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# THEORY OF THE QUANTIZED HALL CONDUCTANCE

B.I. Halperin

Institute for Theoretical Physics
University of California
Santa Barbara, CA 93106

and

Harvard University *
Cambridge, MA 92138

## ABSTRACT

The quantized Hall conductance occurs in two dimensional electron systems at low temperatures and strong magnetic fields. When the Fermi level is between two Landau levels, the longitudinal resistivity $\rho_{xx}$ is observed to vanish, while the Hall conductance $1/\rho_{xy}$ is quantized in precise integer multiples of the unit $e^2/h$. These effects can be explained using an essentially one-electron analysis. Recently, however, "anomalous" Hall conductance plateaus have been discovered at certain simple fractions $\nu$ of the unit $e^2/h$, in a situation where a Landau level is partially filled. Explanation of this effect requires that the electrons form a new type of correlated quantum liquid with a commensurate locking term (i.e., a down-ward cusp) in the energy at the corresponding filling fractions. R.B. Laughlin has constructed an elegant wavefunction with the requisite properties, for $\nu = 1/m$, where m is an odd integer. We propose here trial wavefunctions for other rational fractions $\nu$.

## I. INTRODUCTION

The set of phenomena included in the term quantized Hall conductance together represent some of the most surprising physical discoveries in recent years. The "normal" quantized Hall effect, in which the Hall conductance of a two dimensional electron system has plateaus at integral multiples of the unit $\sigma_0 \equiv e^2/h$, was discovered experimentally by von Klitzing, Dorda and Pepper[1], in 1980. Although some aspects of this phenomenon

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* Permanent address

had been predicted in 1975, by Ando, Matsumoto and Uemura[2], the precision of the quantized Hall plateaus was completely unexpected, and indeed it took about a year before a satisfactory general explanation was forthcoming. Most aspects of the effect, which occur in particular when the Fermi level is between two Landau levels of the sample, can be understood with an essentially single-electron description, in which the electron-electron interaction has only a secondary role.

The "anomalous" quantized Hall effect, which gives Hall conductance plateaus at certain simple rational fractions of $e^2/h$, is in fact an even more remarkable phenomenon than the normal quantized Hall effect. This effect was first discovered by Tsui, Stormer, and Gossard[3], who reported in 1982 a Hall conductance plateau at $\sigma_0/3$, at magnetic fields such that the electrons were all in the first Landau level at a filling factor $\nu \approx 1/3$, in a GaAs accumulation layer with very high mobility and relatively low carrier density. They also saw indications of a similar effect at $\nu = 2/3$. More recent measurements have shown that these Hall plateaus are in fact very flat and are precisely equal to 1/3 and 2/3 of $e^2/h$ in the limit of zero temperature.[4] The longitudinal resistance tends to zero under these conditions, as it does under the conditions of the normal quantized Hall effect. Still more recently, there have been observations of at least some suggestion of an anomaly at filling factors 2/7, 2/5, 3/5, 4/5, 4/3 and 5/3.[5] However no anomaly is observed at filling factor 1/2, or other fractions with even denominator!

The explanation for the anomalous quantized Hall conductance is only just beginning to emerge. It is clear that the electron-electron interaction is playing a central role in this case, and that the electrons in the partially filled Landau level have formed a new type of strongly correlated quantum liquid. This liquid must have a "commensurate locking energy", (i.e., a downward cusp in the energy) at the rational filling factors corresponding to the observed Hall anomalies. Very recently, R. Laughlin[6] has proposed an elegant trial wavefunction, which

exists for filling factors $\nu = 1/m$ or $\nu = 1-(1/m)$, where $m$ is an odd integer, and which appears to have all the requisite properties to explain the corresponding transport anomalies. An explanation of the anomalies at other rational filling factors, requires something more complicated than Laughlin's original wavefunction; some possibilities will be discussed in Section IV E.

In the following section, I shall briefly review the experimental geometry that is used to study the quantized Hall effect. Section III is devoted to the theory of the normal quantized Hall effect, while section IV presents a summary of our current understanding of the fractional Hall steps. An argument for the exactness of the quantized Hall conductance, in both cases, will be presented in section V.

## II. EXPERIMENTAL CONDITIONS

The quantized Hall effect is observed in two-dimensional electron systems, under conditions of very strong magnetic field and low temperatures. The samples, in practice have been inversion layers in semiconductors -- either at a semiconductor-insulator interface as in the original experiments of von Klitzing et al,[1] which utilized a silicon inversion layer, or at a semiconductor heterojunction. The most popular system in the latter category is GaAs-$Ga_xAl_{1-x}As$, but $In_xGa_{1-x}As$-InP has also been used.[7,8] Under the experimental conditions, the conduction electrons are all in the lowest "subband" of the conducting layer,--i.e. they are all trapped in the lowest quantum state for motion perpendicular to the interface--but they are free to move in the x-y plane, parallel to the interface. The magnetic field is applied perpendicular to the plane and a measured current $I_x$ is passed through the sample, along the x-direction. Several leads are attached to measure the voltage drop $V_x$, parallel to the current, and also the Hall voltage $V_y$, perpendicular to the current flow.

## III. THE NORMAL QUANTIZED HALL EFFECT

At relatively high temperatures, the electron accumulation layers show a normal (two-dimensional) Hall effect. The Hall resistance $\rho_{xy} \equiv V_y/I_x$ varies linearly with the magnetic field

B according to the formula

$$\rho_{xy} = B/Nec \qquad (1)$$

where N is the density of carriers, per unit area. The longitudinal resistance $V_x/I_x$ is a weak function of the magnetic field, remaining similar in magnitude to the value at B = 0.

At sufficiently low temperatures and high magnetic fields the situation is quite different. The Hall resistance develops a series of plateaus, as a function of B, with the value of $\rho_{xy}$ on the plateau given by

$$1/\rho_{xy} = n\sigma_0 \qquad (2)$$

where $\sigma_0 = e^2/h$, and n is an integer. As the temperature is lowered, the plateau regions become more and more flat and Eq. (2) is obeyed with greater and greater accuracy. The longitudinal resistance $V_x/I_x$ diminishes rapidly with decreasing temperature in the regions of magnetic field which correspond to the Hall plateaus, and it is clear that at T = 0, a current can flow in these samples without any dissipation. Experimentally, Eq. (2) has been verified at a level of order one part in $10^7$, and the resistivities observed are many orders of magnitude lower than at B = 0, or than the resistivity of copper or of any other metal not a superconductor.[9,10]

Moreover, the fact that Eq. (2) holds precisely, (in the limit $T \to 0$) is true regardless of the precise shape or uniformity of the sample, and without any stringent limitations on the uniformity of the magnetic field.

There are thus three fundamental questions to be answered by an explanation of the normal quantized Hall effect:

1. Why does one see anomalies for Hall conductances near integral multiples of $e^2/h$?

2. Why does the current flow without dissipation ($V_x = 0$) under these conditions?

3. Why does Eq. (1) hold so precisely, independent of details of the material?

The answers to these questions are now rather well understood, although there remain a number of related questions which are still unclear. [See, for example, Section IIIC, below.]

The answers to questions 1 and 2 may in fact be found in the 1975 paper of Ando et. al. written before the experiments of Ref. 1.

Before discussing the general answers to these questions, we will consider the special case of free electrons.

## A. Free Electrons

A heuristic explanation of the normal quantized Hall conductance is easy to produce if one considers an infinite system of free, non-interacting electrons, in a uniform background potential. In the presence of the magnetic field, the energy states of the electron are a series of discrete Landau levels, with energies

$$E_n = \hbar\omega_c(n + 1/2), \tag{3}$$

where $\omega_c$ is the cyclotron frequency

$$\omega_c = eB/m^*c, \tag{4}$$

where $m^*$ is the mass (or effective mass) of the electrons. As one must accommodate a large number of electrons, proportional to the area of the system, in a discrete set of energy levels, it is clear that each Landau level is highly degenerate. The number of electron states per unit area, in a single Landau level, (ignoring the spin degree of freedom) can be shown to be equal to $B/\Phi_0$, where $\Phi_0$ is the flux quantum

$$\Phi_0 = hc/e. \tag{5}$$

If the Fermi level $E_F$ is chosen at random, then except for a coincidence of probability zero, the Fermi level will lie in the gap between two Landau levels. Then at $T = 0$, there will be an integral number n of filled Landau levels, and the density N of electrons is given by

$$N = nB/\Phi_0 . \tag{6}$$

Equation (1) is true for quantum mechanical free electrons as well as for classical electrons. (It is also correct in the presence of the Coulomb interactions between the electrons.) Thus, combining (1), (5) and (6), we are lead immediately to the quantized Hall formula (2).

When the spin degree of freedom is taken into account, there are two sets of electronic levels for each Landau level, one for spin up and one for spin down, each one containing a

density $B/\Phi_0$ of electron states. Then, if the Fermi level is midway between two Landau levels, there are twice as many electrons as before, and the quantized Hall formula (2) holds, with even n. On the other hand, if the Fermi level is chosen to lie in between the spin-up energy and spin-down energy of a certain Landau level, then Eq. (2) will hold with n odd. [In Si inversion layers, there is an additional factor of two from the "valley degeneracy" of the conduction electrons, so that n becomes a multiple of four, when the Fermi level is between two Landau levels.]

Unfortunately, there are several important problems if one attempts to apply this heuristic argument to the actual physical systems. In the first place, although the argument will work for almost all choices of Fermi level, these correspond to a discrete set of carrier densities, for a given magnetic field strength. If, instead, the carrier density is specified, then, with probability unity, the Fermi level will always wind up in the middle of a Landau level, so that the last Landau level is partially occupied, which is precisely the condition where the argument breaks down and where the Hall conductance is not a multiple of $e^2/h$. Because of the requirements of charge neutrality, the total number of electrons in the vicinity of the conducting layer is generally fixed in the experiments. Then in order to explain the observed Hall plateaus, it would be necessary to hypothesize a rather large reservoir of electrons, close to the conducting layer, which can transfer electrons to and from the layer, and keep the density of carriers N equal to $nB/\Phi_0$ over some range of magnetic field B.[11,12]

In addition, this heuristic argument cannot explain the precision of the quantized Hall effect. The argument ignores such factors as effects of the boundary of the sample, deviations from a simple effective mass description, and most important, the effects of impurities, all of which one might naturally expect to cause deviations from Eq. (2), long before the level of one part in $10^{-6}$ is attained.

In fact, in the presence of impurities or other dis-

order, Eq. (1) is no longer exact, and the number of states in a Landau level may not even be precisely defined. Thus the intermediate steps in our heuristic derivation are incorrect, and only the final result, Eq. (2), remains correct. Clearly a more general derivation is necessary.

## B. Effects of Impurities

Let us here review the qualitative effects of a small random potential, due to impurities, on the states of a non-interacting, two-dimensional electron system in strong magnetic field. In this discussion, we shall ignore the effects of the sample boundaries.

The first effect of the disordered potential is to lift the perfect degeneracy of the states in a given Landau level, and to broaden each level into an energy band of finite width. Naively, at least, one would expect the states near the center of each band to be extended in space, while states near the edge of each Landau band will be localized.[2] In addition, if there is a small density of strong impurity potentials, there may be an associated small density of strongly localized states in the gaps between the Landau bands. We shall see that the quantized Hall effect occurs characteristically when the Fermi level is in a region of localized states or in an energy gap.

Now we can address question 2, posed above, why there is no voltage drop parallel to the current $I_x$ at zero temperature. Let us first observe that if we introduce a conductivity tensor $\overleftrightarrow{\sigma}$, by

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E} , \tag{7}$$

where $\vec{E}$ is the electric field and $\vec{j}$ is the current density, then the absence of dissipation may be expressed as the vanishing of the diagonal components of $\overleftrightarrow{\sigma}$,

$$\sigma_{xx} = \sigma_{yy} = 0 , \tag{8}$$

while the off diagonal components are finite

$$\sigma_{yx} = -\sigma_{xy} = 1/\rho_{xy} . \tag{9}$$

Thus the current flows perpendicular to the electric field, and no work is done.

The diagonal portion of the conductivity tensor can be

expressed, via the Kubo formula, entirely in terms of matrix elements of states at the Fermi surface. If these states are localized, then $\sigma_{xx}$ and $\sigma_{yy}$ are found to vanish, at T = 0. However, the off-diagonal elements of $\overleftrightarrow{\sigma}$ are determined by all the states below the Fermi level, and $\sigma_{yx}$ can be non-zero if any of the states below the Fermi level are extended. [We may note that at non-zero temperatures, there will be a finite contribution to $\sigma_{xx}$ and $\sigma_{yy}$ due to phonon-assisted tunneling ("hopping") between localized states near the Fermi level; however, this conductivity vanishes very rapidly at low temperatures and can be made arbitrarily small in practice.[7,13]]

It is plausible that in the presence of weak randomness, $\sigma_{yx}$ should be at least approximately equal to the free-electron quantized Hall value, when the Fermi level is in the middle of the gap between two Landau levels. Furthermore, the value of $\sigma_{yx}$ must remain constant, if the Fermi level is varied within the region of localized states at the edge of a Landau band, since the localized states do not contribute to the current flow. The reasoning above does not prove, however, that the constant $\sigma_{yx}$ is precisely the quantized Hall value. Here two different lines of argument have been used. One line of reasoning utilizes the Kubo formula for $\sigma_{yx}$, and shows that when the Fermi level is midway between two Landau levels, the value of $\sigma_{yx}$ is independent of the disordered potential, provided the strength of the random potential remains below some threshold value.[14-16] The other approach, introduced by Laughlin, uses arguments of Gauge invariance, and seems to be rather more general.[17] In particular, the Laughlin argument easily shows that there are no significant corrections from the lateral edges of the sample, and the argument also handles immediately the case where the voltage difference between the two edges is due in part to a chemical potential difference in addition to an electrostatic potential difference across the sample.[18] Moreover, Laughlin does not assume that the spatially varying potential is weak; he shows quite generally that <u>whenever the Fermi level lies in a gap or a region of localized states</u>, then $\sigma_{xx} = \sigma_{yy} = 0$ <u>and</u>

$\sigma_{yx} = -\sigma_{xy} = ne^2/h$ where n is an integer (possibly zero). The value of n may be quite different than in the absence of the potential however, and may vary in a complicated manner from one plateau to the next, as has been demonstrated quite dramatically in the case of some models with a periodic potential.[19-21] These conclusions remain valid in the presence of the electron-electron interaction, provided that the interaction has not qualitatively changed the nature of the excitations near the Fermi level. The (normal) quantized Hall effect is observed provided that $n \neq 0$.

I shall not reproduce here Laughlin's argument for the exactness of Eq. (2). However, I shall give in Section V another derivation, valid for an infinite system under somewhat restricted circumstances. The reader is also referred to the original literature, and particularly to a number of recent articles which examine in detail the quantized Hall phenomena in various specific models.[18-30]

### C. Fraction of Localized States

An important open question is the proportion of localized versus extended states in each Landau band. For non-interacting electrons, this fraction will determine the width of the magnetic field intervals corresponding to the plateaus of the quantized Hall conductance.

The fraction of localized states, in a given Landau level may depend on the ratio of $\xi$, the correlation length for the disordered potential, to the magnetic length $\ell$, defined by

$$2\pi\ell^2 = \Phi_0/B. \tag{10}$$

[The length $\ell$ is roughly the distance between flux quanta, or the cyclotron radius of an electron in the lowest Landau level.] If the electron-electron interaction is neglected, the fraction of localized states should be independent of the absolute magnitude of the disordered potential, if the potential is small compared to the cyclotron energy. For further discussions of this subject, the reader is referred to various theoretical and experimental investigations in the literature.[7, 22, 26-35]

It can be shown that there will always be at least one energy near the center of each Landau band where the states

remain delocalized, as long as the disorder is not too strong.[14,18]

## IV. THE ANOMALOUS QUANTIZED HALL EFFECT

As mentioned in the introduction, fractional quantized Hall steps have been observed only in samples of very high mobility. (This has been achieved by separating spatially the positive donors from the conducting layer.) Thus it is natural to suppose that the electron-electron interaction is more important than the impurity potentials in lifting the degeneracy of the Landau level in those samples. Thus an explanation of the fractional Hall steps should probably begin by trying to determine the properties of a two-dimensional collection of interacting electrons in a uniform positive background, when the first Landau level is partially filled. The impurities may then be considered afterwards as a perturbation.

The dimensionless measure of the density of the electrons in our system is the filling fraction of the Landau level,

$$\nu \equiv N\Phi_0/B, \tag{11}$$

The natural unit of the potential energy, in the electron system is $e^2/\varepsilon\ell$, where $\varepsilon$ is the background dielectric constant and $\ell$ is the magnetic length defined by Eq. (10). For the magnetic fields employed in the fractional Hall step observations, $B \gtrsim 10^5$ gauss, one finds, using the values $\varepsilon \approx 13$, $m^* \approx 0.07\ m_e$, appropriate to GaAs, that $e^2/\varepsilon\ell \lesssim \hbar\omega_c$. Thus it appears also to be a good starting approximation to ignore the admixture of states above the first Landau level, and to consider an effective Hamiltonian which has only matrix elements of the Coulomb interaction between states of the first Landau level.

In view of the strong magnetic fields involved, we shall also assume that all the electron spins are parallel to the magnetic field, and only one spin state need be considered. [Actually, this assumption may be questioned, because of the unusually small g-factor of electrons in GaAs. See Subsection F, below.]

If we assume that all spins are parallel to the magnetic field, and if we neglect admixtures of higher Landau level,

then the only energy scale in the problem is that of the potential energy $e^2/\varepsilon\ell$, provided that we subtract off the constant kinetic energy per electron of $1/2\,\hbar\omega_c$. Then the energy per particle of the system, measured in units of $e^2/\varepsilon\ell$, can depend only on the single variable $\nu$. One can show, also, that there is an electron-hole symmetry in the problem so that the total energy at filling factor $\nu$ will differ from the energy at $(1 - \nu)$ by a simple quadratic function of $\nu$.

A. Wanted: A Cusp in the Energy

What is necessary to explain the fractional Hall plateaus is that the system of electrons in a uniform positive background have an energetic preference for certain rational values of the filling fraction $\nu$ of the Landau level. Specifically we would like to find a downward cusp in the total energy $\bar{E}$ of the system at the value $\nu = 1/3$, (and any other rationals where an anomaly may be observed.) The existence of a cusp in $\bar{E}$ implies a discontinuity in the chemical potential, $E_F = \partial\bar{E}/\partial N$, at the density in question, which implies, in turn, that the density remains pinned at $\nu = 1/3$ for some range of values of the chemical potential. If we then assume an external reservoir of electrons close to the accumulation layer, which can transfer electrons into the layer, we will find that $\nu = 1/3$ for some range of magnetic fields. Since Eq. (1), for the Hall conductance of electrons in a uniform background, is still valid in the presence of the electron-electron interaction, we have

$$1/\rho_{xy} = \nu e^2/h \qquad (12)$$

and thus, clearly, a Hall conductance plateau at $e^2/3h$.

A discontinuity in $\partial\bar{E}/\partial N$ at $\nu = 1/3$ implies that there is an energy gap in the spectrum of charge carrying excitations, i.e., the energy cost to remove one electron must exceed the energy gained by adding an electron to the system. According to the analysis of Laughlin, the lowest energy charged excitations in this system should be quasiparticles and quasiholes with charge $\pm e/3$.[6] Then, the discontinuity in $\partial\bar{E}/\partial N$ should be three times the energy to create one quasiparticle and one hole, separated in space. The pinning of the density to $\nu = 1/3$ also

suggests that there will be no low frequency phonon-like excitations at long wavelengths.

In the absence of impurities, it is trivial that there is no dissipation associated with the current flow, and thus $\sigma_{xx} = \sigma_{yy} = 0$. The gap in the excitation spectrum also insures that there will be no dissipation in the presence of a weak random potential. (We shall assume the potential to be sufficiently weak so that it does not close up the energy gap.) In a frame of reference moving with the drift velocity $\vec{v}_D = (\vec{E} \times \vec{B})c/B^2$, the Lorentz transformed electric field is zero, and there is no electric current; however the impurities move relative to the frame with velocity $= -\vec{v}_D$. The time dependent potential due to the impurities will produce no excitations and will dissipate no energy, at low temperatures, as a result of the energy gap.

In Section V, we present an argument that the Hall conductance is also unaffected by a weak impurity potential, so that $\sigma_{xy}$ remains precisely equal to $1/3\ e^2/h$, while the density is pinned at $\nu = 1/3$.

The presence of impurities also allows one to explain the Hall plateau without hypothesizing a reservoir to keep the filling fraction pinned at $\nu = 1/3$. We may suppose that small deviations from $\nu = 1/3$ are accomplished by addition of a small density of quasiparticles or quasiholes to the ground state at $\nu = 1/3$. These excitations will be localized by any random potential present, and therefore will not contribute either to the Hall current or to the diagonal part of $\overleftrightarrow{\sigma}$, at $T = 0$.

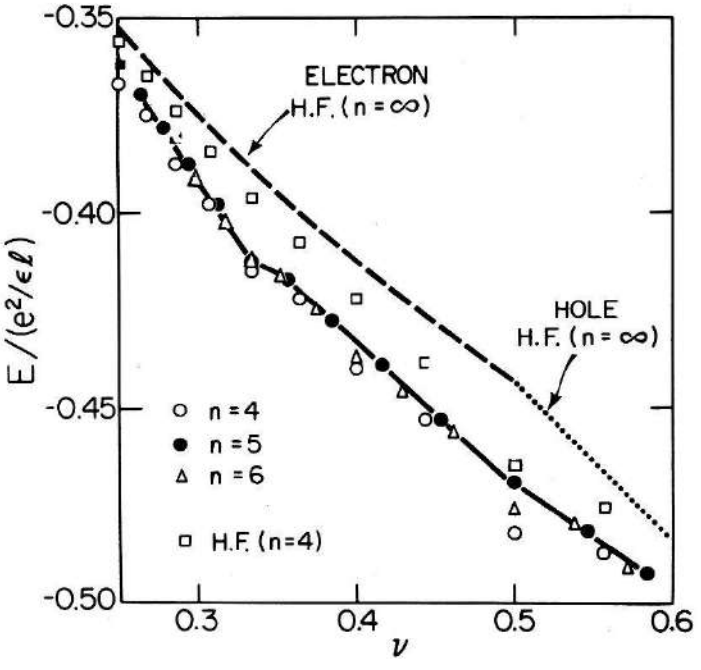
## B. Hartree Fock Approximation

The simplest non-trivial approximation to the energy of an interacting electron system in a uniform positive background potential is the Hartree-Fock approximation. Here, one finds the one-electron eigenstates in a self consistent potential (direct + exchange) that is calculated in turn from the filled eigenstates. When the last Landau level is only partially filled, the uniform electron system is always unstable, in the Hartree Fock approximation, to the formation of charge density waves, at

low temperatures.[36] This suggests that the lowest Hartree Fock state is a Wigner crystal. Hartree-Fock calculations of various crystal structures for electrons in the first Landau level, found a charge density wave ground state which is a triangular lattice of electrons, when the Landau level is less than half full, and a triangular lattice of holes, when the level is more than half full.[37-39]

The energy per electron of these lattices as a function of filling fraction $\nu$, are indicated by the broken and dotted curves in Fig. 1. As can be seen, there is no indication of any singularity in the Hartree Fock energy at $\nu = 1/3$. Furthermore, Yoshioka and Lee have calculated the first non-vanishing correction to the Hartree Fock energy of the crystal.[38] This correction is numerically small, and smooth at $\nu = 1/3$. Thus, if the ground state were indeed a Wigner cyrstal, its

Figure 1. The energies per particle of 2-d electron systems are shown, vs. the fractional filling of the first Landau level. The dashed and dotted lines show energy of the electron and hole crystals resulting from the Hartree-Fock approximation for the infinite system.[38] Open circles, closed circles, and triangles show the ground state energies for n = 4,5, and 6 electrons for $\nu \leq 1/2$ and n = 4,5,6 holes for $\nu > 1/2$. Open squares show the energy of the crystal state for the n = 4 system obtained by the Hartree-Fock approximation. The solid line drawn through the n = 5 ground state energies is a guide to the eye only. (Figure taken from Ref. 40.)



energy would presumable be regular at $\nu = 1/3$ and close in value to the Hartree-Fock approximation.

Even if there were a downward cusp in the energy of the crystal at $\nu = 1/3$, it might be difficult to use this state to explain the anomalous Hall conductance, as one might expect the charge density wave to become pinned by the impurities in a real system. If the charge density wave is not allowed to move with the drift velocity $\vec{E} \times \vec{B}c/B^2$, then the contribution of the electrons to $\sigma_{xy}$ is found to vanish.[21]

Thus one is led to suppose that the true ground state is not a Wigner crystal, but some kind of translationally invariant "liquid", that would not get pinned by the impurities, and whose energy is not well approximated by the Hartree Fock crystal energy.

## C. Solution of Finite Systems

In order to investigate the nature of the ground state, Yoshioka, Halperin and Lee have studied numerically the energy eigenstates of a collection of n electrons, with n = 4,5, and 6, in a rectangular box with "periodic" boundary conditions.[40] The Hamiltonian is restricted to the first Landau level, and only one spin component is considered. The boundary condition requires that there be an integral number m of flux quanta in the box; thus, the filling fraction $\nu$ takes on rational values of the form n/m. The ground state energies per electron for the calculated systems, in the range $.25 \leq \nu < .5$, with a particular choice n/4 for the aspect ratio a/b of the rectangle, are shown as triangles and circles in Fig. 1. (Data for $\nu > 1/2$ are for systems of n holes, obtained by particle-hole symmetry.) The points for n = 5 are connected by a solid line, as a guide to the eye. Also shown, as open squares, are the energies of the Hartree-Fock crystal for n = 4, which are slightly lower than the Hartree-Fock energies of the infinite system (broken and dotted curves).

It is apparent from Fig. 1 that the actual ground state energies are significantly lower than the Hartree Fock energies, in this range of $\nu$. Also, analysis of the ground states

showed that the pair correlation function $g(\vec{r})$ is rather different from that of the Wigner crystal. States resembling the crystal were found at higher energies, for some of the filling factors studied. These results support the view that the true ground state of the system is not a Wigner crystal.

The data in figure 1 show dips at simple rational values of $\nu$, such as $\nu = 1/3$, $2/5$ and $1/2$. Obviously one cannot reliably extrapolate these results to $n = \infty$. It is interesting to note, however, that the energy at $\nu = 1/3$ is essentially independent of n, for $n = 4, 5, 6$, while the data at $\nu = 1/2$ vary strongly, and non-monotonically with n. The data for $n = 5$, connected by the solid curve in Fig. 1, show only a small anomaly at $\nu = 1/2$.

It should be noted here that calculations of small finite systems by R.B. Laughlin had also given indications of a cusp in the energy at a density corresponding to $\nu = 1/3$. However, those calculations did not use periodic boundary conditions, so that extrapolation to infinite systems seems even more difficult in that case.[41]

## D. Laughlin's Wavefunction

Although our previous considerations have led us to expect that the ground state of the system is not a Wigner crystal for the densities of interest, we have not gained much insight into the nature of the ground state wave function, nor the reason it might become pinned at certain rational values of the filling factor. The key idea here has been supplied very recently by Laughlin.[6]

Laughlin has proposed a wave function of the form

$$\psi(\vec{r}_1 \ldots \vec{r}_N) = \left[\prod_{j<k} (z_j - z_k)^m\right]\left[\prod_j e^{-|z_j|^2/4\ell^2}\right] \qquad (13)$$

where $z_j$ is the coordinate of the $j^{th}$ electron, in complex notation:

$$z_j = x_j - iy_j \qquad (14)$$

This wave function obeys Fermi statistics, provided that m is an odd integer. [For even m, the wavefunction obeys Bose statistics.] Furthermore, the wavefunction is entirely made up out of

states in the first Landau level, so that it is an eigenstate of the kinetic energy operator, with the minimum kinetic energy of $1/2\ \hbar\omega_c$ per electron.

It is not immediately obvious that the wavefunction (13) has a uniform electron density. This becomes clear, however if we write the square of the wavefunction in the form

$$|\psi|^2 \equiv e^{-\beta H}, \tag{15}$$

$$\beta H \equiv 2m \sum_{j<k} \ell n|\vec{r}_j - \vec{r}_k| + \frac{1}{2\ell^2} \sum_j |\vec{r}_j|^2. \tag{16}$$

Thus $|\psi|^2$ is just the probability distribution of a classical one component plasma in two dimensions. The first term in (16) represents the logarithmic repulsion between the "charges" in two-dimensions, while the second term is the interaction with a uniform "charged background". It is clear that the equilibrium state of the plasma has the particles spread out over a disk centered at the origin, with a density such as to just neutralize the uniform background within the disk. The density of particles inside the disk is such that

$$\nu = 1/m. \tag{17}$$

It is known from previous numerical studies that the one component plasma forms a crystal at very large values of the parameter m, but that Eq. (15) describes a liquid for $m \lesssim 70$.[6, 42]

We remark that for m = 1, Equation 13 is a Slater determinant that describes a completely filled Landau level. For m = 0, the wavefunction describes a collection of charged bosons that have all condensed into a single Landau orbit localized at the origin. We can further understand the significance of the wavefunction (13) if we fix the positions of all electrons save one and examine the wavefunction as a function of the remaining variable $z_j$. If we move the electron around a close loop of area A, avoiding other electrons in the system, then the phase of the wavefunction changes by an amount $\Delta\phi \sim 2\pi AB/\Phi_0$, when A is large. This is actually required quite generally if we are to avoid having an enormous kinetic energy

for the electron. Thus the wavefunction must contain $AB/\Phi_0$ "vortices" inside the area A -- where a vortex is defined as a point where the wavefunction is zero, and such that the phase changes by $2\pi$ as one circulates the vortex in the negative direction. There must be at least one vortex at the position of each electron $z_k$, for $k \neq j$, in order to satisfy the Pauli principle. The special feature of the Laughlin wavefunction is that there are precisely m vortices at each electron position $z_k$, and no other "wasted" vortices elsewhere in the sample.

It is clear that the m-fold vanishings of the wavefunction, whenever two electrons come together, tend to keep the electrons very well separated, and tend to minimize potential energy of the system. The tendency towards charge neutrality of the classical plasma also tends to eliminate long wavelength fluctuations in the density, which would be costly to the potential energy, although the plasma does not appear to be quite as efficient in this respect as the Hartree Fock solid.

Using various calculations of the pair correlation function of the classical plasma, Laughlin has obtained the expectation value of the potential energy in the wavefunction (13). The result for $m = 3$ is in close agreement with the exact calculations for the systems of four-to-six electrons with periodic boundary conditions (shown in Fig. 1) at $\nu = 1/3$, and is well below the energy of the Hartree Fock solid.

If one tries to vary the electron density from the allowed values $\nu = 1/m$, with m odd, it is no longer possible to construct a wavefunction with precisely m vortices tied to each electron. For $\nu$ slightly less than $1/m$, the simplest way to achieve the necessary electron density is to add a small density of extra vortices, not tied to the positions of the electrons. For example one can add a vortex at point $x_o$, $y_o$ by multiplying the right hand side of Eq. (13) by the product $\prod_j (z_j - z_0)$, where $z_0 = x_0 - iy_0$. The square of the wavefunction is then the probability distribution for a classical plasma with a fixed "charge" of magnitude $1/m$ at point $z_0$. This fictious charge

will be screened by the creation of a "hole" of area $1/m$, in the otherwise uniform distribution of the real electrons. It is clear that this hole will increase the actual potential energy of the system relative to what one would have if the charge density could be uniform. Laughlin has estimated this energy as $0.022e^2/\varepsilon\ell$, for $m = 3$.[43]

It is somewhat more difficult to construct the wavefunction for a system with a deficiency of one vortex line near a specified point $z_0$. [See discussion following Eqs. (22)-(24) below.] Nonetheless it is clear that the system will accumulate an extra charge of $1/m$ electron in the vicinity of the missing fluxoid, and it is again clear that there will be a price in potential energy.

If the filling factor $\nu$ is slightly removed from the value $1/m$, with $m$ odd, then the ground state of the system should consist of a small density of quasiparticles or quasiholes, each of charge $\pm e/m$, held apart by their Coulomb repulsion. Presumably, in an ideal system, at $T = 0$ these quasiparticles or quasiholes will form a regular triangular lattice, whose lattice constant diverges as $|\nu - 1/m|^{-1/2}$ for $\nu \to 1/m$. In the presence of impurities, however, we would expect that quasiparticles or quasiholes will generally become trapped in fluctuations of the potential, at small concentrations.

Finally, one may ask what happens to the wavefunction (13), when we take account of mixing with the higher Landau levels. If all electrons but one are held fixed, in this case, and we consider the dependence on the remaining variable $\vec{r}_j$, we will again find $m$-vortices close to the position of each electron $\vec{r}_k$. In general, however, only one of the vortices will coincide with the position $\vec{r}_k$; the remaining zeroes of the wavefunction being displaced slightly, depending on the positions of all the other electrons in the vicinity of $\vec{r}_k$. Nonetheless, we may characterize the wavefunction as having $m$ vortices bound to each electron.

### E. Other Rational Fractions

Laughlin's results for $\nu = 1/m$ also imply a commensur-

ate locking energy at $\nu = 1 - (1/m)$, in view of the electron-hole symmetry noted above. Similarly, we can understand the observed anomalies at $\nu = 4/3$ and $\nu = 5/3$ by taking into account the second spin state of the electrons. However, Laughlin's results do not directly explain the observed anomalies at $\nu = 2/7$, $2/5$ and $3/5$. Also, there is the possibility that anomalies will be discovered in the future at rational fractions with even denominators. Here we propose some generalizations of Laughlin's ideas, which permit the construction of trial wavefunctions at arbitrary rational filling fractions in the range $0 < \nu < 1$; these wavefunctions will describe a translationally invariant liquid for at least the simplest cases.

One possible generalization of Laughlin's ideas is to group the electrons in the system into bound pairs, and then to construct a wavefunction of the Laughlin type, of even degree m, for the pairs. (Note that the pairs will transform as bosons under interchange of their positions.) This construction leads to a statistically uniform density of electrons with filling factor $\nu = 4/m$. Now if $m = 4p + 2$, with p an integer, this gives

$$\nu = 2/(2p + 1). \tag{18}$$

If $m = 4p$, this construction gives

$$\nu = 1/p \tag{19}$$

for arbitrary integer p, including even values. The possibility that the lowest energy state for $\nu = 1/2$ contains paired electrons is particularly attractive in view of the observed oscillations in the energy of the systems containing four, five and six electrons, which was noted above. It will be most interesting to see if a weak Hall step is eventually observed at $\nu = 1/2$.

In a similar manner one may group the electrons into n-tuplets, and construct a Laughlin wavefunction of degree m for the n-tuplets, where m and n must have the same parity. This construction gives

$$\nu = n^2/m \tag{20}$$

which can, in principle, reach any rational filling fraction. Other possibilities include mixed wavefunctions, of paired and

unpaired electrons, which will be described below.

Of course, we have no guarantee that the wavefunctions constructed as proposed here will yield the lowest energy state for the given value of $\nu$. A quantized Hall plateau will result only if the groundstate at the given $\nu$ is a translationally invariant liquid, and the width of the plateau will be proportional to the energy required to create a quasiparticle or hole (ie, subtract or add a flux quantum) in this groundstate.

To make these ideas more precise we can consider an explicit trial wavefunction of the pair type. Let us write

$$\psi(\vec{r}_1 \ldots \vec{r}_{2N}) = A\tilde{\psi}(\vec{r}_1 \ldots \vec{r}_{2N}) , \tag{21}$$

where A is the antisymmetrization operator, and

$$\tilde{\psi} = [\prod_i e^{-|z_i|^2/4\ell^2}] \; [\prod_{i<j} (z_i - z_j)^s]$$
$$\times [\prod_k (z_{2k-1} - z_{2k})^{-t}] \; [\prod_{k<n} (z_{2k-1} + z_{2k} - z_{2n-1} - z_{2n})^{2u}] \tag{22}$$

where the indices i and j run from 1 to 2N, while k and n run from 1 to N. The quantities s, t, and u are integers, with the conditions $s > 0$, $u \geq 0$, $(s - t) > 0$, and $(s - t)$ odd. Also, either u, or t or both must be $> 0$. The square of the wavefunction $\tilde{\psi}$ may be interpreted as a probability distribution for pairs of classical particles interacting logarithmically with each other and with a uniform background; the various terms in (22) produce, respectively, a logarithmic repulsion between all particles, a reduction of the repulsion between members of a pair, and a repulsion between the centers of gravity of different pairs. One may confirm that $|\tilde{\psi}|^2$ gives rise to an electron filling factor $\nu = 4/m$, with $m = 4s + 2u$.

The wavefunction $\tilde{\psi}$ is antisymmetric under interchange of two electrons that are members of the same pair, and it is symmetric, as required, under interchange of pairs. It is not antisymmetric under the interchange of two electrons that are members of different pairs. However, the antisymmetrization operation will be relatively innocuous, having little effect on the normalization, the potential energy, or the electron density

correlation function, provided that there is little overlap in the regions of configuration space where the wavefunctions $\Im$ and $P_{ij}\Im$ are maximum. (Here $P_{ij}$ interchanges the positions of two electrons from different pairs.) We expect that this will be the case if the pairs in $\Im$ are sufficiently tightly bound, relative to the separation between pairs. On the other hand, we do not wish to bind the pairs too tightly, as this will be costly in the potential energy of the pair. It seems likely that the lowest energies will result from the smallest possible values of t and u.

We remark that for $u \neq 0$, the wavefunction $\Im$ will decrease in a Gaussian fashion, if we attempt to separate the two members of any pair by a distance larger than the mean separation of pairs.

As one example, we may consider the case s = 2, t = 1, u = 1. This gives a reasonable trial wavefunction for $\nu = 2/5$. The choice s = 3, t = 0, u = 1, gives a possible function for $\nu = 2/7$, while s = 1, t = 0, u = 2 gives a wavefunction with $\nu = 1/2$.

To construct a mixed wavefunction of pairs and unpaired electrons, we write $\psi = A\Im'$, where A is the antisymmetrizer, and

$$\psi'(\vec{r}_1 \ldots \vec{r}_{2N+M}) = \Im(\vec{r}_1 \ldots \vec{r}_{2N}) \times \prod_{\mu} e^{-|z_\mu|^2/4\ell^2} \times \prod_{\mu<\nu} (z_\mu - z_\nu)^m \prod_{i,\mu} (z_i - z_\mu)^q \times \prod_{k,\mu} (\tfrac{1}{2}z_{2k-1} + \tfrac{1}{2}z_{2k} - z_\mu)^w \qquad (23)$$

Here $\Im$ is the unsymmetrized pair wavefunction defined in (22), the indices $\mu$ and $\nu$ for the unpaired electrons run from 2N+1 to 2N+M, while i runs from 1 to 2N and k runs from 1 to N as in (22). The exponent m, which controls the correlations between unpaired electrons must be odd; the exponent $p \equiv 4s + 2u$ for the correlation between pairs is automatically even, while the exponent $r \equiv 2q + w$ governing electron-pair correlations can have either parity. The requirement that the pairs and the single electrons cover the same area of space leads to a require-

ment that $N/M = (2m-r)/(p-2r)$. The total number $(2N+M)$ of electrons in this area then leads to a filling factor

$$\nu = \frac{4m + p - 4r}{mp - r^2} . \tag{24}$$

As an example here, we may consider the choice $m = 3$, $r = 5$, and $p > 10$. Now, the wave function for a free electron sees three vortices at every site of another free electron, while there are only five vortices associated with each pair. Thus there is one missing fluxoid for each pair. A single pair is thus a model for the quasiparticle excitation in Laughlin's state of $\nu = 1/3$. (To have the lowest potential energy for the wavefunction, it seems likely that one should choose $q = 2$, $w = 1$, $(s-t) = 1$, and $2s$ as close as possible to the value of $p$.) If we choose $p$ to be large, then the density of pairs will be small, and the value of $\nu$ approaches $1/3$, from above.

The analogy between quasiparticles and quasiholes, suggests that one might also treat the quasihole excitations (extra flux quanta) as bosons, and form analogous functions describing a quantum liquid of quasiholes, added to the simple state $\nu = 1/m$. Indeed, this turns out to be the case. We can construct a wavefunction of the required form by writing

$$\begin{aligned}\psi(\vec{r}_1 \ldots \vec{r}_N) = \int d^2\eta_1 \ldots d^2\eta_M \; & \Big[\prod_{i<j} (z_i - z_j)^m\Big] \\ & \times \Big[\prod_{k<m} (\eta_k^* - \eta_m^*)^p\Big] \Big[\prod_{i,k} (z_i - \eta_k)\Big] \\ & \times \Big[\prod_i e^{-|z_i|^2/4\ell^2}\Big] \Big[\prod_k e^{-\nu|\eta_k|^2/2\ell^2}\Big] , \end{aligned} \tag{25}$$

where $m$ is odd and $p$ is an even integer $> 0$; the indices $i$ and $j$ run from 1 to $N$ while $k$ and $m$ run from 1 to $M$, with $M = N/p$. The complex variables $\eta_k$ represent the positions of the free vortices, or quasiholes, and the electron filling factor is given by $\nu = p/(mp+1)$. Thus, for $p = 2$, $m = 3$, we obtain a state with $\nu = 2/7$, while for $p = 2$, $m = 1$, we obtain a representation of the state $\nu = 2/3$. A derivation of this wavefunction will be given elsewhere.

I note, in closing this section, that Laughlin has also suggested a formal method for constructing wavefunctions with arbitrary rational filling factors of odd denominator.[43] It is not clear what relation there may be between his wavefunctions and the ones proposed above.

## F. Possibility of Reversed Spins

In all of our previous discussion of the partially filled Landau level, we have assumed that only one spin state is occupied. In GaAs, the electron g-factor is 1/4 the free electron value,[44] while $m^* \approx .07\, m_e$. Thus the Zeeman energy is approximately 60 times smaller than the cyclotron energy, and it is similar in scale to the quasiparticle energies of the anomalous Hall state in magnetic fields of order 10T. Hence it cannot be completely ruled out that the ground state for some values of $\nu$ will contain some electrons with reversed spins, when B and N are not too large. [Note that $(e^2/\varepsilon\ell) \propto B^{\frac{1}{2}}$ while $E_{Zeeman} \propto B$.]

As an example, if one half of the electrons have spins antiparallel to the field, we may construct a rather simple state with a total density corresponding to $\nu = 2/5$:

$$\psi = \left[\prod_{i<j}(z_i - z_j)^3\right]\left[\prod_{k<m}(\tilde{z}_k - \tilde{z}_m)^3\right]\left[\prod_{i,k}(z_i - \tilde{z}_k)^2\right] \prod_i e^{-|z_i|^2/4\ell^2} \prod_k e^{-|\tilde{z}_k|^2/4\ell^2} \tag{26}$$

Here z and $\tilde{z}$ are used for the coordinates of spin down electrons and spin up electrons respectively. It will be interesting to compute the energy of this state and compare it with the lowest energy one can find for a spin-alligned $\nu = 2/5$ state.

## V: PRECISION OF THE QUANTIZED HALL CONDUCTANCE

I shall present here an argument for the exactness of the quantized Hall plateaus at integral or fractional values of the quantum $e^2/h$. The argument will be given only for the case of an infinite homogeneous sample. We assume that there is an energy gap in the quasiparticle spectrum and a discontinuity in $\partial E/\partial N$, the derivative of the energy with respect to electron density, for the ideal system without impurity scattering, at a given rational filling factor $\nu_0$, and we assume the disordered

potential to be sufficiently weak so that the energy gap and the discontinuity in $\partial E/\partial N$ persist in the presence of the impurities. Our demonstration will make use of the formula of Widom and Středa,[45,20] which is valid under these circumstances.

Let us consider the change in energy of the sample in the presence of a spatially varying external potential $V_{ext}(\vec{r})$ and a magnetic field $B(\vec{r})$. In the limit of long-wavelength variations, we can write

$$E = \int d^2r \{E[N(\vec{r}), B(\vec{r})] + eV_{ext}(\vec{r}) N(\vec{r})\} + e^2/2 \iint d^2r d^2r' \delta N(\vec{r}) u(\vec{r}-\vec{r}') \delta N(\vec{r}'), \quad (27)$$

where $\delta N(\vec{r})$ is the change in electron density at point $\vec{r}$, and $u(\vec{r}-\vec{r}')$ is the Coulomb interaction between charges at points $\vec{r}$ and $\vec{r}'$. (It is convenient to imagine that there is a ground plane some distance from the electron layer, so that $u(\vec{r}-\vec{r}')$ is cut off at large distances, and a uniform variation of N is possible with a finite cost in energy per electron.) The function E(N,B) is the energy density of an infinite system with constant magnetic field B and constant macroscopic electron density N, in the presence of the random impurity potential plus a uniform positive background charge adjusted to give overall charge neutrality.

Now, under equilibrium conditions, we may write

$$\frac{\delta E}{\delta V_{ext}(\vec{r})} = eN(\vec{r}), \quad \frac{\delta E}{\delta B(\vec{r})} = -M(\vec{r}) \quad (28)$$

where $M(\vec{r})$ is the magnetization density, oriented perpendicular to the plane. Equilibrium conditions, in turn, imply that we have minimized the thermodynamic potential, $E-\mu N$. Ordinarily, this condition means that

$$\frac{\delta E}{\delta N} + eV(\vec{r}) = \mu \quad (29)$$

where $\mu$ is a constant, independent of $\vec{r}$, and

$$V(\vec{r}) = V_{ext}(\vec{r}) + e\int u(\vec{r}-\vec{r}') \delta N(\vec{r}') d^2r' \quad (30)$$

In the present case, however, Eq. (29) must be replaced by

$$N = \nu_0 B/\Phi_0, \quad (31)$$

when $\mu$-eV satisfies the inequality

$$\frac{\partial E^-}{\partial N} < \mu - eV < \frac{\partial E^+}{\partial N} \tag{32}$$

where $\partial E^{\pm}/\partial N$ are the two limiting values of $\partial E/\partial N$, for $\nu \to \nu_0$. From (28) and (31), we see that

$$\left.\frac{\partial M}{\partial V_{ext}}\right|_{\mu,B} = \left.e\frac{\partial N}{\partial B}\right|_{\mu,\ V_{ext}} = \frac{\nu_0 e^2}{hc} \tag{33}$$

when (32) is satisfied.

Now we may consider an equilibrium situation where B is held constant, and where $V_{ext}$ varies periodically in space, with $\nu$ in the energy gap (32). Then using the fact that $\partial N/\partial V_{ext} = 0$ and hence $\delta V = \delta V_{ext}$, we have the desired result

$$\vec{j} = c\vec{\nabla}\times\vec{M} = \nu_0 e^2 h^{-1}\ \hat{z}\times\vec{E}\ . \tag{34}$$

In the actual experiments, the potential is not a periodic function of position, and the chemical potential $\mu$ is not constant across the sample. A gradient of $\mu$ will clearly have no effect on the current in the sample, as long as the Fermi level is inside an energy gap. The gap vanishes at the edge of the sample, however, and there is a contribution from the sample edge which depends sensitively on $\mu$. It may be seen that if the sample is terminated symmetrically, and if $(eV-\mu)$ is the same on the two edges, then there will be no net contribution to the Hall current from the edges. In the more general case one finds that the integrated Hall current is determined by the difference in $\mu$, rather than the difference of the electrostatic potential V between the two edges of the sample. This is fortunate because it is precisely the difference in $\mu$ that would be measured by a voltmeter connected across the sample.[17, 18]

The argument given above cannot be applied directly to the case where the Fermi level lies in a region of localized states, rather than in a true energy gap. In fact, $(\partial M/\partial V)$ is no longer precisely equal to $\nu_0 e^2/hc$ under these conditions. However, as discussed earlier in this review, the localized states do not contribute to the transport current; hence the Hall current remains independent of $\mu$, when the chemical potential

enters a region of localized states.

The analysis becomes somewhat more complicated, if the disorder is increased to the point where there is no longer any region of vanishing density of states between the two Landau levels, or if localized quasiparticle and quasihole states are simultaneously occupied in the case of the anomalous Hall effect. We shall not discuss these situations here.

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