Brane Engineering

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I INTRODUCTION TO BRANES

A Nambu-Goto Action.

By p-branes we will generically mean extended objects with p space and one time dimensions [1] [1]. The dynamics of these objects is defined through the condition of minimizing the worldvolume

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}}. \tag{I.1}$$

For $p = 1$ this is the well known Nambu-Goto action for the bosonic string. The equations of motion that follow (I.1) can be equivalently obtained from the covariant action

$$S = -T_p \int d^{p+1}\xi [\sqrt{-\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - (p-1)\sqrt{-\gamma}], \tag{I.2}$$

with γ^{ij} the worldvolume metric, and $\gamma = |\det \gamma^{ij}|$. The string is special, among extended objects, because only for $p = 1$ the action (I.2) is invariant under Weyl scaling transformations, $\gamma^{ij} \to \Lambda\gamma^{ij}$. In the string case, the action (I.2) can be generalized by replacing the flat Minkowski metric $\eta_{\mu\nu}$ by a generic metric, $g_{\mu\nu}$. Besides, the string can be coupled to other backgrounds, preserving the renormalizability of the theory. These backgrounds correspond to the massless spectrum of the bosonic string, and are the massless spectrum $B_{\mu\nu}(X)$ and the dilaton field $\Phi(X)$,

$$S = -T_1 \int d^2\xi [\sqrt{-\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu} + \epsilon^{ij}\partial_i X^\mu \partial_j X^{nu} B_{\mu\nu}(X) + \frac{1}{4\pi}\frac{\sqrt{-\gamma}}{T_1}\Phi(X)R^{(2)}], \tag{I.3}$$

[1] We apologize for the extremely incomplete set of references included in these notes. More details can be found in a number of general reviews on branes and non perturbative string theory or reference [2], to appear.

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with $R^{(2)}$ the scalar curvature of the worldsheet. In order to preserve Weyl invariance for the sigma model (I.3), we should require the beta functions β_g, β_Φ and β_B to be equal zero. Recalling now the definition of the topological Euler number,

$$\frac{1}{4\pi}\int\sqrt{-\gamma}R^{(2)} = 2 - 2g, \tag{I.4}$$

we observe directly from (I.3) that the vacuum expectation value $< \Phi(X) >$ of the dilaton field defining the background fixes the magnitude of the string coupling constant, g_s. In fact, a Riemann surface Feynman diagram of genus g contains $2g - 2$ string vertices, and contributes to the partition function as $\exp[(2g-2) < \Phi >]$, so that we can define the string coupling constant g_s as

$$g_s = e^{<\Phi>}. \tag{I.5}$$

The term $\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu$ can be interpreted as a minimal coupling of the string to the $B^{\mu\nu}(X)$ field, with the gauge transformations of $B^{\mu\nu}$ defined by

$$\delta B^{\mu\nu} = \partial^\mu \Lambda^\nu - \partial^\nu \Lambda^\mu. \tag{I.6}$$

This can be interpreted claming that the string is a source for the 2-form gauge field $B^{\mu\nu}$. In the generic case of a p-brane extended object, the equivalent to equation (I.3) is

$$S = -T_p \int d^{p+1}\xi[\sqrt{-\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X) +$$

$$\epsilon_{i_1\ldots i_{p+1}} A^{\mu_1\cdots\mu_{p+1}} \partial_{i_1} X^{\mu_1} \ldots \partial_{i_{p+1}} X^{\mu_{p+1}} - (p-1)\sqrt{-\gamma}], \tag{I.7}$$

with A a $p+1$ form, and the p extended object as a source of the field A.

A particularly interresting theory is eleven dimensional supergravity, which contains, in addition to gravitons and gravitinos, a 3-form field, $C_{\mu\nu\rho}$. In this case it is natural to postulate an extended object of dimension two (a membrane) as the source for the $C_{\mu\nu\rho}$ field. The corresponding membrane action would be

$$S = -T_2 \int d^3\xi[\sqrt{-\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X) +$$

$$\epsilon_{ijk} C^{\mu\nu\rho} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho - \sqrt{-\gamma}]. \tag{I.8}$$

However, this action is unfortunately not Weyl invariant, and hence we can not connect (I.8) with the eleven dimensional supergarvity action through a similar argument to the one employed in string theory, where the string action can be related to ten dimensional general relativity equations by impossing Weyl invariance in the form of vanishing beta functions. In the membrane case, the equivalent argument is known as kappa symmetry, and requires an essential ingredient in order to imposse both worldvolume and spacetime supersymmetry. Using kappa symmetry, it is possible to derive, from the membrane action (I.8), the equations of motion of elevan dimensional supergravity. In spite of the fact that eleven dimensional supergravity is not renormalizable, there is one aspect of this theory that makes it, in a certain sense, more fundamental than string theory. In fact, as shown above, the string coupling constant is a free parameter depending on the vacuum expectation value of the dilaton; however, as we move to eleven dimensions this dilaton field becomes part of the eleven dimensional metric tensor, and the string action (I.3) can be obtained through *double dimensional reduction* from the membrane action.

By double dimensional reduction we mean a compactification of both spacetime and worldvolume. More precisely, of the membrane coordinates are decomposed as (X^μ, Y), with $\mu = 0, \ldots, 9$, the double reduction is defined in terms of

$$\begin{aligned} \partial_q Y = \partial_Y C &= 0, \\ Y &= \xi_3, \\ \partial_{1,2} Y &= 0, \\ \partial_3 X &= 0, \end{aligned} \tag{I.9}$$

with ξ_3 the compactified worldvolume coordinate.

If we do not care about trouble related to quantization of membranes, we can continue working some classical facts of membrane dynamics in eleven dimensions. In particular, if we consider the membrane as a source for the 3-form field $C^{\mu\nu\rho}$, we can wonder about its Hodge dual object: in eleven dimensions, the Hodge dual of the strength tensor 4-form, $F^4 = dC$, is a seven form, whose source, to be interpreted as the magnetic dual to the membrane, is an extended object of dimension equal five. We will refer to this object as the M-theory fivebrane. These objects play an interesting role in the dynamics, as ordinary two dimensional membranes can end on them: for a membrane ending on the M-theory fivebrane, the boundary on the fivebrane worldvolume is a string. In the six dimensional worldvolume of the fivebrane, a string is (Hodge) self dual, as it is the source of a 2-form field. Moreover, on the worldvolume of the fivebrane there is a 2-form field, that can couple to the self dual string. Thus, configurations with fivebranes and membranes as the one represented in Figure 1 are allowed, where the membrane can be thought of as an open membrane. Figure 1 determines an special type of boundary conditions on the membrane fields $x^\mu(\xi_i)$. If we take the fivebrane with worldvolume coordinates x^0, x^1, x^2, x^3, x^4 and $x^5$and located at the spacetime positions $x^6 = x^7 = x^8 = x^9 = x^{10} = 0$, then the membrane fields X^μ, with $\mu = 6, \ldots, 10$ are fixed to zero at the boundary of the membrane, so that we are impossing on the membrane fields Dirichlet boundary conditions for X^μ with $\mu = 6, \ldots, 10$.

B Membranes and D-Branes.

The vertex defined in Figure 1 can be dimensionally reduced twice, compactifying one spacetime coordinate and one worldvolume coordinate for the M-theory fivebrane and the membrane. Then, some kind of fourbrane should be expected to arise from the

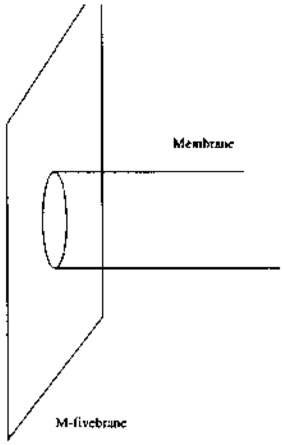

FIGURE 1. A membrane with its boundary on a fivebrane.

fivebrane, besides from a string coming from the membrane; this string will now end on the fourbrane obtained through double dimensional reduction of the fivebrane. If all these steps are consistent, what we will get is a four dimensional object that should now be considered as part of the ten dimensional string spectrum. Moreover, the string will be allowed to end on this four dimensional object. In order to understand the nature of this object we must recall some well known facts about superstring theory. We will focus the discussion on type IIA and type type IIB superstrings. The field content of these theories in the Ramond-Ramond sector is

$$\begin{aligned} \text{IIA} &\to A^1, \quad A^3, \\ \text{IIB} &\to A^2, \quad A^4. \end{aligned} \tag{I.10}$$

As ten dimensional type IIA supergravity can be obtained through dimensional reduction eleven dimensional supergravity, we should search for the four dimensional extended object in type IIA string theory. A four dimensional object is a source for a 5-form field which, in ten dimensions, is the Hodge dual to the 3-form field, A^3. Thus, a good candidate for the double dimensional reduction of the M-theory fivebrane should be a source for the Ramond-Ramond field Hodge dual to A^3. Moreover, on this brane strings can end, which means, by the same argument as above, that on the worldvolume of the brane a 1-form field A^1 must be allowed to exist. As discussed above, the boundary conditions for the string in the $x^6, \ldots, x^{10}$ directions are of Dirichlet type (recall that by double dimensional reduction we have compactified a worldvolume coordinate of the fivebrane). The previous set of properties define what is meant as s Dirichlet-brane; in the example under consideration, a Dirichlet-fourbrane. Hence, the above discussion can be summarized through the diagram in Figure 2.

Moreover, when the sources of all Ramond-Ramond fields in (I.10) are interpreted as

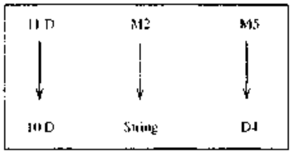

FIGURE 2. Correspondence between eleven dimensional and ten dimensional objects.

Dirichlet, the picture in Figure 3 arises, where the arrows represent the Hodge duality relation in ten dimensions.

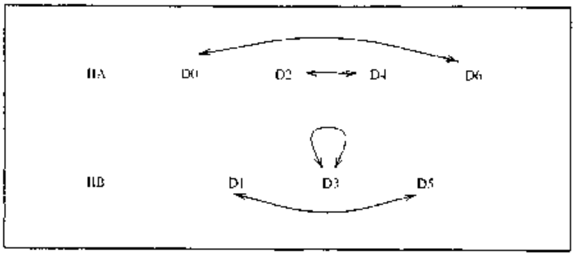

FIGURE 3. Dirichlet-brane spectrum in ten dimensional string theory.

As type IIA and type IIB string theories are, after compactification on S^1 down to nine dimensions. related by T-duality, we can naturally wonder about the action of T-duality on the D-brane spectrum. One of the main abilities of D-branes is defining regions of spacetime where Dirichlet boundary conditions can be impossed for strings. Hence, if a D-fourbrane is located at the $x^5 = \cdots = x^9 = 0$ values, the string worldvolume coordinates at the boundary satisfy the Dirichlet conditions $x^5 = \cdots = x^9 = 0$, while ordinary Neumann boundary conditions. $\partial_{\bar{n}} x^i = 0$ are satisfied for $i = 1, 2, 3, 4$. We can then consider one of the Dirichlet x^j coordinates to live on a circle, S^1. Under T-duality we transform. along this direction, Neumann into Dirichlet boundary conditions, and reciprocally. Thus, under T-daulity along one of the $x^5, \ldots, x^9$ directions we transform the four dimensional brane into a five dimensional brane, and under T-duality along one of the worldvolume directions, $x^1, \ldots, x^4$, we transform a D-fourbarne in type IIA into a D-threebrane in type IIB string theory. The whole set of connections between D-branes under T-duality is given in Figure 4.

The set of tools for brane enginnering is completed once we known the kind of branes that can be obtained in ten dimensions from the M-theory membrane and fivebrane when performing direct dimensional reduction. In this case, the M-theory membrane goes into the D-twobrane of type string theory, and the fivebrane becomes the ten dimensional solitonic fivebrane[2]

The final image arises when $Sl(2; \mathbf{Z})$ duality of type IIB string theory is included:

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[2] The three different categories of branes are classified in terms of their tension: order one for fundamental objects, order $\frac{1}{g_s}$ for Dirichlet branes, and order $\frac{1}{g_s^2}$ for the solitonic fivebrane.

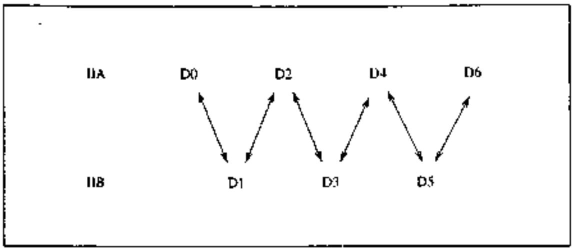

FIGURE 4. T-duality relates type IIA and type IIB D-branes.

$Sl(2;\mathbf{Z})$ S duality exchanges the two form in the R-R sector with the two form field $B_{\mu\nu}$ in the NS-NS sector, apperaing in the string action (I.3); thus, it exchanges the D-string with the fundamental string, and the D-fivebrane with the solitonic fivebrane. The whole discussion is summarized in Figure 5.

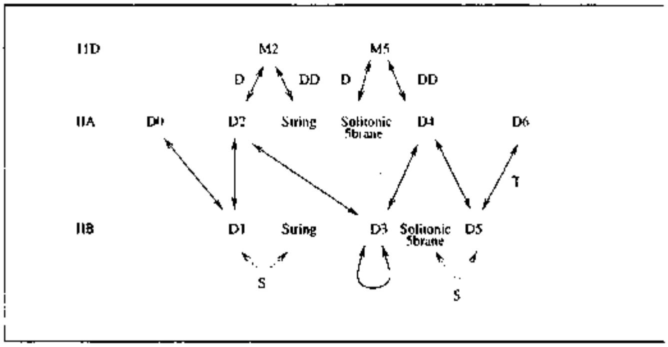

FIGURE 5. Set of correspondences for eleven dimensional and ten dimensional extended objects.

C Brane Vertices.

In this section we will define brane vertices through dimensional reduction and T and S dualities, starting from the generic vertex between a Dirichlet brane and a fundamental string, shown in Figure 6.

This is an allowed vertex in type IIA or type IIB string depending on the even or odd value of p. We will consider the $p = 3$ value of type IIB strings. Then, by S-duality we can pass from a vertex between the D-threebrane (which is self-dual) of type IIB string theory and the string, to a vertex where a D-string ends on the D-threebrane. Now, T-duality transformations in the directions orthogonal to the worldvolumes of the D-3brane and the D-1brane lead to a type IIA vertex between a D-4brane and a D-2brane and, finally, a vertex in type IIB string theory where a D-3brane ends on D-5brane, as shown in Figure 7.

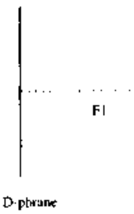

FIGURE 6. A D-brane defined as a boundary for a fundamental string.

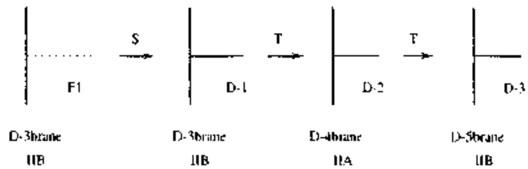

FIGURE 7. T and S-duality determine the allowed ten dimensional vertices.

S-duality on the last vertex of Figure 7 provides a new vertex in type IIB string theory, between the solitonic fivebrane and a D-3brane.

More vertices can yet be obtained upon dimensional reduction. We will now start with the eleven dimensional vertex between the M-theory fivebrane and the membrane. By dimensional reduction, we get the type IIA vertex between the solitonic fivebrane and the D-2brane. Now, we can again perform two T-duality transformations. The result, shown in Figure 8, is a vertex between the solitonic fivebrane and a D-4brane in type IIA.

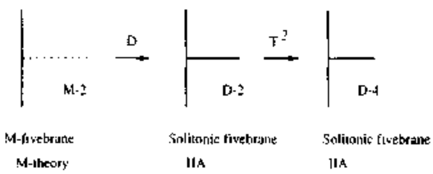

FIGURE 8. A vertex from eleven dimensions.

In the next section we will define brane configurations in flat spacetime using the previous set of brane vertices [3]- [20].

II THREE DIMENSIONAL FIELD THEORIES WITH $N = 4$ SUPERSYMMETRY.

Let us now consider some brane configurations build up using the vertices (5, 3) and $(5^{NS}, 3)$ in type IIB theory [3]. In particular, we will consider solitonic fivebranes, with worldvolume coordinates x^0, x^1, x^2, x^3, x^4 and x^5, located at some definite values of x^6, x^7, x^8 and x^9. It is convenient to organize the coordinates of the fivebrane as

$(x^6, \vec{\omega})$ where $\vec{\omega} = (x^7, x^8, x^9)$. By construction of the vertex, the D-3brane will share two worldvolume coordinates, in addition to time, with the fivebrane. Thus, we can consider D-3branes with worldvolume coordinates x^0, x^1, x^2 and x^6. If we put a D-3brane in between two solitonic fivebranes, at x_2^6 and x_1^6 positions in the x^6 coordinate, then the worldvolume of the D-3brane will be finite in the x^6 direction (see Figure 9).

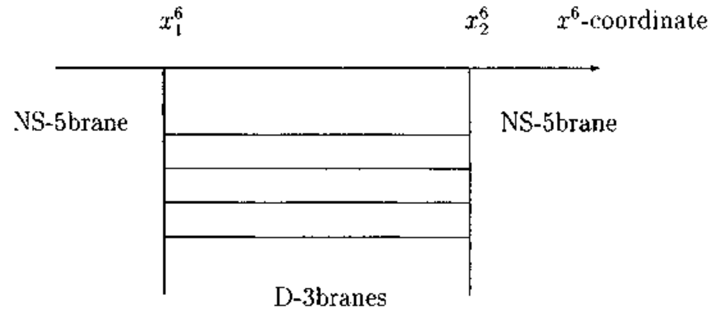

FIGURE 9. Solitonic fivebranes with n Dirichlet threebranes stretching along them.

Therefore, the macroscopic physics, i. e., for scales larger than $|x_2^6 - x_1^6|$, can be effectively described by a $2+1$ dimensional theory. In order to unravel what kind of $2+1$ dimensional theory, we are obtaining through this brane configuration, we must first work out the type of constraint impossed by the fivebrane boundary conditions. In fact, the worldvolume low energy lagrangian for a D-3brane is a $U(1)$ gauge theory. Once we put the D-3brane in between two solitonic fivebranes we imposse Neumann boundary conditions, in the x^6 direction, for the fields living on the D-3brane worldvolume. This means in particular that for scalar fields we imposse

$$\partial_6 \phi = 0 \tag{II.1}$$

and, for gauge fields,

$$F_{\mu 6} = 0, \qquad \mu = 0, 1, 2. \tag{II.2}$$

Thus, the three dimensional $U(1)$ gauge field, A_μ, with $\mu = 0, 1, 2$, is unconstrained which already means that we can interpret the effective three dimensional theory as a $U(1)$ gauge theory for one D-3brane, and therefore as a $U(n)$ gauge theory for n D-3branes. Next, we need to discover the amount of supersymmetry left unbroken by the brane configuration. If we consider Dirichlet threebranes, with worldvolume coordinates x^0, x^1, x^2 and x^6, then we are forcing the solitonic fivebranes to be at positions $(x^6, \vec{\omega}_1)$ and $(x^6, \vec{\omega}_2)$, with $\vec{\omega}_1 = \vec{\omega}_2$. In this particular case, the allowed motion for the D-3brane is reduced to the space $\mathbf{R}^3$, with coordinates x^3, x^4 and x^5. These are the coordinates on the fivebrane worldvolume where the D-3brane ends. Thus, we have defined on the D-3brane three scalar fields. By condition (II.1), the values of these scalar fields can be constrained to be constant on the x^6 direction. What this in practice means is that the two ends of the of the D-3brane have the same x^3, x^4 and x^5 coordinates. Now, if we combine these three scalar fields with the $U(1)$ gauge field A_μ, we get an $N = 4$ vector multiplet in three dimensions. Therefore, we can conclude that our effective three dimensional theory for n parallel D-3branes suspended between two solitonic fivebranes (Figure 1) is a gauge theory with $U(n)$ gauge group, and $N = 4$ supersymmetry. Denoting by $\vec{v}$ the vector (x^3, x^4, x^5), the Coulomb branch of this theory is parametrized by the v_i positions of the n D-3branes (with i labelling each brane). In addition, we have, as discussed in chapter II, the dual photons for each $U(1)$ factor. In this way, we get the hyperkähler structure of the Coulomb branch of the moduli. Hence, a direct way to get supersymmetry preserved by the brane configuration is as follows. The supersymmetry charges are defined as

$$\epsilon_L Q_L + \epsilon_R Q_R, \tag{II.3}$$

where Q_L and Q_R are the supercharges generated by the left and right-moving worldsheet degrees of freedom, and ϵ_L and ϵ_R are ten dimensional spinors. Each solitonic $p$brane, with worldvolume extending along $x^0, x^1, \ldots, x^p$, imposses the conditions

$$\epsilon_L = \Gamma_0 \ldots \Gamma_p \epsilon_L, \;\; \epsilon_R = -\Gamma_0 \ldots \Gamma_p \epsilon_R, \tag{II.4}$$

in terms of the ten dimensional Dirac gamma matrices, Γ_i; on the other hand, the D-$p$branes, with worldvolumes extending along $x^0, x^1, \ldots, x^p$, imply the constraint

$$\epsilon_L = \Gamma_0 \Gamma_1 \ldots \Gamma_p \epsilon_R. \tag{II.5}$$

Thus, we see that NS solitonic fivebrane, with worldvolume located at x^0, x^1, x^2, x^3, x^4 and x^5, and equal values of $\vec{\omega}$, and Dirichlet threebranes with worldvolume along x^0, x^1, x^2 and x^6, preserve eight supersymmetries on the D-3brane worldvolume or, equivalently, $N = 4$ supersymmetry on the effective three dimensional theory.

The brane array just described allows a simple computation of the gauge coupling constant of the effective three dimensional theory: by standard Kaluza-Klein reduction on the finite x^6 direction, after integrating over the (compactified) x^6 direction to reduce the lagrangian to an effective three dimensional lagrangian, the gauge coupling constant is given by

$$\frac{1}{g_3^2} = \frac{|x_6^2 - x_6^1|}{g_4^2}, \tag{II.6}$$

in terms of the four dimensional gauge coupling constant. Naturally, (II.6) is a classical expression that is not taking into account the effect on the fivebrane position at x^6 of the D-3brane ending on its worldvolume. In fact, we can consider the dependence of x^6 on the coordinate $\vec{v}$, normal to the position of the D-3brane. The dynamics of the fivebranes should then be recovered when the Nambu-Goto action of the solitonic fivebrane is minimized. Far from the influence of the points where the fivebranes are located (at large values of x^3, x^4 and x^5), the equation of motion is simply three dimensional Laplace's equation,

$$\nabla^2 x^6(x^3, x^4, x^5) = 0, \tag{II.7}$$

with solution

$$x^6(r) = \frac{k}{r} + \alpha, \tag{II.8}$$

where k and α are constants depending on the threebrane tensions, and r is the spherical radius at the point (x^3, x^4, x^5). From (II.8), it is clear that there is a well defined limit as $r \to \infty$; hence, the difference $x_2^6 - x_1^6$ is a well defined constant, $\alpha_2 - \alpha_1$, in the $r \to \infty$ limit.

Part of the beauty of brane technology is that it allows to obtain very strong results by simply performing geometrical brane manipulations. We will now present one example, concerning our previous model. If we consider the brane configuration from the point of view of the fivebrane, the n suspended threebranes will look like n magnetic monopoles. This is really suggesting since, as described in chapter II, we know that the Coulomb branch moduli space of $N = 4$ supersymmetric $SU(n)$ gauge theories is isomorphic to the moduli space of BPS monopole configurations, with magnetic charge equal n. This analogy can be put more precisely: the vertex $(5^{NS}, 3)$ can, as described above, be transformed into a $(3, 1)$ vertex. In this case, from the point of view of the threebrane, we have a four dimensional gauge theory with $SU(2)$ gauge group broken down to $U(1)$, and n magnetic monopoles. Notice that by passing from the configuration build up ussing $(5^{NS}, 3)$ vertices, to that build up with the $(3, 1)$ vertex, the Coulomb moduli remains the same.

Next, we will work out the same configuration, but now with the vertex $(5, 3)$ made out of two Dirichlet branes. The main difference with the previous example comes from the boundary conditions (II.1) and (II.2), which should now be replaced by Dirichlet boundary conditions. We will choose as worldvolume coordinates for the D-5branes x^0, x^1, x^2, x^7, x^8 and x^9, so that they will be located at some definite values of x^3, x^4, x^5 and x^6. As before, let us denote this positions by $(\vec{m}, x^6)$, where now $\vec{m} = (x^3, x^4, x^5)$. An equivalent configuration to the one studied above will be now a set of two D-5branes, at some points of the x^6 coordinate, that we will again call x_1^6 and x_2^6, subject to $\vec{m}_1 = \vec{m}_2$, with D-3branes stretching between them along the x^6 coordinate, with worldvolume extending again along the coordinates x^0, x^1, x^2 and x^6 (Figure 10). Our task now will

be the description of the effective three dimensional theory on these threebranes. The end points of the D-3branes on the fivebrane worlvolumes will now be parametrized by values of x^7, x^8 and x^9. This means that we have three scalar fields in the effective three dimensional theory. The scalar fields corresponding to the coordinates x^3, x^4, x^5 and x^6 of the threebranes are forzen to the constant values where the fivebranes are located. Next, we should consider what happens to the $U(1)$ gauge field on the D-3brane worldvolume. Impossing Dirichlet boundary conditions for this field is equivalent to

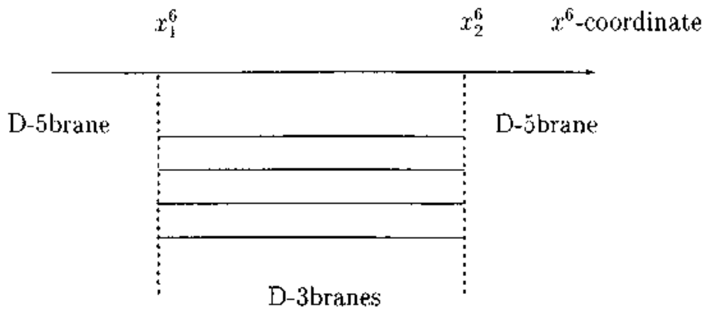

FIGURE 10. Dirichlet threebranes extending between a pair of Dirichlet fivebranes (in dashed lines).

$$F_{\mu\nu} = 0, \quad \mu, \nu = 0, 1, 2, \tag{II.9}$$

i. e., there is no electromagnetic tensor in the effective three dimensional field theory. Before going on, it would be convenient summarizing the rules we have used to impose the different boundary conditions. Consider a D-$p$brane, and let M be its worldvolume manifold, and $B = \partial M$ the boundary of M. Neumann and Dirichlet boundary conditions for the gauge field on the D-$p$brane worldvolume are defined respectively by

$$\begin{aligned} N &\longrightarrow F_{\mu\rho} = 0, \\ D &\longrightarrow F_{\mu\nu} = 0, \end{aligned} \tag{II.10}$$

where μ and ν are directions of tangency to B, and ρ are the normal coordinates to B. If B is part of the worldvolume of a solitonic brane, we will imposse Neumann conditions, and if it is part of the worldvolume of a Dirichlet brane, we will imposse Dirichlet conditions. Returning to (II.9), we see that on the three dimensional effective theory, the only non vanishing component of the four dimensional strenght tensor is $F_{\mu 6} \equiv \partial_\mu b$. Therefore, all together we have four scalar fields in three dimensions or, equivalently, a multiplet with $N = 4$ supersymmetry. Thus, the theory defined by the n suspended D-3branes in between a pair of D-5branes, is a theory of n $N = 4$ massless hypermultiplets.

There exits a different way to interpret the theory, namely as a magnetic dual gauge theory. In fact, if we perform a duality transformation in the four dimensional $U(1)$ gauge theory, and use magnetic variables $*F$, instead of the electric field F, what we get in three dimensions, after impossing D-boundary conditions, is a dual photon, or a magnetic $U(1)$ gauge theory.

The configuration chosen for the worldvolume of the Dirichlet and solitonic fivebranes yet allows a different configuration with D-3branes suspended between a D-5brane and a NS-5brane. This is in fact consistent with the supersymmetry requirements (II.4) and (II.5). Namely, for the Dirichlet fivebrane we have

$$\epsilon_L = \Gamma_0\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9\epsilon_R. \tag{II.11}$$

The solitonic fivebrane imposses

$$\epsilon_L = \Gamma_0\dots\Gamma_5\epsilon_L,\ \ \epsilon_R = -\Gamma_0\dots\Gamma_5\epsilon_R, \tag{II.12}$$

while the suspended threebranes imply

$$\epsilon_L = \Gamma_0\Gamma_1\Gamma_2\Gamma_6\epsilon_R, \tag{II.13}$$

which are easily seen to be consistent. The problem now is that the suspended D-3brane is frozen. In fact, the position (x^3, x^4, x^5) of the end point of the NS-5brane is equal to the position $\vec{m}$ of the D-5brane, and the position (x^7, x^8, x^9) of the end point on the D-5brane is forced to be equal to the position $\vec{\omega}$ of the NS-5brane. The fact that the D-3brane is frozen means that the theory defined on it has no moduli, i. e., posseses a mass gap.

Using the vertices between branes we have described so far we can build quite complicated brane configurations. When Dirichlet threebranes are placed to the right and left of a fivebrane, open strings can connect the threebranes at different sides of the fivebrane. They will represent hypermultiplets transforming as $(k_1, \bar{k}_2)$, with k_1 and k_2 the number of threebranes to the left and right, respectively, of the fivebrane. In case the fivebrane is solitonic, the hypermultiplets are charged with respect to an electric group, while in case it is a D-5brane, they are magnetically charged. Another possibility is that with a pair of NS-5branes, with D-3branes extending between them, and also a D-5brane located between the two solitonic fivebranes. A massless hypermultiplet will now appear whenever the (x^3, x^4, x^5) position of the D-3brane coincides with the $\vec{m} = (x^3, x^4, x^5)$ position of the D-5brane.

So far we have used brane configurations for representing different gauge theories. In these brane configurations we have considered two different types of moduli. For the examples described above, these two types of moduli are as follows: the moduli of the effective three dimensional theory, corresponding to the different positions where the suspended D-3branes can be located, and the moduli corresponding to the different locations of the fivebranes, which are being used as boundaries. This second type of moduli specifies, from the point of view of the three dimensional theory, different coupling constants; hence, we can move the location of the fivebranes, and follow the changes taking place in the effective three dimensional theory. Let us then consider a case with two solitonic branes, and a Dirichlet fivebrane placed between them. Let us now move the NS-5brane on the left of the D-5brane to the right. In doing so, there is a moment when both fivebranes meet, sharing a common value of x^6. If the interpretation of the hypermultiplet we have presented above is correct, we must discover what happens to the hypermultiplet after this exchange of branes has been performed. In order to maintain the hypermultiplet, a new D-3brane should be created after the exchange, extending from the right solitonic fivebrane to the Dirichlet fivebrane. To prove this we will need D-brane dynamics at work. Let us start considering two interpenetrating closed loops, C and C', and suppose electrically charged particles are moving in C, while magnetically charged particles move in C'.

The linking number $L(C,C')$ can be defined using the standard Wilson and 't Hooft loops. Namely, we can measure the electric flux passing through C' or, equivalently, compute $B(C')$, or measure the magnetic flux passing through C, i. e., the Wilson line $A(C)$. In both cases, what we are doing is integrating over C' and C the dual to the field created by the particle moving in C and C', respectively. Let us now extend this simple result to the case of fivebranes. A fivebrane is a source of 7-form tensor field, and its dual is therefore a 3-form. We will call this 3-form H_{NS} for NS-5branes, and H_D for D-5branes. Now, let us consider the worldvolume of the two fivebranes,

$$\begin{aligned} &\mathbf{R}^3 \times Y_{NS}, \\ &\mathbf{R}^3 \times Y_D. \end{aligned} \tag{II.14}$$

We can now define the linking number as we did before, in the simpler case of a particle:

$$L(Y_{NS}, Y_D) = -\int_{Y_{NS}} H^D = \int_{Y_D} H^{NS}. \tag{II.15}$$

The 3-form H^{NS} is locally dB_{NS}. Since we have no sources for H^{NS}, we can use $H^{NS} = dB_{NS}$ globally; however, this requires B to be globally defined, or gauge invariant. In type IIB string theory, B is not gauge invariant; however, on a D-brane we can define the combination $B_{NS} - F_D$, which is invariant, with F_D the two form for the $U(1)$ gauge field on the D-brane. Now, when the D-5brane and the NS-5brane do not intersect, the linking number is obviously zero. When they intersect, this linking number changes, which means that (II.15) should, in that case, be non vanishing. Writing

$$\int_{Y_D} H^{NS} = \int_{Y_D} dB_{NS} - dF_D, \tag{II.16}$$

we observe that the only way to get linking numbers would be adding sources for F_D. These sources for F_D are point like on Y_D, and are therefore the D-3branes with world-volume $\mathbf{R}^3 \times C$, with C ending on Y_D, which is precisely the required appearance of extra D-3branes.

III D-BRANE DESCRIPTION OF SEIBERG-WITTEN SOLUTION.

In the previous example we have considered type IIB string theory and three and fivebranes. Now, let us consider type IIA strings, where we have fourbranes that can be used to define, by analogy with the previous picture, $N = 2$ four dimensional gauge theories [21]. The idea will again be the use of solitonic fivebranes, with sets of fourbranes in between. The only difference now is that the fivebrane does not create a RR field in type IIA string theory and, therefore, the physics of the two parallel solitonic fivebranes does not have the interpretation of a gauge theory, as was the case for the type IIB configuration above described [6].

Let us consider configurations of infinite solitonic fivebranes, with worldvolume coordinates x^0, x^1, x^2, x^3, x^4 and x^5, located at $x^7 = x^8 = x^9 = 0$ and at some fixed value of the x^6 coordinate. In addition, let us introduce finite Dirichlet fourbranes, with worldvolume coordinates x^0, x^1, x^2, x^3 and x^6, which terminate on the solitonic fivebranes; thus, they are finite in the x^6 direction. On the fourbrane worldvolume, we can define a macroscopic four dimensional field theory, with $N = 2$ supersymmetry. This four dimensional theory will, as in the type IIB case considered in previous section, be defined by standard Kaluza-Klein dimensional reduction of the five dimensional theory defined on the D-4brane worldvolume. Then, the bare coupling constant of the four dimensional theory will be

$$\frac{1}{g_4^2} = \frac{|x_6^2 - x_6^1|}{g_5^2}, \tag{III.1}$$

in terms of the five dimensional coupling constant. Moreover, we can interpret as classical moduli parameters of the effective field theory on the dimensionally reduced worldvolume of the fourbrane the coordinates x^4 and x^5, which locate the points on the fivebrane worldvolume where the D-4branes terminate.

In addition to the Dirichlet fourbranes and solitonic fivebranes, we can yet include Dirichlet sixbranes, without any further break of supersymmetry on the theory in the worldvolume of the fourbranes. To prove this, we notice that each NS-5brane imposes the projections

$$\epsilon_L = \Gamma_0 \ldots \Gamma_5 \epsilon_L, \;\; \epsilon_R = -\Gamma_0 \ldots \Gamma_5 \epsilon_R, \tag{III.2}$$

while the D-4branes, with worldvolume localized at x^0, x^1, x^2, x^3 and x^6, imply

$$\epsilon_L = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_6 \epsilon_R. \tag{III.3}$$

Conditions (III.2) and (III.3) can be recombined into

$$\epsilon_L = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_7 \Gamma_8 \Gamma_9 \epsilon_R, \tag{III.4}$$

which shows that certainly sixbranes can be added with no additional supersymmetry breaking.

The solitonic fivebranes break half of the supersymmetries, while the D-6brane breaks again half of the remaining symmetry, leaving eight real supercharges, which leads to four dimensional $N = 2$ supersymmetry.

As we will discuss later on, the sixbranes of type IIA string theory can be used to add hypermultiplets to the effective macroscopic four dimensional theory. In particular, the mass of these hypermultiplets will become zero whenever the D-4brane meets a D-6brane.

One of the main achievements of the brane representations of supersymmetric gauge theories is the ability to represent the different moduli spaces, namely the Coulomb and Higgs branches, in terms of the brane motions left free. For a configuration of k fourbranes connecting two solitonic fivebranes along the x^6 direction, as the one we have described above, the Coulomb branch of the moduli space of the four dimensional theory is parametrized by the different positions of the transversal fourbranes on the fivebranes. When N_f Dirichlet sixbranes are added to this configuration, what we are describing is the Coulomb branch of a four dimensional field theory with $SU(N_c)$ gauge group (in case N_c is the number of D-4branes we are considering), with N_f flavor hypermultiplets. In this brane representation, the Higgs branch of the theory is obtained when each fourbrane is broken into several pieces ending on different sixbranes: the locations of the D-4branes living between two D-6branes determine the Higgs branch. However, we will mostly concentrate on the study of the Coulomb branch for pure gauge theories.

As we know from the Seiberg-Witten solution of $N = 2$ supersymmetric gauge theories, the classical moduli of the theory is corrected by quantum effects. There are two types of effects that enter the game: a non vanishing beta function (determined at one loop) implies the existence, in the assymptotically free regime, of a singularity at the infinity point in moduli space, and strong coupling effects, which imply the existence of extra singularities, where some magnetically charged particles become massless. The problem we are facing now is how to derive such a complete characterization of the quantum moduli space of four dimensional $N = 2$ supersymmetric field theory directly from the dynamics governing the brane configuration. The approach to be used is completely different from a brane construction in type IIA string theory to a type IIB brane configuration. In fact, in the type IIB case, employed in the description of the preceding section of three dimensional $N = 4$ supersymmetric field theories, we can pass from weak to strong coupling through the standard $Sl(2, \mathbf{Z})$ duality of type IIB strings; hence, the essential ingredient we need is to know how brane configurations transform under this duality symmetry. In the case of type IIA string theory, the situation is more complicated, as the theory is not $Sl(2, \mathbf{Z})$ self dual. However, we know that the strong coupling limit of type IIA dynamics is described by the eleven dimensional M-theory; therefore, we should expect to recover the strong coupling dynamics of four dimensional $N = 2$ supersymmetric gauge theories using the M-theory description of strongly coupled type IIA strings.

Let us first start by considering weak coupling effects. The first thing to be noticed, concerning the above described configuration of N_c Dirichlet fourbranes extending along the x^6 direction between two solitonic fivebranes, where only a rigid motion of the transversal fourbranes is allowed, is that this simple image is missing the classical dynamics of the fivebranes. In fact, in this picture we are assuming that the x^6 coordinate on the fivebrane worldvolume is constant, which is in fact a very bad approximation. Of course, one physical requirement we should impose to a brane configuration, as we did in the case of the type IIB configurations of the previous section, is that of minimizing the total worldvolume action. More precisely, what we have interpreted as Coulomb or Higgs branches in term of free motions of some branes entering the configuration, should correspond to zero modes of the brane configuration, i. e., to changes in the configuration preserving the condition of minimum worldvolume action (in other words, changes in the brane configuration that do not constitute an energy expense). The coordinate x^6 can be assumed to only depend on the "normal" coordinates x^4 and x^5, which can be combined into the complex coordinate

$$v \equiv x^4 + ix^5, \tag{III.5}$$

representing the normal to the position of the transversal fourbranes. Far away from the position of the fourbranes, the equation for x^6 reduces now to a two dimensional laplacian,

$$\nabla^2 x^6(v) = 0, \tag{III.6}$$

with solution

$$x^6(v) = k \ln |v| + \alpha, \tag{III.7}$$

for some constants k and α, that will depend on the solitonic and Dirichlet brane tensions. As we can see from (III.7), the value of x^6 will diverge at infinity. This constitutes, as a difference with the type IIB case, a first problem for the interpretation of equation (III.1). In fact, in deriving (III.1) we have used a standard Kaluza-Klein argument, where the four dimensional coupling constant is defined by the volume of the internal space (in this ocasion, the x^6 interval between the two solitonic fivebranes). Since the Dirichlet four branes will deform the solitonic fivebrane, the natural way to define the internal space would be as the interval defined by the values of the coordinate x^6 at v equal to infinity, which is the region where the disturbing effect of the four brane is very likely vanishing, as was the case in the definition of the effective three dimensional coupling in the type IIB case. However, equations (III.6) and (III.7) already indicate us that this can not be the right picture, since these values of the x^6 coordinate are divergent. Let us then consider a configuration with N_c transversal fourbranes. From equations (III.1) and (III.7), we get, for large v,

$$\frac{1}{g_4^2} = -\frac{2kN_c \ln(v)}{g_5^2}, \tag{III.8}$$

where we have differentiated the direction in which the fourbranes pull the fivebrane. Equation (III.8) can have a very nice meaning if we interpret it as the one loop renormalization group equation for the effective coupling constant. In order to justify this interpretation, let us first analyze the physical meaning of the parameter k. From equation (III.7), we notice that if we move in v around a value where a fourbrane is located (that we are assuming is $v = 0$), we get the monodromy transformation

$$x^6 \to x^6 + 2\pi i k. \tag{III.9}$$

This equation can be easily understood in M-theory, where we add an extra eleventh dimension, x^{10}, that we use to define the complex coordinate

$$x^6 + ix^{10}. \tag{III.10}$$

Now, using the fact that the extra coordinate is compactified on a circle of radius R we can, from (III.9), identify k with R. From a field theory point of view, we have a similar interpretation of the monodromy of (III.8), but now in terms of a change in the theta parameter. Let us then consider the one loop renormalization group equation for $SU(N_c)$ $N=2$ supersymmetric gauge theories without hypermultiplets,

$$\frac{4\pi}{g_4^2(u)} = \frac{4\pi}{g_0^2} - \frac{2N_c}{4\pi}\ln\left(\frac{u}{\Lambda}\right), \tag{III.11}$$

with Λ the dynamically generated scale, and g_0 the bare coupling constant. The bare coupling constant can be absorved through a change in Λ; in fact, when going from Λ to a new scale Λ', we get

$$\frac{4\pi}{g_4^2(u)} = \frac{4\pi}{g_0^2} - \frac{2N_c}{4\pi}\ln\left(\frac{u}{\Lambda'}\right) - \frac{2N_c}{4\pi}\ln\left(\frac{\Lambda'}{\Lambda}\right) \tag{III.12}$$

Thus, once we fix a reference scale Λ_0, the dependence on the scale Λ of the bare coupling constant is given by

$$-\frac{2N_c}{4\pi}\ln\left(\frac{\Lambda}{\Lambda_0}\right) \tag{III.13}$$

It is important to distinguish the dependence on Λ of the bare coupling constant, and the dependence on u of the effective coupling. In the brane configuration approach, the coupling constant defined by (III.8) is the bare coupling constant of the theory, as determined by the definite brane configuration. Hence, it is (III.13) that we should compare with (III.8); naturally, some care is needed concerning units and scales. Once we interpret k as the radius of the internal S^1 of M-theory we can, in order to make contact with (III.13), identify g_5^2 with the radius of S^1, which in M-theory units is given by

$$R = gl_s, \tag{III.14}$$

with g the string coupling constant, and l_s the string length. Therefore, (III.1) should be modified to

$$\frac{1}{g_4^2} = \frac{x_2^6 - x_1^6}{gl_s} = -2N_c\ln(v/l_s), \tag{III.15}$$

which is dimensionless. We can fix the scale of the four dimensional theory, Λ, in such a way that $\frac{1}{g_4^2(u;\Lambda)} = -2N_c\ln\left(\frac{u}{\Lambda}\right)$. With this normalization the bare coupling constant at the string scale is given by (III.13) as

$$\frac{4\pi}{g_0^2(l_s^{-1})} = -\frac{2N_c}{4\pi}\ln\left(\frac{l_s^{-1}}{\Lambda}\right) \tag{III.16}$$

Comparing now (III.15) and (III.16), we notice that $\frac{1}{v}$ can be used as the scale of the theory.

Defining now an adimensional complex variable,

$$s \equiv (x^6 + ix^{10})/R, \tag{III.17}$$

and a complexified coupling constant,

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}, \tag{III.18}$$

we can generalize (III.15) to

$$-i\tau_\alpha(v) = s_2(v) - s_1(v), \tag{III.19}$$

for the simple configuration of branes defining a pure gauge theory. Now, we can clearly notice how the monodromy, as we move around $v = 0$, means a change $\theta \to \theta + 2\pi N_c$.

It is important stressing that nothing, a priori, enforces us to interpret the θ-parameter of the four dimensional field theory in terms of the extra dimension coming from M-theory. However, such an interpretation is very natural once the solitonic fivebrane of type IIA string theory is understood as the direct dimensional reduction of the M-theory fivebrane.

Let us now come back, for a moment, to the bad behaviour of $x^6(v)$ at large values of v. A possible way to solve this problem is modifying the configuration of a single pair of fivebranes, with N_c fourbranes extending between them, to consider a larger set of solitonic fivebranes. Labelling this fivebranes by α, with $\alpha = 0, \ldots, n$, the corresponding x^6_α coordinate will depend on v as follows:

$$x^6(v)_\alpha = R\sum_{i=1}^{q_L} \ln|v - a_i| - R\sum_{j=1}^{q_R} \ln|v - b_j|, \tag{III.20}$$

where q_L and q_R represent, respectively, the number of D-4branes to the left and right of the α^{th} fivebrane. As is clear from (III.20), a good behaviour at large v will only be possible if the numbers of fourbranes to the right and left of a fivebrane are equal, $q_L = q_R$, which somehow amounts to compensating the perturbation created by the fourbranes at the sides of a fivebrane. The four dimensional field theory represented now by this brane array will have a gauge group $\prod_\alpha U(k_\alpha)$, where k_α is the number of transversal fourbranes between the $\alpha - 1$ and α^{th} solitonic fivebranes. Now, minimization of the worldvolume action will require not only taking into account the dependence of x^6 on v, but also the fourbrane positions on the NS-5brane, represented by a_i and b_j in (III.20), on the four dimensional worldvolume coordinates x^0, x^1, x^2 and x^3. Using (III.20), and the Nambu-Goto action for the solitonic fivebrane, we get, for the kinetic energy,

$$\int d^4x d^2v \sum_{\mu=0}^{3} \partial_\mu x^6(v, a_i(x^\mu), b_j(x^\mu))\partial^\mu x^6(v, a_i(x^\mu), b_j(x^\mu)). \tag{III.21}$$

Convergence of the v integration implies

$$\partial_\mu(\sum_i a_i - \sum_j b_j) = 0 \tag{III.22}$$

or, equivalently,

$$\sum_i a_i - \sum_j b_j = \text{constant}. \tag{III.23}$$

This "constant of motion" is showing how the average of the relative position between left and right fourbranes must be hold constant. Since the Coulomb branch of the $\prod_\alpha U(k_\alpha)$ gauge theory will be associated with different configurations of the transversal fourbranes, constraint (III.23) will reduce the dimension of this space. As we know from our general discussion on D-branes, the $U(1)$ part of the $U(k_\alpha)$ gauge group can be associated to the motion of the center of mass. Constraint (III.23) implies that the center of mass is frozen in each sector. With no semi-infinite fourbranes to the right, we have that $\sum_i a_i = 0$; now, this constraint will force the center of mass of all sectors to vanish, which means that the field theory we are describing is $\prod_\alpha SU(k_\alpha)$, instead of $\prod_\alpha U(k_\alpha)$. The same result can be derived if we include semi-infinite fourbranes to the left and right of the first and last solitonic fivebranes: as they are infinitely massive, we can assume that they do not move in the x^4 and x^5 directions. An important difference will appear if we consider periodic configurations of fivebranes, upon compactification of the x^6 direction to a circle: in this case, constraint (III.23) is now only able to reduce the group to $\prod_\alpha SU(k_\alpha) \times U(1)$, leaving alive a $U(1)$ factor.

Hypermultiplets in this gauge theory are understood as strings connecting fourbranes on different sides of a fivebrane: therefore, whenever the positions of the fourbranes to the left and right of a solitonic brane become coincident, a massless hypermultiplet arises. As the hypermultiplets are charged under the gauge groups at both sides of a certain $\alpha + 1$ fivebrane, they will transform as $(k_\alpha, \bar{k}_{\alpha+1})$.

However, as the position of the fourbranes on both sides of a fivebrane varies as a function of x^0, x^1, x^2 and x^3, the existence of a well defined hypermultiplet can only be accomplished thanks to the fact that its variation rates on both sides are the same, as follows again from (III.22): $\partial_\mu(\sum_i a_{i,\alpha}) = \partial_\mu(\sum_j a_{j,\alpha+1})$. The definition of the bare massses comes then naturally from constraint (III.23):

$$m_\alpha = \frac{1}{k_\alpha}\sum_i a_{i,\alpha} - \frac{1}{k_{\alpha+1}}\sum_j a_{j,\alpha+1}. \tag{III.24}$$

With this interpretation, the constraint (III.23) becomes very natural from a physical point of view: it states that the masses of the hypermultiplets do not depend on the spacetime position.

The consistency of the previous definition of hypermultiplets can be checked using the previous construction of the one-loop beta function. In fact, from equation (III.19), we get, for large values of v,

$$-i\tau_\alpha(v) = (2k_\alpha - k_{\alpha-1} - k_{\alpha+1})\ln v. \qquad \text{(III.25)}$$

The number k_α of branes in the α^{th} is, as we know, the number of colours, N_c. Comparing with the beta function for $N = 2$ supersymmetric $SU(N_c)$ gauge theory with N_f flavors, we conclude that

$$N_f = k_{\alpha-1} + k_{\alpha+1}. \qquad \text{(III.26)}$$

so that the number of fourbranes (hypemultiplets) at both sides of a certain pair of fivebranes, $k_{\alpha+1} + k_{\alpha-1} \equiv N_f$, becomes the number of flavors.

Notice, from (III.24), that the mass of all the hypermultiplets associated with fourbranes at both sides of a solitonic fivebrane are the same. This implies a global flavor symmetry. This global flavor symmetry is the gauge symmetry of the adjacent sector. This explains the physical meaning of (III.24).

Let us now come back to equation (III.15). What we need in order to unravel the strong coupling dynamics of our effective four dimensional gauge theory is the u dependence of the effective coupling constant, dependence that will contain non perturbative effects due to instantons. It is from this dependence that we read the Seiberg-Witten geometry of the quantum moduli space. Next, we will see how M-theory can be effectively used to find the quantum moduli space.

A M-Theory and Strong Coupling.

From the M-theory point of view, the brane configuration we are considering can be interpreted in a different way. In particular, the D-4branes we are using to define the four dimensional macroscopic gauge theory can be considered as fivebranes wrapping the eleven dimensional S^1. Moreover, the trick we have used to make finite these fourbranes in the x^6 direction can be directly obtained if we consider fivebranes with worldvolume $\mathbf{R}^4 \times \Sigma$, where $\mathbf{R}^4$ is parametrized by the coordinates x^0, x^1, x^2 and x^4, and Σ is two dimensional, and holomorphically embedded in the four dimesional space of coordinates x^4, x^5, x^6 and x^{10}.

In order to get the Riemann surface Σ we will proceed as follows. Let us then define the single valued coordinate t,

$$t \equiv \exp -s, \qquad \text{(III.27)}$$

and define the surface Σ we are looking for through

$$F(t, v) = 0. \qquad \text{(III.28)}$$

From the classical equations of motion of the fivebrane we know the assymptotic behaviour for very large t,

$$t \sim v^k, \qquad \text{(III.29)}$$

and for very small t,

$$t \sim v^{-k} \tag{III.30}$$

Conditions (III.29) and (III.30) imply that $F(t, v)$ will have, for fixed values of t, k roots, while two different roots for fixed v. It must be stressed that the assymptotic behaviour (III.29) and (III.30) corresponds to the one loop beta function for a field theory with gauge group $SU(k)$, and without hypermultiplets. A function satisfying the previous conditions will be of the generic type

$$F(t, v) = A(v)t^2 + B(v)t + C(v), \tag{III.31}$$

with A, B and C polynomials in v of degree k. From (III.29) and (III.30), the function (III.31) becomes

$$F(t, v) = t^2 + B(v)t + \text{ constant}, \tag{III.32}$$

with one undetermined constant. Equation (III.32) is a relation between dimensionless variables, and the constant in (III.32) can be killed through rescalings of v and t. In fact, a change in the constant is equivalent to a rescaling of v through

$$v \to v \left(\frac{\text{const}'}{\text{const}} \right)^{\frac{1}{2k}} \tag{III.33}$$

If we define the dimensionless variable v, relative to the scale of the theory, Λ^{-1}, we observe that a change in Λ is equivalent to (III.33) when the constant is Λ^{2k}. Thus, we can identify the constant in (III.32) with the scale of the theory, Λ^{2k}

In order to kill this constant, we can rescale t to t/constant. The meaning of this rescaling can be easily understood in terms of of the one loop beta function, written as (III.29) and (III.30). In fact, these equations can be read as

$$s = -k \ln \left(\frac{v}{R} \right), \tag{III.34}$$

and therefore the rescaling of R goes like

$$s \to -k \ln \left(\frac{v}{R'} \frac{R'}{R} \right) \tag{III.35}$$

or, equivalently,

$$t \to t \left(\frac{R'}{R} \right)^k \tag{III.36}$$

Thus, and based on the above discussion on the definition of the scale, we observe that the constant in (III.32) defines the scale of the theory. With this interpretation of the constant, we can get the Seiberg-Witten solution for $N = 2$ pure gauge theories, with gauge group $SU(k)$. If $B(v)$ is chosen to be

$$B(v) = v^k + u_2 v^{k-2} + u_3 v^{k-3} + \cdots + u_k, \tag{III.37}$$

we finally get the Riemann surface

$$t^2 + B(v)t + 1 = 0, \tag{III.38}$$

a Riemann surface of genus $k-1$, which is in fact the rank of the gauge group. Moreover, we can now try to visualize this Riemann surface as the worldvolume of the fivebrane describing our original brane configuration: each v-plane can be compactified to $\mathbf{P}^1$, and the transversal fourbranes cna be interpreted as gluing tubes, which clearly represents a surface with $k-1$ handles. This image corresponds to gluing two copies of $\mathbf{P}^1$, with k disjoint cuts on each copy or, equivalently, $2k$ branch points. Thus, as can be observed from (III.38), to each transversal D-4brane there correspond two branch points and one cut on $\mathbf{P}^1$.

If we are interested in $SU(k)$ gauge theories with hypermultiplets, then we should first replace (III.29) and (III.30) by the corresponding relations,

$$t \sim v^{k-k_{\alpha-1}}, \tag{III.39}$$

and

$$t \sim v^{-k-k_{\alpha+1}}, \tag{III.40}$$

for t large and small, respectively. These are, in fact, the relations we get from the beta functions for these theories. If we take $k_{\alpha_1} = 0$, and $N_f = k_{\alpha+1}$, the curve becomes

$$t^2 + B(v)t + C(v) = 0, \tag{III.41}$$

with $C(v)$ a polynomial in v, of degree N_f, parametrized by the masses of the hypermultiplets,

$$C(v) = f \prod_{j=1}^{N_f} (v - m_j), \tag{III.42}$$

with f a complex constant.

Summarizing, we have been able to find a moduli of brane configurations reproducing four dimensional $N = 2$ supersymmetric $SU(k)$ gauge theories. The exact Seiberg-Witten solution is obtained by reduction of the worldvolume fivebrane dynamics on the surface $\Sigma_{\vec{u}}$ defined at (III.38) and (III.40). Obviously, reducing the fivebrane dynamics to $\mathbf{R}^4$ on $\Sigma_{\vec{u}}$ leads to an effective coupling constant in $\mathbf{R}^4$, the $k-1 \times k-1$ Riemann matrix $\tau(\vec{u})$ of $\Sigma_{\vec{u}}$.

Before finishing this section, it is important to stress some peculiarities of the brane construction. First of all, it should be noticed that the definition of the curve Σ, in terms of the brane configuration, requires working with uncompactified x^4 and x^5 directions. This is part of the brane philosophy, where we must start with a particular configuration in flat spacetime. A different approach will consist in directly working with a spacetime $Q \times R^7$, with Q some Calabi-Yau manifold, and consider a fivebrane worldvolume $\Sigma \times \mathbf{R}^4$, with $\mathbf{R}^4 \subset \mathbf{R}^7$, and Σ a lagrangian submanifold of Q. Again, by Mc Lean's theorem, the $N = 2$ theory defined on $\mathbf{R}^4$ will have a Coulomb branch with dimension equal to the first Betti number of Σ, and these deformations of Σ in Q will represent scalar fields in the four dimensional theory. Moreover, the holomorphic top form Ω of Q will define the meromorphic λ of the Seiberg-Witten solution. If we start with some Calabi-Yau manifold Q, we should provide some data to determine Σ (this is what we did in the brane case, with Q non compact and flat). If, on the contrary, we want to select Σ directly from Q, we can only do it in some definite cases, which are those related to the *geometric mirror construction*. Let us then recall some facts about the geometric mirror. The data are

- The Calabi-Yau manifold Q.
- A lagrangian submanifold $\Sigma \to Q$.
- A $U(1)$ flat bundle on Σ.

The third requirement is equivalent to interpreting Σ as a D-brane in Q. This is a crucial data, in order to get from the above points the structure of abelian manifold of the Seiberg-Witten solution. Namely, we frist use Mc Lean's theorem to get the moduli of deformations of $\Sigma \to Q$, preserving the condition of lagrangian submanifold. This space is of dimension $b_1(\Sigma)$. Secondly, on each of these points we fiber the jacobian of Σ, which is of dimension g. This family of abelian varieties defines the quantum moduli of a gauge theory, with $N = 2$ supersymmetry, with a gauge group of rank equal $b_1(\Sigma)$. Moreover, this family of abelian varieties is the moduli of the set of data of the second and third points above, i. e., the moduli of Σ as a D-2brane. In some particular cases, this moduli is Q itself or, more properly, the geometric mirror of Q. This will be the case for Σ of genus equal one, i. e., for the simple $SU(2)$ case. In this cases, the characterization of Σ in Q is equivalent to describing Q as an elliptic fibration. The relation between geometric mirror and T-duality produces a completely different physical picture. In fact, we can, when Σ is a torus, consider in type IIB a threebrane with classical moduli given by Q. After T-duality or mirror, we get the type IIA description in terms of a fivebrane. In summary, it is an important problem to understand the relation of quantum mirror between type IIA and type IIB string theory, and the M-theory strong coupling description of type IIA strings.

But before ending this section we would like to discuss the "rationale" of the M-theory approach. First of all, we have started with a brane configuration in type IIA string theory which allows us to define a theory that in the infrared behaves as $N = 2$ supersymmetric theory, with gauge group $SU(N)$. Preserving the value of the bare coupling constant we can change the separation between the vertical solitonic fivebranes if at the same time we turn on, in appropiate way, the string coupling constant. Through this procedure we can describe the macroscopic gauge theory with $N = 2$ in four dimensions, using strongly coupled type IIA string theory, i. e., using M-theory. Now, in M-theory the brane configuration becomes a single fivebrane. The Coulomb branch of the four dimensional theory is now the moduli of holomorphic curves Σ in $Q = \mathbf{R}^3 \times S^1$, for a fivebrane with worldvolume $\mathbf{R}^4 \times \Sigma$.

B $N = 2$ Models with Vanishing Beta Function.

Let us come back to brane configurations with $n + 1$ solitonic fivebranes, with k_α Dirichlet fourbranes extending between the α^{th} pair of NS-5branes. The beta function, derived in (III.25), is

$$-2k_\alpha + k_{\alpha+1} + k_{\alpha-1}, \tag{III.43}$$

for each $SU(k_\alpha)$ factor in the gauge group. In this section, we will compactify the x^6 direction to a circle of radius L. Imposing the beta function to vanish in all sectors immediately implies that all k_α are the same. Now, the compactification of the x^6 direction does not allow to eliminate all $U(1)$ factors in the gauge group: one of them can not be removed, so that the gauge group is reduced from $\prod_{\alpha=1}^{n} U(k_\alpha)$ to $U(1) \times SU(k)^n$. Moreover, using the definition (III.24) of the mass of the hypermultiplets we get, for periodic configurations,

$$\sum_\alpha m_\alpha = 0. \tag{III.44}$$

The hypermultiplets are now in representations of type $k \otimes \bar{k}$, and therefore consists of a copy of the adjoint representation, and a neutral singlet.

Let us consider the simplest case, of $N = 2$ $SU(2) \times U(1)$ four dimensional theory, with one hypermultiplet in the adjoint representation [6]. The corresponding brane configuration contains a single solitonic fivebrane, and two Dirichlet fourbranes. The mass of the hypermultiplet is clearly zero, and the corresponding four dimensional theory has vanishing beta function. A geometric procedure to define masses for the hypermultiplets is a fibering of the v-plane on the x^6 S^1 direction, in a non trivial way, so that the fourbrane positions are identified modulo a shift in v,

$$\begin{aligned} x^6 &\to x^6 + 2\pi L, \\ v &\to v + m, \end{aligned} \tag{III.45}$$

so that now, the mass of the hypermultiplet, is the constant m appearing in (III.45), as $\sum_\alpha m_\alpha = m$.

From the point of view of M-theory, the x^{10} coordinate has also been compactified on a circle, now of radius R. The (x^6, x^{10}) space has the topology of $S^1 \times S^1$. This space can be made non trivial if, when going around x^6, the value of x^{10} is changed as follows:

$$x^6 \to x^6 + 2\pi L,$$
$$x^{10} \to x^{10} + \theta R, \tag{III.46}$$

and, in addition, $x^{10} \to x^{10} + 2\pi R$. Relations (III.46) define a Riemann surface of genus one, and moduli depending on L and θ for fixed values of R. θ in (III.46) can be understood as the θ-angle of the four dimensional field theory: the θ-angle can be defined as

$$\frac{x_1^{10} - x_2^{10}}{R}, \tag{III.47}$$

with $x_2^{10} = x^{10}(2\pi L)$, and $x_1^{10} = x^{10}(0)$. Using (III.46), we get θ as the value of (III.47). This is the bare θ-angle of the four dimensional theory.

A question inmediately appears concerning the value of the bare coupling constant: the right answer should be

$$\frac{1}{g^2} = \frac{2\pi L}{R}. \tag{III.48}$$

It is therefore clear that we can move the bare coupling constant of the theory keeping fixed the value of R, and changing L and θ. Let us now try to solve this model for the massless case. The solution will be given by a Riemann surface Σ, living in the space $E \times C$, where E is the Riemann surface defined by (III.46), and C is the v-plane. Thus, all what we need is defining Σ through an equation of the type

$$F(x, y, z) = 0, \tag{III.49}$$

with x and y restricted by the equation of E,

$$y^2 = (x - e_1(\tau))(x - e_2(\tau))(x - e_3(\tau)), \tag{III.50}$$

with τ the bare coupling constant defined by (III.47) and (III.48). In case we have a collection of k fourbranes, we will require F to be a polynomial of degree k in v,

$$F(x, y, z) = v^k - f_1(x, y)v^{k-1} + \cdots \tag{III.51}$$

The moduli parameters of Σ are, at this point, hidden in the functions $f_i(x, y)$ in (III.51). Let us denote $v_i(x, y)$ the roots of (III.51) at the point (x, y) in E. Notice that (III.51) is a spectral curve defining a branched covering of E, i. e., (III.51) can be interpreted as a spectral curve in the sense of Hitchin's integrable system. If f_i has a pole at some point (x, y), then the same root $v_i(x, y)$ should go to infinity. These poles have the interpretation of locating the position of the solitonic fivebranes. In the simple case we are considering, with a single fivebrane, the Coulomb branch of the theory will be parametrized by meromorphic functions on E with a simple pole at one point, which is the position of the fivebrane. As we have k functions entering (III.51), the dimension of the Coulomb branch will be k, which is the right one for a theory with $U(1) \times SU(k)$ gauge group.

Now, after this discussion of the model with massless hypermultiplets, we will introduce the mass. The space where now we need to define Σ is not $E \times C$, but the non trivial fibration defined through

$$\begin{aligned} x^6 &\to x^6 + 2\pi L, \\ x^{10} &\to x^{10} + \theta R, \\ v &\to v + m \end{aligned} \tag{III.52}$$

or, equivalently, the space obtained by fibering C non trivially on E. We can flat this bundle over all E, with the exception of one point p_0. Away from this point, the solution is given by (III.51). If we write (III.51) in a factorized form,

$$F(x, y, z) = \prod_{i=1}^{k} (v - v_i(x, y)), \tag{III.53}$$

we can write f_1 in (III.51) as the sum

$$f_1 = \sum_{i=1}^{k} v_i(x, y); \tag{III.54}$$

therefore, f_1 will have poles at the positions of the fivebrane. The mass of the hypermultiplet will be identified with the residue of the differential $f_1\omega$, with ω the abelian differential, $\omega = \frac{dx}{y}$. As the sum of the residues is zero, this means that at the point at infinity, that we identify with p_0, we have a pole with residue m.

IV BRANE DESCRIPTION OF $N = 1$ FOUR DIMENSIONAL FIELD THEORIES.

In order to consider field theories with $N = 1$ supersymmetry, the first thing we will study will be R-symmetry. Let us then recall the way R-symmetries were defined in the case of four dimensional $N = 2$ supersymmetry, and three dimensional $N = 4$ supersymmetry, through compactification of six dimensional $N = 1$ supersymmetric gauge field theories. The $U(1)_R$ in four dimensions, or $SO(3)_R$ in three dimensions, are simply the euclidean group of rotations in two and three dimensions, respectively. Now, we have a four dimensional space Q, parametrized by coordinates t and v, and a Riemann surface Σ, embedded in Q by equations of the type (III.31). To characterize R-symmetries, we can consider transformations on Q which transform non trivially its holomorphic top form Ω. The unbroken R-symmetries will then be rotations in Q preserving the Riemann surface defined by the brane configuration. If we consider only the assymptotic behaviour of type (III.29), or (III.39), we get $U(1)_R$ symmetries of type

$$\begin{aligned} t &\to \lambda^k t, \\ v &\to \lambda v. \end{aligned} \tag{IV.1}$$

This $U(1)$ symmetry is clearly broken by the curve (III.38). This spontaneous breakdown of the $U(1)_R$ symmetry is well understood in field theory as an instanton induced effect. If instead of considering Q, we take the larger space $\hat{Q}$, containing the x^7, x^8 and x^9 coordinates, we see that the $N = 2$ curve is invariant under rotations in the (x^7, x^8, x^9) space.

Let us now consider a brane configuration which reproduces $N = 1$ four dimensional theories [20]. We will again start in type IIA string theory, and locate a solitonic fivebrane at $x^6 = x^7 = x^8 = x^9 = 0$ with, as usual, worldvolume coordinates x^0, x^1, x^2, x^3, x^4 and x^5. At some definite value of x^6, say x^6_0, we locate another solitonic fivebrane, but this time with worldvolume coordinates x^0, x^1, x^2, x^3, x^7 and x^8, and $x^4 = x^5 = x^9 = 0$. As before, we now suspend a set of k D-4branes in between. They will be parametrized by the positions $v = x^4 + ix^5$, and $w = x^7 + ix^8$, on the two solitonic fivebranes. The worldvolume coordinates on this D-4branes are, as in previous cases, x^0, x^1, x^2 and x^3. The effective field theory defined by the set of fourbranes is macroscopically a four dimensional gauge theory, with coupling constant

$$\frac{1}{g^2} = \frac{x^6_0}{gl_s}. \tag{IV.2}$$

Moreover, now we have only $N = 1$ supersymmetry, as no massless bosons can be defined on the four dimensional worldvolume (x^0, x^1, x^2, x^3). In fact, at the line $x^6 = 0$ the only possible massless scalar would be v, since $w = 0$ and $x^9 = 0$, so that we project out x^9 and w. On the other hand, at x^6_0 we have $v = 0$ and $x^9 = 0$ and, therefore, we have projected out all massless scalars. Notice that by the same argument, in the case of two solitonic fivebranes located at different values of x^6 but at $x^7 = x^8 = x^9 = 0$, we have one complex massless scalar that is not projected out, which leads to $N = 2$ supersymmetry in four dimensions. The previous discussion means that v, w and x^9 are projected out as four dimensional scalar fields; however, w and v are still classical moduli parameters of the brane configuration.

Now, we return to a comment already done in previous section: each of the fourbranes we are suspending in between the solitonic fivebranes can be interpreted as a fivebrane wrapped around a surface defined by the eleven dimensional S^1 of M-theory, multiplied by the segment $[0, x^6_0]$. Classically, the four dimensional theory can be defined through dimensional reduction of the fivebrane worldvolume on the surface Σ. The coupling constant will be given by the moduli τ of this surface,

$$\frac{1}{g^2} = \frac{2\pi R}{S}, \tag{IV.3}$$

with S the length of the interval $[0, x^6_0]$, in M-theory units. In $N = 1$ supersymmetric field theories, on the contrary of what takes place in the $N = 2$ case, we have not a classical moduli and, therefore, we can not define a wilsonian coupling constant depending on some mass scale fixed by a vacuum expectation value. This fact can produce some problems, once we take into account the classical dependence of x^6 on v and w. In principle, this dependence should be the same as that in the case studied in previous section,

$$x^6 \sim \tilde{k} \ln v,$$
$$x^6 \sim \tilde{k} \ln w. \tag{IV.4}$$

Using the t coordinate defined in (III.27), equations (IV.4) become

$$t \sim v^k,$$
$$t \sim w^k, \tag{IV.5}$$

for large and small t, respectively, or, equivalently, $t \sim v^k$, $t^{-1} \sim w^k$.

Using the relations in (IV.2) we get, for $k = N_c$,

$$\frac{1}{g^2} \sim -N_c[\ln\left(\frac{v}{l_s}\right) + \ln\left(\frac{w}{l_s}\right)], \tag{IV.6}$$

where we are measuring v and w in string units. Following the same approach as in the $N = 2$ case, we can try read, directly from (IV.6), the one loop beta function of the theory,

$$\mu\frac{\partial g}{\partial \mu} = -\frac{g^3}{16\pi^2}3N_c, \tag{IV.7}$$

or, equivalently,

$$\frac{8\pi^2}{g(\mu)^2} = -N_c \ln\left(\frac{\Lambda}{\mu}\right)^3. \tag{IV.8}$$

In order to relate (IV.6) and (IV.8) we can imposse the relation

$$v = \zeta w^{-1}, \tag{IV.9}$$

where ζ is some constant. Using (IV.9), we get

$$\frac{1}{g(\mu)^2} \sim -N_c \ln\left(\frac{\zeta}{l_s^2}\right), \tag{IV.10}$$

which leads to the relation

$$\frac{\zeta}{l_s^2} = \left(\frac{\Lambda}{\mu}\right)^3. \tag{IV.11}$$

Choosing l_s^{-1} for the renormalization point μ, we get

$$\zeta = \Lambda^3 l_s^5. \tag{IV.12}$$

In other words, the constant ζ in (IV.9) is related to the scale of the $N=1$ theory, Λ. The specific relation will depend on the unit we choose. The previous argument was done in the context of type IIA string theory, without invoquing M-theory. Moreover, (IV.9), together with (IV.29) can be interpreted as defining a fivebrane worldvolume in M-theory of the type $\Sigma \times \mathbf{R}^{3,1}$, where the Riemann surface Σ is defined by

$$
\begin{aligned}
t &= v^{N_c}, \\
\zeta^{N_c} t^{-1} &= w^{N_c}, \\
v &= \zeta w^{-1}.
\end{aligned}
\tag{IV.13}
$$

This is a holomorphic curve of genus zero, in the space $Q = \mathbf{R}^5 \times S^1$. As can be seen from (IV.13), the assymptotic behaviour of the curve is determined by the value of ζ^{N_c}. Thus, the possible vacua of the theory are determined by the N_c different roots ζ of ζ^{N_c}, in agreement with the known value for tr $(-1)^F$.

For a given value of ζ, (IV.13) defines a Riemann surface of genus zero, i. e., a rational curve. This curve is now embedded in the space of (t, v, w) coordinates. We will next observe that these curves, (IV.13), are the result of "rotating" [9] the rational curves in the Seiberg-Witten solution, corresponding to the singular points. However, before doing that let us comment on $U(1)_R$ symmetries. As mentioned above, in order to define an R-symmetry we need a transformation on variables (t, v, w) not preserving the holomorphic top form,

$$
\Omega = dv \wedge dw \wedge \frac{dt}{t} R.
\tag{IV.14}
$$

A rotation in the w-plane, compatible with the assymptotic conditions (IV.5), and defining an R-symmetry, is

$$
\begin{aligned}
v &\to v, \\
t &\to t, \\
w &\to e^{2\pi i/k} w.
\end{aligned}
\tag{IV.15}
$$

Now, it is clear that this symmetry is broken spontaneously by the curve (IV.13). More interesting is an exact $U(1)$ symmetry, that can be defined for the curve (IV.13):

$$
\begin{aligned}
v &\to e^{i\delta} v, \\
t &\to e^{i\delta k} t, \\
w &\to e^{-i\delta} w.
\end{aligned}
\tag{IV.16}
$$

As can be seen from (IV.14), this is not an R-symmetry, since Ω is invariant. Fields charged with respect to this $U(1)$ symmetry should carry angular momentum in the v or w plane, or linear momentum in the eleventh dimension interval (i. e., zero branes) The fields of $N=1$ SQCD do not carry any of these charges, so all fields with $U(1)$ charge should be decoupled from the $N=1$ SQCD degrees of freedom. This is equivalent to the way we have projected out fields in the previous discussion on the definition of the effective $N=1$ four dimensional field theory.

A Rotation of Branes.

A different way to present the above construction is by performing a rotation of branes. We will now concentrate on this procedure. The classical configuration of NS-5branes with worldvolumes extending along x^0, x^1, x^2, x^3, x^4 and x^5, can be modified to a configuration where one of the solitonic fivebranes has been rotated, from the $v = x^4 + ix^5$ direction, to be also contained in the (x^7, x^8)-plane, so that, by moving it a finite angle μ, it is localized in the (x^4, x^5, x^7, x^8) space. Using the same notation as in previous section, the brane configuration, where a fivebrane has been moved to give rise to an angle μ in the (v, w)-plane, the rotation is equivalent to impossing

$$w = \mu v. \qquad \text{(IV.17)}$$

In the brane configuration we obtain, points on the rotated fivebrane are parametrized by the (v, w) coordinates in the (x^4, x^5, x^7, x^8) space. We can therefore imposse the following assymptotic conditions [19]:

$$\begin{aligned} t &= v^k, \qquad w = \mu v, \\ t &= v^{-k}, \qquad w = 0, \end{aligned} \qquad \text{(IV.18)}$$

respectively for large and small t. Let us now assume that this brane configuration describes a Riemann surface, $\hat{\Sigma}$, embedded in the space $(x^6, x^{10}, x^4, x^5, x^7, x^8)$, and let us denote by Σ the surface in the $N = 2$ case, i. e., for $\mu = 0$. In these conditions, $\hat{\Sigma}$ is simply the graph of the function w on Σ. We can interpret (IV.17) as telling us that w on Σ posseses a simple pole at infinity, extending holomorphically over the rest of the Riemann surface. If we imposse this condition, we get that the projected surface Σ, i. e., the one describing the $N = 2$ theory, is of genus zero. In fact, it is a well known result in the theory of Riemann surfaces that the order of the pole at infinity depends on the genus of the surface in such a way that for genus larger than zero, we will be forced to replace (IV.17) by $w = \mu v^a$ for some power a depending on the genus. A priori, there is no problem in trying to rotate using, instead of $w = \mu v$, some higher pole modification of the type $w = \mu v^a$, for $a > 1$. This would provide Σ surfaces with genus different from zero; however, we would immediately find problems with equation (IV.6), and we will be unable to kill all dependence of the coupling constant on v and w. Therefore, we conclude that the only curves that can be rotated to produce a four dimensional $N = 1$ theory are those with zero genus. This is in perfect agreement with the physical picture we get from the Seiberg-Witten solution. Namely, once we add a soft breaking term of the type $\mu \mathrm{tr}\phi^2$, the only points remaining in the moduli space as real vacua of the theory are the singular points, where the Seiberg-Witten curve degenerates.

B QCD Strings and Scales.

The $N = 1$ four dimensional field theory we have described contains, in principle, two parameters. One is the constant ζ introduced in equation (IV.9) which, as we have already mentioned, is, because of (IV.6), intimately connected with Λ, and the radius R of the eleven dimensional S^1. Our first task would be to see what kind of four dimensional dynamics is dependent on the particular value of R, and in what way. The best example we can of course use is the computation of gaugino-gaugino condensates. In order to do that, we should try to minimize a four dimensional suerpotential for the $N = 1$ theory. Following Witten, we will define this superpotential W as an holomorphic function of Σ, and with critical points precisely when the surface Σ is a holomorphic curve in Q. The space Q now is the one with coordinates x^4, x^5, x^6, x^7, x^8 and x^{10} (notice that this second condition was the one used to prove that rotated curves are necesarily of genus equal zero). Moreover, we need to work with a holomorphic curve because of $N = 1$ supersymmetry. A priori, there are two different ways we can think about this superpotential: maybe the simplest one, from a physical point of view, is as a functional defined on the volume of Σ, where this volume is given by

$$\mathrm{Vol}(\Sigma) = J.\Sigma, \tag{IV.19}$$

with J the Kähler class of Q. The other posibility is defining

$$W(\Sigma) = \int_B \Omega, \tag{IV.20}$$

with B a 3-surface such that $\Sigma = \partial B$, and Ω the holomorphic top form in Q. Definition (IV.20) automatically satisfies the condition of being stationary, when Σ is a holomorphic curve in Q. Notice that the holomorphy condition on Σ means, in mathematical terms, that Σ is an element of the Picard lattice of Q, i. e., an element in $H_{1,1}(Q) \cap H_2(Q)$. This is what allows us to use (IV.19), however, and this is the reason for temporarily abandoning the approach based on (IV.19); what we require to W is being stationary for holomorphic curves, but it should, in principle, be defined for arbitrary surfaces Σ, even those which are not part of the Picard group. Equation (IV.20) is only well defined if Σ is contractible, i. e., if the homology class of Σ in $H_2(Q; \mathbf{Z})$ is trivial. If that is not the case, a reference surface Σ_0 needs to be defined, and (IV.20) is modified to

$$W(\Sigma) - W(\Sigma_0) = \int_B \Omega, \tag{IV.21}$$

where now $\partial B = \Sigma \cup \Sigma_0$. For simplicity, we will assume $H_3(Q; \mathbf{Z}) = 0$. From physical arguments we know that the set of zeroes of the superpotential should be related by $\mathbf{Z}_{N_c}$ symmetry, with N_c the number of transversal fourbranes. Therefore, if we choose Σ_0 to be $\mathbf{Z}_k$ invariant, we can write $W(\Sigma_0) = 0$, and $W(\Sigma) = \int_B \Omega$. Let us then take B as the complex plane multiplied by an interval $I = [0, 1]$, and let us first map the complex plane into Σ. Denoting r the coordinate on this complex plane, Σ, as given by (IV.13), is defined by

$$\begin{aligned} t &= r^{N_c}, \\ v &= r, \\ w &= \zeta r^{-1}. \end{aligned} \tag{IV.22}$$

Writing $r = e^{\rho}e^{i\theta}$, we can define Σ_0 as

$$
\begin{aligned}
t &= r^{N_c}, \\
v &= f(\rho)r, \\
w &= \zeta f(-\rho)r^{-1},
\end{aligned} \tag{IV.23}
$$

with $f(\rho) = 1$ for $\rho > 2$, and $f(\rho) = 0$ for $\rho < 1$. The $\mathbf{Z}_k$ transformation $t \to t$, $w \to e^{2\pi i/k}w$ and $v \to v$, is a symmetry of (IV.23) if, at the same time, we perform the reparametrization of the r-plane

$$
\begin{aligned}
\rho &\to \rho, \\
\theta &\to \theta + b(\rho),
\end{aligned} \tag{IV.24}
$$

with $b(\rho) = 0$ for $\rho \geq 1$. and $b(\rho) = -\frac{2\pi}{k}$ for $\rho \leq -1$. Thus, the 3-manifold entering the definition of B, is given by

$$
\begin{aligned}
t &= r^{N_c}, \\
v &= g(\rho, \sigma)r, \\
w &= \zeta g(-\rho, \sigma)r^{-1},
\end{aligned} \tag{IV.25}
$$

such that for $\sigma = 0$ we have $g = 1$, and for $\sigma = 1$, we get $g(\rho) = f(\rho)$. Now, with

$$
\Omega = R dv \wedge dw \wedge \frac{dt}{t}, \tag{IV.26}
$$

we get

$$
W(\Sigma) = N_c R \int_B dv \wedge dw \wedge \frac{dr}{r}. \tag{IV.27}
$$

The dependence on R is already clear from (IV.26). In order to get the dependence on ζ we need to use (IV.25),

$$
W(\Sigma) = N_c R \zeta \int d\sigma d\theta d\rho \left(\frac{\partial g_+}{\partial \sigma} \frac{\partial g_-}{\partial \rho} - \frac{\partial g_+}{\partial \rho} \frac{\partial g_-}{\partial \sigma} \right), \tag{IV.28}
$$

for $g_{\pm} = g(\pm\rho, \sigma)$. Thus we get

$$
W(\Sigma) \sim N_c R \zeta. \tag{IV.29}
$$

Now we should compare (IV.29) with the known value of the gaugino condensate,

$$
W = N_c < \mathrm{tr}\, \lambda\lambda >, \tag{IV.30}
$$

with

$$< \text{tr}\ \lambda\lambda >_j = N_c \Lambda^3 \exp 2\pi i j / N_c \tag{IV.31}$$

the value of the gaugino condensate in the j-vacua. Thus, from (IV.31) we get

$$N_c^2 \Lambda^3 l_s^6 \sim N_c R \zeta, \tag{IV.32}$$

where we have used the string length as reference scale. Notice that our definition of W is the volume of a 3-surface. From (IV.32) and (IV.14), we get the relation

$$l_s = \frac{R}{N_c}. \tag{IV.33}$$

A different way to connect ζ with Λ is defining, in the M-theory context, the QCD string and computing its tension. Following Witten, we will then try an interpretation of ζ independent of (IV.6), by computing in terms of ζ the tension of the QCD string. We will then, to define the tension, consider the QCD string as a membrane, product of a string in $\mathbf{R}^4$, and a string living in Q. Let us then denote by C a curve in Q, and assume that C ends on Σ in such a way that a membrane wrapped on C defines a string in $\mathbf{R}^4$. Moreover, we can simply think of C as a closed curve in Q, going around the eleven dimensional S^1,

$$\begin{aligned} t &= t_0 \exp(-2\pi i \sigma), \\ v &= t_0^{1/N_c}, \\ w &= \zeta v^{-1}. \end{aligned} \tag{IV.34}$$

This curve is a non trivial element in $H_1(Q;\mathbf{Z})$, and a membrane wrapped on it will produce an ordinary type IIA string; however, we can not think that the QCD string is a type IIA string. If $Q = \mathbf{R}^3 \times S^1$, then $H_1(Q;\mathbf{Z}) = \mathbf{Z}$, and curves of type (IV.34) will be the only candidates for non trivial 1-cycles in Q. However, we can define QCD strings using cycles in the relative homology, $H_1(Q/\Sigma;\mathbf{Z})$, i. e., considering non trivial cycles ending on the surface Σ. To compute $H_1(Q/\Sigma;\mathbf{Z})$, we can use the exact sequence

$$H_1(\Sigma;\mathbf{Z}) \rightarrow H_1(Q;\mathbf{Z}) \xrightarrow{\imath} H_1(Q/\Sigma;\mathbf{Z}), \tag{IV.35}$$

which implies

$$H_1(Q/\Sigma;\mathbf{Z}) = H_1(Q;\mathbf{Z})/\imath H_1(\Sigma;\mathbf{Z}). \tag{IV.36}$$

The map $\imath$ is determined by the map defining Σ ($t = v^{N_c}$), and thus we can conclude that, very likely,

$$H_1(Q/\Sigma;\mathbf{Z}) = \mathbf{Z}_{N_c}. \tag{IV.37}$$

A curve in $H_1(Q/\Sigma;\mathbf{Z})$ can be defined as follows:

$$
\begin{aligned}
t &= t_0, \\
v &= t_0^{1/N_c} e^{2\pi i\sigma/N_c}, \\
w &= \zeta v^{-1},
\end{aligned}
\tag{IV.38}
$$

with t_0^{1/N_c} one of the N_c roots. The tension of (IV.38), by construction, is independent of R, because t is fixed. Using the metric on Q, the length of (IV.38) is given by

$$
\left(\frac{\zeta^2 t^{-2/N_c}}{N_c^2} + \frac{t^{2/N_c}}{N_c^2} \right)^{1/2},
\tag{IV.39}
$$

and its minimum is obtained when $t^{2/n} = \zeta$. Thus, the length of the QCD string should be

$$
\frac{|\zeta|^{1/2}}{N_c}.
\tag{IV.40}
$$

which has the right length units, as ζ behaves as $(\text{length})^2$. The relation of (IV.40) with the QCD tension implies

$$
|\zeta|^{1/2} = \Lambda N_c l_s^2
\tag{IV.41}
$$

or, equivalently,

$$
l_s = \frac{N_c^2}{\Lambda}.
\tag{IV.42}
$$

Consistency between (IV.33) and (IV.42) requires

$$
R = \frac{1}{\Lambda N_c}.
\tag{IV.43}
$$

These are not good news, as they imply that the theory we are working with, in order to match QCD, posseses 0-brane modes, with masses of the order of Λ, and therefore we have not decoupled the M-theory modes.

C Domain Walls.

Given the vacuum structure of $N = 1$ supersymmetric QCD, we can define domain walls interpolating between the different vacua. The M-theory picture of these domain walls can be given in terms of a fivebrane with worldvolume $S \times \mathbf{R}^{2,1}$, with S a 3-cycle in $Q \times \mathbf{R}$, with $\mathbf{R}$ standing for one of the space-coordinates, let us say x_3, and such that S in the region defined by $x_3 = +\infty$ coincide with the Riemann sphere Σ_{j+1}. In order to compute the tension the tension of this domain wall we can use the superpotential deirved above, obtaining

$$\Delta W = W_{j+1} - W_j = N_c R\zeta(1 - e^{2\pi i j/N_c}). \tag{IV.44}$$

Therefore, the tension will behave as

$$T \sim R|\zeta|. \tag{IV.45}$$

Using relation (IV.33) we get

$$T \sim N_c \Lambda^3 l_s^6. \tag{IV.46}$$

A crucial aspect of equation (IV.46) is that when considering, as is the case in the large N limit, that the string coupling constant for the QCD string goes like $\frac{1}{N_c}$, then the tension of the domain wall goes like $\frac{1}{g_{string}^{QCD}}$, which is the typical relation for D-branes. In other words, we get the suggesting picture of a relation betweeen the QCD string and domain walls, in perfect parallel to D-branes and open strings. Moreover, we can expect the QCD string ending on the domain wall in a similar way to fundamental strings ending on D-branes. This, in particular, means that very likely we can interpret the domain walls of $N = 1$ supersymmetric QCD as Chan-Paton factors for the QCD string. If this is the case, we should be able to have quarks in the fundamental representation living on the domain wall.

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