Minimal models of diagrams of spaces

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The talk is based on the recent work "On DG-models for diagrams of spaces"

The homotopy groups $\pi_k(X) = [S^k, X]^*$ constitute fundamental invariants for connected spaces, enumerating deformation classes of k-dimensional loops in X. Weak equivalences are characterized as maps $f : X \to Y$ inducing isomorphisms $\pi_k(f)$ for all k
.

Classical homotopy theory emerges from inverting weak equivalences in Top, formalized through the localization

$$\mathsf{Top} \mapsto \mathsf{Top}[W^{-1}] \equiv \mathsf{ho}(\mathsf{Top}),$$

This localization formally adjoins inverses for all weak homotopy equivalences, thus creating the world where we disregard all information that is not preserved by weak equivalences.

The homotopy category $\mathsf{Ho}(\mathsf{Top})$ thus obtained has several equivalent realisations:

- As the category of CW-complexes (spaces obtained by inductively attaching cells along the boundaries) with homotopy classes of maps between them.
- (Combinatorial description) Let Δ be a category of finite ordinals with monotone maps. Then $\mathsf{Set}^{\Delta^{op}} \equiv \mathsf{sSet}$ is the category of simplicial sets.
  There is an adjunction pair $(|-|, \mathsf{Sing}) : \mathsf{sSet} \to \mathsf{Top}$ determined by the values of the geometric realization (left adjoint) $|\Delta^n|$ on simplices (representable $\Delta^{op} \to \mathsf{sSet}$).
  Then, $\mathsf{ho}(\mathsf{Top})$ is equivalent to the homotopy category of simplicial sets obtained by

localization by the maps whose geometric realization are weak equivalences $\mathsf{ho(Top)} \cong \mathsf{ho(sSet)}$

Rational homotopy theory and Sullivan models

Homotopy theory is highly complicated. So one studies it by approximations/completions. Among such approximations are further localizations by larger classes of maps than weak equivalences. Moreover, such approximations can fully determine homotopy types.

- Let us restrict to simply-connected spaces. Everything here works for nilpotent spaces.

The arithmetic square represents a finite simply-connected space X as a homotopy pullback:

$$\begin{array}{ccc} X & \longrightarrow & X^{\wedge} \\ \downarrow & & \downarrow \\ \mathrm{L}_{\mathbb{Q}}X & \longrightarrow & \mathrm{L}_{\mathbb{Q}}X^{\wedge} \end{array}$$

where $X^{\wedge}$ is completion and $\mathrm{L}_{\mathbb{Q}}$ is rationalization.

Rationalization acts as a reflector on the homotopy category:

$$\mathrm{L}_{\mathbb{Q}} : \mathsf{ho}(Top) \to \mathsf{Top}[W_{\mathbb{Q}}^{-1}] \subset \mathsf{ho}(Top)$$

where the codomain is the category of rational homotopy types.

This is equivalent to localizing Top at rational weak equivalences—maps $f : X \to Y$ that induce isomorphisms on rationalized homotopy groups:

$$\pi_k(f) \otimes \mathbb{Q} : \pi_k(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_k(Y) \otimes \mathbb{Q}$$

Thus, rationalization serves as the fundamental approximation of a space's homotopy type.

Homotopy theory employs algebraic topology by constructing algebraic invariants of spaces.

- Classical examples include homotopy groups and graded-commutative cohomology algebras $H^*(X)$.
- Such invariants are functors $I : \mathsf{Ho}(\mathsf{Top}) \to \mathcal{A}$ into categories of algebraic objects.
- An invariant is called $\mathbf{complete}$ if it induces an equivalence between $\mathsf{Ho}(\mathsf{Top})$ and a (subcategory of) $\mathcal{A}$.

Rational homotopy theory turns out to be drastically simpler than the classical homotopy theory.

- Since Serre's seminal computation of the rational homotopy groups of spheres $\pi_*(S^{2n+1}) \otimes \mathbb{Q} = \mathbb{Q}, * = 2n+1$ and zero otherwise; $\pi_*(S^{2n}) \otimes \mathbb{Q} = \mathbb{Q}, * = 2n, 4n-1$ and zero otherise.

it has been recognized that rational homotopy theory itself may admit a concise description through algebraic models.

- Thus the problem was to construct complete rational homotopy invariants using some algebraic category. Such complete invariants are referred to as algebraic models.

The theory of algebraic models for rational homotopy types was initiated by Quillen, and he established the equivalence of homotopy categories

$$\mathsf{ho}(\mathsf{Top}_{\mathbb{Q}}^{\geq 2}) \cong \mathsf{ho}(\mathsf{dgLie}_{\mathbb{Q}}^{\geq 1}) \cong \mathsf{ho}(\mathsf{DGcoA}_{\mathbb{Q}}^{\geq 1})$$

In this approach, the differential graded (dg)-coalgebraic model for a simply-connected rational homotopy type is provided by the singular chain complex $\mathcal{C}_*(X;\mathbb{Q})$.

- For finite homotopy types, one may also consider the cochain dg-algebra $\mathcal{C}^*(X;\mathbb{Q})$ on an equal footing.

However, this latter model is, as is well known, not commutative.

On the other hand, in the context of smooth manifolds one has the de Rham complex $\mathbf{\Omega}^*_{\mathsf{dR}}(X)$.

- This complex is a commutative differential-graded algebra
- possesses an intuitive geometric interpretation
- The connection with homotopy theory is provided by de Rham theorem. This theorem establishes an isomorphism $H^k(\mathbf{\Omega}^*_{\mathsf{dR}}(X)) \cong H^k(X;\mathbb{R})$,

showing that the singular $\mathbb{R}$-valued cohomology $H^*(X;\mathbb{R})$ can be computed via de Rham cohomology.

- An analog of de Rham complex for general spaces is PL de Rham complex

The functor $\mathcal{A}_{\mathsf{PL}} : \mathsf{sSet} \to \mathsf{DGCA}^{op}$

generalizing the de Rham complex was defined in the work of Bousfield-Gugenheim which on a finite ordinal $[n]$ is defined by:

$$\mathcal{A}_{\mathsf{PL}}(\Delta^k) := \mathbb{Q}[t_0^{(0)}, \ldots t_0^{(k)}, dt_1^{(0)}, \ldots, dt_1^{(k)}]/(\Sigma_i t_i^{(0)} = 1, d(t_0^{(i)}) = dt_1^{(i)})$$

The right adjoint is

$$\mathcal{K} : \mathsf{DGCA}^{op} \to \mathsf{sSet},$$

where

$$\mathcal{K}(B) = \mathsf{DGCA}(B, \mathcal{A}_{\mathsf{PL}}(\Delta^*))$$

- A central innovation in this framework was the introduction of minimal models, i.e. certain DGCAs that are equivalent to PL de Rham but are much smaller!

Minimal models

For a commutative differential graded algebra $\mathcal{A}(X)$ (e.g. such as the PL forms $\mathcal{A}_{\mathrm{PL}}(Sing(X))$, a $\mathbf{Sullivan\ minimal\ model}$ is a DGCA $\mathcal{M}(X)$ satisfying specific algebraic conditions, equipped with a quasi-isomorphism $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{A}_{\mathrm{PL}}(X)$.

There are two equivalent approaches to defining minimal models:

Algebraic definition

Let (A, d) be a connected DGCA whose underlying graded-commutative algebra is free; that is, $A \cong \mathsf{Sym}(W)$ for some graded vector space W. Such an algebra is called a Sullivan algebra if W admits a filtration by graded subspaces

$$W(0) \subset W(1) \subset \cdots \subset W(k) \subset \dots$$

with $\bigcup_k W(k) = W$ and $d(W(k)) \subset \mathsf{Sym}(W(k-1))$ for all k. A Sullivan algebra is minimal if its differential is decomposable, meaning $d(W(k)) \subset \mathsf{Sym}(W(k-1)) \cap \mathsf{Sym}^{\geq 2} W$.

Geometric definition

An elementary extension of DGCAs is a pushout:

$$\begin{array}{ccc}
\mathsf{Sym}(\underset{k}{W}^{\vee}) & \longrightarrow & \mathcal{M} \\
\big\downarrow & & \big\downarrow \\
\mathsf{Sym}(\underset{k}{W}^{\vee} \overset{\mathsf{id}}{\to} \underset{k}{W}^{\vee}) & \longrightarrow & \mathcal{M}(W)
\end{array}$$

for some vector space W at degree k.

A Sullivan minimal model $\mathcal{M}$ is such DGCA that is obtained by a sequence of elementary extensions with the base of induction $\mathcal{M}(0) = \mathbb{Q}$ and each W is attached increasingly degree by degree.

- These definitions are equivalent for 1-connected $\mathcal{M}$.

A key feature of the dg approach to rational homotopy theory is that minimal models are constructed by a process dual to reconstructing a homotopy type from its truncations via a Postnikov tower.

Consider

$$X_{n+1} \to X_n \overset{k^{n+2}}{\to} K(\pi_{n+1}(X), n+2)$$

here X_n is n-th truncation of a homotopy type of X.

The k-invariant k^{n+2} of the space X corresponds dually to a homotopy cofiber sequence, which is modelled by

$$\begin{array}{ccc}
\mathsf{Sym}\big(\underset{n+2}{\pi_{n+1}(X)^\vee}\big) & \xrightarrow{\quad \tilde{k}^{n+2} \quad} & \mathcal{M}(X_n) \\
\big\downarrow & & \big\downarrow \\
\mathsf{Sym}\big(\underset{n+1}{\pi_{n+1}(X)^\vee} \overset{\mathrm{id}}{\to} \underset{n+2}{\pi_{n+1}(X)^\vee}\big) & \longrightarrow & \mathcal{M}(X_{n+1})
\end{array}$$

where $\mathcal{M}(X_n)$ is model for n-th truncation and $\tilde{k}^{n+2}$ is a representative of a cocycle $k^{n+2} \in H^{n+2}(X_n; \pi_{n+1}(X))$

- For each space X one may construct its minimal model $\mathcal{M}(X)$ by a sequence of elementary extensions by induction from the PL de Rham complex $\mathcal{A}$:

  $$\mathcal{M}(k) \subset \mathcal{M}(k+1) \subset \ldots \bigcup \mathcal{M}(k) = \mathcal{M},$$

  where $\mathcal{M}(2) = \mathsf{Sym}(H^2(\mathcal{A}))$ with the trivial differential and with $\mathcal{M}(2) \to \mathcal{A}$ being isomorphism on the second cohomology, defined by choosing the representatives of cocycles.

Assume inductively that we have constructed $\mathcal{M}(k)$ so that the map $\mathcal{M}(k) \to \mathcal{A}$ induces an isomorphism on cohomology in degrees $\leq k$ and a monomorphism in degree $k+1$. By applying elementary extensions we kill the cokernel in degree $k+1$ cohomology and eliminate the kernel in degree $k+2$

$$\mathcal{M}(k)(\mathsf{Cok}(H^{k+1}\mathcal{M}(k) \to H^{k+1}\mathcal{A}), \mathsf{ker} H^{k+2}) \equiv \mathcal{M}(k+1)$$

The main result of this theory is equivalence

$$\mathsf{Ho}(\mathsf{sSet})^{\geq 1,\mathsf{nil}}_{\mathsf{fin},\mathbb{Q}} \cong \mathsf{Ho}(\mathsf{DGCA}^{\geq 1}_{\mathsf{fin},\mathbb{Q}})^{op} \cong \mathsf{SullModels}^{\geq 1}_{\mathbb{Q}} / \sim$$

and:

- Homotopy equivalent minimal models must be isomorphic.(rigidity). So the minimal models are complete rational invariants! They are isomorphic iff the the two spaces are rationally homotopy equivalent.
- There is a bijective correspondence between rational $\mathbb{Q}$-finite (nilpotent) homotopy types and isomorphism classes of minimal Sullivan algebras.

Thus, rational homotopy theory becomes entirely algebraic and computable.

On the model structures

The above equivalences can be established using the model structure the category of DGCAs:

Consider the transfer via free-forgetful adjunction

$$\mathsf{Ch}^{\geq 0}(\mathbb{Q})_{\mathsf{inj}} \underset{\mathcal{U}}{\overset{\mathsf{Sym}}{\rightleftarrows}} (\mathsf{DGCA}^{\geq 0}_{\mathbb{Q}})_{\mathsf{trinj}}$$

- In this model structure, the weak equivalences and fibrations are precisely those maps which are weak equivalences and fibrations, respectively, in the underlying injective model structure on cochain complexes, i.e. degreewise surjections and quasi-isomorphisms

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The adjunction:

$$(\mathcal{A}_{PL}, \mathcal{K}) : \mathsf{sSet} \to (\mathsf{DGCA}_{\mathbb{Q}}^{\geq 0})^{op},$$

is Quillen and it yields the derived adjunction at the level of homotopy categories:

$$\mathsf{Ho}(\mathsf{sSet}) \underset{\mathbb{D}\mathcal{K}}{\overset{\mathbb{D}\mathcal{A}_{LP}}{\rightleftarrows}} \mathsf{Ho}((\mathsf{DGCA}_{\mathbb{Q}}^{\geq 0})^{op})$$

The theorem holds:

Theorem 1. [5, Theorem 11.2., Theorem 9.4.] On connected, nilpotent, $\mathbb{Q}$-finite types

1. The unit of the derived adjunction (4) is rationalization

$$X \to (\mathbb{D}\mathcal{K} \circ \mathbb{D}\mathcal{A}_{\mathsf{PL}})(X) \cong \mathrm{L}_{\mathbb{Q}} X.$$

2. The derived adjunction (4) restricts to an equivalence of categories:

$$\begin{array}{ccc}
\mathsf{Ho}(\mathsf{sSet}^{\geq 1,\mathsf{nil}}_{\mathsf{fin}_{\mathbb{Q}}}) & \xrightarrow{\mathbb{D}\mathcal{A}_{\mathsf{PL}}} & \\
\uparrow \downarrow \mathrm{L}_{\mathrm{k}} & & \\
\mathsf{Ho}(\mathsf{sSet}^{\geq 1,\mathsf{nil}}_{\mathsf{fin},\mathbb{Q}}) & \underset{\mathbb{D}\mathcal{K}}{\overset{\cong}{\rightleftarrows}} & \mathsf{Ho}(\mathsf{DGCA}^{\geq 1}_{\mathsf{fin},\mathbb{Q}})^{op}.
\end{array}$$

Why minimal models work so fine?

Because they are bifibrant in the model category $(\mathsf{DGCA}_{\mathbb{Q}}^{\geq 0})^{op}$ and they are "rigid".

The following properties of the class of minimal models provide the above equivalence and in the quest of generalising the theory of minimal models to homotopy types of diagrams of spaces they will prove useful:

- For any fibrant, cohomologically connected $A \in \mathsf{DGCA}_{\mathbb{Q}}^{\geq 0}$ of finite type, there exists a minimal model $\mathcal{M}$ and a quasi-isomorphism $\mathcal{M} \xrightarrow{\sim} A$.
- Minimal models are fibrant-cofibrant objects in $(\mathsf{DGCA}_{\mathbb{Q}}^{\geq 0})_{\mathsf{trinj}}$.
- (Rigidity) For minimal models, homotopy equivalence is an isomorphism. Weakly, any quasi-isomorphism between minimal models is homotopic to an isomorphism.

On the structure of minimal models
As a differential graded-commutative algebra a minimal model $\mathcal{M}(X)$ has a relatively simple form.

As graded-commutative algebra it is free, i.e. there is a graded-vector space V such that

$$\mathcal{M}(X) = \mathsf{Sym}(V)$$

where on the right we have the free symmetric algebra spanned on generators V.

The differential is precisely determined by its value on generators. So we shall use the notation for presentations of minimal models:

$$(V|d : V \to \mathsf{Sym}(V))$$

Invariants from minimal models

All rational invariants of a space X can be extracted from its minimal model $\mathcal{M}(X)$.

For instance, rational homotopy groups and the graded-commutative cohomology $\mathbb{Q}$-algebra of X, can be extracted from the minimal model:

$$(\pi_k(X) \otimes \mathbb{Q})^\vee = V,$$

here $(-)^\vee$ indicates the dual vector space.

For cohomology:

$$H^*(\mathcal{M}(X)) \cong H^*(X; \mathbb{Q})$$

as graded-commutative $\mathbb{Q}$-algebras.

Examples of minimal models of some spaces

- (Eilenberg-Maclane spaces) $\mathcal{M}(K(Z,n)) = (x_n | dx_n = 0)$
- (spheres) $\mathcal{M}(S^n) = (x_n | dx_n = 0)$, if n is odd;
- $\mathcal{M}(S^{2k}) = (x_{2k}, x_{4k-1} | dx_{4k-1} = x_{2k})$
- (Complex projective spaces) $\mathcal{M}(\mathbb{C}P^n) = (x_2, h_{2n+1} | dh_{2n+1} = (x_2)^{n+1})$
- (Quaternion projective spaces) $\mathcal{M}(\mathbb{H}P^n) = (x_4, h_{4n+3} | dh_{4n+3} = (x_4)^{n+1})$
- (some connected sums) $\mathcal{M}(\mathbb{C}P^2 \# \mathbb{C}P^2) = (f_2, g_2, z_3, t_3 | dt_3 = f_2 g_2, dz_3 = f_2^2 - g_2^2)$

- (H-spaces, infinite loop spaces are the same)
  H-spaces have the minimal models with trivial differential $(V|d=0)$ and thus any H-space X rationally splits onto a product

$$X \simeq_{\mathbb{Q}} \Pi_i K(\mathbb{Z}, n_i)$$

- This is not the case when we shall pass to the rational homotopy theory of diagrams of spaces!

Rational homotopy theory of diagrams

In practice, one often must work not merely with homotopy types of spaces, but with **diagrams of spaces.** We recall a few examples of such cases:

When working with **singular spaces**, such as:

- **Orbifolds**: Singular manifolds that locally look like quotients of $\mathbb{R}^n$ by finite group actions (e.g., quotients X/G with fixed points). For instance, $\mathbb{C}P^3$ with an action of $\mathbb{Z}_2$ by $[z_1 : z_2 : z_3 : z_4] \mapsto [z_3 : z_4 : z_1 : z_2]$
  with $\mathbb{C}P^1 \subset \mathbb{C}P^3$ fixed points. These points become singular when one passes to the quotient $\mathbb{C}P^3/\mathbb{Z}_2$.

Working with naive quotient spaces like X/G is often too coarse and fruitless. Instead, one must "resolve" the singularities by constructing a diagram of fixed-point manifolds ($H \leq G \mapsto X^H$), which encodes the full orbifold structure of X. And then one studies the homotopy types of associated diagrams.

Stratifolds: Spaces decomposed into strata, each of which is a manifold.

Figure 1.1: A stratified space (to the left) and separate sketches of its strata (to the right)

here one studies homotopy types of diagrams of the form

$$\mathsf{sd}(P)^{op} \to \mathsf{sSet},$$

where P is a stratification poset and $sd(P)$ its subdivision (category of regular flags $p_0 < p_1 < \dots p_n$)

- **Isotopy classes of embeddings**
  Another example involves isotopy classes of embeddings. A homotopy type of a submanifold embedding $N \to M$ does not, in general, determine its isotopy class (take $S^n \to S^m, n < m$).

The homotopy type of the complement $M \setminus N$ can be highly complex, even when the embedding is homotopically trivial. To capture this information, one can associate the following pushout diagram with each embedding:

$$\begin{array}{ccc} \partial T & \longrightarrow & \overline{M \setminus T} \\ \downarrow & & \downarrow \\ T & \longrightarrow & M \end{array}$$

Here, T is the closed tubular neighbourhood containing N which deformationally retracts onto N (thus has the same homotopy type) and the closure $\overline{M \setminus T}$ deformationally retracts onto the complement $M \setminus N$.

- Below we shall provide an example of a minimal model of such a diagram for $\mathbb{C}P^1 \subset \mathbb{C}P^3$

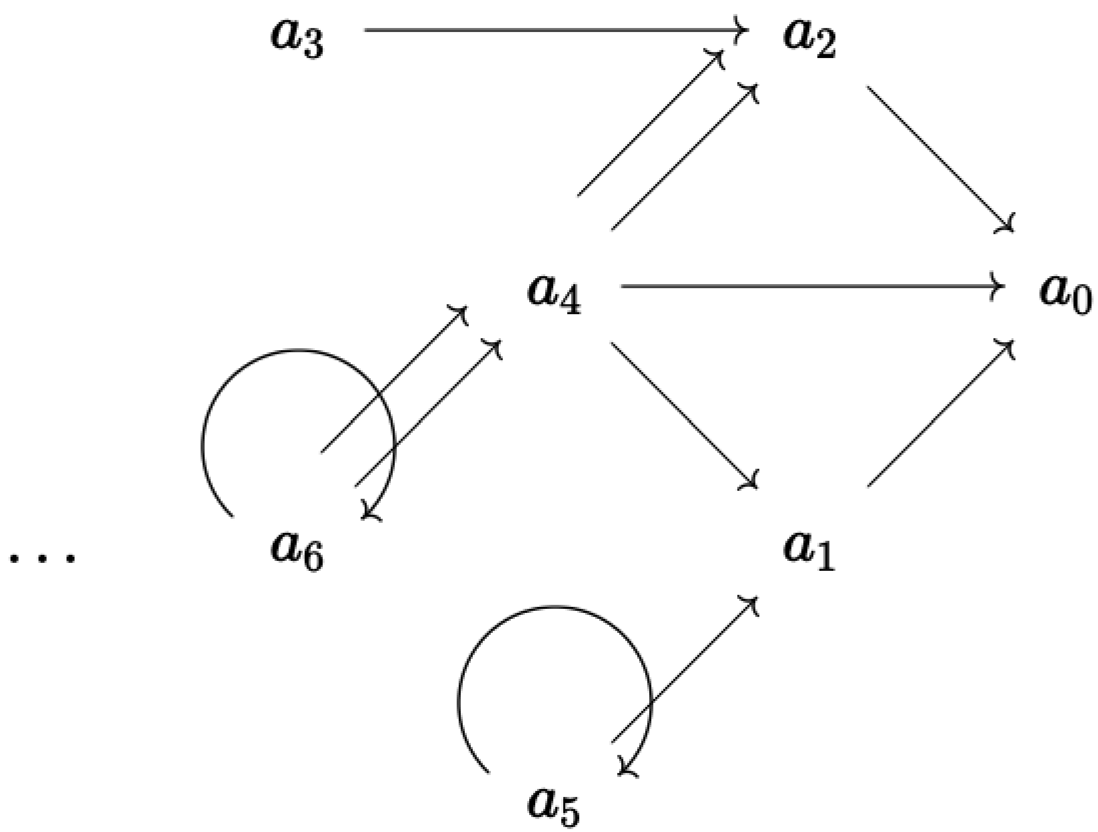
Homotopy theory of diagrams indexed by a category $\mathcal{C}$ is the study of the homotopy category of $\mathcal{C}$-shaped diagrams of spaces:

$$\mathsf{ho}(\mathsf{Fun}(\mathcal{C}^{op}, \mathsf{sSet}))$$

obtained by inverting objectwise weak equivalences.

Our work presents a broad generalization of rational homotopy theory to functors over arbitrary "good" categories $\mathcal{C}$. This framework encompasses equivariant theories for orbifolds, stratified spaces, commutative squares as above, and other structured contexts.

- $\mathcal{C}$ is a good category provided it is finite and inversely directed, with all endomorphisms invertible.



Examples include lots of posets, orbit categories Orb_G of finite groups, with conjugations and inclusions as morphisms...

For a diagram of spaces $\mathcal{X}$ rational homotopy groups are systems of vector spaces

$$\underline{\pi}_n(\mathcal{X}) : (c \in C^{op}) \mapsto \pi_n(\mathcal{X}(c))$$

- Given a system of vector spaces

  $$W : C^{op} \to \mathsf{Vect}_{\mathbb{Q}}$$

  there is an Eilenberg-Maclane diagram $K(W, n)$ for each n with the only nontrivial system of homotopy groups being

  $$\underline{\pi}_n(K(W, n)) = W$$

Lift of the PL de Rham

Define the transferred model structure on the category of C-diagrams of DGCAs

$$\mathsf{DGCA}^{\geq 0}_{\mathbb{Q}}(C)_{\mathsf{trinj}}$$

- Weak equivalences are objectwise quasi-isomorphisms.
- Fibrations are objectwise surjective maps with a kernel that is degreewise injective in $\mathsf{Vect}_{\mathbb{Q}}^{\mathcal{C}}$.
- Cofibrations are defined by the left lifting property with respect to trivial fibrations.

- Fibrant systems $\underline{\mathcal{A}}$ of DGCAs are precisely those, that degreewise are injective as systems of vector spaces

$$\underline{\mathcal{A}}^* : C \to \mathsf{DGCA}_{\mathbb{Q}}$$

- If C is trivial, then all DGCAs are fibrant. But in general this is not the case!
  This discrepancy is responsible for the complications one encounters when building the theory of minimal models for C-diagrams of spaces.
- Minimal models must be fibrant and so this is why the systems of vector spaces of generators shall be injectively resolved by us.

Applying objectwise the PL adjunction we obtain the Quillen adjunction:

$$\mathsf{Fun}(C^{op}, \mathsf{sSet})_{\mathsf{proj}} \underset{\mathcal{K}}{\overset{\mathcal{A}_{PL}}{\rightleftarrows}} (\mathsf{DGCA}^{\geq 0}_{\mathbb{Q}}(C)_{\mathsf{trinj}})^{op}$$

This Quillen adjunction yields equivalence of homotopy categories

$$\mathsf{Ho}(\mathsf{Fun}(\mathcal{C}^{op}, \mathsf{sSet}))^{\geq 1,\mathsf{nil}}_{\mathsf{fin},\mathbb{Q}} \underset{\mathbb{D}\mathcal{K}}{\overset{\mathbb{D}\mathcal{A}_{PL}}{\rightleftarrows}} \mathsf{Ho}(\mathsf{Fun}(\mathcal{C}, \mathsf{DGCA}^{\geq 1}_{\mathsf{fin},\mathbb{Q}})^{op})$$

Now we want to construct classes of minimal models satisfying the conditions ensuring the homotopy category of minimal models is equivalent to the categories above:

1. For any fibrant, cohomologically connected $A \in \mathsf{DGCA}_{\mathbb{Q}}^{\geq 0}(\mathcal{C})$ of finite type, there exists a minimal model $\mathcal{M}$ and a quasi-isomorphism $\mathcal{M} \xrightarrow{\sim} A$.

2. Minimal models are fibrant-cofibrant objects in $\mathsf{DGCA}_{\mathbb{Q}}^{\geq 0}(\mathcal{C})_{\mathsf{trinj}}$.

3. (Rigidity) For minimal models, homotopy equivalence is an isomorphism. Or weaker, any quasi-isomorphism between minimal models is homotopic to an isomorphism.

Minimal models for good diagrams of spaces

- Here one may also define two notions minimal models satisfying the desired conditions

Algebraically minimal models

Definition 6. Let $\mathfrak{M}$ be a system of DGCAs satisfying the following conditions:

1. $\mathfrak{M}$ is a system of Sullivan algebras. That is, $\mathfrak{M}$ is objectwise connected and is isomorphic (as a system of graded-commutative algebras) to $\mathsf{Sym}(\underline{V})$ for a filtered system of cochain complexes $\underline{V} = \bigcup_{k\geq 0} \underline{V}(k)$, where each $\underline{V}(k)$ is given by $\underline{V}(k) = \bigoplus_{i\leq k} \underline{V}^{(i)}$, for a family of fibrant systems of cochain complexes $\underline{V}^{(i)}$. The differential d is nilpotent with respect to this filtration; that is, it satisfies

$$-\bar{d} + \delta = d : \underline{V}^{(k)} \to \underline{V}^{(k)} \oplus \mathsf{Sym}(\underline{V}(k-1)),$$

   where $\bar{d}$ is differential on $\underline{V}^{(k)}$.

2. The value $\mathfrak{M}(\mathsf{last})$ of $\mathfrak{M}$ at the last object in $\mathfrak{C}$, is a Sullivan minimal model.

3. (Decompasability condition) For any $h \in \mathfrak{C}$ the differential decomposes:

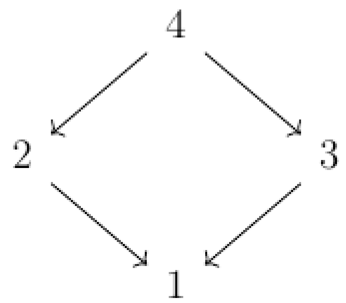
$$d : \bigcap_{h\to h',\ \text{noniso}} \mathsf{ker}(\mathfrak{M}(h) \to \mathfrak{M}(h')) \to \mathfrak{M}^{\geq 2}(h).$$

A system $\mathfrak{M}$ satisfying these conditions is called algebraically minimal, and the conditions themselves are referred to as the algebraic minimality conditions.

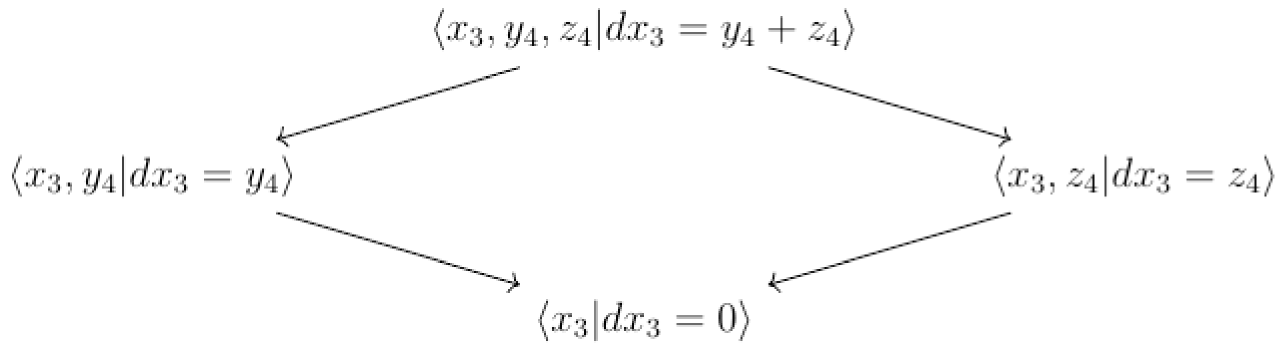
Triantifillou

Example:

Let C be a commutative square category



and consider the following system of DGCAs



this system is isomorphic to the following

$$\mathcal{M}_{\mathsf{alg}} \equiv \mathsf{Sym}(\mathsf{Inj}(\underset{3}{W^\vee}) \xrightarrow{d} \mathsf{Inj}^1(\underset{4}{W^\vee})),$$

where $W^\vee$ denotes the system of vector spaces

$$\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 0 & & 0 \\ \searrow & & \swarrow \\ & (x_3) & \end{array}$$

Here Inj denotes an injective envelope of this system and $\mathsf{Inj}^1(W^\vee) := \mathsf{Inj}(\mathsf{Inj}(W^\vee)/W^\vee)$

This minimal model presents a homotopy type X:

$$X \to K(W,3) \to K(\mathsf{Proj}^2(W),5),$$

with unique k-invariant for $H^5(K(W,3);\mathsf{Proj}^2(W)) \cong \mathbb{Q}$

The homotopy groups of X are $\underline{\pi}_3 = W$, $\underline{\pi}_4 \cong \mathsf{Proj}^2(W)$, with $\mathsf{Proj}^2(W)(4) \cong \mathbb{Q}$ and trivial

$$\begin{array}{ccccc} & & \mathbb{Q} & & \\ & \nearrow & & \nwarrow & \\ 0 & & & & 0 \\ & \nwarrow & & \nearrow & \\ & & 0 & & \end{array}$$

at all other objects:

- !!!! A principal reason for working over “good” categories is that injective envelopes in these settings admit explicit, convenient formulas. Moreover, the symmetric-algebra functor Sym is degreewise injective on any injective system of vector spaces.

Geometrically minimal models

The construction is analogous to the classical situation, only that now, to keep track of bifibrancy of minimal models, we have to injectively resolve dual systems homotopy groups:

Construction 1. [10] Consider the fiber sequence (12) for a $\mathcal{C}$-diagram of spaces. The Hirsch lemma then translates into the following elementary extension: let $\tilde{k}^{n+2} : \underline{\pi}_n(\mathcal{X})^\vee \to \mathcal{M}(\mathcal{X}_n)$ be a representative cocycle of the $(n+2)$-th k-invariant of $\mathcal{X}$. Also assume that $\mathcal{M}(\mathcal{X}_n)$ is fibrant as a system of DGCAs. Using fibrancy of $\mathcal{M}(X_n)$ one can extend $\tilde{k}^{n+2}$ to a map of functors of cochain complexes. Applying Sym then yields the corresponding map for functors of DGCAs:

$$
\begin{array}{ccccccccc}
\underline{\pi}_{n+1} & \hookrightarrow & \underset{n+2}{\mathsf{Inj}}(\underline{\pi}^\vee_{n+1}) & \xrightarrow{d} & \underset{n+3}{\mathsf{Inj}^1}(\underline{\pi}^\vee_{n+1}) & \xrightarrow{d} & \underset{n+4}{\mathsf{Inj}^2}(\underline{\pi}^\vee_{n+1}) & \longrightarrow & \dots \\
& \searrow^{\tilde{k}^{n+2}} & \downarrow{\scriptstyle \tilde{k}^{n+2}} & & \downarrow{\scriptstyle \tilde{k}^{n+3}} & & \downarrow{\scriptstyle \tilde{k}^{n+4}} & & \\
& & \mathcal{M}(X_n)^{n+2} & \xrightarrow[d]{} & \mathcal{M}(X_n)^{n+3} & \xrightarrow[d]{} & \mathcal{M}(X_n)^{n+4} & \longrightarrow & \dots
\end{array}
$$

We therefore obtain a homotopy pushout diagram

$$
\begin{array}{ccc}
\mathsf{Sym}\big(\underset{n+2}{\mathsf{Inj}}(\underline{\pi}^\vee_{n+1}) \xrightarrow{d} \underset{n+3}{\mathsf{Inj}^1}(\underline{\pi}^\vee_{n+1}) \to \dots\big) & \xrightarrow[\tilde{k}^{n+2}]{} & \mathcal{M}(X_n) \\
\big\downarrow & & \big\downarrow \\
\mathsf{Sym}\big(\underset{n+1}{\mathsf{Inj}}(\underline{\pi}^\vee_{n+1}) \xrightarrow{d} \underset{n+2}{\mathsf{Inj}^1(\underline{\pi}^\vee_{n+1}) \oplus \mathsf{Inj}(\underline{\pi}^\vee_{n+1})} \to \dots\big) & \longrightarrow & \mathcal{M}(X_{n+1})
\end{array}
$$

Both classes satisfy the desired properties and we establish:

$$(\mathsf{MinModels}_{\mathbb{Q}}(\mathcal{C})^{op}/\sim) \cong \mathsf{Ho}(\mathsf{Fun}(\mathcal{C}^{op}, \mathsf{sSet}))^{\geq 1,\mathsf{nil}}_{\mathsf{fin},\mathbb{Q}}.$$

for both algebraic and geometric types of models

An example of a geometrically minimal model beyond known equivariant examples is given by the

$$\begin{array}{ccc} \partial T & \longrightarrow & \overline{M \setminus T} \\ \downarrow & & \downarrow \\ T & \longrightarrow & M \end{array}$$

for an embedding $CP^1 \subset CP^3$

and acts trivially on other generators. The underlying system of graded-commutative algebras has the following form:

$$
\begin{array}{ccccc}
 & & \Lambda(\tilde{x}_2, \tilde{y}_3, \tilde{e}_3, \tilde{y}_4, \tilde{e}'_4, \tilde{e}''_4, z_5, h_7) & & \\
 & \swarrow & & \searrow & \\
\Lambda(x'_2, y'_3, e'_3, e'_4) & & & & \Lambda(x''_2, y''_3, e''_3, e''_4) \\
 & \searrow & & \swarrow & \\
 & & \Lambda(x_2, y_3, e_3) & &
\end{array} \tag{82}
$$

$$\mathcal{M}(T) = \left\langle x'_2, y'_3, e'_3, e'_4 | dy'_3 = (x'_2)^2, de'_3 = e'_4 \right\rangle ,$$

$$\mathcal{M}(\partial T) = \left\langle x_2, y_3, e_3 | dy_3 = x_2^2, de_3 = 0 \right\rangle \quad \mathcal{M}(\overline{\mathbb{C}P^3 \setminus T}) = \left\langle x''_2, y''_3, e''_3, e''_4 | dy''_3 = (x''_2)^2, de''_3 = e''_4 \right\rangle$$

$$\mathcal{M}(\mathbb{C}P^3) = \left\langle \tilde{x}_2, \tilde{y}_3, \tilde{e}_3, \tilde{y}_4, \tilde{e}'_4, \tilde{e}''_4, z_5, h_7 \;\middle|\; \begin{array}{l} d\tilde{y}_3 = \tilde{x}_2{}^2, \quad d\tilde{e}_3 = \tilde{e}'_4 + \tilde{e}''_4 \\ d\tilde{y}_4 = 0, \quad dh_7 = \tilde{y}_4^2, \quad d\tilde{e}'_4 = z_5 \end{array} \right\rangle$$

The difference between two approaches

Recall the aforementioned system:

$$\begin{array}{ccc} & \langle x_3, y_4, z_4 | dx_3 = y_4 + z_4 \rangle & \\ \swarrow & & \searrow \\ \langle x_3, y_4 | dx_3 = y_4 \rangle & & \langle x_3, z_4 | dx_3 = z_4 \rangle \\ \searrow & & \swarrow \\ & \langle x_3 | dx_3 = 0 \rangle & \end{array}$$

This is an algebraically minimal model of the fiber
$X \to K(W, 3) \to K(\mathsf{Proj}^2(W), 5)$,
this system is algebraically minimal yet not geometrically non-minimal.

Indeed, the geometrically minimal model can be readily computed:

$$\mathsf{Sym}\left(\underset{3}{\mathsf{Inj}(W^\vee)} \xrightarrow{d} \underset{4}{\mathsf{Inj}^1(W^\vee)} \xrightarrow{d} \underset{5}{\mathsf{Inj}^2(W^\vee)}\right) \otimes \mathsf{Sym}(\underset{4}{\mathsf{Inj}^2(W^\vee)}),$$

the differential on the attached generators in 4-degree is identity sending it to the 5-th degree on the left. Explicitly thys system is given by:

$$\begin{array}{ccc} & \left\langle x_3, y_4, x_4, \overbrace{z_4, z_5} \middle| dx_3 = x_4' + x_4'', dx_4' = z_5, dz_4 = z_5 \right\rangle & \\ \swarrow & & \searrow \\ \langle x_3, y_4 | dx_3 = y_4 \rangle & & \langle x_3, z_4 | dx_3 = z_4 \rangle \\ \searrow & & \swarrow \\ & \langle x_3 | dx_3 = 0 \rangle & \end{array}$$

This is a common thing: geometric minimality tends to force the appearance of redundant generators (these generators do not contribute to the Chern character map)

- These generators can be eliniminated without changing a homotopy type and the result is the algebraically minimal model! (in fact, this is how the existence of algebraically minimal models is established by us)
- Here this means eliminating the $\mathsf{Inj}^2(W^\vee)$ in the 5-th degree and truncating the differential:

$$\mathsf{Sym}\left(\underset{3}{\mathsf{Inj}(W^\vee)} \xrightarrow{d} \underset{4}{\mathsf{Inj}^1(W^\vee)} \xrightarrow{d} \underset{5}{\mathsf{Inj}^2(W^\vee)}\right) \otimes \mathsf{Sym}(\underset{4}{\mathsf{Inj}^2(W^\vee)}),$$

$$\mapsto$$

$$\mathsf{Sym}(\underset{3}{\mathsf{Inj}(W^\vee)} \xrightarrow{d} \underset{4}{\mathsf{Inj}^1(W^\vee)})$$

In case when $G \neq Z/p^k$, Orb_G

When the two approaches coincide?

This is crucial in practice, as it is difficult to recognize geometrically minimal models (we have to construct them "by hand") . In contrast, algebraic axioms can be easily verified.

- If we know in advance, that the model we are constructing must be algebraically minimal — this drastically simplifies calculations!
- So it is important to establish when minimal models (in the geometric sense) must be also algebraically minimal
- The obstruction to general coincidence is cohomological dimension of C!!
- Moreover, this is related to the phenomenon of rational splitting of Hopf-diagrams of spaces onto diagrams of Eilenberg-Maclane type

The following are equivalent:

- $$\mathsf{cd}_{\mathbb{Q}}(\mathcal{C}) \leq 1$$
- All homotopy types of C-diagrams of nilpotent, $\mathbb{Q}$-finite spaces are algebraic.
- Every diagram of nilpotent, $\mathbb{Q}$-finite H-spaces rationally splits as a product of Eilenberg-MacLane diagrams.

Corollary 23. Let G be a finite group. In rational G-equivariant homotopy theory, any $\mathbb{Q}$-finite G Hopf-space rationally splits as a product of Eilenberg–MacLane type diagrams if and only if $G \cong \mathbb{Z}/p^k$. □

- Given a diagram $\mathcal{X}$ of spaces, whether the geometrically minimal model $\mathcal{M}(\mathcal{X})$ is also algebraically minimal is a novel rational invariant of $\mathcal{X}$!
- The obstructions to algebraicity of X lie in the second filtration of the universal coefficient spectral sequence...

- This implies, that if homological dimension of systems of homotopy groups of X is bounded by 1, then the model of X is algebraically minimal

Also

Proposition 17. A diagram $\mathcal{X}$ is algebraic if its homotopy groups are sufficiently remote from each other, in the following precise sense: whenever $\underline{\pi}_*(\mathcal{X})$ is nontrivial in a given degree, $\underline{\pi}_i(\mathcal{X})$ is trivial for all degrees in the interval $* - \mathsf{cd}_{\mathbb{Q}}(\mathcal{C}) \leq i \leq * - 1$.

De Rham complex of diagrams

In classical rational homotopy theory and applications to higher gauge theories it is crucial that the de Rham complex of a smooth manifold $\Omega_{dR}(X)$ presents a real homotopy type of X and that the minimal model of X can be extracted from geometric data:

$$\mathcal{M}(X) \xrightarrow{\sim} \Omega_{dR}(X)$$

- Here the problem occurs: not any diagram of smooth manifolds X is such that the system of de Rham complexes

$$\Omega_{dR}(X) : (c \in C) \mapsto \Omega_{dR}(X(c))$$

is fibrant in our sense!!

Indeed

- Example:

For $C = (2 \to 1)$, consider an inclusion $X(1) \subset X(2)$ and if $X(1)$ is closed, then the system de Rham complexes is fibrant, since we can extend differential forms from closed submanifolds.

For a general arrow $X(1) \to X(2)$ fibrancy of de Rham complex implies degreewise surjectivity $\Omega_{dR}(X(2)) \to \Omega_{dR}(X(1))$

Theorem 28. Consider a good category $\mathcal{C}$ (as defined in Definition 2) and let $\mathcal{X}$ denote a $\mathcal{C}$-shaped presheaf of nilpotent, $\mathbb{Q}$-finite smooth manifolds. The following claims hold:

1. The de Rham complex has the same $\mathbb{R}$-rational homotopy type of $\mathcal{X}$, i.e. there exists an isomorphism in $\mathsf{HoDGCA}_{\mathbb{Q}}^{\geq 0}(\mathcal{C})$:

$$\mathcal{M}(\mathcal{X}) \otimes \mathbb{R} \cong \mathbf{\Omega}^*_{\mathrm{dR}}(\mathcal{X});$$

2. Given that the de Rham complex $\mathbf{\Omega}^*_{\mathrm{dR}}(\mathcal{X})$ is fibrant within the model category $(\mathsf{DGCA}_{\mathbb{Q}}^{\geq 0}(\mathcal{C})_{\mathsf{trinj}})$ of systems of DGCAs, for any geometrically or algebraically minimal model $\mathcal{M}(\mathcal{X})$ of $\mathcal{X}$, a quasi-isomorphism can be established:

$$\mathcal{M}(\mathcal{X}) \otimes \mathbb{R} \xrightarrow{\sim} \mathbf{\Omega}^*_{\mathrm{dR}}(\mathcal{X}); \tag{48}$$

- For all fixed point diagrams $G/H \mapsto X^H$

  where G is abelian (or, more generally, Hamiltonian) the de Rham complex is fibrant!
- Also for diagrams of clean intersections of submanifolds:

$$\begin{array}{ccc} N_1 \cap N_2 & \longrightarrow & N_1 \\ \downarrow & & \downarrow \\ N_1 & \longrightarrow & M \end{array}$$

Assume that they intersect cleanly, meaning that for for each $p \in N_1 \cap N_2$ $T_p(N_1 \cap N_2) = T_p(N_1) \cap T_p(N_2)$. Firstly notice that for N_1 there is a tubular neighbourhood the de Rham complex is fibrant!

Relative minimal models

Relative minimal models, often referred to in the literature as K.S. extensions, serve as algebraic models for fibrations of topological spaces. Consider a Serre fibration $f : X \to Y$ between $\mathbb{Q}$-finite, simply-connected spaces. This yields a cofiber sequence of DGCAs

$$\mathcal{M}(Y) \hookrightarrow \mathcal{M}(f) \twoheadrightarrow \mathcal{M}(F),$$

where F denotes the fiber of f. The map $\mathcal{M}(Y) \hookrightarrow \mathcal{M}(f)$ induced by f is referred to as the relative Sullivan (or relative minimal) model of the fibration (cf. [16, §14]).

here is an example:

Let $S^3 \to S^2$ be a hopf fibration, then its relative minimal model is

$$(x_2, y_3 | dx_2 = 0, dy_3 = x_2^2) \to (x_1, x_2, y_3 | dx_1 = x_2, dy_3 = x_2^2)$$

with the cofiber being the model for S^1:

$$(x_1 | dx_1 = 0)$$

****

- Relative models can be used to classify possible rational homotopy types of fibrations with prescribed fibers and bases...
- We may use our theory to carry out such classifications in equivariant contexts

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Equivariant $\mathbb{H}$-Hopf fibration

Consider the quaternionic Hopf fibration

$$q_{\mathbb{H}}\colon S^7 \subset \mathbb{H}^2 \longrightarrow \mathbb{H}P^1 = S^4, \tag{31}$$

and the following actions:

$$\tau(z_1 + w_1 j,\ z_2 + w_2 j) = (z_1 - w_1 j,\ z_2 - w_2 j), \tag{32}$$

$$\sigma[z_1 + w_1 j : z_2 + w_2 j]_{\mathbb{H}^\times} = [z_1 - w_1 j : z_2 - w_2 j]_{\mathbb{H}^\times}. \tag{33}$$

Here we present quaternions in the form $z+wj$, where $z, w \in \mathbb{C}$. Clearly, the fibration (31) preserves this action. The fixed point subspaces are

$$\{z_1, z_2\colon\ z_i \in \mathbb{C}\} \cong S^3 \tag{34}$$

and

$$\{[z_1, z_2]_{\mathbb{H}^\times} :\ z_i \in \mathbb{C}\} \cong S^2. \tag{35}$$

The induced map on fixed point sets results in the Hopf fibration $S^3 \to S^2$. We compute the relative model for the $\mathbb{Z}_2$-equivariant fibration (31). This model is derived from the induced homomorphism of minimal models of $\mathbb{Z}_2$-equivariant sphere. The computation is straightforward, and we present the result here:

$$
\begin{array}{ccc}
\dfrac{\langle f_2', f_3', h_3', h_4', g_4, g_7\rangle}{(dg_4{=}0,\, df_2'{=}f_3',\, dh_3'{=}(f_2')^2{-}h_4',\, dg_7{=}g_2^2)} & \longrightarrow & \dfrac{\langle x_3', x_4', g_7\rangle}{(dg_7{=}0,\, dx_3'{=}{-}x_4')} \\
\big\downarrow & & \big\downarrow \\
\dfrac{\langle f_2, h_3\rangle}{(dh_3{=}f_2^2)} & \longrightarrow & \dfrac{\langle x_3\rangle}{(dx_3{=}0)}
\end{array}
$$

The vertical map on the left is the model $\mathfrak{M}([S^4/\mathbb{Z}_2])$ of the equivariant 4-sphere with $\mathbb{Z}_2$-action, while the one on the right is the model for the equivariant 7-sphere $\mathfrak{M}([S^7/\mathbb{Z}_2])$. Using the construction of relative models (see also the point (2) of the Theorem 21), the result is given:

$$\begin{array}{ccc} \mathfrak{M}([S^4/\mathbb{Z}_2])(\mathbb{Z}_2/1) & \hookrightarrow & \mathfrak{M}([S^4/\mathbb{Z}_2])(\mathbb{Z}_2/1) \otimes \frac{\langle \psi_1', \psi_2', \zeta_3 \rangle}{(d\psi_1' = \psi_2',\, d\zeta_3 = g_4)} \\ \downarrow & & \downarrow \\ \mathfrak{M}([S^4/\mathbb{Z}_2])(\mathbb{Z}_2/\mathbb{Z}_2) & \hookrightarrow & \mathfrak{M}([S^4/\mathbb{Z}_2])(\mathbb{Z}_2/\mathbb{Z}_2) \otimes \frac{\langle \psi_1 \rangle}{(d\psi_1 = f_2)} \end{array}$$

The cofiber of this map is represented by the system

$$\frac{\langle \psi_1',\ \psi_2',\ \zeta_3 \rangle}{(d\psi_1' = \psi_2',\ d\zeta_3 = 0)} \to \frac{\langle \psi_1 \rangle}{(d\psi_1 = 0)},$$

Relative minimal models appear in the description of fluxes in twisted higher gauge field theories (via the (twisted) Chern character map. In M-theory, one considers 5-brane sigma-models $\phi : \Sigma \to X$ into a spacetime X. Omitting details, the theory (topologically) is determined by a fixed 4-cohomotopy cocycle $c : X \to S^4$, which is a topological part of the C-field and the B-field on the brane's worldvolume.

The C-field gives rise to a twist $c \circ \phi \colon \Sigma \to S^4$, and the B-field on Σ is a cocycle in twisted cohomotopy, represented by the homotopy class of the map $c \circ \phi \to q_{\mathbb{H}}$ in $\mathsf{Top}_{/S^4}$:

$$
\begin{array}{ccc}
\Sigma & \dashrightarrow & S^7 \\
{\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle q_{\mathbb{H}}} \\
X & \xrightarrow{\;\;c\;\;} & S^4
\end{array}
$$

The relative minimal model of $q_{\mathbb{H}}$ is of the form:

$$\frac{\langle g_4,\ g_7\rangle}{(dg_7 = g_4^2,\ dg_4 = 0)} \to \frac{\langle g_4,\ g_7\rangle}{(dg_7 = g_4^2,\ dg_4 = 0)} \otimes \frac{\langle \zeta_3\rangle}{(d\zeta_3 = g_4)}$$

- This encodes the flux fields on the brane and in the bulk via the mapping from the relative minimal model to the de Rham complexes of both the bulk and the brane

$$\left(\frac{\langle\, \overbrace{G_4,\, G_7}^{\text{bulk fluxes}} \,\rangle}{(dG_7 = G_4 \wedge G_4)} \right)_{\text{bulk}}$$

$$\Big\downarrow \;\text{restriction to } \Sigma$$

$$\left(\frac{\langle\, \overbrace{(G_4,\, G_7)}^{\text{bulk fluxes}},\; \overbrace{(H_3)}^{\text{B-field}} \,\rangle}{(dH_3 = G_4|\Sigma)} \right)_{\text{brane}}$$

The same approach can be extended by mapping the $\mathbb{Z}_2$-equivariant relative model of the fibration to the de Rham complex of a singular brane considered as an $\mathbb{Z}_2$-orbifold. In this context, we have four distinct manifolds: the bulk, the singularity, the brane, and the intersection of the singularity with the brane.

$$\frac{\langle (F_2, H_3, W_4), \overbrace{(G_4, G_7)}^{\text{bulk fluxes}} \rangle}{(dG_7 = G_4 \wedge G_4,\ dH_3 = F_2 \wedge F_2 - W_4)}_{\text{bulk}}$$

$$\Big\downarrow \text{restriction to } X^{\mathbb{Z}_2}$$

$$\frac{\langle (F_2|X^{\mathbb{Z}_2}, H_3|X^{\mathbb{Z}_2}) \rangle}{(G_7|X^{\mathbb{Z}_2} = 0,\ G_4|X^{\mathbb{Z}_2} = 0,\ W_4|X^{\mathbb{Z}_2} = 0,\ d(H_3|X^{\mathbb{Z}_2}) = (F_2 \wedge F_2)|X^{\mathbb{Z}_2})}_{\text{singularity}}$$

The singularity is flux-quantized in S^2 — whence the origin of the Chern-Simons terms H_3, F_2.

On the brane, the differential forms data is

$$\frac{\langle (F_2|\Sigma,\ H_3|\Sigma,\ W_4|\Sigma),\ (G_4|\Sigma,\ G_7|\Sigma),\ \overbrace{(\zeta_3)}^{\text{B-field}} \rangle}{(d\zeta_3 = G_4|\Sigma)}_{\text{brane}}$$

$$\Big\downarrow \text{ restriction to } \Sigma \bigcap X^{\mathbb{Z}_2}$$

$$\frac{\langle (F_2|\Sigma \cap X^{\mathbb{Z}_2},\ H_3|\Sigma \cap X^{\mathbb{Z}_2}),\ \overbrace{(\psi_1)}^{\text{new field}} \rangle}{(d\psi_1 = F_2|\Sigma \cap X^{\mathbb{Z}_2})}_{\text{singularity} \bigcap \text{brane}}$$

For any $A \in \mathsf{Vect}^C$ there is an injective resolution

$$A \to A^*$$

of length $\leq cd(C)$

$n \to \cdots \to 1$ are good!

End. Thank you!