Introduction to Stable ∞-Categories

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October 7, 2013

Contents

1 Stable ∞-categories

Remark 1.1. The reference for most of the material in these notes can be found in §1.1 and §1.4 of [1].

Definition 1.2. Let $\mathcal{C}$ be an ∞-category and let $X \in \mathcal{C}$ be an object. We will say that X is a **zero object** of $\mathcal{C}$ if it is both initial and terminal. We will say that $\mathcal{C}$ is pointed if it contains a zero object. In this case the full subcategory spanned by zero objects is contractible. We will say that a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ between pointed ∞-categories is **reduced** if it preserves zero objects. We will denote by $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by reduced functors.

Remark 1.3. Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is naturally enriched in pointed spaces. More precisely, the natural enrichment of $\mathcal{C}$ over spaces lifts to pointed spaces where the base point of $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is given by the contractible subspace of maps which factor through a zero object.

Definition 1.4. Let $\mathcal{C}$ be a pointed ∞-category and let $0 \in \mathcal{C}$ be a zero object. A square of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{1.1}$$

will be called a **fiber sequence** if it is Cartesian and a **cofiber sequence** if it is coCartesian. When such a fiber sequence exists we will say that p **admits a fiber**, and when such a cofiber sequence exists we will say that f **admits a cofiber**.

Remark 1.5. Note that the underlying data of a square of the form 1.1 can be described as a pair of maps $A \xrightarrow{f} B \xrightarrow{p} C$ together with a **null-homotopy** of the composition $p \circ f$. This null-homotopy should be considered as part of the structure, and not a mere condition.

Definition 1.6. Let $\mathcal{C}$ be a pointed ∞-category. We will say that $\mathcal{C}$ is **stable** if

1. Every map admits both a fiber and a cofiber.

2. A square is a fiber sequence if and only if it is a cofiber sequence.

Remark 1.7. If $\mathcal{C}$ is a stable ∞-category then $\mathcal{C}^{\mathrm{op}}$ is stable as well.

Definition 1.8. Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories. We will say that a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a **exact** if it is reduced and preserves fiber/cofiber sequences. We will denote by $\mathrm{Fun}^{\mathrm{Ex}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by exact functors. We will denote by $\mathrm{Cat}^{\mathrm{Ex}}$ the (big) ∞-category of (small) exact ∞-categories and exact functors between them.

Example 1.9.

1. The stable homotopy category described in the last lecture can be identified with the homotopy category of a suitable stable ∞-category of spectra. In fact, the ∞-category of spectra is in some sense the universal stable ∞-category.

2. Let $\mathcal{A}$ be an abelian category with enough projectives/injectives. Then the corresponding derived category $\mathcal{D}(\mathcal{A})$ can be identified with the homotopy category of a suitable stable ∞-category of complexes.

Definition 1.10. Let $\mathcal{C}$ be a pointed ∞-category in which every map has a fiber. Let $\mathcal{M}^{\Omega} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ denote the full subcategory spanned by **Cartesian diagrams** of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & 0 \arrow[d, "p"] \\ 0 \arrow[r] & B \end{tikzcd} \tag{1.2}$$

We have two projection maps

$$\begin{tikzcd} & \mathcal{M}^{\Omega} \arrow[dl, "d_{0,0}"'] \arrow[dr, "d_{1,1}"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd}$$

which are given by restrictions along $\Delta^{\{0\}} \times \Delta^{\{0\}} \subseteq \Delta^1 \times \Delta^1$ and $\Delta^{\{1\}} \times \Delta^{\{1\}} \subseteq \Delta^1 \times \Delta^1$ respectively. Since Cartesian diagrams of the form 1.2 are right Kan extensions of their restriction to $\Delta^1 \coprod_{\Delta^{\{1\}}} \Delta^{\{1\}}$ we conclude that the map $d_{1,1}$ is a trivial Kan fiberation. Choosing a section $\mathcal{C} \longrightarrow \mathcal{M}^{\Omega}$ of $d_{1,1}$ and composing with the projection $d_{0,0} : \mathcal{M}^{\Omega} \longrightarrow \mathcal{C}$ we obtain a functor $\Omega_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ (well defined up to a contractible ambiguity) which we call the **loop functor**.

Similarly, if every map in $\mathcal{C}$ has a **cofiber**, then one can consider the full subcategory $\mathcal{M}^{\Sigma} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by diagrams as above which are **coCartesian**. Again, we have two projection maps

$$\begin{tikzcd} & \mathcal{M}^{\Sigma} \arrow[dl, "d_{0,0}"'] \arrow[dr, "d_{1,1}"] & \\ \mathcal{C} & & \mathcal{C} \end{tikzcd}$$

and the same argument would now show that the restriction map $d_{0,0} : \mathcal{M}^{\Sigma} \longrightarrow \mathcal{M}$ is a trivial Kan fibration. By choosing a section we can obtain a functor $\Sigma_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ (again, well defined up to a contractible ambiguity), which we shall call the **suspension functor**.

Remark 1.11. Let $\mathcal{C}$ be a pointed ∞-category in which every map has both a fiber and a cofiber. Then there exists a natural **adjunction** between $\Sigma_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}$

Now let $\mathcal{C}$ be a stable ∞-category. Since every square of the form 1.2 is Cartesian if and only if it is coCartesian we conclude that both $\Omega_{\mathcal{C}}$ and $\Sigma_{\mathcal{C}}$ are **equivalences**, and are inverse to each other. This fact has a strong consequence on the homotopy category of $\mathcal{C}$: if $A, B \in \mathcal{C}$ are two objects then the set $\pi_0 (\mathrm{Map}_{\mathcal{C}}(A, B))$ of homotopy classes of maps has a natural structure of an abelian group. To see this, choose an object A' and an equivalence $A \simeq \Sigma^2_{\mathcal{C}}(B')$. Then

$$\pi_0 (\mathrm{Map}_{\mathcal{C}}(A, B)) \cong \pi_0 (\mathrm{Map}_{\mathcal{C}}(\Sigma_{\mathcal{C}} A', B')) \cong \pi_0 \left(\Omega^2 \mathrm{Map}_{\mathcal{C}}(A', B)\right) = \pi_2 (\mathrm{Map}_{\mathcal{C}}(A', B))$$

In particular, the homotopy category of $\mathcal{C}$ is naturally enriched in abelian groups (in fact, it is **additive**. This follows from the discussion in the next section).

These abelian mapping groups can be often manipulated and computed using the construction of **long fiber/cofiber sequences**, which we shall now explain. Let $f : A \longrightarrow B$ be a map. Then we have a cofiber sequence

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd}$$

Since $\mathcal{C}$ is stable the map p itself admits a cofiber as well. Hence we can extend the diagram above to

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] & \\
0 \arrow[r] & C \arrow[r, "q"] & A' \arrow[r, equal, "\sim"] & \Sigma_{\mathcal{C}} A
\end{tikzcd}$$

where both squares are coCartesian. In this case the exterior rectangle will be coCartesian as well and so we can identify A' with $\Sigma_{\mathcal{C}} A$. Now the map q admits a cofiber as well. Continuing in this fashion we obtain a diagram on the form:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] & \\
0 \arrow[r] & C \arrow[r, "q"] \arrow[d] & \Sigma_{\mathcal{C}} A \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & \Sigma_{\mathcal{C}} B \arrow[r] & \Sigma_{\mathcal{C}} C
\end{tikzcd}$$

and so we obtain a sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma_{\mathcal{C}} A \longrightarrow \Sigma_{\mathcal{C}} B \longrightarrow \Sigma_{\mathcal{C}} C$$

in which every consecutive pair of maps admits a cofiber sequence structure (and hence also a fiber square structure). This sequence can be infinitely prolonged in both directions in an obvious way (where we use the fact that $\Sigma_{\mathcal{C}}$ is an equivalence). Mapping out of such a sequence (and mapping into such a sequence) will induce a long exact sequence of abelian groups on the corresponding groups of homotopy classes. In essence, this phenomenon is what makes the computation of homotopy classes of maps in stable ∞-categories considerably more tractable. Given all of the above, the following claim is not surprising:

Claim 1.12. *Let $\mathcal{C}$ be a stable ∞-category. Then* $\mathrm{Ho}(\mathcal{C})$ *has a natural structure of a **triangulated category**, where the shift functor is given by $\Sigma_{\mathcal{C}}$ and the distinguished triangles are given by the fiber/cofiber sequences.*

2 Limits and colimits in stable ∞-categories

We will begin with a few exercises:

Exercises 1.

1. Let $\mathcal{C}$ be a stable ∞-category. Show that $\mathcal{C}$ has all pushouts and all pullbacks.

Proof. In light of Remark 1.7 it will be enough to prove the first claim (the second will then follow by applying the same argument to $\mathcal{C}^{\mathrm{op}}$). Consider a diagram of the form

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & B \\ C & \end{tikzcd}$$

We wish to show that this diagram can be extended to a pushout square. Now the map f admits a fiber, and so we can extend the diagram above to a diagram of the form

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d] & A \arrow[r, "g"] \arrow[d, "f"] & B \\ 0 \arrow[r] & C & \end{tikzcd}$$

where the square on the left is a fiber sequence. Now the composition $g \circ h : F \longrightarrow B$ admits a cofiber and since the left square is also a **cofiber** sequence we can extend this diagram as

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d] & A \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

where the external rectangle is coCartesian. Using again the fact that the left square is coCartesian we deduce the right square must be coCartesian as well. $\square$

2. Let $\mathcal{C}$ be a stable ∞-category. Show that a square in $\mathcal{C}$ is a pushout square if and only if it is a pullback square.

Proof. In light of Remark 1.7 it will be enough to prove that every pushout square is a pullback square. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

be a pushout square. Since p admits a cofiber we can extend this diagram as

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \\ C \arrow[r] & D \arrow[r] & E \end{tikzcd}$$

where the right square is coCartesian. Hence we get that the exterior rectangle is coCartesian. This means that both the right square and the exterior rectangle are cofiber sequences and so they are also fiber sequences, i.e. Cartesian. This implies that the left square is Cartesian and we are done. □

3. Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between stable ∞-categories. Show that the following are equivalent:

   (a) f is exact.

   (b) f preserves zero objects and pushout squares.

   (c) f preserves zero objects and pullback squares.

   Proof. Basically the same proof as in exercise (1). □

Corollary 2.1. *Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ has all finite limits and colimits. Furthermore, a functor of stable ∞-categories is exact if and only if it preserves finite colimits, and if and only if it preserves finite limits.*

Proof. This follows from the exercises above together with [2], Theorem 4.4.2.4. □

In light of Corollary 2.1 we may view $\mathrm{Cat}^{\mathrm{Ex}}$ as a full subcategory of the ∞-category of finitely cocomplete pointed ∞-categories (or similarly, as a full subcategory of finitely complete pointed ∞-categories). Our next goal is to characterize these full subcategories in terms of the invertibility of the suspension/loop functor. This result is essential to the discussion of **stabilization** of ∞-categories which we will pursue in the next section.

Theorem 2.2. *Let $\mathcal{C}$ be a pointed ∞-category. Then the following assertions are equivalent:*

1. *$\mathcal{C}$ is stable.*

2. *Every map in $\mathcal{C}$ admits a cofiber and the suspension functor is an equivalence.*

3. *Every map in $\mathcal{C}$ admits a fiber the loop functor is an equivalence.*

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) have already been established. In light of Remark 1.7 it will suffice to prove (2) $\Rightarrow$ (1). Let $\mathcal{C}$ be a pointed ∞-category in which every map damits a cofiber and such that $\Sigma_{\mathcal{C}}$ is an equivalence. We need to show that $\mathcal{C}$ is stable. We will break the claim into three parts:

I Every cofiber sequence in $\mathcal{C}$ is a fiber sequence.

II Every map in $\mathcal{C}$ has a fiber.

III Every fiber sequence in $\mathcal{C}$ is a cofiber sequence.

We start by proving (I). Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{2.1}$$

be a cofiber sequence. We wish to prove that 2.1 is also a fiber sequence. Let $p : \Delta^1 \times \Delta^1 \longrightarrow \mathcal{C}$ denote the above square and let $p_0 : \Delta^1 \coprod_{\Delta\{1\}} \Delta^1 \longrightarrow \mathcal{C}$ denote the restriction of p to the lower and right edges. To prove that p is Cartesian we need to show that the induced map

$$\mathcal{C}_{/p} \longrightarrow \mathcal{C}_{/p_0}$$

is a trivial Kan fibration. For this it will be enough to show that for each $X \in \mathcal{C}$, the induced map

$$\mathcal{C}_{/p} \times_{\mathcal{C}} \{X\} \longrightarrow \mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\}$$

is a weak equivalence of Kan complexes.

Now the Kan complex $\mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\}$ classifies diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{2.2}$$

By taking the cofiber of p we can extend (in an essentially unique manner) every such diagram to a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \\ 0 \arrow[r] & C \arrow[r] & \Sigma_{\mathcal{C}} A \end{tikzcd}$$

where the right square is coCartesian (in light of the cofiber sequence 2.1 we see that the cofiber of p can be identified with $\Sigma_{\mathcal{C}} A$). The exterior rectangle of such a diagram induces a map of the form

$$\Sigma_{\mathcal{C}} X \longrightarrow \Sigma_{\mathcal{C}} A$$

which in turn can be (essentially uniquely) extended to a diagram of the form

$$\begin{tikzcd} \Sigma_{\mathcal{C}} X \arrow[dr] \arrow[drr] \arrow[ddr] & & \\ & \Sigma_{\mathcal{C}} A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\ & \Sigma_{\mathcal{C}} B \arrow[r] & \Sigma_{\mathcal{C}} C \end{tikzcd}$$

Let $q : \Delta^1 \times \Delta^1 \longrightarrow \mathcal{C}$ be the map corresponding to the diagram

$$\begin{tikzcd}
\Sigma_{\mathcal{C}} A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
\Sigma_{\mathcal{C}} B \arrow[r] & \Sigma_{\mathcal{C}} C
\end{tikzcd}$$

and let $q_0 = q|_{\Delta^1 \coprod_{\Delta^{\{1\}}} \Delta^1}$ be the restriction of q to the right and bottom edges. The discussion above yields (a contractible choice of) maps of the form

$$\mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\} \longrightarrow \mathcal{C}_{/q} \times_{\mathcal{C}} \{\Sigma_{\mathcal{C}} X\}$$

Hence we obtain a sequence of maps of the form

$$\begin{tikzcd}
\mathcal{C}_{/p} \times_{\mathcal{C}} \{X\} \arrow[r] \arrow[rr, bend left, dashed] & \mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\} \arrow[r] \arrow[rr, bend right, dashed] & \mathcal{C}_{/q} \times_{\mathcal{C}} \{\Sigma_{\mathcal{C}} X\} \arrow[r] & \mathcal{C}_{/q_0} \times_{\mathcal{C}} \{\Sigma_{\mathcal{C}} X\}
\end{tikzcd}$$

where the dotted arrows indicate the relevant compositions. Unwinding the definitions one can verify that these compositions coincide with the natural maps induced by $\Sigma_{\mathcal{C}}$ (which is an equivalence by our assumptions) and are hence weak equivalences of Kan complexes. By the 2-out-of-6 rule we conclude that all the maps in this diagram are weak equivalences. This shows that 2.1 is a fiber square.

We shall now prove (II). Let $p : B \longrightarrow C$ be a map and let $A \in \mathcal{C}$ be an object such that p admits a cofiber sequence of the form

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \\
C \arrow[r] & \Sigma_{\mathcal{C}} A
\end{tikzcd}$$

(such an A exists because $\Sigma_{\mathcal{C}}$ is an equivalence). From (I) we know that this square is also Cartesian and so we can extend it to a diagram of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \\
0 \arrow[r] & C \arrow[r] & \Sigma_{\mathcal{C}} A
\end{tikzcd}$$

Since the right square and the exterior rectangle are Cartesian we deduce that the left square is Cartesian as well. This proves (2).

It is left to prove (III). Consider the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. From (II) we know that $\mathcal{C}^{\mathrm{op}}$ has all cofibers. Since $\Sigma_{\mathcal{C}}$ is an equivalence we get that $\Sigma_{\mathcal{C}^{\mathrm{op}}}$ is an equivalence as well. Applying (I) to $\mathcal{C}^{\mathrm{op}}$ we deduce that every fiber sequence in $\mathcal{C}$ is a cofiber sequence. This finishes the proof of Theorem 2.2.

$\square$

3 Stabilization of ∞-categories

Let $\mathrm{Cat}_*^{\mathrm{fincolim}}$ denote the (big) ∞-category of pointed finitely cocomplete small ∞-categories and finite-colimit-preserving functors between them. In light of Theorem 2.2 and Corollary 2.1 we may think of $\mathrm{Cat}^{\mathrm{Ex}}$ as a full subcategory of $\mathrm{Cat}_*^{\mathrm{fincolim}}$ which is characterized by the suspension functor being an equivalence. In this case one should naturally ask whether this inclusion admits a left or right adjoint. In other words, given a pointed ∞-category $\mathcal{C}$ with finite limits, does there exists a universal stable ∞-category admitting a map from or to $\mathcal{C}$? Informally, can we naturally **stablize** $\mathcal{C}$?

Similarly, let $\mathrm{Cat}_*^{\mathrm{finlim}}$ denote the (big) ∞-category of pointed finitely complete small ∞-categories and finite-limit-preserving functors between them. In light of Theorem 2.2 we may think of $\mathrm{Cat}^{\mathrm{Ex}}$ as a full subcategory of $\mathrm{Cat}_*^{\mathrm{finlim}}$ which is characterized by the loop functor being an equivalence. The same natural question arises in this case as well.

The answer to both these questions is yes. For the case of $\mathrm{Cat}_*^{\mathrm{fincolim}}$ the inclusion $\mathrm{Cat}^{\mathrm{Ex}} \subseteq \mathrm{Cat}_*^{\mathrm{fincolim}}$ admits a **left adjoint**. For the case of $\mathrm{Cat}_*^{\mathrm{finlim}}$ the inclusion $\mathrm{Cat}^{\mathrm{Ex}} \subseteq \mathrm{Cat}_*^{\mathrm{finlim}}$ admits a **right** adjoint.

Let us try to explain why this is the case. Note that in both cases, the full subcategory of stable ∞-categories is characterized by certain natural operations being equivalences. There are in principle two ways to make a transformation invertible. The **left way** is to **formally add inverses**. Given a pointed ∞-category $\mathcal{C}$ with finite colimits, one can formally invert the suspension functor by taking the colimit of the sequence

$$\mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \ldots$$

in the ∞-category $\mathrm{Cat}_*^{\mathrm{fincolim}}$. We will denote this colimit by $\mathrm{Sp}^{\Sigma}(\mathcal{C})$. In fact, $\mathrm{Sp}^{\Sigma}(\mathcal{C})$ will coincide with the corresponding colimit in Cat_∞, namely, the objects of $\mathrm{Sp}^{\Sigma}(\mathcal{C})$ will be pairs (X, n) where $X \in \mathcal{C}$ and $n \in \mathbb{N}$ and the mapping spaces will be given by

$$\mathrm{Map}_{\mathrm{Sp}^{\Sigma}(\mathcal{C})}((X,n),(Y,m)) = \operatorname*{colim}_k \mathrm{Map}_{\mathcal{C}}\left(\Sigma_{\mathcal{C}}^{k-n} X, \Sigma_{\mathcal{C}}^{k-m} Y\right)$$

This construction will yield a **left adjoint** to the inclusion $\mathrm{Cat}^{\mathrm{Ex}} \subseteq \mathrm{Cat}_*^{\mathrm{fincolim}}$. More precisely, we have a unit map

$$\Sigma_{\mathcal{C}}^{\infty} : \mathcal{C} \longrightarrow \mathrm{Sp}^{\Sigma}(\mathcal{C})$$

given by $X \mapsto (X, 0)$, which satisfies the following universal property: for every stable ∞-category $\mathcal{D}$, pre-composition with $\Sigma_{\mathcal{C}}^{\infty}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{Ex}}\left(\mathrm{Sp}^{\Sigma}(\mathcal{C}), \mathcal{D}\right) \longrightarrow \mathrm{Fun}_*^{\mathrm{fincolim}}(\mathcal{C}, \mathcal{D})$$

We will call $\mathrm{Sp}^{\Sigma}(\mathcal{C})$ the ∞-category of **Σ-spectrum objects in** $\mathcal{C}$. When $\mathcal{C}$ is the ∞-category of finite pointed spaces this construction recovers the ∞-category of **finite spectra** (which are all suspension spectra of finite spaces up to a shift).

Dually, if $\mathcal{C}$ is a pointed ∞-category with finite limits then one can try to invert $\Omega_{\mathcal{C}}$ **from the right**, by choosing for each object an $\Omega_{\mathcal{C}}$-inverse. Formally, this will translate to taking the limit of the tower

$$\mathcal{C} \xleftarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xleftarrow{\Omega_{\mathcal{C}}} \dots$$

in the ∞-category $\mathrm{Cat}_*^{\mathrm{finlim}}$. We will denote this limit by $\mathrm{Sp}^{\Omega}(\mathcal{C})$. In fact, $\mathrm{Sp}^{\Omega}(\mathcal{C})$ will coincide with the corresponding limit in Cat_{∞}, namely, an object of $\mathrm{Sp}^{\Omega}(\mathcal{C})$ is given by sequence $\{X_n\}$ of objects of $\mathcal{C}$ together with equivalences $X_n \simeq \Omega_{\mathcal{C}} X_{n+1}$ and maps will be given by a compatible families of maps. This construction will yield a **right adjoint** to the inclusion $\mathrm{Cat}^{\mathrm{Ex}} \subseteq \mathrm{Cat}_*^{\mathrm{finlim}}$. More precisely, we have a counit map

$$\Omega_{\mathcal{C}}^{\infty} : \mathrm{Sp}^{\Omega}(\mathcal{C}) \longrightarrow \mathcal{C}$$

given by $\{X_n\} \mapsto X_0$, which satisfies the following universal property: for every stable ∞-category $\mathcal{D}$, composition with $\Omega_{\mathcal{C}}^{\infty}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{Ex}}\left(\mathcal{D}, \mathrm{Sp}^{\Omega}(\mathcal{C})\right) \longrightarrow \mathrm{Fun}_*^{\mathrm{finlim}}(\mathcal{D}, \mathcal{C})$$

In the case of $\mathcal{C}$ being the category of pointed spaces, this construction recovers the usual notion of an **Ω-spectrum**. We shall hence call $\mathrm{Sp}^{\Omega}(\mathcal{C})$ the ∞-category of **Ω-spectrum objects in** $\mathcal{C}$.

What is the relation between these two constructions? In general, given a pointed ∞-category with finite limits and finite colimits, the two constructions need not coincide. However, there is a context in which the two will be **closely related**. This is the context of **presentable ∞-categories** which we will address next.

4 Presentable stable ∞-categories

Let $\mathrm{Pr}_*^{\mathrm{L}}$ denote the ∞-category of pointed presentable ∞-categories and **left** functors between them (i.e. functors which admit right adjoints) and $\mathrm{Pr}_*^{\mathrm{R}}$ the ∞-category of pointed presentable ∞-categories and **right** functors between them (i.e. functors which admit left adjoints). The categories $\mathrm{Pr}_*^{\mathrm{L}}$ and $\mathrm{Pr}_*^{\mathrm{R}}$ are naturally opposite to each other (and can be considered as the two sides of the ∞-category of (pointed) presentable ∞-categories and **adjunctions** as morphisms).

The adjoint functor theorem for presentable ∞-categories tells us that a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ between presentable ∞-categories is a left functor if and only if it preserves all colimits and is a right functor if and only if it is accessible and preserves all limits. In particular, if $\mathcal{C}, \mathcal{D}$ are stable presentable ∞-categories then any left functors between them and any right functor between them is exact. We will denote by $\mathrm{Pr}_{\mathrm{Ex}}^{\mathrm{L}} \subseteq \mathrm{Pr}_*^{\mathrm{L}}$ the full subcategory spanned by exact ∞-categories and similarly by $\mathrm{Pr}_{\mathrm{Ex}}^{\mathrm{R}} \subseteq \mathrm{Pr}_*^{\mathrm{R}}$.

We can hence contemplate the question of **stablization** inside the ∞-category $\mathrm{Pr}_*^{\mathrm{L}}$ or $\mathrm{Pr}_*^{\mathrm{R}}$. The following observation shows that the answer should be similar to the discussion in the previous section:

Observation 4.1. *Let $\mathcal{C}$ be a pointed presentable ∞-category. Then the following are equivalent:*

1. *$\mathcal{C}$ is stable.*
2. *$\Sigma_{\mathcal{C}}$ is an equivalence.*
3. *$\Omega_{\mathcal{C}}$ is an equivalence.*

From Observation 4.1 we see that in order to perform the stabilization process inside the world of pointed presentable ∞-categories one just needs to invert either the suspension or the loop functor. As above, this can be done from the left or from the right. However, since $\mathrm{Pr}^{\mathrm{L}}_{*}$ and $\mathrm{Pr}^{\mathrm{R}}_{*}$ are opposite to each other, it will be enough to understand just one of these procedures. In this case the right option has an advantage, and that is that limits in $\mathrm{Pr}^{\mathrm{R}}_{*}$ can be computed just as limits in Cat_{∞} (where the same is not true for colimits in $\mathrm{Pr}^{\mathrm{L}}_{*}$, not even filtered ones).

Now the functor $\Omega_{\mathcal{C}}$ has a left adjoint $\Sigma_{\mathcal{C}}$ we see that $\Omega_{\mathcal{C}}$ is a **right functor**, i.e., a legitimate morphism in $\mathrm{Pr}^{\mathrm{R}}_{*}$. As above, we can invert it by taking the inverse limit of the tower

$$\mathcal{C} \xleftarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xleftarrow{\Omega_{\mathcal{C}}} ...$$

in the ∞-category $\mathrm{Pr}^{\mathrm{R}}_{*}$. Fortunately, this procedure is the same as computing the limit in Cat_{∞}, i.e., it will coincide with $\mathrm{Sp}^{\Omega}(\mathcal{C})$ described above. However, we are now guaranteed that $\mathrm{Sp}^{\Omega}(\mathcal{C})$ will be a presentable ∞-category and that the projection map

$$\Omega^{\infty}_{\mathcal{C}} : \mathrm{Sp}^{\Omega}(\mathcal{C}) \longrightarrow \mathcal{C}$$

will be a right functor of presentable ∞-categories. Now if $\mathcal{D}$ is any stable **presentable** ∞-category then composition with $\Omega^{\infty}_{\mathcal{C}}$ induces an equivalence of categories

$$\mathrm{Fun}^{\mathrm{R}}\left(\mathcal{D}, \mathrm{Sp}^{\Omega}(\mathcal{C})\right) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C})$$

What about inverting the suspension functor from the left? The duality between $\mathrm{Pr}^{\mathrm{R}}_{*}$ and $\mathrm{Pr}^{\mathrm{L}}_{*}$ means that we can automatically get such a dual result with no extra work. Namely, the left adjoint

$$\Sigma^{\infty}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Sp}^{\Omega}(\mathcal{C})$$

of $\Omega^{\infty}_{\mathcal{C}}$ will also exhibits $\mathrm{Sp}^{\Omega}(\mathcal{C})$ as a stabilization of $\mathcal{C}$ **from the left** in the ∞-category $\mathrm{Pr}^{\mathrm{L}}_{*}$. In other words, if $\mathcal{D}$ is any stable presentable ∞-category then pre-composition with $\Sigma^{\infty}_{\mathcal{C}}$ induces an equivalence of categories

$$\mathrm{Fun}^{\mathrm{L}}\left(\mathrm{Sp}^{\Omega}(\mathcal{C}), \mathcal{D}\right) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \tag{4.1}$$

Remark 4.2. Note that when $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{*}$ is also **compactly generated**, i.e. it is of the form $\mathrm{Ind}(\mathcal{C}^0)$ where $\mathcal{C}^0$ is a small pointed ∞-category with finite colimits, then one can attempt to left-stabilize $\mathcal{C}$ by first left-stabilizing $\mathcal{C}_0$ using the construction $\mathrm{Sp}^{\Sigma}(\mathcal{C}_0)$ of the previous section, and then considering its

Ind-completion $\mathrm{Ind}\left(\mathrm{Sp}^{\Sigma}(\mathcal{C}_0)\right)$. This construction will yield again a stable presentable ∞-category satisfying the same universal property 4.1 as $\mathrm{Sp}^{\Omega}(\mathcal{C})$. We will hence deduce that

$$\mathrm{Ind}\left(\mathrm{Sp}^{\Sigma}(\mathcal{C}_0)\right) \simeq \mathrm{Sp}^{\Omega}(\mathcal{C})$$

Definition 4.3. Let $\mathcal{S}_*$ be the presentable ∞-category of pointed spaces. Then we denote $\mathrm{Sp}^{\Omega}(\mathcal{S}_*)$ simply by Sp^{Ω}. We will refer to it as the **∞-category of spectra**.

Remark 4.4. Applying Remark 4.2 to the case $\mathcal{C} = \mathcal{S}_*$ and $\mathcal{C}_0 = \mathcal{S}_*^{\mathrm{fin}}$ (see below) we obtain the following result: the ∞-category Sp^{Ω} of Ω-spectra can be identified with the Ind-completion of the ∞-category of (shifts of) finite suspension spectra.

Remark 4.5. Applying 4.1 to the case of $\mathcal{C} = \mathcal{S}_*$ we get that for every stable presentable ∞-category $\mathcal{D}$ there are natural equivalences

$$\mathrm{Fun}^{\mathrm{L}}\left(\mathrm{Sp}^{\Omega}, \mathcal{D}\right) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}_*, \mathcal{D}) \simeq \mathcal{D} \tag{4.2}$$

where the composition can be obtained by evaluating at the **sphere spectrum**. This can be phrased as follows: the ∞-category Sp^{Ω} is the **free stable presentable ∞-category generated from one object**.

Corollary 4.6. *Substituting $\mathcal{D} = \mathrm{Sp}^{\Omega}$ in 4.2 above we obtain*

$$\mathrm{Fun}^{\mathrm{L}}\left(\mathrm{Sp}^{\Omega}, \mathrm{Sp}^{\Omega}\right) \simeq \mathrm{Sp}^{\Omega}$$

*This means that Sp^{Ω} carries a natural monoidal structure. This monoidal structure is in fact a **symmetric monoidal** structure, known as the **smash product** of spectra.*

5 Reduced excisive functors and other models for spectra

Definition 5.1. Let $\mathcal{C}, \mathcal{D}$ be pointed ∞-categories such that $\mathcal{C}$ has finite colimits and $\mathcal{D}$ has finite limits. We will say that a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is **excisive** if it sends pushout squares to pullback squares. We will denote by $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by reduced excisive functors.

Let $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{S}_*$ denote the minimal pointed full subcategory which contains $S^0 \in \mathcal{S}_*$ and is closed under finite colimits. Alternatively, $\mathcal{S}_*^{\mathrm{fin}}$ is the ∞-category of pointed spaces which are equivalent to a finite pointed simplicial set. The ∞-category $\mathcal{S}_*^{\mathrm{fin}}$ has the following important property: $\mathcal{S}_*$ is the free pointed ∞-category with finite colimits generated by S^0. Put formally, if $\mathcal{C}$ is any pointed ∞-category with finite colimits then evaluation at S^0 induces an equivalence of ∞-categories

$$\mathrm{Fun}_*^{\mathrm{fincolim}}(\mathcal{S}_*, \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}$$

Corollary 5.2. *Let $\mathcal{D}$ be a stable ∞-category. Then*

$$\mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}\right) \simeq \mathrm{Fun}_*^{\mathrm{fincolim}}\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D}\right) \simeq \mathcal{D}$$

where the last equivalence is given by evaluating at S^0.

In other words, the operation $\mathcal{C} \mapsto \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$ maps every stable ∞-category $\mathcal{C}$ to itself. Furthermore, if $\mathcal{C}$ is any pointed ∞-category with finite limits, then $\mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$, while might not be equivalent to $\mathcal{C}$, will still be pointed and will have finite limits (which are computed objectwise). Hence we can consider the operation $\mathcal{C} \mapsto \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$ as a functor from $\mathrm{Cat}_*^{\mathrm{finlim}}$ to itself (endowed with a natural transformation to the identity given by evaluating at S^0).

A key observation now is that for every $\mathcal{C} \in \mathrm{Cat}_*^{\mathrm{finlim}}$, the ∞-category $\mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$ is **stable**. To prove this, we see that in view of Theorem 2.2 it will suffice to show that the loop functor on $\mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$ is an equivalence. Now note that the loop (like any limit) is applied objectwise, i.e., by composing with $\Omega_{\mathcal{C}}$. Hence we see that pre-composition with the suspension functor $\Sigma_{\mathcal{S}_*^{\mathrm{fin}}}$ gives an inverse to the loop functor.

The above discussion leads one the believe that the construction

$$\mathcal{C} \mapsto \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$$

might yield a right adjoint to the inclusion $\mathrm{Cat}^{\mathrm{Ex}} \subseteq \mathrm{Cat}_*^{\mathrm{finlim}}$. Indeed, this is in fact the case. More precisely, we claim that if $\mathcal{C}$ is a pointed ∞-category with finite limits and $\mathcal{D}$ is a stable ∞-category then composition with the natural map

$$\mathrm{ev}_{S^0} : \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right) \longrightarrow \mathcal{C}$$

yields an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{Ex}}\left(\mathcal{D}, \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)\right) \longrightarrow \mathrm{Fun}_*^{\mathrm{finlim}}\left(\mathcal{D}, \mathcal{C}\right)$$

To prove this, one can observe that

$$\mathrm{Fun}^{\mathrm{Ex}}\left(\mathcal{D}, \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)\right) \simeq \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathrm{Fun}_*^{\mathrm{finlim}}\left(\mathcal{D}, \mathcal{C}\right)\right) \simeq \mathrm{Fun}_*^{\mathrm{finlim}}\left(\mathcal{D}, \mathcal{C}\right)$$

since $\mathrm{Fun}_*^{\mathrm{finlim}}\left(\mathcal{D}, \mathcal{C}\right) \simeq \mathrm{Exc}_*\left(\mathcal{D}, \mathcal{C}\right)$ is stable.

We can hence consider $\mathrm{Exc}_*\left(\mathcal{D}, \mathcal{C}\right)$ as another way to stabilize a pointed finitely complete ∞-category $\mathcal{C}$. Following Lurie, we will denote

$$\mathrm{Sp}(\mathcal{C}) \stackrel{\mathrm{def}}{=} \mathrm{Exc}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$$

Definition 5.3. When $\mathcal{C} = \mathcal{S}_*$ is the ∞-category of pointed spaces we will denote $\mathrm{Sp}(\mathcal{C})$ simply by Sp.

Remark 5.4. If $\mathcal{C}$ is presentable then $\mathrm{Sp}(\mathcal{C})$ will be an accessible localization of the presentable ∞-category $\mathrm{Fun}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right)$. The localization functor

$$\mathcal{L} : \mathrm{Fun}_*\left(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}\right) \longrightarrow \mathrm{Sp}(\mathcal{C})$$

admits an explicit formula in term of the Goodwille derivative:

$$\mathcal{L}(f)(X) = \operatorname*{colim}_n \Omega^n_{\mathcal{C}} f\left(\Sigma^n X\right)$$

Remark 5.5. The definition of Sp posses at least one conceptual advantage on the one of Sp^Ω above, and that is that one can obtain a more direct description of the smash product in terms of **Day convolution** of functors. However, one needs to be a bit careful because the Day convolution of two excisive functors needs not be excisive. Instead, one will need to apply the localization functor of Remark 5.4 to the result.

In light of the uniqueness of right adjoints we conclude that we must have a natural equivalence

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp}^\Omega(\mathcal{C})$$

It is worth spelling out what this equivalence is. Let $f : \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{C}$ be a reduced excisice functor. Then for every pushouts square of the form

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & \star \arrow[d] \\ \star \arrow[r] & S^{n+1} \end{tikzcd}$$

The induced square

$$\begin{tikzcd} f\left(S^n\right) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & f\left(S^{n+1}\right) \end{tikzcd}$$

will be a pullback square. We will hence obtain a natural equivalence in $\mathcal{C}$

$$\varphi_n : f\left(S^n\right) \xrightarrow{\simeq} \Omega_{\mathcal{C}} f\left(S^{n+1}\right)$$

We conclude that the square of objects $\{f\left(S^n\right)\}$ together with the maps φ_n determine an object of $\mathrm{Sp}^\Omega(\mathcal{C})$. It is a formal consequence of uniqueness of adjoint functors that this functor induces an equivalence

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp}^\Omega(\mathcal{C}) \tag{5.1}$$

In particular, a reduced excisive functor can be **uniquely reconstructed from its values on the spheres**, together with suitable structure maps.

It can be conceptually convenient to be able to rap all the information of an Ω-spectrum in a form of a **functor** (satisfying certain conditions), such that the equivalence 5.1 will be induced by a suitable restriction. This can be done as follows. Let $\mathrm{Top}_*^{\mathrm{fin}}$ be the **topological category** of finite pointed CW complexes and continuous maps, so that $\mathcal{S}_*^{\mathrm{fin}}$ is equivalent to the coherent nerve of $\mathrm{Top}_*^{\mathrm{fin}}$.

Let $\mathrm{Sph} \subseteq \mathrm{Top}_*^{\mathrm{fin}}$ be the topological subcategory whose objects are the point $* \in \mathrm{Top}_*^{\mathrm{fin}}$ together with and all the spheres $S^n \in \mathrm{Top}_*^{\mathrm{fin}}$. For every $k \geq 0$ we have

$$\mathrm{Map}_{\mathrm{Sph}}(S^n, S^{n+k}) = S^k \subseteq \mathrm{Map}_{\mathrm{Top}_*^{\mathrm{fin}}}\left(S^n, S^{n+k}\right)$$

embedded via the adjoint to the natural homeomorphism $S^k \wedge S^n \cong S^{n+k}$, and all other mapping spaces contain only the constant map.

We now observe that if $\mathcal{C}$ is a pointed ∞-category with finite limits then a pointed functor

$$\mathrm{N}(\mathrm{Sph}) \longrightarrow \mathcal{C}$$

can be identified with a sequence of objects $\{X_n\}$ in $\mathcal{C}$ together with pointed maps $S^n \longrightarrow \mathrm{Map}_{\mathcal{C}}(X_n, X_{n+1})$, which in turn can be identified with morphisms in $\mathcal{C}$ of the form $\varphi_n : X_n \longrightarrow \Omega_{\mathcal{C}} X_{n+1}$. Hence we conclude that the category $\Omega^{\mathcal{C}}(\mathcal{C})$ can be identified with full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathrm{Sph}), \mathcal{C})$ consisting of those functors for which the maps φ_n are equivalences. We can call such functors **Ω-functors**.

One can think of the $\mathrm{Sp}(\mathcal{C})$ and $\mathrm{Sp}^{\Omega}(\mathcal{C})$ as two extremes attempts to model the stabilization of $\mathcal{C}$. In $\mathrm{Sp}(\mathcal{C})$ we do in some sense a more canonical construction, but at the price of having a big object at hand. With $\mathrm{Sp}^{\Omega}(\mathcal{C})$ we keep only the very necessary information, but at a price of maybe loosing direct access to some interesting invariants and constructions, for example smash products. These two extremes admit many intermediate steps - for many intermediate subcategories $\mathrm{Sph} \subseteq \mathcal{B} \subseteq \mathrm{Top}_*^{\mathrm{fin}}$ we may expect that the restriction functor

$$\mathrm{Exc}_*\left(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C}\right) \longrightarrow \mathrm{Fun}_*(\mathcal{B}, \mathcal{C})$$

to be fully faithful with an easy to identify essential image. For example, by adding to Sph the morphisms generated by the action of the symmetric group Σ_n on $S^n = \overbrace{S^1 \wedge ... \wedge S^1}^{n}$ we obtain a subcategory $\mathrm{Sph}^{\Sigma} \subseteq \mathrm{Top}_*^{\mathrm{fin}}$ such that the restriction functor

$$\mathrm{Exc}_*\left(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C}\right) \longrightarrow \mathrm{Fun}_*\left(\mathrm{Sph}^{\Sigma}, \mathcal{C}\right)$$

is fully-faithful and its essential image can be identified with **symmetric** Ω-spectrum objects in $\mathcal{C}$, which is known to be an equivalent construction. Another choice is to add the morphisms generated by the action of the orthogonal group $O(n)$ on S^n, which leads to the model of **orthogonal** Ω-spectrum objects in $\mathcal{C}$. Naturally, many other choices are also possible.

Remark 5.6. If one takes $\mathcal{B} \subseteq \mathrm{Top}_*^{\mathrm{fin}}$ to be the full subcategory generated by **discrete objects** then the restriction map

$$\mathrm{Exc}_*\left(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C}\right) \longrightarrow \mathrm{Fun}_*(\mathcal{B}, \mathcal{C})$$

will not be fully-faithful: it will remember from every spectrum object only its corresponding **connective cover**.

References

[1] Lurie, J. *Higher Algebra*, `http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf`.

[2] Lurie, J., *Higher Topos Theory*, Annals of Mathematics Studies, 170, Princeton University Press, 2009, `http://www.math.harvard.edu/~lurie/papers/highertopoi.pdf`.