THE ATIYAH-BOTT FIXED POINT FORMULA

Ezra Getzler

We would like to describe the influence that this theorem has had in the development of several techniques of differential geometry and topology: the development of zeta-function renormalization, the formulation of the equivariant Atiyah-Singer index theorem, and the localization theorem for equivariant differential forms. For some further bibliographic references, see for example the bibliography of [2].

Let us summarize Atiyah and Bott's first proof of their theorem [**40**], which differs a little from the proof in [**42**]. Let φ be a diffeomorphism of a compact manifold M, and let $T \in \operatorname{Hom}(\mathcal{E}, \varphi^*\mathcal{E})$ be a bundle isomorphism of a bundle $\mathcal{E}$ over M which covers φ. Denote also by T the bounded operator on $L^2(M, \mathcal{E})$ induced by $T \in \operatorname{Hom}(\mathcal{E}, \varphi^*\mathcal{E})$. If Δ is a positive elliptic differential operator on $\mathcal{E}$, then the trace

$$\zeta_T(s) = \operatorname{Tr}\bigl(T \cdot \Delta^{-s}\bigr) \tag{1}$$

has a meromorphic extension to the whole complex plane, and is holomorphic at $s = 0$. Atiyah and Bott observed that $\operatorname{Tr}_\zeta(T) = \zeta_T(0)$ is a renormalized trace of T.

Now, take an elliptic complex

$$0 \longrightarrow \mathcal{E}_0 \xrightarrow{D_0} \dots \xrightarrow{D_{p-1}} \mathcal{E}_p \longrightarrow 0, \tag{2}$$

with automorphisms T_i of $\mathcal{E}_i$ covering a diffeomorphism φ of M, and such that $T_{i+1} \cdot D_i = D_i \cdot T_i$. Denote by H_i the cohomology space $\ker(D_i)/\operatorname{im}(D_{i-1})$, and by τ_i the endomorphism induced by the bundle map T_i on H_i. Choose any Hermitian metrics on the bundles $\mathcal{E}_i$, and form the positive second-order elliptic operators $\Delta_i = D_{i+1}^* D_i + D_{i-1} D_i^*$. We have the basic formula

$$\sum_{i=0}^{p} (-1)^i \operatorname{Tr}(\tau_i) = \sum_{i=0}^{p} (-1)^i \operatorname{Tr}\bigl(T_i \cdot \Delta_i^{-s}\bigr). \tag{3}$$

Analytically continuing to $s \longrightarrow 0$, we see that

$$\sum_{i=0}^{p} (-1)^i \operatorname{Tr}(\tau_i) = \sum_{i=0}^{p} (-1)^i \operatorname{Tr}_\zeta(T_i). \tag{4}$$

Since φ has non-degenerate fixed points, the right-hand side is actually quite easy to calculate, and one obtains the Atiyah-Bott fixed point formula.

The renormalized trace may just as well be defined using the heat operator: it is just the constant coefficient a_0 in the asymptotic expansion

$$\mathrm{Tr}\big(T \cdot e^{-t\Delta}\big) \sim \sum_{k=0}^{\infty} a_k \, t^{k-\dim(M)/2}. \tag{5}$$

This leads to a formula for the index of a Dirac operator which has been exploited by McKean and Singer [5] and Patodi [6], among others, culminating in Bismut's formula for the index of a family of Dirac operators [4].

The fixed point formula has inspired a number of other theorems. In the first of these, one replaces the group of automorphisms $\mathbb{Z}$ by a compact Lie group G acting on the manifold M and on two bundles $\mathcal{E}$ and $\mathcal{F}$ over M, and the elliptic complex by an invariant elliptic operator $\mathcal{D} : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{F})$. (It would be quite interesting to permit non-compact Lie groups here, but it is not known how this might be done: see also our final remarks.)

The kernel $\ker(\mathcal{D})$ and cokernel $\mathrm{coker}(\mathcal{D})$ of the operator $\mathcal{D}$ are finite-dimensional representations of G, whose difference $\mathrm{ind}(\mathcal{D}) = \ker(\mathcal{D}) - \mathrm{coker}(\mathcal{D})$ is an element of the virtual representation ring $R(G)$. This element was calculated by Atiyah and Segal [1], using Segal's theory of equivariant K-theory and the Atiyah-Singer index theorem. However, in the special case where $\mathcal{D}$ is a generalized Dirac operator, there is a proof due to Bismut [3], extending earlier work of Gilkey and Patodi, which starts with the analogue of (3): if $\mathrm{ind}_g(\mathcal{D})$ is the character of $\mathrm{ind}(\mathcal{D}) \in R(G)$ at $g \in G$, then

$$\mathrm{ind}_g(\mathcal{D}) = \mathrm{Tr}\big(ge^{-t\mathcal{D}^*\mathcal{D}}\big) - \mathrm{Tr}\big(ge^{-t\mathcal{D}\mathcal{D}^*}\big) \quad \text{for } g \in G. \tag{6}$$

Another direction in which the influence of the fixed point formula was immediately felt is Bott's discovery of the localization principal for equivariant differential forms [**43**]. Let M be a compact manifold, and let X be a vector field whose flow is periodic, and such that the zeroes of X are discrete. Stated in its simplest form, Bott proved the following result.

Theorem. *If ω is a differential form on M such that $d\omega = \iota(X)\omega$, then*

$$\int_M \omega = \sum_{X(x)=0} \frac{\omega(x)}{\det^{1/2}(L_x(X))},$$

where $L_x(X) \in \mathrm{End}(T_xM)$ is the endomorphism of T_xM induced by the Lie derivative with respect to X at the zero x.

This theorem is proved by a remarkable elementary calculation, resembling a higher dimensional version of the proof of Cauchy's theorem. However, in the intoduction to [**43**], Bott states that he originally came upon the result as an application of the fixed point formula, and only later found the simple proof.

Like Cauchy's theorem, the localization theorem has many important applications. Bott's original application is to prove the following result on characteristic numbers. If M has dimension n, let Φ be an invariant polynomial on the Lie algebra $\mathfrak{so}(n)$: thus, Φ is a polynomial in the Pontryagin classes and the Euler class. Let $\Phi(M) = \langle \Phi(TM), [M] \rangle$ be the corresponding characteristic number.

Theorem. *If Φ is a homogeneous invariant polynomial of degree $k \leq \ell$, then*

$$\sum_{X(x)=0} \frac{\Phi(L_x(X))}{\det^{1/2}(L_x(X))} = \begin{cases} \Phi(M), & k = \ell, \\ 0, & k < \ell. \end{cases}$$

The localization theorem has been greatly generalized. In particular, it has been seen to fit together beautifully with the theory of equivariant differential forms initiated by H. Cartan (see Chapter 7 of [2].) Also, Baum and Bott proved an analogue of the result which holds in the algebraic context [**49**]. For another recent application, see the work of Bott and Taubes on the elliptic genus [**91**]. However, many mysteries remain, the most reknowned being to understand whether Harish-Chandra's formula for the character of a discrete series representation, which seems to be a formal consequence of the the Atiyah-Bott fixed point formula might follow from a suitable generalization.

References

1. M. F. Atiyah and G. B. Segal, *The index of elliptic operators II*, Ann. Math. **87** (1968), 531–545.
2. N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin-Heidelber-New York, 1992.
3. J.-M. Bismut, *The Atiyah-Singer theorems: a probabilistic approach. II*, J. Funct. Anal. **57** (1984), 329–348.
4. J.-M. Bismut, *The index theorem for families of Dirac operators: two heat equation proofs*, Invent. Math. **83** (1986), 91–151.
5. H. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geom. **1** (1967), 43–69.
6. V. K. Patodi, *An analytic proof of the Riemann-Roch-Hirzebruch theorem*, J. Diff. Geom. **5** (1971), 251–283.