to generate a high-dimensional Greenberger-Horne-Zeilinger state!

So far, no graphs other than K_4 is known with a matching index at least 3.

Simpler case: No destructive interference

- Destructive interference allows intricate cancellation of impure terms
- When there is no destructive interference, the GHZ property is simply:
  - All perfect matchings in the graph are monochromatic.

Simpler case: No destructive interference

- Destructive interference allows intricate cancellation of impure terms
- When there is no destructive interference, the GHZ property is simply:
  - All perfect matchings in the graph are monochromatic.

Why?

Simpler case: No destructive interference

- Destructive interference allows intricate cancellation of impure terms
- When there is no destructive interference, the GHZ property is simply:
  - All perfect matchings in the graph are monochromatic.

Why?

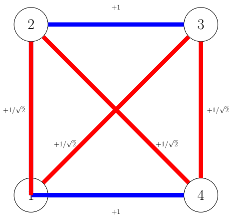
- Given an unweighted coloured graph with such a property, it is easy to assign weights such that the graph is perfectly monochromatic.

Simpler case: No destructive interference

- Destructive interference allows intricate cancellation of impure terms
- When there is no destructive interference, the GHZ property is simply:
  - All perfect matchings in the graph are monochromatic.

Why?

- Given an unweighted coloured graph with such a property, it is easy to assign weights such that the graph is perfectly monochromatic.

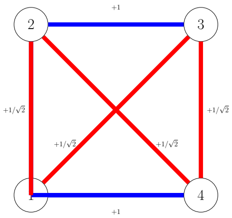


Simpler case: No destructive interference

- Destructive interference allows intricate cancellation of impure terms
- When there is no destructive interference, the GHZ property is simply:
  - All perfect matchings in the graph are monochromatic.

Why?

- Given an unweighted coloured graph with such a property, it is easy to assign weights such that the graph is perfectly monochromatic.



1

(unweighted) GHZ graphs

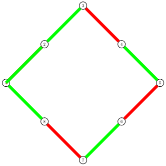


Figure: A couloured graph
C_8

(unweighted) GHZ graphs

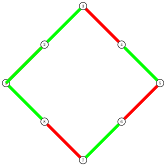


Figure: A couloured graph C_8

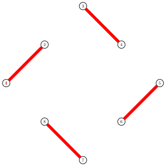


Figure: First perfect matching

(unweighted) GHZ graphs

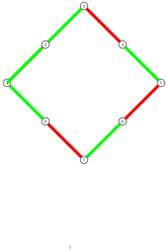


Figure: A couloured graph C_8

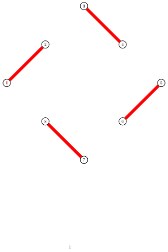


Figure: First perfect matching

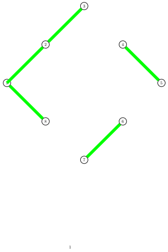


Figure: Second perfect matching

(unweighted) A coloured graph which is not GHZ

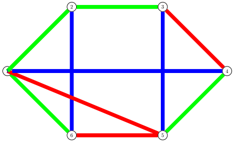
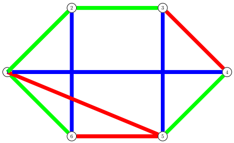


Figure: 6 vertex graph with four perfect matchings

(unweighted) A coloured graph which is not GHZ



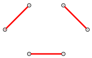


Figure: PM
1

Figure: 6 vertex graph with four
perfect matchings

(unweighted) A coloured graph which is not GHZ

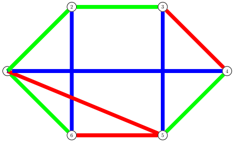


Figure: 6 vertex graph with four perfect matchings

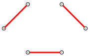


Figure: PM 1

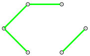


Figure: PM 2

(unweighted) A coloured graph which is not GHZ

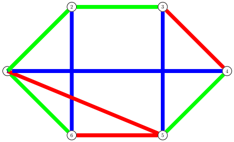


Figure: 6 vertex graph with four perfect matchings

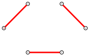


Figure: PM 1

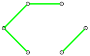


Figure: PM 2

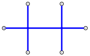


Figure: PM 3

(unweighted) A coloured graph which is not GHZ

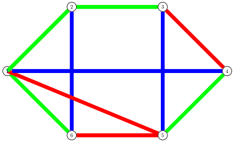


Figure: 6 vertex graph with four perfect matchings

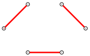


Figure: PM 1

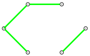


Figure: PM 2

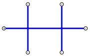


Figure: PM 3

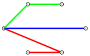


Figure: PM 4

Bounding (unweighted) matching index

Theorem 2 (Bogdanov17, Thomason78)

If G is not isomorphic to K_4, then $\mu(G) \leq 2$ and $\mu(K_4) = 3$.

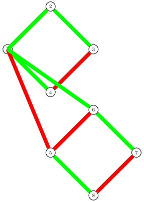
- **Proof idea:** Assume there is GHZ graph with three colours (say red, blue, green) towards a contradiction.

Union of perfect matchings is a Hamiltonian cycle

- The union of any two perfect matchings is a disjoint union of cycles.

Union of perfect matchings is a Hamiltonian cycle

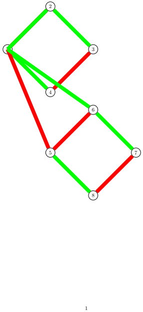
- The union of any two perfect matchings is a disjoint union of cycles.
- Consider the union of a red perfect matching and green perfect matching

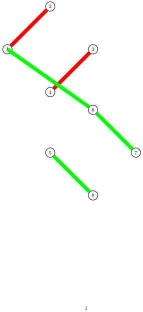


1

Union of perfect matchings is a Hamiltonian cycle

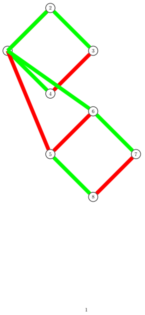
- The union of any two perfect matchings is a disjoint union of cycles.
- Consider the union of a red perfect matching and green perfect matching

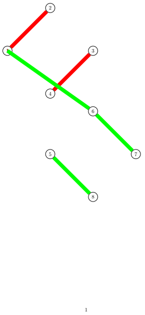




Union of perfect matchings is a Hamiltonian cycle

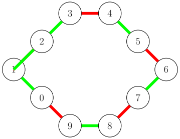
- The union of any two perfect matchings is a disjoint union of cycles.
- Consider the union of a red perfect matching and green perfect matching



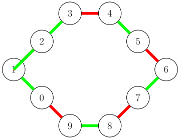


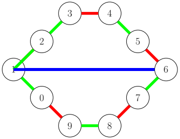
- Therefore, union of any two perfect matchings is a Hamiltonian cycle

A blue can't split the hamiltonian cycle into two even parts

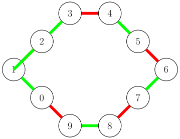


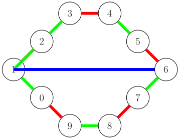
A blue can't split the hamiltonian cycle into two even parts

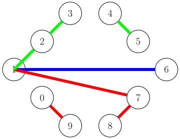




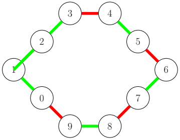
A blue can't split the hamiltonian cycle into two even parts

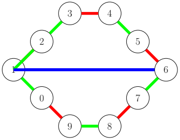


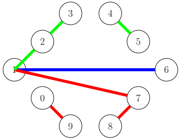




A blue can't split the hamiltonian cycle into two even parts

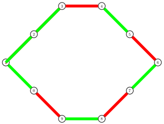




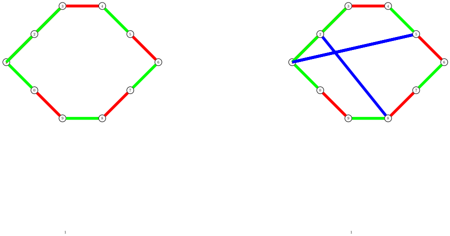


Therefore, all blue edges split the Hamiltonian cycle into two odd parts

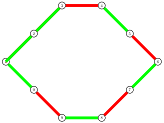
Not all edges can split the Hamiltonian cycle into odd parts

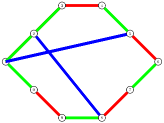


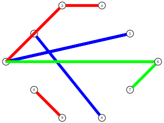
Not all edges can split the Hamiltonian cycle into odd parts



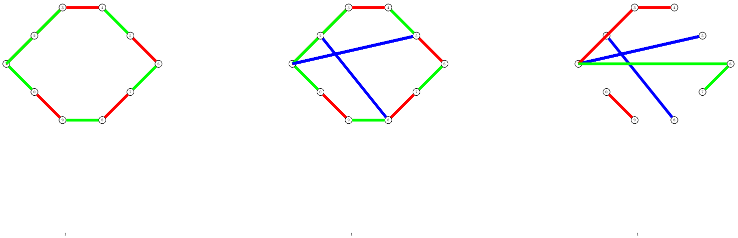
Not all edges can split the Hamiltonian cycle into odd parts







Not all edges can split the Hamiltonian cycle into odd parts



Therefore, the (unweighted) matching index is at most 2 when graph is not isomorphic to K_4.

Structure of Type 2 graphs

- Type 0 graphs. $\mu(G) = 0$. They have no perfect matching, therefore $\bar{\mu}(G) = 0$.

Structure of Type 2 graphs

- Type 0 graphs. $\mu(G) = 0$. They have no perfect matching, therefore $\bar{\mu}(G) = 0$.
- Type 1 graphs. $\mu(G) = 1$.

Structure of Type 2 graphs

- Type 0 graphs. $\mu(G) = 0$. They have no perfect matching, therefore $\bar{\mu}(G) = 0$.
- Type 1 graphs. $\mu(G) = 1$.
- Type 2 graphs. $\mu(G) = 2$.

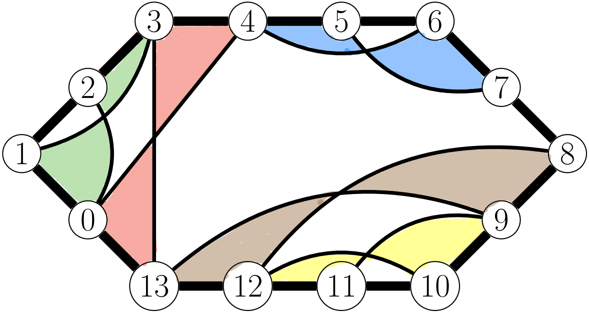
Structure of Type 2 graphs

- Type 0 graphs. $\mu(G) = 0$. They have no perfect matching, therefore $\bar{\mu}(G) = 0$.
- Type 1 graphs. $\mu(G) = 1$.
- Type 2 graphs. $\mu(G) = 2$.
- K_4 is the only Type 3 graph. $\mu(K_4) = 3$ and $\bar{\mu}(K_4) = 3$.

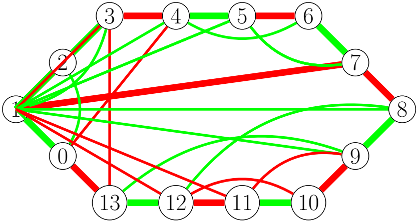
Structure of Type 2 graphs

- Type 0 graphs. $\mu(G) = 0$. They have no perfect matching, therefore $\bar{\mu}(G) = 0$.
- Type 1 graphs. $\mu(G) = 1$.
- Type 2 graphs. $\mu(G) = 2$.
- K_4 is the only Type 3 graph. $\mu(K_4) = 3$ and $\bar{\mu}(K_4) = 3$.

It is sufficient to consider the matching covered graphs of Type 2 graphs.



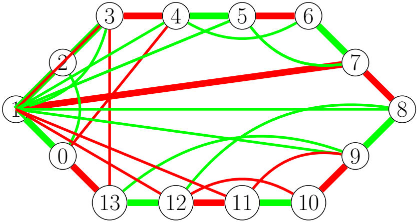
Structure of Type 2 graphs



- Drum is a cycle in the order $(i, i+1, j, j-1)$, where i, j are of different parity.

1

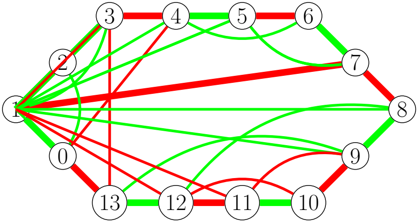
Structure of Type 2 graphs



- Drum is a cycle in the order $(i, i+1, j, j-1)$, where i, j are of different parity.
- All non-cycle edges must be part of exactly one drum

1

Structure of Type 2 graphs



- Drum is a cycle in the order $(i, i+1, j, j-1)$, where i, j are of different parity.
- All non-cycle edges must be part of exactly one drum
- These drums don't intersect each other.

1

Weighted matching index of Type 2 graphs

Theorem 3

If $\mu(G) \neq 1$, *then* $\bar{\mu}(G) = \mu(G)$. *There are graphs with* $\mu(G) = 1$ *and* $\bar{\mu}(G) \neq \mu(G)$

We can only expect high dimensional GHZ graphs when $\mu(G) = 1$

Weighted matching index of Type 2 graphs

Theorem 3

If $\mu(G) \neq 1$, *then* $\bar{\mu}(G) = \mu(G)$. *There are graphs with* $\mu(G) = 1$ *and* $\bar{\mu}(G) \neq \mu(G)$

We can only expect high dimensional GHZ graphs when $\mu(G) = 1$

Proof idea:

- Recall Type 0, Type 3 graphs satisfy above.

Weighted matching index of Type 2 graphs

Theorem 3

If $\mu(G) \neq 1$, *then* $\bar{\mu}(G) = \mu(G)$. *There are graphs with* $\mu(G) = 1$ *and* $\bar{\mu}(G) \neq \mu(G)$

We can only expect high dimensional GHZ graphs when $\mu(G) = 1$

Proof idea:

- Recall Type 0, Type 3 graphs satisfy above.
- We prove that the presence of certain gadget imply $\bar{\mu}(G) = 2$

Weighted matching index of Type 2 graphs

Theorem 3

If $\mu(G) \neq 1$, then $\bar{\mu}(G) = \mu(G)$. There are graphs with $\mu(G) = 1$ and $\bar{\mu}(G) \neq \mu(G)$

We can only expect high dimensional GHZ graphs when $\mu(G) = 1$

Proof idea:

- Recall Type 0, Type 3 graphs satisfy above.
- We prove that the presence of certain gadget imply $\bar{\mu}(G) = 2$
- We then use the structural characterisation for Type 2 graphs and show that it should contain such a gadget.
- We will now look at an an example of Type 1 graph for which $\bar{\mu}(G) \neq \mu(G)$

Weighted matching index of Type 2 graphs

Theorem 3

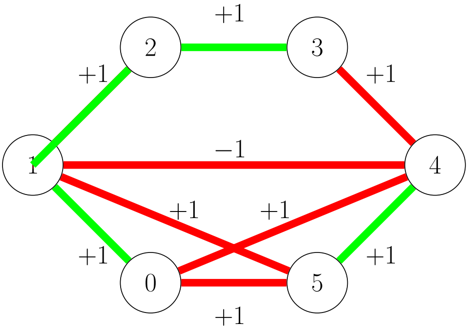
If $\mu(G) \neq 1$*, then* $\bar{\mu}(G) = \mu(G)$*. There are graphs with* $\mu(G) = 1$ *and* $\bar{\mu}(G) \neq \mu(G)$

We can only expect high dimensional GHZ graphs when $\mu(G) = 1$

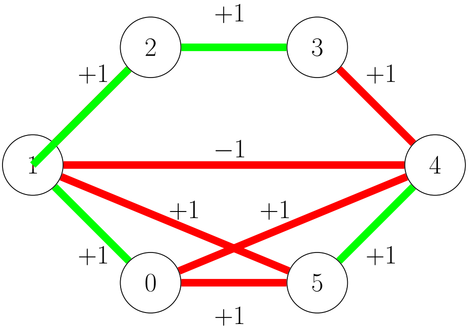
Proof idea:

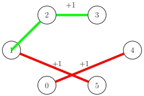
- Recall Type 0, Type 3 graphs satisfy above.
- We prove that the presence of certain gadget imply $\bar{\mu}(G) = 2$
- We then use the structural characterisation for Type 2 graphs and show that it should contain such a gadget.
- We will now look at an an example of Type 1 graph for which $\bar{\mu}(G) \neq \mu(G)$

Type 1 graph with $\bar{\mu}(G) = 2$

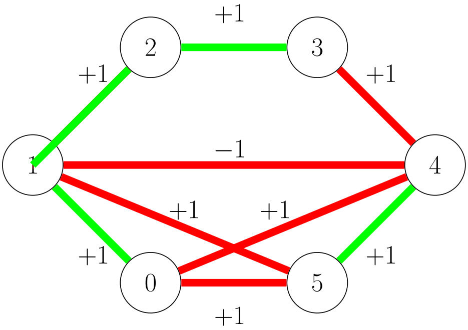


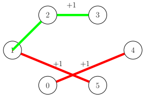
Type 1 graph with $\bar{\mu}(G) = 2$

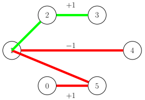




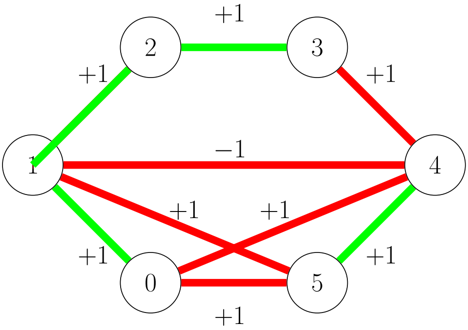
Type 1 graph with $\bar{\mu}(G) = 2$

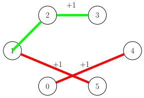


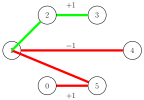


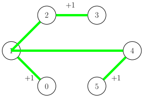


Type 1 graph with $\bar{\mu}(G) = 2$

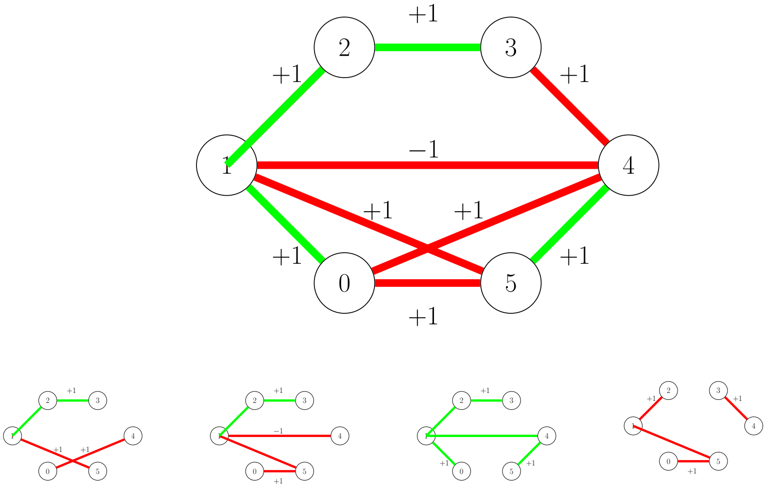




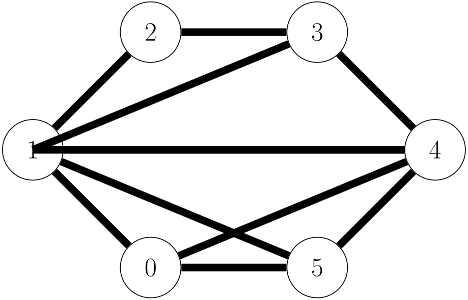




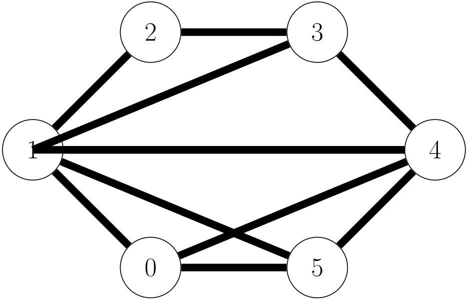
Type 1 graph with $\bar{\mu}(G) = 2$

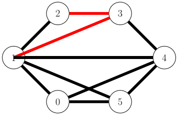


Why is it is a Type 1 graph?

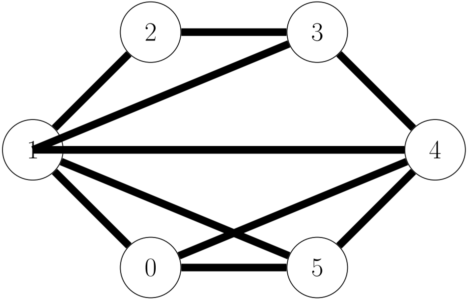


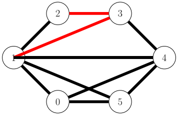
Why is it is a Type 1 graph?

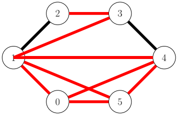




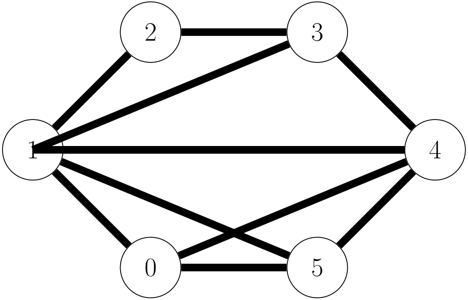
Why is it is a Type 1 graph?

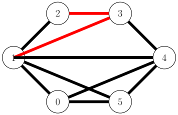


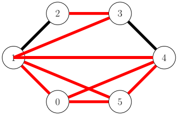


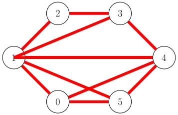


Why is it is a Type 1 graph?









Krenn-Gu Conjecture

Question 4

Are there graphs with a higher matching index?

Krenn-Gu Conjecture

Question 4

Are there graphs with a higher matching index?

Conjecture 5 (Krenn and Gu)

There are no graphs other than K_4 *with a matching index at least* 3

- If true, we save huge computational efforts and get insights into quantum resource theory

Krenn-Gu Conjecture

Question 4

Are there graphs with a higher matching index?

Conjecture 5 (Krenn and Gu)

There are no graphs other than K_4 with a matching index at least 3

- If true, we save huge computational efforts and get insights into quantum resource theory
- If false, we can find a high-dimensional Greenberger-Horne-Zeilinger state!

KG Conjecture for Sparse Graphs

Theorem 6 (C**G**I, MFCS 2024)

Krenn-Gu conjecture is true for (sub-)cubic graphs and graphs with vertex connectivity at most 2*.*

KG Conjecture for Sparse Graphs

Theorem 6 (C**GI**, MFCS 2024)

Krenn-Gu conjecture is true for (sub-)cubic graphs and graphs with vertex connectivity at most 2*.*

Theorem 7 (C**GI**2024)

The minimum counter example to Krenn-Gu conjecture matching index is 4*-vertex connected.*

KG Conjecture for Sparse Graphs

Theorem 6 (C**GI**, MFCS 2024)

Krenn-Gu conjecture is true for (sub-)cubic graphs and graphs with vertex connectivity at most 2*.*

Theorem 7 (C**GI**2024)

The minimum counter example to Krenn-Gu conjecture matching index is 4*-vertex connected.*

Useful for automated GHZ graph searches using tools like PyTheus.

Proof Ideas

Scaling lemma:

- Reformulation of the conjecture in more general terms

Proof Ideas

Scaling lemma:

- Reformulation of the conjecture in more general terms
- This is handy for reductions.

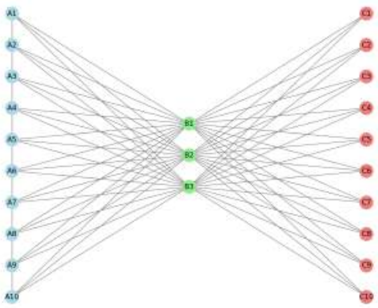
Proof Ideas

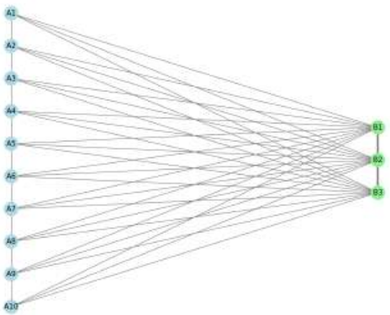
Scaling lemma:

- Reformulation of the conjecture in more general terms
- This is handy for reductions.

Graph reduction:

- Capture all weights on smaller components:
- Requires very delicate constructions adjusting weights such that GHZ property is retained





But do we have any bounds on the dimension?

Conjecture 8 (CKA, Quantum 2022)

It is not possible to generate a $n > 4$ vertex GHZ experiment graph (without multi-edges) with dimension $> n/2$

- True for graphs with ≤ 8 vertices [CKA2022]

But do we have any bounds on the dimension?

Conjecture 8 (CKA, Quantum 2022)

It is not possible to generate a $n > 4$ *vertex GHZ experiment graph (without multi-edges) with dimension* $> n/2$

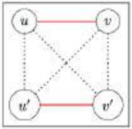
- True for graphs with ≤ 8 vertices [CKA2022]

Theorem 9 (C**G**, Quantum 2024)

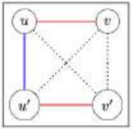
It is not possible to generate a $n > 4$ *vertex simple GHZ experiment graph with dimension* $\geq n/\sqrt{2}$

Bounds on the dimension

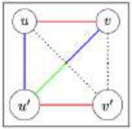
We find a list of contradicting forbidden structures which are unavoidable in high dimensional GHZ graphs (if they exist).



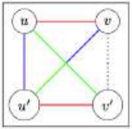
(a) Possible



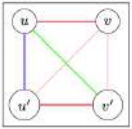
(b) Possible



(c) Possible

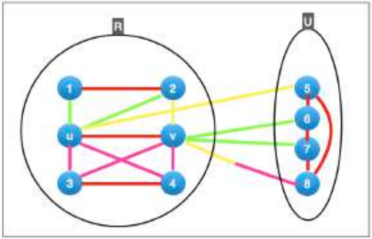


(d) Not possible



(e) Not possible

However, high-dimensional GHZ graphs also require a lot of intricate structures to be present in them.



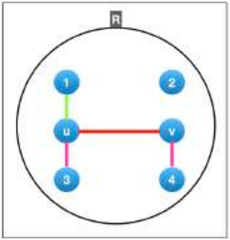


Table: Current state-of-the-art results, conjectures and our results

Destructive Interfer..	**Multi Edges**	**Bichromatic Edges**	**Dimension**	**Status**
✗	✓	✓	$n \geq 4,\ d \leq 3$	Proved in [KGZ17]
✗	✓	✓	$n \geq 4,\ d \leq 3$	Characterized in [C**G**23]
✓	✓	✗	$n = 6,\ d < 3$ $n = 8,\ d < 4$	[CKA22]
✓	✓	✗	$n > 4,\ d < \frac{n}{2}$	CKA conjecture
✓	✗	✓	$n > 4,\ d < \frac{n}{\sqrt{2}}$	[C**G**24]
✓	✓	✓	$n > 4,\ d \leq 2$	KG conjecture
✓	✓	✓	$n = 4,\ d \leq 3$	[Mantey24]
✓	✓	✓	$n > 4,\ d \leq 2$	For sparse graphs [C**G**I24]