CESC Cargèse, Corsica, France July 2017

Geometry of "flux attachment" in the fractional quantum Hall effect.

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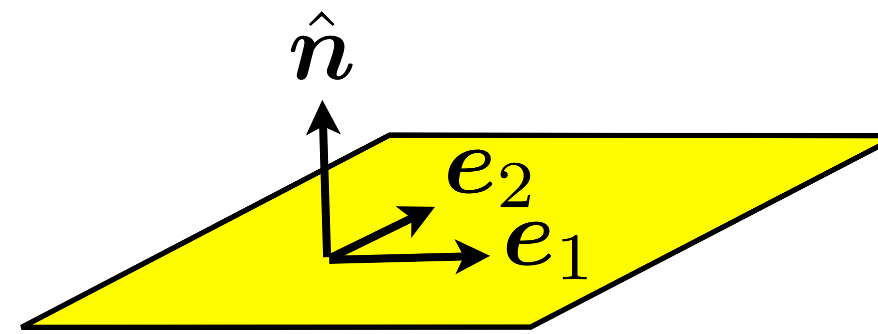
- Flux attachment as emergent gauge-field with geometry
- Geometry and energy of the "flux attachment" that forms "composite particles"

Supported by DOE DE-SC-0002140 and W. M. Keck Foundation

- landau level geometry
- laughlin state, unsolved problems
- why is QHE (gapped 2+1) well represented by 2d cft?
- status of projector models with conformal block ground states

- The quantum Hall fluid in a Landau level on the flat plane

$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j)$$



- Put it on the flat Euclidean plane representing a lattice plane in a crystal

$$\boldsymbol{x} = x^1\boldsymbol{e}_1 + x^2\boldsymbol{e}_2 \equiv x^a\boldsymbol{e}_a$$

$$p_a \equiv \boldsymbol{e}_a \cdot \boldsymbol{p} = -i\hbar\frac{\partial}{\partial x^a} - eA_a(\boldsymbol{x})$$

using covariant/contravariant spatial indices with summation convention to make any metrics explicit

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$$

Euclidean metric of the plane

- time-reversal symmetry is broken by a uniform magnetic flux density through plane:

$$\epsilon^{ab}\partial_a A_b(\boldsymbol{x}) = B$$

antisymmetric 2d Levi-Civita symbol

$$\boldsymbol{e}_a \times \boldsymbol{e}_b = \epsilon_{ab}\hat{\boldsymbol{n}}$$

- Define orientation of plane so $eB > 0$ (so ϵ_{ab} is odd under time-reversal)

$$\boxed{H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j)}$$

$$[x^a, x^b] = 0 \qquad [x^a, p_b] = i\hbar\delta^a_b \qquad [p_a, p_b] = i\hbar eB\epsilon_{ab}$$

Kronecker symbol
(not metric)

- symmetries: spatial translation symmetry
- spatial inversion symmetry plays a fundamental role in the QHE: we will impose

$$\boxed{\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})}$$

- full O(2) rotational symmetry defined by a metric is **<u>not</u>** a fundamental symmetry of the QHE, and will not be required.

$$\boxed{H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j)}$$

$$\boxed{\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})}$$

$$[x^a, x^b] = 0 \qquad [x^a, p_b] = i\hbar\delta^a_b \qquad [p_a, p_b] = i\hbar eB\epsilon_{ab}$$

- spatial translation and inversion symmetries

$$\boxed{\begin{array}{l} \boldsymbol{x}_i \mapsto \pm\boldsymbol{x}_i + \boldsymbol{c} \\ \boldsymbol{p}_i \mapsto \pm\boldsymbol{p}_i \end{array}}$$

- decomposition of the 2D Heisenberg algebra

Landau orbit radius vector

$$\bar{R}^a = (eB)^{-1}\epsilon^{ab}p_b$$

Landau orbit guiding center

$$R^a = x^a - \bar{R}^a$$

$$\boxed{\begin{array}{l} [\bar{R}^a, \bar{R}^b] = i\epsilon^{ab}\ell_B^2 \\ [R^a, R^b] = -i\epsilon^{ab}\ell_B^2 \\ [R^a, \bar{R}^b] = 0 \end{array}}$$

$2\pi\ell_B^2$ is the area per quantum of magnetic flux

$$\Phi_0 = h/e$$

$$\boxed{H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j)}$$

$$\boxed{\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})}$$

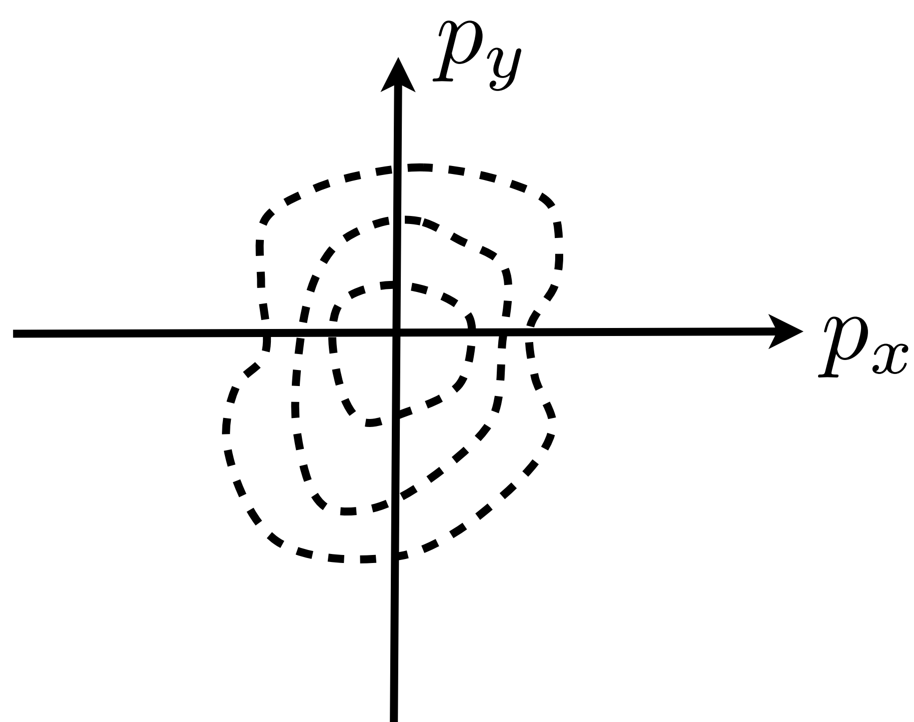
$$\boxed{\begin{aligned} &[\bar{R}^a, \bar{R}^b] = i\epsilon^{ab}\ell_B^2 \\ &[R^a, R^b] = -i\epsilon^{ab}\ell_B^2 \\ &[R^a, \bar{R}^b] = 0 \end{aligned}}$$

$$x_i^a = R_i^a + \bar{R}_i^a$$
$$p_{ai} = -(eB)\epsilon_{ab}\bar{R}_i^b$$

- will also specify that $\varepsilon(p)$ = $\varepsilon(p_1, p_2)$ is an *entire* function of each component of p (e.g. a polynomial) and
- has a unique minimum at p = 0 with no other stationary points (this ensures a simple Landau-level structure)

- semiclassical Landau quantization
- like phase space

$$[p_x, p_y] = i\hbar e B$$



There is no reason that Landau orbits with different index n should be congruent (have the same shape)

semiclassical Landau levels are localised on contours of constant $\varepsilon(\boldsymbol{p})$ that enclose momentum-space area $2\pi\hbar e B(n+\frac{1}{2})$

(analog of Bohr-Sommerfeld quantization)

- Quantum treatment: $[p_x, p_y] = i\hbar e B$

usual model

(harmonic oscillator, separable)

$$h = \frac{1}{2m}\left((p_x)^2 + (p_y)^2\right)$$

generic model

(bivariate non-separable function of non-commuting coordinates)

$$h = \varepsilon(p_x, p_y)$$

- To be well-defined, $\varepsilon(\boldsymbol{p})$ must have an absolutely convergent expansion in **p** - $\mathbf{p}_0$ for all (c-number) $\mathbf{p}_0$, so must be an entire function of both p_x and p_y, e.g. a polynomial,

- macroscopic degeneracy of Landau levels

$$[R^a, \varepsilon(\boldsymbol{p})] = 0$$

$$\varepsilon(\boldsymbol{p})|\Psi_{n\alpha}\rangle = E_n|\Psi_{n\alpha}\rangle$$

basis within degenerate Landau level

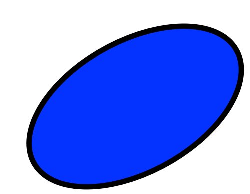
- one independent state for each quantum of magnetic flux through the plane

- Coherent states of a Landau orbit are defined by a guiding-center metric
- The coherent state centered at the origin is defined by

$$\varepsilon(\boldsymbol{p})|\Psi_n(\mathbf{0},\tilde{g})\rangle = E_n|\Psi_n(\mathbf{0},\tilde{g})\rangle$$
$$\tilde{g}_{ab}R^aR^b|\Psi_n(\mathbf{0},\tilde{g})\rangle = \surd(\det\tilde{g})(\ell_B)^2|\Psi_n(\mathbf{0},\tilde{g})\rangle$$

- we can always choose the metric to be unimodular (determinant = 1), as it just defines a complex structure z(x,y)

coherent states minimize the uncertainty of the guiding center



- factorization of a metric to define a complex structure

$$\frac{1}{2}\left(\tilde{g}_{ab} + i\epsilon_{ab}\right) = e_a^* e_b \qquad \det \tilde{g} = 1$$

a complex vector

example

$\tilde{g}_{ab} = \delta_{ab}$

$\boldsymbol{e} = \frac{1}{\sqrt{2}}(1, i)$

(Euclidean metric)

$$z_{\tilde{g}}(x, y) = e_a x^a / \ell_B$$

a unimodular metric defines a (dimensionless) complex structure (up to a U(1) ambiguity)

$$z \mapsto e^{i\phi} z$$

- The Schrödinger wavefunction of a Landau level coherent state defined by a metric has the form

$$\psi_n(x,y) = f_n(z^*;\tilde{g})e^{-\frac{1}{2}z^*z}$$

a holomorphic function of z* that depends of the choice of guiding-center metric to define the coherent state

- so far, we have complete freedom of choice to choose this metric: is there a "natural choice"?

- Yes, the natural choice of metric is provided by the "Hall viscosity tensor" of the Landau orbit:

$$\langle\Psi_{n\alpha}|\tfrac{1}{2}\{p_a,p_b\}|\Psi_{n\alpha}\rangle = 2\pi\hbar\eta^{(n)}_{ab}$$

- viscosity is the linear relation between stress and flow-velocity gradient. Stress is a mixed-index tensor (momentum current density) which is traceless in gapped incompressible quantum fluids

$$\sigma^a_b = \eta^{ac}_{bd}\partial_c v^d \qquad \eta^{ac}_{bd} = \epsilon^{ae}\epsilon^{bf}\eta^H_{\{be\}\{df\}} \qquad \eta^{ac}_{bd} = -\eta^{ca}_{db}$$

stress velocity gradient Traceless condition dissipationless

$$\eta^H_{\{ab\}\{cd\}} = \tfrac{1}{2}\left(\eta_{ac}\epsilon_{bd} + \eta_{ad}\epsilon_{bc} + \eta_{bc}\epsilon_{ad} + \eta_{bd}\epsilon_{ac}\right)$$

rank-4 tensor odd under time-reversal

rank-2 symmetric Hall-viscosity tensor even under time-reversal

odd under time-reversal

- every Landau level has a “natural metric” proportional to its Hall viscosity tensor, which characterises its shape

- It also has a natural effective mass tensor defined by its reponse to polarization:

$$\mathcal{L}(\boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{p} - \varepsilon(\boldsymbol{p})$$

Lagrangian operator

$$\mathcal{L}(\boldsymbol{v})|\Psi_{n\alpha}(\boldsymbol{v})\rangle = L_n(\boldsymbol{v})|\Psi_{n\alpha}(\boldsymbol{v})\rangle$$

$$L_n(\boldsymbol{v}) = -E_n + \tfrac{1}{2} m_{ab}^{(n)} v^a v^b + O(v^4)$$

- finally, it also has a topological spin

- special case, rotationally invariant Landau levels:

$$[\varepsilon(\boldsymbol{p}), \tilde{g}^{ab} p_a p_b] = 0$$

$$\psi_n(x,y) \propto (z^*)^n e^{-\frac{1}{2} z^* z}$$ general n, coherent state

$$\psi_0(x,y) \propto e^{-\frac{1}{2} z^* z}$$ n=0, coherent state

$$\psi(x,y) = f(z) e^{-\frac{1}{2} z^* z}$$ n=0, general state

This structure is non-generic, only applies to case where $\varepsilon(\boldsymbol{p})$ has rotational symmetry.

- Filled LLL (rotational symmetry)

$$\Psi = \prod_{i<j}(z_i - z_j)\prod_i e^{-\frac{1}{2}z_i^* z_i}$$

inspiration for Laughlin

$$\Psi_L^{(m)} = \prod_{i<j}(z_i - z_j)^m \prod_i e^{-\frac{1}{2}z_i^* z_i}$$

conformal block

- quartic polynomial case

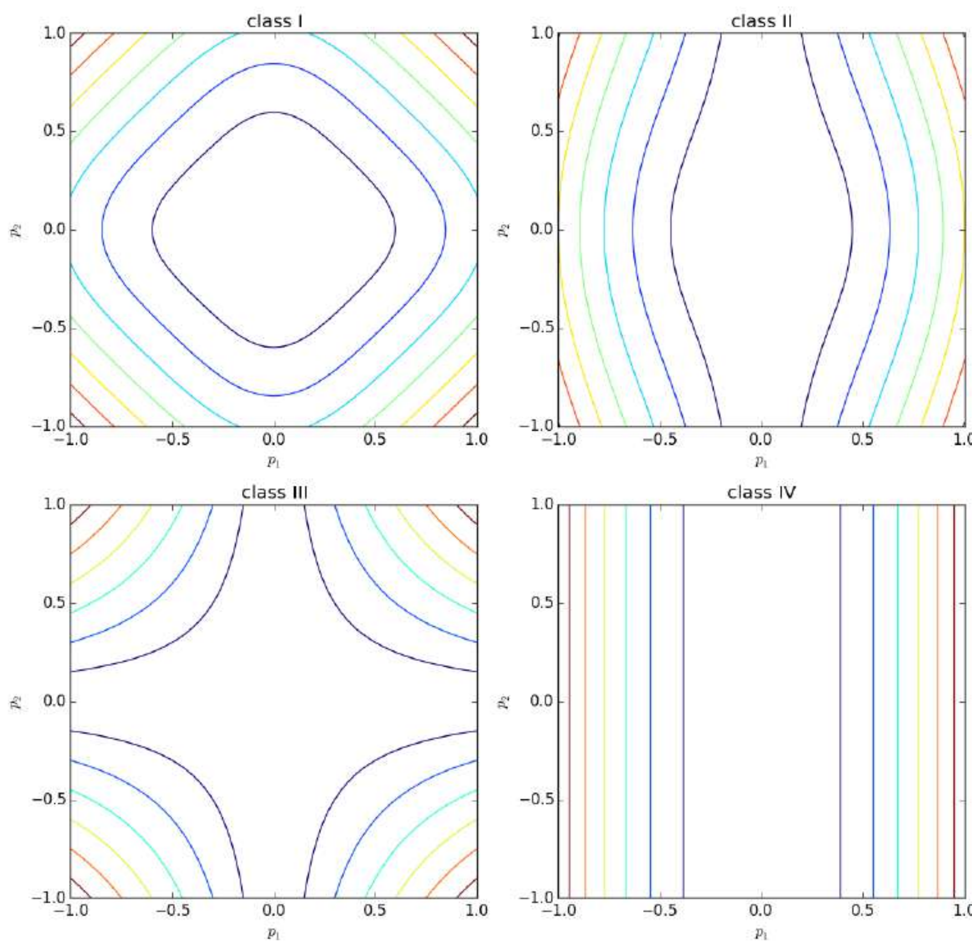


Figure 2.7: Contour plots in the momentum space of a typical member in each class of the quartic term. The quartic terms are I) $\{5p_1^2 + p_2^2, p_1^2 + 5p_2^2\}$, II) $\{p_1^2, p_1^2 + p_2^2\}$, III) $\{p_1^2, p_2^2\}$ and IV) p_1^4. Notice that only the contours for class I are closed.

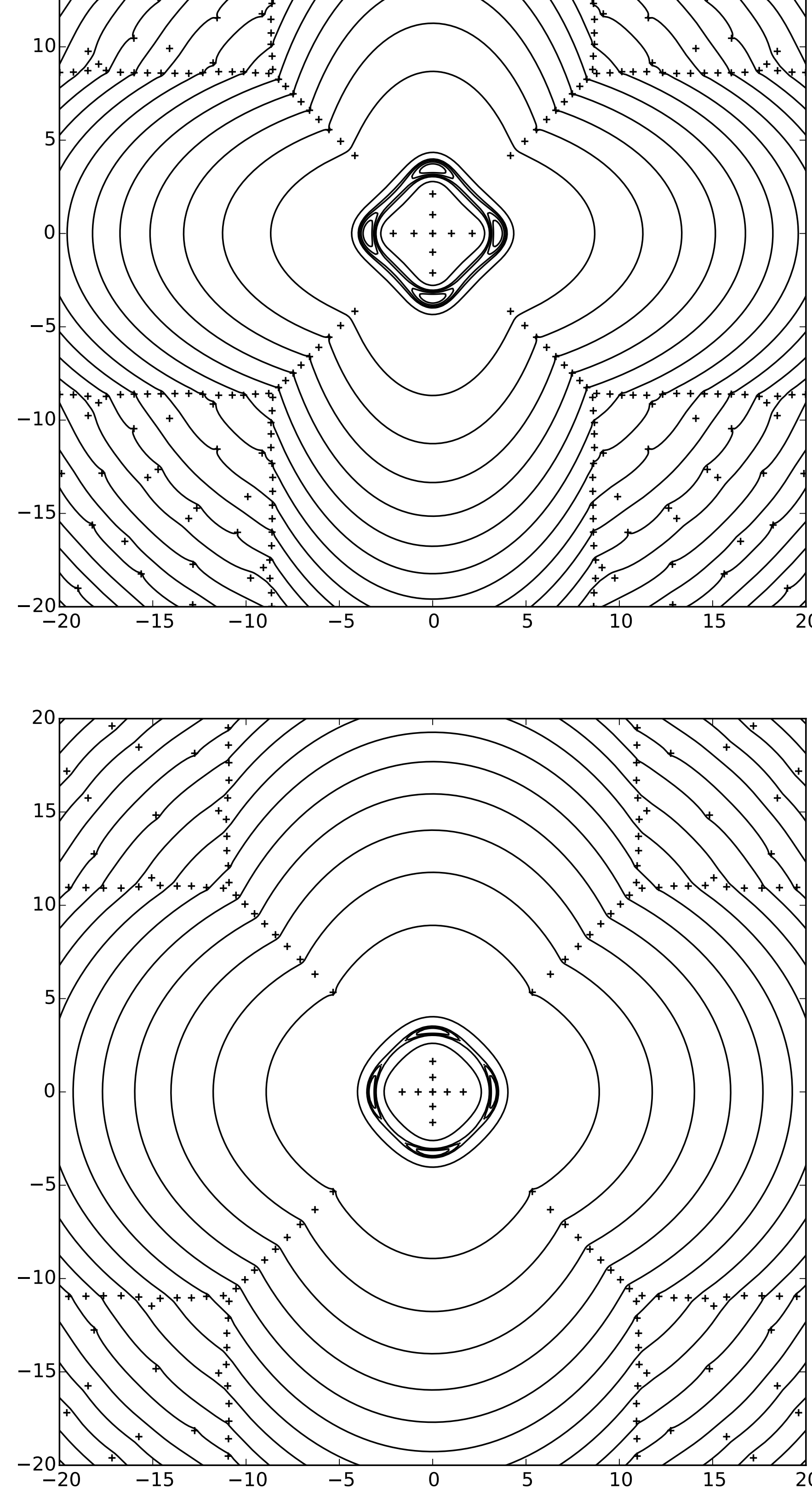


Figure 3.5: Contour plots of $\ln|\Psi_{10}(z,z^*)|^2$ where $\Psi_{10}(z,z^*)$ is the coherent state in the 10th Landau level for class I quartic term (2.45) with $c-1=4>2$ (top) and $c-1=1<2$ (bottom). Both plots show piece-wise contours with the four spikes and the line charges as branch cuts. Another common feature is the existence of four maxima along the directions of the central cross and four saddle points along those of the spikes. Despite the similarities, the two plots also show qualitatively different

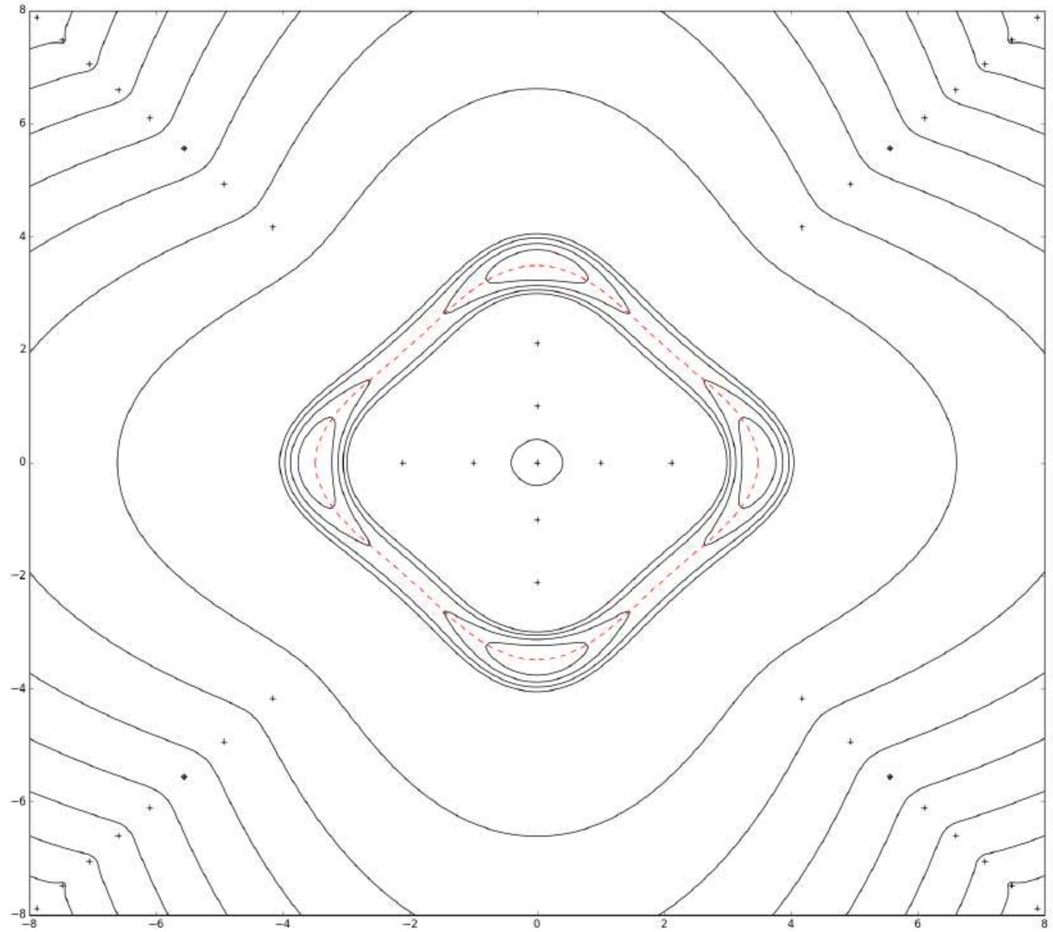


Figure 3.6: A “ridge” (dashed) defined as the boundary between two regions where the gradient field flows to the origin (inside) or infinity (outside). The roots are those of the coherent state in the 10th Landau level for class I quartic term with $c-1=2$. Contours are also shown. Notice that the local maxima and the saddle points are all right on the ridge. The area enclosed by the ridge is πn, corresponding to an area of $2\pi\hbar eB$ on the momentum plane, which is consistent with (shifted) semiclassical quantization.

- The effective continuum Hamiltonian is

$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j)$$

- The model has 2D inversion symmetry if

$$\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$$

- The only role played by the Euclidean metric of the inertial background frame is the non-relativistic criterion

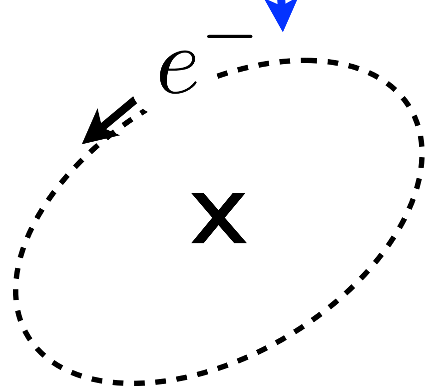
$$\delta_{ab} v^a v^b \ll c^2 \qquad v^a(\boldsymbol{p}) = \frac{\partial \varepsilon}{\partial p_a}$$

Generic model with translation and inversion symmetry only, no rotational symmetry

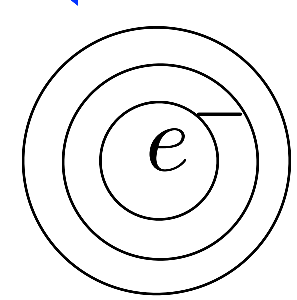
$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j) \qquad \varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$$

affected by elastic degrees of freedom

- two distinct unrelated sources of geometry



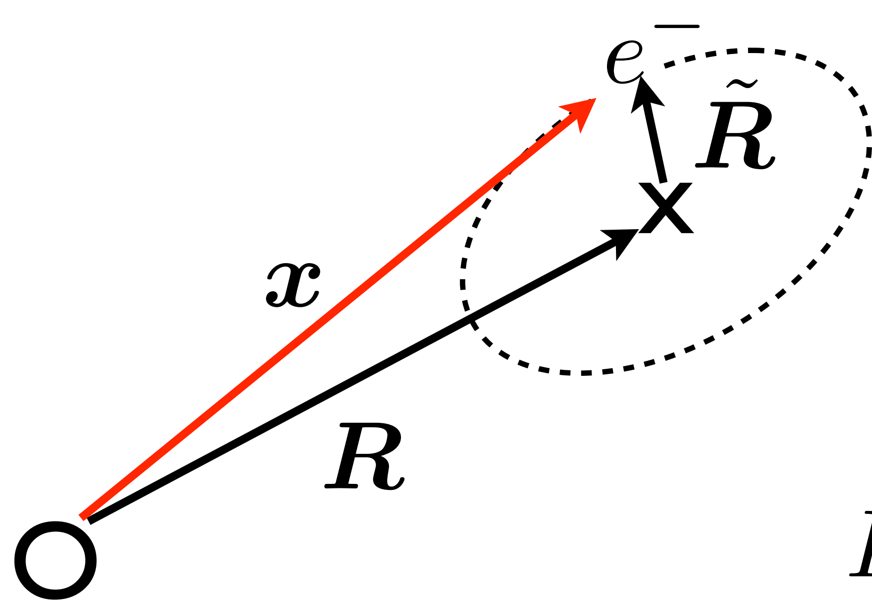
shape of Landau orbit around guiding center



equipotentials around point charge (from 3D dielectric tensor)

- The "holomorphic lowest Landau level wavefunction" is a property of a SO(2) rotationally-invariant system:

$$\boldsymbol{x} = \boldsymbol{R} + \tilde{\boldsymbol{R}}$$



$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$[\tilde{R}^a, \tilde{R}^b] = i\ell_B^2 \epsilon^{ab}$$

$$[R^a, \tilde{R}^b] = 0$$

angular momentum

$$L = \frac{\hbar}{2\ell_B^2}\delta_{ab}(R^a R^b - \tilde{R}^a \tilde{R}^b)$$

$$= \tfrac{1}{2}\hbar \left(a^\dagger a - b^\dagger b \right)$$

guiding center   Landau level

Two sets of ladder operators:

- Now write the Laughlin state as a Heisenberg state, not a Schrödinger wavefunction:

$$|\Psi_L\rangle \propto \prod_{i<j} (a_i^\dagger - a_j^\dagger)^m |0\rangle \quad a_i|0\rangle = 0 \qquad a^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$b_i|0\rangle = 0$ lowest Landau level condition

In the Heisenberg form, we see that the LLL condition is quite incidental to the Laughlin state, which involves guiding-center correlations

- The fundamental form of the Laughlin state does not reference the details of the Landau level in any way:

$$|\Psi_L(\tilde{g})\rangle \propto \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0\rangle \quad a_i|0\rangle = 0 \qquad a^\dagger = \frac{\omega_a R^a}{\sqrt{2}\ell_B}$$

$$\omega_a^*\omega_b = \tfrac{1}{2}\left(\tilde{g}_{ab} + i\epsilon_{ab}\right) \qquad \det \tilde{g} = 1$$

a unimodular Euclidean-signature metric that parameterizes the Laughlin state

- The historical identification of this metric with the Euclidean metric is unnecessary unless there is SO(2) symmetry.

- Topological states of matter have been a major theme in the recent developments in understanding novel quantum effects.
- key questions are: why do they occur, what features of materials favor such states, and how can we understand the energetics that drives their emergence.
- I will principally discuss the fractional quantum Hall effect, but this is a general question

- thirty years after its experimental discovery and theoretical description in terms of the Laughlin state, the fractional quantum Hall effect remains a rich source of new ideas in condensed matter physics.

- The key concept is "**flux attachment**" that forms "**composite particles**" and leads to topological order.

- Recently, it has been realized that flux attachment also has interesting **geometric** properties

- I will talk about what has been interesting me for the last few years:

- What is an “incompressible quantum fluid” such as the one described by Laughlins wavefunction.
- What “fluid dynamics” describes it?

$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \int \frac{d^2\boldsymbol{q}}{2\pi}\tilde{V}(\boldsymbol{q}) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{x}_i - \boldsymbol{x}_j)}$$

- Landau level form-factor

$$f_n(\boldsymbol{q}) = \langle \psi_{n\alpha} | e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{R})} | \psi_{n\alpha} \rangle$$

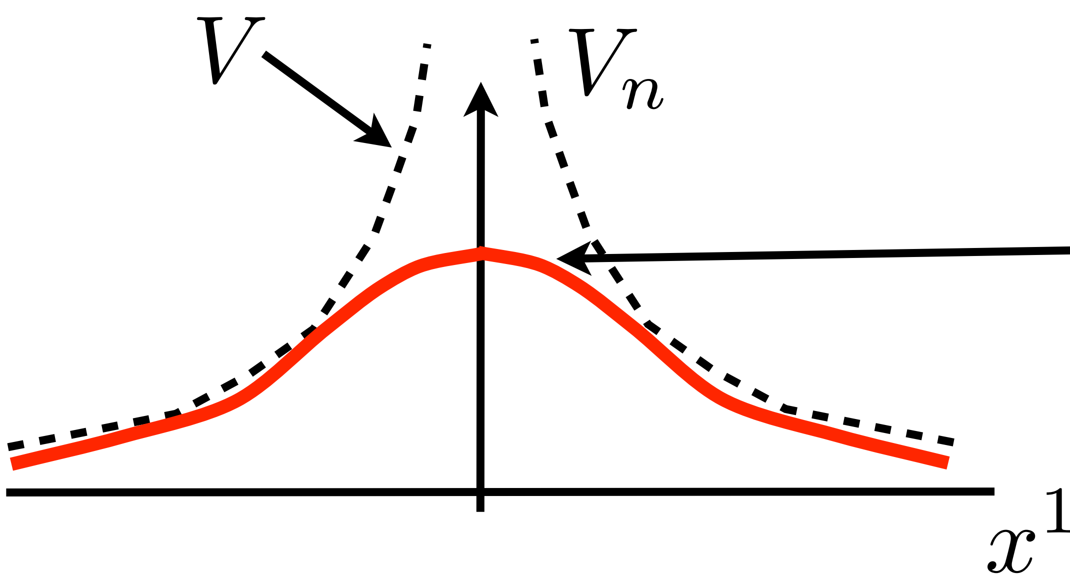
This is a rapidly-decreasing (Gaussian) function of q

- After projection into (any) single Landau level

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$V_n(\boldsymbol{x}) = \int \frac{d^2\boldsymbol{q}\ell_B}{2\pi}\tilde{V}(\boldsymbol{q})|f_n(\boldsymbol{q})|^2$$

Fourier transform of bare (e.g. Coulomb) interaction



a very smooth function:
in fact it is an <u>entire function</u> of both x^1 and x^2

The "entire" property is needed because $(R_i^1 - R_j^1)$ and $(R_i^2 - R_j^2)$ do not commute

- Girvin-Macdonald-Platzman Lie algebra

$$\rho(\boldsymbol{q}) = \sum_{i=1}^{N} e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}$$

$$[\rho(\boldsymbol{q}), \rho(\boldsymbol{q}')] = 2i\sin(\tfrac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2)\rho(\boldsymbol{q}+\boldsymbol{q}')$$

- q = 0 generator = N, is in kernel.

$$H = \int \frac{d^2\boldsymbol{q}\ell^2}{2\pi}\tilde{V}_n(\boldsymbol{q})(\tfrac{1}{2}\rho(\boldsymbol{q})\rho(-\boldsymbol{q}))$$

rapidly-decreasing (Gaussian) function at large q
(Fourier transform of an entire function)

- “compactification” on the torus

$$\boldsymbol{L} \in \{m\boldsymbol{L}_1 + n\boldsymbol{L}_2\} \qquad\qquad \boldsymbol{L}_1 \times \boldsymbol{L}_2 = 2\pi N_\Phi \ell_B^2$$

Bravais Lattice of periodic translations

- reciprocal lattice (discrete set of allowed wavevectors)

$$e^{i\boldsymbol{q}\cdot\boldsymbol{L}} = 1 \qquad\qquad \boldsymbol{L} = L^a \boldsymbol{e}_a \qquad\qquad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$$

Euclidean metric

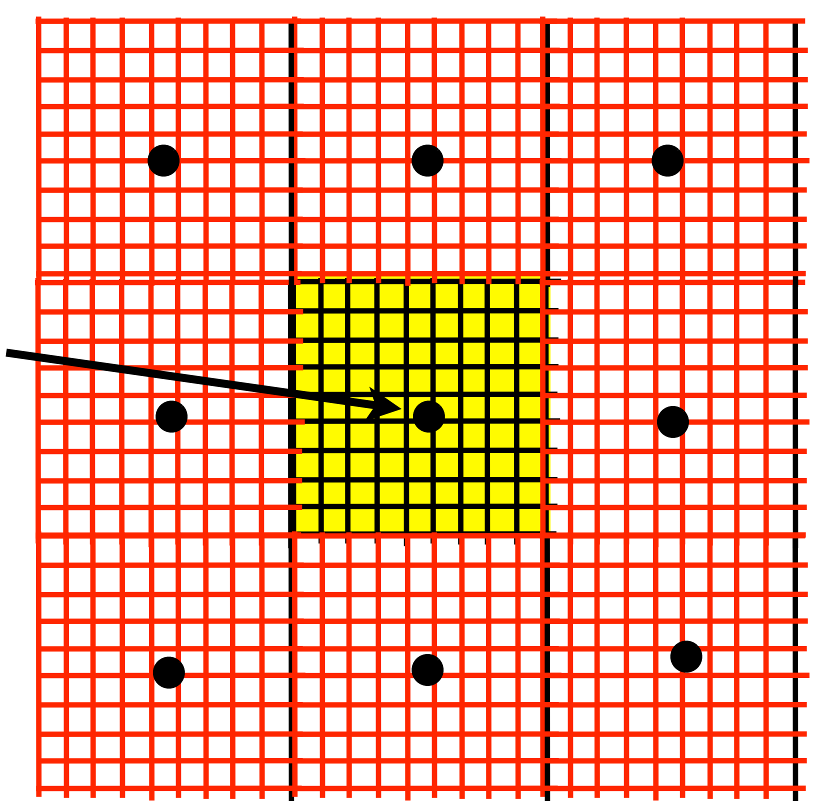
$$q_a \ell_B \in \left\{ \frac{\epsilon_{ab} L^b}{N_\Phi \ell_B} \right\} \qquad\qquad \boxed{\left(e^{i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\right)^{N_\Phi} = 1}$$

- reciprocal vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are equivalent if

$$q_1 - q_2 = N_\Phi q$$

- There are $(N_\Phi)^2$ distinct reciprocal vectors in a “Brillouin zone”

$\boldsymbol{q} = 0$



- a reciprocal vector $\boldsymbol{q}$ is <u>even</u> if $\frac{1}{2}\boldsymbol{q}$ is also an allowed reciprocal vector

- the inversion-symmetric pbc is

$$\left(e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}\right)^{N_\Phi}|\Psi\rangle = \eta(\boldsymbol{q})^{N_\Phi}|\Psi\rangle$$

$\eta(\boldsymbol{q}) = 1$ if $\boldsymbol{q}$ is even, -1 if not.

- why is there a Brillouin zone?

- The pbc means the allowed translations compatible with the pbc are $\boldsymbol{L}/N_\Phi$

- In fact, we can with full generality work only on this lattice.

$$\langle\psi_1|\psi_2\rangle = \int \frac{dz\wedge dz^*}{2\pi i}(f_1(z))^* f_2(z) e^{-z^*z}$$

naive formula based in the idea that these are Schrödinger wavefunctions

$$= \frac{1}{N_\Phi}\sum_{\boldsymbol{x}}{}' (f_1(z))^* f_2(z) e^{-z^*z}$$

sum over $\boldsymbol{x} = \boldsymbol{L}/N_\Phi$ in the unit cell

- for $N = p\bar{N}, \quad N_\Phi = q\bar{N}$ $\qquad \gcd(p,q) = 1$

$$\left(\prod_{i=1}^{N} e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}\right)^q |\Psi_\alpha\rangle = (\eta(\boldsymbol{q}))^{pq} \left(e^{i\boldsymbol{Q}\times\boldsymbol{q}\ell_B^2}\right)^q |\Psi_\alpha\rangle$$

many-body translation quantum $\boldsymbol{Q}$ number takes $(\bar{N})^2$ distinct values

If $\bar{N}$ is even (odd), one (four) of these have inversion symmetry

α is a q-fold exact topological degeneracy many-body $\boldsymbol{Q}$ lives in a “Brillouin zone” of $\bar{N} = \gcd(N, N_\Phi)$ points

$\bar{N}$ even $\qquad$ $\bar{N}$ odd

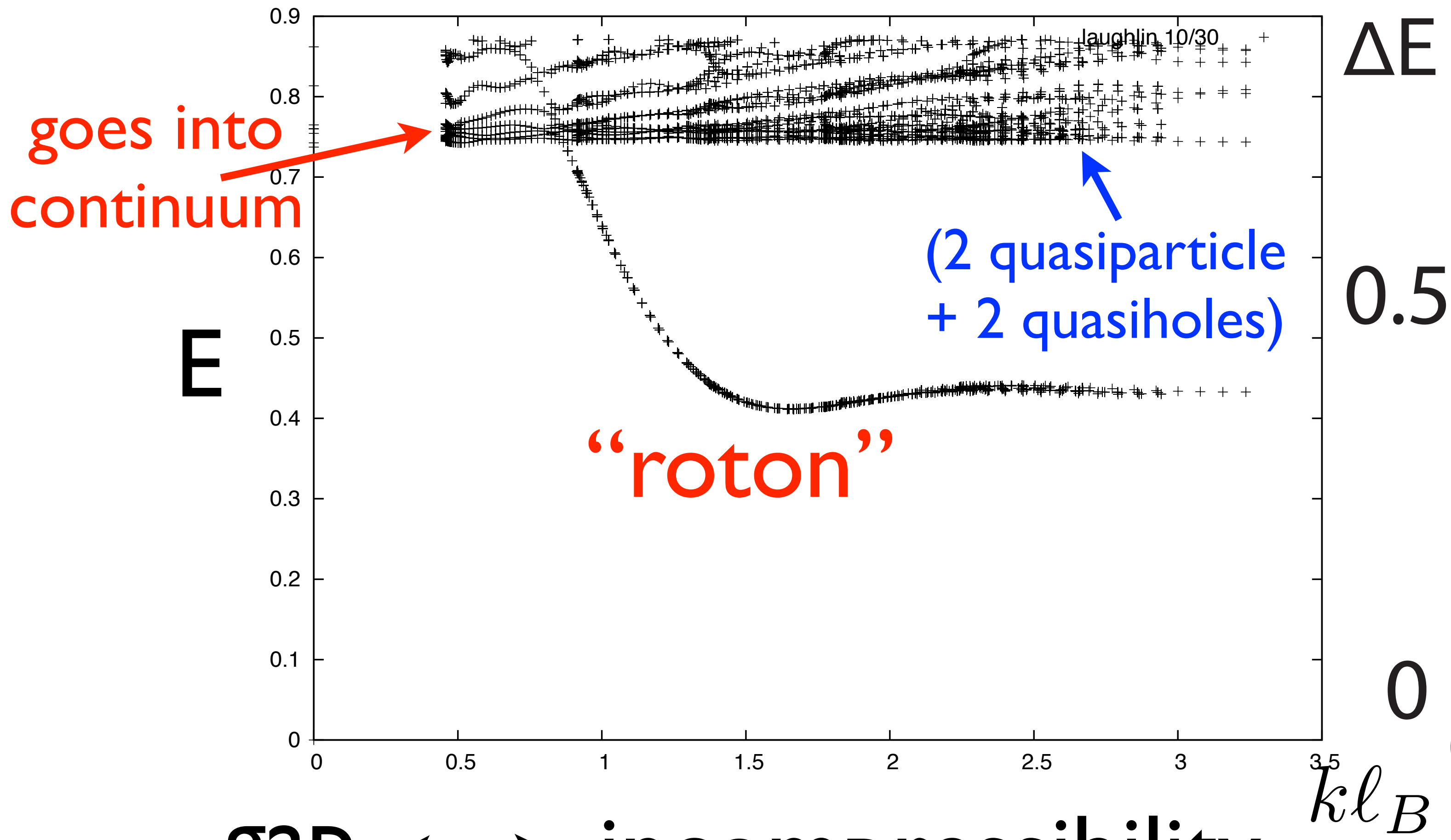
- quantum Hall states always have inversion-symmetric $\boldsymbol{Q}$.

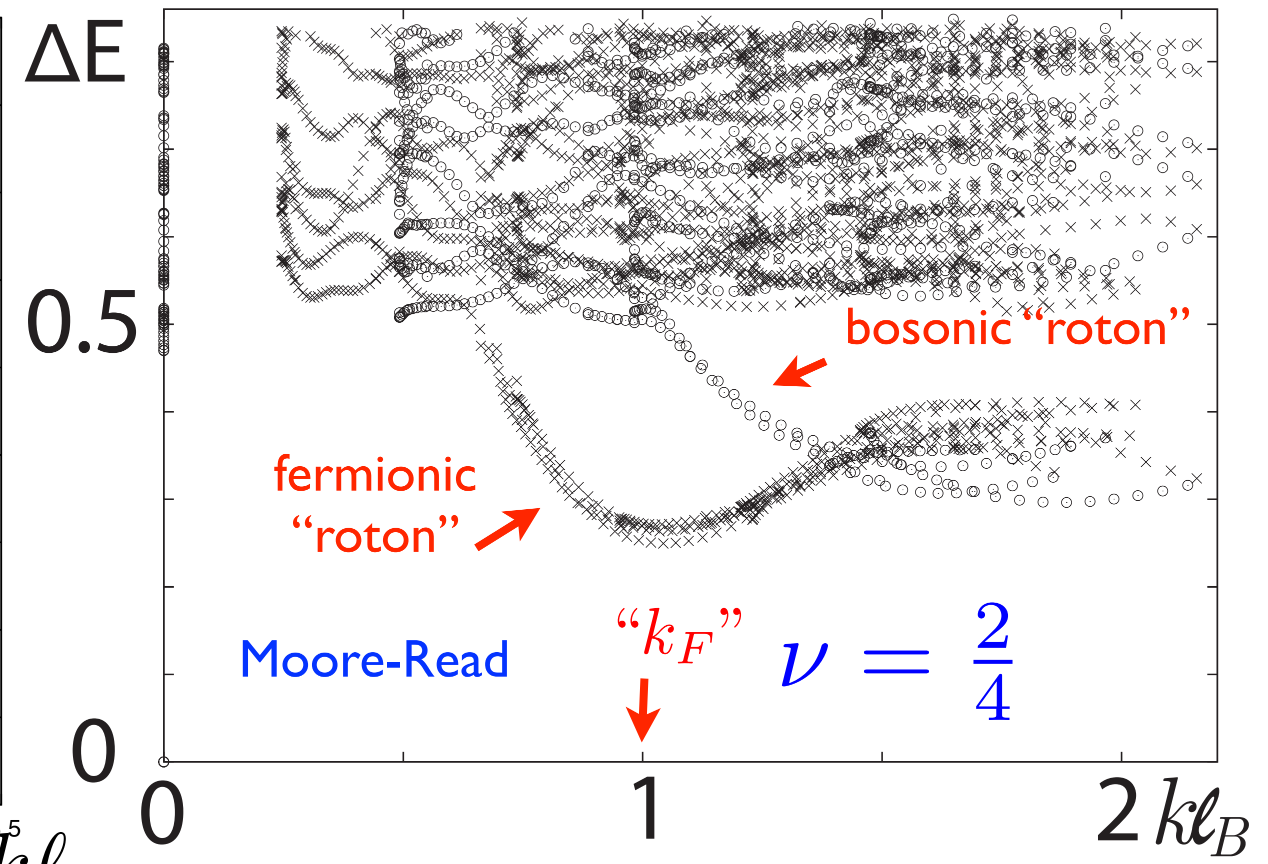
- If they occur only at $\boldsymbol{Q}$ =0, the elementary droplet has p particles with flux attachment q

- If they occur on the zone boundary, p and q must be <u>doubled</u> and $\bar{N}$ halved (e. g., Moore-Read state) $\quad p/q = 1/2 \rightarrow 2/4$

The physical FQH (as opposed to algebraic) p, q obey

$$\boxed{\gcd(p, q) \leq 2}$$





gap $\longleftrightarrow$ incompressibility

Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar$k of a quasiparticle-quasihole pair is proportional to its **electric dipole moment p$_e$**

$$\hbar k_a = \epsilon_{ab} B p_e^b$$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: **doesn't transmit pressure!**

(origin of Virasoro algebra in FQHE ?)

$$\langle \rho(\boldsymbol{q}) \rangle = N\eta(\boldsymbol{q})^{N_\Phi} \delta^P_{\boldsymbol{q},0}$$
$$= \nu N_\Phi \eta(\boldsymbol{q})^{N_\Phi} \delta^P_{\boldsymbol{q},0}$$

expectation with a translationally- invariant inversion-symmetric density matrix

$$\delta^P_{\boldsymbol{q}_1,\boldsymbol{q}_2} = \frac{1}{(N_\Phi)^2} {\sum_{\boldsymbol{q}}}' e^{i\boldsymbol{q}\times(\boldsymbol{q}_1 - \boldsymbol{q}_2)}$$

= 1 is q_1 and q_2 are equivalent,
= 0 if not

$$\delta\rho(\boldsymbol{q}) = \rho(\boldsymbol{q}) - N\eta(\boldsymbol{q})^{N_\Phi} \delta^P_{\boldsymbol{q},\boldsymbol{0}}$$

- Guiding-center structure function (2-point function)

$$\langle \delta\rho(\boldsymbol{q}) \delta\rho(-\boldsymbol{q}) \rangle = N_\Phi S_{\mathrm{gc}}(\boldsymbol{q})$$

$$S_\infty = \nu(1 + \xi\nu)$$

± 1 for fermions/bosons

- for an uncorrelated state

$$S_{\mathrm{gc}}(\boldsymbol{q}) - S_\infty = 0 \quad (\delta^P_{\boldsymbol{q},0} = 0)$$

- for an completely uncorrelated (mixed) state

$$S_{\mathrm{gc}}(\boldsymbol{q}) - S_\infty = 0 \quad (\delta^P_{\boldsymbol{q},0} = 0)$$

- for a correlated pure state $S_{\mathrm{gc}}(\boldsymbol{q}) - S_\infty$ is a rapidly-decreasing function away from the center of the (geometric) Brillouin zone (defined by a metric)

$$\lim_{\lambda\to 0} S_{\mathrm{gc}}(\lambda\boldsymbol{q}) = 0$$

- for gapped FQH state (topologically degenerate multiplet)

$$\lim_{\lambda\to 0} S_{\mathrm{gc}}(\lambda\boldsymbol{q}) \to \lambda^4 \Gamma^{\{\{ab\},\{cd\}\}} q_a q_b q_c q_d \ell_B^2$$

First discovered by GMP

$\Gamma^{\{\{ab\},\{cd\}\}}$ is positive,

satisfies a bound involving the Hall viscosity tensor $\gamma_H^{[\{ab\},\{cd\}]}$

This bound seems to be saturated in conformal-block models

- For a (commutative) 2D liquid with density n_0

$$S(\boldsymbol{q}) = \frac{1}{N}\sum_{i\neq j}\langle e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\rangle$$

$$g(\boldsymbol{r}) - 1 = \frac{1}{2\pi n_0}\int \frac{d^2\boldsymbol{q}}{2\pi} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}(S(\boldsymbol{q}) - 1)$$

pair correlation function

standard structure factor

- For the guiding-center liquid, there is a self-duality

$$S_{\mathrm{gc}}(\boldsymbol{q}) - S_\infty = \xi \int \frac{d^2\boldsymbol{q}'\ell_B^2}{2\pi} e^{i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\,(S_{\mathrm{gc}}(\boldsymbol{q}') - S_\infty)$$

± 1 (fermion/boson)

$$P_{ij} = \frac{1}{N_\Phi}\sum_{\boldsymbol{q}}{}' e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)}$$

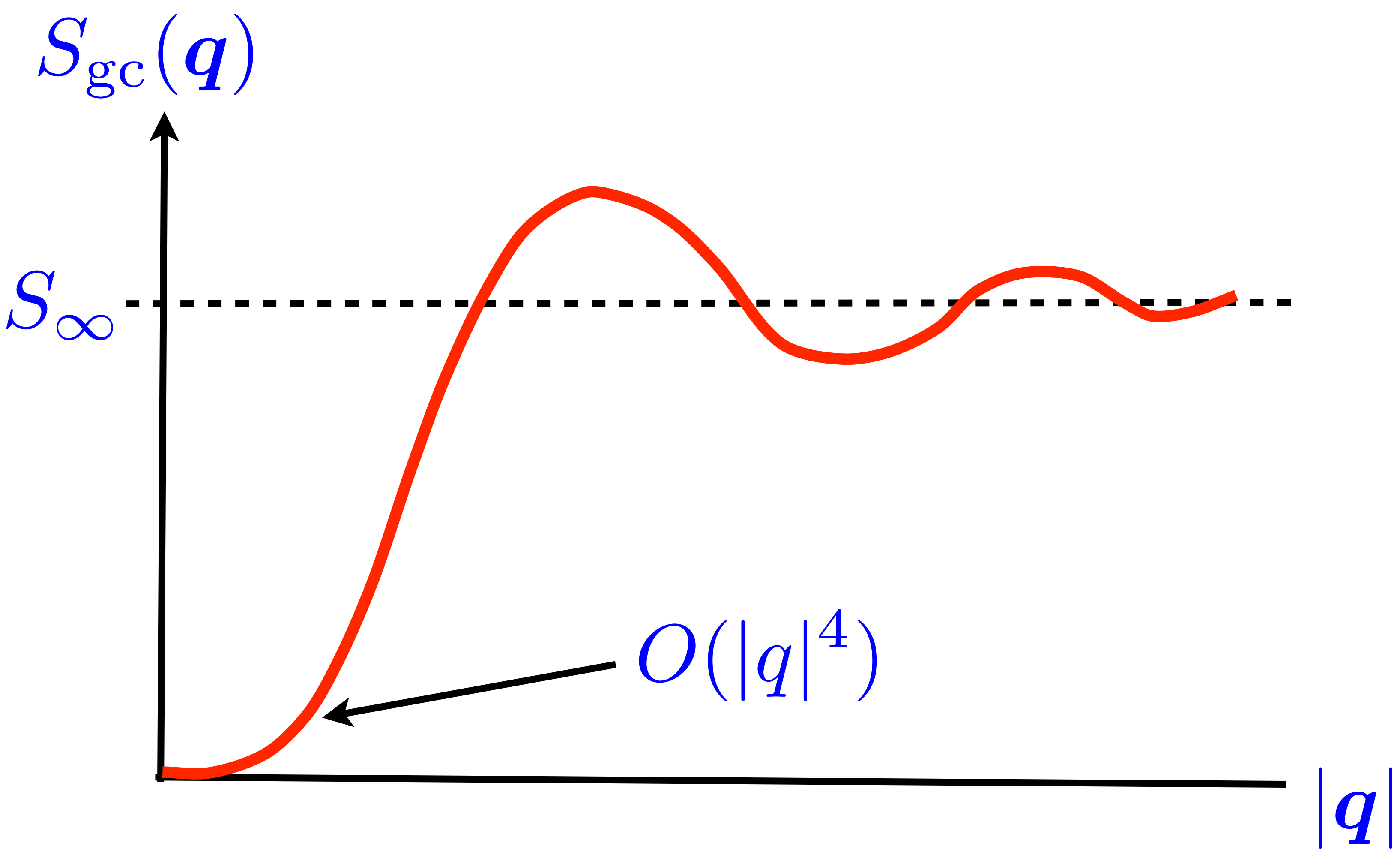
- rotationally-invariant states have a global metric, are eigenstates of

$$L = \frac{1}{2\ell_B^2}\tilde{g}_{ab}\sum_i R_i^a R_i^b$$

- $S_{\mathrm{gc}}(\boldsymbol{q})$ is a function of $q^2 = \tilde{g}^{ab}q_a q_b$

analytic on real-q^2 axis

- conjecture: for conformal block states $S_{\mathrm{gc}}(\boldsymbol{q})$ is an entire function of q^2


$S_{\rm gc}(\boldsymbol{q})$
S_∞
$O(|q|^4)$
$|\boldsymbol{q}|$

$$H = \tfrac{1}{2}\Gamma\left(\sum_{i<j} 2\ln(\frac{1}{|z_i - z_j|}| + \sum_i \tfrac{1}{2}|z_i|^2\right)$$

- Laughlin state and 2D OCP (log plasma)

$$|\Psi|^2 \propto \left(\prod_{i<j} |z_i - z_j|^2 \prod_i e^{-\frac{1}{2}|z_i|^2}\right)^{\frac{1}{2}\Gamma}$$

Boltzmann factor of Plasma has no branch cuts when Γ is an even integer

$$S(\boldsymbol{q}) = \frac{q^2}{2\pi\Gamma n_0 + (1 - \frac{1}{4}\Gamma)q^2 + \frac{1}{48}q^4/n_0 + O(q^6)}$$

Debye-Huckel pole near $q^2 = -\dfrac{2\pi\Gamma n_0}{1 - \frac{1}{4}\Gamma}$ (small Γ)

$$S(\boldsymbol{q}) = 1 - e^{-\frac{1}{2}|q|^2\ell^2}\left(1 - \nu^{-1} S^L_{\mathrm{gc}}(\boldsymbol{q})\right)$$

$\Gamma = 2m$

$\nu = \frac{1}{m}$

entire

- conjecture: The OCP at even integer has a pair correlation function that decays more rapidly that any exponential, (e..g as a Gaussian) and hence hs a structure factor that is and entire function of $|q|^2$.

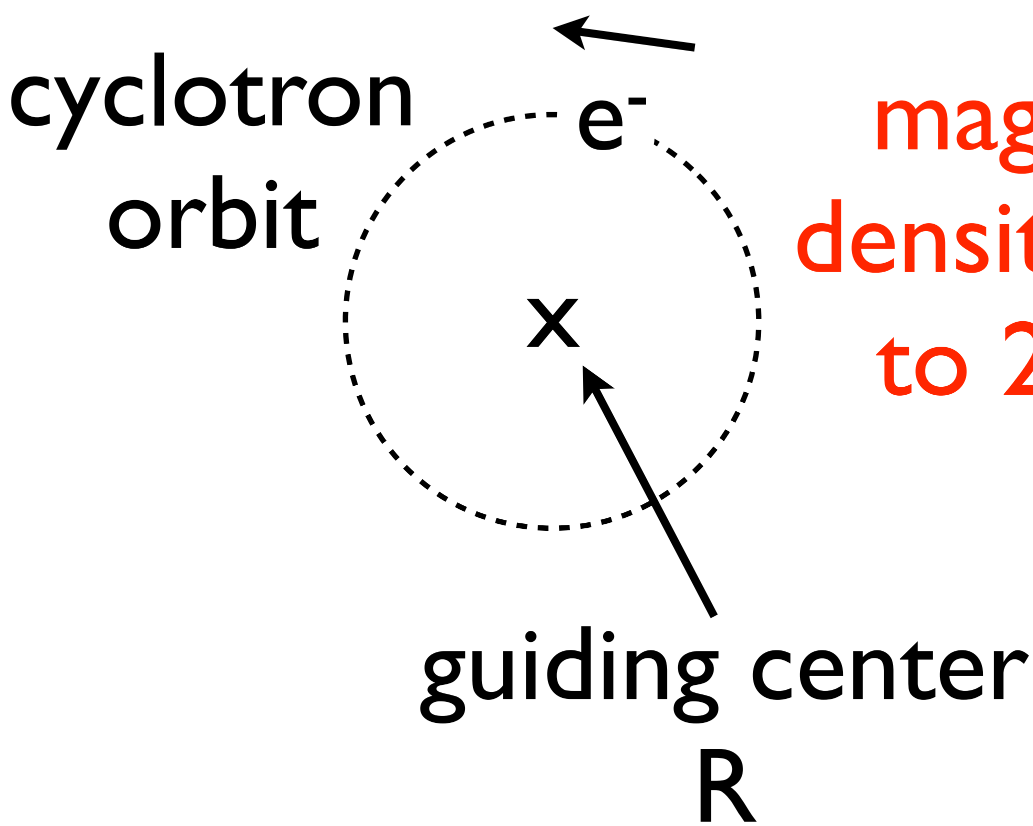
- This implies that the guiding-center structure factor of the Laughlin states are also entire functions of $|q|^2$.

- In turn, this suggests that the guiding-center structure factors of all conformal-block model FQH states are entire functions of $|q|^2$.

However, "generic" rotationally-invariant FQH states may be expected to have singularities in the structure factor off the real -positive $|q|^2$ axis

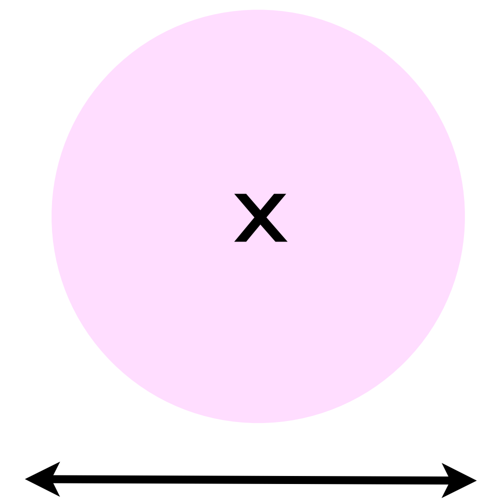
- electron in 2D Landau orbit (bound to 2D surface)

Becomes a "fuzzy object" after kinetic energy is quantized



magnetic flux density B normal to 2D surface

=



$$\ell_B = \left(\frac{\hbar}{|eB|}\right)^{\frac{1}{2}}$$

$$\boxed{[R^x, R^y] = -i\ell_B^2}$$

non-commutative geometry

$$\Psi = \prod_{i<j}(z_i - z_j)^3 \prod_i e^{-\frac{1}{2}z_i^* z_i}$$

Laughlin 1983

- elegant wavefunction, describes topologically-ordered fluid with fractional charge fractional statistics excitations

- exact ground state of modified model keeping only short range part of coulomb repulsion

- Validity confirmed by numerical exact diagonalization

30 years later:
unanswered question:
we know it works, but why?

my answer:
hidden geometry

some widespread misconceptions about the Laughlin state

- "it describes particles in the lowest Landau level"

No Landau level was specified: all specifics of the Landau level are hidden in the form of $U(\boldsymbol{r}_{12})$

- "It is a Schrödinger wavefunction"

Non-commutative geometry has no Schrödinger representation (this requires classical locality); it only has a Heisenberg representation.

- "Its shape is determined by the shape of the Landau orbit"

The interaction potential $U(\boldsymbol{r}_{12})$ determines its geometry (shape)

- "It has no continuously-tunable variational parameter"

Its geometry is a continuously-variable variational parameter

$$g^{ab} = \frac{1}{2}\langle\Psi|\{(R_i^a - R_j^a),(R_i^b - R_j^b)\}|\Psi\rangle$$

inverse metric

Fundamental symmetries of the incompressible quantum fluid

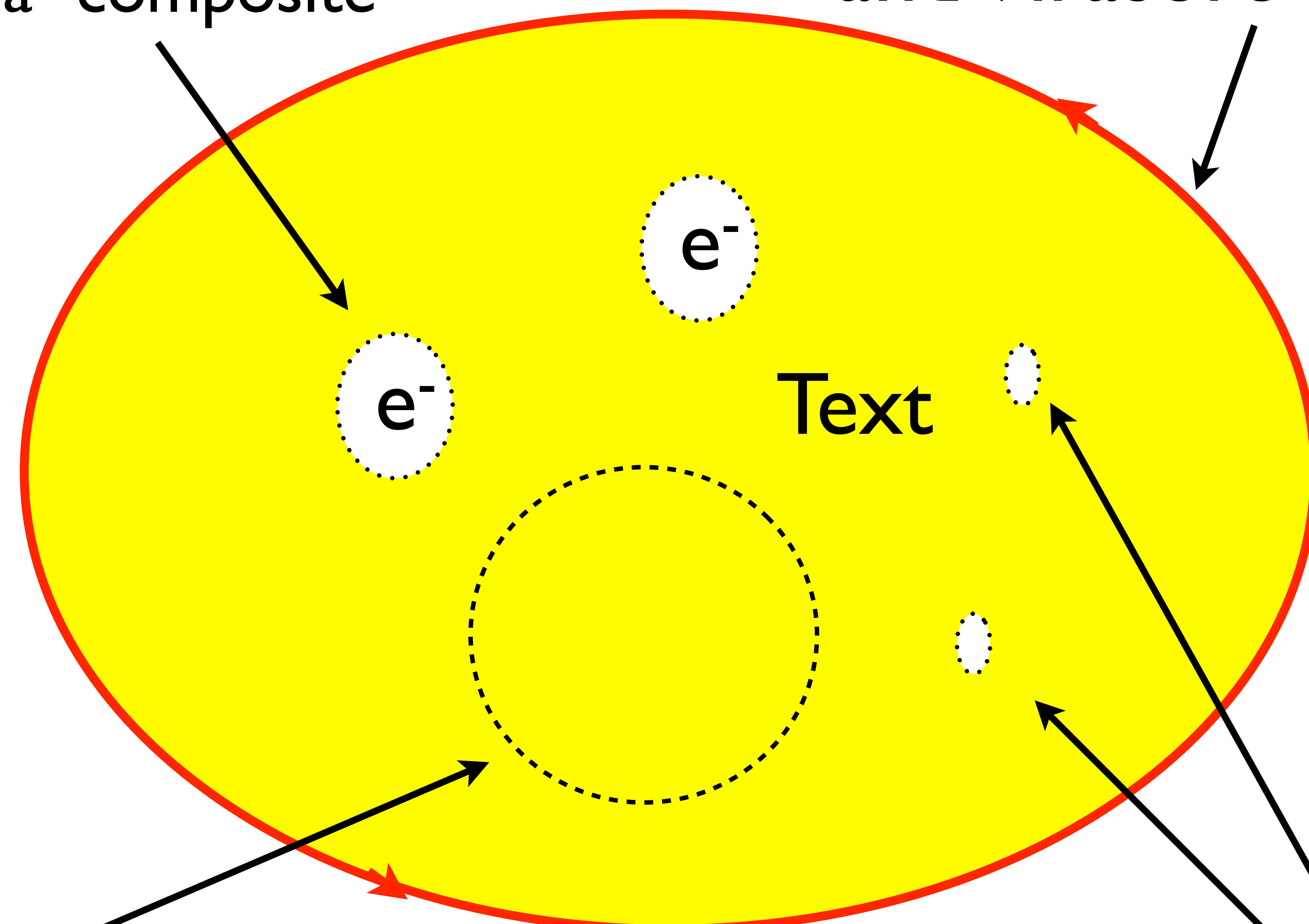
$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j) \qquad [R_i^a, R_i^b] = i\epsilon^{ab}\ell_B^2$$

- Particle-number conservation
- translations $\boldsymbol{R} \to \boldsymbol{R} + \boldsymbol{a}$
- spatial inversion $\boldsymbol{R} \to -\boldsymbol{R}$
- Nothing else! (no rotation or Galilean symmetry)

• Anatomy of Laughlin state

electron with "flux attachment" to form a "composite boson"

Chiral edge mode with chiral anomaly and Virasoro anomaly



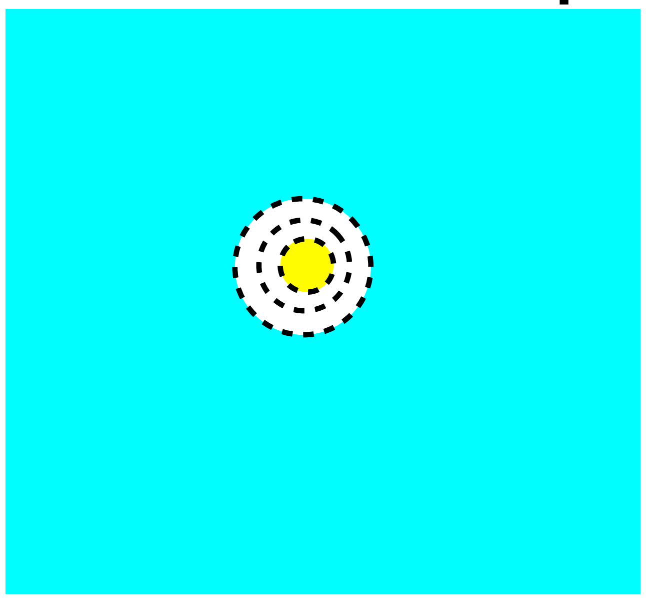
geometric edge dipole moment determined by Hall viscosity

Topological and geometric bulk properties revealed by entanglement spectrum of cut

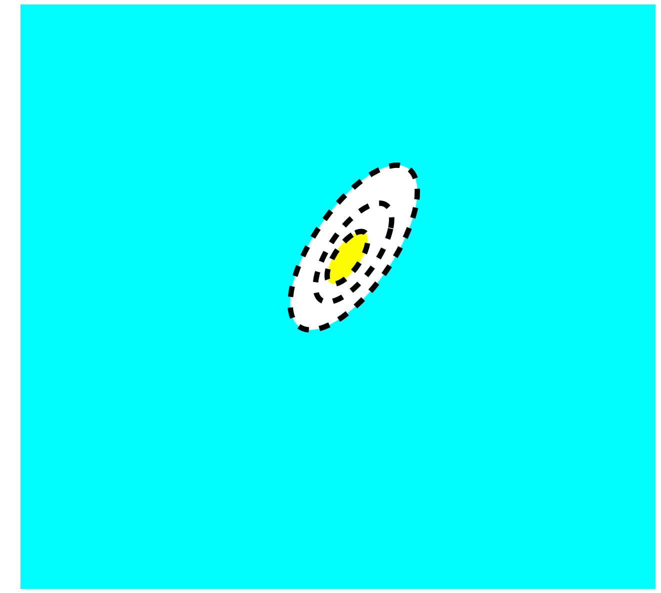
fractionally-charged e/3 quasiholes obeying (Abelian) fractional statistics

- the essential unit of the 1/3 Laughlin state is the electron bound to a correlation hole corresponding to "units of flux", or three of the available single-particle states which are exclusively occupied by the particle to which they are "attached"
- In general, the elementary unit of the FQHE fluid is a "composite boson" of p particles with q "attached flux quanta"
- This is the analog of a unit cell in a solid....

- The Laughlin state is parametrized by a unimodular metric: what is its physical meaning?



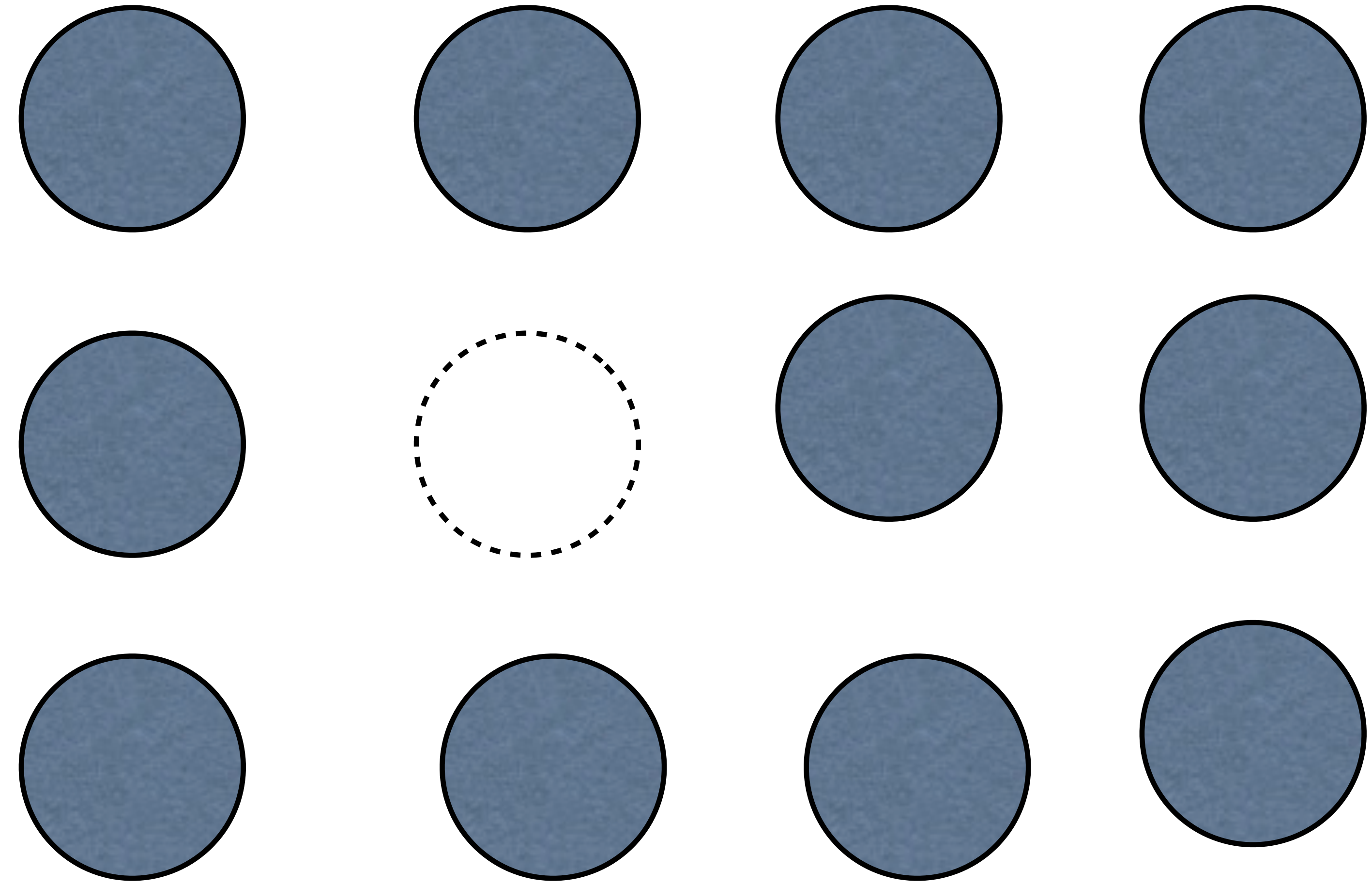
correlation holes in two states with different metrics



- In the $\nu = 1/3$ Laughlin state, each electron sits in a correlation hole with an area containing 3 flux quanta. The metric controls the *shape* of the correlation hole.

- In the $\nu = 1$ filled LL Slater-determinant state, there is <u>no</u> correlation hole (just an exchange hole), and this state does <u>**not**</u> depend on a metric

- quantum solid
- unit cell is correlation hole
- defines geometry



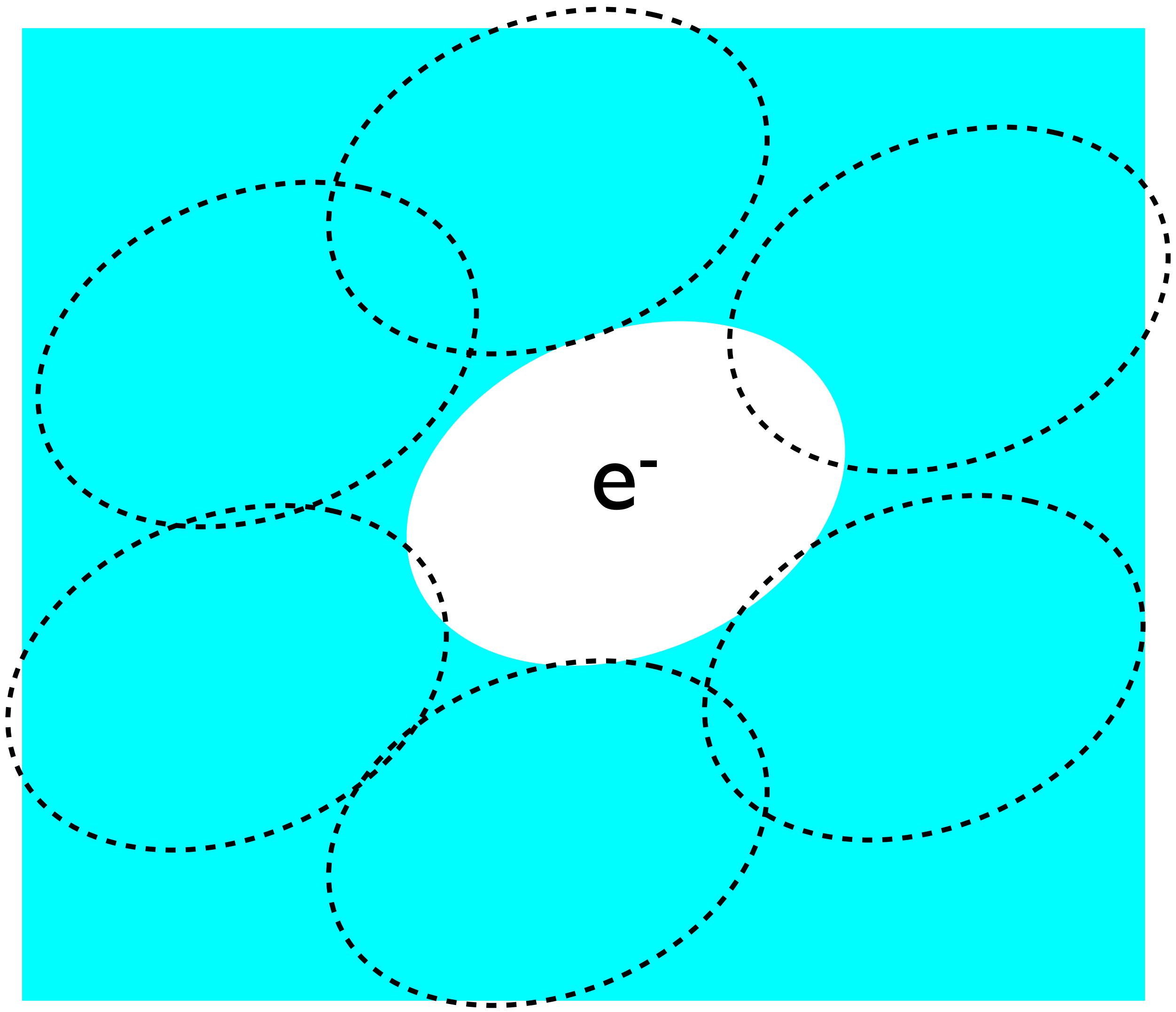
- repulsion of other particles make an attractive

potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

but no broken symmetry

- similar story in FQHE:



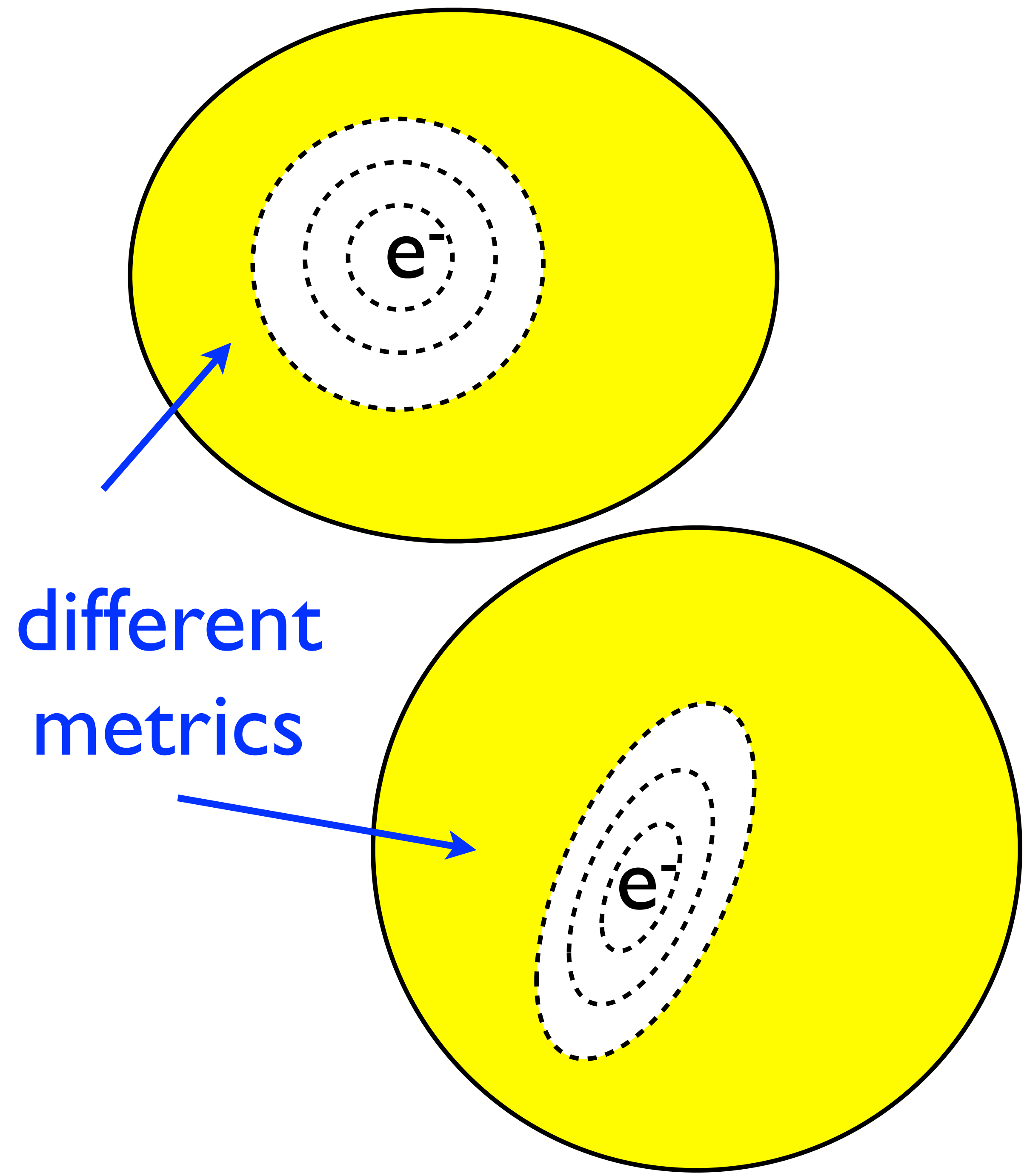
- "flux attachment" creates correlation hole
- defines an emergent geometry
- potential well must be strong enough to bind electron
- continuum model, but similar physics to Hubbard model
- new physics: Hall viscosity, geometry...........

- composite boson: if the central orbital of a basis of eigenstates of L(g) is filled, the next two are empty

$$|\Psi_L^3\rangle = \prod_{i<j} \left(a_i^\dagger - a_j^\dagger\right)^3 |0\rangle$$

$$L(g)|\psi_m\rangle = (m + \tfrac{1}{2})|\psi_m\rangle$$

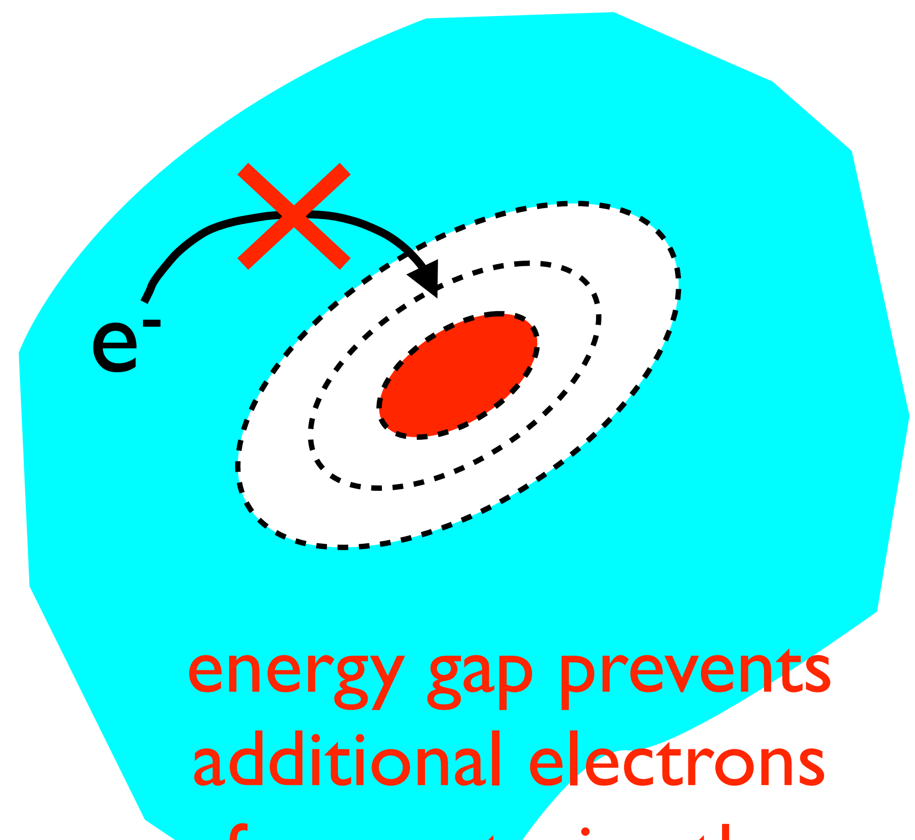
- this correlation hole is equivalent to "attachment of three flux quanta" or vortices that travel with the particle, generating a Berry phase that cancels the Bohm-Aharonov phase and transmutes Fermi to Bose exchange statistics.



- this shape of the corelation hole - and hence its correlation energy - varies with the metric g_{ab}

- Origin of FQHE incompressibility is analogous to origin of **Mott-Hubbard gap** in lattice systems.
- There is an energy gap for putting an **extra particle** in a quantized region that is **already occupied**

- **On the lattice** the "quantized region" is an atomic orbital with a fixed shape
- **In the FQHE** only the **area** of the "quantized region" is fixed. The **shape** must adjust to minimize the correlation energy.

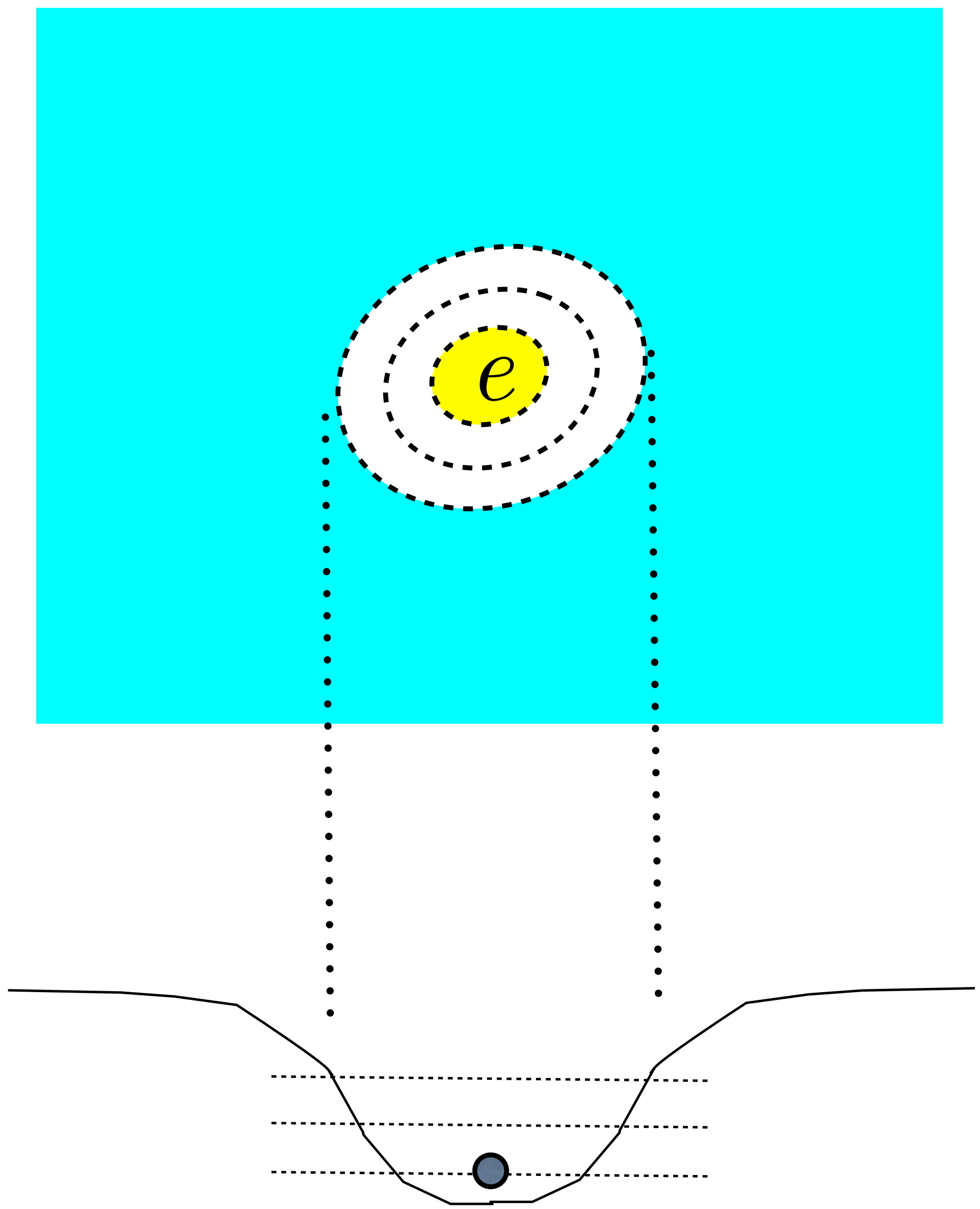


energy gap prevents additional electrons from entering the region covered by the composite boson

$1/3$ Laughlin state

If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center



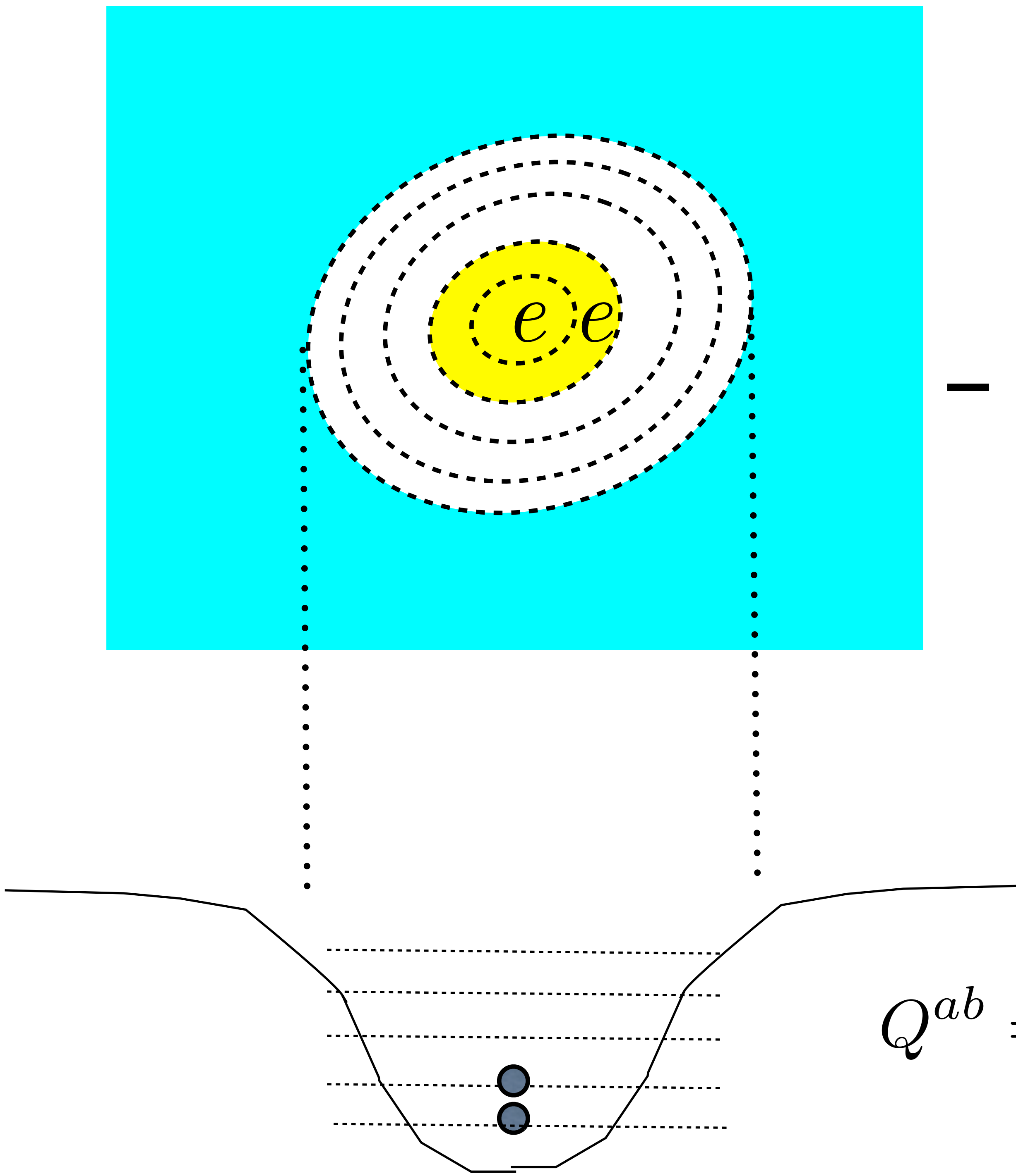
It has a "spin"

	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$		
	1	0	0	.....	$L = \frac{1}{2}$
$-$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.....	$- \; L = \frac{3}{2}$

$$s = -1$$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

$2/5$ state



$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$			
1	1	0	0	0	.....
$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	.....

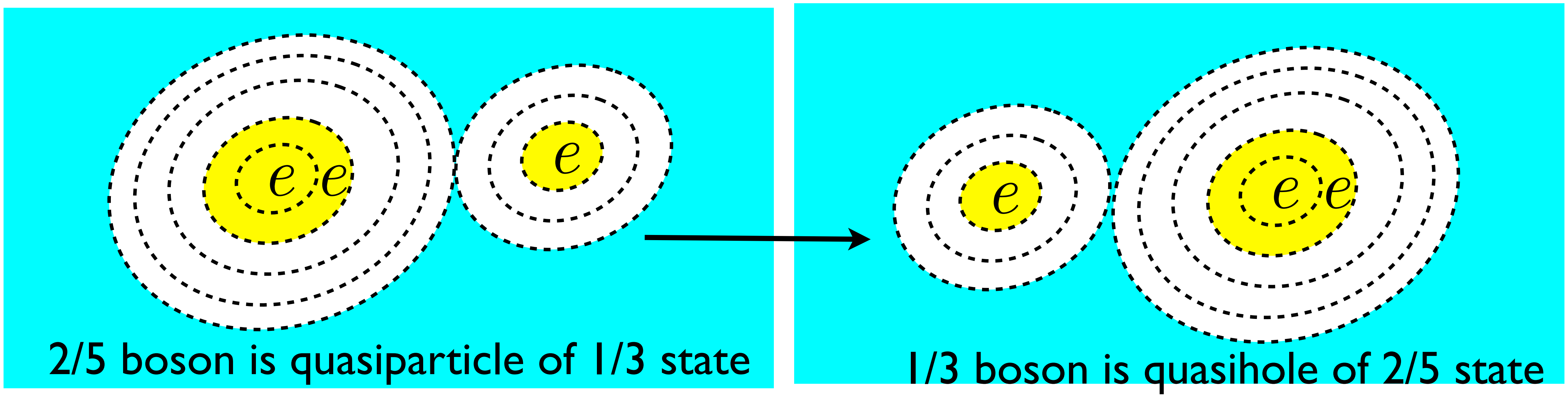
$$L = 2$$
$$- L = 5$$
$$s = -3$$

$$L = \frac{g_{ab}}{2\ell_B^2}\sum_i R_i^a R_i^b$$

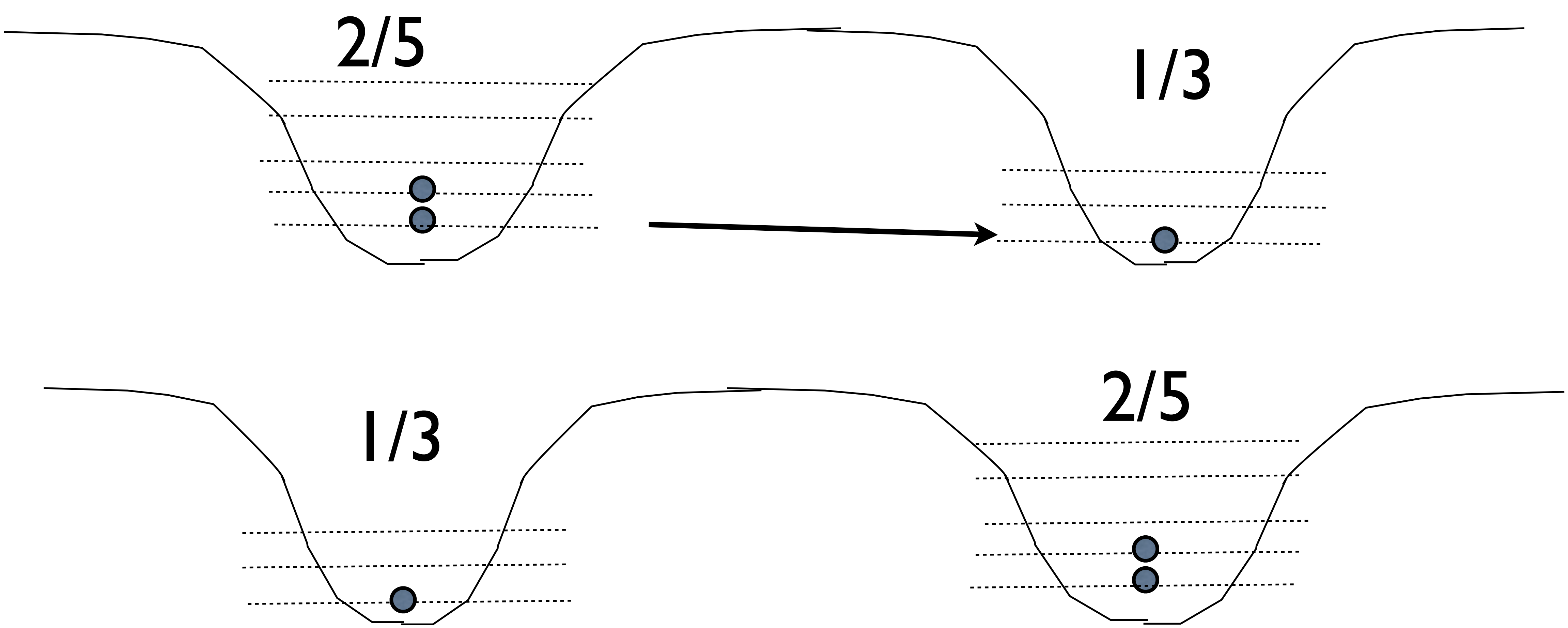
$$Q^{ab} = \int d^2r\, r^a r^b \delta\rho(r) = s\ell_B^2 g^{ab}$$

second moment of neutral
composite boson
charge distribution

hopping of a "composite fermion" (electron + 2 flux quanta)



Jain's "pseudo Landau levels"

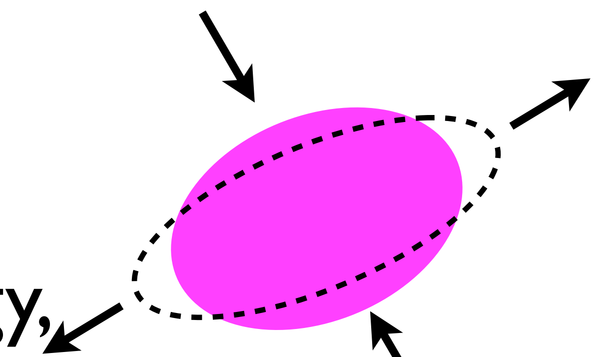


- The composite boson behaves as a neutral particle because the Berry phase (from the disturbance of the the other particles as its “exclusion zone” moves with it) cancels the Bohm-Aharonov phase

- It behaves as a boson provided its statistical spin cancels the particle exchange factor when two composite bosons are exchanged

p particles	$(-1)^{pq} = (-1)^p$	fermions
q orbitals	$(-1)^{pq} = 1$	bosons

- The metric (shape of the composite boson) has a preferred shape that minimizes the correlation energy, but fluctuates around that shape



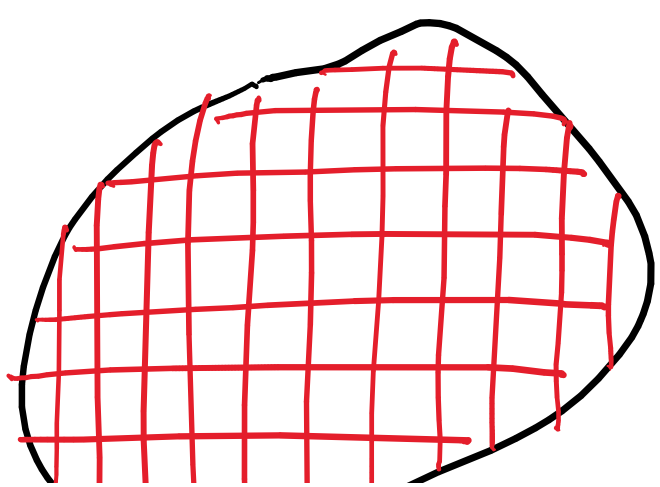
$\delta E \propto (\text{distortion})^2$

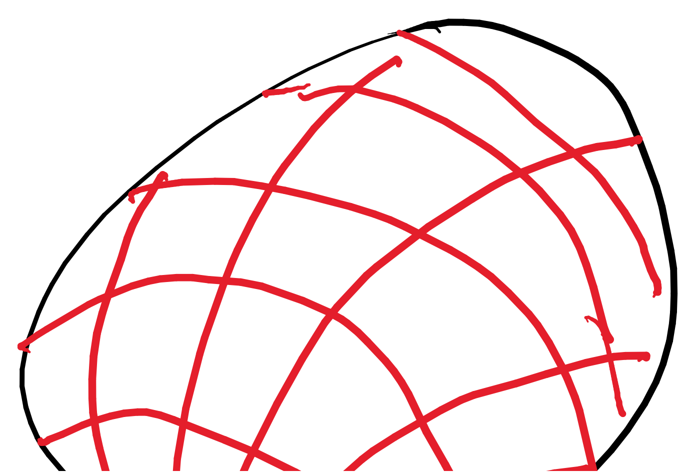
- The zero-point fluctuations of the metric are seen as the O(q^4) behavior of the "guiding-center structure factor" (Girvin et al, (GMP), 1985)

- long-wavelength limit of GMP collective mode is fluctuations of (spatial) metric (analog of "graviton")

FDMH, Phys. Rev. Lett. **107**, 116801 (2011)

- An “intrinsic metric” measures lengths in dimensionless units, like unit cells in a solid
- to describe the “intrinsic metric tensor” we need a coordinate system
- It could be the Euclidean Laboratory frame, but doesn’t have to be!





Laughlin's model wavefunction has provided the inspiration for the modern understanding of the fractional quantum Hall effect

$$\Psi \propto \prod_{i<j}(z_i - z_j)^m \prod_i e^{-\frac{1}{4}z_i^* z_i/\ell_B^2}$$

- It has a striking holomorphic form that is generally attributed to "Lowest Landau Level physics"

- It has a natural interpretation in terms of "flux attachment"

- It involves a "complex structure" $z = x+iy$ that defines a unimodular metric on a Riemann surface

- It has the rotational symmetry of this metric, and has been recognized to be mathematically equivalent to a "conformal block" of a 2D conformal field theory

I will give a somewhat heretical reinterpretation of the Laughlin state

- Despite what Laughlin told us, its holomorphic structure has nothing to do with the electrons being in the "Lowest Landau Level"

- It should not be regarded as a "wavefunction", but as a Heisenberg state of guiding centers, which obey a "quantum geometry"

- It was proposed as a "trial wavefunction" with no apparent variational parameter: it does in fact have such a parameter: its metric.

- Perhaps one of the most surprising (and very fruitful) aspects of the Laughlin state is its connection to conformal field theory.
- Its "conformal block" property was noticed as an empirical observation, but has never really been explained.
- Incompressible (bulk) FQHE states are **essentially unlike** gapless cft's (the conformal group here is the "(2+0)d" conformal orthogonal group, not the "(1+1)d" Lorentz variant)

- The conformal orthogonal group CO(2) is a profound local extension of the global SO(2) rotation group (that can be regarded as "the rotation group on steroids" !)

- non-generic "Toy models" with CFT properties are particularly simple to treat, because the CFT makes their generic topological properties easy to expose, but the topological properties <u>do not require</u> conformal invariance

- I will argue that SO(2) rotational invariance is a "toy model" feature that should not be part of a fundamental theory of the FQHE, just as the SO(3) and Galilean invariance of the free electron gas should not be part of the theory of metals.

- The "standard model" for the QHE is usually taken to be the Galileian-invariant Newtonian-dynamics model $p_a = -i\hbar\frac{\partial}{\partial x^a} - eA_a(\boldsymbol{x})$

$$H = \sum_i \frac{1}{2m}\delta^{ab} p_{ia}p_{ib} + \sum_{i<j} \frac{e^2}{4\pi\epsilon_0\epsilon}\frac{1}{d(\boldsymbol{x}_i, \boldsymbol{x}_j)}$$

$$d(\boldsymbol{x}_1, \boldsymbol{x}_2)^2 = \delta_{ab}(x_1^a - x_2^a)(x_1^b - x_2^b)$$

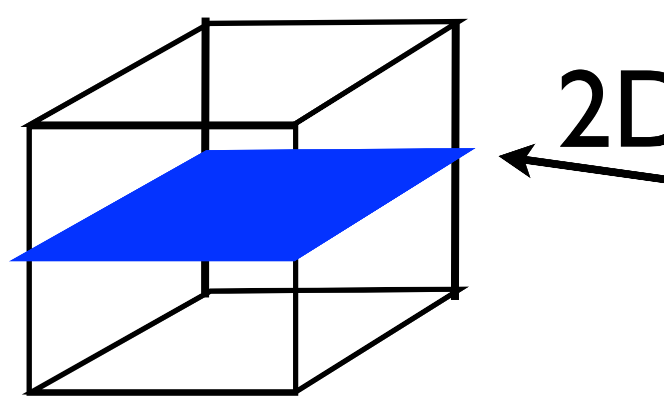
Euclidean metric of 2D plane

(derived from the spatial metric of an inertial frame in which the plane on which the electrons move non-relativistically is embedded)

Cartesian coordinates $\boldsymbol{x} = x^a \boldsymbol{e}_a \quad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$

- However, the continuous translational symmetry "plane" on which the electrons move is an emergent symmetry of a low-density of electrons moving on a crystal lattice plane, and generically does NOT have the rotational invariance of Newtonian dynamics

- The only generic point symmetry of a crystal plane is 2D inversion (180° rotation in plane)



2D plane of epitaxial quantum well embedded in 3D crystal

- The effective continuum Hamiltonian is

$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j)$$

- The model has 2D inversion symmetry if

$$\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$$

- The only role played by the Euclidean metric of the inertial background frame is the non-relativistic criterion

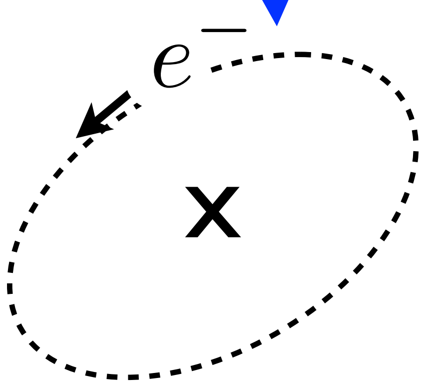
$$\delta_{ab} v^a v^b \ll c^2 \qquad\qquad v^a(\boldsymbol{p}) = \frac{\partial \varepsilon}{\partial p_a}$$

Generic model with translation and inversion symmetry only, no rotational symmetry

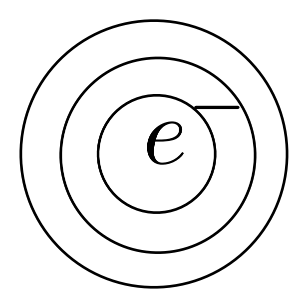
$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j) \qquad \varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$$

affected by elastic degrees of freedom

- two distinct unrelated sources of geometry



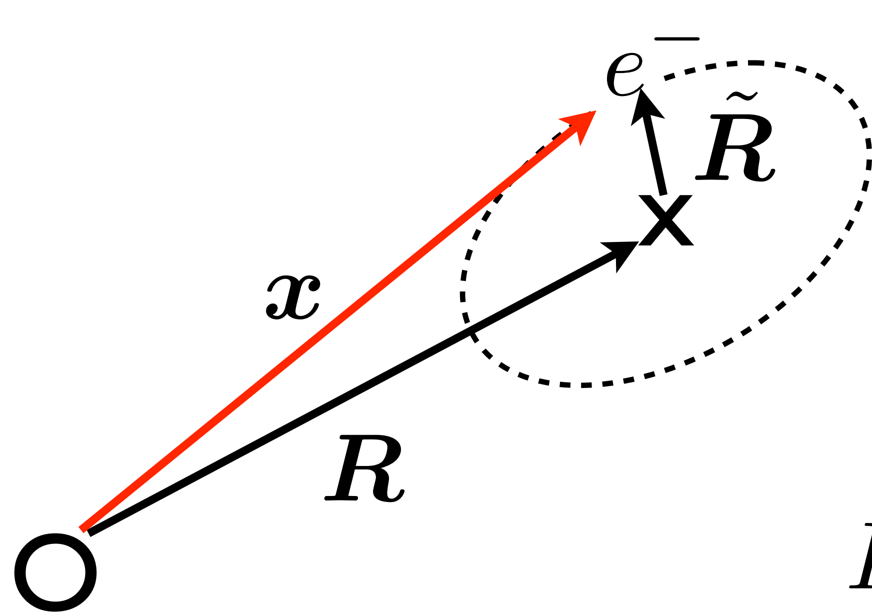
shape of Landau orbit around guiding center



equipotentials around point charge (from 3D dielectric tensor)

- The "holomorphic lowest Landau level wavefunction" is a property of a SO(2) rotationally-invariant system:

$$\boldsymbol{x} = \boldsymbol{R} + \tilde{\boldsymbol{R}}$$



$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$
$$[\tilde{R}^a, \tilde{R}^b] = i\ell_B^2 \epsilon^{ab}$$
$$[R^a, \tilde{R}^b] = 0$$

angular momentum

$$L = \frac{\hbar}{2\ell_B^2}\delta_{ab}(R^a R^b - \tilde{R}^a \tilde{R}^b)$$
$$= \tfrac{1}{2}\hbar\left(a^\dagger a - b^\dagger b\right)$$

guiding center   Landau level

<u>Two</u> sets of ladder operators:

- Now write the Laughlin state as a Heisenberg state, not a Schrödinger wavefunction:

$$|\Psi_L\rangle \propto \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0\rangle \quad a_i|0\rangle = 0 \qquad a^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$b_i|0\rangle = 0$ lowest Landau level condition

In the Heisenberg form, we see that the LLL condition is quite incidental to the Laughlin state, which involves guiding-center correlations

- The fundamental form of the Laughlin state does not reference the details of the Landau level in any way:

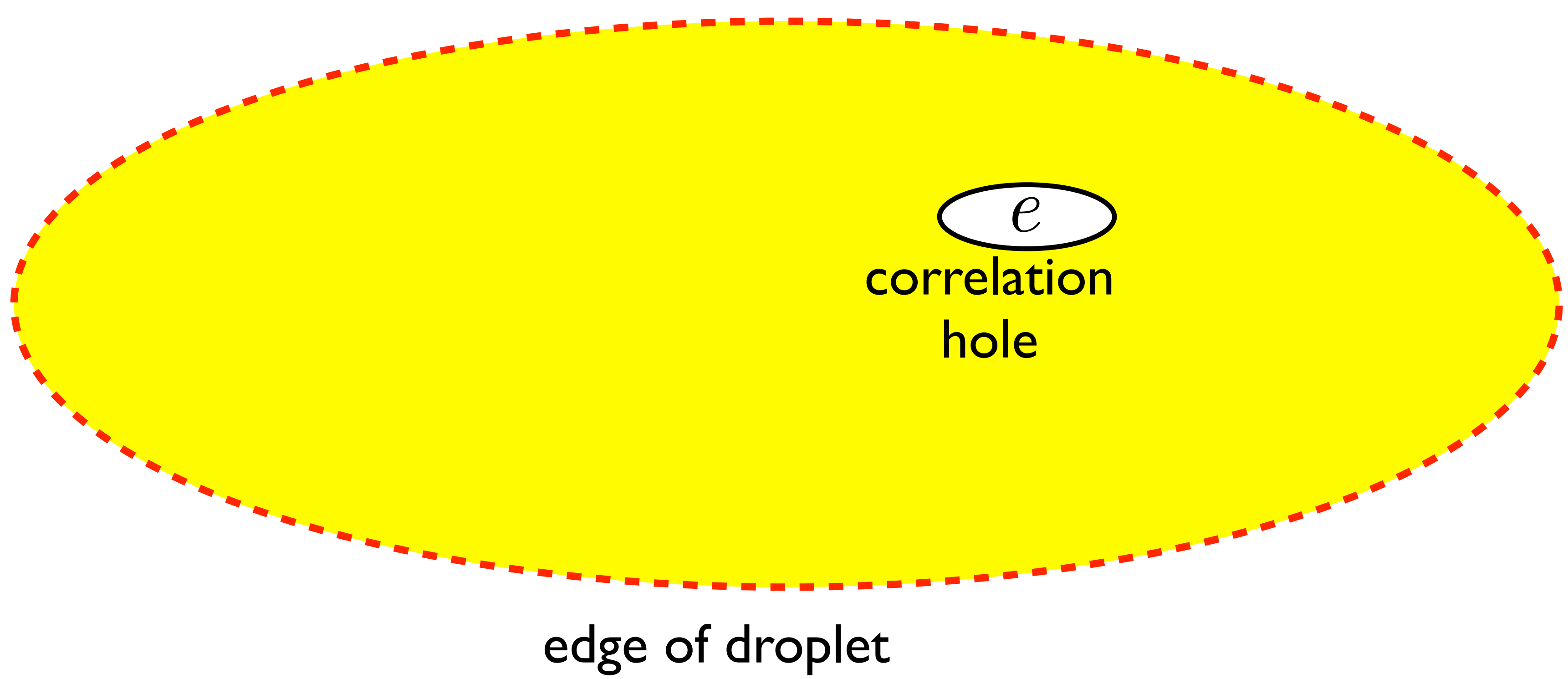
$$|\Psi_L(\tilde{g})\rangle \propto \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0\rangle \quad a_i|0\rangle = 0 \qquad a^\dagger = \frac{\omega_a R^a}{\sqrt{2}\ell_B}$$

$$\omega_a^*\omega_b = \tfrac{1}{2}\left(\tilde{g}_{ab} + i\epsilon_{ab}\right) \qquad \det\tilde{g} = 1$$

a unimodular Euclidean-signature metric that parameterizes the Laughlin state

- The historical identification of this metric with the Euclidean metric is unnecessary unless there is SO(2) symmetry.

- The original form of the Laughlin state is a finite-size droplet of N particles on the infinite plane.
- Somewhat confusingly, in this droplet state the metric parameter fixes both the shape of the droplet state **and** the shape of the correlation hole around each particle formed by “flux attachment”:



- to remove the edge, compactify on the torus with N_Φ flux quanta:

- An unnormalized holomorphic single-particle state has the form

$$|\psi\rangle = \prod_{i=1}^{N_\Phi} \sigma(a^\dagger - w_i)|0\rangle, \qquad \sum_{i=1}^{N_\Phi} w_i = 0$$

Weierstrass sigma function

$$\sigma(z) = z \prod_{L\neq 0} (1 - \tfrac{z}{L}) \exp(\tfrac{z}{L} + \tfrac{1}{2}(\tfrac{z}{L})^2)$$

Filled Landau level $N = N_\Phi$

$$|\Psi_{\text{filledLL}}\rangle = \sigma(\textstyle\sum_i a_i^\dagger) \displaystyle\prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)|0\rangle$$

independent of choice of metric, after normalization

- Laughlin state on torus $(\nu = 1/m, \quad m > 1)$

$$|\Psi_L^m(\tilde{g})\rangle \propto \left(\prod_{j=1}^{m} \sigma(\sum_i a_i^\dagger - w_j)\right) \prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)^m |0\rangle$$

Topological degeneracy parametrized by w_j $\quad \sum_{j=1}^{m} w_j = 0$

- Unlike the filled LL state, the Laughlin state does depend on the metric, which characterizes the shape of the correlation hole (flux attachment).

- The Laughlin state is indeed a variational trial state, we must choose its metric to minimize the correlation energy

$$H = \frac{1}{N_\Phi}\sum_{\boldsymbol{q}}\left(\frac{\tilde{V}(\boldsymbol{q})|f_n(\boldsymbol{q})|^2}{2\pi\ell_B^2}\right)\sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)}$$

Fourier transform of interaction

Landau-level form-factor

reciprocal vector compatible with pbc

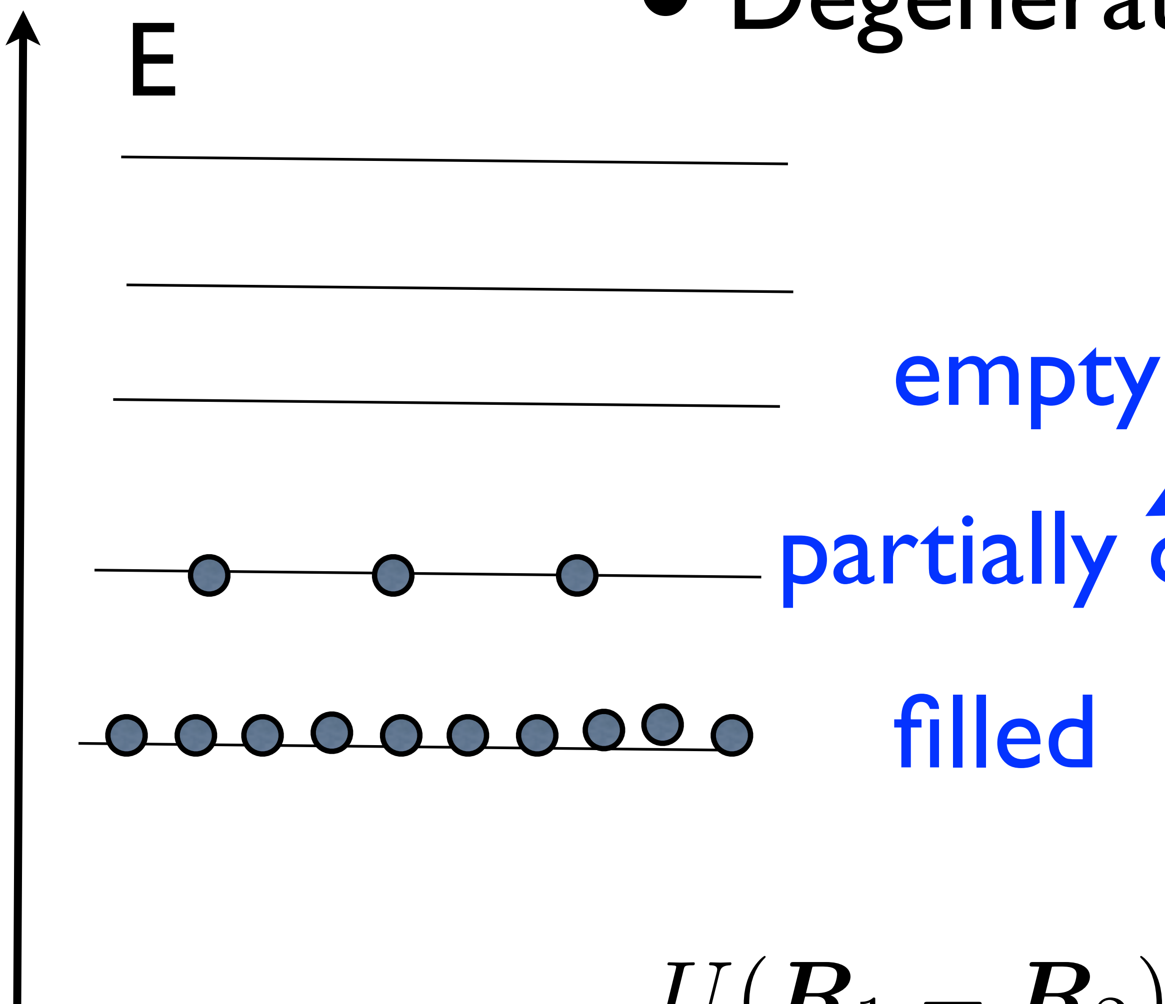
- Note that the residual two-body interaction between guiding centers always has 2D inversion symmetry.

- The Laughlin states are also the exact zero-energy ground states of the metric-dependent "pseudopotential" interaction

$$H(\tilde{g}) = \frac{1}{N_\Phi} \sum_{\boldsymbol{q}} \left(\sum_{m'<m} V_{m'} L_{m'}(q^2 \ell_B^2) e^{-\frac{1}{2} q^2 \ell_B^2} \right) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)}$$

$$q^2 \equiv \tilde{g}^{ab} q_a q_b$$

- Degenerate (flat) Landau levels

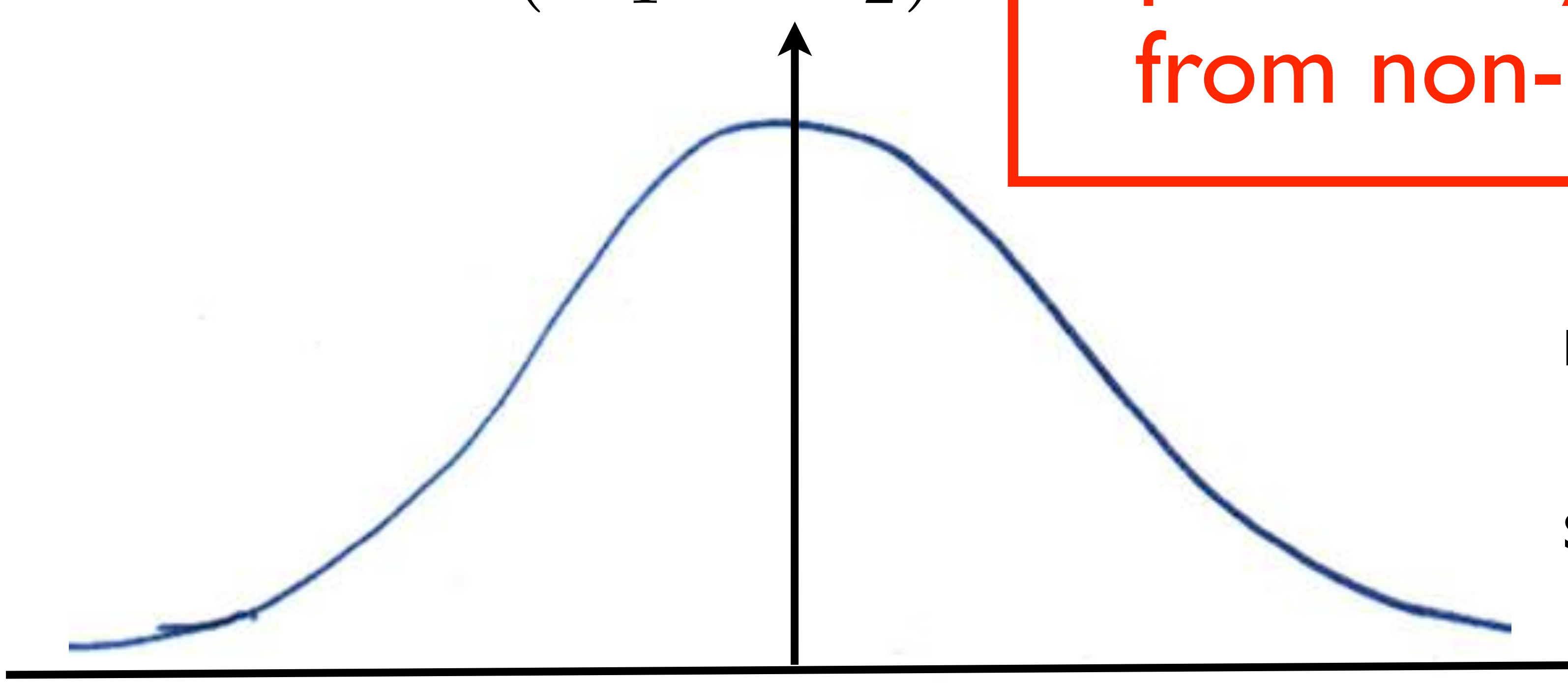


$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$

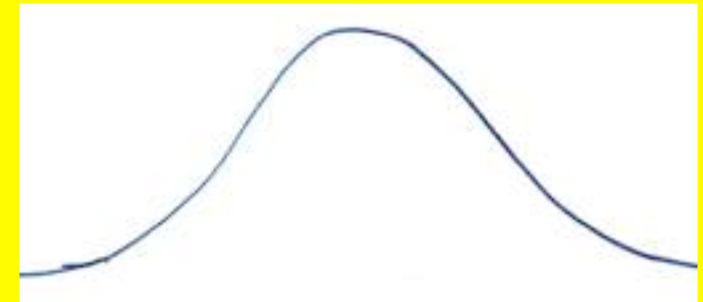
quantum dynamics comes from non-commutativity

$U(\boldsymbol{R}_1 - \boldsymbol{R}_2)$



effective Coulomb repulsion is analytic at origin because of smoothing by Landau-orbit form factor

This is the **entire** problem: nothing other than this matters!



$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$

like phase-space, has Heisenberg uncertainty principle

- H has translation and inversion symmetry

$$[(R_1^x + R_2^x), (R_1^y - R_2^y)] = 0$$

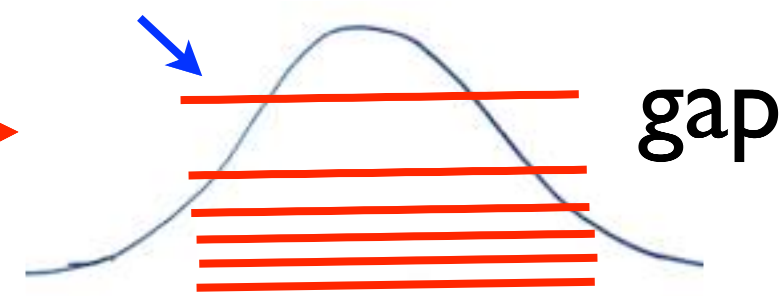
$$[H, \textstyle\sum_i \boldsymbol{R}_i] = 0$$

- generator of translations and electric dipole moment!

$$[(R_1^x - R_2^x), (R_1^y - R_2^y)] = -2i\ell_B^2$$

- relative coordinate of a pair of particles behaves like a single particle

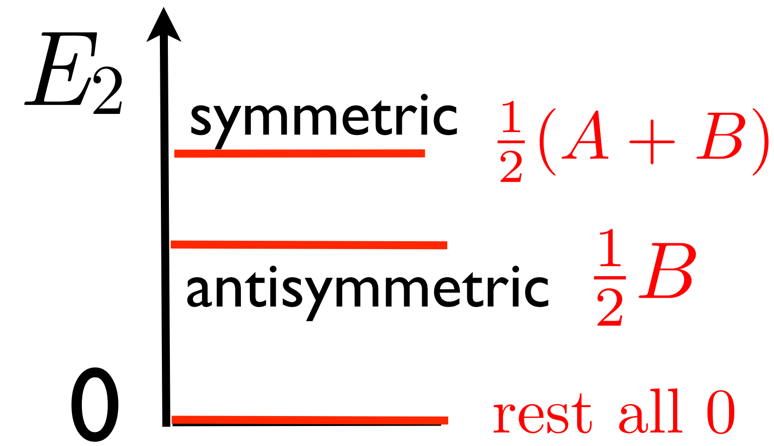
want to avoid this state



two-particle energy levels

- Solvable model! ("short-range pseudopotential")

$$U(r_{12}) = \left(A + B\left(\frac{(r_{12})^2}{\ell_B^2}\right)\right) e^{-\frac{(r_{12})^2}{2\ell_B^2}}$$



- Laughlin state

$$|\Psi_L^m\rangle = \prod_{i<j}\left(a_i^\dagger - a_j^\dagger\right)^m |0\rangle$$

$$a_i|0\rangle = 0 \qquad a_i^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$$E_L = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

maximum density null state

- m=2: (bosons): all pairs avoid the symmetric state E_2 = ½(A+B)
- m=3: (fermions): all pairs avoid the antisymmetric state E_2 = ½B

- Furthermore, the local electric charge density of the fluid with $\nu = p/q$ is determined by a combination of the magnetic flux density and the Gaussian curvature of the metric

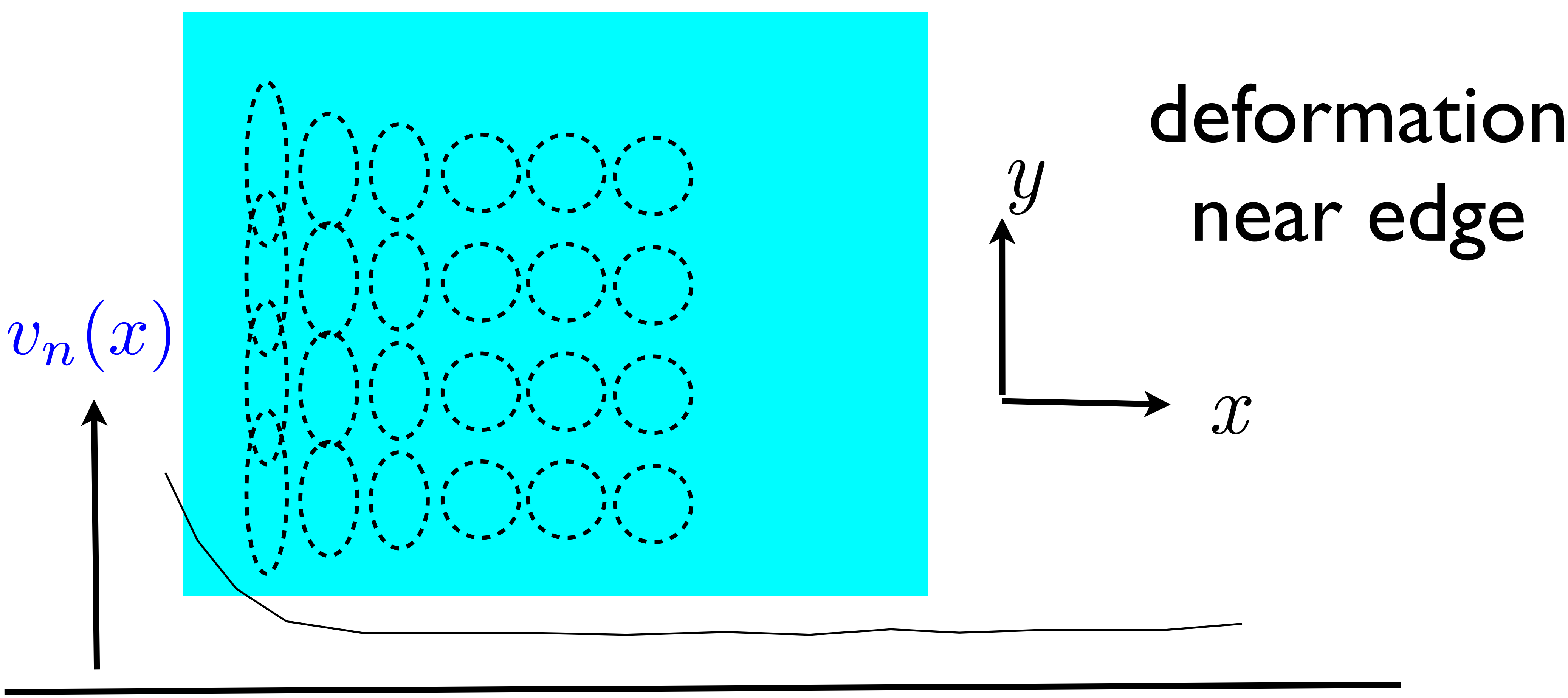
$$J_e^0(\boldsymbol{x}) = \frac{e}{2\pi q}\left(\frac{peB}{\hbar} - sK_g(\boldsymbol{x})\right)$$

Topologically quantized "guiding center spin"

Gaussian curvature of the metric

- In fact, it is locally determined, if there is an inhomogeneous slowly-varying substrate potential

$$H = \sum_i v_n(\boldsymbol{R}_i) + \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$



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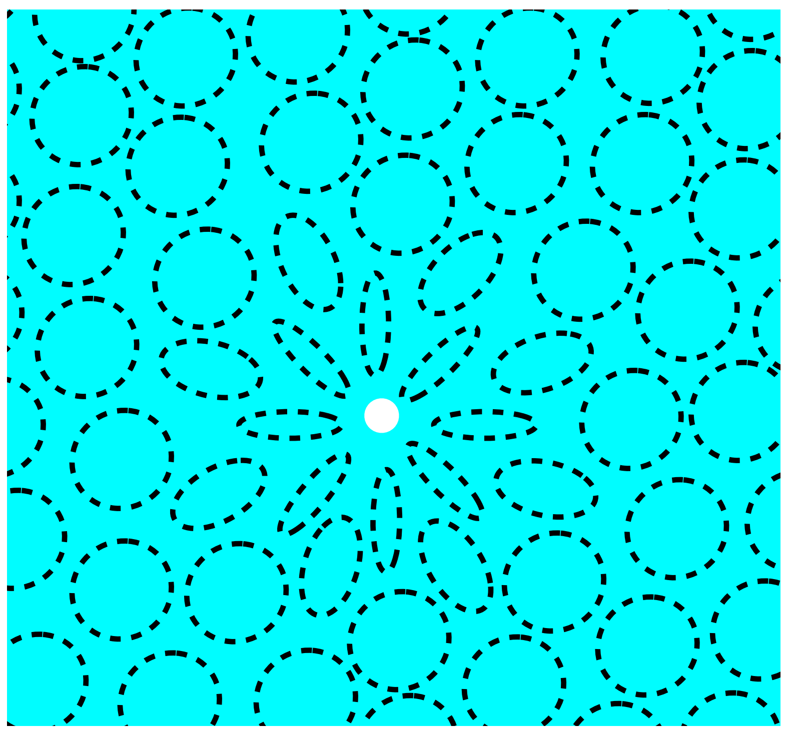

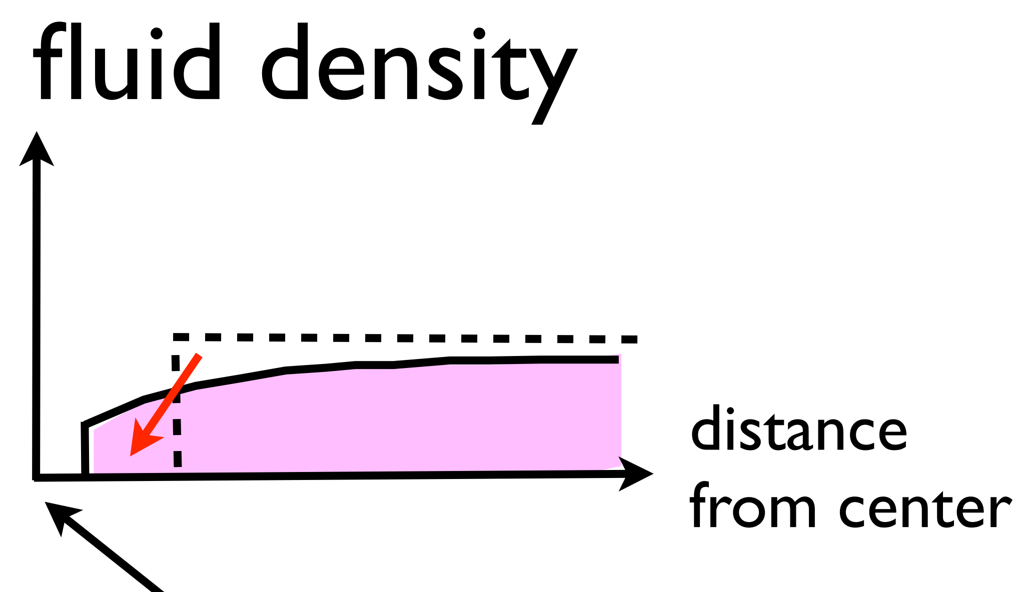

in an effective theory,
core of quasihole may collapse
into a cone singularity of the metric.

One final result

- In the "trivial" non-topologically-ordered integer QHE (due to the Pauli principle)

$$\tilde{c} = \nu = \text{Chern number}$$

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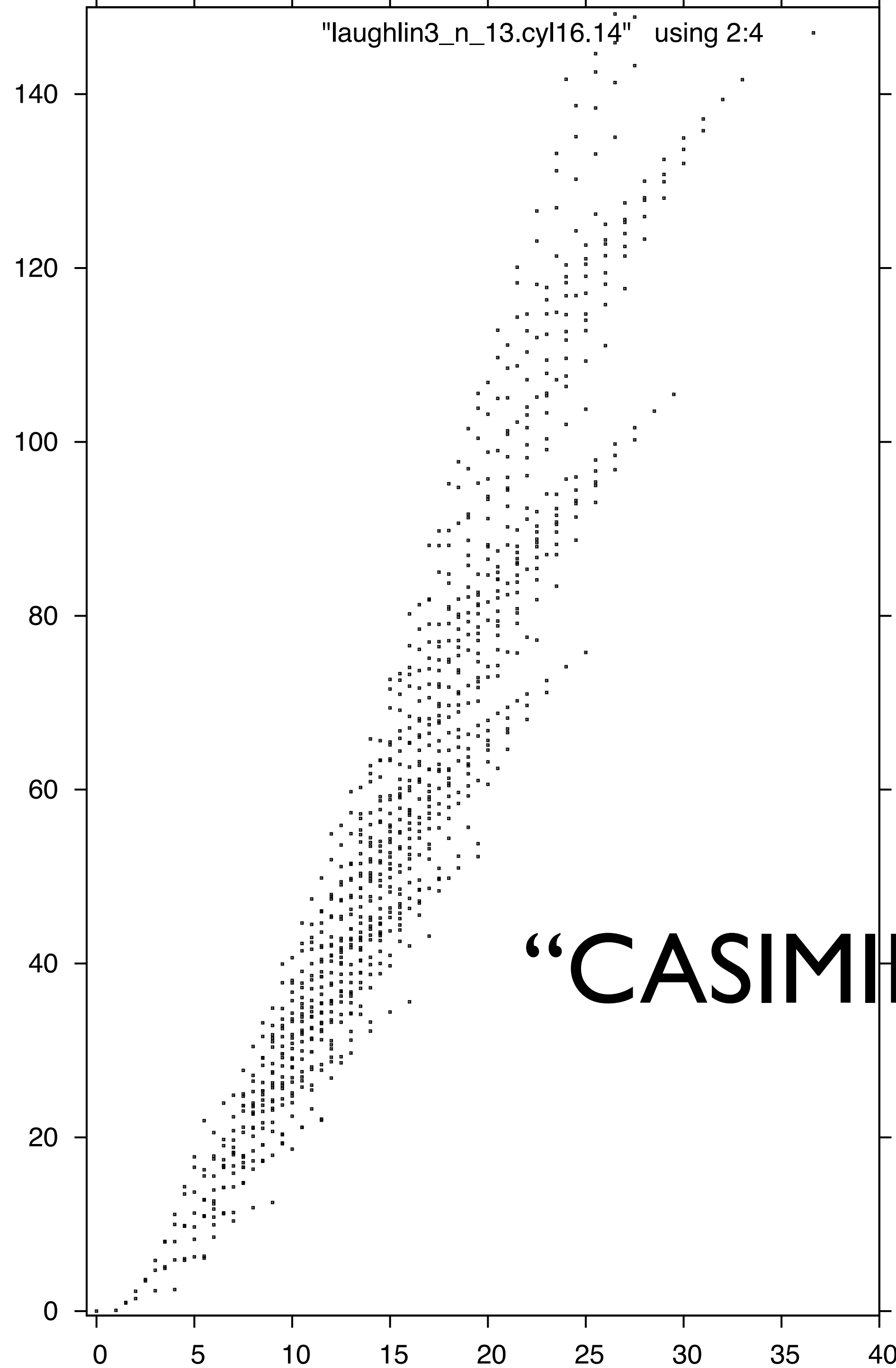

ξ

m

ORBITAL CUT

$$\frac{P_a L^a}{2\pi} = \frac{\sum_\alpha m_\alpha e^{-\xi_\alpha}}{\sum_\alpha e^{-\xi_\alpha}} = \eta_H^{cd}\epsilon_{ac}\epsilon_{bd}\frac{L^a L^b}{2\pi\ell_B^2}$$

$$+\tfrac{1}{24}(\tilde{c}-\nu) - h$$

signed conformal anomaly (chiral stress-energy anomaly)

chiral anomaly

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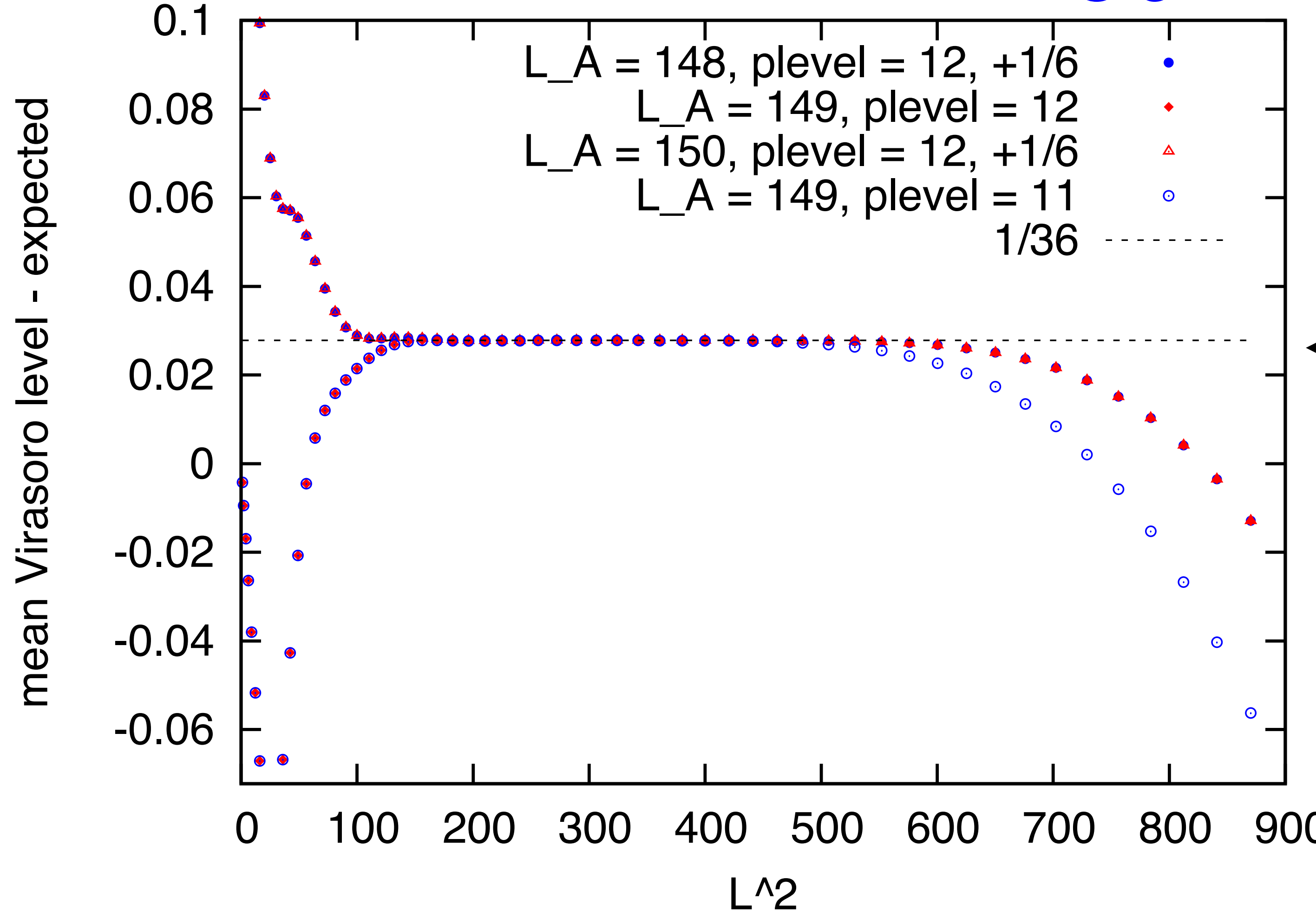
Yeje Park, Z Papic, N Regnault

$$\frac{1}{24}\left(\tilde{c}-\nu\right)=\frac{1}{24}\left(1-\tfrac{1}{3}\right)=\frac{1}{36}$$



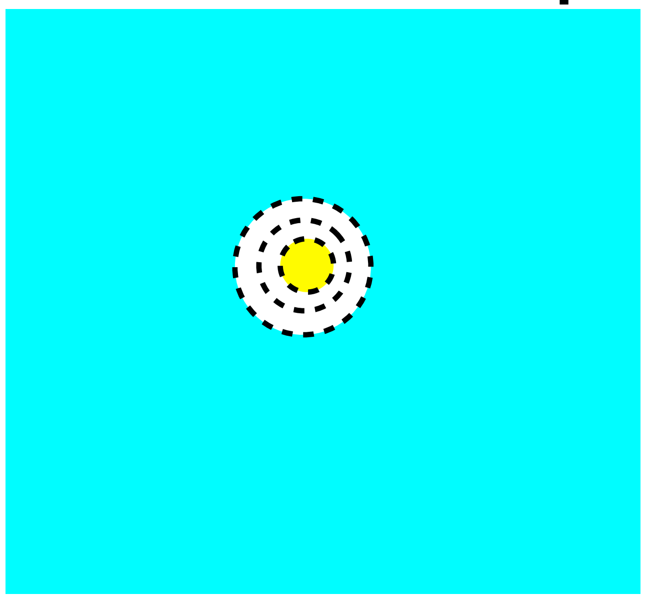
$\leftarrow \frac{1}{36}$

Matrix-product state calculation on cylinder with circumference L
("plevel" is Virasoro level at which the auxialliary space is truncated)

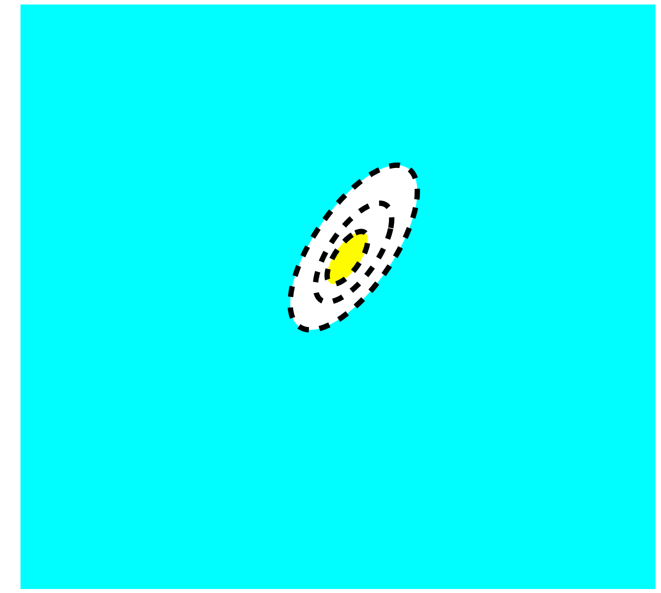
- In a 2D Landau level, we apparently start from a Schrödinger picture, but end with a "quantum geometry" which is no longer correctly described by Schrödinger wavefunctions in real space because of "quantum fuzziness" (non-locality)
- It remains correctly described by the Heisenberg formalism in Hilbert space.

- the essential unit of the 1/3 Laughlin state is the electron bound to a correlation hole corresponding to "units of flux", or three of the available single-particle states which are exclusively occupied by the particle to which they are "attached"
- In general, the elementary unit of the FQHE fluid is a "composite boson" of p particles with q "attached flux quanta"
- This is the analog of a unit cell in a solid....

- The Laughlin state is parametrized by a unimodular metric: what is its physical meaning?



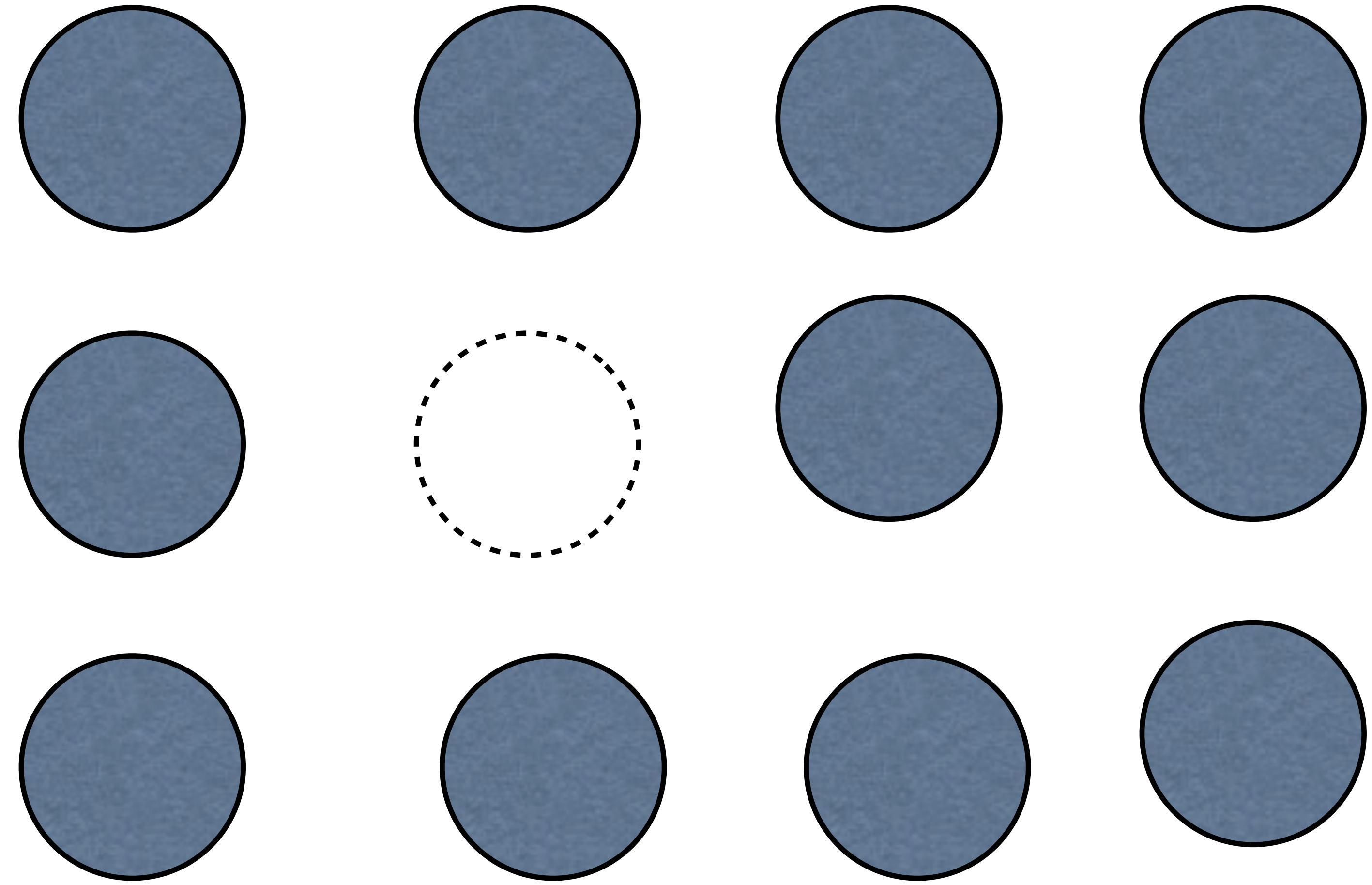
correlation holes in two states with different metrics



- In the $\nu = 1/3$ Laughlin state, each electron sits in a correlation hole with an area containing 3 flux quanta. The metric controls the *shape* of the correlation hole.

- In the $\nu = 1$ filled LL Slater-determinant state, there is <u>no</u> correlation hole (just an exchange hole), and this state does <u>**not**</u> depend on a metric

- quantum solid
- unit cell is correlation hole
- defines geometry



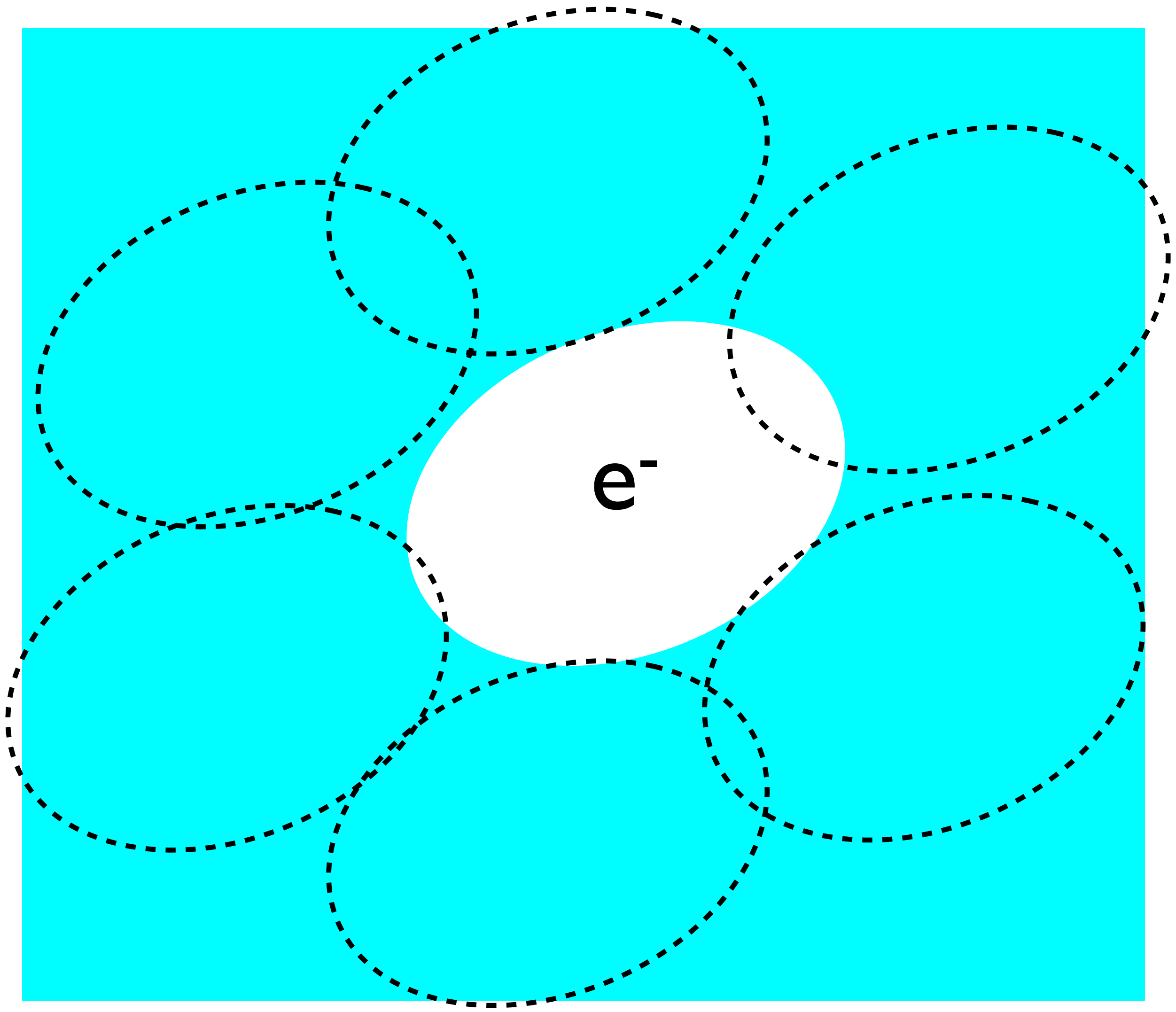
- repulsion of other particles make an attractive

potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

but no broken symmetry

- similar story in FQHE:



- "flux attachment" creates correlation hole
- defines an emergent geometry
- potential well must be strong enough to bind electron

- continuum model, but similar physics to Hubbard model

- new physics: Hall viscosity, geometry...........

- The composite boson fluid covers the plane, and provide an intrinsic dimensionless spatial distance measure on the plane, analogous to measuring distances in lattice units in the solid.
- The effective field theory should only involve a connection compatible with the intrinsic spatial metric, not the connection compatible with the Euclidean metric.

- space-time connection compatible with a time-dependent intrinsic spatial metric $g_{ab}(\boldsymbol{x}, t)$

$$\nabla_\mu f_a = \partial_\mu f_a - \Gamma^b_{\mu a} f_b$$
$$\Gamma^a_{\mu b} = \tfrac{1}{2} g^{ac} \left(\partial_\mu g_{bc} + \delta^d_\mu \left(\partial_b g_{cd} - \partial_c g_{bd}\right)\right)$$

- unusual feature, connection 1-form carries only spatial indices $\Gamma^a{}_b = \Gamma^a_{\mu b} dx^\mu$

- Geometric Chern-Simons 3-form is analog of gravitational CS form, but trace is over spatial indices

$$\Gamma^a{}_b \wedge d\Gamma^b{}_a + \tfrac{2}{3}\Gamma^a{}_b \wedge \Gamma^b{}_c \wedge \Gamma^c{}_a \qquad = 2\omega \wedge d\omega$$

spin connection

- conserved Gaussian curvature current of intrinsic metric:

$$g_{ab} = \sqrt{g}\,\tilde{g}_{ab}$$

unimodular part

$$\boxed{\begin{aligned} J^{\mu}_{g} &= \tfrac{1}{2}\left(\delta^{\mu}_{a}\partial_t - \delta^{\mu}_{0}\partial_a\right)\left(\partial_b \tilde{g}^{ab} + \tilde{g}^{ab}\partial_b \ln\sqrt{g}\right) \\ &\quad + \tfrac{1}{8}\epsilon^{\mu\nu\lambda}\epsilon_{ac}\tilde{g}_{bd}\left(\partial_\nu \tilde{g}^{ab}\right)\left(\partial_\lambda \tilde{g}^{cd}\right) \qquad \text{(Brioschi formula)} \end{aligned}}$$

$$\partial_\mu J^{\mu}_{g} = 0$$

- <u>any</u> non-singular time-dependent symmetric spatial tensor field can define a conserved Gaussian curvature current

- three dynamical ingredients $g_{ab},\ v^a, P^a$:
  - a "dynamic emergent 2D spatial metric" $g_{ab}(\boldsymbol{x},t)$ with $g \equiv \det\ g$, and Gaussian curvature current $J^\mu_g = \epsilon^{\mu\nu\lambda}\partial_\nu\omega_\lambda(\boldsymbol{x},t)$
  - a flow velocity field $v^a(\boldsymbol{x},t)$
  - an electric polarization field $P^a(\boldsymbol{x},t)$
  - a composite boson current $J^\mu_b = \sqrt{g}(\boldsymbol{x},t)\,(\delta^\mu_0 + v^a(\boldsymbol{x},t)\delta^\mu_a)$

here a is a 2D spatial index, and μ is a (2+1D) space-time index. The fluid motion is non-relativistic relative to the preferred inertial rest frame of the crystal background

- effective bulk action: $\sigma_H = \dfrac{(pe)^2}{2\pi\hbar K}$

$U(1)$ Chern-Simons field

$U(1)$ condensate field

"spin connection" of the metric

$$S = \int d^2x dt\, \mathcal{L}_0 - \mathcal{H}$$

$$\mathcal{L}_0 = \frac{\hbar}{4\pi}\epsilon^{\mu\nu\lambda}\left(K^{-1} b_\mu \partial_\nu b_\lambda + \beta\omega_\mu\partial_\nu\omega_\lambda\right)$$
$$+ J_b^\mu\left(\hbar(\partial_\mu\varphi - b_\mu - S\omega_\mu) + peA_\mu\right)$$

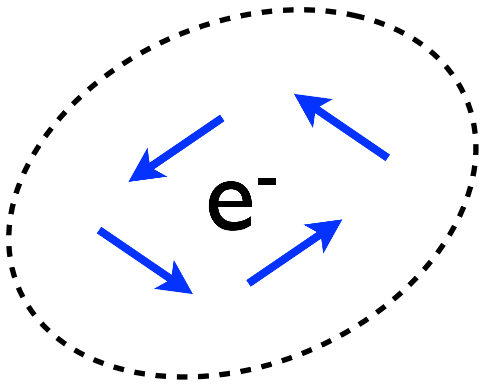
$$\mathcal{H} = \sqrt{g}\left(\varepsilon(\boldsymbol{v}, B) - U(g, B, P) - (E_a + \epsilon_{ab}v^b B)P^a\right)$$

kinetic energy of flow

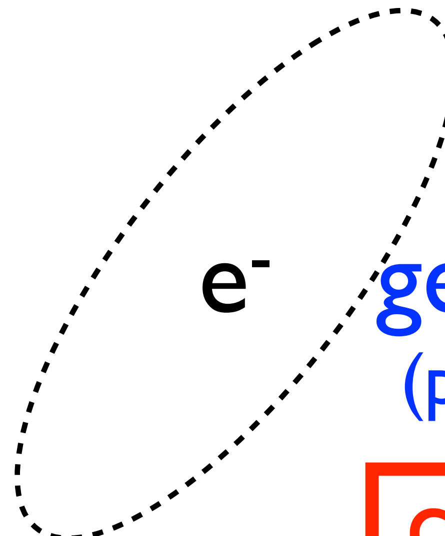
metric-dependent correlation energy

- shape of correlation hole (flux attachment) fluctuates, adapts to environment (electric field gradients)

S



new property:
"spin" couples
to curvature

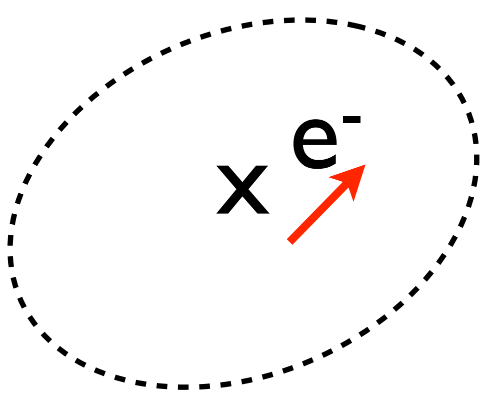


geometric distortion
(preserving inversion symmetry)

$\boxed{\text{shape=metric}}$

$\boxed{\text{creates "curvature" of metric}}$

- polarizable, B x electric dipole = momentum, origin of "inertial mass"

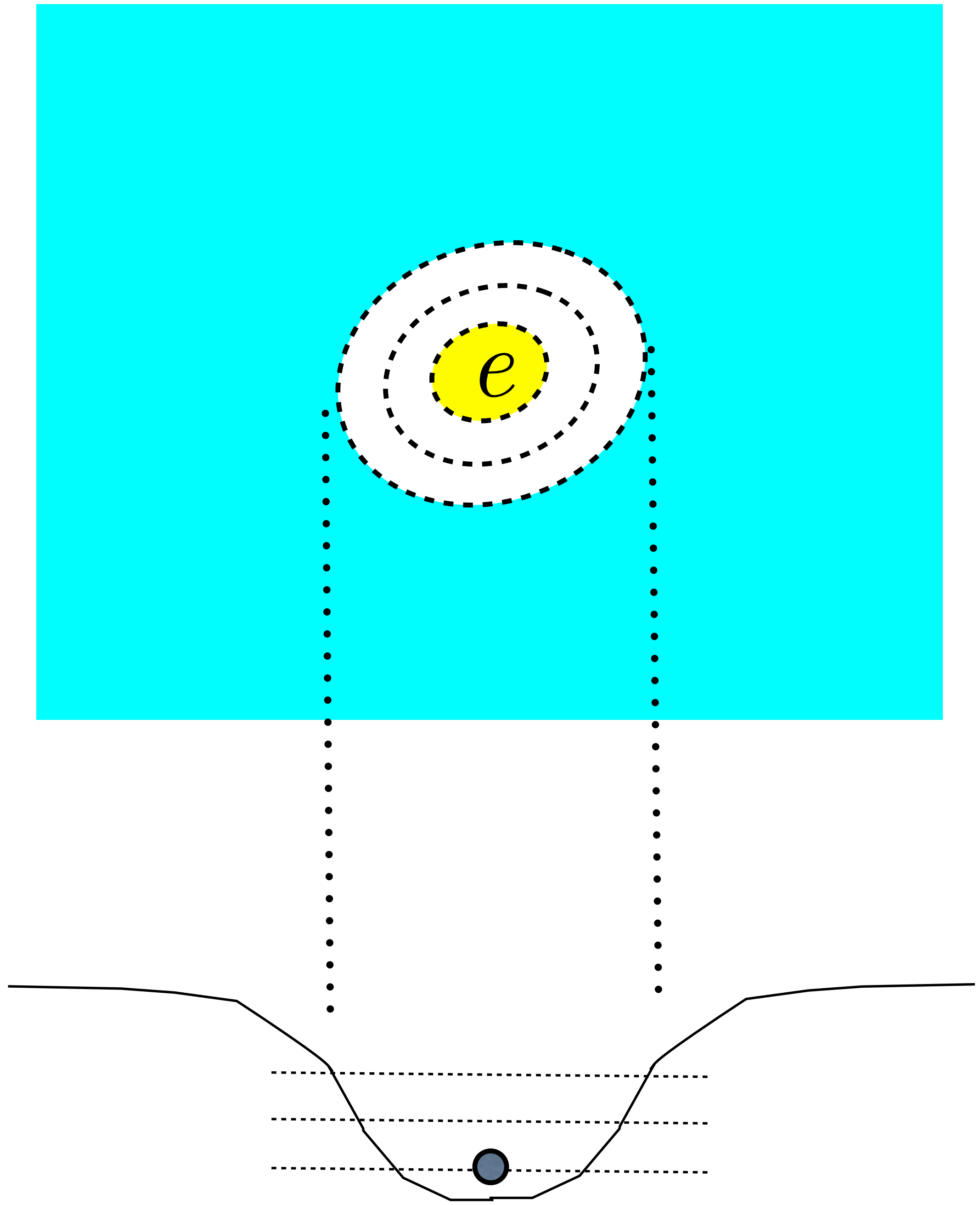


electric polarizability

$1/3$ Laughlin state

If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center



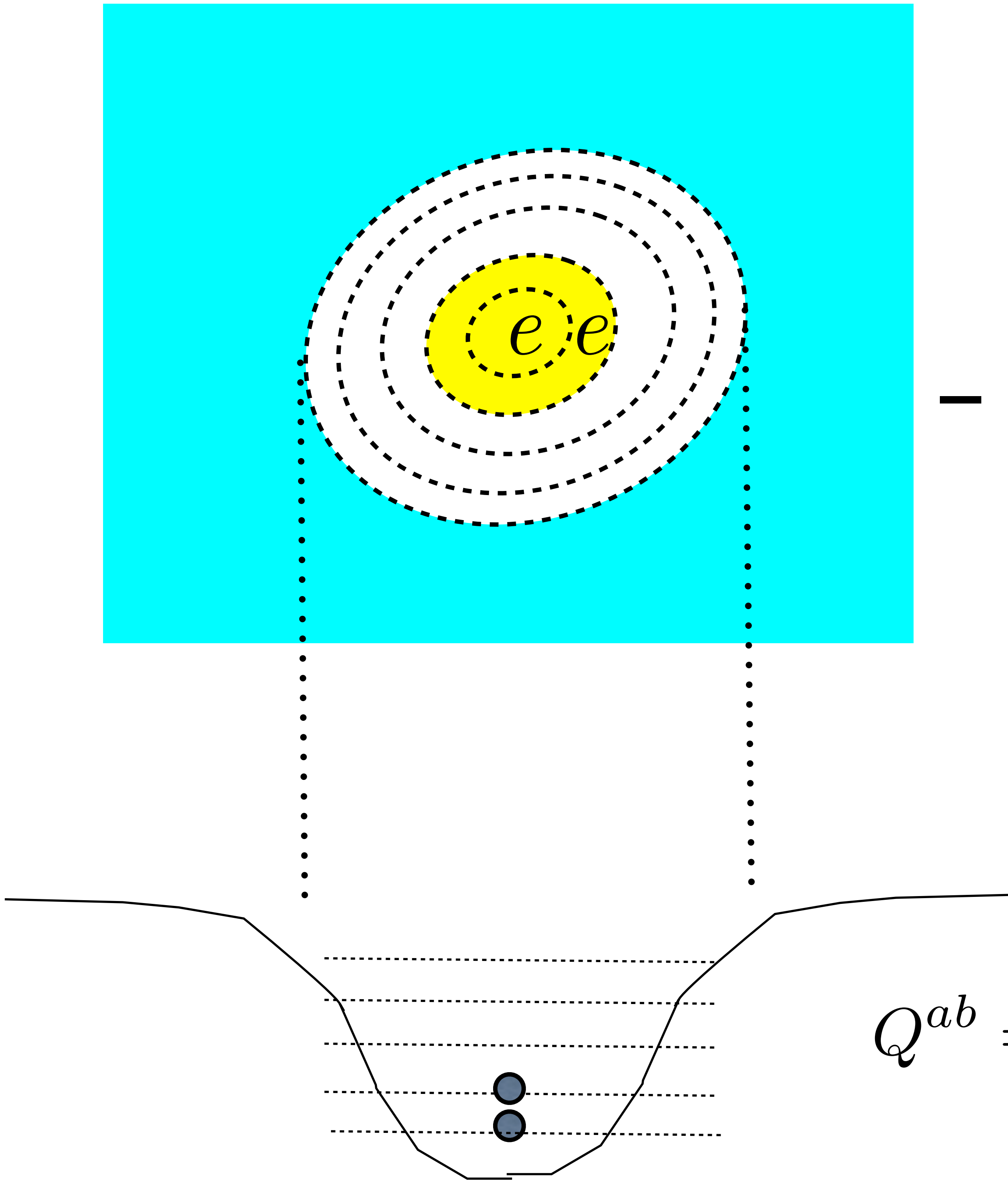
It has a "spin"

$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$		
1	0	0	.....	$L = \frac{1}{2}$
$-$ $\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	.....	$-\ L = \frac{3}{2}$

$$s = -1$$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

$2/5$ state



$$\begin{array}{ccccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & & \\ \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} \end{array} \cdots\cdots \qquad L = 2$$

$$- \begin{array}{ccccc} \boxed{\frac{2}{5}} & \boxed{\frac{2}{5}} & \boxed{\frac{2}{5}} & \boxed{\frac{2}{5}} & \boxed{\frac{2}{5}} \end{array} \cdots\cdots \qquad - L = 5$$

$$s = -3$$

$$L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b$$

$$Q^{ab} = \int d^2r\, r^a r^b \delta\rho(r) = s\ell_B^2 g^{ab}$$

second moment of neutral
composite boson
charge distribution

- Furthermore, the local electric charge density of the fluid with $\nu = p/q$ is determined by a combination of the magnetic flux density and the Gaussian curvature of the metric

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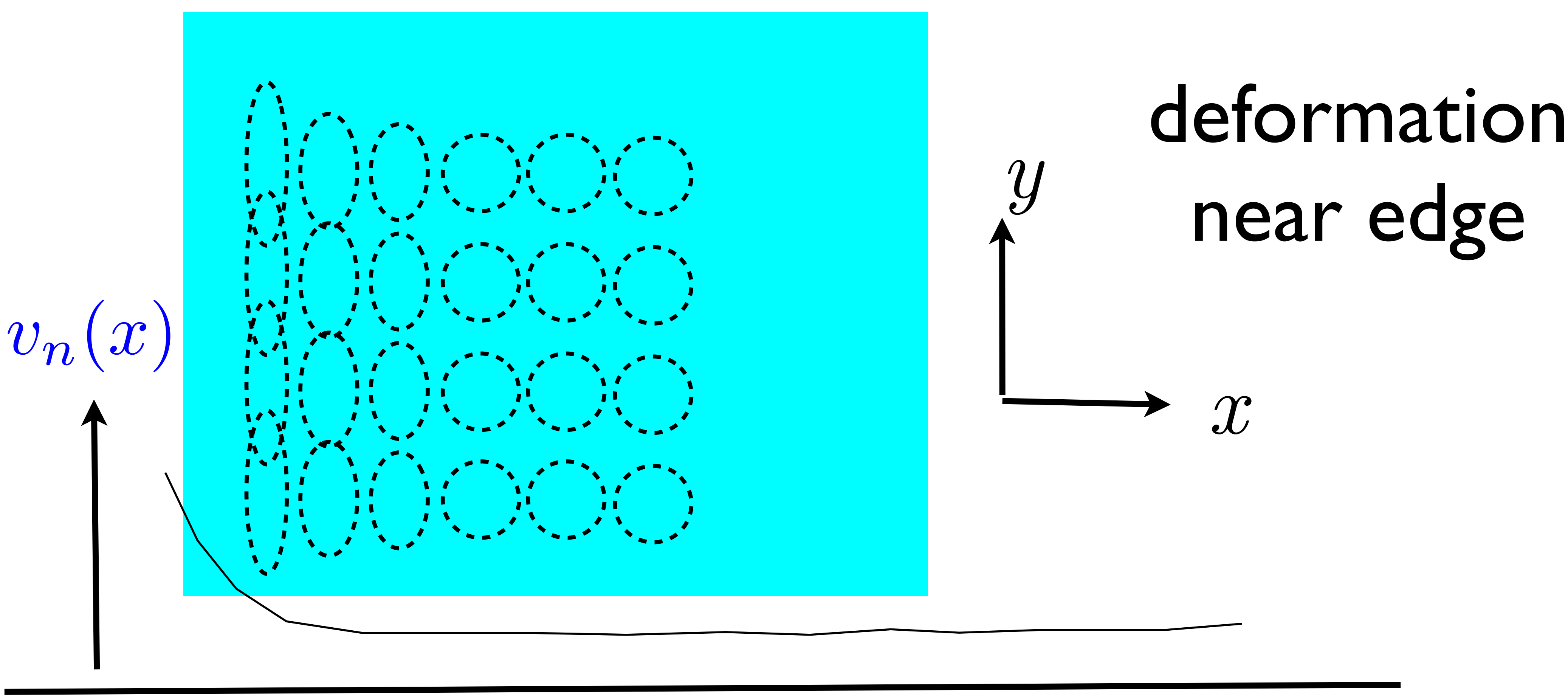

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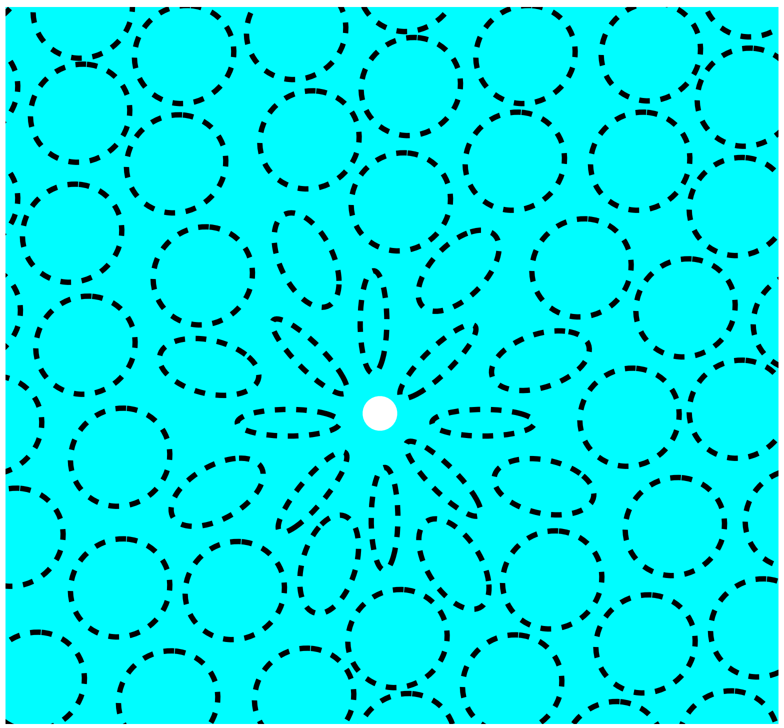

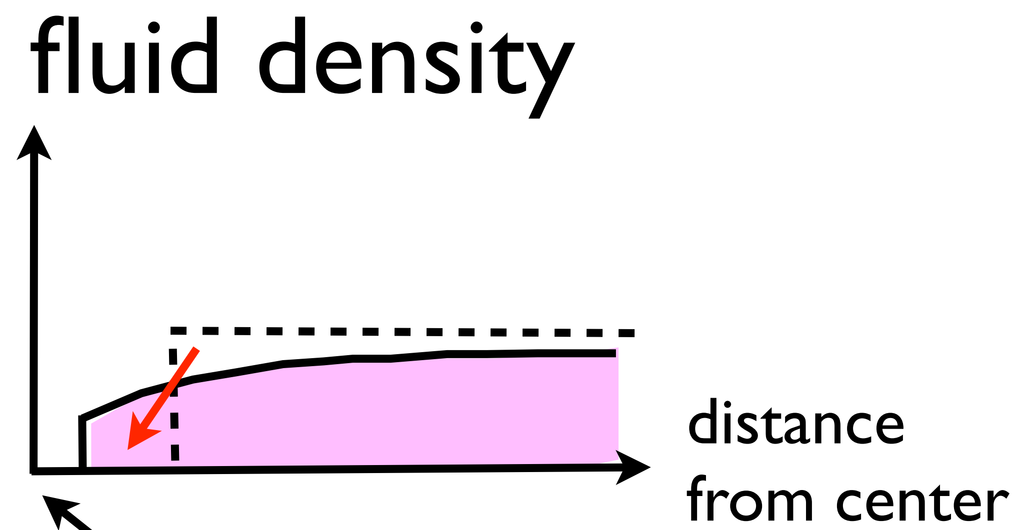

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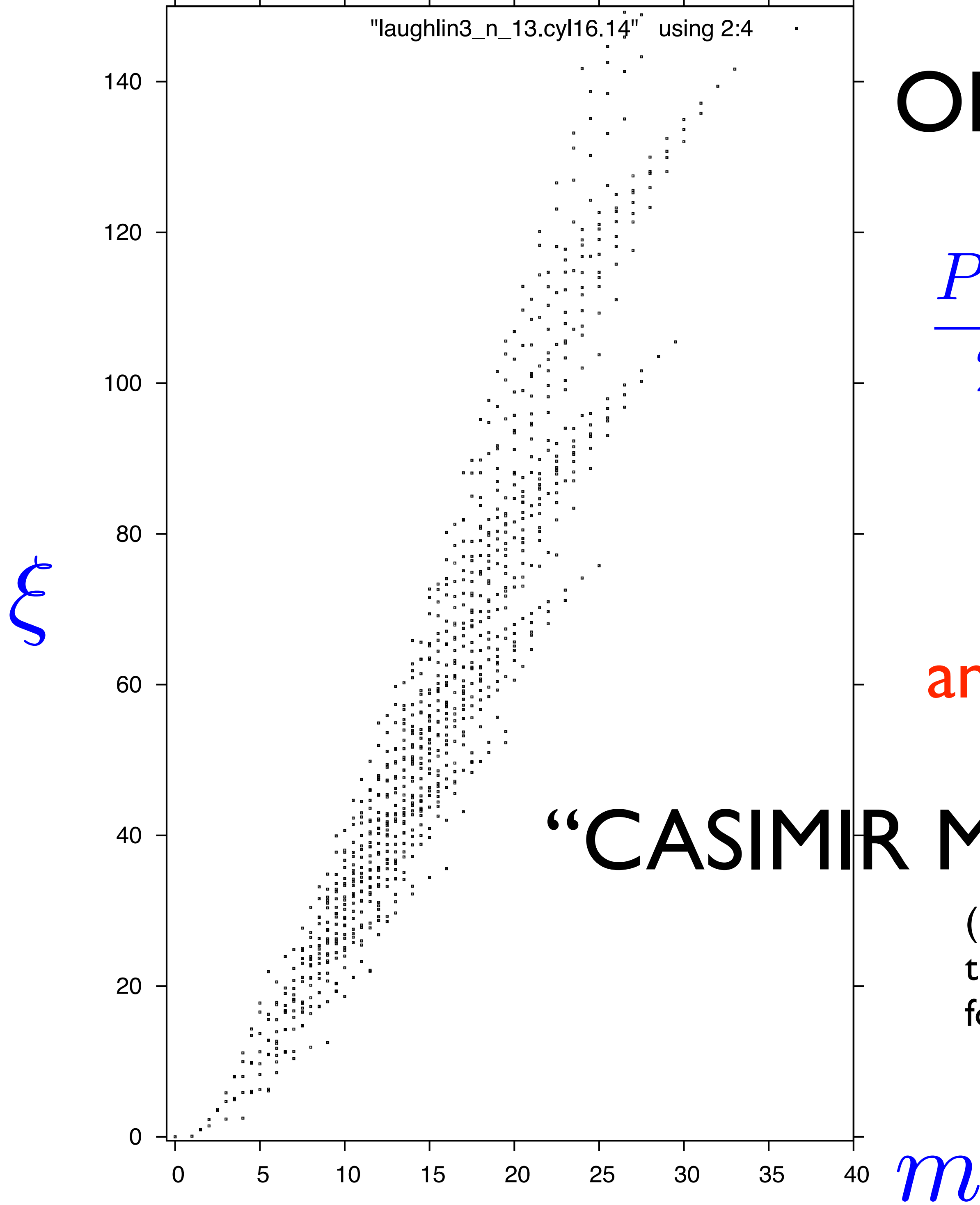

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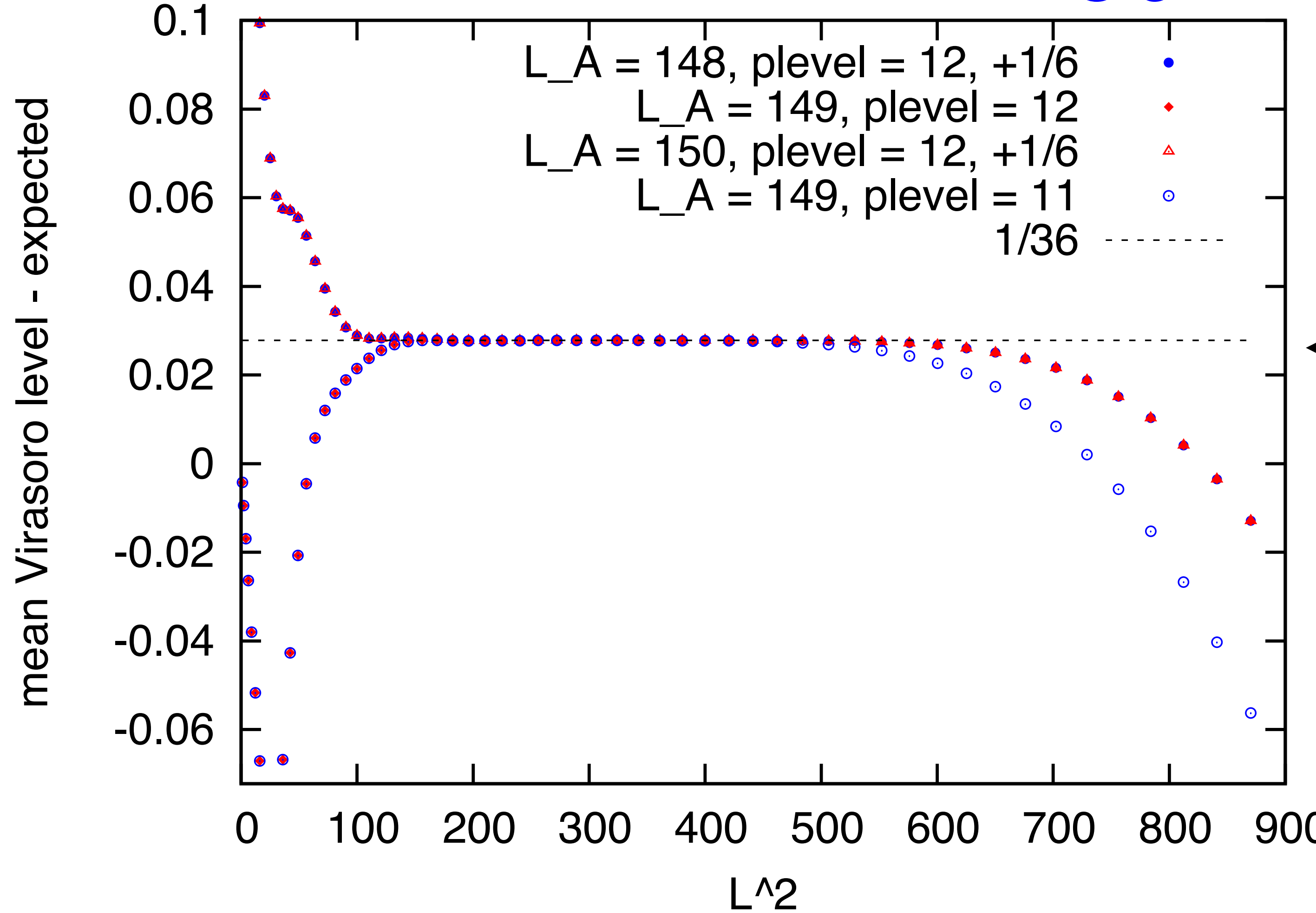

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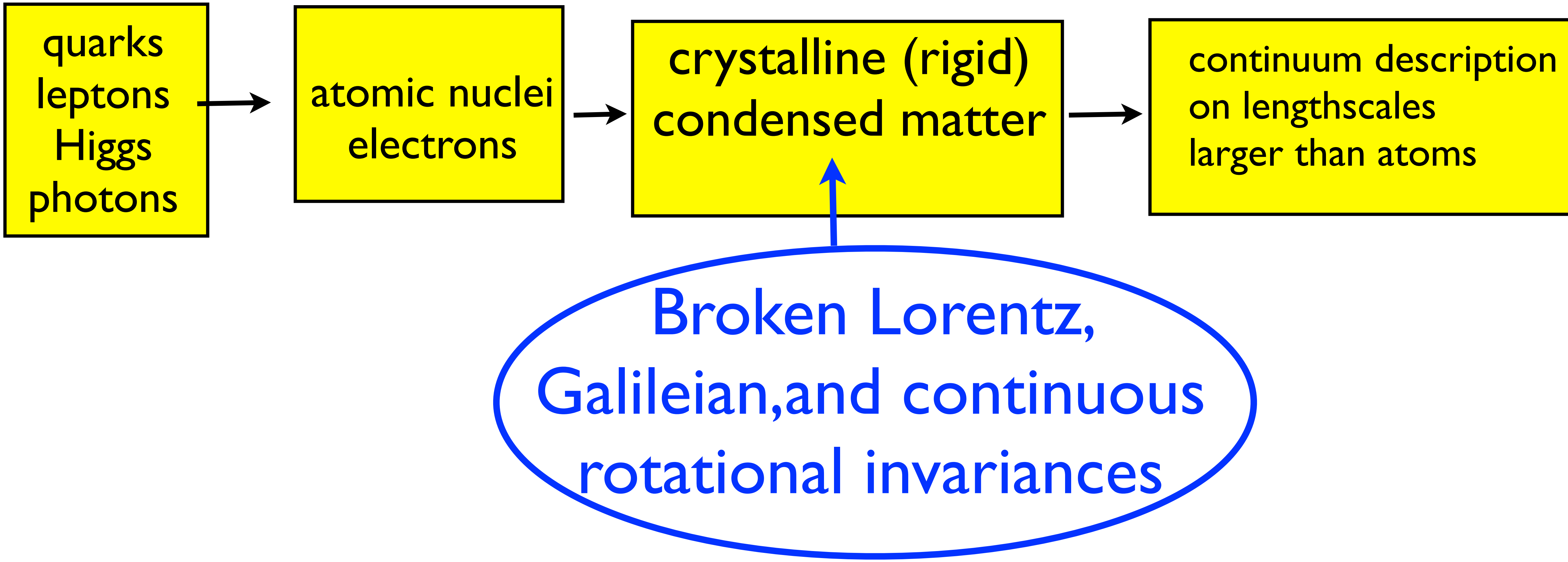


- A very effective approach for understanding the essential physics is to remove all unnecessary non-generic ingredients from the description.

- Only **translation** and (possibly) **inversion** symmetries are generic in a continuum description of phenomena in homogeneous crystalline condensed matter on a larger-than-atomic scale

- The essential property of the uniform incompressible FQHE fluids is unbroken spatial inversion symmetry and a gap for excitations that carry an electric dipole moment (= momentum)
- The momentum gap means that these fluids do not transmit forces through their bulk, unlike "classical incompressible fluids"

- Top-level model (Schrödinger):

$$p_i = -i\hbar \boldsymbol{\nabla}_r - e\boldsymbol{A}(\boldsymbol{r})$$

$$\boldsymbol{\nabla}_r \times \boldsymbol{A}(\boldsymbol{r}) = B$$

$$\boxed{H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V_0(\boldsymbol{r}_i - \boldsymbol{r}_j)}$$

not necessarily quadratic (**no** Galilean invariance should be assumed)

bare Coulomb interaction controlled by (possibly anisotropic) dielectric tensor of medium (no rotational invariance should be assumed)

- model has inversion symmetry if $\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$, but even this need not be assumed

$$\boldsymbol{r} = r^a \boldsymbol{e}_a$$

displacement (contravariant index)

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$$

orthonormal basis of tangent vectors of 2D plane: $a = 1,2$

Euclidean metric of 2D plane

$$p_a = \boldsymbol{e}_a \cdot \boldsymbol{p}$$

dynamical momentum (covariant index)

- Two independent Heisenberg algebras:

$$\boxed{\begin{aligned} [p_a, p_b] &= i\hbar e B \epsilon_{ab} \\ [r^a, p_b] &= i\hbar \delta^a_b \\ [r^a, r^b] &= 0 \end{aligned}} \xrightarrow{\text{organize as}} \boxed{\begin{aligned} [\bar{R}^a, \bar{R}^b] &= i\ell_B^2 \epsilon^{ab} \\ [R^a, \bar{R}^b] &= 0 \\ [R^a, R^b] &= -i\ell_B^2 \epsilon^{ab} \end{aligned}}$$

antisymmetric (2D Levi-Civita) symbol

$$\bar{R}^a = \hbar^{-1} \epsilon^{ab} p_b \ell_B^2$$

Landau orbit radius vector

$$R^a = r^a - \bar{R}^a$$

Landau orbit guiding-center displacement

$$2\pi \ell_B^2 = \frac{2\pi\hbar}{eB} > 0$$

quantum area (per h/e flux quantum)

- **Note: origin of guiding-center displacement has a gauge ambiguity under $\boldsymbol{A}(\boldsymbol{r}) \mapsto \boldsymbol{A}(\boldsymbol{r})$ + constant**

- Landau quantization $\varepsilon(\boldsymbol{p})|\Psi_n\rangle = E_n|\Psi_n\rangle$

discrete spectrum of macroscopically-degenerate Landau levels

- Project residual interaction in a single partially occupied "active" Landau level, all other dynamics is frozen by Pauli principle when gap between Landau levels dominates interaction potential

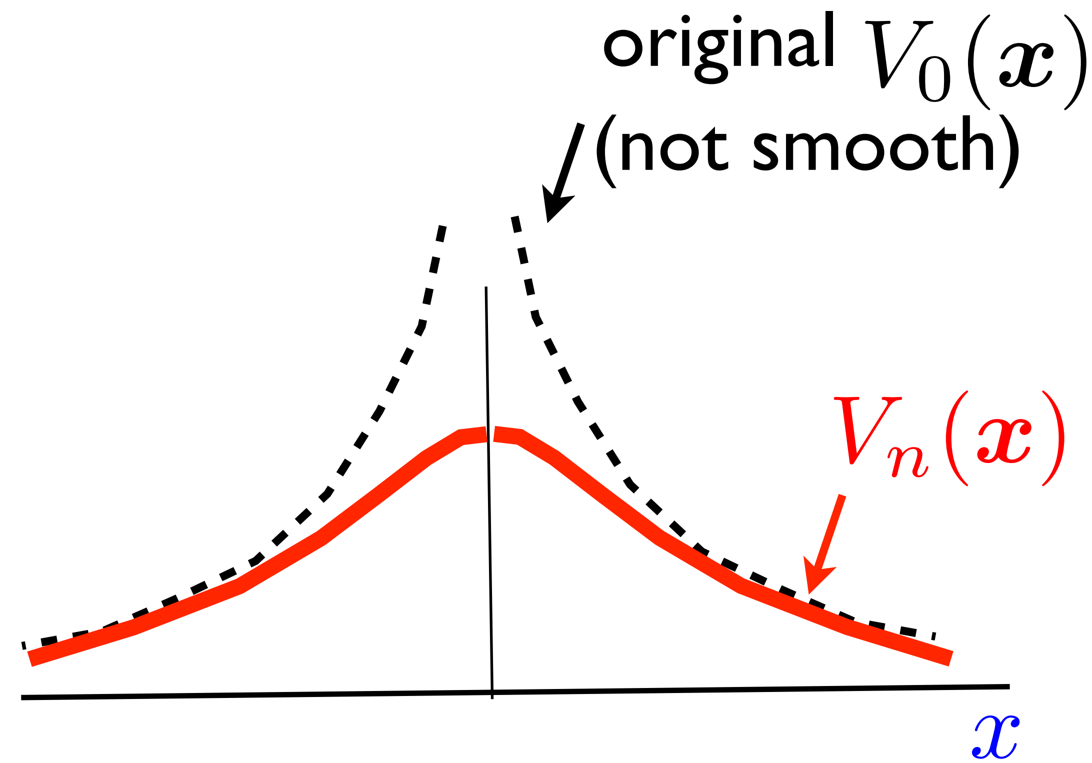
$$\boxed{\begin{gathered}[R^a, R^b] = -i\ell_B^2\epsilon^{ab} \\ H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)\end{gathered}}$$

residual problem is non-commutative quantum geometry!

$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

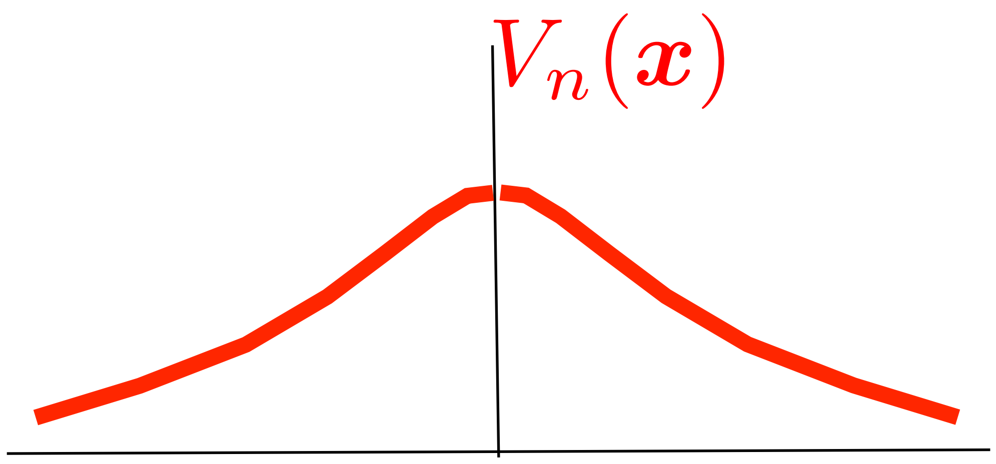
Identical quantum particles
(fermions or bosons)



We now have the final form of the problem:

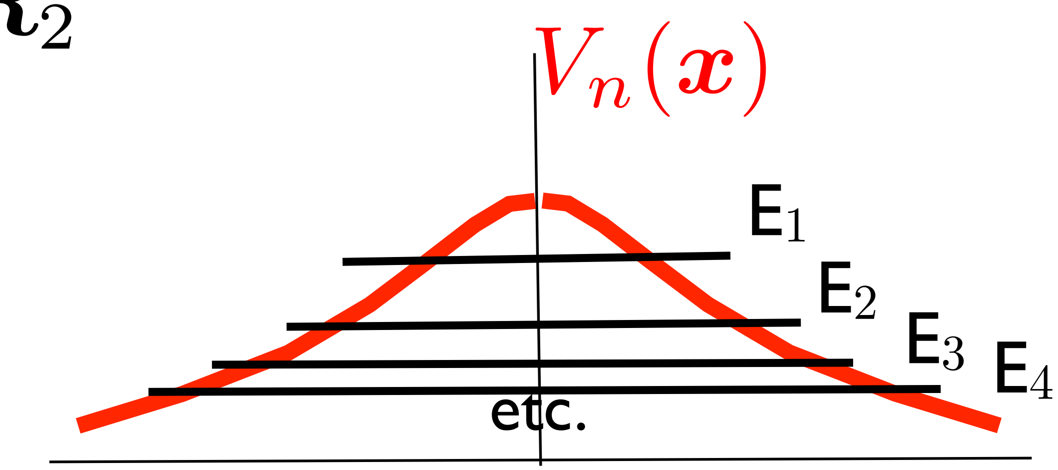
- The potential $V_n(\boldsymbol{x})$ is a **very smooth** (in fact entire) function that depends on the form- factor of the partially-occupied Landau level
- The essential clean-limit symmetries are translation and inversion: $\boldsymbol{R}_i \mapsto \boldsymbol{a} \pm \boldsymbol{R}_i$

$$\boxed{\begin{aligned}&[R^a, R^b] = -i\ell_B^2\epsilon^{ab}\\ &H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)\end{aligned}}$$



- The quadratic expansion of this even function around the origin defines a natural "interaction metric"
- The problem is often simplified by giving it a continuous rotation symmetry that respects this metric, but this is non-generic, and not necessary.
- This metric and a rotation symmetry are important in model FQH wavefunctions based on cft, which have a stronger conformal invariance property.

- It is straightforward to solve the two-body Hamiltonian: $\boldsymbol{R}_{12} = \boldsymbol{R}_1 - \boldsymbol{R}_2$



equivalent to a one-particle problem

$$[R^a_{12}, R^b_{12}] = 2i\ell_B^2\epsilon^{ab}$$

$$H = V_n(\boldsymbol{R}_{12})$$

- If there is a rotational symmetry, the energy levels (called "pseudopotentials") completely characterize the interaction potential.

- a large gap between energy levels favors **flux attachment** with a shape close to that of the "interaction metric"

- The non-commutative quantum geometry does not have a Schrödinger representation because there is no orthogonal local basis within its Hilbert space

- We can create an unfaithful Schrödinger representation by adding back a now-unphysical copy of the Landau orbit-degrees of freedom:

$$[R_i^a, R_i^b] = -i\epsilon^{ab}\ell_B^2 \quad [\bar{R}_i^a, R_i^b] = 0 \quad [\bar{R}_i^a, \bar{R}_i^b] = i\epsilon^{ab}\ell_B^2$$

physical

unphysical

local basis

$$e^{i\boldsymbol{k}\cdot(\boldsymbol{R}+\bar{\boldsymbol{R}})}|\boldsymbol{r}\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|\boldsymbol{r}\rangle$$

$$[e^{i\boldsymbol{k}\cdot(\boldsymbol{R}+\bar{\boldsymbol{R}})}, e^{i\boldsymbol{k}'\cdot(\boldsymbol{R}+\bar{\boldsymbol{R}})}] = 0$$

Projection into physical basis of holomorphic states

$$P_0(g) = \int \frac{d^2\boldsymbol{q}\ell_B^2}{2\pi} e^{-\frac{1}{4}(g^{ab}q_a q_b)\ell_B^2} e^{i\boldsymbol{q}\cdot\bar{\boldsymbol{R}}}$$

- This looks like just mapping the quantum geometry back into a "lowest-Landau-level problem"
- But important new features appear when the problem is "compactified on the torus" by imposing (quasi)periodic boundary conditions

- Guiding-center translation operator

$$t(\boldsymbol{d}) = e^{i\boldsymbol{d}\times\boldsymbol{R}/\ell_B^2} \qquad t(\boldsymbol{d})\boldsymbol{R} = (\boldsymbol{R}+\boldsymbol{d})t(\boldsymbol{d})$$

- Periodic Boundary Conditions on a primitive region with flux N_Φ in units h/e :

$$t(\boldsymbol{L})|\Psi\rangle = \eta(\boldsymbol{L})^{N_\Phi}|\Psi\rangle$$

$$\eta(\boldsymbol{L}) = 1 \quad (\tfrac{1}{2}\boldsymbol{L} \in \{\boldsymbol{L}'\})$$

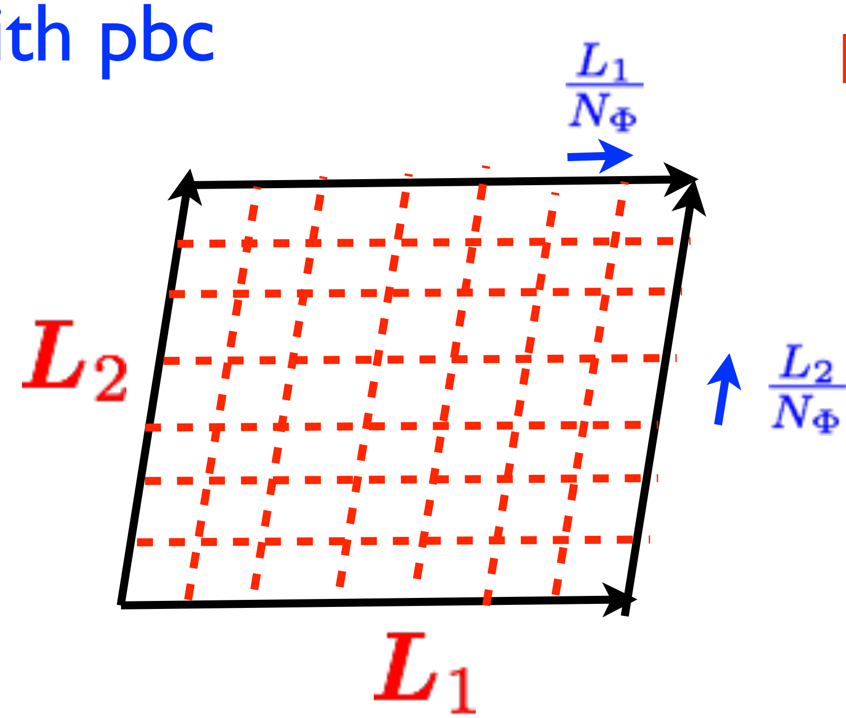
$$\eta(\boldsymbol{L}) = -1 \quad (\tfrac{1}{2}\boldsymbol{L} \notin \{\boldsymbol{L}'\})$$

$$[t(\tfrac{\boldsymbol{L}}{N_\Phi}), t(\boldsymbol{L}')] = 0$$

smallest translation compatible with pbc

$$\boldsymbol{L} = m\boldsymbol{L}_1 + n\boldsymbol{L}_2$$

Bravais lattice of periodic translations



NOTE: choice of a basis $L_1,\ L_2$ is a "modular choice"

- On the Torus (pbc) there are two distinct type of bases:

(a). Geometry-independent, orthonormal bases that depend on a modular choice:

$$t(\tfrac{\boldsymbol{L}_1}{N_\Phi})|\Psi_0(\boldsymbol{L}_1)\rangle = -|\Psi_0(\boldsymbol{L}_1)\rangle$$
$$|\Psi_m(\boldsymbol{L}_1,\boldsymbol{L}_2)\rangle = \left(-t(\tfrac{\boldsymbol{L}_2}{N_\Phi})\right)^m|\Psi_0(\boldsymbol{L}_1)\rangle$$
$$|\Psi_{m+N_\Phi}(\boldsymbol{L}_1,\boldsymbol{L}_2)\rangle = |\Psi_m(\boldsymbol{L}_1,\boldsymbol{L}_2)\rangle$$

This is a standard "Landau basis"

- The other type of basis is a non-orthogonal basis of modular-invariant geometry-dependent holomorphic states:

$$|\Psi\rangle = F(a^\dagger)|0\rangle \qquad a|0\rangle = 0$$

$$F(z) = \prod_{i=1}^{N_\Phi} \tilde{\sigma}(z - w_i) \qquad \sum_i w_i = 0$$

zeroes

$$\tilde{\sigma}(z) = e^{\frac{1}{2}C_2 z^2}\sigma(z)$$

"almost holomorphic modular invariant"

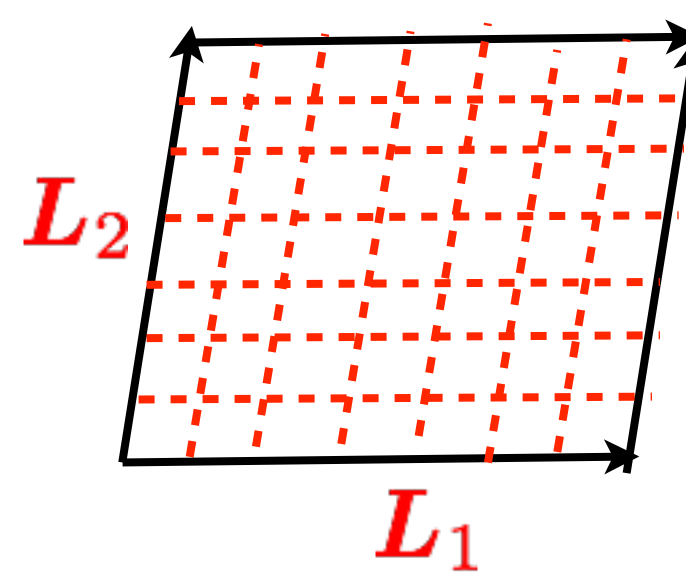
Weierstrass sigma function
(modular invariant)

- Holomorphic Laughlin states

$$F(\{z_i\}) = \left(\prod_{i<j} \tilde{\sigma}(z_i - z_j)\right)^m \prod_{k=1}^{m} \tilde{\sigma}(\textstyle\sum_i z_i - w_k) \qquad \sum_k w_k = 0$$

- Also get very useable forms for other states (composite fermion Fermi liquid states)

- A surprise: only need to evaluate holomorphic states on the lattice $z \in \frac{L}{N_\Phi}$ of $(N_\Phi)^2$ points in the primitive region of the torus!



- On the torus, the Heisenberg algebra is compactified to the unitaries

$$U(\boldsymbol{q}) = e^{i\boldsymbol{q}\cdot\boldsymbol{R}}, \qquad e^{i\boldsymbol{q}\cdot\boldsymbol{L}} = 1$$ (discrete set of reciprocal vectors q)

- To construct the unfaithful Schrödinger representation, we now use compactified dual variables: $\bar{U}(\boldsymbol{q}) = e^{i\boldsymbol{q}\cdot\bar{\boldsymbol{R}}}$,

$$[\bar{U}(\boldsymbol{q})U(\boldsymbol{q}), \bar{U}(\boldsymbol{q}')U(\boldsymbol{q}')] = 0$$

$$\bar{U}(\boldsymbol{q})U(\boldsymbol{q})|\boldsymbol{x}\rangle = e^{i\boldsymbol{q}\cdot\boldsymbol{x}}|\boldsymbol{x}\rangle \qquad \boldsymbol{x} \in \{\frac{\boldsymbol{L}}{N_\Phi}\}$$

- We get a modular-invariant lattice-based Schrödinger representation on $(N_\Phi)^2$ sites (square of one-particle Hilbert space dimension)

- A corollary: based on the naive lowest-Landau-level interpretation, one expects that the overlap between two holomorphic states must be calculated as

$$\langle\Psi_1|\Psi_2\rangle = \int_{\Box} \frac{dz \wedge dz^*}{2\pi i} f_1(z)^* f_2(z) e^{-z^* z}$$

primitive region of torus

- In fact

$$\langle\Psi_1|\Psi_2\rangle = \frac{1}{N_\Phi} \sum_z f_1(z)^* f_2(z) e^{-z^* z}$$

primitive lattice sum

This allows a lattice-based Monte Carlo evaluation of model state properties on the torus, which would not have been guessed in the LLL picture

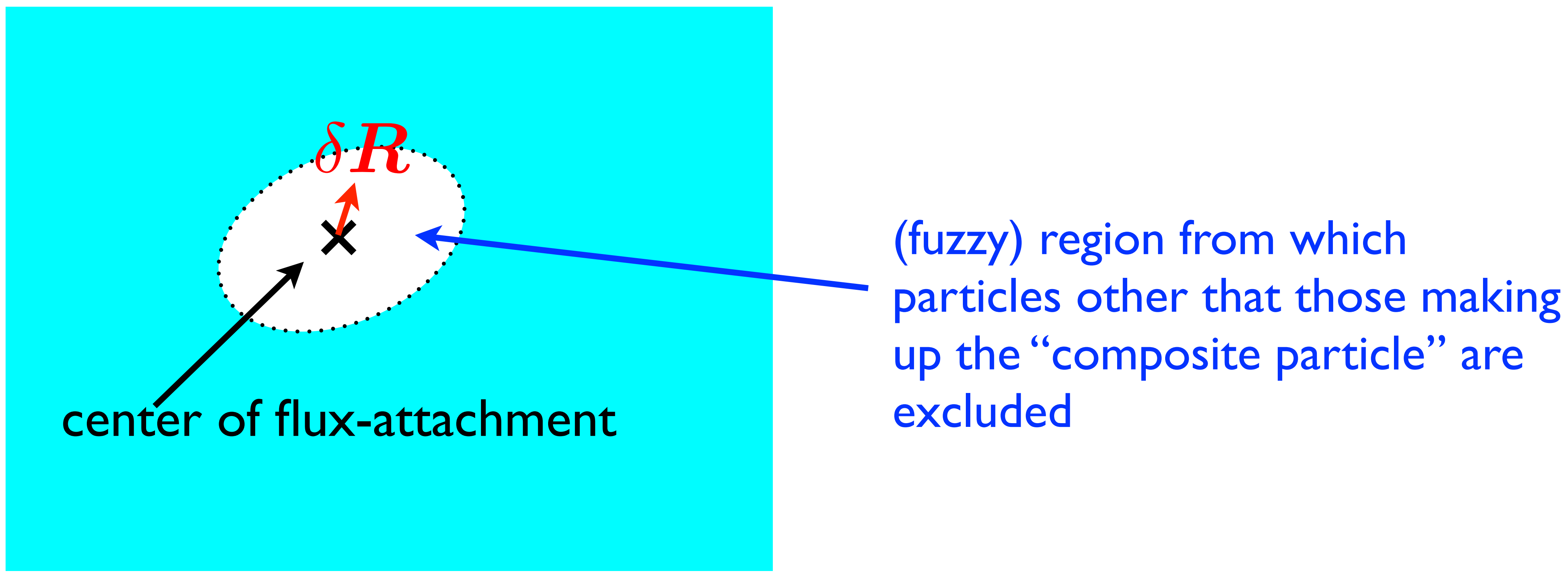
- Furthermore, the projections of the lattice sites into the physical (holomorphic) space is essentially a coherent state representation, and all operators have a diagonal representation on the lattice.
- A very economical Monte-Carlo method for model wavefunction properties (with <u>huge</u> speed-ups compared to previous continuum methods) is obtained and has been tested! (with Ed Rezayi, Scott Geraedts, Jie Wang)

summary

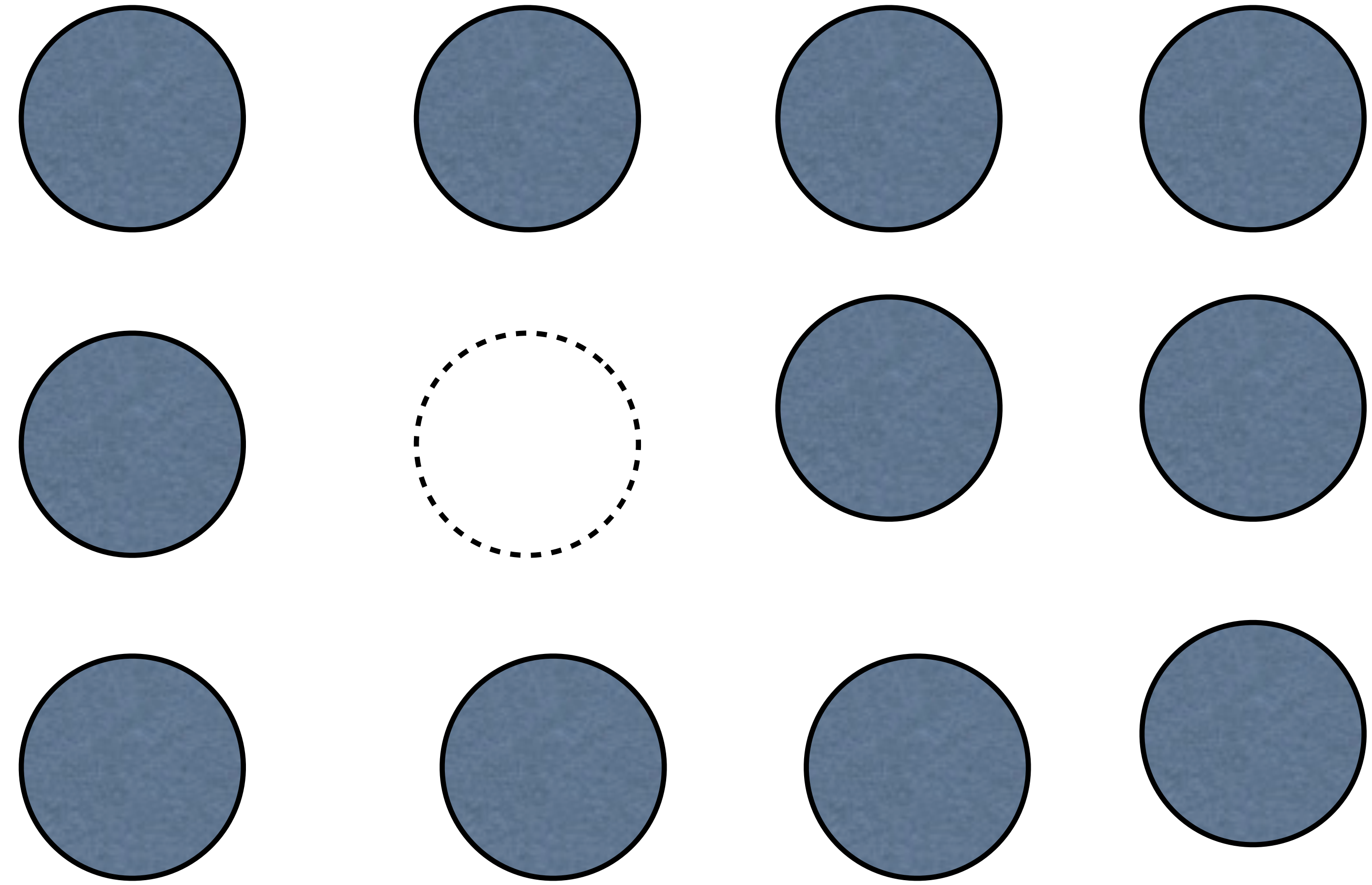
- While the "lowest Landau level wavefunction" (LLLWF) interpretation of model FQH and CFL states is entrenched in the "common wisdom" it is misleading. The system projected into a Landau level is a "quantum geometry" with no faithful Schrödinger representation.
- modular-invariant holomorphic model states have an intrinsic metric that is the shape of "flux attachment", fixed by the interaction in translationally-invariant systems
- A new lattice-based approach to systems on the torus is obtained after discarding the LLLWF interpretation. Modular invariance is a key property.

- Flux attachment is a gauge condensation that removes the gauge ambiguity of the guiding centers, giving each one a "natural" origin, so they define a physical electric dipole moment of the "composite particle" in which they are bound by the "attached flux".

- This is analogous to how the "the vector potential becomes an observable" (in a hand-waving way) in the London equations for a superconductor.



- quantum solid
- unit cell is correlation hole
- defines geometry



- repulsion of other particles make an attractive

potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

- In Maxwell's equations, the momentum density is

$$\pi_i = \epsilon_{ijk} D^j B_k \qquad D^i = \epsilon_0 \delta^{ij} E_j + P^i$$

- The momentum of the condensed matter is

$$\boldsymbol{p} = \boldsymbol{d} \times \boldsymbol{B}$$

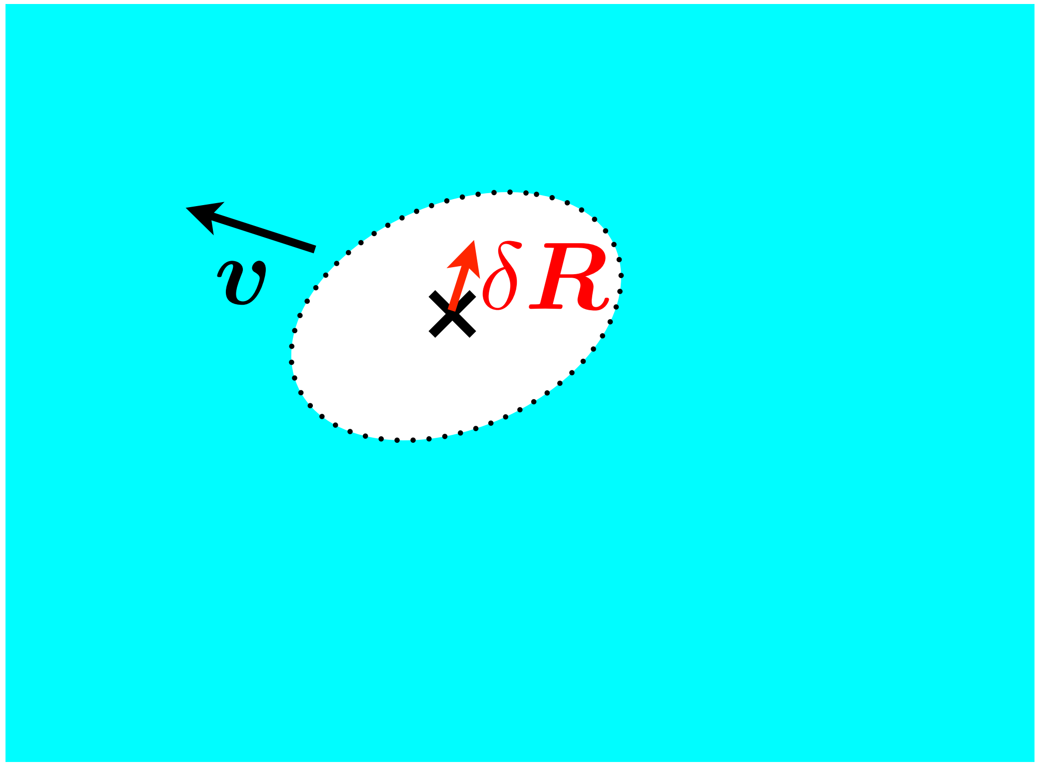
electric dipole moment

- in 2D the guiding-center momentum then is

$$p_a = eB\epsilon_{ab}\delta R^b$$

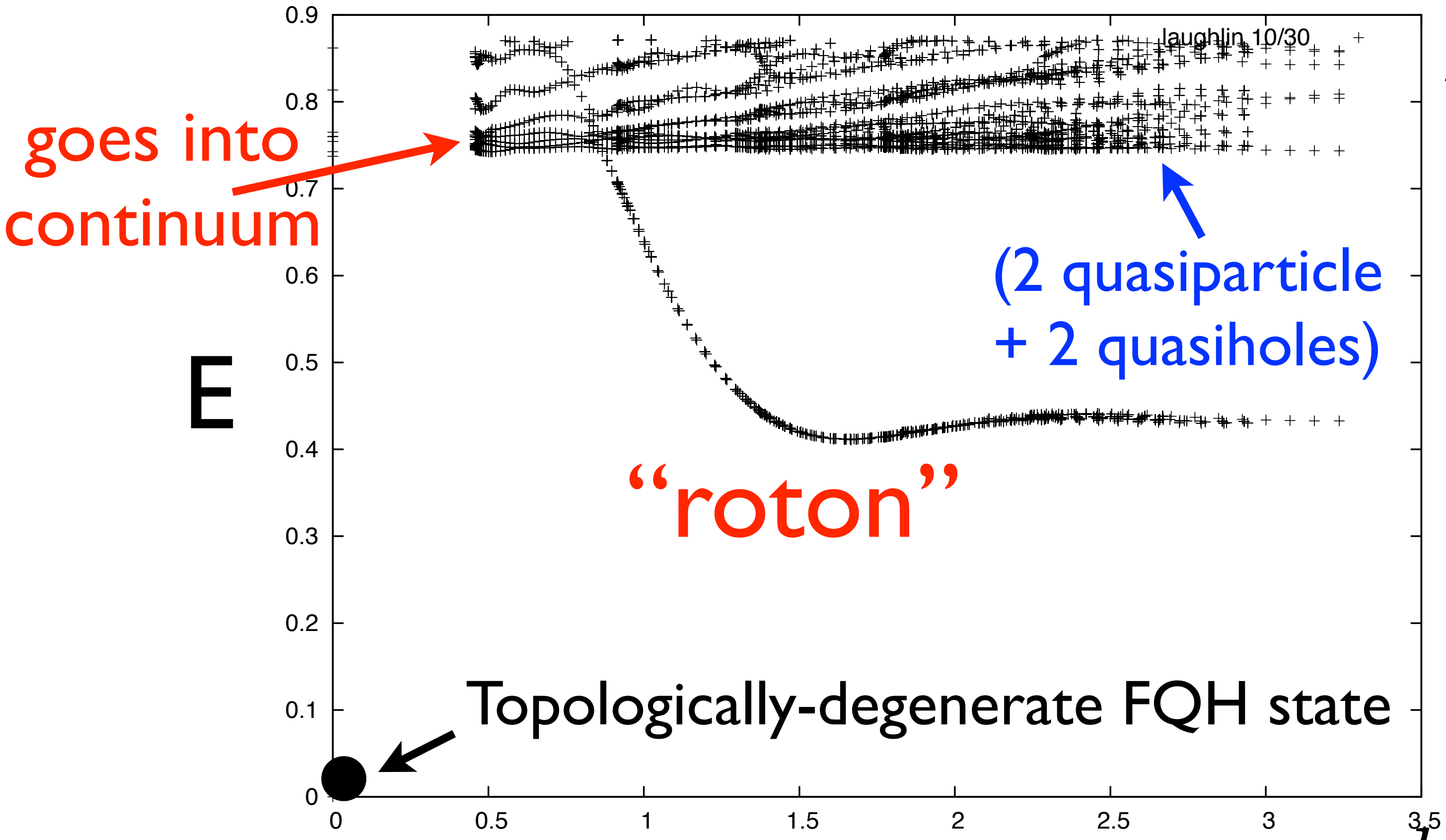
- The electrical polarization energy of the dielectric composite particle then gives its energy-momentum dispersion relation, with no involvement of any "Newtonian inertia" involving an effective mass

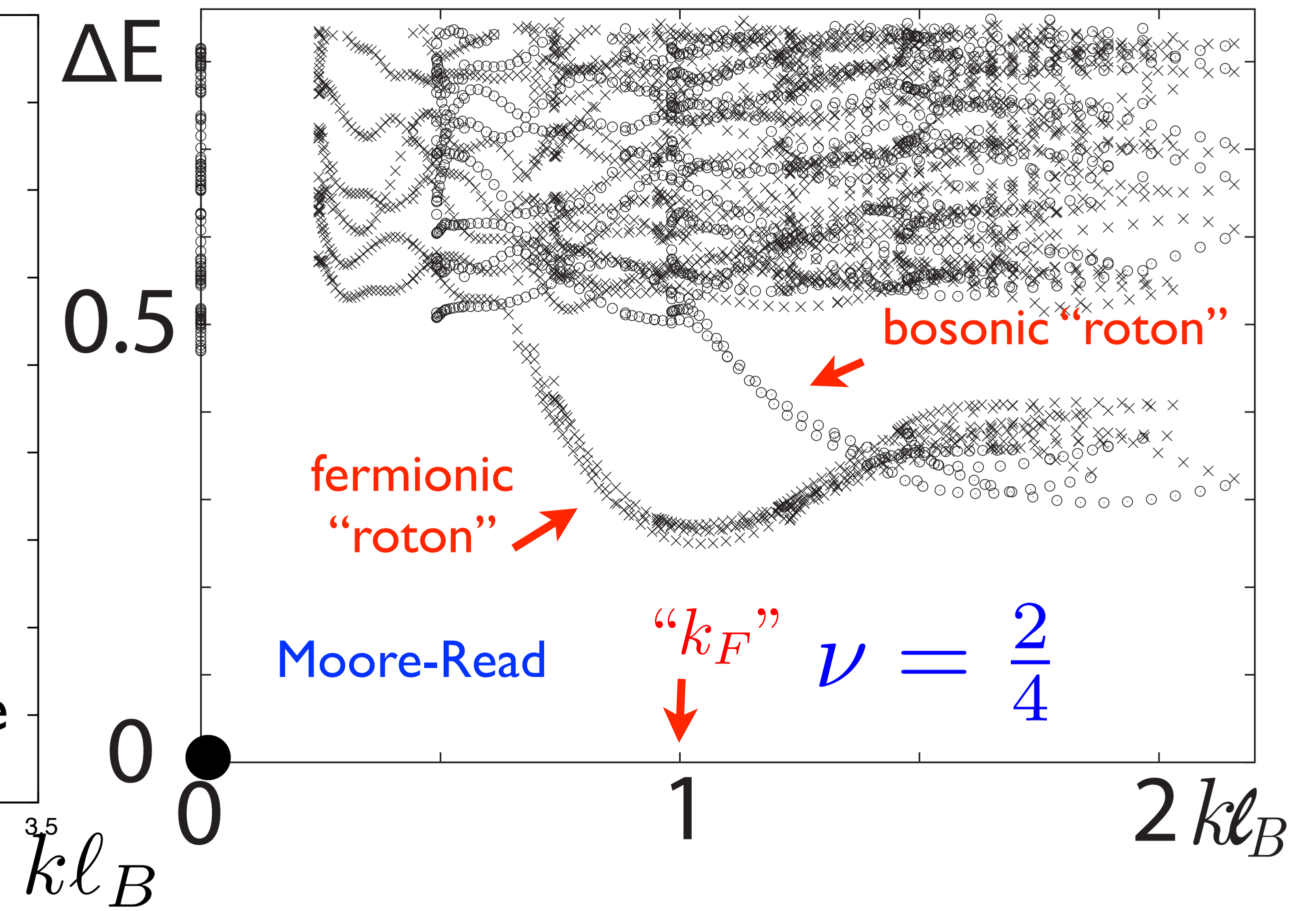
- The Berry phase generated by motion of the "other particles" that "get out of the way" as the vortex-like "flux-attachment" moves with the particle(s) it encloses can be formally-described as a Chern-Simons gauge field that cancels the Bohm-Aharonov phase, so that the composite object propagates like a neutral particle.



- If the composite particle is a boson, it condenses into the zero-momentum **(zero electric dipole-moment)** inversion-symmetric state, giving an incompressible-fluid Fractional Quantum Hall state, with an energy gap for excitations that carry momentum or electric dipole moment ("quantum incompressibility", no transmission of pressure through the bulk).

- All FQH states have an elementary unit (analogous to the unit cell of a crystal) that is a composite boson under exchange.
- It may be sometimes be useful to describe this boson as a a bound state of composite fermions (with their own preexisting flux attachment) bound by extra flux (Jain's picture)





gap $\longleftrightarrow$ incompressibility

Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

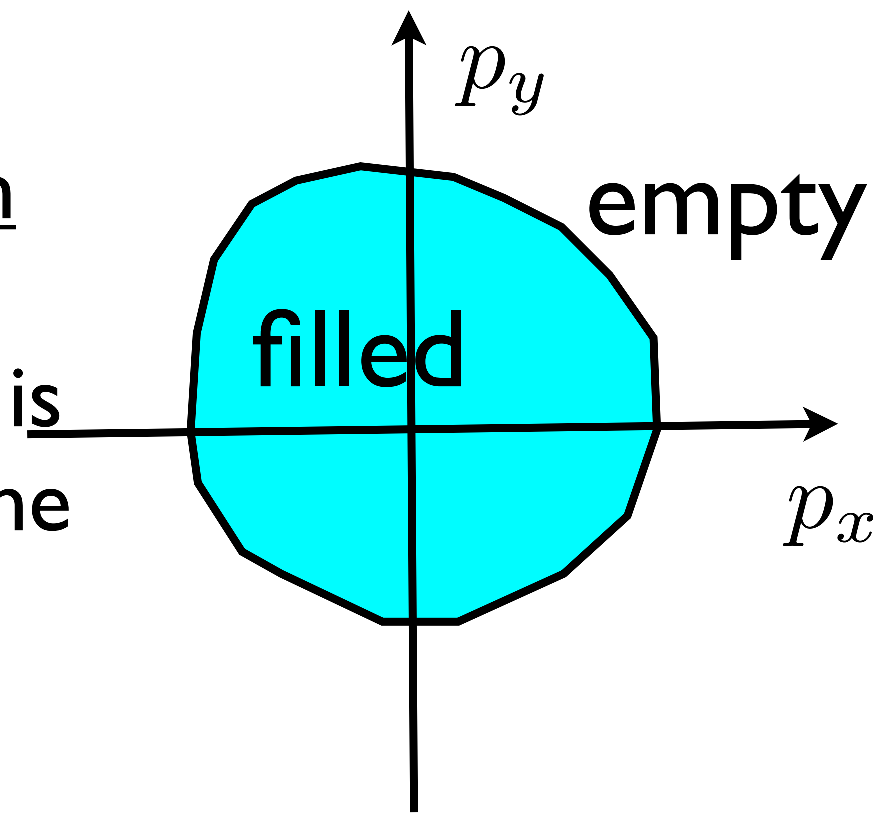
Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar$k of a quasiparticle-quasihole pair is proportional to its **electric dipole moment $\mathbf{p}_e$**

$$\hbar k_a = \epsilon_{ab} B p_e^b$$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: fluid **does not transmit pressure through bulk!**

- The composite particle may also be a fermion. Then one gets a Fermi surface in momentum-space = electric dipole space, and a gapless anomalous Hall effect which is quantized when the Berry phase cancels the Bohm-aharonov phase. (HLR-type state)



- There must be a distribution of dipole moments (or momentum) of the composite fermions, centered at the inversion-symmetric zero-moment state which has lowest energy. These are quantized by a pbc, and no two composite fermions can have the same diople moment.

- Fermi surface quasiparticle formulas for anomalous Hall effect (FDMH 2006)
- in 2D:

$$\sigma_H = \frac{e^2}{2\pi\hbar}(n + \frac{\phi}{2\pi})$$

Integer determined at edge

$$e^{i\phi}$$

Berry phase for moving a quasiparticle around Fermi surface (arc)

- holomorphic representations of guiding-center states

$$\frac{R^a}{\sqrt{2}\ell_B} = w^a a^\dagger + w^a a \qquad [a, a^\dagger] = 1$$

$$(w_a)^* w_a = \tfrac{1}{2}(g_{ab} + i\epsilon_{ab}) \qquad w_a = g_{ab} w^b \quad \det g = 1$$

- This is the Girvin-Jach formalism, except they implicitly assumed the metric g_{ab} was the Euclidean metric of the plane. **In fact, it is a <u>free choice</u>, not fixed by the any physics of the problem.**

- Then, once a metric (i.e., a complex structure) has been chosen, a one-particle state can be described as

$$|\Psi\rangle = f(a^\dagger)|0\rangle \qquad a|0\rangle = 0$$

holomorphic

- Both the "vacuum" $|0\rangle$ and the function f(z) vary as the metric is changed (a Bogoliubov transformation)

- Normalization/overlap:

$$\langle\Psi_1|\Psi_2\rangle = \int \frac{dz \wedge dz^*}{2\pi i} f_1(z)^* f_2(z) e^{-z^* z}$$

- When compactified on the torus with flux N_Φ, the modular-invariant formulation is

$$f(z) \propto \prod_{i=1}^{N_\Phi} \tilde{\sigma}(z - w_i) \qquad \sum_i w_i = 0$$

Bravais lattice in complex plane

$$\tilde{\sigma}(z|\{L\}) = e^{\frac{1}{2}C_2(\{L\})z^2}\sigma(z|\{L\})$$

"almost holomorphic modular invariant" (Eisenstein series)

Weierstrass sigma function

- In the Heisenberg-algebra reinterpretation

$$|\Psi\rangle = \prod_{i=1}^{N_\Phi} \tilde{\sigma}(a_i^\dagger - w_i)|0\rangle \qquad \sum_i w_i = 0$$

one particle
$N = 1$

- The filled Landau level is

$$|\Psi\rangle = \left(\prod_{i<j} \tilde{\sigma}(a_i^\dagger - a_j^\dagger) \tilde{\sigma}(\textstyle\sum_i a_i^\dagger) \right) |0\rangle$$

filled Level
$N = N_\Phi$

- The Laughlin states are

$\nu = \frac{1}{m}$ Laughlin state
$N_\Phi = mN$

$$|\Psi\rangle = \left(\prod_{i<j} \tilde{\sigma}(a_i^\dagger - a_j^\dagger)^m \right) \prod_{k=1}^{m} \tilde{\sigma}(\textstyle\sum_i a_i^\dagger - w_k)|0\rangle \quad \displaystyle\sum_{k=1}^{m} w_k = 0.$$

- A previously unknown (?) identity that I recently guessed and found was indeed true, and which dramatically transforms calculations on torus (e.g., orders of magnitude Monte-Carlo speedup)

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{\square} \frac{dz \wedge dz^*}{2\pi i} f_1(z)^* f_2(z) e^{-z^* z}$$

$$= \frac{1}{N_\Phi} \sum_z{}' \qquad z \in \{ \tfrac{mL_1 + nL_2}{N_\Phi} \}$$

$(N_\Phi)^2$ points

replace integral over fundamental region by a modular-invariant finite sum

- with Ed Rezayi, I found a remarkable clean composite Fermi liquid model state on the flat torus, inspired by a construction by Jain on the sphere.
- On the torus, the state is precisely equivalent to the usual treatments of the Fermi gas with a pbc.
- It is very accurate as compared to exact diagonalization of the Coulomb interaction, and amazingly "almost" (99.99%) particle-hole symmetric at $\nu = 1/2$.

- Composite Fermi liquid (HLR-like) at $\nu = \dfrac{1}{m}$

gives Chern-Simons

gives bf? / Z2

$$f(\{z_i\}) = \prod_{i<j} \tilde{\sigma}(z_i - z_j)^{(m-2)} \det_{ij} M_{ij} \prod_{k=1}^{m} \tilde{\sigma}(\textstyle\sum_i z_i - w_k)$$

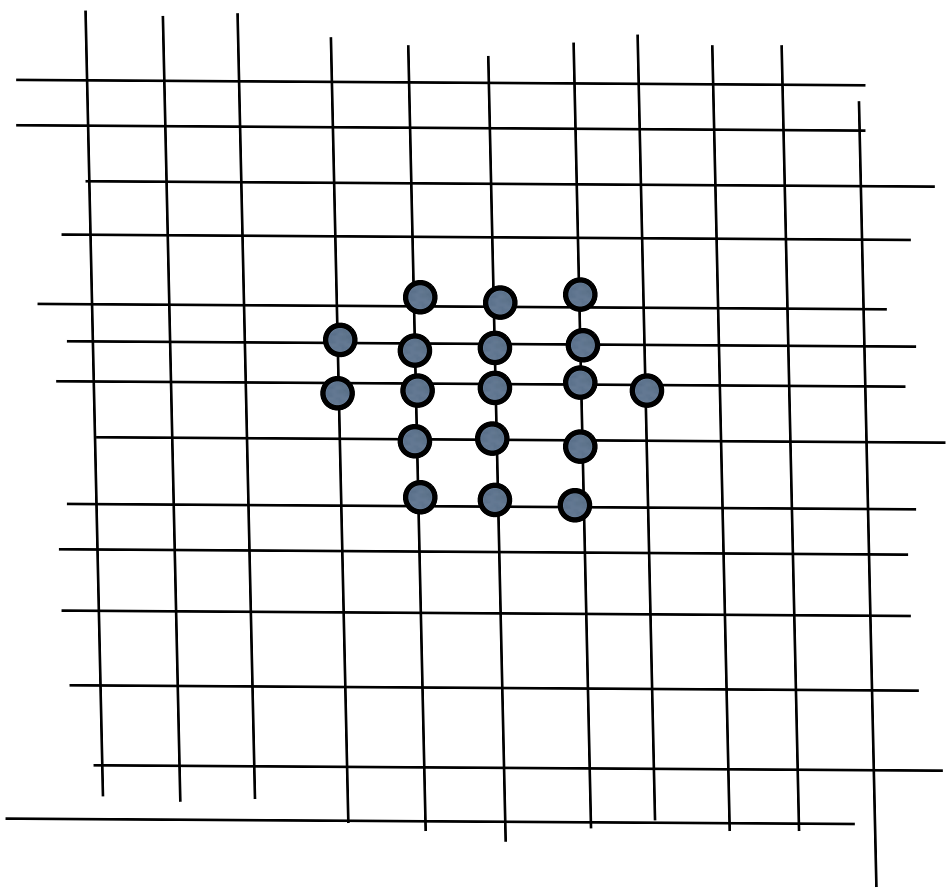
$$\sum_{\alpha=1}^{m} w_\alpha = \sum_{j=1}^{N} d_j = N\bar{d}$$

Fermi (Bose) for m even (odd)

$$M_{ij}(\{z_k\};\{d_k\}) = e^{d_j^* z_i/m} \prod_{k\neq i}{}' \tilde{\sigma}(z_i - z_j - d_i + \bar{d})$$

a set of dipole moments $\quad d_i \in \dfrac{L}{N}$ (particle number, not flux)

- There are vastly more possible choices of dipole “occupations” than independent states: The “good” ones are clusters that minimize $\sum_i |d_i - \bar{d}|^2$



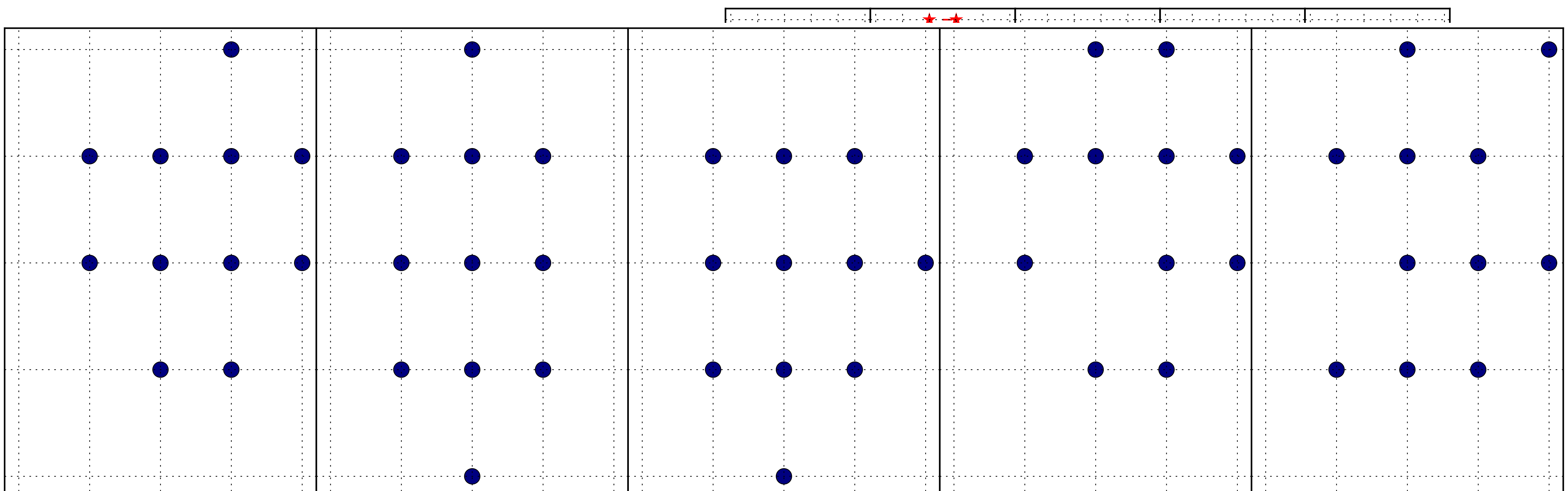
Computing ph symmetry

(with Scott Geraedts)

model state is numerically **<u>very</u>** close to p-h symmetry when k’s are clustered

```
# Z_{COM}  overlap with PH-conjugate
overlap
0 0.999998870263 1.1297367517e-06
1 0.999999369175 6.3082507884e-07
2 0.99999860296 1.39704033186e-06
3 0.99999860296 1.3970403312e-06
4 0.999999369175 6.30825078063e-07
5 0.999998870263 1.12973675237e-06
6 0.999999369175 6.30825079173e-07
7 0.99999860296 1.39704032942e-06
8 0.99999860296 1.39704032909e-06
9 0.999999369175 6.30825078507e-07
```

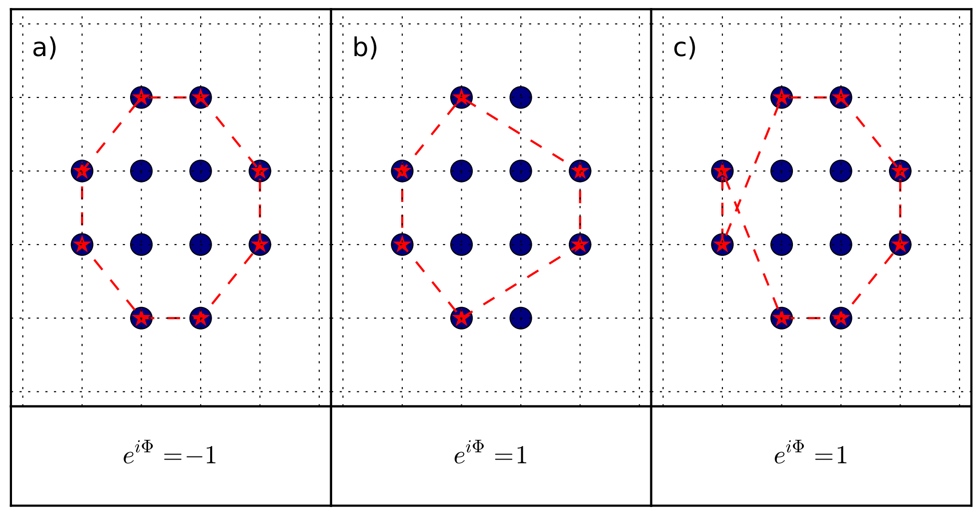
- particle-hole symmetry, and Kramers Z_2 structure (Scott Geraedts and Jie Wang)

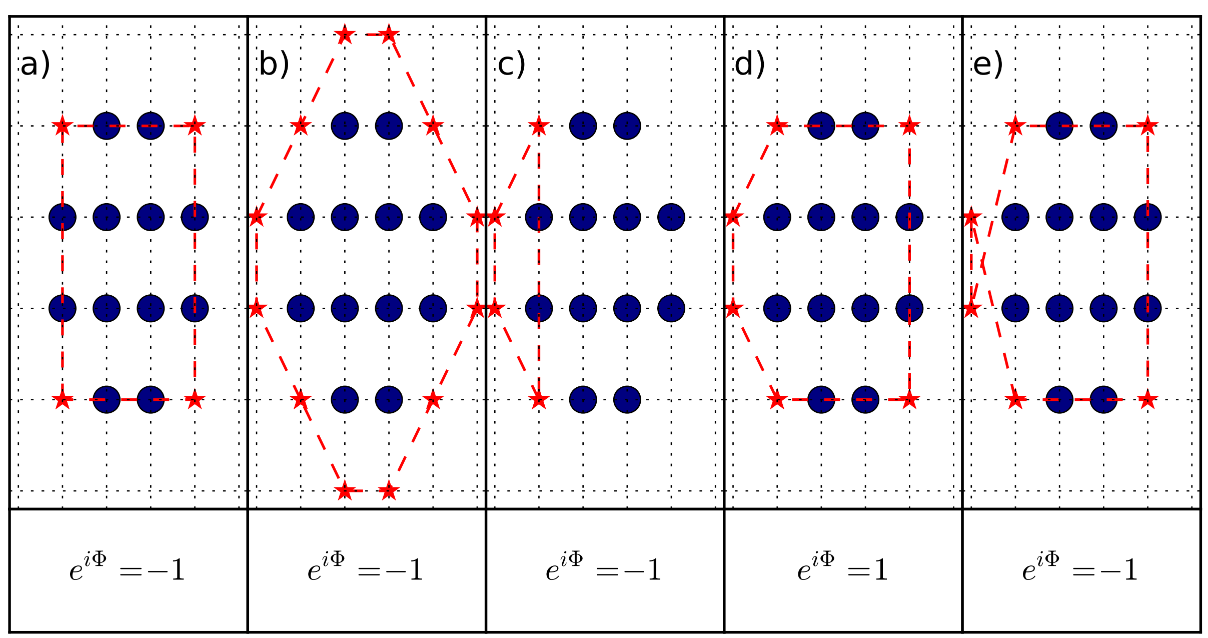


ED ΔE	0.0	ED ΔE	0.000404	ED ΔE	0.000268	ED ΔE	0.005555	ED ΔE	0.0
overlap	0.981955	overlap	0.736717	overlap	0.992142	overlap	0.226429	overlap	0.276141
ΔE	0.000446	ΔE	0.001419	ΔE	0.000531	ΔE	0.009271	ΔE	0.011863
$\langle H^2 \rangle - \langle H \rangle^2$	6.81819e-06	$\langle H^2 \rangle - \langle H \rangle^2$	6.5675e-06	$\langle H^2 \rangle - \langle H \rangle^2$	4.88251e-06	$\langle H^2 \rangle - \langle H \rangle^2$	1.58755e-05	$\langle H^2 \rangle - \langle H \rangle^2$	3.72072e-05
PH symmetry	0.993543	PH symmetry	0.987538	PH symmetry	0.995312	PH symmetry	0.856664	PH symmetry	0.964223

A many-body ansatz for Berry phase

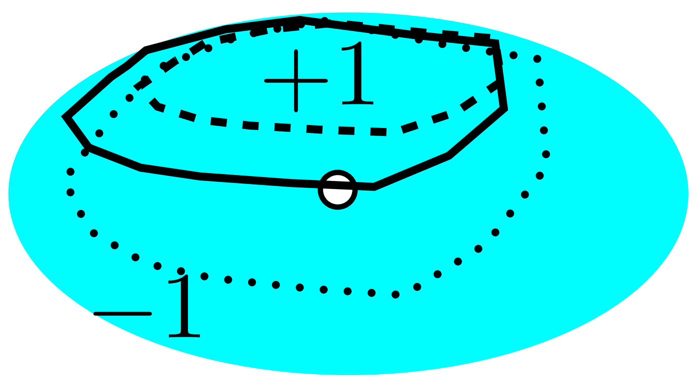
we confirmed all paths are real, by Kramers





These are ED results on exact Coulomb interaction states, with the exact particle hole symmetry, with occupation patterns obtained by finding the model states they have high overlap with

- is there an analog of Dirac cone point ?

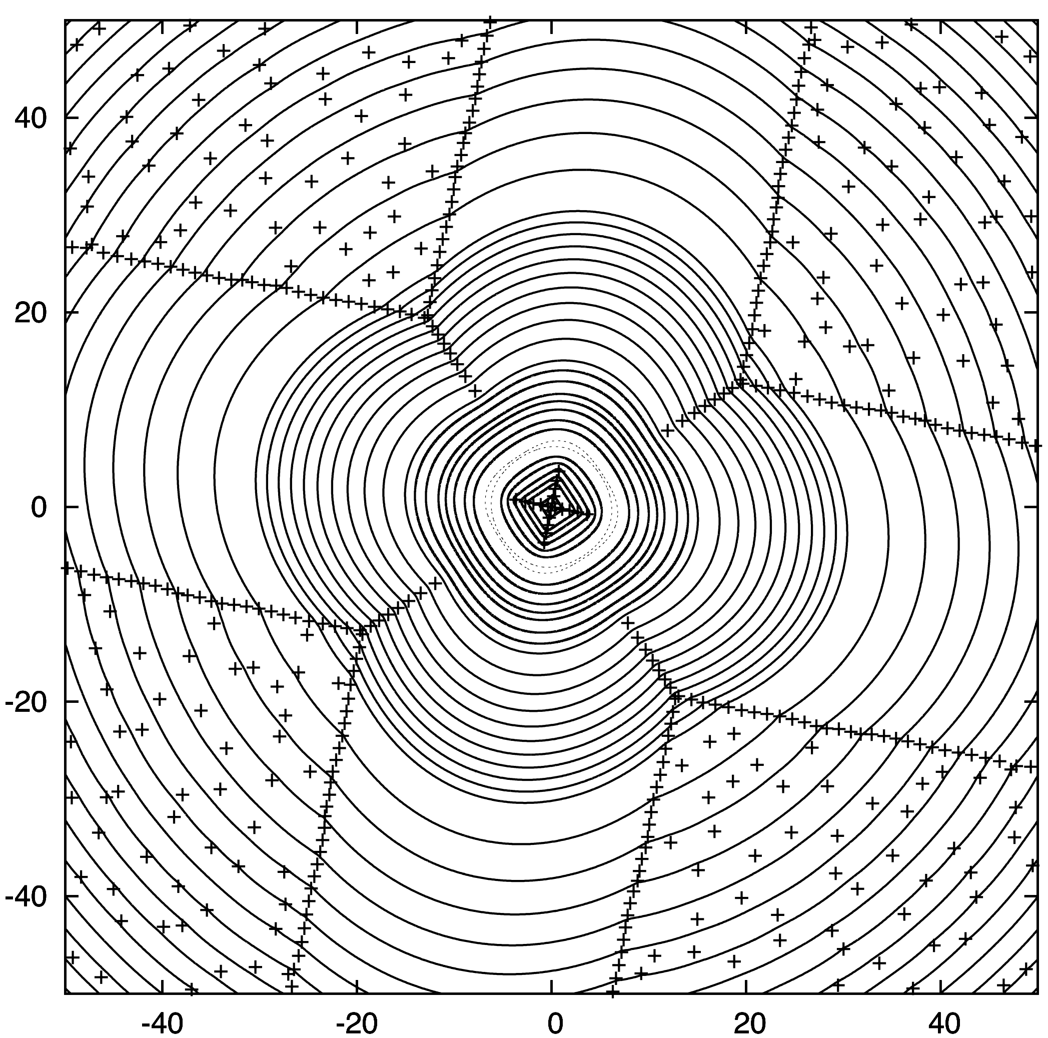


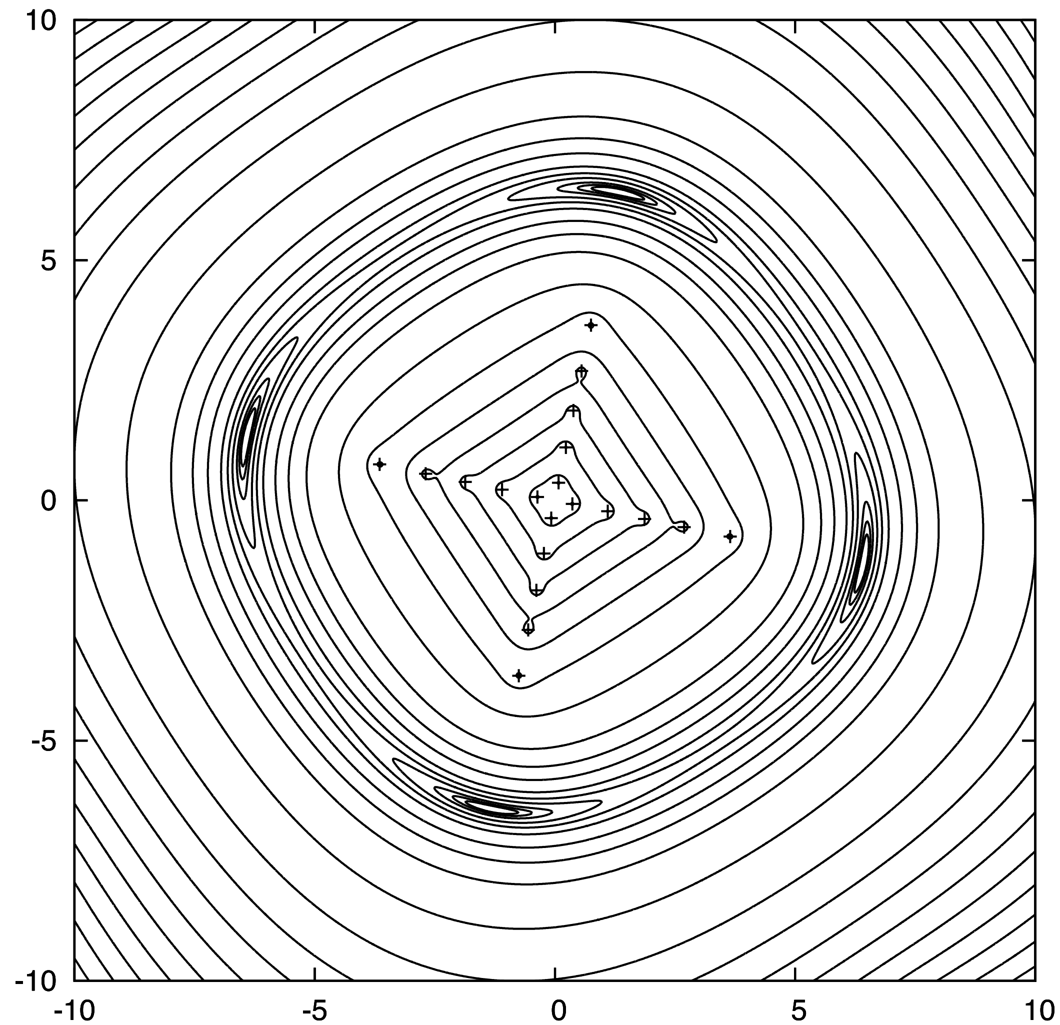
State with the quantum numbers of an inversion symmetric Fermi sea with a single hole at the center (has an even number of particles)

A state on the Torus with these quantum numbers is a parity doublet

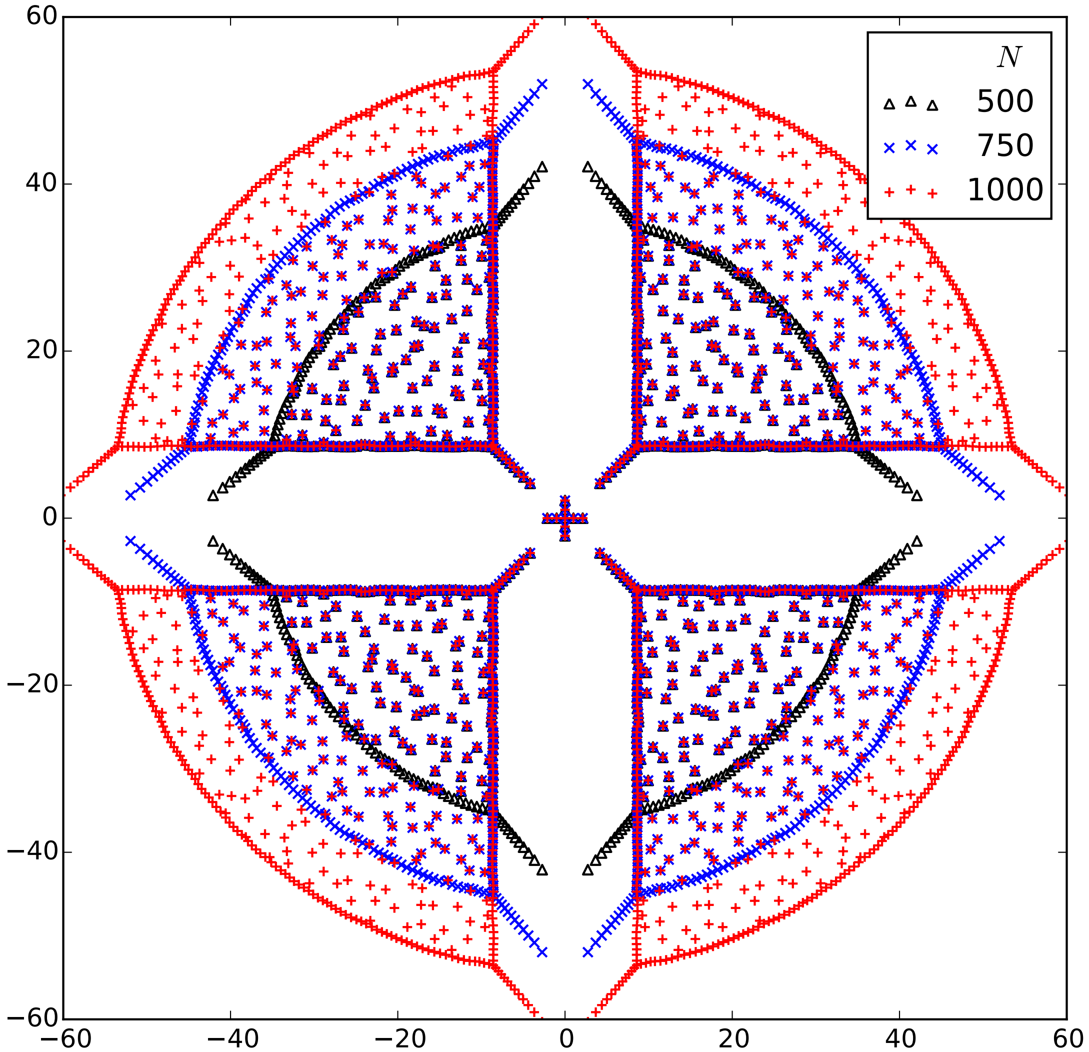
- as a hole is moved into the bulk, the ansatz must fail as if goes through the inversion-symmetric point!

- quadratic + quartic dispersion (Yu Shen + FDMH)





- non-polynomial landau orbit states



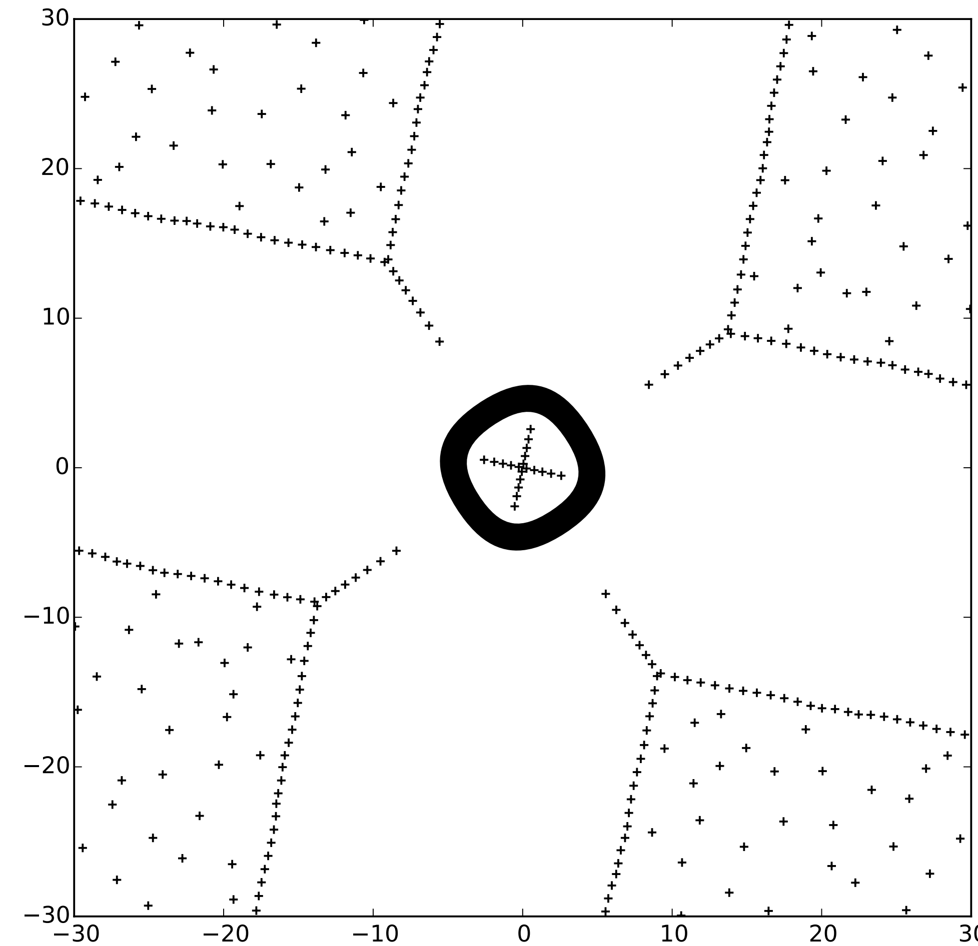


FIG. 7. Black region accounting for 90 percent of the total weight (probability) of Ψ_{20}, the coherent state in the 20th Landau level of the Hamiltonian $p_x^2 + p_y^2 + 2p_x^4 + 3p_y^4 + 4\{p_x^2, p_y^2\} + \{p_x, p_y^3\}$. The annulus is bounded by contours at the same value of the amplitude $|\Psi_{20}|$ and cleanly separates the central zeroes from the rest of the structure.