Brown-Gitler Spectra – Brown-Gitler spectra were introduced by E.H. Brown, Jr. and Samuel Gitler [1] to study higher order obstructions to immersions of manifolds, but immediately found wide applicability in a variety of areas of homotopy theory, most notably in the stable homotopy groups of spheres ([9] and [4]), in studying homotopy classes of maps out of various classifying spaces ([3], [10], and [8]), and, as might be expected, in studying the Immersion Conjecture for manifolds ([2] and [5]).

The mod p homology $H_*X = H_*(X, \mathbb{Z}/p\mathbb{Z})$ comes equipped with a natural right action of the Steenrod algebra $\mathcal{A}$ which is *unstable*: at the prime 2, for example, this means

$$0 = \mathrm{Sq}^i : H_nX \to H_{n-i}X, \qquad 2i > n.$$

Write $\mathcal{U}_*$ for the category of all unstable right modules over $\mathcal{A}$. This category has enough projectives; indeed, there is an object $G(n)$, $n \geq 0$, of $\mathcal{U}_*$ and a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}_*}(G(n), M) \cong M_n$$

where M_n is the vector spaces of elements of degree n in M. The module $G(n)$ can be explicitly calculated. For example, if $p = 2$ and $x_n \in G(n)_n$ is the universal class, then the evaluation map $\mathcal{A} \to G(n)$ sending θ to $x_n\theta$ defines an isomorphism

$$\Sigma^n\mathcal{A}/\{\mathrm{Sq}^i \ : \ 2i > n\}\mathcal{A} \cong G(n).$$

These are the *dual Brown-Gitler modules.*

This pleasant bit of algebra can be only partly reproduced in algebraic topology. For example, for general n there is no space whose (reduced) homology is $G(n)$; specifically, if $p = 2$, the module $G(8)$ cannot support the structure of an unstable *coalgebra* over the Steenrod algebra. However, after stabilizing, this objection does not apply and we have the following result from [1],[4],[6]: there is a unique p-complete spectrum $T(n)$ so that $H_*T(n) \cong G(n)$ and for all pointed CW complexes Z, the map

$$[T(n), \Sigma^\infty Z] \to \bar{H}_nZ$$

sending f to $f_*(x_n)$ is surjective. Here $\Sigma^\infty Z$ is the suspension spectrum of Z, the symbol [,] denotes stable homotopy classes of maps, and $\bar{H}$ is reduced homology. The spectra $T(n)$ are the *dual Brown-Gitler spectra.* The *Brown-Gitler spectra* themselves can be obtained by the formula

$$B(n) = \Sigma^n DT(n)$$

where D denotes the Spanier-Whitehead duality functor. The suspension factor is a normalization introduced to put the bottom cohomology class of $B(n)$ in degree 0. An easy calculation shows that $B(2n) \simeq B(2n+1)$ for all primes and all $n \geq 0$.

For a general spectrum X and $n \not\equiv \pm 1$ modulo $2p$, the group $[T(n), X]$ is naturally isomorphic to the group $D_nH_*\Omega^\infty X$ of homogeneous elements of degree n in the Cartier-Dieudonné module $D_*H_*\Omega^\infty X$ of the abelian Hopf algebra $H_*\Omega^\infty X$. In fact, one way to construct the Brown-Gitler spectra is to note that the functor

$$X \mapsto D_{2n}H_*\Omega X$$

is the degree $2n$ group of an extraordinary homology theory; then $B(2n)$ is the p-completion of the representing spectrum. See [6]. This can be greatly, but not completely, destabilized. See [7].

References

[1] Brown, Edgar H., Jr; Gitler, Samuel, "A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra," *Topology* 12 (1973), 283–295.

[2] Brown, E.H., Jr; Peterson, F.P. "A universal space for normal bundles of n-manifolds, *Comment. Math. Helv.* 54 (1979), no. 3, 405–430.

[3] Carlsson, Gunnar, "G.B. Segal's Burnside ring conjecture for $(\mathbb{Z}/2)^k$, *Topology* 22 (1983) no. 1, 83-103.

[4] Cohen, Ralph L., "Odd primary infinite families in stable homotopy theory", *Mem. Amer. Math Soc.* 30 (1981), no. 242.

[5] Cohen, Ralph L., "The immersion conjecture for differentiable manifolds", *Ann. of Math. (2)* 122 (1985), no. 2, 237-328.

[6] Goerss, P.; Lannes, J.; Morel, F., "Hopf algebras, Witt vectors, and Brown-Gitler spectra", *Algebraic Topology (Oaxtepec, 1991)*, 111–128, *Contemp. Math. 146*, Amer. Math. Soc., Providence, RI, 1993.

[7] Goerss, Paul; Lannes, Jean; Morel, Fabien, "Vecteurs de Witt non-commutatifs et représentabilité de l'homologie modulo p", *Invent. Math.* 108 (1992), no. 1, 163–227.

[8] Lannes, Jean, "Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire", *Inst. Hautes Études Sci. Publ. Math*, no. 75, (1992), 135-244.

[9] Mahowald, Mark, "A new infinite family in ${}_2\pi_*^s$, *Topology* 16 (1977), no. 3, 249-256.

[10] Miller, Haynes, "The Sullivan conjecture on maps from classifying spaces", *Ann. of Math. (2)* 120 (1984), no. 1, 39-87.