Workshop on NC and Gen. Geom. in Physics 09/2024

Sheaf Topos Theory as a Setting for Physics

based on

"Field Theory via Higher geometry I: Smooth sets of Fields"

arXiv:2312.16301 (~120 p.)

– w Hisham Sati

and soon

Part II (synthetic/infinitesimal), Part III (fermionic/super)

Part IV (higher)

2311.11026 short intro

– w Hisham Sati, Urs Schreiber.

see also

[2403.16456, 2406.11304, 2408.09921]

Problem #1

Lagrangian Field Theory (Variational problems)

- In (bosonic) classical field theory:

(i) "smooth space" of fields

$\mathcal{F} := C^\infty(M, N)$ (generally $\Gamma_M(F)$ etc.)

(ii) "smooth" (local) action functional

$$S : \mathcal{F} \longrightarrow \mathbb{R}$$
$$\phi \longmapsto \int_M L(\phi) = \int_M L(\phi, \partial\phi, \ldots, \partial^k\phi) \cdot dx^1 \cdots dx^n$$

with extrema $\phi_0 \in \mathcal{F}_0 \hookrightarrow \mathcal{F}$

calc. of variations

s.t. $\frac{\partial}{\partial t}(S \circ \phi_t)\Big|_{t=0} = 0$ ($\Leftrightarrow$ .... $EL(\phi_0) = 0$)

$\forall$ "smooth paths"

$\phi_t : \mathbb{R}^1 \longrightarrow \mathcal{F}$ through $\phi_0 \in \mathcal{F}$.

$\downarrow$

(!) usually taken to be $\phi_t \in C^\infty(M \times \mathbb{R}^1, N)$

Similarly $\phi^k \in C^\infty(M \times \mathbb{R}^k, N)$

Needs

(i) "Smooth" structure on mapping spaces $\tilde{F}$

($\phi_t : \mathbb{R}^1 \to \tilde{F}$ smooth)

(ii) Local action functionals

$S : \tilde{F} \longrightarrow \mathbb{R}$ are smooth

($\Rightarrow$ $S \circ \phi_t : \mathbb{R} \to \mathbb{R}$ smooth)

Category of Gen. Smooth Spaces.

(iii) Rigorous Calc. of Variations

(Infinite jet bundles)

(iv) Extrema $F_0 \hookrightarrow \tilde{F}$ form smooth subspace.

($\Rightarrow$ smooth local observables on classical configurations) problem #2

Fermionic fields

- In physical field theories, there exist "odd" "fermionic" fields ψ s.t. $\boxed{\psi^a \cdot \psi^b = -\psi^b \cdot \psi^a}$

Dirac electron field $\int_M \{\bar\psi \gamma^\mu \partial_\mu \psi\}$

- **Example:** Fermionic Particle on a line $\mathbb{R}$ (time)

$$S_f = \int_{\mathbb{R}} \psi \cdot \frac{d\psi}{dt} \cdot dt$$

lw

$$EL(\psi) = \frac{d\psi}{dt} = 0$$

Note: If ψ not fermionic $\Rightarrow$ $S_f = \int \frac{d}{dt}(\psi^2) \cdot dt$ trivial!

Q: What is the field space F_{ferm}?

Attempts:

(i) $C^\infty(\mathbb{R}, \mathbb{R})$? X not anticom, action trivial

(ii) $\mathrm{Hom}_{SMan}(\mathbb{R}^1, \mathbb{R}_{odd}) \cong \{0\}$ X trivial

$\mathrm{Hom}_{SAlg}(\mathcal{O}(\mathbb{R}_{odd}), \mathcal{O}(\mathbb{R}^1))$

(ii) $\mathrm{Hom}_{\mathrm{SMan}}(\mathbb{R}^1 \times \underbrace{\mathbb{R}^{\mathrm{Aux}}_{\mathrm{odd}}}_{\theta}, \mathbb{R}_{\mathrm{odd}})$ $\quad$ [c.f. $\phi_t \in C^\infty(M \times \mathbb{R}_t, N)$]

$$\underset{\mathrm{Set}}{\cong} \mathrm{Hom}_{\mathrm{sAlg}}\Big(\mathcal{O}(\mathbb{R}_{\mathrm{odd}}), \mathcal{O}(\mathbb{R}^1 \times \underset{\theta}{\mathbb{R}^{\mathrm{Aux}}_{\mathrm{odd}}})\Big)$$

$$\cong \mathrm{Hom}_{\mathrm{sAlg}}\Big(\overset{\substack{\mathbb{R}\oplus\mathbb{R}\cdot b \\ \text{s-vect}}}{\mathbb{R}[\overset{\mathrm{odd}}{b}]}, C^\infty(\mathbb{R}^1) \otimes \mathbb{R}[\theta]\Big)$$

$$b \longmapsto \theta \cdot f(t)$$

$$\cong \Big\{ f(t) \cdot \underbrace{\theta}_{\mathrm{odd}} \;\Big|\; f(t) \in C^\infty(\mathbb{R}^1) \Big\}$$

$$\rotatebox{90}{$\in$} \; \text{"}\varphi_{\mathrm{odd}}\text{"}$$

$$\cong C^\infty(\mathbb{R}^1) \cdot \theta$$ non-trivial and "odd"

but then $\varphi_{\mathrm{odd}} \cdot \frac{d}{dt} \varphi_{\mathrm{odd}} = f \frac{df}{dt} \cdot \theta^2 = 0$

$\Rightarrow S_f = 0.$ trivial

(iv) Similarly, $\mathrm{Hom}_{\mathrm{SMan}}(\mathbb{R}^1 \times \underbrace{\mathbb{R}^{2\ \mathrm{Aux}}_{\mathrm{odd}}}_{\theta_1, \theta_2}, \mathbb{R}_{\mathrm{odd}})$

$$\cong C^\infty(\mathbb{R}^1)\cdot\theta_1 \oplus C^\infty(\mathbb{R}^1)\cdot\theta_2$$

So $\psi_{\mathrm{odd}} = f_1(t)\cdot\theta_1 + f_2(t)\cdot\theta_2$

$$S(\psi^{\mathrm{odd}}) = \cdots = \int\left(f_1(t)\cdot\frac{df_2}{dt} - f_2\cdot\frac{df_1}{dt}\right)\theta_1\cdot\theta_2$$

$\neq 0, \neq d(\cdots)$

- Takeaway: — S_f $\mathcal{O}(2)$-polynomial)

  $\rightsquigarrow$ need 2 "auxiliary coord." $\mathbb{R}^{2\ \mathrm{aux}}_{\mathrm{odd}}$

  — Higher polynomials

  $\rightsquigarrow$ more auxiliary coords !!

Q: • What is ψ? What is its "field space" $\mathcal{F}_f$?

• Also need smooth structure!

Super Smooth Sets

(Topos Theory)

- **Basic principle** [Grothendieck, Lawvere, Lurie, Schreiber]
  (alg. geom) (synthetic logic) (conversion) (∞-version in field th…)

✗ **Not** set of points + structure

✓ **But** purely operationally, by giving meaning to
'probe the would-be space.'

In particular, • what are smooth paths in field space?
• → $\mathbb{R}^{0|2}$-parametrized elements?
→ $\mathbb{R}^{0|2}$ → →

Definition

A super smooth set $\mathcal{X}$ is a (pre) sheaf

$$\mathcal{X} : \mathrm{SCart}^{op}_{f.d.} \longrightarrow \mathrm{Set}$$

on SCart (w.r.t. the open coverage).
(Diff. good)

$\mathbb{R}^{p|q}$ w/ $\mathcal{O}(\mathbb{R}^{p|q}) := C^\infty(\mathbb{R}^p) \otimes \mathbb{R}[\theta_1, \ldots, \theta_q]$.

Intuition on Super Smooth Sets

On objects

(i) For $\Sigma \in \mathrm{SCart}$, (probe)

$$\mathcal{X}(\Sigma) \in \mathrm{Set}$$

the "(smooth) Σ-shaped plots in $\mathcal{X}$"

$$\phi_\Sigma \in \mathcal{X}(\Sigma) \equiv \mathrm{Plots}(\Sigma, \mathcal{X})$$

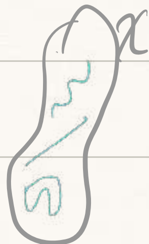
$$\leftrightsquigarrow$$

$$\text{"}\ \Sigma \xrightarrow[\Sigma\text{-plot}]{\phi_\Sigma} \mathcal{X}\ \text{"}$$

e.g.

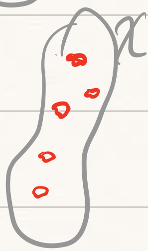
- $\Sigma = *$ $\rightsquigarrow$ $\mathcal{X}(*) \equiv \mathrm{Plots}(*, \mathcal{X})$

  the "points in $\mathcal{X}$"



- $\Sigma = \mathbb{R}^1$ $\rightsquigarrow$ $\mathcal{X}(\mathbb{R}^1) \equiv \mathrm{Plots}(\mathbb{R}^1, \mathcal{X})$

  the "smooth lines in $\mathcal{X}$"



- $\Sigma = \mathbb{R}^{0|1}_{\ominus} = \mathbb{R}_{odd}$ $\rightsquigarrow$ $\mathcal{X}(\mathbb{R}^{0|1})$

  the "odd-points (lines) lines in $\mathcal{X}$"

On morphisms

(ii) For $f : \Sigma' \longrightarrow \Sigma$ in $\mathrm{SCart}_{\mathfrak{I}}$ [SCart$_{\mathfrak{I}}$]

(smooth map of probes)

$$f^* \equiv \mathcal{X}(f) : \underset{\text{Plots}(\Sigma, \mathcal{X})}{\overset{\shortparallel}{\mathcal{X}(\Sigma)}} \longrightarrow \underset{\text{Plots}(\Sigma', \mathcal{X})}{\mathcal{X}(\Sigma')} \quad \text{in Set}$$

the "precomposition of Σ-plots"

$$`\Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}'$$

$$\downarrow$$

$$`\Sigma' \xrightarrow{f} \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}' \qquad (f^*\Phi_\Sigma)$$

(— Functoriality :

a) $(\mathrm{id}_\Sigma)^* = \mathrm{id}_{\mathcal{X}(\Sigma)}$ b) $(f \circ g)^* = g^* \circ f^*$

$\Longleftrightarrow$

Consistent Interpretation

$\Sigma \xrightarrow{\mathrm{id}_\Sigma} \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X} \;=\; \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}$

$(\Sigma'' \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma) \xrightarrow{\Phi_\Sigma} \mathcal{X} \;=\; \Sigma'' \xrightarrow{f} (\Sigma' \xrightarrow{g} \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X})$)

Super smooth maps between super smooth sets

SKIP

Definition

(Gen. super smooth spaces).

The category of super smooth sets is

$$\mathrm{sSmSet} := \mathrm{Sh}(\mathrm{sCart})$$

hw morphisms nat. transformations

- A 'smooth' map $P : X \longrightarrow Y$ should map

(Smooth) Σ-plots of X to Σ-plots of Y:

$$P_\Sigma : \underbrace{X(\Sigma)}_{\text{set}} \longrightarrow \underbrace{Y(\Sigma)}_{\text{set}}$$

for each $\Sigma \in \mathrm{SMan}$.

(• It should do so consistently with pullback of probes:

$$\begin{tikzcd} X(\Sigma) \arrow[r, "P_\Sigma"] \arrow[d, "f^*_X"'] \arrow[dr, phantom, "\circlearrowright"] & Y(\Sigma) \arrow[d, "f^*_Y"] \\ X(\Sigma') \arrow[r, "P_{\Sigma'}"'] & Y(\Sigma') \end{tikzcd}$$

commutes $\forall f : \Sigma' \to \Sigma$)

Example: Super manifolds as super smooth sets

- There is a functor

$$
\begin{aligned}
y : \mathrm{SMan} &\longrightarrow \mathrm{sSmSet} \\
M &\longmapsto y(M) = \mathrm{Hom}_{\mathrm{sMan}}(-, M)
\end{aligned}
$$

- **Consistency:** Let $\mathcal{X} \in \mathrm{sSmSet}$ and $\Sigma \in \mathrm{sCart}$.

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{sSmSet}}(y(\Sigma), \mathcal{X}) &\xrightarrow[\text{Yoneda lemma}]{\sim} \mathcal{X}(\Sigma) \equiv \mathrm{Plot}(\Sigma, \mathcal{X}) \\
P &\longmapsto P(\mathrm{id}_\Sigma)
\end{aligned}
$$

$$
\left[\begin{aligned}
y(M)(M) = \mathrm{Plots}(M, M) &\longmapsto \mathcal{X}(M) = \mathrm{Plots}(M, \mathcal{X}) \\
\mathrm{id}_M &\longmapsto P_M(\mathrm{id}_M)
\end{aligned}\right]
$$

- In particular, for $\mathcal{X} = y(N)$ then

$$
\mathrm{Hom}_{\mathrm{sSmSet}}(y(\Sigma), y(N)) \cong y(N)(\Sigma) := \mathrm{Hom}_{\mathrm{sMan}}(\Sigma, N)
$$

$$
\boxed{y : \mathrm{sMan} \xhookrightarrow[\text{f.f.}]{\text{f.d.}} \mathrm{sSm\,Set}}
$$

Super Smooth sets of fields

For any $M, N \in \mathrm{sMan}$ — e.g. · M, N s.d. manifolds "σ-model"
· $\mathbb{R}, \mathbb{R}_{odd}$ fermionic particle

- Want super smooth set $F : \mathrm{sCartSp}^{op} \longrightarrow \mathrm{Set}$.

  $\exists$ obvious assignment

$$\mathbb{R}^{p|q} \longmapsto \mathrm{Hom}_{\mathrm{sMan}}(M \times \mathbb{R}^{p|q}, N) \cong \mathrm{Hom}_{\mathrm{sAlg}}(\mathcal{O}(N), \mathcal{O}(M) \otimes \mathcal{O}(\mathbb{R}^{p|q}))$$

(indeed a sheaf.)

Fact

- For any $X, Y \in \mathrm{sSmSet}$ $\exists$ $[X, Y] \in \mathrm{sSmSet}$ "Internal Hom"

- For M, N bosonic,

$$F(\mathbb{R}^1) = C^\infty(M \times \mathbb{R}^1, N) \xhookrightarrow{\ \mathrm{set}\ } \mathrm{Hom}_{\mathrm{Set}}(\mathbb{R}^1, C^\infty(M, N))$$

$$\phi(x,t) \longmapsto \hat{\phi}(t)(-) = \phi(-, t)$$

recovers 'smooth path in F'.

simply as the $\mathbb{R}^1$-plots

of field space $[M, N]$

• For $\mathbb{R}$, $\mathbb{R}_{odd}$,

$$F(\mathbb{R}^{0|2}) = \mathrm{Hom}_{SMan}(\mathbb{R} \times \underbrace{\mathbb{R}^{0|2}}_{\text{"aux" } \theta_1, \theta_2}, \mathbb{R}_{odd})$$

recovers "fermion fields w/ auxiliary coords"

as $\mathbb{R}^{0|2}$-plot of field space

$[\mathbb{R}, \mathbb{R}^{odd}]$. ! has no "points"

$\Rightarrow$ $S_f = \int \psi \partial_t \psi$ non-trivial as

map of S-sets.

! Takeaway !

Classical field theory naturally takes place in the topos of spaces probeable by $\mathbb{R}^{p|q}$

⊛ ∃ Generalizations:

(in disks) • $D^n(k)$ ⇝ i) $T\tilde{F}$ $\cong [M, TN] \cong \Gamma_M(VF)$

$[D^1(1), F]$

ii) $T\tilde{F}_{EL} \hat{=}$ "Jacobi fields"

iii) lag. $\Phi \hookrightarrow F$

(simplices) • Δ^k ⇝ super/smooth ∞-groupoids

$$F(\Delta^1) = \{\phi \xrightarrow{\sim} \phi'\}$$

(finite) gauge equiv

$$F(\Delta^2) = \left\{ \begin{tikzcd} \phi \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \phi' \end{tikzcd} \right\}$$

(finite) gauge of gauge.