GTP 04/10/2023

Classical field theory in smooth sets

based on

"Field Theory via Higher geometry I: Smooth sets of Fields"

- joint work w/ Hisham Sati, Urs Schreiber

Plan of the talk

i) Recall standard description of classical field theory.

ii) Collect desired properties of ambient category.

iii) Describe "generalized smooth spaces" as those which 'can be smoothly probed by f.d. manifolds'.

$\rightsquigarrow$ Def. of smooth sets.

iv) Indicate how this sheaf topos of smooth sets satisfies (ii) and naturally hosts many more field theoretic notions.

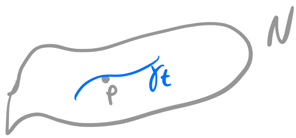
Critical locus in finite dimensions

'Kinematics' 'Dynamics'

- Let $N \in \mathrm{Man}^{\mathrm{fd.}}$, and $f : N \longrightarrow \mathbb{R}$.

  Then $p \in N$ is crit. pt. for f if

  $$\frac{\partial}{\partial t} f(\gamma_t)\Big|_{t=0} = 0 \qquad \forall\ \gamma_t : \mathbb{R} \longrightarrow N$$

  $$\iff df\big|_p = 0$$

  "$\frac{\partial f(p)}{\partial x^i} dx^i = 0$"

- $$\mathrm{Crit}(f) = \left\{ p \in N \ \Big|\ \frac{\partial}{\partial t} f(\gamma_t)\Big|_{t=0} = 0 \right\}$$

  $\forall \gamma_t : \mathbb{R} \longrightarrow N$ ...

- Alternatively,

$$\begin{tikzcd}
\mathrm{Crit}(f) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & N \arrow[d, "df"] \\
N \arrow[r, "0"'] & T^*N
\end{tikzcd}$$

in Set!

- **Note**: Crit(f) might not have a f.d. Man str.

Classical field theory and onshell fields

Kinematics

- Let "spacetime" be $M \in \mathrm{Man}^{\text{f.d.}}$ and "field bundle" be a f.d. fiber bundle $F \longrightarrow M$.

- The (offshell) $\overset{\text{(set)}}{\text{space}}$ of fields is given by

$$\tilde{F} := \Gamma_M(F) = \left\{ M \xrightarrow{\Phi} \overset{F}{\underset{M}{\downarrow}} \right\}$$

e.g. $F = M \times \mathbb{R} \longrightarrow M$

$\Gamma_M(F) \cong C^\infty(M, \mathbb{R})$

Dynamics

- A Lagrangian density is a 'smooth'

$$\tilde{F} \xrightarrow{L} \Omega^d(M)$$

$$\Phi \longmapsto L(\Phi) = \text{"} \bar{L}(\Phi) \cdot dx^1 \dots dx^n \text{"}$$

- The action functional is the 'smooth'

$$S : \tilde{F} \xrightarrow{L} \Omega^d(M) \xrightarrow{\int_M} \mathbb{R}$$

$$\Phi \longmapsto \int_M L(\Phi)$$

- A Lagrangian is <u>local</u> if it 'smoothly' factors as

$$\tilde{F} \xrightarrow{j^\infty} \Gamma_M(J^\infty_M F) \xrightarrow{L} \Omega^d(M)$$

Bundle map

$J^\infty_M F \longrightarrow \wedge^d T^* M$, over M

jet prolongation.

infinite jet bundle

- A field $\phi \in \mathcal{F}$ is "on-shell" if

it is a critical point of $S := \int_M \mathcal{L}$,

$$`\delta S = \frac{\delta S}{\delta \phi}\delta\phi'$$

$$\frac{\partial}{\partial t} S(\phi_t)\Big|_{t=0} = 0$$

vanishes $\forall$ 'smooth' $\phi_t : \mathbb{R}^1 \longrightarrow \mathcal{F}$.

For local $\mathcal{L}$

- Since $\frac{\partial}{\partial t} S(\phi_t)\Big|_{t=0} \overset{\text{int. by parts}}{=} \cdots\cdots = \int_M \langle \mathrm{EL}(\phi), \underbrace{\partial_t \phi_t|_{t=0}}_{\in \Gamma_M(V\mathcal{F})} \rangle$,

equivalently

$$\mathrm{EL}_a(\phi) = \sum (-1)^I \frac{\partial}{\partial x^I}\left(\frac{\partial \mathcal{L}(\phi)\,\partial\phi)}{\partial(\partial_I \phi)}\right)$$

$$\mathrm{EL}(\phi) = 0$$

where
$$\mathrm{EL} : \mathcal{F} \longrightarrow \Gamma_M(\Lambda^d T^*M \otimes V^*\mathcal{F})$$
$$\phi \longmapsto \mathrm{EL}(\phi)$$

is the Euler-Lagrange diff. op.

- i.e. $\underset{\text{a set}}{\mathrm{Crit}(S) \cong \mathcal{F}_{\mathrm{EL}=0}} \hookrightarrow \mathcal{F}$

$$\begin{tikzcd}
\mathrm{Crit}(S) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "(pb)"] & \mathcal{F} \arrow[d, "\mathrm{EL}"] \\
\mathcal{F} \arrow[r, "0"'] & T^*_{\mathrm{Var}}\mathcal{F}
\end{tikzcd}$$

$T^*_{\mathrm{Var}}\mathcal{F} := \Gamma_M(\Lambda^d T^*M \otimes V^*\mathcal{F})$ "Variational cotangent bundle"

Desiderata of "generalized smooth spaces"

- Mapping spaces between f.d. mfds should naturally be smooth spaces.

- '$J^\infty F = \lim J^k F$' should exist as such.

- $\Gamma_M(J^\infty F) = \left\{ M \rightarrow J^\infty F \atop \overline{\overline{\quad M \quad}} \right\}$ $\rightarrow$

- $j^\infty : \Gamma_M(F) \longrightarrow \Gamma_M(J^\infty F)$ $\rightarrow$
  $\phi \longmapsto j^\infty \phi$

- Smooth bundle maps $\begin{array}{ccc} J^\infty F & \xrightarrow{L} & \wedge^d T^* M \\ & \searrow & \downarrow \\ & & M \end{array}$

  should give smooth Lagrangian densities / actions.

  $L(\phi) = L \circ J^\infty(\phi)$, $S = \int_M \circ L$

- Smooth $\phi_t : \mathbb{R}^1 \longrightarrow F$ well dep. such that

  $S(\phi_t) : \mathbb{R}^1 \longrightarrow F \longrightarrow \mathbb{R}$ is smooth.

$$\mathbb{R}^1 \xrightarrow{\phi_t} \tilde{F} \xrightarrow{S} \mathbb{R}$$

- $\frac{\partial}{\partial t} S(\phi_t)\big|_{t=0} = 0 \iff EL(\phi) = 0$

- The on-shell space of fields has a smooth subspace structure.

i.e.
$$\begin{tikzcd} \mathrm{Crit}(S) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "(pb)"] & \tilde{F} \arrow[d] \\ \tilde{F} \arrow[r] & T^*\tilde{F} \end{tikzcd}$$
exists

in generalized smooth spaces

- F.d. manifolds should be faithfully viewed as smooth spaces.

⌜ Since if $M = *$, then bundle is $N \to *$

and $\tilde{F} \subseteq N \in \mathrm{Man}^{f.d.}$ . . . . ⌟

Defining smooth sets

- **Basic idea:**

✗ Not set of points + structure

√ But purely operationally, by giving meaning to 'smoothly probe the would-be space.'

In particular what are smooth paths in field space?

- Let $\Sigma \in \mathrm{Man}^{\mathrm{fd.}}$ and $\mathcal{X}$ the would be space to be smoothly probed.

One thinks of a 'smooth map' → Not yet defined.

$$`\Sigma \xrightarrow[\Sigma\text{-plot}]{} \mathcal{X}'$$

as a smooth Σ-shaped plot in $\mathcal{X}$.

Intuitive consistency $\Rightarrow$ definition

- Minimum consistency requirements:

(i) For each probe $\Sigma \in \mathrm{Man}^{f.d.}$, $\exists$ set of smooth Σ-plots of $\mathcal{X}$

$$\mathrm{Plots}(\Sigma, \mathcal{X}) \equiv \mathcal{X}(\Sigma) \in \mathrm{Set}.$$

e.g. $-\ \Sigma = *$ $\rightsquigarrow$ $\mathrm{Plots}(*, \mathcal{X}) \equiv \mathcal{X}(*)$
the "points of $\mathcal{X}$"

$-\ \Sigma = \mathbb{R}^1$ $\rightsquigarrow$ $\mathrm{Plots}(\mathbb{R}^1, \mathcal{X}) \equiv \mathcal{X}(\mathbb{R}^1)$
the "smooth lines in $\mathcal{X}$"

. . . . .

(ii) For any $f: \Sigma' \longrightarrow \Sigma$ smooth map of mfds, there should exist a 'composite plot'

$$\begin{tikzcd} \Sigma' \arrow[r, "f"] \arrow[rr, bend left, dashed, "f^* \phi_\Sigma"] & \Sigma \arrow[r, "\phi_\Sigma"] & \mathcal{X} \end{tikzcd}$$

for any Σ-plot $\phi_\Sigma \in \mathrm{Plots}(\Sigma, \mathcal{X})$.

i.e., a 'pullback' map

$$f^*: \mathcal{X}(\Sigma) = \mathrm{Plots}(\Sigma, \mathcal{X}) \longrightarrow \mathcal{X}(\Sigma') = \mathrm{Plots}(\Sigma', \mathcal{X})$$

such that

a) $(\mathrm{id}_\Sigma)^* = \mathrm{id}_{\mathcal{X}(\Sigma)}$

b) $(f \circ g)^* = g^* \circ f^*$

$$\Sigma'' \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma \xrightarrow{\phi_\Sigma} \mathcal{X}$$

- Conditions (i) and (ii) say that the consistent assignment of plots for $\mathcal{X}$ forms a functor

$$\mathcal{X}: \mathrm{Man}_{f.d.}^{op} \longrightarrow \mathrm{Set}.$$

i.e. $\mathcal{X} \in \mathrm{PSh}(\mathrm{Man})$

Glueing plots

Condition (iii): For $j : \coprod_{i \in I} \Sigma_i \longrightarrow \Sigma$ an open cover

the pullback

$$\mathrm{Plots}(\Sigma, X) \xrightarrow{\ j^*\ } \left\{ \left(\phi_{\Sigma_i} \in \mathrm{Plots}(\Sigma_i, X) \right)_{i \in I} \;\middle|\; \begin{array}{c} \phi_{\Sigma_i} = \phi_{\Sigma_j} \\ \text{on } \Sigma_i \cap \Sigma_j \end{array} \right\} \subset \mathrm{Plots}\left(\coprod_i \Sigma_i, X\right)$$

$$(\Sigma \xrightarrow{\Phi} X) \longmapsto \left(\Sigma_i \xrightarrow{\iota_i} \Sigma \xrightarrow{\phi} X \right)_{i \in I}$$

is bijective.

The "sheaf condition wrt. open covers".

- Conditions (i), (ii) and (iii) give

$$X \in \mathrm{Sh}(\mathrm{Man}) \hookrightarrow \mathrm{PSh}(\mathrm{Man})$$

Def.

A smooth set X is a sheaf on $\mathrm{Man}_{f.d}$

$$X : \mathrm{Man}_{f.d.}^{op} \longrightarrow \mathrm{Set}$$

wrt open coverage.

Smooth maps between gen. sm. spaces

- A 'smooth' map $f : X \to Y$ should map smooth plots to smooth plots, so a function

$$f_\Sigma : \underbrace{X(\Sigma)}_{\text{set}} \longrightarrow \underbrace{Y(\Sigma)}_{\text{set}}$$

for each $\Sigma \in \mathrm{Man}$.

- It should do so consistently w/ pullback of probes:

$$\begin{tikzcd}
X(\Sigma) \arrow[r, "f_\Sigma"] \arrow[d, "f^*_X"'] \arrow[dr, phantom, "\circlearrowright"] & Y(\Sigma) \arrow[d, "f^*_Y"] \\
X(\Sigma') \arrow[r, "f_{\Sigma'}"'] & Y(\Sigma')
\end{tikzcd}$$

commutes $\forall f : \Sigma' \to \Sigma$

Def.

The category of smooth sets (Gen. smooth spaces) is

$$\boxed{\mathrm{SmSet} := \mathrm{Sh}(\mathrm{Man})},$$

sheaves over the site of fd Man. wrt open coverage.

Example: Smooth manifolds as Smooth Sets

- There is a functor

$$
\begin{aligned}
y : \mathrm{Man} &\longrightarrow \mathrm{SmSet} \\
M &\longmapsto y(M) = \mathrm{Hom}_{\mathrm{Man}}(-, M)
\end{aligned}
$$

- **Consistency:** Let $\mathcal{X} \in \mathrm{SmSet}$ and $M \in \mathrm{Man}$.

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{SmSet}}(y(M), \mathcal{X}) &\xrightarrow{\ \sim\ } \mathcal{X}(M) = \mathrm{Plot}(M, \mathcal{X}) \\
\mathcal{P} &\longmapsto \mathcal{P}_M(\mathrm{id}_M)
\end{aligned}
$$

Yoneda lemma

$$
\left[\ \begin{aligned}
y(M)(M) = \mathrm{Plots}(M, M) &\longmapsto \mathcal{X}(M) = \mathrm{Plots}(M, \mathcal{X}) \\
\mathrm{id}_M &\longmapsto \mathcal{P}_M(\mathrm{id}_M)
\end{aligned}\ \right]
$$

- In particular, for $\mathcal{X} = y(N)$

$$
\mathrm{Hom}_{\mathrm{SmSet}}(y(M), y(N)) \cong y(N)(M) := \mathrm{Hom}_{\mathrm{Man}}(M, N)
$$

i.e. $\quad y : \mathrm{Man} \xhookrightarrow{\ \text{f.f.}\ } \mathrm{Sm\,Set}$

Simplifying the site

- Any $\Sigma \in \mathrm{Man}$ admits an Cartesian open cover
  by charts $\{\phi_i : \mathbb{R}^{n}_i \longrightarrow \Sigma\}$.
- $j : \mathrm{CartSp} \hookrightarrow \mathrm{Man}_{fd.}$ is a full subcat.

- Sheaf condition (iii) implies equivalence

$$j^* : \mathrm{Sh}(\mathrm{Man}) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{Cart})$$

$$\mathcal{X} \longmapsto j^*\mathcal{X} = \mathcal{X}|_{\mathrm{Cart}}$$

- Thus define instead [Schreiber 13'],

$$\mathrm{SmoothSets} := \mathrm{Sh}(\mathrm{Cart})$$

wrt open coverage.

Smooth sets of fields

- Consider the case $F = M \times N \longrightarrow M$, so

$$\Gamma_M(M \times N) \cong C^\infty(M, N) = \mathrm{Hom}_{\mathrm{Man}}(M, N)$$

- Want smooth set $\mathcal{F} : \mathrm{CartSp} \longrightarrow \mathrm{Set}$.

  $\exists$ obvious assignment

$$\mathbb{R}^k \longmapsto C^\infty(M \times \mathbb{R}^k, N) = \mathrm{Hom}_{\mathrm{Man}}(M \times \mathbb{R}^k, N)$$

indeed a sheaf.

- In particular

$$\mathcal{F}(\mathbb{R}^1) = C^\infty(M \times \mathbb{R}^1, N) \xhookrightarrow{\ \mathrm{set}\ } \mathrm{Hom}_{\mathrm{Set}}(\mathbb{R}^1, C^\infty(M, N))$$

$$\phi(x, t) \longmapsto \hat{\phi}(t)(-) := \phi(-, t)$$

naturally a 'smooth curve in $\mathcal{F}$'.

- For arbitrary $F \xrightarrow{\pi} M$, $\Gamma_M(F)$ is smooth set

via

$$\mathcal{F}(\mathbb{R}^k) := \left\{ \begin{array}{ccc} & & F \\ & \nearrow & \downarrow \pi \\ \mathbb{R}^k \times M & \xrightarrow{pr_2} & M \end{array} \right\}$$

Fact

- For any $X, Y \in \mathrm{SmSet}$ $\exists$ $[X, Y] \in \mathrm{SmSet}$.

- Lagrangian $L : \Gamma_M(F) \longrightarrow \Omega^d(M) = \Gamma_M(\wedge^d T^*M)$

  is smooth if it defines

  $$L : F \longrightarrow \Omega^d_{\mathrm{Vert}}(M) \cong \Omega^d(M) \hat{\otimes} C^\infty(-)$$

- Integration over M is smooth.

  $$\int_M : \Omega^d_{\mathrm{Vert}}(M) \longrightarrow Y(\mathbb{R})$$

  $$\omega_{\mathbb{R}^k} \longmapsto \left(\int_M \omega_{\mathbb{R}^k}\right)(x) := \int_M \omega_{\mathbb{R}^k}(x)$$

- Thus $S = \int_M \circ L$ is smooth, if L is.

- Are "local" functionals smooth?

  $$L[\phi] = \bar{L}(\phi, \partial\phi, \dots) \cdot dx^1 \cdots dx^d$$

Infinite Jet Bundle

Facts

Prop [Takens 79']

- $J^\infty_M F = \lim J^k F$ is a smooth "Frèchet manifold"

  lw coords $\{\underbrace{x^\mu}_{M}, \underbrace{u^a}_{F}, \underbrace{u^a_\mu}_{J^1F}, \underbrace{u^a_{\mu_1\mu_2}}_{J^2F}, \ldots\}$ ...

[Takens 79', Schreiber Khavkine 17']

Prop

- Smooth Frèchet maps $J^\infty F \to \mathbb{R}$ are (locally) pullbacks of $J^k F \to \mathbb{R}$.

- **Lemma** [GSS 23']

  $\Longrightarrow$ $J^\infty F \to N$ for $N \in \mathrm{Man} \hookrightarrow \mathrm{FrMan}$

  $\Longrightarrow$ of $J^k F \to N$.

- E.g. a bundle map $L : J^\infty F \longrightarrow \Lambda^d T^* M$, $\searrow M \circlearrowleft$

  is (locally)

  $$L = \bar{L}(x^\mu, u^a, u^a_{\mu_1}, \ldots, u^a_{\mu_1 \ldots \mu_k}) \cdot dx^1 \wedge \cdots \wedge dx^d$$

- The (set) jet prolongation $j^\infty : \Gamma_M(F) \longrightarrow \Gamma_M(J^\infty F)$ gives

  [By thm'? maps smoothly]

  locally $\mathcal{L}(\phi) = L \circ j^\infty(\phi) = \bar{L}(x^\mu, \phi^a, \phi^a_{\mu}, \ldots) \cdot dx^1 \cdots dx^d$

  — Is it smooth?

Jet bundle as smooth set

Thm [Khosik 92'J]

- The functor
$$\begin{aligned} y : \mathrm{FrMan} &\hookrightarrow \mathrm{SmSet} \\ G &\longmapsto \mathrm{Hom}_{\mathrm{FrMan}}(-, G)\big|_{\mathrm{CartSp}} \end{aligned}$$
is fully faithful.

- Since $y(J^\infty F) \longrightarrow y(M)$ bundle in SmSet.

$$\Gamma_M(J^\infty F)(\mathbb{R}^k) := \left\{ \begin{tikzcd} & y(J^\infty F) \arrow[d] \\ y(\mathbb{R}^k) \times y(M) \arrow[ur, "\tilde{\phi}^k"] \arrow[r] & y(M) \end{tikzcd} \right\}_{\mathrm{SmSet}} \Longleftrightarrow \left\{ \begin{tikzcd} & J^\infty F \arrow[d] \\ \mathbb{R}^k \times M \arrow[ur] \arrow[r] & M \end{tikzcd} \right\}_{\mathrm{FrMan}}$$

[GSS 23']

- It follows that j^∞ defines smooth map
$$j^\infty : \mathcal{F} \longrightarrow \Gamma_M(J^\infty F)$$

- Also bundle map $L : y(J^\infty F) \longrightarrow y(\Lambda^d T^*M)$ defines smooth
$$\begin{aligned} \Gamma_M(J^\infty F) &\longrightarrow \Omega^d_{\mathrm{Vert}}(M) \\ \tilde{\phi}^k &\longmapsto L \circ \tilde{\phi}^k \end{aligned}$$

Local Lagrangian action is smooth

- Composing the maps of smooth sets

$$h : \mathcal{F}^\infty \xrightarrow{j^\infty} \Gamma_M(J^\infty F) \xrightarrow{L} \Omega^d_{\mathrm{Vert}}(M)$$

$$\phi_t \longmapsto \tilde{\phi}_t = j^\infty \phi_t \longmapsto L \circ j^\infty \phi_t$$

is smooth.

- Similarly, $S = \int_M \circ\, h : \mathcal{F} \to y(\mathbb{R})$ is smooth.

- In particular, if $\phi_t \in \mathcal{F}(\mathbb{R}^1)$ then

$$S(\phi_t) \in y(\mathbb{R})(\mathbb{R}) \cong C^\infty(\mathbb{R}, \mathbb{R})$$

$\rightsquigarrow$ derivative makes sense.

Prop [GSS 23']

- For any local action $S : \mathcal{F} \longrightarrow Y(\mathbb{R})$

(i) $\phi \in \mathcal{F}(*)$ is crit. iff $\mathcal{E}L(\phi) = 0$

(ii) $\phi^k \in \mathcal{F}(\mathbb{R}^k)$ is crit., i.e.

$$\frac{\partial}{\partial t} S(\phi^k_t)\Big|_{t=0} = 0 \in C^\infty(\mathbb{R}^k, \mathbb{R})$$

$\forall \; \phi^k_t \in \mathcal{F}(\mathbb{R}^k \times \mathbb{R}^1_t)$ w $\phi^k_{t=0} = \phi^k$.

iff.

$$\mathcal{E}L(\phi^k) = 0$$

(iii) Critical plots form a smooth set,

$$\mathcal{F}_{\mathcal{E}L=0} \cong \mathrm{Crit}(S) \hookrightarrow \mathcal{F}.$$

$$\begin{tikzcd}
\mathrm{Crit}(S) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "(pb)"] & \mathcal{F} \arrow[d, "\mathcal{E}L"] \\
\mathcal{F} \arrow[r] & T^*_{\mathrm{Var}}\mathcal{F}
\end{tikzcd} \quad \text{in SmSet}$$

$= \Gamma_M(\wedge^d T^*M \otimes V^*F)$

- If $M = *$, $F = N \times *$ reduces to

$$\begin{tikzcd} \mathrm{Crit}(S) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "(\mathrm{pb})"] & N \arrow[d, "dS"] \\ N \arrow[r, "0"] & T^*N \end{tikzcd}$$

in smooth sets.

List of extra fruits

- p-form (local) currents $\mathcal{P}$ on $\mathcal{F}$ are smooth.
- Charges $\int_{\Sigma^p} \circ \mathcal{P} : \mathcal{F} \longrightarrow \mathbb{R}$ are smooth.
- On-shell cons. currents/charges are smooth on $\mathcal{F}_{\mathcal{E}_L = 0} \hookrightarrow \mathcal{F}$.
- Natural notions of diffeomorphisms, smooth v. fields, tang. vectors on $\mathcal{F} = \Gamma_M(F)$.

  $\rightsquigarrow$ local diffs, local v. fields.
- Noether's 1st and 2nd Theorems are about local ($\Rightarrow$ smooth) symmetries and currents.

Outlook on series

II: Perturbation Theory / Infinitesimal neighborhoods:

How to define these precisely?

$\rightsquigarrow$ Euclid site for

[Model for SDG]

'Infinitesimally thickened points'

- Natural def. of tang. vectors / bundles
- Jet bundles w/o Fréchet theory
- Classifying spaces of diff. forms
- Def. of Cauchy surface
- Inf. neighborhood exists in field space $D_\phi \hookrightarrow F$ around a field $\phi \in F(X)$.

$\rightsquigarrow$ Perturbation theory is restriction

$$D_\phi \hookrightarrow F \xrightarrow{L} \Omega^d_{Vert}(M)$$

⋮

II - III: Fermions e.g. $S = \int_{\mathbb{R}^1} \psi \cdot \frac{d}{dt} \psi \cdot dt$

What is the symbol ψ?

✗? $C^\infty(\mathbb{R}, V) = \mathrm{Hom}_{\mathrm{Man}}(\mathbb{R}, V)$ for $V \in \mathrm{Vect}_{\mathbb{R}}$

✗? $\mathrm{Hom}_{\mathrm{SMan}}(\mathbb{R}, \Pi V)$

A: Arbitrary $\mathbb{R}^{p|q}$-plot of smooth super mapping space $[\mathbb{R}, \Pi V] \in \mathrm{Sh}(\mathrm{SCart})$

$X : \mathrm{ThSCart} \longrightarrow \mathrm{Set}$

- Jet bundles of odd bundles
- local fermionic/bosonic Lagrangians, Variations, on-shell super space of fields

III: • (Higher) Gauge fields: (Connections on (higher)-bundles)

Non-perturbative smooth space of fields

$\rightsquigarrow$ smooth ∞-groupoids

$X : \mathrm{ThSCart} \longrightarrow \mathrm{sSet}$ "good" quotients

III: Gauge symmetries obstruct existence Cauchy surface

$\rightsquigarrow$ $\nexists$ symplectic phase space

$\rightsquigarrow$ $\nexists$ geometric quantization

$\rightsquigarrow$ Need "weakly/homotopically" equiv. theory w/o gauge sym.

i.e. w/ symplc phase space

$\rightsquigarrow$ Higher/Derived smooth/super geometry

'$X: \mathrm{dg}\,\mathrm{SCart}^{\geq 0} \longrightarrow \mathrm{sSet}$'

(Or similar/equivalent formulations)

⌜'Good' quotients/intersections

$\leftrightsquigarrow$ (Non Pert.) BV/BRST formalism,

'Gauge fixing', (Geom.) Quantization ⌟