M-theory and Geometry of Loops on the Four Sphere

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Five superstring theories, rich gauge geometry

Type I	Type IIA	Type IIB
Heterotic $E_8 \times E_8$		Heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$

The gauge data of heterotic theories are well understood.

Our focus: the coupled Type II form sector.

- RR potentials and their magnetic duals: odd degrees in IIA, even degrees in IIB.
- NS B_2, its dual B_6, and the dilaton-dual potential B_8 in IIA.
- Their curvature and gauge-transformation laws mix the fields.

[SSS09], differential twisted String geometry; [BDKOR06], democratic Type IIA gauge algebra.

The coupled local Type IIA system

Massless conventions. $F = F_2 + F_4 + F_6 + F_8 + F_{10}$; products are wedges.

$$\begin{aligned} \mathrm{d}H_3 &= 0, & \mathrm{d}F &= H_3 F, \\ \mathrm{d}H_7 &= \tfrac{1}{2}F_4^2 - F_2 F_6, & & \\ \mathrm{d}H_9 &= H_3 H_7 + \tfrac{1}{2}F_4 F_6 - \tfrac{3}{2}F_2 F_8. & & \end{aligned}$$

With $C = C_1 + C_3 + \cdots$, the RR gauge law is

$$\delta C = \mathrm{d}\Lambda - H_3 \Lambda, \qquad \delta B_2 = \mathrm{d}\Sigma_1.$$

Even the dilaton-dual potential transforms with RR/NS fields:

$$\begin{aligned} \delta_\Sigma B_8 = \mathrm{d}\Sigma_7 + \big[& H_7 + F_2 C_5 - \tfrac{1}{2} F_4 C_3 \\ & + \tfrac{1}{2} B_2 (F_4 C_1 - F_2 C_3) \big] \Sigma_1 . \end{aligned}$$

[BDKOR06], (2.13)–(2.34), in wedge notation with $H_7 = -H_7^{\mathrm{lit}}$, $H_9 = 2H_9^{\mathrm{lit}}$; full law in the backup slide.

The global higher-gauge-field question

Twisted K-theory supplies a global framework for the RR sector with an NS twist. The jointly coupled sector asks for more.

How do the coupled RR, NS, and dilaton-dual form fields glue into one integral higher gauge field, with local curvature and descent data reproducing their Bianchi identities and gauge transformations, and with globally defined, anomaly-compatible Wess–Zumino couplings to the relevant branes?

Hypothesis H and the M-circle

Hypothesis H: the M-theory C-field, extending the higher gauge field of leading-order 11D supergravity, is topologically classified by 4-cohomotopy:

$$[c] \in [Y^{11}, S^4] = \pi^4(Y^{11}), \qquad \mathrm{d}G_4 = 0, \quad \mathrm{d}G_7 = \tfrac{1}{2} G_4 \wedge G_4.$$

For a principal circle bundle $P \to X$, there is an adjunction

$$\mathrm{Map}(P, S^4) \simeq \mathrm{Map}_{/BS^1}(X, \mathrm{Cyc}\, S^4), \qquad \mathrm{Cyc}\, S^4 := LS^4 /\!\!/ S^1.$$

Rationally, cyclification transgresses the C-field flux algebra:

$$\tau(g_4) = h_3, \qquad \tau(g_7) = -f_6, \qquad H_3 = \int_{S^1_M} G_4.$$

The remaining components give F_4, H_7; the circle class gives F_2.
[GSS26], §2.2–2.3. Classical, source-free character; gravitational shifts suppressed.

Plain cyclification misses the D0 tadpole

The direct descendants are F_2, H_3, F_4, F_6, H_7. There is no independent F_8 generator.

String theory also requires the D0 Gauss law:

$$\mathrm{d}F_8 = H_3 \wedge F_6, \qquad [H_3 \wedge F_6] = 0 \text{ on } X^9.$$

Here there are no explicit D0 sources. On a closed X^9, integrating gives the corresponding tadpole cancellation condition.

What mechanism governs the reduction of the *globally completed* C-field under compactification?

Our proposal

Reduction selects a subspace $A \subset LS^4$, with coefficient $A /\!\!/ S^1$. The first geometric candidate comes from the loop-energy filtration.

[GSS26], introduction, Thm. 3.8 and Prop. 3.10. Selection of a subspace is an extra physical proposal.

Smallness through Morse–Bott theory

On the round unit sphere, use the loop energy

$$E(\gamma) = \tfrac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, \mathrm{d}t.$$

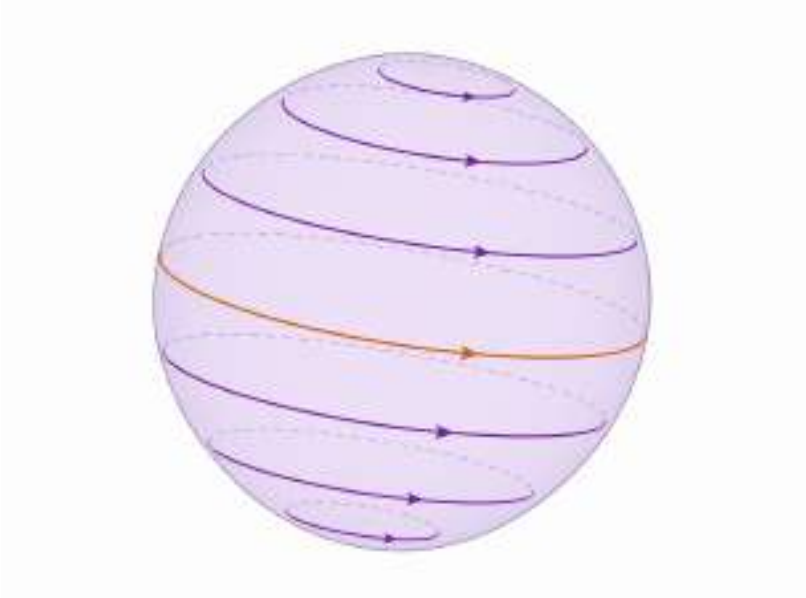
Critical loops	Energy	Morse index
Constants: S^4	0	0
k-fold great circles: $V_2(\mathbb{R}^5)$	$2\pi^2 k^2$	$3(2k-1)$

$$F_1 := E^{-1}([0,a]), \quad 2\pi^2 < a < 8\pi^2, \qquad \mathrm{SmlCyc}\, S^4 \simeq F_1 /\!\!/ S^1.$$

Keep the constants and the first critical family.

[GSS26], Def. 3.3 and Lem. 3.4; project Morse–Bott calculations.

The first attachment is a family of round circles



One spherical slice of the family.
Each circle is traversed once.

The descending directions translate a great circle to smaller round circles.

At the boundary, each circle collapses to a constant loop.

$$F_1 \simeq D(\zeta) \cup_{S(\zeta)} S^4,$$

$$\zeta_{(x,v)} = \langle x, v \rangle^{\perp}.$$

$$\gamma(t) = w + \sqrt{1 - |w|^2}\,(x \cos 2\pi t + v \sin 2\pi t),$$

(x, v) orthonormal, $w \perp x, v$, $|w| \leq 1$. This is an equivariant model of the first energy stage.

[GSS26], §3.2. Illustration: generated schematic of a family of round circles.

An algebraic presentation of small cyclification

$$\boxed{\operatorname{SmlCyc} S^4 \simeq \operatorname{hocolim}\left(Q^3 \xleftarrow{q} \mathcal{I} \xrightarrow{(p,\, b_{q^* L_Q})} S^4 \times BU(1)\right).}$$

$$Q^3 = \{[y] \in \mathbb{CP}^4 : y \cdot y = 0\}, \qquad L_Q = \mathcal{O}_Q(1),$$
$$\mathcal{I} = \{([y], n) \in Q^3 \times S^4 : y \cdot n = 0\}.$$

- Q^3 encodes the oriented circle plane $\Pi = \langle \operatorname{Re} y, \operatorname{Im} y \rangle$.
- $\mathcal{I}$ pairs this plane with a unit normal $n \in \Pi^\perp$: the boundary data for a circle collapsing to n.
- $r_0([y], n) = (n, [y])$ attaches that boundary to the constant-loop stratum: p selects $n \in S^4$, while $b_{q^* L_Q}$ retains the phase line in $BU(1)$.

[M2–3], polarization geometry and the explicit attachment. $[y]$ in the last line is its image in $\mathbb{CP}^\infty$.

Round loops recover the D0 tadpole

The real character of $\mathrm{SmlCyc}\, S^4$ gives, on a spatial X^9,

$$\mathrm{d}H_3 = 0, \qquad \mathrm{d}F_2 = 0,$$
$$\mathrm{d}F_4 = H_3 \wedge F_2, \qquad \mathrm{d}F_6 = H_3 \wedge F_4,$$
$$\mathrm{d}F_8 = H_3 \wedge F_6,$$
$$\mathrm{d}H_7 = \tfrac{1}{2} F_4 \wedge F_4 - F_2 \wedge F_6.$$

The small-loop restriction recovers the full massless RR/NS gauge sector on X^9, including the missing D0 Gauss law.

It retains H_7 as well. The dilaton-dual extension comes later, from minimal cyclification.

[GSS26], Prop. 3.10. Wedge products and source-free conventions throughout.

How twisted K-theory emerges

Golub–Sati–Schreiber, arXiv:2609.36048

$$\mathrm{SmlCyc}\, S^4 \xrightarrow{\Phi} BU /\!\!/ BU(1) \longrightarrow K(\mathbb{Z}, 3).$$

- The comparison preserves the NS twist.
- Rationally, it retains the RR tower and forgets H_7.
- Integrally, the universal comparison has an essential factor of two in its lowest RR normalization.

Small-cyclified cohomotopy is a nonabelian (unstable) enhancement of twisted K-theory.

[GSS26], Thm. 3.17, Thm. 3.21, Cor. 3.23.

The D0 tadpole is the lifting obstruction

A global test for smallness. For $\dim X \leq 9$,

$$c : X \to \mathrm{Cyc}\, S^4 \text{ lifts to } \mathrm{SmlCyc}\, S^4 \quad \Longleftrightarrow \quad c^* o_9 = 0,$$

$$o_9 \in H^9(\mathrm{Cyc}\, S^4; \mathbb{Z}), \qquad (o_9)_{\mathbb{R}} \propto [H_3 \wedge F_6].$$

On a connected oriented X^9, D0 tadpole cancellation is exactly the condition for a small-loop lift.

This tests existence. The choices of lift retain further integral information.

[GSS26], Thm. 3.8. Source-free sector.

A global coefficient for massive Type IIA

Romans mass is the constant RR zero-form flux $F_0 = m$ (away from D8 sources), deforming massless IIA. Here $m \in \mathbb{Z}_{>0}$.

Ordinary M-circle reduction does not generate it. Our mechanism changes the phase attachment at loop collapse:

$$\boxed{\mathcal{C}_m := \mathrm{hocolim}\Big(Q^3 \xleftarrow{q} \mathcal{I} \xrightarrow{r_m} S^4 \times BU(1)\Big),}$$

$$\boxed{r_m([y], n) = \big(n, [y \otimes \psi^{\odot m}]\big).}$$

- Q^3 records the circle plane; $\mathcal{I}$ adds its collapse point n.
- Choose $0 \neq \psi \in \ell_{y,n}$, the common-spinor line; $T = \ell^*$. The phase component classifies $q^* L_Q \otimes T^m$.
- At $m = 0$, $r_0([y], n) = (n, [y])$ recovers small cyclification.

[M1–3], explicit attachment; spinor notation in backup 2. Ordinary reduction: [AMT22]; other proposed M-theory origins: [Hull98].

Romans identities and torsion NS data

For the specified character of $\mathcal{C}_m$,

$$F_0 = m, \qquad \mathrm{d}H_3 = 0, \qquad \mathrm{d}F_2 = mH_3,$$
$$\mathrm{d}F_{2j} = H_3 \wedge F_{2j-2} \qquad (2 \leq j \leq 5),$$
$$\mathrm{d}H_7 = mF_8 - F_2 \wedge F_6 + \tfrac{1}{2} F_4 \wedge F_4.$$

The universal integral NS class satisfies

$$H^3(\mathcal{C}_m; \mathbb{Z}) \cong \mathbb{Z}/m, \qquad h_m = \delta c_1(T), \qquad mh_m = 0,$$

with a *specified trivialization* of its m-th power.

A source-free gauge completion; 11D descent, gravitational refinements and D8 interfaces need additional data.

[M1], completion theorem and § "All RR/NS Bianchi identities"; [M2], conclusions.

The minimal family carrying the NS class

Carrying the NS class: the transgression $h_3 = \int_{S^1} \mathrm{ev}^* \omega_4$ restricts nontrivially to the loop family. We require a primitive unit: a family $\sigma : S^3 \to A$ with

$$\int_{S^1 \times S^3} \mathrm{ev}_\sigma^* \omega_4 = 1, \qquad \int_{S^4} \omega_4 = 1.$$

Minimal in the specified geometric class: three parameters of distinct, once-traversed round circles, all their phases, and one limiting constant loop.

$$A_{\min} \cong (S^1 \times \mathbb{R}^3)^+ \hookrightarrow LS^4, \qquad \mathrm{MinCyc}\, S^4 := A_{\min} /\!\!/ S^1.$$

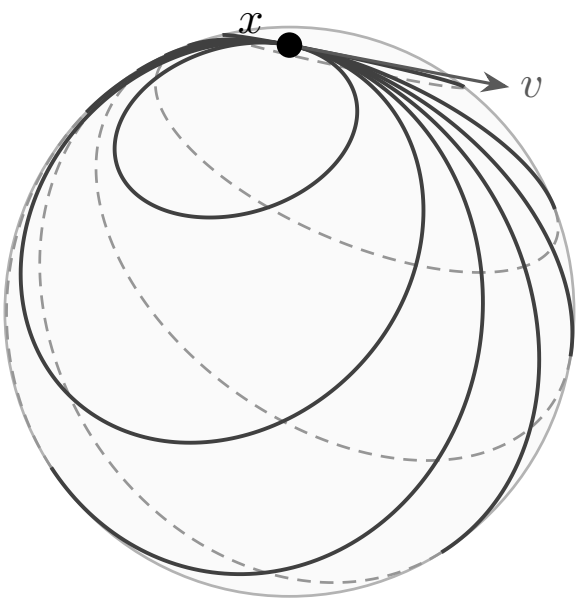
The selection principle

Retain just enough round-loop geometry to sweep S^4 once and support the primitive NS three-class.

[B], definition and characterization of minimal cyclification. $+$ denotes one-point compactification. Minimality is within this family class.

An explicit minimal family

A target-sphere slice S^2



Curves: nonconstant round circles
Point x: the single constant loop c_x

Supplied figure [B], shown in an S^2 slice.

Fix $x \in S^4$ and a unit tangent v at x.

All circles pass through x with the same oriented tangent v.

Stereographic projection from x turns them into parallel lines in $\mathbb{R}^4$.

The parameter is $y \in \langle x, v \rangle^\perp \cong \mathbb{R}^3$; as $|y| \to \infty$, the loop becomes c_x.

Why the Borel quotient is a wedge

The circle action is free away from the cap:

$$(S^1 \times \mathbb{R}^3) /\!\!/ S^1 \simeq \mathbb{R}^3.$$

At the constant loop it is trivial:

$$\{c_x\} /\!\!/ S^1 \simeq BS^1.$$

The boundary attachment is nullhomotopic, hence

$$\boxed{\mathrm{MinCyc}\, S^4 \simeq BS^1 \vee S^3.}$$

BS^1: the circle phase / RR two-class $\qquad$ S^3: the primitive NS three-class

[B], topology and homotopy types. Ordinary orbits give S^3 and lose the BS^1 summand.

The Borcherds algebra follows from the wedge

$$\mathfrak{l}_{\mathbb{R}}(BS^1) = \mathbb{L}(\varphi_1)/([\varphi_1, \varphi_1]),$$
$$\mathfrak{l}_{\mathbb{R}}(S^3) = \mathbb{L}(\eta_2).$$

Wedges give free products of these Lie models:

$$\boxed{\mathfrak{l}_{\mathbb{R}}(\mathrm{MinCyc}\, S^4) \simeq \frac{\mathbb{L}(\varphi_1, \eta_2)}{([\varphi_1, \varphi_1])} = \mathfrak{bor}^{+}_{\mathrm{IIA}}.}$$

All displayed Lie differentials vanish; subscripts are homological degrees.

Hypothesis H supplies the loops; minimal NS support supplies the wedge; the wedge supplies Borcherds.

[B], main theorem; positive IIA presentation in [GH11] and [HJL12]. Over $\mathbb{R}$ (also $\mathbb{Q}$).

Mixed Whitehead brackets generate the hierarchy

Lie element	Degree	Flux
$\varphi_1,\ \eta_2$	$1, 2$	$F_2,\ H_3$
$(\mathrm{ad}\,\eta_2)^k\varphi_1$	$2k+1$	F_{2k+2}
$\eta_6 = [\varphi_3, \varphi_3]$	6	H_7
$\eta_8 = [\eta_2, \eta_6]$	8	H_9
Two independent elements	10	$H_{11}^A,\ H_{11}^B$

The new flux H_9 is dual to the dilaton, with identity

$$\boxed{\mathrm{d}H_9 = H_3 \wedge H_7 + \tfrac{1}{2} F_4 \wedge F_6 - \tfrac{3}{2} F_2 \wedge F_8.}$$

The Maurer–Cartan equation packages the entire tower.
[B], low-degree Borcherds elements and Bianchi identities; [GH11].

Three geometries, three levels of completion

$$\mathrm{MinCyc}\, S^4 \longrightarrow \mathrm{SmlCyc}\, S^4 \longrightarrow \mathrm{Cyc}\, S^4.$$

Coefficient	What its character retains
$\mathrm{Cyc}\, S^4$	Direct circle descendants, including H_7
$\mathrm{SmlCyc}\, S^4$	Also F_8 and the RR tower
$\mathrm{MinCyc}\, S^4$	Also H_9 (dual dilaton) and the hierarchy

$$\mathfrak{l}_{\mathbb{R}}(\mathrm{SmlCyc}\, S^4) \simeq \mathfrak{bor}^{+}_{\mathrm{IIA}}/\langle \eta_8 \rangle.$$

Smaller spaces can have more homotopy: allowing more loop families can fill and kill a charge.

[B], main theorem and comparison; [GSS26], the small-cyclification character.

Integral homotopy: free and torsion charges

n	$\pi_n(\mathrm{MinCyc}\, S^4)$	$\pi_n(\mathrm{SmlCyc}\, S^4)$	$\pi_n(\mathrm{Cyc}\, S^4)$
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
4	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
5	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_2^{\oplus 2}$
6	$\mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2$
7	$\mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2^{\oplus 2}$
8	$\mathbb{Z} \oplus \mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z} \oplus \mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z}_2^{\oplus 4}$
9	$\mathbb{Z} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_{24} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2^{\oplus 3}$	$\mathbb{Z}_{24} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2^{\oplus 2}$

The minimal family carries the primitive NS class and all torsion and free charge-group types of full cyclification in this range.

$\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. The inclusion need not reach every charge: the free π_6 map is $\times 2$; a $\mathbb{Z}_2$ summand in π_7 is killed.

[B], proposition "Homotopy groups of the minimal cyclification". Charges here are detected on linking spheres.

Geometric interpretation of minimal cocycles

Theorem (geometric cocycle classification)

For a smooth spatial manifold X^9, homotopy classes of maps $X \to \mathrm{MinCyc}\, S^4$ are in natural bijection with

$$\left(P \xrightarrow{\pi} X, \quad N^6 \hookrightarrow X, \quad \nu_N \cong \mathbb{R}^3, \quad \tau : N \to P|_N\right)$$

modulo structured cobordism: a principal circle bundle, a properly embedded framed codimension-three submanifold, and a section over it.

$$h = \mathrm{PD}_X[N], \qquad g = \mathrm{PD}_P[\tau(N)], \qquad \pi_! g = h.$$

[B], Pontryagin–Thom classification. The section gives a framed codimension-four graph upstairs.

The geometry organizes the gauge hierarchy

Hypothesis H: S^4

$\Downarrow$ circle reduction

$LS^4 /\!\!/ S^1$

$\Downarrow$ small round circles

Joint RR/NS completion, including the D0 tadpole

$\Downarrow$ minimal family retaining the NS class

$$BS^1 \vee S^3 \quad \Longrightarrow \quad \mathfrak{bor}^{+}_{\mathrm{IIA}}$$

Massive gluing gives a global Romans gauge coefficient.
Minimal cocycles have a geometric interpretation as lifted NS flux sheets.

[GSS26]; [B]; [M1–4]; [A3].

From Type IIA to Type IIB

So far we have focused on Type IIA, whose relation to M-theory is the most transparent.

Type IIB remains an open problem.

The next question

How do these geometric mechanisms extend to the global completion of Type IIB?

[FSS18], T-duality in rational homotopy theory.

Compactification of the quantum C-field

M-theory still lacks a full quantum formulation. The general mechanism reducing its globally completed C-field under compactification remains unknown.

Even for the M-circle, naive C-field reduction misses the D0 tadpole required by Type IIA superstring theory:

$$\mathrm{d}F_8 = H_3 \wedge F_6.$$

Small and minimal cyclification recover this constraint geometrically, without a direct appeal to superstring theory.

Beyond the circle

What generalizes this mechanism to other closed compactification manifolds? The question remains open even for tori.

[GSS26]; [B].

Background-independent origins

Our constructions so far concern fields on a chosen spacetime background.

From the worldsheet perspective, geometric backgrounds arise as local presentations within the global moduli of SCFTs.

String theory extends beyond any one such presentation.

The broader question

What are the background-independent origins of the global completions that our geometric mechanisms reveal?

Worldsheet perspective: [Witten93], Phases of $N = 2$ theories; [Hull05], Non-geometric string backgrounds.

Thank you

Thank you for your attention!

Questions and discussion

References and project sources

[GSS26] N. Golub, H. Sati, U. Schreiber,
On the Derivation of Twisted K-Theory from M-Theory,
arXiv:2609.36048 (2026).

[B] Same authors, *Flux-Quantization of Massive Borcherds Type IIA SuGra from Hypothesis H*, supplied draft.

[GH11] J. Greitz, P. S. Howe, *Maximal supergravity in* $D = 10$, arXiv:1103.5053.

[HJL12] M. Henneaux, B. Julia, J. Levie, E_{11}, *Borcherds algebras and maximal supergravity*, arXiv:1007.5241.

[SSS09] Sati–Schreiber–Stasheff, *Twisted differential String and Fivebrane structures*, arXiv:0910.4001.

[BDKOR06] Bergshoeff et al., *IIA Ten-forms and the Gauge Algebras of Maximal Supergravity Theories*, hep-th/0602280.

[M1–4], [A2–3] Project sources, figures and additional background references are listed in the source bundle.

Visual reference: D. Grady, Differential cohomology and applications (2018), supplied PDF.

The full massless dilaton-dual gauge law

All products are wedges. In the conventions of the opening,

$$\begin{aligned}\delta B_8 = {} & \mathrm{d}\Sigma_7 + H_7\Sigma_1 \\ & + F_2\big(\tfrac{3}{2}\Lambda_6 - \Lambda_4 B_2 + \tfrac{1}{4}\Lambda_2 B_2^2 + C_5\Sigma_1 - \tfrac{1}{2}C_3\Sigma_1 B_2\big) \\ & + F_4\big(-\tfrac{1}{2}\Lambda_4 + \tfrac{1}{2}\Lambda_2 B_2 - \tfrac{1}{4}\Lambda_0 B_2^2 \\ & \qquad - \tfrac{1}{2}C_3\Sigma_1 + \tfrac{1}{2}C_1\Sigma_1 B_2\big).\end{aligned}$$

$$\delta C = \mathrm{d}\Lambda - H_3\Lambda, \qquad F = \mathrm{d}C - H_3 C, \qquad \delta B_2 = \mathrm{d}\Sigma_1.$$

The accompanying six-form transformation keeps H_7 invariant.

Relative to [BDKOR06], our B_6, H_7 have the opposite sign and our B_8, H_9, Σ_7 are twice theirs. The RR and B_2 conventions agree.

[BDKOR06], (2.13)–(2.34), with $G_{(0)} = 0$. Component antisymmetrization converted to differential forms.

The common-spinor line and ψ

Hermitian Clifford matrices Γ_a act on the $\mathrm{Spin}(5)$ spinor space $\mathbb{C}^4$:

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\delta_{ab}I, \qquad y \cdot \Gamma := \sum_{a=1}^{5} y_a\Gamma_a.$$

For $([y], n) \in \mathcal{I}$, the null vector y specifies the circle plane and the perpendicular unit vector n its collapse point.

$$\boxed{\ell_{y,n} = \{\psi \in \mathbb{C}^4 : (y \cdot \Gamma)\psi = 0,\ (n \cdot \Gamma)\psi = \psi\}.}$$

Why a line? $\ker(y \cdot \Gamma)$ is two-dimensional; $n \cdot \Gamma$ splits it into one-dimensional ± 1 eigenspaces. We take the positive one.

ψ **is any nonzero vector spanning this line.** The lines form a bundle $\ell \to \mathcal{I}$; $T := \ell^*$ is its dual.

$\psi^{\odot m}$ means the symmetric m-fold tensor power. Projectivization removes the choice of scale; no global choice of ψ is needed.

[M3], incidence flag, common-spinor lines and the explicit projective attachment.