

Spinor Structure of Space-Times in General Relativity. I

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(Received 8 January 1968)

In order to define spinor fields on a space-time M , it is necessary first to endow M with some further structure in addition to its Lorentz metric. This is the spinor structure. The definition and the elementary implications of the existence of a spinor structure are discussed. It is proved that a necessary and sufficient condition for a noncompact space-time M to admit a spinor structure is that M have a global field of orthonormal tetrads.

INTRODUCTION

It is sometimes convenient to formulate general properties which any space-time¹ should have if it is to be of physical interest. For example, one would normally reject space-times having closed timelike curves,² and possibly also those having incomplete geodesics.³ It has been argued,⁴ based on a recent paper of Aharonov and Susskind,⁵ that another property we might require of a physically reasonable space-time M is that M admit globally defined spinor fields.⁶ Now, in order that spinors may be defined on M , it is necessary that M satisfy certain global conditions.⁴ In the case of space-times (but not, in general, for other signatures or dimensions), these conditions have the following consequence: *Every noncompact⁷ space-time on which spinors may be defined carries a global field of orthonormal tetrads.* This, our main result, is proved in Sec. II.

In Sec. I we review the definition of a spinor field and the basic properties of space-times which admit such fields.

The Appendix contains a proof that every space-time is paracompact.

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¹ By a space-time we understand a mathematical object: a 4-manifold with a C^∞ metric of signature $(+, -, -, -)$.

² See, for example, H. Reichenbach, *The Philosophy of Space and Time* (Dover Publications, Inc., New York, 1958), Sec. 21.

³ See C. W. Misner, *J. Math. Phys.*, **4**, 924 (1963); R. Geroch, "What is a Singularity in General Relativity?," *Ann. Phys. (N.Y.)* (to be published).

⁴ R. Penrose, "The Structure of Space-Time" in *Battelle Rencontres in Mathematics and Physics: Seattle, 1967*, C. DeWitt, Ed. (W. A. Benjamin, Inc., New York, 1968).

⁵ Y. Aharonov and L. Susskind, *Phys. Rev.* **158**, 1237 (1967).

⁶ Note that we are not concerned with the existence of spinor fields having special properties, but merely with whether or not the notion of a spinor field is well defined.

⁷ This assumption is quite reasonable from the physical point of view because every compact space-time is known to have closed timelike curves. "Compact" and "noncompact" refer, of course, to the entire 4-dimensional space-time manifold.

I. SPINOR STRUCTURE

Let P be a point of a space-time M . A spinor⁸ at P consists, at least, of a rule which assigns to each⁹ orthonormal tetrad w at P an array of complex numbers $\xi^A \dots \xi^{B'} \dots(w)$, this array given only up to an over-all sign. The individual numbers in each array are labeled by indices $A, \dots, B', \dots, C, \dots, D', \dots$, whose range is 1, 2. To complete the definition we must specify how such an array is to change under a change in the choice of tetrad. Let v and w be two tetrads at P , and suppose that the Lorentz transformation L which carries v to w preserves the time direction and the spatial parity, i.e., that L is in the restricted Lorentz group \mathcal{L}_0 . Since⁸ the group $SL(2, C)$ of unimodular complex 2×2 matrices is the universal covering space of the (doubly connected) group \mathcal{L}_0 , there corresponds to the element L precisely two elements $\pm U^A_B$ of $SL(2, C)$.¹⁰ We now require that the arrays associated with the tetrads v and w be related as follows:

$$\xi^A \dots \xi^{B'} \dots(w) = \pm U^A_E \dots \bar{U}^{B'}_{F'} \dots (U^{-1})^G_C \dots (U^{-1})^{H'}_{D'} \xi^E \dots \xi^{F'} \dots(v). \quad (1)$$

A sign ambiguity still remains in Eq. (1). It is the condition that this "two-valuedness" can be consistently eliminated over the entire space-time M that places some restrictions on the global structure of M .

Let us first of all complete our definition of a spinor at the single point P . Fix a reference tetrad w at P . We now deal, not with the set Ψ of all tetrads at P , but rather with the collection Ψ' of all pairs (v, α) ,

⁸ See, for example, F. A. E. Pirani, in *Lectures in General Relativity, Brandeis Summer Institute in Theoretical Physics, 1964* (Prentice Hall, Inc., Englewood Cliffs, N.J., 1965). See also Ref. 4.

⁹ We shall be concerned only with those tetrads having a certain preassigned temporal and spatial orientation.

¹⁰ This ambiguity in sign cannot be removed in a continuous way. The situation is similar to the sign ambiguity in choosing a normal vector field to a Möbius band embedded in Euclidean 3-space.

where $v \in \Psi$ and α is a path in Ψ from v to w .¹¹ The pairs (v, α) and (u, β) are considered to represent the same point of Ψ if $u = v$ and if the path α can be continuously distorted into the path β while keeping its endpoints fixed. In other words, Ψ is the universal covering manifold of the six-dimensional manifold Ψ . (Intuitively, we may think of Ψ as a collection of tetrads at P , but such that a tetrad is considered to be a different object on rotation through 360° , while it is left unchanged by a rotation through 720° .) The crucial property of Ψ is that any two elements v and w of Ψ are transformed into each other by a *unique* element U^A_B of $SL(2, C)$. We may now define a spinor at P as a rule which assigns to each element v of Ψ an array $\xi^A \dots B' \dots(v)$ of complex numbers such that if $v, w \in \Psi$ are related by $U^A_B \in SL(2, C)$, then

$$\xi^A \dots B' \dots(w) = U^A_E \dots \bar{U}^{B'}_{F'} \dots (U^{-1})^G_C \dots (\bar{U}^{-1})^{H'}_{D'} \xi^E \dots F' \dots(v). \quad (2)$$

This definition would serve also to define spinor fields over all of Minkowski space, since a tetrad at any point can be referred to the origin by means of parallel transport.

Unfortunately, in a general space-time M , it is not possible to set up a continuous correspondence between the tangent space of each point of M and that of a single fixed point $P \in M$. What is required in order to define spinor fields on M is to do globally what we have just done locally at P . One must take the covering space of the 6-manifold of tetrads at each point of M and then piece together this 4-dimensional collection of 6-manifolds. The process of "piecing together" results in a fiber bundle.

Let the space-time M have a given time and space orientation, and let B be the principal fiber bundle¹² of oriented orthonormal tetrads on M . That is, B is a 10-manifold. Each point of B consists of a tetrad at a single point of M . The group of B is the restricted Lorentz group \mathcal{L}_0 , while the fiber over a point $P \in M$ is the collection Ψ of tetrads at P having the prescribed temporal and spatial orientation. A spinor field on M could now be defined (up to sign) as a mapping, with the appropriate transformation properties, from B into arrays of complex numbers. To correct the sign ambiguity, we must have a fiber bundle whose fiber is not Ψ , but rather the universal covering space Ψ

of Ψ . A *spinor structure*¹³ on M is defined as a second principal fiber bundle \tilde{B} [with group $SL(2, C)$] over M , along with a 2-1 mapping $\varphi: \tilde{B} \rightarrow B$, such that:

- (1) φ maps each fiber of \tilde{B} into a single fiber of B ;
- (2) φ commutes with the group operations. That is, for each $U \in SL(2, C)$, $\varphi \circ U = \Lambda(U) \circ \varphi$, where $\Lambda: SL(2, C) \rightarrow \mathcal{L}_0$ is the covering mapping of the restricted Lorentz group.¹⁴

A space-time M , if it has any spinor structure at all, does not have a unique one unless M is simply connected.

Given a spinor structure \tilde{B} , a spinor field on M may be defined,¹⁵ just as before, as a mapping ξ of \tilde{B} into arrays of complex numbers such that ξ transforms as in Eq. (2).

The mere fact that a spinor formalism is useful in some calculations is a rather unsatisfactory reason to include the existence of a spinor structure among the "reasonable physical conditions" on space-times. However, an argument due to Penrose,⁴ based on a gedanken experiment of Aharonov and Susskind,⁵ makes the assumption of a spinor structure somewhat more plausible. The Aharonov-Susskind apparatus consists of a box which may be separated into two halves. If we join the two halves together, a current flows from one half to the other. If we now separate the halves, rotate one through 360° while keeping the other fixed, and then rejoin the halves, the direction of current flow is reversed. Thus the Aharonov-Susskind boxes behave somewhat like a spinor in that they are capable of keeping track of a relative rotation through 360° . This experiment demonstrates that the

¹³ See Ref. 4. Essentially because of the way that the Lorentz group is embedded in the general linear group $GL(4, R)$, there is a one-to-one correspondence between the spinor structures on a (space- and time-oriented) space-time M and the spin structures on the tangent bundle of the underlying manifold of M . See J. Milnor, *L'enseignement math.*, 9, 198 (1963); also "Remarks Concerning Spin Manifolds," in *Differential and Combinatorial Topology*, S. S. Cairns, Ed. (Princeton University Press, Princeton, N.J., 1965), p. 55.

¹⁴ It is well known that a necessary and sufficient condition that a space- and time-oriented space-time M have spinor structure is that the second Stiefel-Whitney class of M vanish. See Ref. 13.

¹⁵ Penrose (Ref. 4) has introduced a generalization of this definition which may be more convenient when conformal transformations are contemplated. Define an orthotetrad at P as a collection of four vectors $k^a_{(i)}$, $i = 1, 2, 3, 4$, at P which satisfy

$$k^a_{(i)} k^b_{(j)} g_{ab} = \lambda \eta_{ij},$$

where λ is a positive number and where η_{ij} is the matrix diagonal $(+1, -1, -1, -1)$. Now define a spinor as a rule which assigns to each orthotetrad k at P an array of complex numbers $\xi^A \dots B' \dots(k)$ such that (a) under a Lorentz transformation the array behaves as in Eq. (2), and (b) under the transformation $k^a_{(i)} \rightarrow \mu k^a_{(i)}$ the array transforms as follows:

$$\xi^A \dots B' \dots(\mu k) = \mu^{-n/2} \xi^A \dots B' \dots(k).$$

Here n is the conformal weight of the spinor.

¹¹ Our Ψ is essentially the collection of spin frames at P . See E. T. Newman and R. Penrose, *J. Math. Phys.* 3, 566 (1962).

¹² See, for example, N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton, N.J., 1951).

two-valuedness of spinors can be realized physically by macroscopic systems. If such Aharonov-Susskind boxes could be moved throughout space-time (i.e., along curved world lines and over regions of non-vanishing curvature) and be intercompared with one another—all without their losing track of the “spin-orientation” information—we would, in principle, have an experimental means to measure the spinor structure of our own universe.

Two conditions, each necessary and sufficient for the existence of a spinor structure, can be obtained fairly easily from the definitions.¹⁶ These conditions provide some insight into the global implications for a space-time of our requirement that it carry spinor fields.

Let B be the bundle of oriented frames of the space-time M . Then M has a spinor structure if and only if the fundamental groups¹⁷ of B and M are related as follows:

$$\pi_1(B) \approx \pi_1(M) \oplus \pi_1(\Psi). \quad (3)$$

[Recall that $\pi_1(\Psi) = Z_2$.] To see the implications of this equation, assume for the moment that Eq. (3) holds. We may then take a (double) covering space of B , this space obtained by “unwrapping Ψ ,” i.e., by annihilating $\pi_1(\Psi)$ while leaving $\pi_1(M)$ unchanged. Such a covering space turns out to be just a spinor structure \tilde{B} on M . Thus the existence of a spinor structure is equivalent to the statement that we may “unwrap each of the fibers on the bundle of frames without at the same time unwrapping any of the underlying manifold M .”

Consider next a closed curve γ in B which lies entirely in the fiber over one point P of M . Suppose γ is so chosen that it cannot be contracted to a point while remaining in the fiber (i.e., γ corresponds to a rotation of a tetrad at P through 360°). We now permit γ to be distorted throughout the entire bundle B . That it still be impossible to contract γ to a point is a necessary and sufficient condition that M admit a spinor structure (assuming, once again, that M is space- and time-oriented). This is just the result we should have expected intuitively. Consider a (one-index) spinor attached¹⁸ to a frame at $P \in M$. When the frame is rotated through 360° at P , the spinor reverses its sign. This rotation of the frame defines a curve in the fiber of B over the point P . Suppose now that this curve can be contracted, in B , to a point. We then arrive at a contradiction because a “sign change” (which occurs when the curve is in the fiber

over P) has been continuously distorted to “no sign change” (which occurs when the curve has finally been contracted to a point).

We mention one further property of spinor structures: A space- and time-oriented space-time has a spinor structure if and only if each of its covering manifolds does.¹⁹ It is of interest to compare spinor structure and other global properties of space-times with respect to behavior under taking a covering manifold. It is known,²⁰ for example, that (1) the experimental evidence on nonconservation of C , P , and CP in elementary-particle reactions, (2) the CPT theorem, and (3) the strong principle of equivalence²¹ together imply that our universe must be orientable. However, from one point of view this result places no restrictions whatever on the underlying manifold of our universe: any space-time (whether orientable or not) has a covering manifold which, while representing exactly the same physical universe, is necessarily orientable. We see that the question of the existence of spinor fields is very different from the question of the existence of an orientation in that we cannot “create” spinor structure merely by taking a covering manifold.

We conclude Sec. I with an example of a space-time which has no spinor structure. In fact, we may just as easily display what is, in a sense, to be clarified shortly—the “generic” example. Let A_1 and A_2 each be a copy of the cross product of the closed unit 2-disc (polar coordinates θ_i and $r_i \leq 1$, $i = 1, 2$) with the 2-dimensional plane (polar coordinates φ_i and ρ_i). We now identify the boundaries (each diffeomorphically $S^1 \times R^2$) of A_1 and A_2 as follows: the point $(\theta_1, r_1, \varphi_1, \rho_1)$ of A_1 is identified with the point $(\theta_2, r_2, \varphi_2, \rho_2)$ of A_2 if

$$\begin{aligned} \theta_1 &= \theta_2, & \varphi_1 &= \varphi_2 + m\varphi_1, \\ r_1 &= r_2 = 1, & \rho_1 &= \rho_2, \end{aligned}$$

where m is a fixed nonnegative integer. That is, for each m we define a 4-manifold (without boundary) M_m . (In fact,²² the M_m are precisely the collection of all R^2 bundles over S^2 . In particular, $M_2 =$ tangent

¹⁶ See Refs. 4 and 13.
¹⁷ See, for example, A. H. Wallace, *Introduction to Algebraic Topology* (Pergamon Press, Inc., New York, 1957).
¹⁸ That is, ξ , by definition, changes as the frame w is changed, so that the components of ξ relative to w are constant.

¹⁹ This is so essentially because the spinor structure involves the second homotopy group of M , while the operation of taking a covering space acts only on the first homotopy group.
²⁰ R. Geroch, “Singularities in the Spacetime of General Relativity,” Ph.D. thesis, Dept. of Physics, Princeton University, 1967 (unpublished). The same type of argument has been used by Ya. B. Zeldovich and I. D. Novikov (“The Topology of the Universe: Restrictions from Elementary Particle Physics,” submitted to *Zh. Eksp. Teor. Fiz. Pis'ma Redaktsiya*, English transl.: *JETP Letters*) to obtain a slightly different result.
²¹ R. H. Dicke, *Science* **129**, 621 (1951).
²² See Ref. 12, p. 96.

bundle of S^2 .) We remark that each M_m may be given a metric of Lorentz signature.²³

If m is odd, then, no matter what Lorentz metric is placed on M_m , M_m has no spinor structure. To verify this, choose a closed curve γ which lies entirely in the fiber over the point $\rho_1 = 0, r_1 = 0$, of M_m , and which is not homotopically zero in the fiber. Observe that if m is odd, then γ may be distorted to a point in the bundle of frames of M_m by sliding it over the 2-sphere $\rho_1 = 0, \rho_2 = 0$.

Our example is the generic one in the following sense. Let M be a space-time which cannot be given a spinor structure. Then there must be some closed curved γ lying in the fiber over a point $P \in M$ such that, while γ is not homotopically zero in the fiber, it can be contracted to a point in the entire bundle of frames. Contract γ to a point, while at the same time projecting its path into M . We thus obtain the image S in M of a 2-sphere. Suppose²⁴ that S can be so chosen that it is smooth and does not intersect itself. Then we may select a neighborhood V of S which is topologically a 2-dimensional vector bundle over S^2 . Since it contains S as a subset, V itself does not admit a spinor structure, and must therefore be just one of the M_m for m odd. That is, the space-time M contains an open subset which is diffeomorphic to one of the M_m, m odd.

II. GLOBAL SYSTEMS OF TETRADS

Our main result reexpresses the existence of a spinor structure in more familiar terms.

Theorem: Let M be a noncompact⁷ space-time.¹ Then M has a spinor structure if and only if there exists on M a global system of (orthonormal) tetrads.²⁵

Proof: Suppose first of all that M has a spinor structure \tilde{B} . Since M is a space-time, it is paracompact (see the Appendix). Therefore²⁶ M may be triangulated.²⁷ (Since M is noncompact, there will necessarily

²³ See L. Markus, Ann. Math. 62, 411 (1955). In fact, this result of Markus has been strengthened slightly (Ref. 4): Every noncompact 4-manifold may be given a time-oriented Lorentz metric which has, in addition, no closed timelike curves.

²⁴ In fact, we need only require for this argument that there be some 2-submanifold S of M such that S is diffeomorphically an S^2 , and such that S , considered as an element of $\pi_2(M)$, is not in the kernel of the homomorphism Δ in the (exact) homotopy sequence of the bundle B (Ref. 12, p. 91):

$$\dots \rightarrow \pi_2(B) \rightarrow \pi_2(M) \xrightarrow{\Delta} \pi_1(\Psi) \rightarrow \dots$$

Can such an S always be found?

²⁵ We actually prove slightly more than this. If M has spinor structure \tilde{B} , then the global system of tetrads (considered now as a cross section T of B) can be so chosen that T may be lifted to a cross section of \tilde{B} .

²⁶ See J. H. C. Whitehead, Ann. Math., 41, 809 (1940).

²⁷ S. S. Cairns, *Introductory Topology* (Ronald Press, New York, 1961), p. 76.

be an infinite number of simplices.) Choose²⁸ a sequence M_1, M_2, \dots of subcomplexes of M such that:

(1) Each M_i is, as a topological space, a compact 4-manifold with boundary. (It follows that each M_i must consist of only a finite number of simplices.)

(2) $M_i \supset M_{i-1}, i = 2, 3, \dots$; and $\bigcup_i M_i = M$.

(3) Each connected component of $M_i - M_{i-1}$ contains at least one boundary 3-simplex, $i = 2, 3, \dots$.

The underlying group of the bundle \tilde{B} is $SL(2, C)$. But $SL(2, C)$ is topologically $R^3 \times S^3$, and so its homotopy groups are easily calculated:

$$\pi_q(SL(2, C)) \approx \pi_q(R^3) \oplus \pi_q(S^3) \approx \pi_q(S^3).$$

It is convenient to list these groups here for later reference:

$$\begin{aligned} \pi_1(SL(2, C)) &= 0, & \pi_2(SL(2, C)) &= 0, \\ \pi_3(SL(2, C)) &= Z. \end{aligned}$$

The proof consists of constructing a cross section of the principal fiber bundle \tilde{B} . We proceed inductively, extending a cross section from one M_i to the next. Let us suppose, therefore, that we are given a cross section of \tilde{B} over $J \equiv M_{i-1}$, and that we wish to extend this cross section to $K \equiv M_i$ (in case $i = 1$, set $J \equiv \phi$). Denote by K^j ($j = 0, 1, 2, 3, 4$) the j -skeleton of K (i.e., the union of all simplices of K of dimension less than or equal to j).

It follows²⁹ from the vanishing of the first two homotopy groups of $SL(2, C)$ that our given cross section over J can be extended uniquely (up to homotopy) to a cross section C over $J \cup K^2$, and further that C can be extended (though not, in general, uniquely) to a cross section over $J \cup K^3$. Since $\pi_2(SL(2, C)) \neq 0$, however, a given cross section over $J \cup K^3$ cannot, in general, be extended over $J \cup K^4 = K$. One would like to use the freedom available in selecting an extension of C over $J \cup K^3$ to find one such extension C' having the property that C' can be further extended over $J \cup K^4$. However, such a C' does not in general exist. In fact, the cross section C determines³⁰ an element α of the cohomology group $H^4(K, J; Z)$ (a "characteristic class"). The vanishing

²⁸ Subcomplexes M_i which satisfy these conditions may be constructed in the following way. Let $\sigma_1, \sigma_2, \dots$ denote the 4-simplices of M . Define

$$\begin{aligned} M_0 &\equiv \phi, \\ M_i &\equiv \bigcup_{k \in \Gamma_i} \sigma_k, \quad i = 1, 2, \dots, \end{aligned}$$

where

$$\Gamma_i = \{\text{positive integers } k \text{ such that there exists a compact set } C \text{ with } \partial C \in (M_{i-1} \cup \sigma_k) \text{ and with } \sigma_k \subset C\}.$$

²⁹ Ref. 12, p. 149.

³⁰ Ref. 12, p. 174.

of α is a necessary and sufficient condition for the existence of a C' which can be further extended over $J \cup K^4 = K$.

Our three conditions on the M_i have as a consequence, however, that $H^4(K, J; Z)$ vanishes identically.³¹ We conclude that $\alpha = 0$, and therefore that a cross section over K exists.

We have shown that any cross section of \tilde{B} over J can be extended to a cross section over K . Setting $K = M_1$ and $J = \phi$, we obtain a cross section over M_1 . Setting $K = M_2$ and $J = M_1$, we extend this cross section from M_1 to $M_2 \supset M_1$. Continuing in this way, we obtain finally a cross section of \tilde{B} over all of M . The projection of our cross section from \tilde{B} to B yields the required global system of tetrads.

The converse—if M has a global system of tetrads, then it has a spinor structure—is an immediate consequence of the definition of spinor structure.

This theorem depends critically on the vanishing of homotopy groups of the spinor group $SL(2, C)$. The first homotopy group vanishes essentially because a spinor structure is defined by the property that its fiber is the universal covering space of the fiber of the bundle of frames. (Taking the universal covering space automatically annihilates the first homotopy group.) The second homotopy group vanishes for all the spin groups (in fact, for all Lie groups). The third homotopy group fails to vanish, but at this point we are sufficiently close to the dimension of the manifold that the obstruction to extending a cross section can be made to vanish. Thus the dimension of the manifold enters in an essential way. In fact, the theorem is true in four dimensions, uninteresting in lower dimensions (in this case, every orientable manifold is parallelizable³²), and false in higher dimensions.

We remark on one application of our theorem, an application to the problem of classifying space-times according to their global structure. The homotopy classes³³ of metrics of Lorentz signature on a given

manifold M are in one-to-one correspondence with homotopy classes of nowhere-vanishing vector fields on M . Let us agree that a space-time, to be of physical interest, must admit a spinor structure. Then we may, according to the theorem, take M to be parallelizable, and so its tangent bundle is the cross product $M \times R^4$. Therefore the homotopy classes of Lorentz metrics on M are in one-to-one correspondence with the collection Λ of homotopy classes of maps $\lambda: M \rightarrow S^3$. It might be interesting to see if, given M , the homotopy classes Λ may be characterized in some fairly simple way. Such a characterization would be a start toward a classification of the "homotopy classes of space-times": Which global properties of space-times are invariants of the homotopy class?

ACKNOWLEDGMENTS

I wish to thank R. Penrose for valuable discussions throughout the course of this work, as well as for reading the manuscript and suggesting several improvements.

This work was carried out under a U.S. Air Force Office of Scientific Research postdoctoral fellowship.

APPENDIX

The following theorem makes it possible to simplify the statement of many other theorems. In dealing with a manifold M which carries a metric of Lorentz signature, it is not necessary to specify that, in addition, M have a countable basis (or, equivalently, that M be paracompact).

Theorem: Let M be a (connected, Hausdorff) 4-manifold with a C^∞ metric of signature $(+, -, -, -)$. Then the topology of M has a countable basis.³⁴

Proof: Fix once and for all a countable basis O_i ($i = 1, 2, \dots$) for the open sets of R^4 and, for each i , a point $q_i \in O_i$.

Let P be a point of M . The exponential map³⁵ "exp" is a continuous function from a subset of T_P , the tangent space at P , into M . Let t_P denote the union of all open subsets of T_P on which "exp" is defined and is an open map.³⁶ We first show that "exp" is an

³¹ That is, each 4-simplex Σ of $K - J$ can be represented as the coboundary of some 3-chain \mathcal{C} lying entirely in $K - J$. To see this, let γ be a curve from Σ to the boundary of $K - J$, so chosen that $\gamma \cap (K^2 \cup J) = \phi$. Then define \mathcal{C} to be the (properly counted) sum of the 3-simplices which intersect γ .

³² E. Stiefel, *Comm. Math. Helv.*, 8, 3 (1936).

³³ The idea of looking at homotopy classes of metrics was considered by D. Finkelstein and C. W. Misner (*Ann. Phys. (N.Y.)* 6, 230 (1959)). Their approach differs from the one contemplated here, however, in the following way. Finkelstein and Misner consider space-times which are topologically R^4 . Consequently, it is necessary to impose a boundary condition (asymptotic flatness) to prevent the "twists" in the metric from being lost to infinity. We impose no boundary conditions, but as a result we must require that the manifold have non-Euclidean topology (an "O-geon" in the Finkelstein-Misner terminology) if there is to be even the possibility of more than one homotopy class of Lorentz metrics. In addition, our homotopy classes do not have a group structure, as do those of Finkelstein-Misner.

³⁴ The theorem actually proved here is somewhat stronger: Every n -manifold with a C^1 connection has a countable basis. Surprisingly enough, it is *not* true that every manifold with a conformal Lorentz metric (i.e., a Lorentz metric given at each point only up to an arbitrary nonzero factor) has a countable basis. A counterexample is the 2-manifold defined as the Cartesian product of two "long lines" [J. G. Hocking and G. S. Young, *Topology* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1961), p. 55]. The null directions (which define the conformal metric) are specified as lying along the coordinate axes at each point.

³⁵ See, for example, N. J. Hicks, *Notes on Differential Geometry* (D. Van Nostrand, Inc., Princeton, N.J., 1965), p. 131.

³⁶ A mapping is said to be open if the image of each open set is again an open set.

open map on t_P . Let V be an arbitrary open subset of t_P , and let V_α be a covering of t_P by open sets on each of which "exp" is an open map. For each α , $V \cap V_\alpha$ is open in V_α , and therefore $\exp(V \cap V_\alpha)$ is open in M . It follows that

$$\exp(V) = \bigcup_\alpha \exp(V \cap V_\alpha)$$

is open in M . Since V is arbitrary, "exp" is an open map on t_P . Define $S_P \equiv \exp(t_P) \subset M$.

Now let v be a tetrad at P . The tetrad v determines a natural diffeomorphism $\varphi_v: T_P \rightarrow R^4$, where, for each vector $\xi \in T_P$, $\varphi_v(\xi)$ is defined as the 4-tuple of components of ξ relative to v . We define the composition $\theta \equiv \exp \circ \varphi_v^{-1}$, a continuous map from a subset of R^4 into M .

Whenever the integer i is such that $q_i \in \theta^{-1}(S_P)$, we define

$$P_i = \theta(q_i) \in M, \\ U_i \equiv \theta(O_i \cap \theta^{-1}(S_P)) \subset M.$$

At each of the points P_i we define a tetrad v_i by parallel transport of the tetrad v along the geodesic⁸⁷ from P to P_i .

We now show that the U_i are a basis for the open sets of S_P . Since the O_i cover R^4 , the U_i cover S_P . Since θ is an open map on $\theta^{-1}(S_P)$, each U_i is an open subset of M . Let U be any open subset of S_P . Then $\theta^{-1}(U)$ is open in R^4 . Since the O_i are a basis for R^4 ,

⁸⁷ In case there are several geodesics from P to P_i , let us select, for definiteness, that one which has been used (in the exponential map) to define P_i .

we may select a collection Γ of integers such that $\bigcup_{i \in \Gamma} O_i = \theta^{-1}(U)$. Therefore, $\bigcup_{i \in \Gamma} U_i = U$. Since U is arbitrary, the U_i are a basis for S_P .

To summarize, given any pair (P, v) , where v is a tetrad at the point P , we have defined an open neighborhood S_P of P in M , a countable basis U_i for S_P , a countable dense set P_i in S_P , and a tetrad v_i at each P_i . We now repeat our construction for each pair (P_i, v_i) . That is, we define an open neighborhood S_{P_i} of P_i , a basis U_{ij} for S_{P_i} , a dense set P_{ij} in S_{P_i} , and a tetrad v_{ij} at each P_{ij} (i fixed, $j = 1, 2, \dots$). Applying the same construction to the pairs (P_{ij}, v_{ij}) , and so on, we obtain finally a countable collection $U_\alpha \equiv \{U_i, U_{ij}, U_{ijk}, \dots\}$ of open sets of M and a countable collection $P_\alpha \equiv \{P_i, P_{ij}, P_{ijk}, \dots\}$ of points of M .

Since the U_α form a basis for each of the S_{P_β} , they also form a basis for $S \equiv \bigcup_\alpha S_{P_\alpha}$. We must finally show that $S = M$.

Let Q be any point in the closure of S . There is some neighborhood U of Q such that no geodesic segment in U has a pair of conjugate points in U . Since Q is in the closure of S , $U \cap S \neq \emptyset$. Therefore, since the P_α are dense in S , there must be some $P_\beta \in U$. But no geodesic segment has a pair of conjugate points in U , and so $S_{P_\beta} \supset U$. We have shown that $Q \in U \subset S_{P_\beta} \subset S$, i.e., that S contains each point of its closure. But S is also open in M , because it is the union of the open sets S_{P_α} . Since M is connected, it follows that $S = M$. This completes the proof.

New Solutions of the Einstein-Maxwell Equations from Old*

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(Received 14 December 1967)

Methods are discussed with which one may derive theorems which allow one to generate new solutions of the Einstein-Maxwell equations from old ones. The old solutions used to generate new ones must admit at least one nonnull Killing vector and may be required to satisfy other conditions, depending on the theorem derived. Examples of derivable theorems are shown; these theorems are used in turn to show how generation of new solutions is accomplished. Examples of the latter are shown, such as generation of Brill or electrified NUT space from the Schwarzschild solution, generation of a new twisted Melvin universe from flat space, and generation of a new generalization of the Ozsvath-Schücking metric. Possible physical interpretations, uses, and extensions of this type of theorem are discussed.

1. ASSUMED METRIC AND ITS EINSTEIN-MAXWELL EQUATIONS

We outline a method, suitable for use on a wide class of metrics, by which one can derive theorems which in turn can be used to obtain new solutions of

the Einstein-Maxwell equations from old. In Sec. 1, we find the form of the Einstein-Maxwell equations for the assumed class of metrics; in Sec. 2 we show how to derive these theorems. Section 3 presents examples of theorems, including one (Theorem 2)