THE C^*-ALGEBRAIC FORMALISM OF QUANTUM MECHANICS

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ABSTRACT. In this paper, by examining the more tangible, more physically intuitive classical mechanics, we aim to motivate more natural axioms of quantum mechanics than those usually given in terms of Hilbert spaces. Specifically, we plan to replace the assumptions that observable are self-adjoint operators on a separable Hilbert space and states are normalized vectors in that Hilbert space with more natural assumptions about the observables and states in terms of C^*-algebras. Then, with these assumptions in place, we plan to derive the aforementioned assumptions about observables and states in terms of Hilbert spaces from our new assumptions about observables and states in terms of C^*-algebras.

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1. INTRODUCTION

If you are familiar with the usual mathematical formulation of quantum mechanics (i.e., the *states* are elements of a separable Hilbert space and the *observables* self-adjoint linear operators on that space), then you have certainly realized that this mathematical formulation is starkly different from the mathematical formulation used in Classical Mechanics. In classical mechanics, at a very early stage one is usually introduced to the Newtonian formalism of classical mechanics, and as the student progresses through his or her study of physics, in particular mechanics, they are eventually introduced to the Hamiltonian formalism of classical mechanics and they are shown that this formalism is in fact equivalent to the Newtonian formalism. I see this development as rather aesthetic in nature, as opposed to what is usually done with quantum mechanics: we start with very intuitive physical axioms of a theory (those in the Newtonian formalism), and eventually these laws are shown

Date: August 30, 2009.

to be equivalent to the more mathematically convenient Hamiltonian formalism. However, as is usually done with quantum mechanics, we immediately begin by introducing the typical axioms (i.e., where states are elements of a Hilbert space and observables are self-adjoint operators on that space), which, while mathematically convenient, they have *absolutely no physical intuitive justification whatsoever*. I view this as a significant problem.

In this paper, I aim to introduce *equivalent* axioms of quantum mechanics, which are much more natural than the axioms usually taken. To motivate these more natural axioms, we shall first examine the mathematical formalism of classical mechanics. In particular, we shall prove results about the observables and states in classical mechanics. We then examine these results about observables and states to determine why these results are not in agreement with experimental observation. Once it is determined why this characterization of states and observables is not compatible with our physical world, we take these results about observables and states from classical mechanics, and modify them just slightly so as to be compatible with the physical laws that govern our world, and then taking the resulting statements as *axioms* of quantum theory. Once we have those axioms in place, we will show that these axioms actually *imply* that our observables are the self-adjoint operators on some separable Hilbert space and that states are normalized elements in that Hilbert space.

2. A Brief Look at Classical Mechanics

In order to motivate more natural axioms of a quantum theory (as mentioned in the abstract), I first wish to examine (superficially) the mathematical formulation of classical mechanics (in the Hamiltonian sense). In any theory of mechanics, we must come to grips with two ubiquitous concepts: the notion of a *state* and the notion of an *observable*.

In Hamiltonian mechanics, we describe the *state* of a system by an point (q, p)[1] in a two dimensional symplectic manifold M, known as phase space (usually, we identify the classical system with phase space itself). Now, in all real physical systems, a particles position and momentum must remain bounded, and hence, for the remainder of the paper, we shall assume that M is *compact*.

It is an experimental fact that we can never measure something with *infinite* precision; however, there are such things that we can, in principle, measure to an arbitrary precise degree. We call such things *classical observables*.

We would like to come up with a mathematically precise, physically motivated way to characterize these classical observables. A first natural requirement is that observables depend on the state of the system, that is, observables better be functions of q and p. Secondly, we better require that these functions be real-valued. Thirdly, we must require that there is some way to make the error, when we measure an observable in the laboratory, arbitrarily small. Let us assume (by virtue of experimental fact, in the classical realm of course), that we can always measure q and p arbitrarily precisely.[2] Now, say I want to measurable something, namely a real valued function of q and p, call it f, and I want to measure f with error less

[1]We shall just restrict ourselves to the case where our system consists of one particle. For our purposes, there is no loss of generality.

[2]Of course, we mean to imply that the measurements of q and p are simultaneous, which, in classical mechanics, is perfectly acceptable.

than some $\varepsilon > 0$. Now, I know that I can make the error in q and p arbitrarily small, so, if there is some maximum error in q and p, call it δ, so that when I plug in my measured values of q and p into f, my experimental value of f will be within ε of the true value of f, then f will be observable. But of course, this is just the definition of a continuous function! Thus, the natural definition for an observables in classical mechanics can be stated as follows:

Definition 2.1 (Classical Observables)**.** The *classical observables* on M are exactly the continuous real-valued functions on M.

Hereafter, we shall simply denote the set of all observables on M as $\mathcal{O} = \mathcal{C}^0(M, \mathbb{R})$. Now that we have a concrete way of viewing observables on M, we can define some obvious structure:

Definition 2.2. Let $S = (p, q) \in M$, $f, g \in \mathcal{O}$, and $a \in \mathbb{R}$. Then, define

(1) $(f + g)(S) \equiv f(S) + g(S)$
(2) $(af)(S) \equiv af(S)$
(3) $(fg)(S) \equiv f(S)g(S)$
(4) $\|f\| = \sup\{|f(S)|\,|S = (p, q) \in M\}$
(5) $(f^*)(S) = \overline{f(S)}$

With these simple definitions, we have the following theorem:

Theorem 2.1 (Properties of Classical Observables)**.** *The set of observables $\mathcal{O}$ of a* classical system *are exactly the self-adjoint elements of a separable commutative unital C^*-algebra $\mathcal{A}$.*

Proof. Step 1: Construct $\mathcal{A}$.
Define $\mathcal{A} \equiv C^0(M, \mathbb{C})$ and equip $\mathcal{A}$ with the operations given in Definition 2.2.

Step 2: Relegate the trivial work to the reader.
Except for proving that $\mathcal{A}$ is separable and complete, everything is just a matter of checking and we leave it to the reader.

Step 3: Find the limit of a Cauchy sequence.
To prove that $\mathcal{A}$ is complete, let $f_n \in \mathcal{A}$ be a Cauchy sequence. It follows that, for each $x \in M$, $f_n(x)$ is Cauchy in $\mathbb{C}$. But $\mathbb{C}$ is complete, so define $f : M \to \mathbb{C}$ such that $f(x) = \lim f_n(x)$. We now show that f_n converges to f. Let $\varepsilon > 0$, and choose $N' \in \mathbb{N}$, so that if $n > m \geq N'$, it follows that $\|f_n(x) - f_m(x)\| < \frac{\varepsilon}{2}$ for all $x \in M$. Then, pick $N \geq N$ so that whenever $n \geq N$, it follows that $\|f_M(x_0) - f(x_0)\| < \frac{\varepsilon}{2}$ for a fixed $x_0 \in M$. Then, whenever $n \geq N$,

$$|f(x_0) - f_n(x_0)| \leq |f_n(x_0) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon.$$

And so f_n converges to f.

Step 4: Show the limit of a Cauchy sequence is continuous.
To prove f is continuous, fix $x_0 \in M$, let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ so that whenever $n \geq N$, it follows that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in M$. Then, let $n \geq N$, and choose $\delta > 0$ so that whenever $d(x, x_0) < \delta$,[3] it follows that $|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$. Then, whenever $d(x, x_0) < \delta$, it follows that

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon.$$

Thus, $f \in \mathcal{A}$, and so $\mathcal{A}$ is complete.

[3]Here, d is the metric on M.

Step 5: Prove that $\mathcal{A}$ is separable.
By the Stone-Weierstrass Theorem[4], the set of all polynomials in q and p with coefficients with rational real and imaginary part form a countable dense subset of $\mathcal{A}$, and hence $\mathcal{A}$ is separable. □

This theorem will serve as our guide for axiomizing quantum mechanics for the remainder of the paper. Eventually, we will take the above theorem (with a slight modification) as an *axiom* of the observables of quantum mechanics.

Now we wish to do something similar with the states of a classical system. That is to say, we would like to examine the mathematical description of states in classical mechanics, and arrive at a result that we can hopefully take as an axiom for our theory of quantum mechanics. There is a natural way of viewing states in a classical system as linear functionals on $\mathcal{O}$.

Definition 2.3. Let $S = (p, q) \in M$ be a state. Then, we define $S : \mathcal{A} \to \mathbb{C}$ such that, for $f \in \mathcal{A}$:

$$S(f) = f(S)$$

Note that, we will not distinguish between classical states and the corresponding linear functional, which we also refer to as states; which one we are referring to will be clear from context.

It is easy to prove some trivial results about these states:

Theorem 2.2 (Properties of Classical States)**.** *Let $S \in M$ be a state. Then, $S : \mathcal{A} \to \mathbb{C}$ is normalized positive linear functional on $\mathcal{A}$.*

Proof. Except normalization, these properties are all trivial to check. We note first that

$$\|S\| = \sup\{|S(f)| \,|\, \|f\| = 1\} \geq |S(\mathbf{1})| = 1.$$

Secondly, for $f \in \mathcal{A}$,

$$|S(f)| = |f(S)| \leq \sup\{|f(S)| \,|S \in M\} = \|f\|,$$

and so $\|S\| \leq 1$, from which it follows that $\|S\| = 1$, and hence S is normalized. □

At this point, we have fully characterized both the observables and states in classical mechanics. The characterization of the observables is given in Theorem 2.1 and the characterization of the states is given in Theorem 2.2. The idea now is to examine Theorems 1.1 and 1.2 and determine what it is about them that is incompatible with quantum mechanics, and to figure out in what way we can modify them so that they are consistent with the way in which our world actually works.

3. What's Wrong with Classical Mechanics

Before we attempt at "fixing" our notions of states and observables for classical mechanics, we first want to gain a more enlightening view of states when viewed as linear functionals. A nice theorem, due to Riesz and Markov[5], actually characterizes these states very nicely:

[4]See [11], pg. 175.
[5]See [10] pg. 130.

Theorem 3.1 (Riesz-Markov Theorem). *Let X be a locally compact Hausdorff space, and let S be a state on $C^0(X, \mathbb{R})$. Then, there exists a unique Borel probability measure μ_S on X such that, for all $f \in C^0(X, \mathbb{R})$*

$$S(f) = \int_X f d\mu_S$$

When viewed in the light of the Riesz-Markov Theorem, it makes sense to view $S(f)$ as *the expected value of the observable f in the state S*. Physically, if we measure f many times in the laboratory, and our particle is in the state S, then our results *should* average to the value $S(f)$. With this intuition in mind, it makes sense to define the variance of an observable with respect to a state:

Definition 3.1 (Variance). Let $S \in M$ be a state and let $f \in \mathcal{O}$. Then, the *variance of f with respect to S* is defined as

$$\sigma_S(f)^2 \equiv S\left[(f - S(f))^2\right]$$

The reader may check, that for our states defined the way they are (i.e., $S(f) = f(S)$), $\sigma_S(f) = 0$ for all $f \in \mathcal{O}$. Keeping our experimental knowledge of quantum mechanics in mind, we know that this is *not the case for all states.* A good counterexample is a particle in a square well. We would like to develop our theory so that the ground "state" of this particle is to be considered a state in the mathematical theory. Unfortunately, with the ability of hindsight, we know that the variance of the position in this state is *nonzero*, and so if it is to be included in our notion of a state, we must modify the classical definition of a state.

To include such states, we must now throw away our notion that our states are points living in a symplectic manifold, in which case, it makes no sense to define them as linear functionals of the form $S(f) = f(S)$; however, it is natural, and still mathematically possible, to take the set of all states of a *quantum* system to be the normalized positive linear functionals on the algebra of observables (just as in the classical case). We still have to come back to this and make this formal, however, because we have not yet defined the notion of an observable for a quantum system.

Now for the observables. It is *an experimental fact* that

$$\sigma_S(p)\sigma_S(q) \geq \frac{1}{2}\hbar \tag{3.1}$$

for any state S (we refer the reader to any standard textbook on quantum mechanics, e.g., [12]). We would like such a result to be derivable from our mathematical theory, and the following derivation suggests that we should take our algebra observables to be *noncommutative*, in contrast to the classical case.

For the following argument, we shall assume all the properties of the observables stated in Theorem 2.1 except for commutativity. Let $A, B \in \mathcal{O}$ and fix some state S. Without loss of generality, we may assume that $S(A) = S(B) = 0$ (because we could just as well take the observables $A - S(A)$ and $B - S(B)$). Thus,

$$\sigma_S(A)^2\sigma_S(B)^2 = S(A^2)S(B^2)$$

Now, $(\alpha A + i\beta B)^* = \alpha A - i\beta B$ for $\alpha, \beta \in \mathbb{R}$ (here, we have used the fact that A, B are self-adjoint). So, by positivity of states, we have that

$$\begin{aligned} S\left((\alpha A - i\beta B)(\alpha A + i\beta B)\right) &= S(\alpha^2 A^2 + i\alpha\beta AB - i\alpha\beta BA + \beta^2 B^2) \\ &= S(A^2)\alpha^2 + S\left(i[A, B]\right)\alpha\beta + S(B^2)\beta^2 \geq 0, \end{aligned}$$

where $[A, B] \equiv AB - BA$ is the *commutator* of A and B. Defining

$$M = \begin{bmatrix} S(A^2) & \frac{1}{2}S\left(i[A,B]\right) \\ \frac{1}{2}S\left(i[A,B]\right) & S(B^2) \end{bmatrix} \text{ and } \alpha = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

we see that the above inequality becomes

$$\alpha^T M \alpha \geq 0$$

Thus, M is positive-definite, and hence

$$\det M = S(A^2)S(B^2) - \frac{1}{4}S\left(i[A,B]\right)^2 \geq 0$$

and hence

$$\sigma_S(A)\sigma_S(B) \geq \frac{1}{2}\left|S\left([A,B]\right)\right|$$

We immediately see that the equation 3.1 is derivable from the above equation if $[q,p] = \alpha\hbar\mathbf{1}$ where $\alpha \in \mathbb{C}$ has norm 1. Of course, we must also have that (because all observables must be self-adjoint)

$$[q,p]^* = (qp - pq)^* = pq - qp = -[q,p]$$

so $\alpha^* = -\alpha$, so $\alpha = \pm i$. In the end, it makes no difference whether we take $\alpha = i$ or $\alpha = -i$, so we might as well take $\alpha = i$. Thus, we see that if our theory takes the observables to be a *noncommutative algebra*, in particular, with the relation $[q,p] = i\hbar\mathbf{1}$, then equation 3.1 will be derivable in this theory. This suggests modifying Theorem 2.1 only slightly, removing the requirement that the algebra be commutative, and taking this as a *definition* of observables in quantum mechanics. We now make this formal:

Axiom 3.1 (Quantum Observables)**.** The set of observables $\mathcal{O}$ of a *quantum system* are exactly the self-adjoint elements of a separable (noncommutative) unital C^*-algebra $\mathcal{A}$.

The reader should compare Axiom 3.1 to Theorem 2.1. Note how little we are changing between the classical and quantum.

And using the justification given at the beginning of this section, we make the following definition (which itself is just a slight modification of Theorem 2.2):

Axiom 3.2 (Quantum States)**.** The set of states $\mathcal{S}$ of a *quantum system* is the set of all positive linear functionals ψ on $\mathcal{A}$ such that $\psi(\mathbf{1}) = 1$.[6]

So what have we accomplished so far? We first took a formulation of classical mechanics, namely Hamiltonian mechanics, which itself can be shown to be equivalent to the extremely physically intuitive Newtonian formalism of mechanics, and examined its mathematical properties, specifically its mathematical properties relating to the classification of states and observables. We then took this mathematical characterization of states and observables contained in Theorems 2.1 and 2.2, and tried to figure out why these characterizations are incompatible with what we know about quantum mechanics. We eventually determined that to make these characterizations of states and observables compatible with quantum mechanics (with the benefit of hindsight of course), we should modify them slightly in the manner presented in Axioms 3.1 and 3.2.

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[6]In the classical case, we had that all our states were normalized. However, we do not wish to assume this here, because this presupposes that our states are bounded, something we don't need to assume. This can actually be *proven* (see Proposition 4.2), as you will see later.

What has been presented up to this point has mostly been all physical and mathematical justification for taking as assumptions Axioms 3.1 and 3.2. We now seek to build the theory of Quantum Mechanics from the ground up with these two fundamental assumptions, assumptions that arise naturally from the study of classical mechanics. Before we attempt to do this, however, a fair amount of machinery is needed to be built up from the theories of C^*-algebra and Hilbert spaces. Instead of going into the details of how all this might be built up, we simply state the results we shall need throughout the rest of the paper. The reader who is already familiar with the theories of C^*-algebras and Hilbert spaces, may skip to section 6 and use the following two sections as reference when necessary. The unfamiliar reader should be warned that the following two sections are not intended to teach, but merely provide a library of useful results that will be used within the paper.

4. The Theory of C^*-Algebras

Proposition 4.1. *Let $\mathcal{A}$ be a unital C^*-algebra and let ψ be a bounded linear functional on $\mathcal{A}$. Then, if $\|\psi\| = \psi(\mathbf{1})$, then for $\mathbf{A} \in \mathcal{A}$, $\psi(\mathbf{A}^*) = \psi(\mathbf{A})^*$.*

Proposition 4.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let ψ be a linear functional on $\mathcal{A}$. Then, ψ is positive iff $\|\psi\| = \psi(\mathbf{1})$.*

Proposition 4.3. *The states on a unital C^*-algebra $\mathcal{A}$ separate the elements of $\mathcal{A}$.*

Proposition 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, and let $\pi : \mathcal{A} \to \mathcal{B}$ be a *-homomorphism. Then, $\|\pi(\mathbf{A})\| \leq \|\mathbf{A}\|$. Furthermore, if π is a *-isomorphism, then, $\|\pi(\mathbf{A})\| = \|\mathbf{A}\|$.*

5. The Theory of Hilbert Spaces

Notation. For the rest of the paper, unless otherwise stated, a sesquilinear form on a vector space will be denoted $(\cdot|\cdot)$, with the sesquilinear form conjugate linear in the *first coordinate.*

Notation. We shall denote by $\mathcal{BL}(V)$ the set of all bounded linear operators on a normed vector space V and by $\mathcal{L}(V)$ the set of all linear operators on a vector space V.

We would eventually like to prove that our observables are actually operators on a Hilbert space. Before we attempt to do this, however, we better first show that the bounded linear operators on a Hilbert space do in fact form a C^*-algebra. To do this, we must first come up with an involution to put on our space of operators. The notion of "adjointing" is a natural one; however, we must first prove that this notion is well-defined and makes sense:

Proposition 5.1. *Let H be a Hilbert space. Then, for $\mathbf{A} \in \mathcal{BL}(H)$, there exists a unique $\mathbf{A}^* \in \mathcal{BL}(H)$ such that $(\mathbf{Ay}|\mathbf{x}) = (\mathbf{y}|\mathbf{A}^*\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in H$.*

We now show that the set of all bounded operators on a Hilbert spaces, equipped with pointwise addition and scalar multiplication, "adjointing" as the involution, and the usual operator norm forms a C^*-algebra:

Proposition 5.2 (Bounded Operators on a Hilbert Space Form a C^*-Algebra)**.** *Let H be a Hilbert space. Then, $\mathcal{BL}(H)$ with pointwise addition, scalar multiplication, multiplication, the unary operation of "adjointing" for an involution, and the operator norm make $\mathcal{BL}(H)$ into a C^*-algebra.*

Proposition 5.3 (Direct Sum of Hilbert Spaces)**.** *Let I be an index set and let $\{H_i | i \in I\}$ be a collection of Hilbert spaces, each with inner product $(\cdot|\cdot)_i$. Define H to be the set of all functions $f : I \to \bigcup_{i \in I} H_i$ such that:*

(1) For $i \in I, f(i) \in H_i$.
(2) There is a countable set $S_f \subseteq I$ such that $f(I \backslash S_f) = \{\mathbf{0}\}$
(3) $\sum_{i \in S_f} \|f(i)\|^2 < \infty$.

Define addition and scalar multiplication pointwise, and define an inner product on H such that, for $f, g \in H$,

$$(g|f) = \sum_{i \in S_f \cup S_g} (g(i)|f(i))_i \, .$$

Then, H with this inner product is a Hilbert space.

Notation. The set H in the above proof is usually denoted $\oplus_{i \in I} H_i$. To denote an element in $\oplus_{i \in I} H_i$, we shall use the notation $\oplus_{i \in I} f_i$ or $\oplus_{i \in I} f(i)$.

Proposition 5.4. *Let H be an inner product space. Then, H is separable iff there exists an orthonormal basis of H.*

Proposition 5.5. *Let H be a separable Hilbert space, let $\{\mathbf{e_n} | n \in \mathbb{N}\}$ and $\{\mathbf{f_n} | n \in \mathbb{N}\}$ be countable orthonormal bases for H, which exist by virtue of Proposition 5.4, and let $\mathbf{A} \in \mathcal{L}(H)$ be trace-class. Then,*

$$\sum_{n \in \mathbb{N}} (\mathbf{e_n} | \mathbf{A} \mathbf{e_n}) = \sum_{n \in \mathbb{N}} (\mathbf{f_n} | \mathbf{A} \mathbf{f_n}) < \infty.$$

Proposition 5.6. *Let V be a normed vector space, let W be a Banach space, and let U be a dense subspace of V. Then, for $\mathbf{A} \in \mathcal{BL}(U, W)$, there exists a unique $\widetilde{\mathbf{A}} \in \mathcal{BL}(V, W)$ such that $\widetilde{A}|_U = A$.*

6. Quantum Mechanics from the Ground Up

Notation. For the duration of this section, unless otherwise noted, the symbols $\mathcal{A}$, $\mathcal{O}$, and $\mathcal{S}$ have the meanings given to them in Axioms 3.1 and 3.2.

Before we begin, we first state three of the Dirac-Von Neumann Axioms of quantum mechanics[7] that we shall derive from Axioms 3.1 and 3.2.

Statement 6.1 (Dirac-Von Neumann Axiom 1)**.** To each quantum system, we associate a separable Hilbert space H over $\mathbb{C}$.

Statement 6.2 (Dirac-Von Neumann Axiom 2)**.** Every observable is a self-adjoint operator on H.

Statement 6.3 (Dirac-Von Neumann Axiom 3)**.** The states of a quantum system are exactly the positive operators of trace 1.

Before we get down to business, we first have to prove some elementary properties of $\mathcal{A}$ and $\mathcal{S}$ using the firepower of the previous two sections.

[7]See [14], pg. 66.

Lemma 6.1. *$\mathcal{S}$ is a weak-* compact subset of $\mathcal{A}^*$.*

Proof. We first show that $\mathcal{S}$ is closed in the weak-* topology. Let $\psi_n \in \mathcal{S}$ be a sequence converging to $\psi \in \mathcal{A}^*$. We wish to show that ψ is positive. Let $\mathbf{A} \in \mathcal{A}$ be positive. Then, by definition of the weak-* topology,

$$\lim \psi_n(\mathbf{A}) = \psi(\mathbf{A}).$$

But, each $\psi_n(\mathbf{A}) \geq 0$, and hence $\psi(\mathbf{A}) \geq 0$ for every positive $\mathbf{A} \in \mathcal{A}$. Thus, ψ is positive, and by the same reasoning $\psi(\mathbf{1}) = \lim \psi_n(\mathbf{1}) = 1$, and hence $\mathcal{S}$ is closed. However, $\mathcal{S} \subseteq \{\phi \in \mathcal{A}^* | \, \|\phi\| \leq 1\} \equiv B$ because $\|\psi\| = 1$ for $\psi \in \mathcal{S}$. But, by the Banach-Alaoglu Theorem[8], B is compact in the weak-* topology. Thus, because $\mathcal{S}$ is a closed subset of B, $\mathcal{S}$ must also be compact in the weak-* topology. □

Theorem 6.2. *$\mathcal{S}$ is separable in the weak-* topology.*

Proof. Step 1: Introduce notation.
Because $\mathcal{A}$ is separable, we can pick a countable dense subset of $\mathcal{A}$. Enumerate the elements in this countable dense subset $\mathbf{A_n}$. Define $X = \{\phi \in \mathcal{A}^* | \, \|\phi\| \leq 1\}$, $D = \{z \in \mathbb{C} | |z| \leq 1\}$, and $Y = \prod_{n \in \mathbb{N}} D$. Equip X with the weak-* topology.

Step 2: Define a homeomorphism from X to a subset of Y.
Define $h : X \to Y$ such that, for $\phi \in X$,

$$h(\phi) = \left(\frac{1}{\|\mathbf{A_0}\|} \phi(\mathbf{A_0}), \ldots, \frac{1}{\|\mathbf{A_n}\|} \phi(\mathbf{A_n}), \ldots \right) \in Y.$$

Step 3: Prove h is continuous.
To prove that h is continuous, let $\phi_i \in X$ be a net converging to $\phi \in X$. Then, by definition of the weak-* topology, for $n \in \mathbb{N}$, $\phi_i(\mathbf{A_n})$ converges to $\phi(\mathbf{A})$ in $\mathbb{C}$, and hence

$$h(\phi_i) = \left(\frac{1}{\|\mathbf{A_0}\|} \phi_i(\mathbf{A_0}), \ldots, \frac{1}{\|\mathbf{A_n}\|} \phi_i(\mathbf{A_n}), \ldots \right)$$

converges to

$$(\phi(\mathbf{A_0}), \ldots, \phi(\mathbf{A_n}), \ldots)$$

in Y. Thus, h is continuous.

Step 4: Prove h is injective.
Suppose $h(\phi) = h(\psi)$. Then, $\phi(\mathbf{A_n}) = \phi(\mathbf{A_n})$ for all $n \in \mathbb{N}$. Now, we would like to show that for an arbitrary $\mathbf{A} \in \mathcal{A}, \phi(\mathbf{A}) = \psi(\mathbf{A})$. So let $\mathbf{A} \in \mathcal{A}$, and by density, let $\mathbf{A_{n_k}}$ be a sequence of the $\mathbf{A_n}$s converging to $\mathbf{A}$. Then, by continuity of ϕ and ψ, we have that

$$\phi(\mathbf{A}) = \phi\left(\lim_{k\to\infty} \mathbf{A_{n_k}}\right) = \lim_{k\to\infty} \phi(\mathbf{A_{n_k}}) = \lim_{k\to\infty} \psi(\mathbf{A_{n_k}}) = \psi\left(\lim_{k\to\infty} \mathbf{A_{n_k}}\right) = \psi(\mathbf{A}).$$

Thus, $\phi = \psi$, and so h is injective.

Step 5: Prove h is a homeomorphism from X to a subspace of Y.
Now, $h : X \to h(X)$ is continuous and bijective, so to prove that $h : X \to h(X)$ is a homeomorphism, it suffices to show that $h^{-1} : h(X) \to X$ is continuous, or equivalently, that h itself is closed. So let $S \subseteq X$ be closed. Now, by the Banach-Alaoglu Theorem[9], X is compact, so S itself is compact. Now, because h is continuous, it follows that $h(S)$ is compact, and hence closed. Thus, $h : X \to h(X)$ is a homeomorphism. Of course, $h(X)$ is a metric space, so X is metrizable.

[8] See [9], pg. 66.
[9] See [9], pg. 66.

STEP 6: CONCLUDE $\mathcal{S}$ IS SEPARABLE.
$\mathcal{S} \subseteq X$, so $\mathcal{S}$ is of course also metrizable. Let d be the metric on $\mathcal{S}$ that induces the weak-* topology on X. Define $S_n = \left\{B_{\frac{1}{n}}(\phi) | \phi \in X\right\}$. Each S_n is an open cover of $\mathcal{S}$. Now, by Lemma 6.1, S is compact, so for each S_n, we can pick a finite subcover T_n. For each T_n, let C_n be the set of the centers of the balls in T_n. Note that each C_n is finite. Define $C = \cup_{n\in\mathbb{N}} C_n$. Because each C_n is finite, it follows that C is at most countable. To show that C is dense, let $\phi \in \mathcal{S}$. We wish to construct a sequence in C converging to ϕ. Now, each T_n covers X, so pick center of the open ball that ϕ is contained in: call it ϕ_n. It follows that $d(\phi, \phi_n) < \frac{1}{n}$, and hence ϕ_n converges to ϕ. Thus, C is dense, and hence X is separable. □

Now, with the appropriate machinery in place, we plan to link our definitions of quantum observables and states, which were given in terms of C^*-algebras, to the usual definitions of quantum observables and states, which are given in terms of Hilbert spaces.

The following theorem is the key that links everything together, and it, along with the Gelfand-Naimark Theorem, is one of the two primary results of this paper.

Theorem 6.3 (Gelfand-Naimark-Segal Theorem). *Let $\psi \in \mathcal{S}$. Then, there exists a separable Hilbert space H_ψ over $\mathbb{C}$ and a *-representation $\pi_\psi : \mathcal{A} \to \mathcal{BL}(H_\psi)$ such that:*

(1) There exists a cyclic vector $\mathbf{x}_\psi \in H$ of π_ψ.
(2) There is some positive $\mathbf{\Psi} \in \mathcal{BL}(H_\psi)$ with $\mathrm{Tr}\,\Psi = 1$ such that, for $\mathbf{A} \in \mathcal{A}$, $\psi(\mathbf{A}) = (\mathbf{x}_\psi | \pi_\psi(\mathbf{A})\mathbf{x}_\psi) = \mathrm{Tr}\,[\mathbf{\Psi}\pi_\psi(\mathbf{A})]$.
(3) Every *-representation π of $\mathcal{A}$ into a Hilbert space H over $\mathbb{C}$ with cyclic vector $\mathbf{x}$ such that $\psi(\mathbf{A}) = (\mathbf{x}|\pi(\mathbf{A})\mathbf{x})$ for $\mathbf{A} \in \mathcal{A}$ is equivalent to π_ψ in the sense that, for every such representation, there exists a unitary transformation $U : H \to H_\psi$ such that $U\mathbf{x} = \mathbf{x}_\psi$ and for $\mathbf{A} \in \mathcal{A}, \pi_\psi(\mathbf{A}) = U\pi(\mathbf{A})U^{-1}$.

Proof. STEP 1: DEFINE A SEMIDEFINITE SESQUILINEAR FORM ON $\mathcal{A}$.
We first wish to turn $\mathcal{A}$ into an inner product space. To do this, we first define a sesquilinear form on $\mathcal{A}$. Define, for $\mathbf{A}, \mathbf{B} \in \mathcal{A}$:

$$(\mathbf{A}|\mathbf{B}) \equiv \psi(\mathbf{A}^*\mathbf{B})$$

The reader may check that this is linear in the first argument, and conjugate symmetric (conjugate symmetry follows from Propositions 4.2 and 4.1). We would like this to be an inner product; however, while it certainly is nonnegative, it isn't necessarily positive definite. To turn it into an inner product product, we plan to show that the set

$$\mathcal{I} = \{\mathbf{B} \in \mathcal{A} | \forall \mathbf{A} \in \mathcal{A}, (\mathbf{A}|\mathbf{B}) = 0\}$$

is a subspace of $\mathcal{A}$. Then, we would like to show that the above defined sesquilinear form is a well-defined inner product on the quotient space $\mathcal{A}/\mathcal{I}$.

STEP 2: SHOW THAT $\mathcal{A}/\mathcal{I}$ IS AN INNER PRODUCT SPACE WITH THE INNER PRODUCT $(\cdot|\cdot)$ EXTENDED NATURALLY TO THE QUOTIENT SPACE.
The reader may check for themselves that $\mathcal{I}$ is in fact a left-sided ideal. Now, $\mathcal{I}$ is a subspace of $\mathcal{A}$, so if we temporarily forget about the extra structure on $\mathcal{A}$, we can construct the quotient space $\mathcal{A}/\mathcal{I}$ by the usual means. We define the sesquilinear

form on the *quotient space* in the obvious way: for $\mathbf{A}+\mathcal{I}, \mathbf{B}+\mathcal{I} \in \mathcal{A}/\mathcal{I}$, we define

$$(\mathbf{A}+\mathcal{I}|\mathbf{B}+\mathcal{I}) \equiv (\mathbf{A}|\mathbf{B}).$$

Once again, this is obviously linear in the first argument and conjugate symmetric, so all that we have to prove is well-definedness and positive-definiteness. To prove well-definedness, let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{A}$, and suppose $\mathbf{A}+\mathcal{I} = \mathbf{C}+\mathcal{I}$ and $\mathbf{B}+\mathcal{I} = \mathbf{D}+\mathcal{I}$. Thus, we may write $\mathbf{C} = \mathbf{A}+\mathbf{I}_1$ and $\mathbf{D} = \mathbf{B}+\mathbf{I}_2$ for $\mathbf{I}_1, \mathbf{I}_2 \in \mathcal{I}$. Then,

$$(\mathbf{C}|\mathbf{D}) = (\mathbf{A}+\mathbf{I}_1|\mathbf{B}+\mathbf{I}_2) = (\mathbf{A}|\mathbf{B}) + (\mathbf{I_1}|\mathbf{B}) + (\mathbf{A}|\mathbf{I}_2) + (\mathbf{I}_1|\mathbf{I}_2) = (\mathbf{A}|\mathbf{B})$$

and so the sesquilinear form is well-defined. To prove positive-definiteness, suppose $(\mathbf{A}+\mathcal{I}|\mathbf{A}+\mathcal{I}) = 0$. It immediately follows that $(\mathbf{A}|\mathbf{A}) = 0$. By the Cauchy-Schwarz Inequality (whose proof does not require positive-definiteness), for $\mathbf{B} \in \mathcal{A}$

$$|(\mathbf{B}|\mathbf{A})|^2 \leq (\mathbf{A}|\mathbf{A})(\mathbf{B}|\mathbf{B}) = 0,$$

and hence, for $\mathbf{B} \in \mathcal{A}$, $(\mathbf{B}|\mathbf{A}) = 0$, and so $\mathbf{A} \in \mathcal{I}$. Thus, $\mathbf{A}+\mathcal{I} = \mathbf{0}+\mathcal{I}$ and hence $(\cdot|\cdot)$ is a well-defined inner product on $\mathcal{A}/\mathcal{I}$. In fact, this actually proves that

$$\mathcal{I} = \{\mathbf{A} \in \mathcal{A}|\, (\mathbf{A}|\mathbf{A}) = 0\},$$

a result that we will use later in the proof.

Step 3: Complete $\mathcal{A}/\mathcal{I}$ to a Hilbert space, H_ψ.
This inner product induces a norm, and hence a metric on $\mathcal{A}/\mathcal{I}$. Complete the metric space $\mathcal{A}/\mathcal{I}$ in to a Hilbert space H_ψ over $\mathbb{C}$. We now proceed to construct our representation.

Step 4: Construct a *-homomorphism $\phi : \mathcal{A} \to \mathcal{BL}(\mathcal{A}/\mathcal{I})$.
We first define a *-homomorphism $\phi : \mathcal{A} \to \mathcal{BL}(\mathcal{A}/\mathcal{I})$ defined such that, for $\mathbf{A} \in \mathcal{A}$, ϕ sends $\mathbf{A}$ to the operator that sends $\mathbf{B}+\mathcal{I} \in \mathcal{A}/\mathcal{I}$ to $\mathbf{AB}+\mathcal{I}$. That is,

$$\phi(\mathbf{A})(\mathbf{B}+\mathcal{I}) = \mathbf{AB}+\mathcal{I}.$$

We first prove well-definedness of the "operator" $\phi(\mathbf{A})$. Suppose $\mathbf{B}+\mathcal{I} = \mathbf{C}+\mathcal{I}$. Then, $\mathbf{C} = \mathbf{B}+\mathbf{I}$ for some $\mathbf{I} \in \mathcal{I}$. Thus,

$$\begin{aligned}\phi(\mathbf{A})(\mathbf{C}+\mathcal{I}) = \mathbf{AC}+\mathcal{I} &= \mathbf{A}(\mathbf{B}+\mathbf{I})+\mathcal{I} \\ &= \mathbf{AB}+\mathbf{AI}+\mathcal{I} = \mathbf{AB}+\mathcal{I} = \phi(\mathbf{A})(\mathbf{B}+\mathcal{I}),\end{aligned}$$

where we have used the fact that $\mathcal{I}$ is a left-ideal so that we know $\mathbf{AI} \in \mathcal{I}$. The reader may check that $\phi(\mathbf{A})$ is actually linear. To prove that $\phi(\mathbf{A})$ is bounded, it suffices to show that the set $\left\{\|\phi(\mathbf{A})(\mathbf{B}+\mathcal{I})\|^2 \mid \|\mathbf{B}+\mathcal{I}\| = 1\right\}$ is bounded above, so let $\mathbf{B}+\mathcal{I} \in \mathcal{A}/\mathcal{I}$ be of norm 1. For convenience, let us first define

$$\psi_{\mathbf{B}}(\mathbf{A}) \equiv \psi(\mathbf{B}^*\mathbf{AB}).$$

The reader may check that $\psi_{\mathbf{B}}$ is a positive linear functional on $\mathcal{A}$ (this follows trivially from the fact that ψ is a positive linear functional on $\mathcal{A}$). Then,

$$\begin{aligned}\|\phi(\mathbf{A})(\mathbf{B}+\mathcal{I})\|^2 = \|\mathbf{AB}+\mathcal{I}\|^2 = (\mathbf{AB}|\mathbf{AB}) = \psi(\mathbf{B}^*\mathbf{A}^*\mathbf{AB}) = \psi_{\mathbf{B}}(\mathbf{A}^*\mathbf{A}) \\ \leq \|\psi_{\mathbf{B}}\|\,\|\mathbf{A}^*\mathbf{A}\|_{\mathcal{A}} = \psi_{\mathbf{B}}(\mathbf{1})\,\|\mathbf{A}\|_{\mathcal{A}}^2 = \|\mathbf{A}\|_{\mathcal{A}}^2,\end{aligned}$$

where we have applied Proposition 4.2 and used the subscript $\mathcal{A}$ to denote the norm on $\mathcal{A}$ that makes it into a C^*-algebra, to distinguish it from the semi-norm on $\mathcal{A}$ induced by the semi-definite sesquilinear form. Thus, $\phi(\mathbf{A})$ is well-defined and $\phi(\mathbf{A}) \in \mathcal{BL}(\mathcal{A}/\mathcal{I})$.

Step 5: Show that ϕ is actually a *-homomorphism.
To show that ϕ is a *-homomorphism, for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}$ we see that

$$\begin{aligned}\phi(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathcal{I}) &= (\mathbf{A}+\mathbf{B})\mathbf{C}+\mathcal{I} = \mathbf{AC}+\mathbf{BC}+\mathcal{I} \\ &= (\mathbf{AC}+\mathcal{I})+(\mathbf{AB}+\mathcal{I}) = \phi(\mathbf{A})(\mathbf{C}+\mathcal{I})+\phi(\mathbf{B})(\mathbf{C}+\mathcal{I}) \\ &= (\phi(\mathbf{A})+\phi(\mathbf{B}))(\mathbf{C}+\mathcal{I}),\end{aligned}$$

and so $\phi(\mathbf{A}+\mathbf{B}) = \phi(\mathbf{A})+\phi(\mathbf{B})$. Similarly, we obtain $\phi(\mathbf{AB}) = \phi(\mathbf{A})\phi(\mathbf{B})$ and $\phi(\mathbf{1}) = \mathbf{1}$. To show that $\phi(\mathbf{A}^*) = \phi(\mathbf{A})^*$, we see that

$$\begin{aligned}(\mathbf{C}+\mathcal{I}|\phi(\mathbf{A}^*)(\mathbf{B}+\mathcal{I})) &= (\mathbf{C}+\mathcal{I}|\mathbf{A}^*\mathbf{B}+\mathcal{I}) = (\mathbf{C}|\mathbf{A}^*\mathbf{B}) = \psi(\mathbf{C}^*\mathbf{A}^*\mathbf{B}) \\ &= \psi\left((\mathbf{AC})^*\mathbf{B}\right) = (\mathbf{AC}|\mathbf{B}) = (\mathbf{AC}+\mathcal{I}|\mathbf{B}+\mathcal{I}) \\ &= (\phi(\mathbf{A})(\mathbf{C}+\mathcal{I})|\mathbf{B}+\mathcal{I}).\end{aligned}$$

Thus, by Proposition 5.1, it follows that $\phi(\mathbf{A}^*) = \phi(\mathbf{A})^*$. Thus, ϕ is a *-homomorphism.

Step 6: Extend ϕ to a *-representation π_ψ of $\mathcal{A}$.
Now, by the usual process of completion, $\mathcal{A}/\mathcal{I}$ is dense in H_ψ, thus, by Proposition 5.6, there exists a unique bounded linear operator, call it $\pi_\psi(\mathbf{A})$, such that $\pi_\psi(\mathbf{A})|_{\mathcal{A}/\mathcal{I}} = \phi(\mathbf{A})$. This defines a function $\pi_\psi : \mathcal{A} \to \mathcal{BL}(H_\psi)$. It follows from the fact that ϕ is a *-homomorphism that π_ψ is a *-representation.

Step 7: Construct the cyclic vector.
Define $\mathbf{x}_\psi \equiv \mathbf{1}+\mathcal{I} \in H_\psi$. Then,

$$\pi(\mathcal{A})\mathbf{x}_\psi = \{\mathbf{A}+\mathcal{I}|\mathbf{A} \in \mathcal{A}\} = \mathcal{A}/\mathcal{I}.$$

But, by the completion process, $\mathcal{A}/\mathcal{I}$ is dense in H, and so $\mathbf{x}_\psi$ is a cyclic vector of π_ψ.

Step 8: Prove H_ψ is separable.
Because $\mathcal{A}$ is separable, it follows that $\pi_\psi(\mathcal{A})\mathbf{x}_\psi$ is separable. But, $\pi_\psi(\mathcal{A})\mathbf{x}_\psi$ is dense in H_ψ, and hence H_ψ is separable.

Step 9: Prove the first equality of (2).
We see that, for $\mathbf{A} \in \mathcal{A}$,

$$\begin{aligned}(\mathbf{x}_\psi|\pi_\psi(\mathbf{A})\mathbf{x}_\psi) &= (\mathbf{1}+\mathcal{I}|\pi_\psi(\mathbf{A})(\mathbf{1}+\mathcal{I})) = (\mathbf{1}+\mathcal{I}|\mathbf{A}+\mathcal{I}) = (\mathbf{1}|\mathbf{A}) = \psi(\mathbf{1}^*\mathbf{A}) \\ &= \psi(\mathbf{A}).\end{aligned}$$

and so the first equality of (2) is proved.

Step 10: Prove the second equality of (2).
Because H_ψ is separable, by Proposition 5.4, H_ψ has a countable orthonormal basis. Denote the basis by $\{\mathbf{e_n}|n \in \mathbb{N}\}$. Now, write

$$\mathbf{x}_\psi = \sum_{n\in\mathbb{N}} c_n\mathbf{e_n}$$

for $c_n \in \mathbb{C}$, and define $\mathbf{\Psi} \in \mathcal{BL}(H_\psi)$ such that

$$(\mathbf{e_n}|\mathbf{\Psi}\mathbf{e_m}) = c_m^* c_n.$$

It follows that

$$\begin{aligned}\psi(\mathbf{A}) &= \left(\sum_{m\in\mathbb{N}} c_m\mathbf{e_m}\middle|\pi_\psi(\mathbf{A})\left(\sum_{n\in\mathbb{N}} c_n\mathbf{e_n}\right)\right) = \left(\sum_{m\in\mathbb{N}}\mathbf{e_m}\middle|\sum_{n\in\mathbb{N}}[c_n\pi_\psi(\mathbf{A})\mathbf{e_n}]\right)\\ &= \sum_{n\in\mathbb{N}}\left[\left(\sum_{m\in\mathbb{N}} c_m\mathbf{e_m}\middle|c_n\pi_\psi(\mathbf{A})\mathbf{e_n}\right)\right] = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}[c_m^*c_n(\mathbf{e_m}|\pi_\psi(\mathbf{A})\mathbf{e_n})]\\ &= \sum_{n\in\mathbb{N}}\left[\left(\mathbf{e_m}\middle|\sum_{m\in\mathbb{N}}[c_m^*c_n\pi_\psi(\mathbf{A})\mathbf{e_n}]\right)\right]\\ &= \sum_{n\in\mathbb{N}}\left[\left(\mathbf{e_m}\middle|\pi_\psi(\mathbf{A})\left(\sum_{m\in\mathbb{N}}[c_m^*c_n\mathbf{e_n}]\right)\right)\right] = \sum_{n\in\mathbb{N}}[(\mathbf{e_m}|\pi_\psi(\mathbf{A})\mathbf{\Psi e_m})]\\ &= \mathrm{Tr}\,[\pi_\psi(\mathbf{A})\mathbf{\Psi}] = \mathrm{Tr}\,[\mathbf{\Psi}\pi_\psi(\mathbf{A})].\end{aligned}$$

Using the fact that $\psi(\mathbf{1}) = 1$, we easily see that $\mathrm{Tr}\,\Psi = 1$. To prove that $\mathbf{\Psi}$ is positive, we define $\mathbf{B} \in \mathcal{BL}(H)$ such that

$$(\mathbf{e_m}|\mathbf{Be_n}) = \begin{cases} c_n^* & \text{if } m = 1\\ 0 & \text{otherwise}\end{cases}.$$

It follows that

$$(\mathbf{e_n}|\mathbf{B}^*\mathbf{Be_m}) = (\mathbf{Be_n}|\mathbf{Be_m}) = (c_n^*\mathbf{e_1}|c_m^*\mathbf{e_1}) = c_m^*c_n,$$

and hence $\mathbf{\Psi} = \mathbf{B}^*\mathbf{B}$. Thus, $\mathbf{\Psi}$ is positive.

Step 11: Define the unitary transformation of property (3) on $\pi(\mathcal{A})\mathbf{x}$.

Let H be a Hilbert space[10] over $\mathbb{C}$ and let $\pi : \mathcal{A} \to \mathcal{BL}(H)$ be a *-representation of $\mathcal{A}$ with cyclic vector $\mathbf{x}$ such that, for $\mathbf{A} \in \mathcal{A}$,

$$\psi(\mathbf{A}) = (\mathbf{x}|\pi(\mathbf{A})\mathbf{x}).$$

We first define $\widetilde{U} : \pi(\mathbf{A})\mathbf{x} \to H_\psi$ and then aim to extend it uniquely to all of H by Proposition 5.6. First of all, for $\mathbf{v} \in \pi(\mathbf{A})\mathbf{x}$ write $\mathbf{v} = \pi(\mathbf{A})\mathbf{x}$ for some $\mathbf{A} \in \mathcal{A}$. Then, define

$$\widetilde{U}\mathbf{v} = \widetilde{U}(\pi(\mathbf{A})\mathbf{x}) = \pi_\psi(\mathbf{A})\mathbf{x}_\psi.$$

Step 12: Prove this transformation is well-defined.

We first show that $\widetilde{U}$ is well-defined, i.e., $\widetilde{U}\mathbf{v}$ does not depend on our choice of $\mathbf{A} \in \mathcal{A}$ such that $\pi(\mathbf{A})\mathbf{x} = \mathbf{v}$ because such an $\mathbf{A}$ might not be unique. So suppose $\mathbf{A}' \in \mathcal{A}$ is such that $\pi(\mathbf{A}')\mathbf{x} = \mathbf{v}$. It follows that

$$\begin{aligned}\psi(\mathbf{A}^*\mathbf{A}') &= (\mathbf{x}|\pi(\mathbf{A}^*\mathbf{A}')\mathbf{x}) = (\mathbf{x}|\pi(\mathbf{A}^*)\pi(\mathbf{A}')\mathbf{x}) = (\mathbf{x}|\pi(\mathbf{A}^*)\pi(\mathbf{A})\mathbf{x})\\ &= (\mathbf{x}|\pi(\mathbf{A}^*\mathbf{A})\mathbf{x}) = \psi(\mathbf{A}^*\mathbf{A}).\end{aligned}$$

Similarly,

$$\psi(\mathbf{A}^*\mathbf{A}) = \psi\left(\mathbf{A}'^*\mathbf{A}\right) = \psi\left(\mathbf{A}'^*\mathbf{A}'\right).$$

Thus,

$$\begin{aligned}(\mathbf{A}-\mathbf{A}'|\mathbf{A}-\mathbf{A}') &= \psi\left((\mathbf{A}-\mathbf{A}')^*(\mathbf{A}-\mathbf{A}')\right)\\ &= \psi(\mathbf{A}^*\mathbf{A}) - \psi(\mathbf{A}^*\mathbf{A}') - \psi\left(\mathbf{A}'^*\mathbf{A}\right) + \psi\left(\mathbf{A}'^*\mathbf{A}'\right) = 0,\end{aligned}$$

[10]The inner product on H will also be denoted $(\cdot|\cdot)$. This should cause no confusion.

and so $\mathbf{A} - \mathbf{A}' \in \mathcal{I}$. But then,

$$\widetilde{U}\left(\pi\left(\mathbf{A}'\right)\mathbf{x}\right) = \pi_\psi\left(\mathbf{A}'\right)\mathbf{x}_\psi = \mathbf{A}' + \mathcal{I} = \mathbf{A} + \mathcal{I} = \widetilde{U}\left(\pi\left(\mathbf{A}\right)\mathbf{x}\right),$$

and so $\widetilde{U}$ is well-defined.

STEP 13: PROVE THIS TRANSFORMATION IS LINEAR AND BOUNDED. It is easy to check that $\widetilde{U}$ is linear. To check that is is bounded, we see that

$$\begin{aligned}\left\|\widetilde{U}\left(\pi\left(\mathbf{A}\right)\mathbf{x}\right)\right\|^2 &= \left\|\pi_\psi\left(\mathbf{A}\right)\mathbf{x}_\psi\right\|^2 = \left\|\mathbf{A} + \mathcal{I}\right\|^2 = \left(\mathbf{A} + \mathcal{I} | \mathbf{A} + \mathcal{I}\right) = \left(\mathbf{A}|\mathbf{A}\right) \\ &= \psi\left(\mathbf{A}^*\mathbf{A}\right) = \left(\mathbf{x}|\pi\left(\mathbf{A}^*\mathbf{A}\right)\mathbf{x}\right) = \big(\mathbf{x}|\pi\left(\mathbf{A}\right)^*\pi\left(\mathbf{A}\right)\mathbf{x}\big) \\ &= \left(\pi\left(\mathbf{A}\right)\mathbf{x}|\pi\left(\mathbf{A}\right)\mathbf{x}\right) = \left\|\pi\left(\mathbf{A}\right)\mathbf{x}\right\|^2,\end{aligned}$$

and so $\widetilde{U} \in \mathcal{BL}\left(\pi\left(\mathcal{A}\right)\mathbf{x}, H_\psi\right)$.

STEP 14: EXTEND THIS TRANSFORMATION TO ALL OF H.
Because $\pi\left(\mathcal{A}\right)\mathbf{x}$ is dense in H, by Proposition 5.6, there exists a unique $U \in \mathcal{BL}(H, H_\psi)$ such that $U|_{\pi(\mathcal{A})\mathbf{x}} = \widetilde{U}$.

STEP 15: PROVE THAT U IS UNITARY.
To prove U is injective, it suffices to show that $U\mathbf{v} = \mathbf{0} + \mathcal{I}$ implies $\mathbf{v} = 0$ for $\mathbf{v} \in \pi\left(\mathcal{A}\right)\mathbf{x}$ by linearity of U, continuity of U, and denseness of $\pi\left(\mathcal{A}\right)\mathbf{x}$. So let $\mathbf{v} \in \pi\left(\mathcal{A}\right)\mathbf{x}$ and write $\mathbf{v} = \pi\left(\mathbf{A}\right)\mathbf{x}$ for $\mathbf{A} \in \mathcal{A}$. If $U\mathbf{v} = 0$, then

$$\pi_\psi\left(\mathbf{A}\right)\mathbf{x}_\psi = \mathbf{A} + \mathcal{I} = \mathbf{0} + \mathcal{I}.$$

Thus, $\mathbf{A} \in \mathcal{I}$. So

$$0 = \left(\mathbf{A}|\mathbf{A}\right) = \psi\left(\mathbf{A}^*\mathbf{A}\right) = \left(\mathbf{x}|\pi\left(\mathbf{A}^*\mathbf{A}\right)\mathbf{x}\right) = \left(\pi\left(\mathbf{A}\right)\mathbf{x}|\pi\left(\mathbf{A}\right)\mathbf{x}\right) = \left(\mathbf{v}|\mathbf{v}\right).$$

Thus, by positive-definiteness, $\mathbf{v} = \mathbf{0}$, and hence U is injective.
To prove U is surjective, by linearity of U, continuity of U, denseness of $\pi_\psi\left(\mathcal{A}\right)\mathbf{x}$ in H_ψ, and denseness of $\pi\left(\mathcal{A}\right)\mathbf{x}$ in H, it suffices to show that for $\mathbf{w} \in \pi_\psi\left(\mathcal{A}\right)\mathbf{x}_\psi$, there exists $\mathbf{v} \in \pi\left(\mathcal{A}\right)\mathbf{x}_\psi$ such that $U\mathbf{v} = \mathbf{w}$. Write $\mathbf{w} = \pi_\psi\left(\mathbf{A}\right)\mathbf{x}_\psi \in \pi_\psi\left(\mathcal{A}\right)\mathbf{x}_\psi$ for $\mathbf{A}$. Then, pick $\mathbf{v} = \pi\left(\mathbf{A}\right)\mathbf{x} \in \pi\left(\mathcal{A}\right)\mathbf{x}$. It is easy to show that $U\mathbf{v} = \mathbf{w}$, and hence U is surjective, and hence bijective.
To prove that U is unitary, it suffices to show that

$$\left(\mathbf{w}|\mathbf{v}\right) = \left(U\mathbf{w}|U\mathbf{v}\right)$$

for $\mathbf{v}, \mathbf{w} \in \pi\left(\mathcal{A}\right)\mathbf{x}$, by density of $\pi\left(\mathcal{A}\right)\mathbf{x}$ and continuity of U. Write $\mathbf{v} = \pi\left(\mathbf{A}\right)\mathbf{x}$ and $\mathbf{w} = \pi\left(\mathbf{B}\right)\mathbf{x}$ for $\mathbf{A}, \mathbf{B} \in \mathcal{A}$. Then,

$$\begin{aligned}\left(\mathbf{w}|\mathbf{v}\right) &= \left(\pi\left(\mathbf{B}\right)\mathbf{x}|\pi\left(\mathbf{A}\right)\mathbf{x}\right) = \left(\mathbf{x}|\pi\left(\mathbf{B}^*\mathbf{A}\right)\mathbf{x}\right) = \psi\left(\mathbf{B}^*\mathbf{A}\right) = \left(\mathbf{B}|\mathbf{A}\right) \\ &= \left(\mathbf{B} + \mathcal{I}|\mathbf{A} + \mathcal{I}\right) = \left(\pi_\psi\left(\mathbf{B}\right)\mathbf{x}_\psi|\pi_\psi\left(\mathbf{A}\right)\mathbf{x}_\psi\right) \\ &= \left(U\left(\pi\left(\mathbf{B}\right)\mathbf{x}\right)|U\left(\pi\left(\mathbf{A}\right)\mathbf{x}\right)\right) = \left(U\mathbf{w}|U\mathbf{v}\right).\end{aligned}$$

Thus, U is a unitary transformation.

STEP 16: PROVE U SATISFIES THE PROPERTIES LISTED IN (3)
We see that

$$U\mathbf{x} = U\left(\pi\left(\mathbf{1}\right)\right)\mathbf{x} = \pi_\psi\left(\mathbf{1}\right)\mathbf{x}_\psi = \mathbf{x}_\psi.$$

Let $\mathbf{B} + \mathcal{I} \in \mathcal{A}/\mathcal{I}$. Then, for $\mathbf{A} \in \mathcal{A}$

$$\begin{aligned}U\pi\left(\mathbf{A}\right)U^{-1}\left(\mathbf{B} + \mathcal{I}\right) &= U\pi\left(\mathbf{A}\right)\pi\left(\mathbf{B}\right)\mathbf{x} = U\pi\left(\mathbf{AB}\right)\mathbf{x} = \pi_\psi\left(\mathbf{AB}\right)\mathbf{x}_\psi \\ &= \pi_\psi\left(\mathbf{A}\right)\pi_\psi\left(\mathbf{B}\right)\mathbf{x}_\psi = \pi_\psi\left(\mathbf{A}\right)\left(\mathbf{B} + \mathcal{I}\right).\end{aligned}$$

Thus, once again, by density, continuity, and linearity, it follows that

$$\mathbf{A} \in \mathcal{A}, \pi_\psi(\mathbf{A}) = U\pi(\mathbf{A})U^{-1}.$$

This completes the proof. □

Now, with the Gelfand-Naimark-Segal Theorem in place, we are finally ready to prove the theorem that gives us the three Dirac-von Neumann axioms stated at the beginning of this section: the Gelfand-Naimark Theorem.

Theorem 6.4 (Gelfand-Naimark Theorem)**.** *There exists a separable Hilbert space H over $\mathbb{C}$ such that:*

(1) There exists an isometric *-isomorphism $\pi : \mathcal{A} \to H$.
(2) $\psi \in \mathcal{S}$ iff there exists a positive $\mathbf{\Psi} \in \mathcal{BL}(H)$ such that $\operatorname{Tr}\Psi = 1$ *and* $\psi(\mathbf{A}) = \operatorname{Tr}[\mathbf{\Psi}\pi(\mathbf{A})]$.

Proof. Step 1: Construct H.
By Theorem 6.2, there is a countable subset $\mathcal{F}$ of $\mathcal{S}$ that is dense in $\mathcal{S}$ in the weak-* topology. By the Gelfand-Naimark-Segal Theorem, for each $\psi \in \mathcal{F}$, there exists a *-representation π_ψ into a Hilbert space H_ψ satisfying properties (1), (2), and (3) of Theorem 6.3. Denote the cyclic vector of π_ψ by $\mathbf{x}_\psi$ as in Theorem 6.3. Define $H = \oplus_{\psi\in\mathcal{F}} H_\psi$[11].

Step 2: Prove H is separable.
By the Gelfand-Naimark-Segal Theorem, each H_ψ is separable, so because there are countably many H_ψs, and each H_ψ is separable, it follows that H itself is separable.

Step 3: Construct the isometric *-isomorphism.
Define $\pi : \mathcal{A} \to \mathcal{BL}(H)$ such that $\pi(\mathbf{A})$ is an operator on H that sends $\oplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi \in H$ to $\oplus_{\psi\in\mathcal{F}}[\pi_\psi(\mathbf{A})\mathbf{v}_\psi] \in H$. That is,

$$\pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right) = \bigoplus_{\psi\in\mathcal{F}}[\pi_\psi(\mathbf{A})\mathbf{v}_\psi].$$

Step 4: Prove that $\pi(\mathbf{A})$ is linear.
It is easy to check linearity:

$$\begin{aligned}
\pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi + \bigoplus_{\psi\in\mathcal{F}}\mathbf{w}_\psi\right) &= \pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}[\mathbf{v}_\psi + \mathbf{w}_\psi]\right)\\
&= \bigoplus_{\psi\in\mathcal{F}}[\pi_\psi(\mathbf{A})(\mathbf{v}_\psi + \mathbf{w}_\psi)]\\
&= \bigoplus_{\psi\in\mathcal{F}}[\pi_\psi(\mathbf{A})\mathbf{v}_\psi + \pi_\psi(\mathbf{A})\mathbf{w}_\psi]\\
&= \bigoplus_{\psi\in\mathcal{F}}[\pi_\psi(\mathbf{A})\mathbf{v}_\psi] + \bigoplus_{\psi\in\mathcal{F}}[\pi_\psi(\mathbf{A})\mathbf{w}_\psi]\\
&= \pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right) + \pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{w}_\psi\right).
\end{aligned}$$

You can check homogeneity similarly.

[11] See Proposition 5.3 and Definition 8.15.

Step 5: Prove that π is a *-homomorphism.
We leave it to the reader that π respects addition. For multiplication, we have

$$\begin{aligned}\pi(\mathbf{AB})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right) &= \bigoplus_{\psi\in\mathcal{F}}\left[\pi_\psi(\mathbf{AB})\mathbf{v}_\psi\right] = \bigoplus_{\psi\in\mathcal{F}}\left[\pi_\psi(\mathbf{A})\pi_\psi(\mathbf{B})\mathbf{v}_\psi\right] \\ &= \pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}\left[\pi_\psi(\mathbf{A})\mathbf{v}_\psi\right]\right) = \pi(\mathbf{A})\pi(\mathbf{B})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right).\end{aligned}$$

And for involution:

$$\begin{aligned}\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{w}_\psi|\pi(\mathbf{A}^*)\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right)\right) &= \left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{w}_\psi|\bigoplus_{\psi\in\mathcal{F}}\left[\pi_\psi(\mathbf{A}^*)\mathbf{v}_\psi\right]\right) \\ &= \left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{w}_\psi|\bigoplus_{\psi\in\mathcal{F}}\left[\pi_\psi(\mathbf{A})^*\mathbf{v}_\psi\right]\right) \\ &= \sum_{\psi\in\mathcal{F}}\left[\left(\mathbf{w}_\psi|\pi_\psi(\mathbf{A})^*\mathbf{v}_\psi\right)_\psi\right] \\ &= \sum_{\psi\in\mathcal{F}}\left[\left(\pi_\psi(\mathbf{A})\mathbf{w}_\psi|\mathbf{v}_\psi\right)_\psi\right] \\ &= \left(\bigoplus_{\psi\in\mathcal{F}}\left[\pi_\psi(\mathbf{A})\mathbf{w}_\psi\right]|\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right) \\ &= \left(\pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{w}_\psi\right)|\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right)\end{aligned}$$

Thus, by Proposition 5.1, $\pi(\mathbf{A}^*) = \pi(\mathbf{A})^*$, and hence π is a *-homomorphism.

Step 6: Prove that $\pi(\mathbf{A})$ is bounded
To prove that $\pi(\mathbf{A})$ is bounded, we see that

$$\begin{aligned}\left\|\pi(\mathbf{A})\left(\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right)\right\|^2 &= \left\|\bigoplus_{\psi\in\mathcal{F}}\left[\pi_\psi(\mathbf{A})\mathbf{v}_\psi\right]\right\|^2 = \sum_{\psi\in\mathcal{F}}\|\pi_\psi(\mathbf{A})\mathbf{v}_\psi\|^2 \\ &\leq \|\mathbf{A}\|^2\sum_{\psi\in\mathcal{F}}\|\mathbf{v}_\psi\|^2 = \|\mathbf{A}\|^2\left\|\bigoplus_{\psi\in\mathcal{F}}\mathbf{v}_\psi\right\|^2,\end{aligned}$$

where we have applied Proposition 4.4. Thus, $\pi(\mathbf{A}) \in \mathcal{BL}(H)$.

Step 7: Prove that $\mathcal{F}$ separates elements in $\mathcal{A}$.
We now show that $\mathcal{F}$ separates elements in $\mathcal{A}$. Equivalently, we may show that for nonzero $\mathbf{A} \in \mathcal{A}$, there exists a $\psi \in \mathcal{F}$ such that $\psi(\mathbf{A}) \neq 0$. So let $\mathbf{A} \in \mathcal{A}$ be nonzero. Now, by Proposition 4.3, there exists $\psi \in \mathcal{S}$ such that $\psi(\mathbf{A}) \neq 0$. Now, because $\mathcal{F}$ is dense in $\mathcal{S}$ in the weak-* topology, we can find a sequence $\psi_n \in \mathbf{F}$ such that $\psi_n(\mathbf{A})$ converges to $\psi(\mathbf{A}) \neq 0$. Thus, we can find some N so that

$$|\psi_N(\mathbf{A}) - \psi(\mathbf{A})| < |\psi(\mathbf{A})|,$$

and hence $\psi_N(\mathbf{A}) \neq 0$. Thus, $\mathcal{F}$ separates the elements of $\mathcal{A}$.

STEP 8: PROVE THAT π IS INJECTIVE.
Because π is linear, to prove π is injective, it suffices to show that $\pi(\mathbf{A}) = \mathbf{0}$ implies $\mathbf{A} = \mathbf{0}$. So suppose $\pi(\mathbf{A}) = \mathbf{0}$. This implies that each $\pi_\psi(\mathbf{A}) = \mathbf{0}$. We proceed by contradiction: suppose $\mathbf{A} \neq \mathbf{0}$. Then, because $\mathcal{F}$ separates the elements of $\mathcal{A}$, there is some $\psi \in \mathcal{F}$ such that

$$0 \neq \psi(\mathbf{A}) = (\mathbf{x}_\psi | \pi_\psi(\mathbf{A}) \mathbf{x}_\psi) = (\mathbf{x}_\psi | \mathbf{0}) = 0 :$$

a contradiction. Thus, π is injective, and hence $\pi : \mathcal{A} \to \pi(\mathbf{A})$ is a *-isomorphism.

STEP 8: PROVE THAT ϕ IS AN ISOMETRY.
Because π is a *-isomorphism, by Proposition 4.4, $\|\pi(\mathbf{A})\| = \|\mathbf{A}\|$. Thus, $\pi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{BL}(H)$, with H separable, and $\pi : \mathcal{A} \to \pi(\mathcal{A})$ is an *-isomorphic isometry.

STEP 9: PROVE (2).
It is easy to verify that if $\Psi \in \mathcal{BL}(H)$ is positive $\operatorname{Tr} \Psi = 1$, then $\operatorname{Tr}[\Psi\pi(\mathbf{A})]$ defines a state on $\mathcal{A}$. To prove the other direction, let $\psi \in \mathcal{S}$ and let ψ_n be a sequence in $\mathcal{F}$ converging to ψ in the weak-* topology. Now, for each ψ_n, define $\mathbf{\Psi_n} \in \mathcal{BL}(H)$ such that $\mathbf{\Psi_n}$ maps every vector in H_ϕ for $\phi \in \mathcal{F}$ not equal to ψ_n to $\mathbf{0}$ and agrees with the positive operator given in (2) of the Gelfand-Naimark-Segal Theorem on H_{ψ_n}. It is easy to check that $\mathbf{\Psi_n}$ is positive, $\operatorname{Tr} \mathbf{\Psi_n} = 1$, and $\operatorname{Tr}[\Psi_n \pi(\mathbf{A})] = \psi_n(\mathbf{A})$. Thus, if we define $\mathbf{\Psi}$ to be the limit of $\mathbf{\Psi_n}$, we have that

$$\psi(\mathbf{A}) = \lim \psi_n(\mathbf{A}) = \operatorname{Tr}[\mathbf{\Psi}_n \pi(\mathbf{A})] = \operatorname{Tr}[\mathbf{\Psi} \pi(\mathbf{A})].$$

Similarly, we also have that $\operatorname{Tr} \mathbf{\Psi} = 1$ and that Ψ is positive.

This completes the proof. □

Thus, finally, we have proven, from our C^*-algebraic axioms of quantum mechanics given in Axioms 3.1 and 3.2, that: to every quantum system there is an associated separable Hilbert space, the observables are the self-adjoint operators on this space, and that the states are positive operators of trace 1.

7. Closing Comments

A little should be said about this last statement, namely, the identification of states with positive operators of trace 1, as this is not the typical formulation of states. Although we do not have room to do so here, one can define the notion of a *pure state*, and then, prove that every pure state is a projection operator onto a one dimensional subspace. Thus, we can identify the pure states with elements in the Hilbert space, that is, we identify the pure state with the element in the Hilbert space of norm 1 contained in the one dimensional subspace the operator projects onto. This is the usual presentation of states in quantum mechanics; however, as it turns out, the manner of looking at states in the Hilbert space formulation given in Statement 6.3 is more mathematically convenient, and essentially the same as the usual presentation when the state is a pure state.

The reader should take note that this is far from a complete treatment. For one thing, there are still three other Dirac-von Neumann axioms (see [14], pgs. 66-73) that need to be proven from equivalent axioms in terms of C^*-algebra. In fact, it is clear from their statement, that we will actually need to assume more than we have if we wish to prove them. This, however, should not be seen as a problem. One should not expect to be able to derive all of quantum mechanics from two relatively simple axioms. However, I personally know of no natural axioms (although I'm sure they exist) that would fit nicely into our C^*-algebraic framework of quantum mechanics that would make the remaining three Dirac-von Neumann axioms provable. The second thing the reader should take note of is that, essentially, we have only outlined the general mathematical framework of quantum mechanics. In particular, we have not axiomized, for example, the canonical commutation relation, or equivalent. Of course, one would need to assume the canonical commutation relation in some form, or an equivalent statement, if they wished to develop all of quantum mechanics. We have not implemented this assumption in our axiomatic formulation for the same reason we have not assumed axioms that make the remaining three Dirac-von Neumann axioms provable: it was just simply not the purpose of this paper. Thus, the work contained in this paper is far from a complete treatment: three Dirac-von Neumann axioms remain to be proven, and the canonical commutation relation was not implemented in the theory. Nevertheless, it is the author's opinion that the ability to prove three of the Dirac-von Neumann axioms from extremely natural axioms taken from a study of classical mechanics is quite aesthetically pleasing.

8. Appendix: Definitions

I have decided to relegate most definitions to this appendix, with the idea in mind that most of the readers are probably familiar with most of the following definitions and it is best not to break the flow of the paper to present a definition the reader is probably already familiar with. In any case, most objects I use in the paper are defined below (in alphabetical order), so that the unfamiliar reader may reference them when needed.

Definition 8.1 (*-Algebra)**.** A **-algebra* an associative algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a unary operation $^* : \mathcal{A} \to \mathcal{A}$ such that for $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

(i) $(\mathbf{A} + \mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$
(ii) $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$
(iii) $(\alpha\mathbf{A})^* = \alpha^*\mathbf{A}^*$
(iv) $(\mathbf{A}^*)^* = \mathbf{A}$

Definition 8.2 (*-Homomorphism)**.** A **-homomorphism* is an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between two (unital) *-algebras $\mathcal{A}$ and $\mathcal{B}$ such that for $\mathbf{A} \in \mathcal{A}, \phi(\mathbf{A}^*) = \phi(\mathbf{A})^*$.

Definition 8.3 (*-Isomorphism)**.** A **-isomorphism* is a bijective *-homomorphism.

Definition 8.4 (*-Representation)**.** A **-representation* of a *-algebra $\mathcal{A}$ is a *-homomorphism from $\mathcal{A}$ to the *-algebra of bounded operators on a Hilbert space.

Notation. We will typically use the notation $\mathcal{BL}(V)$ to denote the C^*-algebra[12] of bounded operators on a normed vector space V.

Definition 8.5 (Adjoint)**.** Let H be a Hilbert space over $\mathbb{C}$ and let $\mathbf{A} \in \mathcal{BL}(H)$. Then, the *adjoint* of $\mathbf{A}$, denoted $\mathbf{A}^*$, is the unique bounded linear operator on H, whose existence is guaranteed by Proposition 5.1, such that $(\mathbf{Ay}|\mathbf{x}) = (\mathbf{y}|\mathbf{A}^*\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in H$.

Definition 8.6 (Algebra)**.** An *algebra* is a vector space $\mathcal{A}$ over a field F with an additional binary operation $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}$ and $\alpha, \beta \in \mathcal{A}$:

[12] See Proposition 5.2.

(i) (Left Distributivity) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$
(ii) (Right Distributivity) $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$
(iii) (Compatibility with Scalars) $(\alpha\mathbf{A}) \cdot (\beta\mathbf{B}) = (\alpha\beta)(\mathbf{A} \cdot \mathbf{B})$

Notation. This new binary operation is typically called *multiplication* and is often denoted by juxtaposition.

Definition 8.7 (Algebra Homomorphism)**.** An *algebra homomorphism* is a function $\phi : \mathcal{A} \to \mathcal{B}$ between two (unital) algebras $\mathcal{A}$ and $\mathcal{B}$ over a field F such that for $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $\alpha \in \mathbb{F}$:
(i) $\phi(\mathbf{A} + \mathbf{B}) = \phi(\mathbf{A}) + \phi(\mathbf{B})$
(ii) $\phi(\mathbf{AB}) = \phi(\mathbf{A})\,\phi(\mathbf{B})$
(iii) $\phi(\alpha\mathbf{A}) = \alpha\phi(\mathbf{A})$
(iv) (Only if $\mathcal{A}$ and $\mathcal{B}$ are unital) $\phi(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$

Definition 8.8 (Algebra Isomorphism)**.** A *algebra isomorphism* is a bijective algebra homomorphism.

Definition 8.9 (Associative Algebra)**.** We say that an algebra is *associative* iff the multiplication is associative.

Definition 8.10 (Banach Algebra)**.** A *Banach algebra* is an associative algebra $\mathcal{A}$ over a normed field F equipped with a norm $\|\cdot\|$ such that:
(i) (Completeness) The resulting normed linear space is complete.
(ii) (Compatibility with Norm) For $\mathbf{A}, \mathbf{B} \in \mathcal{A}$, $\|\mathbf{AB}\| \le \|\mathbf{A}\|\,\|\mathbf{B}\|$.

Definition 8.11 (C^*-Algebra)**.** A *C^*-algebra* is a *-algebra $\mathcal{A}$ that is also a Banach algebra such that for $\mathbf{A} \in \mathcal{A}$, $\|\mathbf{AA}^*\| = \|\mathbf{A}\|^2$.

Definition 8.12 (Commutative Algebra)**.** We say that an algebra is *commutative* iff the multiplication is commutative.

Definition 8.13 (Compact Operator)**.** Let V be a normed vector space and let $\mathbf{A} \in \mathcal{L}(V)$. Then, we say that $\mathbf{A}$ is *compact* iff $\mathbf{A}$ is bounded and the image of every bounded subset of V under $\mathbf{A}$ is relatively compact.

Definition 8.14 (Cyclic Vector of a C^*-Representation)**.** Let $\mathcal{A}$ be a C^*-algebra, let H be a Hilbert space, and let $\pi : \mathcal{A} \to \mathcal{BL}(H)$ be a *-representation of $\mathcal{A}$. Then, we say that $\mathbf{x} \in H$ is a *cyclic vector* of π iff the set $\{\pi(\mathbf{A})\,\mathbf{x} | \mathbf{A} \in \mathcal{A}\} = [\pi(\mathcal{A})]\,\mathbf{x}$ is dense in H.

Definition 8.15 (Direct Sum of Hilbert Spaces)**.** Let I be an index set and let $\{H_i | i \in I\}$ be a collection of Hilbert spaces. Then, the Hilbert space constructed in Proposition 5.3 is the *direct sum of the collection of H_is.*

Notation. For a collection $\{H_i | i \in I\}$, we shall typically denote their direct sum as $\oplus_{i \in I} H_i$.

Definition 8.16 (Finite Rank Operator)**.** Let V be a vector space and let $\mathbf{A} \in \mathcal{L}(V)$. Then, $\mathbf{A}$ is of *finite rank* iff the dimension of the range of $\mathbf{A}$ is finite.

Definition 8.17 (Generation of a C^*-Algebra)**.** Let $\mathcal{A}$ be a unital C^*-algebra and let S be a finite subset of $\mathcal{A}$. Then, the C^*-algebra *generated* by S is the closure of the set of all polynomials of elements in $S \cup S^*$ equipped with the operations from the original C^*-algebra.

Definition 8.18 (Ideal of an Algebra)**.** Let $\mathcal{A}$ be an algebra. We say that a subset $\mathcal{I}I \subseteq \mathcal{A}$ is a left/right *ideal* iff I is a subspace of $\mathcal{A}$ (when considered just as a vector space) and for $\mathbf{A} \in \mathcal{I}$ and $\mathbf{B} \in \mathcal{A}$, $\mathbf{BA} \in \mathcal{I}/\mathbf{AA} \in \mathcal{I}$.

Definition 8.19 (Multiplicative Linear Functional)**.** A *multiplicative linear functional* on an algebra $\mathcal{A}$ over $\mathbb{C}$ is a algebra homomorphism from $\mathcal{A}$ to $\mathbb{C}$.

Definition 8.20 (Normal Element)**.** Let $\mathcal{A}$ be a *-algebra and let $\mathbf{A} \in \mathcal{A}$. Then, we say that $\mathbf{A}$ is normal iff A commutes with $\mathbf{A}^*$.

Definition 8.21 (Normalized)**.** Let V be a normed vector space and let $\mathbf{v} \in V$. Then, we say that $\mathbf{v}$ is normalized iff $\|\mathbf{v}\| = 1$.

Definition 8.22 (Positivity of an Element of a *-Algebra)**.** Let $\mathcal{A}$ be a *-algebra. Then, we say that $\mathbf{A} \in \mathcal{A}$ is *positive* iff $\mathbf{A} = \mathbf{B}^*\mathbf{B}$ for some $\mathbf{B} \in \mathcal{A}$.

Definition 8.23 (Positivity of a Linear Functional)**.** Let $\mathcal{A}$ be a *-algebra an let $\phi : \mathcal{A} \to \mathbb{C}$ be a linear functional. Then, we say that ϕ is *positive* iff $\phi(\mathbf{A}) \geq 0$ for all positive $\mathbf{A} \in \mathcal{A}$.

Definition 8.24 (Self-Adjoint)**.** Let $\mathcal{A}$ be a *-algebra and let $\mathbf{A} \in \mathcal{A}$. Then, we say that $\mathbf{A}$ is *self-adjoint* iff $\mathbf{A} = \mathbf{A}^*$.

Definition 8.25 (Spectrum)**.** Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let $\mathbf{A} \in \mathcal{A}$. Then, the *spectrum* of $\mathbf{A}$, denoted $\sigma_{\mathcal{A}}(\mathbf{A})$, is the set of complex numbers λ such that $\mathbf{A} - \lambda\mathbf{1}$ is not invertible in $\mathcal{A}$.

Notation. When it causes no confusion, we may omit the subscript on $\sigma_{\mathcal{A}}(\mathbf{A})$.

Definition 8.26 (State)**.** Let $\mathcal{A}$ be a normed *-algebra and let ψ be a linear functional on $\mathcal{A}$. Then, we say that ψ is a *state* iff ψ is positive and normalized.

Definition 8.27 (Trace)**.** Let H be a separable inner product space and let $\mathbf{A} \in \mathcal{L}(H)$ be trace-class. By Proposition 5.4, H has a countable orthonormal basis $\{\mathbf{e_n} | n \in \mathbb{N}\}$. Then, the *trace* of $\mathbf{A}$ is defined as $\sum_{n\in\mathbb{N}} (\mathbf{e_n}|\mathbf{Ae_n})$, which is finite and well-defined by Proposition 5.5.

Notation. For $\mathbf{A} \in \mathcal{L}(H)$ trace-class, we denote the trace of $\mathbf{A}$ by $\operatorname{Tr}\mathbf{A}$.

Definition 8.28 (Trace-Class)**.** Let H be a separable inner product space and let $\mathbf{A} \in \mathcal{L}(H)$. Then, we say that $\mathbf{A}$ is *trace-class* iff $\mathbf{A} = \mathbf{BC}$ for Hilbert-Schmidt operators $\mathbf{B}$ and $\mathbf{C}$.

Definition 8.29 (Unital Algebra)**.** We say that an algebra is *unital* iff there exists a multiplicative identity.

Acknowledgments. It is a pleasure to thank my mentors, Catalin Carstea, Thomasz Zamojski, and Brent Werness for their contributions to this paper, without which, it would not have been possible.

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