8.821 F2008 Lecture 8: Large N Counting

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1   Introduction

Today we'll continue our discussion of large N scaling in the t'Hooft limit of quantum matrix models. In particular, we'll review the N counting for vacuum diagrams and learn how to do N counting for correlators. Our perspective on the large N limit continues to be one of convenience. It is one of several assumptions that enable the simplest realization of holography and gauge/gravity duality. Next time we'll continue talking about simplifying assumptions and turn our attention to conformal invariance.

2   Vacuum t'Hooft counting

Let's recall our starting point. We were thinking about matrix field theories, field theories in which all the fields are N x N matrices in addition to whatever other labels they may have. In particular, we write our theory schematically in terms of one big field Φ which we think of as potentially including scalars ϕ, gauge fields A_μ, and fermions ψ_α all of which are N x N matrices. The schematic Lagrangian is

$$L = \frac{1}{g^2} \text{Tr} \left(\partial \Phi \partial \Phi - V(\Phi) \right) \tag{1}$$

where $V(\Phi) \sim \Phi^2 + \Phi^4 + ...$ and all products should be thought of as matrix products. This schematic representation is meant to convey two pieces of information. First, that the Lagrangian is invariant under $U(N)$ rotations which act on the matrix fields by conjugation: $\Phi \rightarrow U\Phi U^{-1}$. Second, that the fields are scaled so that the coupling g^2 sits out in front of the entire Lagrangian. In particular, we have to very careful about field normalizations when doing large N counting for correlation functions. The t'Hooft limit is the limit where $N \rightarrow \infty$ and $g^2 \rightarrow 0$ while $\lambda = g^2 N$ remains fixed. λ is the t'Hooft coupling.

Note that we are adopting a different normalization condition than in the previous lecture on large N counting (there we used the field $\tilde{\Phi} = (1/g)\Phi$ which had kinetic terms with the usual field theory normalization). With our current normalization the propagator $\langle \Phi\Phi \rangle$ goes like $g^2 = \frac{\lambda}{N}$. The propagator in the double line notation appears in Fig. 1. Remember that for the purposes of
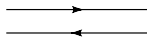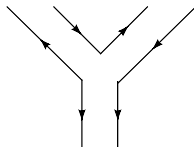
Figure 1: Propagator, λ/N

Figure 2: Three point vertex, N/λ

large N counting we are instructed to think only about counting powers of N, so we will suppress positions of fields, spin indices, etc in our analysis.

In a generic matrix theory there will be three point (Fig. 2) and four point (Fig. 3) interaction vertices. Both types of vertices come with factors of $1/g^2 = N/\lambda$ from the overall $1/g^2$ multiplying the Lagrangian. A general diagram consists of propagators, interaction vertices, and index loops, and gives a contribution

$$\text{diagram} \sim \left(\frac{\lambda}{N}\right)^{\text{no. of prop.}} \left(\frac{N}{\lambda}\right)^{\text{no. of int. vert.}} N^{\text{no. of index loops}}. \tag{2}$$

For example, the planar diagram in Fig. 4 has 4 three point vertices, 6 propagators, and 4 index loops giving the final result $N^2\lambda^2$.

We learned last time how we to associate a triangulated surface with a Feynman diagram in two ways. The first way is to let the interaction vertices of the diagram be the vertices of the triangulation. In this case the index loops of the diagram bound pieces of surface. The alternative is the dual triangulation in which we place vertices in the middle of the index loops of the Feynman diagram. Two vertices are connected if their respective index loops share a propagator in the Feynman diagram. Don't be confused by multiple uses of the word vertex. There are interaction vertices of various kinds in the Feynman diagrams and these correspond to vertices in the triangulation only in the first formulation.

If $E = $ no. of prop., $V = $ no. of int. vert., and $F = $ no. of index loops then we showed last time that the diagram gives a contribution $N^{F-E+V}\lambda^{E-V}$. The letters refer to the first way to triangulate

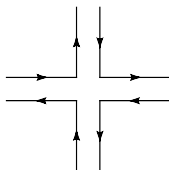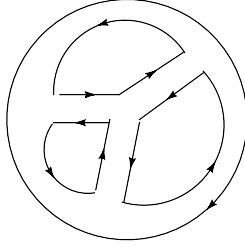Figure 3: Four point vertex, N/λ

Figure 4: This diagram consists of 4 three point vertices, 6 propagators, and 4 index loops

the surface in which interaction vertices are triangulation vertices. Then we interpret E as the number of edges, F as the number of faces, and V as the number of vertices in the triangulation. In the dual triangulation there are dual faces $\tilde{F}$, dual edges $\tilde{E}$, and dual vertices $\tilde{V}$. The relationship between the original and dual variables is $E = \tilde{E}$, $V = \tilde{F}$, and $F = \tilde{V}$. In the dual formulation we found that a diagram contributes $N^{\tilde{F}-\tilde{E}+\tilde{V}}\lambda^{\tilde{E}-\tilde{F}}$. It's not a coincidence that the powers of N agree in both formulations. The exponent $\chi = F - E + V = \tilde{F} - \tilde{E} + \tilde{V}$ is the Euler character and it is a topological invariant of two dimensional surfaces. In general it is given by $\chi = 2 - 2h - b$ where h is the number of handles (the genus) and b is the number of boundaries. Note that the exponent of λ, $E - V$ or $\tilde{E} - \tilde{F}$ is not a topological invariant and depends on the triangulation (Feynman diagram).

Before we continue with the analysis its worth pointing out that there are other large N limits one might want to consider. The motivation for the t'Hooft limit comes in part from thinking about radiative corrections which grow with N and decrease with g. This suggests some sort of balancing act might be possible which keeps the theory non-trivial in the large N limit. We encode this scaling by fixing $\lambda = g^2 N$ as we make N large. Other interesting possibilities are provided by critical points of the matrix model. Suppose when $g = g_c$ the matrix model is at a critical point. Then there may be a sensible large N limit in which we fix $|g - g_c|^\alpha N$ as we make N large. One example of this is matrix quantum mechanics with an inverted oscillator potential.

Back to the t'Hooft limit. Because the N counting is topological (depending only on χ) we can sensibly organize the perturbation series for the free energy in terms of surfaces! Because we're computing only vacuum diagrams for the moment, the surfaces we're considering have no boundaries $b = 0$ and are classified by their number of handles h. $h = 0$ is the two dimensional sphere, $h = 1$ is the torus, and so on. We may write the free energy as

$$\ln Z = \sum_{h=0}^{\infty} N^{2-2h} \sum_{\ell=0}^{\infty} c_{\ell,h}\lambda^\ell = \sum_{h=0}^{\infty} N^{2-2h}\mathfrak{F}_h(\lambda) \tag{3}$$

were the sum over surfaces is explicit.

Now we can see some similarities between this expansion and perturbative string expansions. By writing $N^{2-2h} = (1/N)^{2h-2}$ we can identify the string coupling $g_s = g(\lambda)/N$ where we can allow some function of λ in addition to the pure N dependence. Because of the proliferation of coupling constants, let's rename the g^2 in our matrix model g_{YM}^2 in anticipation of the case of most interest to us when the matrix model is a gauge theory. The t'Hooft coupling is now written $\lambda = g_{YM}^2 N$.

If $1/N$ plays the role of the string coupling g_s (so that in the large N limit joining and splitting of strings is suppressed) then its reasonable to ask what plays the role of the worldsheet coupling. It turns out that we can think of λ as a sort of chemical potential for edges in our triangulation. Looking back at our diagram counting we can see that if λ becomes large then diagrams with lots of edges are important. Thus large λ encourages a smoother triangulation of the world sheet which we might interpret as fewer quantum fluctuations on the worldsheet. More formally, we expect a relation of the form $\lambda^{-1} \sim \alpha'$ which encodes our intuition about large λ suppressing fluctuations.

This story is very general in the sense that all matrix models define something like a theory of two dimensional quantum gravity via these random triangulations. The connection makes even more sense when we remember all the extra labels we've been suppressing on our field Φ. For example, the position labeling where the field Φ sits plays the role of embedding coordinates on the worldsheet. Other indices (spin, etc.) indicate further worldsheet degrees of freedom. We may think of the matrix model partition function as giving the "thermodynamics" of the string theory in the sense that it provides the functions $\mathfrak{F}_h$ entering the free energy. However, the microscopic details of the theory are not so easily discovered. In particular, it's not at all clear what microscopic action we should integrate over moduli space to obtain the functions $\mathfrak{F}_h$.

As a final check on the non-triviality of the theory in the large N limit, let's see if the world sheet coupling λ will generically run in a matrix model. In $d = 4$ the one loop behavior is basically $\mu \partial_\mu g_{YM} = \beta_g \sim g_{YM}^3 N = g_{YM}\lambda$ and this implies the beta function for λ is $\mu \partial_\mu \lambda = \beta_\lambda = 2g_{YM}N\beta_g \sim \lambda^2$. Thus λ can still run in the large N limit and the theory is non-trivial. If $d \neq$ then the beta function for g now looks like $\beta_g \sim g_{YM}^3 N + \frac{4-d}{2}g_{YM}$ and we find that $\beta_\lambda \sim \lambda^2 + \frac{4-d}{2}\lambda$ which is still finite in the large N limit.

3   Correlation functions

Let's now consider how to calculate correlation functions of operators, and in particular, how to do large N counting for such correlation functions. We will consider only "single trace" operators, operators $\mathcal{O}(x)$ that look like

$$\mathcal{O}(x) = c(k,N)\text{Tr}(\Phi_1(x)...\Phi_k(x)) \tag{4}$$

and which we will often abbreviate as $\text{Tr}(\Phi^k)$. There are two annoying complications now. We must be careful about how we normalize the fields Φ and we must be careful about how we normalize the operator $\mathcal{O}$. The normalization of the fields will continue to be such that the Lagrangian takes the form $L = \frac{1}{g_{YM}^2}\mathcal{L} = \frac{N}{\lambda}\mathcal{L}$ which $\mathcal{L}(\Phi)$ containing no explicit factors of N. To fix the normalization of $\mathcal{O}$ (to determine the constant $c(k,N)$) we will adopt the convention that $\langle \mathcal{O}\mathcal{O} \rangle_c \sim N^0$ where the subscript c stands for connected. This convention is nice because states like $\mathcal{O}|\text{vac}\rangle$ will have finite norm (given by the correlation function) in the large N limit.

To use our convention to determine $c(k,N)$ we need to know how to insert single trace operators into the t'Hooft counting. We do this by associating a new vertex with each single trace operator. This vertex has k legs where k propagators can be attached and looks like a big squid. An example of such a new vertex appears in Fig. 5 which corresponds to the insertion of the operator $\text{Tr}(\Phi^6)$.

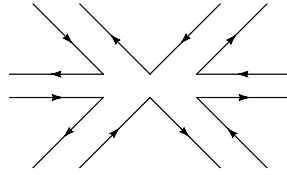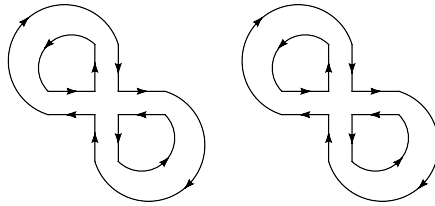Figure 5: New vertex for an operator insertion of $\mathrm{Tr}(\Phi^k)$ with $k = 6$

Figure 6: Disconnected diagram contributing to the correlation function $\langle \mathrm{Tr}(\Phi^4)\mathrm{Tr}(\Phi^4)\rangle$

For the moment we don't associate any explicit factors of N with the new vertex. Let's consider the example $\langle \mathrm{Tr}(\Phi^4)\mathrm{Tr}(\Phi^4)\rangle$. We need to draw two four point vertices for the two single trace operators in the correlation function. How are we to connect these vertices with propagators? The dominate contribution actually comes from disconnected diagrams like the one shown in Fig. 6. The leading disconnected diagram has four propagators and six index loops and so gives a factor $\lambda^4 N^2 \sim N^2$. One the other hand, the leading connected diagram shown in Fig. 7 has four propagators and four index loops and so only gives a contribution $\lambda^4 \sim N^0$. The fact that disconnected diagrams win in the large N limit is general and goes by the name "large N factorization". It says that single trace operators are basically classical objects in the large N limit $\langle \mathcal{O}\mathcal{O}\rangle \sim \langle \mathcal{O}\rangle\langle \mathcal{O}\rangle$. As an aside, another convenient way to draw the connected diagram in Fig. 7 is shown in Fig. 8 where we have deformed the two four point operator insertion vertices so that they are "ready for contraction" making it easier to draw propagators.

Figure 7: Connected diagram contributing to the correlation function $\langle \mathrm{Tr}(\Phi^4)\mathrm{Tr}(\Phi^4)\rangle$
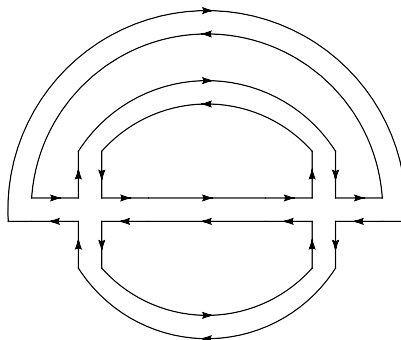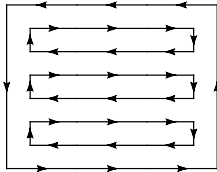
Figure 8: A redrawing of the connected diagram shown in Fig. 7

The leading connected contribution to the correlation function is independent of N and so $\langle \mathcal{O}\mathcal{O}\rangle_c \sim c^2 N^0$. Requiring that $\langle \mathcal{O}\mathcal{O}\rangle_c \sim N^0$ means we can just set $c = 1$. Note that the MAGOO review normalizes their single trace operators with an explicit $1/N$ which is wrong. Having fixed the normalization of $\mathcal{O}$ we can now determine the N dependence of higher order correlation functions. For example, the leading connected diagram for $\langle \mathcal{O}^3\rangle$ where $\mathcal{O} = \text{Tr}(\Phi^2)$ is just a triangle and contributes a factor $\lambda^3 N^{-1} \sim N^{-1}$. In fact, quite generally we have $\langle \mathcal{O}^n\rangle_c \sim N^{2-n}$ for the leading contribution.

The states $\mathcal{O}|\text{vac}\rangle$ are called glueball states, and they correspond to excitations of the theory that are free at large N. These states also make sense from the string worldsheet point of view where the operator $\mathcal{O}$ is like an integrated vertex operator.

We now want to study the generating function of correlation functions of single trace operators. Consider a compound operator like $\prod_A \mathcal{O}_A$ where each $\mathcal{O}_A \sim \text{Tr}(\Phi^{k_A})$ is a single trace operator with $k_A \ll N$. This is a multi-trace operator because it is the product of single trace operators. We can write a generating function for correlation functions like $\langle \prod_A \mathcal{O}_A\rangle$ using the operator $\Delta S = N \sum_A \lambda_A \mathcal{O}_A$ where the overall factor of N is meant to normalize ΔS in the same way as the Lagrangian. Remember that we're suppressing all indices including spacetime locations of fields meaning there may be integrals involved in the proper definition of ΔS. The generating function is defined as

$$Z = e^{-F} = \langle e^{-\Delta S}\rangle \tag{5}$$

and when calculating Z we must remember that each $\mathcal{O}_A$ acts just like an extra vertex with the normalization we defined above.

As a simple check of the AdS/CFT correspondence we would like to determine the large N scaling of F. The naive answer is just that $F \sim N^2$ because there are N^2 degrees of freedom, however this counting may fail for example when the theory is in a confining phase. Let us therefore assume the theory is not in a confining phase and calculate the large N behavior of F more carefully. In the free theory we do indeed have $F \sim N^2$ just by our naive counting of degrees of freedom. What about in t'Hooft limit? We may lump the interactions in L in with the operators in ΔS since they are all single trace with an overall constant set by N. Then we can write the generating function as

$$Z = \langle e^{-N\mathcal{O}}\rangle_{free} \approx e^{-N\langle \mathcal{O}\rangle_{free}} \tag{6}$$

because in the t'Hooft limit we have seen that single trace operators behave as if they are classical. Using our previous formula for the N dependence of correlation functions we find that $\langle \mathcal{O}\rangle \sim N$ so that $F \sim N^2$ because of the overall factor of N.

Figure 9: A quark vacuum bubble and a quark vacuum bubble with "gluon" exchange
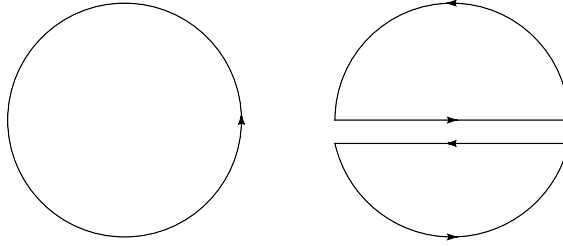
Let's check this result using the GKPW formula relating the partition functions of field theory and supergravity

$$Z = \langle e^{-\Delta S} \rangle = \exp\left(-S_{SUGRA}\right). \tag{7}$$

We can determine the N dependence of $S_{SUGRA} \sim \frac{1}{G_N^5} \int d^5 x \sqrt{g} R$ by scaling the action in terms of L the AdS scale. In five dimensions the Newton constant has units of length cubed, and so the correct scaled form of the action is $S_{SUGRA} \sim \frac{L^3}{G_N^5} S_{\text{dimensionless}}$. Using the fact that $L^3/G_N^5 \sim N^2$ we discover that S_{SUGRA} scales in the same way as F from the field theory. This fact follows from the relation $L^4/(\alpha')^4 = \lambda$ which we found by studying the RR soliton, from the ten dimensional Newton constant in string units $G_N^{10} = g_s^2(\alpha')^4$, and from the definition $\lambda = g_{YM}^2 N = g_s N$. The assumption of a non-confining ground state on the field theory side translates into the absence of additional scales in the bulk spacetime that would invalidate our scaling argument.

4   Simple generalizations

We can generalize the analysis performed so far without too much effort. One possibility is the addition of fields, "quarks", in the fundamental of $U(N)$. We can add fermions $\Delta L \sim \bar{q}\gamma^\mu D_\mu q$ or bosons $\Delta L \sim |D_\mu q|^2$. Because quarks are in the fundamental of $U(N)$ their propagator consists of only a single line. When using Feynman diagrams to triangulate surfaces we now have the possibility of surfaces with boundary. Two quark diagrams are shown in Fig. 9 both of which triangulate a disk. Notice in particular the presence of only a single outer line representing the quark propagator. We can conclude that adding quarks into our theory corresponds to admitting open strings into the string theory. We can also consider "meson" operators like $\bar{q}q$ or $\bar{q}\Phi^k q$ in addition to single trace operators.

Another direction for generalization is to consider different matrix groups such as $SO(N)$ or $Sp(N)$. The adjoint of $U(N)$ is just the fundamental times the anti-fundmental. However, the adjoint representations of $SO(N)$ and $Sp(N)$ are more complicated. For $SO(N)$ the adjoint is given by the anti-symmetric product of two fundamentals (vectors), and for $Sp(N)$ the adjoint is the symmetric product of two fundamentals. Also, lines in the double line formalism no longer have arrows. As a consequence the lines in the propagator for the matrix field can join directly or cross and then join as shown in Fig. 10. In the string language the worldsheet can now be unoriented, an example being given by a matrix field vacuum bubble where the lines cross giving rise to the worldsheet $\mathbb{RP}^2$.

Figure 10: Propagator for SO(N) ($+$) or Sp(N) ($-$) matrix models
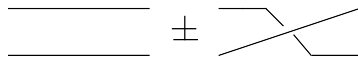
5 Large N vector models

Everything we've said for large N matrix models can be contrasted with the story for large N vector models. The difference between these two classes of models is that large N matrix models have N^2 degrees of freedom while large N vector models have only N degrees of freedom. This changes the scaling of the coupling in the large N limit. The Lagrangian of a large N vector model might look something like $L = \partial\vec{\phi}\cdot\partial\vec{\phi} - NV(\vec{\phi}\cdot\vec{\phi}/N)$ where the whole second term has the form $NV \sim m^2\vec{\phi}\cdot\vec{\phi} - \lambda_v/N(\vec{\phi}\cdot\vec{\phi})^2$. This form of the Lagrangian amounts to many more interactions than in the large N matrix model. The coupling $g_v = \lambda_v/N \sim 1/N$ (λ_v is not the t'Hooft coupling!) goes to zero much faster than the coupling in the matrix model $g_{YM} \sim 1/\sqrt{N}$. Because of the different scaling, the only interactions in the standard large N vector model are "cactus" diagrams. This means that modulo some self energy corrections the theory is essentially free. There also don't appear to be any strings.

6 Preview: the need for strings

As a preview of coming attractions, let's see what matrix models can tell us about the appearance of string theory in gauge theory. Recall that we said every SUGRA perturbation in AdS$_5 \times$ S$_5$ corresponds to some single trace operator in $\mathcal{N} = 4$ SYM. In particular, we found a correspondence between symmetric scalar KK modes in SUGRA and chiral primary operators in the $\mathcal{N} = 4$ theory. However, this map from SUGRA modes to $\mathcal{N} = 4$ operators is not onto, in fact it represents a very small subset of the $\mathcal{N} = 4$ operators. We would like to know what the rest of the $\mathcal{N} = 4$ operators correspond to in the gravity theory. A clue comes from looking at single trace operators like

$$\mathcal{O} = \mathrm{Tr}(XXXXYXXDXX...YX) \tag{8}$$

which we think of as a string of J letters (with $1 \ll J \ll N$). Because of the cyclic property of the trace, this string of letters looks a lot like a sequence of operator insertions on a closed string!