

Spinor Structure of Space-Times in General Relativity. II

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Spinor fields can only be defined on a space-time which has been given a spinor structure. A number of conditions (some sufficient, others necessary and sufficient) for the existence of a spinor structure are derived. By applying one or another of these conditions, it is shown that many well-known solutions of Einstein's equations do have spinor structure. The question of the existence of spinor structure depends only on the topology of the underlying manifold, not on the (time- and space-oriented) metric. It is shown that, nonetheless, a certain "threshold" of curvature must be exceeded before there can be even the possibility of a space-time's having no spinor structure.

INTRODUCTION

Before spinor fields may be defined on a space-time M , it is first necessary to endow M with some further structure—called the spinor structure—in addition to the Lorentz metric. Certain space-times cannot be given any spinor structure at all, and, furthermore, even when there exists a spinor structure it will not, in general, be unique.¹ However, there is a gedanken experiment,² based on the quantum-mechanical properties of spin- $\frac{1}{2}$ particles, by which one could, in principle, determine a definite spinor structure for our own universe. Thus, the existence of a spinor structure appears, on physical grounds, to be a reasonable condition to impose on any cosmological model in general relativity. In the present paper we shall investigate the global restrictions this condition places on possible cosmological models.

It was shown in I that a necessary and sufficient condition that a noncompact space-time M have spinor structure is that M may be given a global system of orthonormal tetrads. While this criterion represents a strong condition to be satisfied by cosmological models, it is not always the most convenient way to decide whether or not a given space-time has spinor structure. The Schwarzschild solution, for example, has spinor structure, but it is not immediately clear that this space-time may be given a system of tetrads. [It would not do, for example, to choose two of the vectors to lie along the r and t axes (in the usual coordinates), for then the other two vectors would have to lie in the 2-spheres $r = \text{const}$, $t = \text{const}$, which is impossible.] In Sec. I we develop some

criteria for the existence of spinor structure, based on the neighborhoods of certain 2-spheres in M . With each such 2-sphere S we associate an index, defined as the number of times that S intersects a surface obtained by slightly deforming S . That this index be even for each S in M is a necessary and sufficient condition that M have spinor structure. We may also characterize the existence of spinor structure in terms of the type of the Weyl tensor. It is shown that a space-time whose Weyl tensor is everywhere type $[1, 1, 1, 1]$, $[2, 1, 1]$, $[3, 1]$, or $[4]$ necessarily has spinor structure. In the "generic" case, however, we would expect that the Weyl tensor would be algebraically general everywhere except in certain regions of lower dimensionality. It turns out that the behavior of these regions in the large is sufficient to determine whether or not the space-time has spinor structure. Finally, we show that every space-time which arises from initial-value data—that is, every space-time which has a Cauchy surface—also has spinor structure. The goal of Sec. I is to obtain a list of criteria which may be used to test in practice whether a given space-time has spinor structure (or, equivalently, a global system of tetrads). Most common solutions of Einstein's equations satisfy one or another of the conditions in Sec. I.

In Sec. II we discuss the relation of spinor structure to the amount of curvature present in the space-time. That there should be any relation at all is somewhat surprising, for the existence or nonexistence of a spinor structure is a property only of the underlying manifold, independent of the metric (provided only that the metric is time and space oriented). We display a curvature integral over 2-spheres: that this integral be less than a certain value is a sufficient (but not necessary) condition for the existence of spinor structure. Intuitively, we may think of the integral as representing a threshold condition on the curvature: the threshold must be exceeded before there is even the possibility that the space-time have no spinor structure. This integral represents one of the few

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¹ See, for example, R. Penrose, "The Structure of Space-Time," in *Battelle Recontres in Mathematics and Physics: Seattle, 1967*, C. DeWitt and J. A. Wheeler, Eds. (W. A. Benjamin, Inc., New York, 1968); R. Geroch, *J. Math. Phys.* **9**, 1739 (1968); this paper will be referred to hereafter as I.

² Y. Aharonov and L. Susskind, *Phys. Rev.* **158**, 1237 (1967); Ref. 1.

situations in which, without imposing symmetries, the curvature of a space with an indefinite metric is known to have a bearing on the global structure of the space.

I. CRITERIA FOR THE EXISTENCE OF SPINOR STRUCTURE

By a space-time M we understand a 4-dimensional manifold with a metric of signature $(+, -, -, -)$. We shall assume, furthermore, that M is noncompact and both space and time oriented.³ The noncompactness assumption is reasonable from the physical point of view because every compact space-time is known^{4,5} to have closed timelike curves. If, on the other hand, M were not space and time oriented, then we could always find a covering space of M —representing exactly the same physical universe—which is.⁵

Let S be a fixed (not necessarily spacelike) 2-surface in M which is topologically equivalent to a 2-sphere, but which may cross itself at isolated points.⁶ By slightly deforming S , we obtain another 2-sphere S' in M . In general, two 2-dimensional surfaces in a 4-dimensional manifold will intersect in a region of dimension zero, i.e., in points. Choose S' so that it intersects S only at isolated, nondegenerate points $p_k, k = 1, 2, \dots, m$, that is, so that S and S' have no common tangent vectors at their points of intersection. Let us assign an orientation to S , whence S' , which is obtained by deforming S , is also assigned a definite orientation. Now at each intersection point p_k , the vectors tangent to the oriented surfaces S and S' span the set of all vectors at p_k , and so define an orientation of this 4-dimensional vector space. Define $\iota(p_k)$ to be $+1$ if this orientation is the same as that of M , and -1 if it is the opposite. The index of the surface S is defined by⁷

$$I(S) \equiv \sum_{k=1}^m \iota(p_k).$$

If we reverse the orientation originally assigned to S , the orientation of S' is also reversed, and so the index is unchanged. Furthermore, the index of S is independent of the distortion by which S' is obtained

³ A space-time M is said to be *time oriented* if the light cones of M are divided into two systems, past and future, and *space oriented* if any collection of three independent spacelike vectors at a point which are all orthogonal to a single timelike vector is assigned a definite parity, $+1$ or -1 .

⁴ E. Kronheimer and R. Penrose, Proc. Cambridge Phil. Soc. 63, 481 (1967).

⁵ R. Geroch, J. Math. Phys. 8, 782 (1967).

⁶ We shall regard such a crossing point as representing two distinct points of S , each of which must be treated independently of the other. More precisely, S represents a mapping from a 2-sphere into M , not just the image of such a mapping.

⁷ This index is well known in homology theory. See, for example, P. S. Aleksandrov, *Combinatorial Topology* (Graylock Press, Albany, N.Y., 1960), Vol. 3, p. 73.

from S , for under any further deformation of S' new points of intersection are created in pairs whose ι -values are -1 and $+1$. Finally, the index must be continuous under deformations of S , and so, since $I(S)$ takes only integral values, the index is invariant under such deformations. Note also that the definition of the index does not involve the metric on M .⁸

Alternatively, the index of S may be characterized in the following way. Consider any vector field ξ^a which is nonvanishing in a neighborhood of S and which is tangent to S only at nondegenerate points. Define a new 2-surface S' , obtained from S by moving a small distance along the trajectories of ξ^a . The intersection points of S and S' now correspond precisely to the points at which ξ^a is tangent to S . Thus, the index of S is equal to the number of times (properly counted with regard to sign) that ξ^a is tangent to S .

The relationship between the index and spinor structure is given by the following result: *A necessary and sufficient condition that M have spinor structure is that the index $I(S)$ be even for every 2-surface S , topologically a 2-sphere, in M .* To prove this statement, we make use of the fact (see I) that there exists a neighborhood of S which is topologically equivalent to just one of a certain collection of 4-dimensional manifolds $M_m, m = 0, 1, 2, \dots$. (If S happens to cross itself, then any neighborhood of S will overlap itself. We must then count each point in the overlap region twice, that is, we treat the neighborhood as though it did not intersect itself.) The M_m are defined as follows. Let A_1 and A_2 be two copies of the Cartesian product of a closed unit 2-disc (polar coordinates θ_i and $r_i, i = 1, 2$) and a 2-dimensional plane (Cartesian coordinates x_i and y_i). Join A_1 and A_2 across their boundaries ($r_1 = 1$ and $r_2 = 1$, respectively) by identifying the point $(\theta_1, r_1, x_1, y_1)$ of A_1 and the point $(\theta_2, r_2, x_2, y_2)$ of A_2 whenever⁹

$$\begin{aligned} r_1 = r_2 = 1, \quad x_1 = x_2 \cos(m\theta_1) - y_2 \sin(m\theta_1), \\ \theta_1 = \theta_2, \quad y_1 = y_2 \cos(m\theta_1) + x_2 \sin(m\theta_1). \end{aligned}$$

The 2-sphere S in M is to correspond to the 2-sphere S_m in M_m given by $x_i = y_i = 0$. It was shown in I that

⁸ The various properties of the index are most easily understood by considering a lower-dimensional case: closed curves on a 2-dimensional manifold. There is, however, a curious property of intersections of 2-surfaces in a 4-dimensional manifold which is not present in the 2-dimensional case. The surfaces in Minkowski space given (in the usual coordinates) by $y = z^2 - t^2, x = tz$ and $y = -z^2 + t^2, x = -zt$ intersect only at the origin. If, however, we distort either surface slightly, the number of intersections increases to two: there is no way—as there would be in the analogous situation in two dimensions—slightly to distort the surfaces so that they do not intersect.

⁹ The corresponding expressions were given incorrectly in I: the term $m\theta_1$ there should have been $m\theta_2$.

the space-time M has spinor structure if and only if, for every S in M , the corresponding M_m has an even value for m . Consider the vector field in M_m whose components in the coordinate patches A_1 and A_2 are

$$\begin{aligned} \xi^{r_1} &= 0, \quad \xi^{x_1} = (r_1)^2 \cos(m\theta_1) \\ &\quad + [1 - (r_1)^2], \\ \xi^{\theta_1} &= (r_1)^2 [1 - (r_1)^2], \quad \xi^{y_1} = (r_1)^2 \sin(m\theta_1), \end{aligned}$$

and

$$\begin{aligned} \xi^{r_2} &= 0, \quad \xi^{x_2} = 1, \\ \xi^{\theta_2} &= 0, \quad \xi^{y_2} = 0, \end{aligned}$$

respectively. The field ξ^a is tangent to S_m at the m (nondegenerate) points $r_1 = 2^{-\frac{1}{2}}$, $x_1 = y_1 = 0$, $\theta_1 = \pi/m, 3\pi/m, \dots, (2m - 1)\pi/m$, and so the index of S_m is simply m (or $-m$, depending on the orientation chosen for M_m). Thus, the index of S is even precisely when m is even, and our theorem follows.

Choosing ξ^a to be everywhere timelike, we see that the index of S is determined by the number of times that the light cone "cuts across" S . In particular, if S is spacelike everywhere, then its index is necessarily zero, because a timelike ξ^a will never be tangent to S . Consider, for example, the Schwarzschild solution. The 2-spheres $r = \text{const}$, $t = \text{const}$, in the usual coordinates, are spacelike. Furthermore, every 2-sphere S in the Schwarzschild solution may be continuously deformed to one of these 2-spheres, perhaps described several times. (In other words, the second homotopy group has a set of spacelike generators.) Therefore, the index of every S is even, and so the Schwarzschild solution has a spinor structure. As another application of our theorem, we see that any space-time in which every S may be contracted to a point (i.e., whose second homotopy group vanishes) necessarily has spinor structure,¹⁰ for the index, which is invariant under continuous deformations of S , clearly vanishes when S is a small sphere in some coordinate patch. In particular, the Robertson-Walker models, the Gödel solution, the fluid-ball solutions, and the plane waves all have spinor structure.¹¹

A further criterion for the existence of spinor structure can be obtained in terms of the algebraic properties of the Weyl tensor. It is known that if the

Weyl tensor C_{abcd} is type¹² [1, 1, 1, 1], [2, 1, 1], or [3, 1] at a point p , then C_{abcd} determines four orthonormal vectors at p . (For types [1, 1, 1, 1] and [2, 1, 1], the vectors are completely determined by the principal null directions, but this is not the case for type [3, 1].) These vectors are uniquely defined up to sign, but are not assigned any particular ordering. That is to say, if we select one of the vectors to be labeled η^a and keep track of η^a around some closed curve, then, on our return to the starting point, it may be that η^a now coincides with one of the other vectors of the tetrad. Furthermore, the vectors may change discontinuously when the type of the Weyl tensor changes. If, for example, C_{abcd} is type [1, 1, 1, 1] in some region R_1 and type [2, 1, 1] in some other region R_2 , then the tetrad defined by the Weyl tensor will not be continuous across the boundary region between R_1 and R_2 .

We first show that the space-time M must have spinor structure provided its Weyl tensor is everywhere type [1, 1, 1, 1], [2, 1, 1], or [3, 1]. We may take M to be simply connected (i.e., such that every closed curve in M may be contracted to a point) because the universal covering space of M , which is always simply connected, has spinor structure if and only if M does (see I). At every point of M we have an orthonormal tetrad of vectors defined by the Weyl tensor. If we carry any vector of this tetrad continuously around some closed curve γ in M , then we must return with the same element of the tetrad. This property certainly holds in the limit as γ is contracted to a point, and is invariant under continuous deformations of γ , and so, since M is simply connected, it holds for all closed curves γ . The directions defined by the Weyl tensor therefore constitute a global system of orthonormal tetrads, and so M must have a spinor structure.

In particular, certain of the Robinson-Trautman solutions¹³ are everywhere type [2, 1, 1] or [3, 1]. We conclude that these space-times have a spinor structure.

A similar, but slightly more complicated, argument suffices to show that a space-time must also have spinor structure if its Weyl tensor is everywhere type [4]. In this case the Weyl tensor defines, at each point, a null direction l^a and a pair of orthogonal null 2-planes containing l^a . We may therefore choose a triad of independent vector fields x^a , y^a , and l^a

¹⁰ This result follows directly from the usual characterization of spinor structure in terms of Stiefel-Whitney classes. See, for example, J. Milnor, *L'Enseignement Math.* 9, 198 (1963); K. Bichteler, *J. Math. Phys.* 9, 813 (1968).

¹¹ Each of these spaces have topology R^4 , except for certain of the Robertson-Walker models (those with positive spatial curvature) whose topology is $S^3 \times R$.

¹² We shall adopt Penrose's notation for the six types of Weyl tensors. The relation to Petrov's notation is given by [1, 1, 1, 1] = I, [2, 1, 1] = II, [3, 1] = III, [2, 2] = 0, [4] = N, [-] = 0. For a discussion of the classification of the Weyl tensor, see, for example, R. Penrose, *Ann. Phys. (N.Y.)* 10, 171 (1960); R. Penrose, W. Rindler, and R. Geroch, *The Spinor Approach to Space-Time* (Cambridge University Press, to be published).

¹³ I. Robinson and A. Trautman, *Proc. Roy. Soc. (London)* 265A, 463, 1962.

on M , where x^a lies in one of the 2-planes defined by the Weyl tensor and y^a lies in the other. To construct an orthonormal tetrad on M , we first choose an arbitrary unit timelike vector field t^a . Two of the spacelike vectors of the tetrad are obtained by projecting x^a and y^a into the space orthogonal to t^a at each point, while the third spacelike vector is that vector orthogonal to the other three. Thus, a type [4] space-time must have spinor structure.

It is not possible to construct an orthonormal tetrad in this way from a type [2, 2] or [-] Weyl tensor. Of course, a similar analysis could be carried out for the stress-energy tensor—or, indeed, for any tensor which has been defined on M . However, the Weyl tensor is perhaps the most convenient object for this purpose because it is always available and, except in very special cases, is nonzero.

In the case of a "generic" space-time, we would not expect that the Weyl tensor would be the same type throughout the entire space. The collection of all tensors at a point which have both the symmetries of the Weyl tensor and vanishing contractions is ten-dimensional. The tensors of type [1, 1, 1, 1] form a ten-dimensional subset of this collection, those of type [2, 1, 1] an eight-dimensional subset, and those of type [3, 1], [2, 2], [4], and [-] subsets of six dimensions or fewer. Hence, in the generic case, we would expect that the Weyl tensor C_{abcd} would be type [1, 1, 1, 1] everywhere except in some 2-dimensional region D . Let S be any 2-sphere in M , chosen to intersect D in nondegenerate isolated points. If we move in a small circle on S about one such point, the tetrad defined by the Weyl tensor will undergo a rotation (relative to a fixed tetrad at the intersection point) which, in the generic case, is through an angle of 2π . Adding these rotations over all of S , we obtain the total rotation of the tetrad. It was shown in I that a necessary and sufficient condition that the index of S be even (i.e., that there exist a tetrad in a neighborhood of S) is that this total rotation be an even multiple of 2π . Thus, M will have spinor structure if and only if D intersects every 2-sphere S in M an even number of times (with nondegenerate intersection points).

Consider, for example, the algebraically general Weyl solutions.¹⁴ The Weyl tensor is type [1, 1, 1, 1] everywhere except on the symmetry axis ($r = 0$ in the usual coordinates), where C_{abcd} is type [2, 2]. In the full four-dimensional space-time, the symmetry axis defines a 2-surface. The tetrad defined by C_{abcd} , on being taken around the axis, undergoes one complete rotation. (This property follows from the axial

symmetry: it is not necessary to calculate the Riemann tensor explicitly.) But every 2-sphere S in the Weyl solution which intersects the symmetry axis at nondegenerate points does so an even number of times. (The sphere about the singular region, for example, intersects $r = 0$ at one positive and one negative value of z .) Thus, the Weyl solutions have spinor structure.

Finally, consider a space-time M which may be expressed in the form of a topological product of a 3-surface with the real line such that each of the embedded 3-surfaces is spacelike. That is to say, suppose that there is some scalar field t on M such that (i) $\nabla_a t$ is timelike (and nonzero) everywhere, and (ii) the spacelike 3-surfaces Q_t , given by $t = \text{const}$, are topologically identical where, for any t and t' , Q_t is mapped onto $Q_{t'}$ by following along the trajectories of $\nabla^a t$. Since M is space oriented, the surface Q_0 is oriented. But every oriented 3-manifold may be given a global system of triads.¹⁵ Choose the elements of this triad to be covariant vectors in M (lowering indices with the metric on M) and extend the vector fields to all of M by imposing the requirement that the Lie derivative of each vector of the triad with respect to $\nabla^a t$ be zero. We thus obtain four independent vector fields on M such that, at each point, three of the vectors are spacelike and orthogonal to the timelike vector $\nabla_a t$. Finally, orthogonalize the tetrad (e.g., by the Gram-Schmidt orthogonalization procedure). Thus, every space-time which satisfies the above "topological-product" condition has spinor structure. In particular, every space-time which has a Cauchy surface, which is globally hyperbolic, or which is asymptotically simple with null J is known¹⁶ to satisfy the topological-product condition. We conclude that any one of these three conditions implies the existence of a spinor structure.

Suppose we begin with initial-value data for Einstein's equations (with or without sources) on a spacelike 3-surface Q . This initial data defines some space-time M . It may be that M can be further extended (as is the case, for example, in the Reissner-Nordström solution), but the metric in the extension will not be determined uniquely by the data on Q . That is, ${}^-Q$ is necessarily a Cauchy surface for the space-time M , and so M must have a spinor structure. Thus, we see that the initial data on Q will never

¹⁴ See, for example, N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton, N.J., 1951), p. 204.

¹⁶ In fact, every space-time with a Cauchy surface is a topological product (R. Penrose, "An Analysis of the Structure of Space-Time," preprint; S. W. Hawking, "Singularities and the Geometry of Space-Time," preprint; R. Geroch, "The Domain of Dependence," to be published in *J. Math. Phys.*); global hyperbolicity is completely equivalent to the existence of a Cauchy surface ("The Domain of Dependence"); and asymptotic simplicity with null J is sufficient (but not necessary) for global hyperbolicity.

¹⁴ See, for example, J. L. Synge, *Relativity: The General Theory* (North-Holland Publ. Co., Amsterdam, 1960), p. 312.

suffice to "predict" that at some later time no spinor structure will be possible. The absence of spinor structure can occur only in a space-time which has been extended in some way not defined by initial data.

II. SPINOR STRUCTURE AND CURVATURE

In our discussion of spinor structure so far we have been concerned only with the topological properties of the space-time M . The relationship of spinor structure to the Weyl tensor, for example, did not directly involve the amount of curvature present, but rather merely the qualitative behavior of certain directions constructed from the metric. That the existence of spinor structure should be related to the topological properties of tensor fields is not particularly surprising, for it is only the topology of the underlying manifold—not the choice of metric—which determines whether or not there will be a spinor structure. We shall now show that the degree of curvature is also related to the question of the existence of spinor structure. A certain minimum amount of curvature—expressed in the form of a surface integral—is necessary if there is to be even the possibility of a space-time's having no spinor structure.

The integral expression itself, while too complicated to be of much use in deciding whether or not a space-time has spinor structure, is of interest primarily because it provides a definite link between spinor structure and curvature. Thus, certain manifolds—the M_{2n+1} of Sec. I for example—are themselves sufficiently "twisted" already that any metric defined on them must contain at least a certain minimum amount of curvature. It may even turn out to be true that all exact source-free solutions of Einstein's equations have spinor structure. (As far as the author is aware, all known solutions whose global structure has been analyzed do have spinor structure.)

Consider a (noncompact, space- and time-oriented) space-time M , and let S be a 2-surface, topologically a 2-sphere, in M . The idea is to attempt to construct a tetrad on S using a certain kind of "parallel transport" in M . Such a construction must succeed if the curvature is sufficiently small in a neighborhood of S . Choose a point p of S and a one-parameter family $\gamma_s(w)$ of curves on S , where $s \in [0, 1]$ labels the individual curves and $w \in [0, 1]$ is a parameter along each curve. The curves are to all begin and end at p , and each point of S (except p) is to lie on exactly one curve. The curves $s = 0$ and $s = 1$ correspond to "zero curves" which remain at p (Fig. 1). Let ω^a and σ^a denote the vectors tangent to the lines $s = \text{const}$ and $w = \text{const}$, respectively, where ω^a and σ^a are

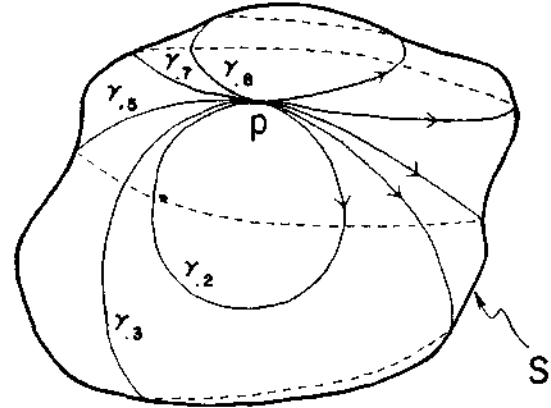


FIG. 1. A one-parameter family of curves covering the 2-sphere S . Each curve begins and ends at p , and each point of S (except p) lies on exactly one curve.

normalized by the conditions

$$\omega^a \nabla_a w = \sigma^a \nabla_a s = 1.$$

Our construction proceeds as follows. Choose an arbitrary unit timelike vector field t^a on M . At p , choose a triad¹⁷ x_a^i of spacelike vectors which, together with t^a , form an orthonormal tetrad at p . For each value of s , we transport the triad x_a^i along the curve γ_s , according to the equation

$$\omega^b \nabla_b x_a^i = -t^a (x_a^c \omega^b \nabla_b t_c). \tag{1}$$

Under the transport (1) the x_a^i remain orthogonal to t^a and orthogonal to each other. On returning to p , we have a new tetrad whose timelike vector coincides with t^a , but whose spacelike vectors will in general be different from our original x_a^i . Let $\mathcal{R}_a^b(s)$ denote the corresponding rotation matrix:

$$\begin{aligned} x_a^i|_{w=1;s} &= \mathcal{R}_a^b(s)(x_b^i|_{w=0;s=0}), \\ \mathcal{R}_a^b(s)\mathcal{R}_c^d(s) &= \delta_a^d. \end{aligned} \tag{2}$$

Thus, for each value of s , we obtain a rotation at p , and so we define a curve $\mathcal{R}_a^b(s)$ in the rotation group. For $s = 0$ and $s = 1$, the rotation is just the identity, and so $\mathcal{R}_a^b(s)$ represents a closed curve, beginning and ending at the identity element of the rotation group. The tangent to this curve is obtained by taking the derivative of the first equation (2) with respect to s :

$$\mathcal{R}_{a\gamma} \frac{d}{ds} \mathcal{R}_\beta^\gamma = (x_a^\alpha \sigma^b \nabla_b x_{\alpha\beta})_{w=1;s} = \int_{\gamma_s} d w P_{\alpha\beta}. \tag{3}$$

¹⁷ Greek letters are triad indices (range 1, 2, 3) which label individual spacelike vectors, while Latin letters are tensor indices. The Greek indices are raised and lowered using the unit 3×3 matrix.

where

$$P_{\alpha\beta} = P_{[\alpha\beta]} = \omega^c \nabla_c (x_\alpha^a \sigma^b \nabla_b x_{\alpha\beta}). \quad (4)$$

Expanding (4) and using the transport equation (1) and the fact (which follows from our construction) that the Lie derivative of σ^a with respect to ω^a vanishes, we have

$$P_{\alpha\beta} = 2x_\alpha^a x_\beta^b \sigma^c \omega^{d1} [(\nabla_c t_b)(\nabla_d t_a) + R_{abcd}]. \quad (5)$$

The closed curves in the rotation group are divided into two classes: those (such as a small loop, or a rotation through angle 4π) which may be contracted to a point, and those (such as a rotation through 2π) which cannot. If our curve $\mathcal{R}_\alpha^\beta(s)$ is of the latter type, then there is an essential 2π twist in our tetrad system on S ; in this case, it will not be possible to find a tetrad in a neighborhood of S , and so M will not have spinor structure. We may characterize the curve $\mathcal{R}_\alpha^\beta(s)$ in terms of a length using the standard invariant metric on the rotation group:

$$L = 2^{-\frac{1}{2}} \int_0^1 ds \left[\left(\frac{d}{ds} \mathcal{R}_{\alpha\beta} \right) \left(\frac{d}{ds} \mathcal{R}^{\alpha\beta} \right) \right]^{\frac{1}{2}}. \quad (6)$$

Whenever the length L is less than 2π , the curve $\mathcal{R}_\alpha^\beta(s)$ may always be contracted to a point. (The length is exactly 2π for a 360° rotation about a single axis.) We conclude that there will necessarily be a tetrad system in a neighborhood of S provided $L < 2\pi$.

To obtain an upper bound for L which is independent of the coordinate grid (s, w) , we first substitute Eq. (3) into (6):

$$\begin{aligned} L &= 2^{-\frac{1}{2}} \int_0^1 ds \left[\left(\mathcal{R}_{\alpha\mu} \frac{d}{ds} \mathcal{R}^{\beta\mu} \right) \left(\mathcal{R}^{\alpha\nu} \frac{d}{ds} \mathcal{R}_{\beta\nu} \right) \right]^{\frac{1}{2}} \\ &= 2^{-\frac{1}{2}} \int_0^1 ds \left[\left(\int_0^1 dw P_{\alpha\beta} \right) \left(\int_0^1 dw P^{\alpha\beta} \right) \right]^{\frac{1}{2}} \\ &\leq 2^{-\frac{1}{2}} \int_0^1 ds \int_0^1 dw [P_{\alpha\beta} P^{\alpha\beta}]^{\frac{1}{2}}. \end{aligned} \quad (7)$$

But

$$\begin{aligned} P_{\alpha\beta} P^{\alpha\beta} &= 4(g^{\alpha\beta} - t^a t^b)(g^{b\alpha} - t^b t^a) \sigma^{[c} \omega^{d1} \sigma^{r} \omega^{s]} \\ &\quad \times [(\nabla_c t_b)(\nabla_d t_a) + R_{abcd}] [(\nabla_r t_q)(\nabla_s t_p) + R_{pqrs}] \\ &\leq 8\sigma^{[c} \omega^{d1} \sigma^{r} \omega^{s]} [(\nabla_c t_b)(\nabla_d t_a)(\nabla_r t^b)(\nabla_s t^a) \\ &\quad + R_{abcd} R^{ab}_{rs} - 2t^b t^p R_{abcd} R^a_{prs}]. \end{aligned} \quad (8)$$

Finally, substituting (8) into (7) and introducing the surface element of S , $dS^{ab} = \sigma^{[c} \omega^{b1} ds dw$, we obtain

$$\begin{aligned} L &\leq \int_S 2[(R_{abcd} R^{ab}_{rs} - 2t^b t^p R_{abcd} R^a_{prs}) dS^{cd} dS^{rs}]^{\frac{1}{2}} \\ &\quad + \int_S 2[(\nabla_c t_b)(\nabla_r t^b)(\nabla_d t_a)(\nabla_s t^a) dS^{cd} dS^{rs}]^{\frac{1}{2}}. \end{aligned} \quad (9)$$

Equation (9) still depends on the arbitrary unit time-like vector field t^a .¹⁸ We may eliminate this dependence, at least formally, by defining $\Psi(S)$ to be the minimum value of the right side of (9) over all possible choices of t^a . (Note that the right side is always nonnegative.) Unfortunately, the quantity Ψ depends on S in a nonlocal way because of the second integral on the right in (9): it appears, in fact, that Ψ cannot be expressed as a single integral over S . Thus, our relationship between spinor structure and curvature is the following: A sufficient (but not necessary) condition that M have spinor structure is that every S may be so deformed that $\Psi(S) < 2\pi$.

It follows immediately that every flat space-time has spinor structure. (By "flat" we mean only that the Riemann tensor vanishes: M could still have a quite complicated topology.) We may, without loss of generality, take M to be simply connected. Choosing t^a to be (covariantly) constant over M , we see from (9) that $\Psi(S) = 0$ for every S , and, therefore, that M has spinor structure. In particular, none of the 4-manifolds M_{2n+1} of Sec. I can be given a flat metric.

¹⁸ If the metric of space-time were positive-definite, then we could parallel transport the entire tetrad around each curve γ_s , and so obtain a formula which does not involve an arbitrary vector field. This procedure does not work, however, in the indefinite case because there is no invariant positive-definite length defined for curves in the Lorentz group.