A Model Category Structure for Differential Graded Coalgebras

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June 7, 1999

1 The structure of coalgebras

In this section we lay the groundwork for the calculations of the next. The first main result is this: every coalgebra is the filtered colimit of its finite dimensional sub-coalgebras. The proof we give is essentially that of Sweedler [7], modified for the case of differential graded coalgebras. We then use this to give a formula for the cofree coalgebra on a differential graded vector space; this will be used in the proof of Theorem 2.1. The calculation of that result is the only part of the proof of the existence of the model category structure which is not essentially formal.

Let $C_*\mathcal{M}_k$ denote the category of non-negatively graded chain complexes over a field k. The boundary map has degree -1. This category has a symmetric monoidal structure with

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential

$$\partial(x \otimes y) = \partial(x) \otimes y + (-1)^p x \otimes \partial(y)$$

with $x \otimes y \in C_p \otimes D_q$. The switch map $T : C \otimes D \to D \otimes C$ must also have a sign:

$$T(x \otimes y) = (-1)^{pq}(y \otimes x).$$

It is this tensor product we use for coalgebras, comodules, and so on.

Let $\mathcal{CA}$ denote the category of coassociative and counital coalgebras in $C_*\mathcal{M}_k$. We do not assume cocommutavity, although, in section 2, we will discuss a restricted category of coalgebras where we make some assumptions about degree zero. We also do not assume that the unit map $\epsilon : C \to k$ is an isomorphism in degree zero; at this point C_0 is simply a coassociative coalgebra. Given a fixed coalgebra C, there is a notion of right comodule over C; we will write

$$\psi_M = \psi : M \to M \otimes C$$

for the comodule structure map.

There are also the auxiliary categories of graded vector spaces, coalgebras in graded vector spaces, and comodules in graded vector spaces. There are forgetful functors from the above categories which neglect the boundary map.

The following observation is standard.

*The authors were partially supported by the National Science Foundation.

Lemma 1.1 *Let C be a coalgebra in graded vector spaces and let M be a right comodule in graded vector spaces over C. If $x \in M$ is a homogeneous element, then there is a finite dimensional sub-comodule $N \subseteq M$ so that $x \in N$.*

Proof. Compare [7], §2.2. Let $\{c_i\}$ be a homogeneous basis for C. Then we may write

$$\psi(x) = \sum_i x_i \otimes c_i.$$

Note that $x_i = 0$ except for finitely many i. Let N be the span of the x_i in M. Since $x = (1 \otimes \epsilon)\psi(x)$, one has

$$x = \sum_i x_i \epsilon(c_i) \in N.$$

To show N is a sub-comodule one computes

$$\begin{aligned}\sum_i \psi(x_i) \otimes c_i &= (\psi \otimes 1)\psi(x) \\ &= (1 \otimes \Delta_C)\psi(x) \\ &= \sum_j x_j \otimes \Delta(c_j) \\ &= \sum_{j,i} x_j \otimes c_{ij} \otimes c_i\end{aligned}$$

for some $c_{ij} \in C$. Then

$$\psi(x_i) = \sum_j x_j \otimes c_{ij} \in N \otimes C.$$

This completes the proof.

If C is any coalgebra in graded vector spaces, then $C^* = \{\mathrm{Hom}_k(C_n, k)\}$ is a non-positively graded algebra, and if M is a right comodule over C in graded vector spaces, then M is also a left C^* module; indeed, if $\phi \in C^*$ and $x \in M$, then

$$\phi \cdot x = \sum x_i \langle \phi, c_i \rangle$$

where $\psi(x) = \sum x_i \otimes c_i$. Not every left C^* module arises this way, because of Lemma 1.1.

Lemma 1.2 *Let C be a coalgebra in graded vector spaces and $x \in C$ a homogeneous element. Then there is a finite dimensional sub-coalgebra $D \subseteq C$ so that $x \in D$. Furthermore, it can be assumed that $D_n = 0$ for n larger than the degree of x.*

Proof. Again, compare [7], §2.2. The coalgebra C has an obvious structure as a right comodule over itself. Thus Lemma 1.1 supplies a finite-dimensional sub-comodule $N \subseteq C$ with $x \in N$. If C is cocommutative, then N is in fact already a sub-coalgebra (see [2]); however, in the more general case, one must proceed as follows.

Let $J \subseteq C^*$ be the annihilator ideal of N; note that J is the kernel of the map of rings

$$C^* \longrightarrow \mathrm{End}_k(N)$$

from C^* to the endomorphism of N determined by the left module structure on N. Since N is finite dimensional, C^*/J is a finite dimensional vector space. Let

$$D = J^\perp = \{y \in C \ : \langle \phi, y\rangle = 0 \text{ for all } \phi \in J\}.$$

Note that $x \in N \subseteq D$. We now show that D is a finite dimensional sub-coalgebra of C.

Since C^*/J is finite dimensional, $[C^*/J]^*$ is a finite dimensional coalgebra. Since $D = C \cap [C^*/J]^* \subseteq C^{**}$, we have that D is finite dimensional; therefore, this result follows from Lemma 1.3 below.

If we want $D_n = 0$ for n greater than the degree of x, simply take the sub-vector space of the D constructed above generated by the homogeneous elements of degree less than or equal to that of x. This is also a sub-coalgebra.

Lemma 1.3 *Let C be a coalgebra in graded vector spaces and $I \subseteq C^*$ a two-sided ideal. Then $I^\perp \subseteq C$ is a sub-coalgebra.*

Proof. The argument is identical to that of Proposition 1.4.3 of [7].

Lemma 1.3 has the following immediate consequences, which will be useful later. If $V \subseteq C$ is a sub-vector space, then we also have an orthogonal complement $V^\perp \subseteq C^*$. Note that $(V^\perp)^\perp = V$. If $D \subseteq C$ is a sub-coalgebra, then $D^\perp \subseteq C^*$ is a two-sided ideal.

Lemma 1.4 *Let C be a coalgebra either in graded vector spaces or in chain complexes, and let $\{C_i\}$ be a set of sub-coalgebras of C, in the appropriate category. Then*

1. *$\cap_i C_i$ is a sub-coalgebra of C in the appropriate category; and*
2. *$\sum_i C_i$ is a sub-coalgebra of C in the appropriate category.*

Proof. If C is a coalgebra in chain complexes, then $\cap C_i$ and $\sum C_i$ are sub-chain complexes; thus, it is sufficient to argue the graded case. However,

$$\cap C_i = \cap (C_i^\perp)^\perp = (\sum C_i^\perp)^\perp.$$

and

$$\sum C_i = (\cap C_i^\perp)^\perp.$$

This brings us to the following result, crucial to all that follows.

Proposition 1.5 *Let $C \in \mathcal{CA}$ be a coalgebra in chain complexes and let $x \in C$ be a homogeneous element. Then there is a finite dimensional differential graded coalgebra $D \subseteq C$ so that $x \in C$.*

Proof. Suppose the degree of x is n. We define an ascending sequence of sub-coalgebras

$$D(n) \subseteq D(n-1) \subseteq \cdots \subseteq D(0) \subseteq C$$

with the properties that

1. $x \in D(n)$;
2. each $D(k)$ is finite-dimensional;
3. $D(n)_m = 0$ for $m > n$ and $D(k-1)_m = D(k)_m$ for $m \geq k$; and
4. $\partial(D(k)_k) \subseteq D(k-1)_{k-1}$.

Then $D = D(0)$ is the desired coalgebra. Note that $D(n)$ is supplied by Lemma 1.2. If $D(k)$ has been constructed, choose a basis for $\{y_i\}$ for $D(k)_k$ and use Lemma 1.2 to produce finite dimensional sub-coalgebras $D(y_i) \subseteq C$ so that $\partial y_i \in D(y_i)$ and $D(y_i)_m = 0$ for $m \geq k$. Then set

$$D(k-1) = D(k) + \sum_i D(y_i).$$

This is a sub-coalgebra by the previous result, and the proposition follows.

This last result immediately implies the next. Note that the forgetful functor from $\mathcal{CA}$ to differential graded vector spaces makes all colimits.

Corollary 1.6 *Every differential graded coalgebra over a field k is the (right) filtered colimit of its finite dimensional sub-coalgebras.*

Now let **Algf** be the category of profinite non-positively graded differential algebras over k, with unit. The objects of **Algf** can either be regarded as filtered diagrams of non-positively graded finite dimensional differential k-algebras or as complete topological differential k algebras with a neighborhood base of 0 consisting of two-sided ideals of finite codimension and closed under the differential. Then morphisms are either pro-morphisms of diagrams of finite dimensional dgas or continuous dga morphisms. The following is now a formal consequence of Corollary 1.6 and the fact that the dual of a finite dimensional algebra is a coalgebra. This implies that the continuous dual of a profinite algebra is a coalgebra. Compare [2].

Proposition 1.7 *Linear duality defines an anti-equivalence between the category $\mathcal{CA}$ of differential graded coalgebras and the category* **Algf** *of profinite differential graded algebras.*

The functor back takes the continuous dual of a profinite algebra.

An implication of the previous result is the following.

Proposition 1.8 *The category $\mathcal{CA}$ has all small limits and colimits.*

Proof. Indeed, as mentioned above, the forgetful functor from $\mathcal{CA}$ to differential graded vector spaces makes all colimits. To prove the existence of limits, it is sufficient, by the previous result, to show that the category **Algf** has all colimits.

Let $A : I \to \mathbf{Algf}$ be diagram of profinite dg-algebras and let B be the colimit of this diagram in the category of dg-algebras. This is not yet a profinite algebra. Define a neighborhood base of 0 in B to be the set of two-sided ideals J which can be realized as the kernel of map of algebras $B \to C$ such that C is a finite dimensional dg-algebra and so that for each $i \in I$ the composite $A_i \to B \to C$ is continuous. Then the colimit is the completion of B with respect to this topology.

In light of Corollary 1.6, the following technical result will have many applications:

Lemma 1.9 *Let $\{C_\alpha\}$ be a right filtered diagram of coalgebras in $\mathcal{CA}$ and let $D \in \mathcal{CA}$ be finite dimensional. Then the natural map*

$$\operatorname{colim}_\alpha \operatorname{Hom}_{\mathcal{CA}}(D, C_\alpha) \to \operatorname{Hom}_{\mathcal{CA}}(D, \operatorname{colim}_\alpha C_\alpha)$$

is an isomorphism.

Proof. The forgetful functor to differential graded vector spaces makes colimits, so this map is an injection. To prove it is a surjection, note that any coalgebra map $f : D \to \operatorname{colim}_\alpha C_\alpha$ factors as a morphism of $D \to C_\alpha$ of differential graded vector spaces, since D is finite dimensional. Since f is a coalgebra map, there must be a morphism $C_\alpha \to C_\beta$ so that composite $D \to C_\alpha \to C_\beta$ is a coalgebra map, again since D is finite dimensional. This completes the proof.

For the next result we need the following observation. The forgetful functor from **Algf** to the category of differential graded algebras has a left adjoint given by profinite completion: to a differential graded algebra A one assigns the diagram $\hat{A} = \{A/I_\alpha\}$ where I_α runs over all two-sided ideals of A of finite codimension and closed under the differential.

We can now prove:

Proposition 1.10 *The forgetful functor from $\mathcal{CA}$ to differential graded vector spaces has a right adjoint S.*

Proof. Let V be a differential graded vector space. If V is finite dimensional, let $S(V)$ be the continuous dual of the profinite completion of the tensor algebra on V^*. Then

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{CA}}(C, S(V)) &\cong \operatorname{Hom}_{\mathbf{Algf}}(\widehat{Tens(V^*)}, C^*) \\
&\cong \operatorname{Hom}_{alg}(Tens(V^*), C^*) \\
&\cong \operatorname{Hom}_{\mathcal{M}_k^*}(V^*, C^*) \\
&\cong \operatorname{Hom}_{\mathcal{M}_k}(C, V)
\end{aligned}$$

where $\mathcal{M}_k^*$ is the category of profinite k-vector spaces. Thus $S(V)$ has the desired property. For general V, define

$$S(V) = \operatorname{colim}_\alpha S(V_\alpha) \tag{1.1}$$

where V_α runs over the finite dimensional sub-vector spaces of V. Then if $C = \mathrm{colim}_\beta C_\beta$ is a coalgebra in $\mathcal{CA}$ written as the colimit of its finite dimensional sub-dg-coalgebras, one has, by Lemma 1.9,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{CA}}(C, S(V)) &\cong \lim_\beta \mathrm{Hom}_{\mathcal{CA}}(C_\beta, S(V)) \\
&\cong \lim_\beta \mathrm{colim}_\alpha \mathrm{Hom}_{\mathcal{CA}}(C_\beta, S(V_\alpha)) \\
&\cong \lim_\beta \mathrm{colim}_\alpha \mathrm{Hom}_{\mathcal{M}_k}(C_\beta, V_\alpha) \\
&\cong \mathrm{Hom}_{\mathcal{M}_k}(C, V).
\end{aligned}$$

This completes the proof.

The preceding argument gives only moderate insight into the structure of $S(V)$ – we did obtain the formula Equation 1.1 and the fact that if V is finite dimensional, then the dual of $S(V)$ is the profinite completion of the tensor algebra on V^*. We spend the rest of this section delving more deeply into the structure of $S(V)$. For model category theoretic reasons, we're actually interested in $C \times S(V)$ for an arbitrary coalgebra C. Here is a preliminary reduction.

Lemma 1.11 *Suppose A and B are differential graded coalgebras and we have presentations*

$$\mathrm{colim}_\alpha A_\alpha \cong A \qquad \text{and} \qquad \mathrm{colim}_\beta B_\beta \cong B$$

of A and B as filtered colimits of sub-coalgebras. Then the natural map

$$\mathrm{colim}_{\alpha,\beta}(A_\alpha \times B_\beta) \to A \times B$$

is an isomorphism.

Proof. If $D = \mathrm{colim}\, D_\gamma$ is written as the colimit of its finite dimensional sub-differential graded coalgebras, then by Lemma 1.9,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{CA}}(D, \mathrm{colim}_{\alpha,\beta} A_\alpha \times B_\beta) &\cong \lim_\gamma \mathrm{colim}_{\alpha,\beta} \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, A_\alpha \times B_\beta) \\
&\cong \lim_\gamma \mathrm{colim}_{\alpha,\beta} [\mathrm{Hom}_{\mathcal{CA}}(D_\gamma, A_\alpha) \times \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, B_\beta)].
\end{aligned}$$

Now, because we are working with sub-coalgebras, the maps

$$\mathrm{Hom}_{\mathcal{CA}}(D_\gamma, A_\alpha) \to \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, A_{\alpha'})$$

and

$$\mathrm{Hom}_{\mathcal{CA}}(D_\gamma, B_\beta) \to \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, B_{\beta'})$$

are inclusions, so this last colimit is a union of sets and, hence, one can interchange the product and colimit and again use Lemma 1.9.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{CA}}(D, \mathrm{colim}_{\alpha,\beta} A_\alpha \times B_\beta) &\cong \lim_{\gamma}[\mathrm{colim}_\alpha \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, A_\alpha) \times \mathrm{colim}_\beta \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, B_\beta)] \\ &\cong \lim_{\gamma}[\mathrm{Hom}_{\mathcal{CA}}(D_\gamma, \mathrm{colim}_\alpha A_\alpha) \times \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, \mathrm{colim}_\beta B_\beta))] \\ &\cong \lim_{\gamma} \mathrm{Hom}_{\mathcal{CA}}(D_\gamma, A \times B) \\ &\cong \mathrm{Hom}_{\mathcal{CA}}(D, A \times B).\end{aligned}$$

In particular, this last result implies that if C is any coalgebra in $\mathcal{CA}$ and V is a differential graded vector space, then there is an isomorphism

$$\mathrm{colim}_{\alpha,\beta}(C_\alpha \times S(V_\beta)) \cong C \times S(V) \tag{1.2}$$

where C_α runs over the finite dimensional sub-dg-coalgebras of C and V_β runs over the finite dimensional sub-dg-vector spaces of V. Here we use Equation 1.1. We reduce further.

Let $S(n)$ denote the differential graded vector space which is of dimension 1 over k concentrated in degree n. Let $D(n)$ be the differential graded vector space with $D(n)_p = 0$ unless $p = n$ or $n+1$, $D(n)_n = D_{n+1} = k$ and the boundary map is the identity. Every finite dimensional differential vector space can be written (non-canonically) as a finite product of differential vector spaces of the form $S(n)$ or $D(n)$. Since the functor S is a right adjoint, it preserves products; thus to understand $C \times S(V)$ we are reduced, in some sense, to understanding $C \times S(S(n))$ and $C \times S(D(n))$. This is the purpose of the next result.

If V is a differential graded vector space concentrated in non-negative degrees, then $Tens(V^*)$ has a natural structure of a differential graded algebra in non-positive degrees. However, more is true. If A is a differential graded algebra concentrated in non-positive degrees and M is a differential graded A bi-module, also in non-positive degrees, then we can form the tensor algebra

$$Tens_A(M) = \bigoplus_{n\geq 0} M \otimes_A M \otimes_A \cdots \otimes_A M$$

where, in the nth summand, M appears n times. (If $n = 0$, we have only A.) This has an obvious structure as a differential graded A-algebra. In particular, if W is a differential graded vector space in non-positive degrees, we can form

$$Tens_A(A \otimes W \otimes A) = \bigoplus_{n\geq 0} A \otimes W \otimes A \cdots A \otimes M \otimes A$$

where, in the nth summand, W appears n times. The functor

$$W \longmapsto Tens_A(A \otimes W \otimes A)$$

is left adjoint to the forgetful functor from A algebras of dg-vector spaces; hence, there is a natural isomorphism

$$Tens_A(A \otimes W \otimes A) \cong A \sqcup Tens(W)$$

all in the category of dg-algebras.

If A is an algebra, let $(X) \subseteq A$ denote the two-sided ideal of A generated by a subset X of A.

Lemma 1.12 *Let C be a finite dimensional coalgebra and V a differential graded vector space which is in non-negative degrees and finite dimensional in each degree.*

1. *If $V_0 = 0$, then $[C \times S(V)]^* \cong Tens_{C^*}(C^* \otimes V^* \otimes C^*)$.*
2. *If $V_0 = k$ with generator x, then*

$$[C \times S(V)]^* \cong \lim_{q(x)} Tens_{C^*}(C^* \otimes V^* \otimes C^*)/(q(x), \partial q(x)).$$

where $q(x) \in k[x]$ runs over all monic polynomials and there is a projection

$$Tens_{C^*}(C^* \otimes V^* \otimes C^*)/(q(x), \partial q(x)) \to Tens_{C^*}(C^* \otimes V^* \otimes C^*)/(p(x), \partial p(x))$$

if and only if $p(x)$ divides $q(x)$.

Proof. In either case, $[C \times S(V)]^*$ is the profinite completion of the dg-algebra

$$C^* \sqcup Tens(V^*) \cong Tens_{C^*}(C^* \otimes V^* \otimes C^*).$$

In the first case, let

$$I_n \subseteq Tens_{C^*}(C^* \otimes V^* \otimes C^*)$$

be the ideal of elements of degrees less than $-n$. Then the system

$$Tens_{C^*}(C^* \otimes V^* \otimes C^*)/I_n$$

is cofinal among quotients of $Tens_{C^*}(C^* \otimes V^* \otimes C^*)$ of finite dimension; therefore, this algebra is its own profinite completion.

In the second case, the system of algebras

$$Tens_{C^*}(C^* \otimes V^* \otimes C^*)/(I_n, q(x), \partial q(x))$$

is again cofinal among quotients of of finite dimension; hence

$$S(V)^* = \lim Tens_{C^*}(C^* \otimes V^* \otimes C^*)/(I_n, q(x), \partial q(x)) = \lim_{q(x)} Tens_{C^*}(C^* \otimes V^* \otimes C^*)/(q(x), \partial q(x)).$$

Example 1.13 One can use Lemma 1.12 to calculate $S(S(0)) = S(k)$ where k is in degree zero, at least in the case where k is perfect. Let $\bar{k}$ be the algebraic closure of k and G the Galois group of $\bar{k}$ over k. Then write $[a]$ for the orbit of $a \in \bar{k}$ and $k(a)$ for k adjoin a. Then, as profinite algebras,

$$S(k)^* \cong \prod_{[a]} k(a)[[z]]$$

where $[a]$ runs over the distinct orbits of the action of G on $\bar{k}$.

2 The model category structure

The purpose of this section is to prove that the category $\mathcal{CA}$ of non-negatively graded differential coalgebras over a field k has the structure of a closed model category where the weak equivalences are quasi-isomorphisms and the cofibrations are simply inclusions. This is, in fact, a cofibrantly generated model category, and we end the section by specifying the generating cofibrations.

We begin with the key algebraic result. Let S be the cofree coalgebra functor of section 1, and let $D(n)$ be the non-negatively graded chain complex with $D(n)_p = 0$ unless $p = n$ or $n+1$, $D(n)_n = D_{n+1} = k$ and the boundary map is the identity. This is the same object that was labelled $D(n)$ in section 1.

Theorem 2.1 *For all coalgebras C and all non-negative integers n, the natural projection*

$$C \times S(D(n)) \to C$$

is a quasi-isomorphism.

Proof. Let $\{C_\alpha\}$ be the filtered system of finite dimensional sub-coalgebras of C. Then Lemma 1.11 supplies an isomorphism

$$\mathrm{colim}_\alpha [C_\alpha \times S(D(n))] \cong C \times S(D(n)).$$

Since the colimit is filtered and made in differential graded vector spaces, it is sufficient to discuss the case where C is finite dimensional. If $n > 0$, Lemma 1.12.1 supplies the answer. Thus we are reduced to the case $n = 0$. Then Lemma 1.12.2 writes $C \times S(D(0))$ as a filtered colimit:

$$C \times S(D(0)) \cong \mathrm{colim}_{q(x)} [Tens_{C^*}(C^* \otimes D(0)^* \otimes C^*)/(q(x), \partial q(x))]^*.$$

The algebras in this diagram for which $q(x)$ is divisible by x form a cofinal sub-diagram, so can be used to compute this colimit. We will show that each of these algebras is quasi-isomorphic to C^*.

To simplify notation, let $A = C^*$ and $M = A \otimes D(0)^* \otimes A$. Let $x \in D(0)^*_0$ be a generator and $y = \partial x$ be the generator in degree 1. To begin, $D(0)^*$ has a chain contraction h given by $h(y) = x$ and $h(x) = 0$. This gives a chain contraction for M by the formula

$$h(a \otimes z \otimes b) = (-1)^{|a|}(a \otimes h(z) \otimes b).$$

This extends to a chain homotopy $H : Tens_A(M) \to Tens_A(M)$ from the identity to the composite

$$Tens_A(M) \xrightarrow{\epsilon} Tens_A(0) = A \xrightarrow{\eta} Tens_A(M)$$

where η is the unit map and ϵ is the augmentation induced by the zero map $M \to 0$. To define H, write

$$Tens_A(M) = \bigoplus_{n \geq 0} M \otimes_A \cdots \otimes_A M.$$

Then H restricted to A (the summand $n = 0$) is the zero map, and on the nth summand with $n > 0$,

$$H(x_1 \otimes \cdots x_n) = h(x_1) \otimes x_2 \otimes \cdots \otimes x_n.$$

Let I be the augmentation ideal in $Tens_A(M)$; that is, I is the kernel of ϵ Note that we have the formula, for $a, b \in Tens_A(M)$:

$$H(ab) = \begin{cases} H(a)b, & a \in I, \\ aH(b), & a \in A. \end{cases}$$

Since every element $a \in Tens_A(M)$ can be written uniquely as

$$a = (a - \eta\epsilon(a)) + \eta\epsilon(a),$$

with $a - \eta\epsilon(a) \in I$ and $\eta\epsilon(a) \in A$, we can can combine this observation into a single formula:

$$H(ab) = H(a - \eta\epsilon(a))b + \eta\epsilon(a)H(b) \tag{2.1}$$

Now suppose $J \subseteq I \subseteq Tens_A(M)$ is two-sided ideal contained in the augmentation ideal so that H restricts to a chain homotopy on J. Then the induced map on $Tens_A(M)/J$ is a chain homotopy between the identity and

$$Tens_A(M)/J \xrightarrow{\epsilon} Tens_A(0) = A \xrightarrow{\eta} Tens_A(M)/J.$$

So let $q(x)$ be a polynomial in x divisible by x. Then $q(x)$ is in the augmentation ideal, so the two-sided ideal $J = (q(x), \partial q(x))$ is in augmentation ideal. Thus we need only show that H restricts to J.

Every element of J can be written as a sum

$$aq(x)b + c\partial q(x)d.$$

Applying H to this sum and using Equation 2.1, we see that H restricts to J if and only if $H(q(x))$ and $H(\partial q(x))$ are in J. However, $q(x) = xq_0(x)$, since $q(x)$ is divisible by x. Because x is in the augmentation ideal, Equation 2.1 implies

$$H(q(x)) = H(x)q_0(x) = h(x)q_0(x) = 0.$$

Similarly, y is in the augmentation ideal, so

$$H(\partial q(x)) = H(yq_0(x) + x\partial q_0(x)) = h(y)q_0(x) + h(x)\partial q_0(x) = xq_0(x) = q(x).$$

This completes the argument.

Theorem 2.1 has the following corollary:

Proposition 2.2 *Let V be a non-negatively graded differential vector space with $H_*V = 0$. Then for all $C \in \mathcal{CA}$, the projection*

$$C \times S(V) \to C$$

is a quasi-isomorphism.

Proof. The dg-vector space V can be written as a sum of objects isomorphic to some $D(n)$. Thus V can be written as a filtered colimit of $V \cong \mathrm{colim}_\alpha V_\alpha$ so that $H_*V_\alpha = 0$. Then Equation 1.1 and Lemma 1.11 imply that

$$\mathrm{colim}_\alpha[C \times S(V_\alpha)] \cong C \times S(V).$$

Since the colimit is filtered and made in vector spaces we are reduced to the case where V is finite dimensional. In that case, write down an isomorphism

$$V \cong D(n_1) \times D(n_2) \times \cdots D(n_k).$$

Since S is a right adjoint we have

$$C \times S(V) \cong C \times S(D(n_1)) \times \cdots \times S(D(n_k))$$

and the result follows by induction from Theorem 2.1.

We now formally specify the classes of maps of our model category structure:

Definition 2.3 A morphism $f : C \to D$ of differential graded coalgebras is

1. a weak equivalence if it is a quasi-isomorphism; that is, if $H_*f : H_*C \cong H_*D$;
2. a cofibration if it is a degree-wise injection of graded vector spaces;
3. a fibration if it has the right lifting property with respect to acyclic cofibrations.

As is customary, we use the shorthand acyclic cofibration for a morphism which is at once a weak equivalence and a cofibration. There is also a notion of acyclic fibration. A morphism $X \to Y$ has the right lifting property with respect to a morphism $A \to B$ if every lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

has a solution so that both triangles commute.

The following result is why went to such difficulty with Theorem 2.1.

Lemma 2.4 *1.) Let V be a non-negatively graded differential vector space so that $H_*V = 0$. Then for all coalgebras $C \in \mathcal{CA}$ the projection*

$$C \times S(V) \to C$$

is an acyclic fibration with the right lifting property with respect to all cofibrations.

2.) Any morphism $f : D \to C$ is $\mathcal{CA}$ can be factored

$$D \xrightarrow{j} X \xrightarrow{q} C$$

where j is a cofibration and q is an acyclic fibration with the right lifting property with respect to all fibrations.

Proof. We begin with part 1. By Proposition 2.2, the projection is a weak equivalence; therefore, we need only show it is a fibration. This will follow if we show that the map has the right lifting property with respect to all cofibrations. Consider a lifting problem in $\mathcal{CA}$

$$\begin{array}{ccc} A & \longrightarrow & C \times S(V) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ B & \longrightarrow & C \end{array}$$

where the morphism i is a cofibration. By an adjointness argument, this is equivalent to a lifting problem in differential graded vector spaces

$$\begin{array}{ccc} A & \longrightarrow & V \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ B & \longrightarrow & 0. \end{array}$$

Since i is a cofibration and $V \to 0$ is an acyclic fibration in the standard model category structure on differential graded vector spaces (see [5]), a solution to the lifting problem exists.

For part 2, regard D as a differential graded vector space and choose an inclusion of dg-vector spaces $i : D \to V$ with $H_* V = 0$. Define $X = C \times S(V)$, $q : X \to C$ to be the projection and $j : D \to X$ to be

$$j = (f, i^*) : D \to C \times S(V)$$

where i^* is adjoint to $i : D \to V$. By part 1, q is an acyclic fibration of the required sort; thus we need only show that j is an inclusion. Consider the composition

$$D \xrightarrow{j} C \times S(V) \xrightarrow{\pi_2} S(V) \xrightarrow{\epsilon} V$$

where π_2 is projection onto the second factor and ϵ is the counit of the adjunction. This composition is i, hence an injection; therefore, j is an injection, as required.

There is another factorization required by the model category structure – the "acylic cofibration–fibration" factorization. This is more formal, using the now-standard technique pioneered by Bousfield [1]. The crucial input is supplied by Lemmas 2.5 and 2.6 to follow; the factorization is done by the small object argument in Lemma 2.7.

Lemma 2.5 *Let $j : C \to D$ be an acylic cofibration and $x \in D$ a homogeneous element. Then the is a sub-coalgebra $B \subseteq D$ so that*

1. *$x \in B$;*
2. *B has a countable homogeneous basis; and*
3. *$C \cap B \to B$ is an acyclic cofibration in $\mathcal{CA}$.*

Proof. We recursively define sub-dg-coalgebras

$$B(1) \subseteq B(2) \subseteq \cdots \subseteq D$$

so that $x \in B(1)$, each $B(n)$ is finite dimensional, and the induced map of dg-vector spaces

$$B(n-1)/[C \cap B(n-1)] \to B(n)/[C \cap B(n)]$$

is zero in homology. Then we may set B to be the union of the $B(n)$.

The sub-dg-coalgebra $B(1)$ is supplied by Proposition 1.5. Suppose that $B(n-1)$ has been constructed. Since $B(n-1)$ is finite dimensional we may choose a finite set

$$z_i + (C \cap B(n-1)) \in B(n-1)/[C \cap B(n-1)]$$

of homogeneous cycles so that the resulting homology classes span $H_*(B(n-1)/[C \cap B(n-1)])$. For each index i, use Proposition 1.5 to select a finite dimensional sub-coalgebra of $A(z_i) \subseteq D$ so that $z_i \in A(z_i)$. Then set

$$B(n) = B(n-1) + \sum_i A(z_i).$$

This is again finite dimensional and has the requisite properties.

Lemma 2.6 *A morphism $q : X \to Y$ in $\mathcal{CA}$ is a fibration if and only if q has the right lifting property with respect to all acyclic cofibrations $A \to B$ so that B has a countable homogeneous basis.*

Proof. The necessity of this lifting property is a consequence of Definition 2.3. For sufficiency, suppose we have a lifting problem

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ D & \longrightarrow & Y \end{array}$$

where j is an arbitrary acyclic cofibration. We solve this problem by a Zorn's Lemma argument.

Define Ω to be the set of pairs $(\bar{D}, g)$ where $\bar{D}$ fits into a sequence of acyclic cofibrations in $\mathcal{CA}$

$$C \xrightarrow{\subseteq} \bar{D} \xrightarrow{\subseteq} D$$

and $g : \bar{D} \to X$ is a solution to the restricted lifting argument. Order Ω by setting $(\bar{D}_1, g_1) \leq (\bar{D}_2, g_2)$ if $\bar{D}_1 \subseteq \bar{D}_2$ and g_2 restricts to g_1. Since $(C, f) \in \Omega$ the set is non-empty and any chain has an upper bound given by the union. Let $(E, g) \in \Omega$ be a maximal element. We show $E = D$. Note that $E \to D$ is an acyclic cofibration.

Let $x \in D$. By the previous lemma, there is a sub-dg-coalgebra $B \subseteq D$ so that $x \in B$, B has a countable basis, and $E \cap B \to B$ is an acyclic cofibration. Then the induced lifting problem

$$\begin{array}{ccccc} E \cap B & \xrightarrow{\subseteq} & E & \xrightarrow{g} & X \\ \downarrow & & & \nearrow & \downarrow{\scriptstyle q} \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

has a solution, by hypothesis. Hence g can be extended over $E + B$. Furthermore, the Meyer-Vietoris sequence

$$\cdots \to H_q(E \cap B) \to H_q(E) \oplus H_q(B) \to H_q(E + B) \to H_{q-1}(E \cap B) \to \cdots$$

shows that $E \to E + B$ is an acylic cofibration. By the maximality of (E, g) we must then have $E = E + B$ and $x \in E$, as required.

For the following result, the class of morphisms generated by a stipulated list of morphisms A is the smallest class containing A and closed under coproducts, cobase change, directed colimits, retracts, and isomorphisms.

Lemma 2.7 *Any morphism $C \to D$ in $\mathcal{CA}$ can be factored*

$$C \xrightarrow{i} X \xrightarrow{p} D$$

where i is an acyclic cofibration and p is a fibration. Furthermore i is in the class or morphisms generated by the acyclic cofibrations $A \to B$ so that B has a countable homogeneous basis.

Proof. Since colimits in $\mathcal{CA}$ are made in differential graded vector spaces, the class of acyclic cofibrations is closed under cobase change and directed colimits. In light of Lemma 2.6 the small object argument, over an ordinal whose cardinality larger than that of the first infinite cardinal, now applies. See [1], where such arguments first appeared, and [4], among many references.

We can now prove the main result.

Theorem 2.8 *With the definitions of weak equivalence, fibration, and cofibration given in Definition 2.3, the category $\mathcal{CA}$ becomes a model category.*

Proof. The category $\mathcal{CA}$ has all small limits and colimits, by Proposition 1.8. Weak equivalences satisfy the two-out-of-three axiom by inspection; similarly, the classes of weak equivalences, cofibrations, and fibration are closed under retracts. The factorization axiom follows from Lemmas 2.4.2 and 2.7. The "acyclic cofibration – fibration" half of the lifting axiom is the definition of fibration. Thus we need only show that any acyclic fibration has the right lifting property with respect to all cofibrations. Let $p : C \to D$ be an acyclic fibration. By Lemma 2.4.2 we may factor p as

$$C \xrightarrow{j} X \xrightarrow{q} D$$

where j is a cofibration and q is an acyclic fibration with the right lifting property with respect to all cofibrations. Note that j is also a weak equivalence. Thus there is a solution to the lifting problem

$$\begin{array}{ccc} C & \xrightarrow{=} & C \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ X & \xrightarrow{q} & D \end{array}$$

as p has the right lifting property with respect to all acyclic cofibrations. This solution shows that p is a retract of q and, since q has the right lifting property with respect to all cofibrations, so does p.

We are also going to prove that $\mathcal{CA}$ is a cofibrantly generated model category. See §2.1 of [4] for the definitions and implications. We have

Lemma 2.9 *1.) A morphism $p : C \to D$ in $\mathcal{CA}$ is an acylic fibration if and only if it has the right lifting property with respect to all cofibrations $A \to B$ with B finite dimensional.*

2.) The cofibrations $A \to B$ with B finite dimensional generate the class of cofibrations in $\mathcal{CA}$.

Proof. Part 1 is proved by the evident variation of the Zorn's Lemma argument given in Lemma 2.6. Proposition 1.5 substitutes from Lemma 2.5 in the argument.

To prove part 2, let $i : C \to D$ be any cofibration $\mathcal{CA}$. Use part 1 and the small object argument to factor i as

$$C \xrightarrow{j} X \xrightarrow{q} D$$

where j is generated by the cofibrations $A \to B$ with B finite dimensional and q is an acyclic fibration. Then the solution to the lifting problem

$$\begin{array}{ccc} C & \xrightarrow{j} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ D & \xrightarrow{=} & D \end{array}$$

shows i is a retract of j and, hence, in the class of morphisms generated by inclusions of finite dimensional coalgebras.

We can now state:

Proposition 2.10 *The class of cofibrations in $\mathcal{CA}$ is generated by all cofibrations $A \to B$ with B finite dimensional, and the class of acyclic cofibrations is generated by all acyclic cofibrations $C \to D$ so that D has a countable homogeneous basis. The category $\mathcal{CA}$ is cofibrantly generated.*

Proof. The statements about generating cofibrations and acyclic cofibrations follow from Lemma 2.9 and Lemma 2.7, respectively. Since both types of morphisms have targets which are small with respect to long enough directed colimits, the result follows.

We also have the following result, which is technically convenient:

Proposition 2.11 *The cofree coalgebra functor S preserves fibrations and weak equivalences.*

Proof. The forgetful functor from $\mathcal{CA}$ to dg-vector spaces preserves weak equivalences and cofibrations; hence, S preserves fibrations and weak equivalences between fibrant objects. However, every dg-vector space is fibrant.

3 Space-like coalgebras

The normalized chains on a topological space form a coassociative differential graded coalgebra, using the Alexander-Whitney diagonal; however, the degree zero part of such a coalgebra supports a very special structure. We isolate and study this structure.

In the category $\mathcal{CA}$ of differential graded coalgebras, the one dimensional coalgebra k concentrated in degree zero is the terminal object. If $C \in \mathcal{CA}$, we define the set $X(C)$ of points in C by the equation

$$X(C) = \mathrm{Hom}_{\mathcal{CA}}(k, C). \tag{3.1}$$

This is the set of elements $x \in C_0$ so that $\Delta_C(x) = x \otimes x$. Note that if X is a set, then the vector space $k[X]$ generated by X is a coalgebra with diagonal

$$k[\Delta_X] : k[X] \to k[X \times X] \cong k[X] \otimes k[X]$$

and there is a natural isomorphism $X \cong X(k[X])$.

Definition 3.1 A differential graded coalgebra $C \in \mathcal{CA}$ is *space-like* if the natural map of coalgebras

$$k[X(C)] \to C_0$$

is an isomorphism. Let $\mathcal{CA}_+$ denote the full sub-category of $\mathcal{CA}$ of space-like coalgebras.

For example, the chains on a space are space-like – hence the name.

Note that any sub-coalgebra of a space-like coalgebra is space-like. Hence the fundamental structure result of coalgebras – that any coalgebra is the filtered colimit of its finite dimensional sub-coalgebras – applies equally well to $\mathcal{CA}_+$.

Lemma 3.2 *The inclusion functor $\mathcal{CA}_+ \to \mathcal{CA}$ has a right adjoint Φ. Furthermore, Φ commutes with filtered colimits.*

Proof. In fact one can give a formula for Φ:

$$\Phi(C) = \sum_\alpha C_\alpha$$

where C_α runs over the space-like sub-dg-coalgebras of C. Then, since the image of a space-like coalgebra is a space-like coalgebra, $\Phi(C)$ has the requisite adjointness properties.

To see that Φ commutes with filtered colimits, let $\{C_\alpha\}$ be a filtered diagram of coalgebras and D a finite dimensional space-like coalgebra. Then we have, using Lemma 1.9,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{CA}_+}(D, \mathrm{colim}_\alpha \Phi(C_\alpha)) &\cong \mathrm{colim}_\alpha \mathrm{Hom}_{\mathcal{CA}_+}(D, \Phi(C_\alpha)) \\
&\cong \mathrm{colim}_\alpha \mathrm{Hom}_{\mathcal{CA}}(D, C_\alpha) \\
&\cong \mathrm{Hom}_{\mathcal{CA}}(D, \mathrm{colim}_\alpha C_\alpha) \\
&\cong \mathrm{Hom}_{\mathcal{CA}_+}(D, \Phi(\mathrm{colim}_\alpha C_\alpha)).
\end{aligned}$$

This immediately implies the following result.

Lemma 3.3 *1.) The category $\mathcal{CA}_+$ has all small limits and colimits.*

2.) The forgetful functor from $\mathcal{CA}_+$ to differential graded vector spaces has a right adjoint S_+. Also, if $V = \mathrm{colim}_\alpha V_\alpha$ is written as a filtered colimit of finite dimensional sub-dg-vector spaces, then

$$\mathrm{colim}_\alpha\, S_+(V_\alpha) \cong S_+(V)$$

Proof. The forgetful functor to differential graded vector spaces makes all colimits. For limits, if $\{C_i\}$ is a diagram in $\mathcal{CA}_+$, then the limit in that category is Φ applied to the limit in $\mathcal{CA}$. The right adjoint is given by the formula $S_+ = \Phi \circ S$, where S is the right adjoint to the forgetful functor from $\mathcal{CA}$ to dg-vector spaces.

We would now like to give the analog of Lemma 1.12 in the category of space-like coalgebras. Part 1 of that Lemma remains unchanged. Thus we need only worry about part 2. Suppose that C is a finite dimensional space-like coalgebra and V is a differential graded vector space, finite dimensional in each degree and $V_0 \cong k$ with generator x. We define certain ideals

$$I_{q(x)} \subseteq Tens_{C^*}(C^* \otimes V^* \otimes C^*)$$

by

$$I_{q(x)} = (q(x), \partial q(x), [a,x], \partial[a,x])$$

where $q(x) \in k[x]$ is a monic polynomial, $a \in (C_0)^*$ and $[b,z] = bz - zb$ is the commutator. There is an inclusion

$$I_{q(x)} \subseteq I_{p(x)}$$

if and only if $p(x)$ divides $q(x)$.

Lemma 3.4 *If C is a finite dimensional space-like coalgebra and V is a differential graded vector space, finite dimensional in each degree and $V_0 \cong k$ with generator x, then in $\mathcal{CA}_+$,*

$$C \times S_+(V) \cong \lim_{q(x)} Tens_{C^*}(C^* \otimes V^* \otimes C^*)/I_{q(x)}$$

where $q(x)$ runs over the monic polynomials in $k[x]$ that split completely into distinct factors over k.

Proof. Define a profinite algebra $A = \lim_\alpha A_\alpha$ where A_α runs over the finite dimensional algebra quotients of

$$Tens_{C^*}(C^* \otimes V^* \otimes C^*) \cong C^* \sqcup Tens(V^*)$$

so that A_α^* is a space-like coalgebra. Then the claim is that the continuous dual of A is $C \times S_+(V)$. This follows from the following calculation, where D is a finite dimensional space-like coalgebra and $A^\sharp$ is the continuous dual of A. We use Lemma 1.9 for the first isomorphism.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{CA}_+}(D, A^\sharp) &\cong \mathrm{colim}_\alpha\, \mathrm{Hom}_{\mathcal{CA}_+}(D, A_\alpha^*) \\
&\cong \mathrm{colim}_\alpha\, \mathrm{Hom}_{k-alg}(A_\alpha, D^*) \\
&\cong \mathrm{Hom}_{k-alg}(C^* \sqcup Tens(V^*), D^*) \\
&\cong \mathrm{Hom}_{k-alg}(C^*, D^*) \times \mathrm{Hom}_{C_* \mathcal{M}_k}(V^*, D^*) \\
&\cong \mathrm{Hom}_{\mathcal{CA}_+}(D, C) \times \mathrm{Hom}_{C_* \mathcal{M}_k}(D, V) \\
&\cong \mathrm{Hom}_{\mathcal{CA}_+}(D, C \times S_+(V)).
\end{aligned}$$

So let D be a finite-dimensional space-like coalgebra and $B = D^*$. Then the degree-zero subalgebra of $B_0 \subseteq B$ is isomorphic to the semi-simple commutative algebra $k^{X(D)}$; in particular, B_0 is commutative and and algebra map $k[x] \to B_0$ must factor

$$k[x] \to k[x]/(q(x)) \xrightarrow{\subseteq} B_0$$

where $q(x)$ is monic and generates the kernel. Since every sub-k-algebra of B_0 is of the form k^Y for some quotient set of $X(D)$, we must have that $q(x)$ splits completely into distinct factors over k. Conversely, if $q(x)$ so splits $k[x]/(q(x)) \cong k^Y$ for some set Y.

Let $I_n \subseteq Tens_{C^*}(C^* \otimes V^* \otimes C^*)/I_{q(x)}$ denote the elements of degree less than $-n$. Then the algebras

$$Tens_{C^*}(C^* \otimes V^* \otimes C^*)/(I_n, I_{q(x)})$$

are cofinal among all finite dimensional dg-algebra quotients which map to duals of space-like coalgebras. The result follows.

Example 3.5 Let k be the one-dimensional differential vector space concentrated in degree zero. Then

$$S_+(k)^* \cong k^k.$$

Compare Example 1.13. Note this implies that $S_+(k) \cong k[k]$.

To produce a model category structure on the category $\mathcal{CA}_+$ we follow the rubric of the previous section. The only part argument of that section which was not a formal consequence of the properties of coalgebras that are inherited by space-like coalgebras is the analog of Theorem 2.1. This we now prove.

Theorem 3.6 *For all space-like coalgebras C and all non-negative integers n, the natural projection*

$$C \times S_+(D(n)) \to C$$

is a quasi-isomorphism.

Proof. The strategy of the proof is the same. Only the case $n = 0$ requires thought, and we may reduce to the case where C is finite dimensional. Then we again, set $A = C^*$, $M = A \otimes D(0)^* \otimes A$ and $x \in D(0)_0^*$ the generator in degree 0. Then we must show that the chain contraction H on $Tens_A(M)$ to the inclusion of the unit $A \to Tens_A(M)$ descends to a chain contraction on $Tens_A(M)/I_{q(x)}$ where $q(x)$ is as in Lemma 3.4. Again we may assume that x divides $q(x)$. In the end we must show that H sends the generators of $I_{q(x)}$ back into $I_{q(x)}$. However, using Equation 2.1 we have

$$H(q(x)) = 0 \qquad \text{and} \qquad H(\partial q(x)) = q(x)$$

as before, and

$$H([a, x]) = [a, H(x)] = 0$$

and

$$H(\partial[a, x]) = H([\partial a, x] + [a, y]) = [a, H(y)] = [a, x].$$

We would now like to propose the following result.

Theorem 3.7 *The category $\mathcal{CA}_+$ of space-like coalgebras has a closed model category structure where a morphism $f : C \to D$ is*

1. *a weak equivalence if it is a quasi-isomorphism;*

2. *a cofibration if it is a level-wise inclusion; and*

3. *a fibration if it has the right lifting property with respect to all acyclic cofibrations.*

Proof. In outline, the proof is the same as that of Theorem 2.8. We touch on the highlights. The analog of Proposition 2.2 is a formal consequence of Theorem 3.6 and Lemma 1.11, which applies equally well to $\mathcal{CA}_+$. The analog of Lemma 2.4 is now formal. Lemma 2.5 for $\mathcal{CA}_+$ is a consequence of Proposition 1.5 and the fact that every sub-coalgebra of a space-like coalgebra is space-like; from this one deduces the analogs of Lemmas 2.6, and 2.7. This completes the existence of the factorizations, and the proof of Theorem 2.8 goes through verbatim.

This model category is also cofibrantly generated. Indeed, we have, exactly as in Proposition 2.10:

Proposition 3.8 *The class of cofibrations in $\mathcal{CA}_+$ is generated by all cofibrations $A \to B$ with B finite dimensional, and the class of acyclic cofibrations is generated by all acyclic cofibrations $C \to D$ so that D has a countable homogeneous basis. The category $\mathcal{CA}_+$ is cofibrantly generated.*

One also has, exactly as in Proposition 2.11

Proposition 3.9 *The cofree coalgebra functor S_+ from differential graded vector spaces to $\mathcal{CA}_+$ preserves weak equivalences.*

Remark 3.10 There are other varieties of coalgebras one might consider. The standard example (see [6] and also [3], although the later is graded over the integers) is that of connected coalgebras. A coalgebra C is connected if it has a unique simple sub-coalgebra D. (See [7] for definitions). The dual of graded coalgebra is a complete local k-algebra with residue field a finite extension of k. A differential graded coalgebra is connected if C_0 is connected. It is relatively straightforward to produce a model category structure on connected differential coalgebras; the methods of [6] apply. If one wants to use the ideas proposed here, we note that the forgetful functor from connected dg-coalgebras to dg-vector spaces has a right adjoint S_{con} and if $D(0)^*$ has generators x and $y = \partial x$, then

$$S_{con}(D(0))^* = \lim_n Tens_k(x,y)/(x^n, \partial(x^n)).$$

From this one easily deduces the analogs of Lemma 1.12 and Theorem 2.1.

Another variety of coalgebras that arises naturally are the "étale" ones. Let k be a perfect field and $\bar{k}$ its algebraic closure, a coalgebra C is étale if $\bar{k} \otimes_k C$ is space-like. For example, if E is a finite field extension of k, then E^* is étale, but space-like if and only of $k = E$. Again, there is a category of differential graded étale coalgebras: we require C_0 be étale. The right adjoint S_{et} now has

$$S_{et}(D(0)) = \lim_{q(x)} Tens_k(x,y)/(q(x), \partial q(x))$$

where $q(x)$ is a monic polynomial which splits into distinct factors over $\bar{k}$. The analog of Theorem 2.1 in this case can be deduced from Theorem 3.6 by tensoring up over $\bar{k}$, or by direct calculation.

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