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AdS/CFT Correspondence

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Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence is a conjectured equivalence between a quantum field theory in d spacetime dimensions with conformal scaling symmetry and a quantum theory of gravity in $(d+1)$-dimensional anti-de Sitter space. The most promising approaches to quantizing gravity involve superstring theories, which are most easily defined in 10 spacetime dimensions, or M-theory which is defined in 11 spacetime dimensions. Hence, the AdS/CFT correspondences based on superstrings typically involve backgrounds of the form $\mathrm{AdS}_{d+1} \times Y_{9-d}$ while those based on M-theory involve backgrounds of the form $\mathrm{AdS}_{d+1} \times Y_{10-d}$, where Y are compact spaces.

The examples of the AdS/CFT correspondence discussed in this article are dualities between (super)conformal nonabelian gauge theories and superstrings on $\mathrm{AdS}_5 \times Y_5$, where Y_5 is a five-dimensional Einstein space (i.e., a space whose Ricci tensor is proportional to the metric, $R_{ij} = 4g_{ij}$). In particular, the most basic (and maximally supersymmetric) such duality relates $\mathcal{N} = 4\,\mathrm{SU}(N)$ super Yang–Mills (SYM) and type IIB superstring in the curved background $\mathrm{AdS}_5 \times S^5$.

There exist special limits where this duality is more tractable than in the general case. If we take the large-N limit while keeping the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ fixed (g_{YM} is the Yang–Mills coupling strength), then each Feynman graph of the gauge theory carries a topological factor N^χ, where χ is the Euler characteristic of the graph. The graphs of spherical topology (often called "planar"), to be identified with string tree diagrams, are weighted by N^2; the graphs of toroidal topology, to be identified with string one-loop diagrams, by N^0, etc. This counting corresponds to the closed-string coupling constant of order N^{-1}. Thus, in the large-N limit the gauge theory becomes "planar," and the dual string theory becomes classical. For small $g_{\mathrm{YM}}^2 N$, the gauge theory can be studied perturbatively; in this regime the dual string theory has not been very useful because the background becomes highly curved. The real power of the AdS/CFT duality, which already has made it a very useful tool, lies in the fact that, when the gauge theory becomes strongly coupled, the curvature in the dual description becomes small; therefore, classical supergravity provides a systematic starting point for approximating the string theory.

There is a strong motivation for an improved understanding of dualities of this type. In one direction, generalizations of this duality provide the tantalizing hope of a better understanding of quantum chromodynamics (QCD); QCD is a non-abelian gauge theory that describes the strong interactions of mesons, baryons, and glueballs, and has a conformal symmetry which is broken by quantum effects. In the other direction, AdS/CFT suggests that quantum gravity may be understandable as a gauge theory. Understanding the confinement of quarks and gluons that takes place in low-energy QCD and quantizing gravity are well acknowledged to be two of the most important outstanding problems of theoretical physics.

Some Geometrical Preliminaries

The d-dimensional sphere of radius L, S^d, may be defined by a constraint

$$\sum_{i=1}^{d+1} (X^i)^2 = L^2 \qquad [1]$$

on $d+1$ real coordinates X^i. It is a positively curved maximally symmetric space with symmetry group $\mathrm{SO}(d+1)$. We will denote the round metric on S^d of unit radius by $\mathrm{d}\Omega_d^2$.

The d-dimensional anti-de Sitter space, AdS_d, may be defined by a constraint

$$(X^0)^2 + (X^d)^2 - \sum_{i=1}^{d-1}(X^i)^2 = L^2 \qquad [2]$$

This constraint shows that the symmetry group of AdS_d is $SO(2, d-1)$. AdS_d is a negatively curved maximally symmetric space, that is, its curvature tensor is related to the metric by

$$R_{abcd} = -\frac{1}{L^2}[g_{ac}g_{bd} - g_{ad}g_{bc}] \qquad [3]$$

Its metric may be written as

$$ds^2_{AdS} = L^2\left(-(y^2+1)dt^2 + \frac{dy^2}{y^2+1} + y^2\, d\Omega^2_{d-2}\right) \qquad [4]$$

where the radial coordinate $y \in [0, \infty)$, and t is defined on a circle of length 2π. This space has closed timelike curves; to eliminate them, we will work with the universal covering space where $t \in (-\infty, \infty)$. The boundary of AdS_d, which plays an important role in the AdS/CFT correspondence, is located at infinite y. There exists a subspace of AdS_d called the Poincaré wedge, with the metric

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - (dx^0)^2 + \sum_{i=1}^{d-2}(dx^i)^2\right) \qquad [5]$$

where $z \in [0, \infty)$.

A Euclidean continuation of AdS_d is the Lobachevsky space (hyperboloid), $\boldsymbol{L}_d$. It is obtained by reversing the sign of $(X^d)^2, dt^2$, and $(dx^0)^2$ in [2], [4], and [5], respectively. After this Euclidean continuation, the metrics [4] and [5] become equivalent; both of them cover the entire $\boldsymbol{L}_d$. Another equivalent way of writing the metric is

$$ds^2_L = L^2\left(d\rho^2 + \sinh^2\rho\, d\Omega^2_{d-1}\right) \qquad [6]$$

which shows that the boundary at infinite ρ has the topology of S^{d-1}. In terms of the Euclideanized metric [5], the boundary consists of the $\boldsymbol{R}^{d-1}$ at $z=0$, and a single point at $z=\infty$.

The Geometry of Dirichlet Branes

Our path toward formulating the AdS_5/CFT_4 correspondence requires introduction of Dirichlet branes, or D-branes for short. They are soliton-like "membranes" of various internal dimensionalities contained in type II superstring theories. A Dirichlet p-brane (or Dp brane) is a $(p+1)$-dimensional hyperplane in $(9+1)$-dimensional spacetime where strings are allowed to end. A D-brane is much like a topological defect: upon touching a D-brane, a closed string can open up and turn into an open string whose ends are free to move along the D-brane. For the endpoints of such a string the $p+1$ longitudinal coordinates satisfy the conventional free (Neumann) boundary conditions, while the $9-p$ coordinates transverse to the Dp brane have the fixed (Dirichlet) boundary conditions, hence the origin of the term "Dirichlet brane." The Dp brane preserves half of the bulk supersymmetries and carries an elementary unit of charge with respect to the $(p+1)$-form gauge potential from the Ramond–Ramond (RR) sector of type II superstring.

For this article, the most important property of D-branes is that they realize gauge theories on their world volume. The massless spectrum of open strings living on a Dp brane is that of a maximally supersymmetric U(1) gauge theory in $p+1$ dimensions. The $9-p$ massless scalar fields present in this supermultiplet are the expected Goldstone modes associated with the transverse oscillations of the Dp brane, while the photons and fermions provide the unique supersymmetric completion. If we consider N parallel D-branes, then there are N^2 different species of open strings because they can begin and end on any of the D-branes. N^2 is the dimension of the adjoint representation of U(N), and indeed we find the maximally supersymmetric U(N) gauge theory in this setting.

The relative separations of the Dp branes in the $9-p$ transverse dimensions are determined by the expectation values of the scalar fields. We will be interested in the case where all scalar expectation values vanish, so that the N Dp branes are stacked on top of each other. If N is large, then this stack is a heavy object embedded into a theory of closed strings which contains gravity. Naturally, this macroscopic object will curve space: it may be described by some classical metric and other background fields including the RR $(p+2)$-form field strength. Thus, we have two very different descriptions of the stack of Dp branes: one in terms of the U(N) supersymmetric gauge theory on its world volume, and the other in terms of the classical RR charged p-brane background of the type II closed superstring theory. The relation between these two descriptions is at the heart of the connections between gauge fields and strings that are the subject of this article.

Coincident D3 Branes

Gauge theories in $3+1$ dimensions play an important role in physics, and as explained above, parallel D3 branes realize a $(3+1)$-dimensional U(N) SYM theory. Let us compare a stack of D3 branes with the RR-charged black 3-brane classical solution where the metric assumes the form

$$\mathrm{d}s^2 = H^{-1/2}(r)\left[-f(r)(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2\right] + H^{1/2}(r)\left[f^{-1}(r)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5{}^2\right] \quad [7]$$

where $i = 1, 2, 3$ and

$$H(r) = 1 + \frac{L^4}{r^4}, \qquad f(r) = 1 - \frac{r_0{}^4}{r^4}$$

The solution also contains an RR self-dual 5-form field strength

$$F = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}(H^{-1}) + 4L^4\,\mathrm{vol}(S^5) \quad [8]$$

so that the Einstein equation of type IIB supergravity, $R_{\mu\nu} = F_{\mu\alpha\beta\gamma\delta}F_\nu{}^{\alpha\beta\gamma\delta}/96$, is satisfied.

In the extremal limit $r_0 \to 0$, the 3-brane metric becomes

$$\mathrm{d}s^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2}\left(-(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2}\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5^2\right) \quad [9]$$

Just like the stack of parallel, ground-state D3 branes, the extremal solution preserves 16 of the 32 supersymmetries present in the type IIB theory. Introducing $z = L^2/r$, one notes that the limiting form of [9] as $r \to 0$ factorizes into the direct product of two smooth spaces, the Poincaré wedge [5] of AdS_5, and S^5, with equal radii of curvature L. The 3-brane geometry may thus be viewed as a semi-infinite throat of radius L which, for $r \gg L$, opens up into flat $(9+1)$-dimensional space. Thus, for L much larger than the string length scale, $\sqrt{\alpha'}$, the entire 3-brane geometry has small curvatures everywhere and is appropriately described by the supergravity approximation to type IIB string theory.

The relation between L and $\sqrt{\alpha'}$ may be found by equating the gravitational tension of the extremal 3-brane classical solution to N times the tension of a single D3 brane:

$$\frac{2}{\kappa^2}L^4\,\mathrm{vol}(S^5) = N\frac{\sqrt{\pi}}{\kappa} \quad [10]$$

where $\mathrm{vol}(S^5) = \pi^3$ is the volume of a unit 5-sphere, and $\kappa = \sqrt{8\pi G}$ is the ten-dimensional gravitational constant. It follows that

$$L^4 = \frac{\kappa}{2\pi^{5/2}}N = g_{\mathrm{YM}}^2\,N\alpha'^2 \quad [11]$$

where we used the standard relations $\kappa = 8\pi^{7/2}g_{st}\alpha'^2$ and $g_{\mathrm{YM}}^2 = 4\pi g_{st}$ [10]. Thus, the size of the throat in string units is $\lambda^{1/4}$. This remarkable emergence of the 't Hooft coupling from gravitational considerations is at the heart of the success of the AdS/CFT correspondence. Moreover, the requirement $L \gg \sqrt{\alpha'}$ translates into $\lambda \gg 1$: the gravitational approach is valid when the 't Hooft coupling is very strong and the perturbative field-theoretic methods are not applicable.

Example: Thermal Gauge Theory from Near-Extremal D3 Branes

An important black hole observable is the Bekenstein–Hawking (BH) entropy, which is proportional to the area of the event horizon. For the 3-brane solution [7], the horizon is located at $r = r_0$. For $r_0 > 0$ the 3-brane carries some excess energy E above its extremal value, and the BH entropy is also nonvanishing. The Hawking temperature is then defined by $T^{-1} = \partial S_{\mathrm{BH}}/\partial E$.

Setting $r_0 \ll L$ in [9], we obtain a near-extremal 3-brane geometry, whose Hawking temperature is found to be $T = r_0/(\pi L^2)$. The eight-dimensional "area" of the horizon is

$$A_h = (r_0/L)^3 V_3 L^5\,\mathrm{vol}(S^5) = \pi^6 L^8 T^3 V_3 \quad [12]$$

where V_3 is the spatial volume of the D3 brane (i.e., the volume of the x^1, x^2, x^3 coordinates). Therefore, the BH entropy is

$$S_{\mathrm{BH}} = \frac{2\pi A_h}{\kappa^2} = \frac{\pi^2}{2}N^2 V_3 T^3 \quad [13]$$

This gravitational entropy of a near-extremal 3-brane of Hawking temperature T is to be identified with the entropy of $\mathcal{N} = 4$ supersymmetric U(N) gauge theory (which lives on N coincident D3 branes) heated up to the same temperature.

The entropy of a free U(N) $\mathcal{N} = 4$ supermultiplet – which consists of the gauge field, $6N^2$ massless scalars, and $4N^2$ Weyl fermions – can be calculated using the standard statistical mechanics of a massless gas (the blackbody problem), and the answer is

$$S_0 = \frac{2\pi^2}{3}N^2 V_3 T^3 \quad [14]$$

It is remarkable that the 3-brane geometry captures the T^3 scaling characteristic of a conformal field theory (CFT) (in a CFT this scaling is guaranteed by the extensivity of the entropy and the absence of dimensionful parameters). Also, the N^2 scaling indicates the presence of $O(N^2)$ unconfined degrees of freedom, which is exactly what we expect in the $\mathcal{N}=4$ supersymmetric U(N) gauge theory. But what is the explanation of the relative factor of 3/4 between S_{BH} and S_0? In fact, this factor is not a contradiction but rather a prediction about the strongly coupled $\mathcal{N}=4$ SYM theory at finite temperature. As we argued above, the supergravity calculation of the BH entropy, [13], is relevant to the $\lambda \to \infty$ limit of the $\mathcal{N}=4$ SU(N) gauge theory, while the free-field calculation, [14], applies to the $\lambda \to 0$ limit. Thus, the relative factor of 3/4 is not a discrepancy: it relates two different limits of the theory. Indeed, on general field-theoretic grounds, we expect that in the 't Hooft large-N limit, the entropy is given by

$$S = \frac{2\pi^2}{3} N^2 f(\lambda) V_3 T^3 \qquad [15]$$

The function f is certainly not constant: perturbative calculations valid for small $\lambda = g_{\mathrm{YM}}^2 N$ give

$$f(\lambda) = 1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{3/2} + \cdots \qquad [16]$$

Thus, the BH entropy in supergravity, [13], is translated into the prediction that

$$\lim_{\lambda \to \infty} f(\lambda) = \frac{3}{4} \qquad [17]$$

The Essentials of the AdS/CFT Correspondence

The AdS/CFT correspondence asserts a detailed map between the physics of type IIB string theory in the throat of the classical 3-brane geometry, that is, the region $r \ll L$, and the gauge theory living on a stack of D3 branes. As already noted, in this limit $r \ll L$, the extremal D3 brane geometry factors into a direct product of $\mathrm{AdS}_5 \times S^5$. Moreover, the gauge theory on this stack of D3 branes is the maximally supersymmetric $\mathcal{N}=4$ SYM.

Since the horizon of the near-extremal 3-brane lies in the region $r \ll L$, the entropy calculation could have been carried out directly in the throat limit, where $H(r)$ is replaced by L^4/r^4. Another way to motivate the identification of the gauge theory with the throat is to think about the absorption of massless particles. In the D-brane description, a particle incident from asymptotic infinity is converted into an excitation of the stack of D-branes, that is, into an excitation of the gauge theory on the world volume. In the supergravity description, a particle incident from the asymptotic (large r) region tunnels into the $r \ll L$ region and produces an excitation of the throat. The fact that the two different descriptions of the absorption process give identical cross sections supports the identification of excitations of $\mathrm{AdS}_5 \times S^5$ with the excited states of the $\mathcal{N}=4$ SYM theory.

Maldacena (1998) motivated this correspondence by thinking about the low-energy ($\alpha' \to 0$) limit of the string theory. On the D3 brane side, in this low-energy limit, the interaction between the D3 branes and the closed strings propagating in the bulk vanishes, leaving a pure $\mathcal{N}=4$ SYM theory on the D3 branes decoupled from type IIB superstrings in flat space. Around the classical 3-brane solutions, there are two types of low-energy excitations. The first type propagate in the bulk region, $r \gg L$, and have a cross section for absorption by the throat which vanishes as the cube of their energy. The second type are localized in the throat, $r \leq L$, and find it harder to tunnel into the asymptotically flat region as their energy is taken smaller. Thus, both the D3 branes and the classical 3-brane solution have two decoupled components in the low-energy limit, and in both cases, one of these components is type IIB superstrings in flat space. Maldacena conjectured an equivalence between the other two components.

Immediate support for this identification comes from symmetry considerations. The isometry group of AdS_5 is SO(2, 4), and this is also the conformal group in $3+1$ dimensions. In addition, we have the isometries of S^5 which form SU(4) $\sim$ SO(6). This group is identical to the R-symmetry of the $\mathcal{N}=4$ SYM theory. After including the fermionic generators required by supersymmetry, the full isometry supergroup of the $\mathrm{AdS}_5 \times S^5$ background is SU(2, 2|4), which is identical to the $\mathcal{N}=4$ superconformal symmetry. We will see that, in theories with reduced supersymmetry, the S^5 factor is replaced by other compact Einstein spaces Y_5, but AdS_5 is the "universal" factor present in the dual description of any large-N CFT and makes the SO(2, 4) conformal symmetry a geometric one.

The correspondence extends beyond the supergravity limit, and we must think of $\mathrm{AdS}_5 \times Y_5$ as a background of string theory. Indeed, type IIB strings are dual to the electric flux lines in the gauge theory, providing a string-theoretic setup for calculating correlation functions of Wilson loops. Furthermore, if $N \to \infty$ while $g_{\mathrm{YM}}^2 N$ is held fixed and finite, then there are string scale corrections to the supergravity limit (Maldacena 1998, Gubser *et al.* 1998, Witten 1998) which proceed in powers of $\alpha'/L^2 = (g_{\mathrm{YM}}^2 N)^{-1/2}$. For finite N, there are also string loop corrections in powers of $\kappa^2/L^8 \sim N^{-2}$. As expected, with $N \to \infty$ we can take the classical limit of the string theory on $AdS_5 \times Y_5$. However, in order to understand the large-N gauge theory with finite 't Hooft coupling, we should think of $AdS_5 \times Y_5$ as the target space of a two-dimensional sigma model describing the classical string physics.

Correlation Functions and the Bulk/Boundary Correspondence

A basic premise of the AdS/CFT correspondence is the existence of a one-to-one map between gauge-invariant operators in the CFT and fields (or extended objects) in AdS. Gubser *et al.* (1998) and Witten (1998) formulated precise methods for calculating correlation functions of various operators in a CFT using its dual formulation. A physical motivation for these methods comes from earlier calculations of absorption by 3-branes. When a wave is absorbed, it tunnels from asymptotic infinity into the throat region, and then continues to propagate toward smaller r. Let us separate the 3-brane geometry into two regions: $r \gtrsim L$ and $r \lesssim L$. For $r \lesssim L$ the metric is approximately that of $AdS_5 \times \boldsymbol{S}^5$, while for $r \gtrsim L$ it becomes very different and eventually approaches the flat metric. Signals coming in from large r (small $z = L^2/r$) may be considered as disturbing the "boundary" of AdS_5 at $r \sim L$, and then propagating into the bulk of AdS_5. Discarding the $r \gtrsim L$ part of the 3-brane metric, the gauge theory correlation functions are related to the response of the string theory to boundary conditions at $r \sim L$. It is therefore natural to identify the generating functional of correlation functions in the gauge theory with the string theory path integral subject to the boundary conditions that $\phi(\boldsymbol{x}, z) = \phi_0(\boldsymbol{x})$ at $z = L$ (at $z = \infty$ all fluctuations are required to vanish). In calculating correlation functions in a CFT, we will carry out the standard Euclidean continuation; then on the string theory side, we will work with $\boldsymbol{L}_5$, which is the Euclidean version of AdS_5.

More explicitly, we identify a gauge theory quantity W with a string-theory quantity Z_{string}:

$$W[\phi_0(\boldsymbol{x})] = Z_{\text{string}}[\phi_0(\boldsymbol{x})] \qquad [18]$$

W generates the connected Euclidean Green's functions of a gauge-theory operator $\mathcal{O}$,

$$W[\phi_0(\boldsymbol{x})] = \left\langle \exp \int d^4 x \phi_0 \mathcal{O} \right\rangle \qquad [19]$$

Z_{string} is the string theory path integral calculated as a functional of ϕ_0, the boundary condition on the field ϕ related to $\mathcal{O}$ by the AdS/CFT duality. In the large-N limit, the string theory becomes classical which implies

$$Z_{\text{string}} \sim e^{-I[\phi_0(\boldsymbol{x})]} \qquad [20]$$

where $I[\phi_0(\boldsymbol{x})]$ is the extremum of the classical string action calculated as a functional of ϕ_0. If we are further interested in correlation functions at very large 't Hooft coupling, then the problem of extremizing the classical string action reduces to solving the equations of motion in type IIB supergravity whose form is known explicitly. A simple example of such a calculation is presented in the next subsection.

Our reasoning suggests that from the point of view of the metric [5], the boundary conditions are imposed not quite at $z = 0$, which is the true boundary of $\boldsymbol{L}_5$, but at some finite value $z = \epsilon$. It does not matter which value it is since the metric [5] is unchanged by an overall rescaling of the coordinates $(z, \boldsymbol{x})$; thus, such a rescaling can take $z = L$ into $z = \epsilon$ for any ϵ. The physical meaning of this cutoff is that it acts as a UV regulator in the gauge theory. Indeed, the radial coordinate z is to be considered as the effective energy scale of the gauge theory, and decreasing z corresponds to increasing the energy. A safe method for performing calculations of correlation functions, therefore, is to keep the cutoff on the z-coordinate at intermediate stages and remove it only at the end.

Two-Point Functions and Operator Dimensions

In the following, we present a brief discussion of two-point functions of scalar operators in CFT_d. The corresponding field in $\boldsymbol{L}_{d+1}$ is a scalar field of mass m whose Euclidean action is proportional to

$$\frac{1}{2} \int d^d x \, dz \, z^{-d+1} \left[(\partial_z \phi)^2 + \sum_{a=1}^{d} (\partial_a \phi)^2 + \frac{m^2 L^2}{z^2} \phi^2 \right] \qquad [21]$$

In calculating correlation functions of vertex operators from the AdS/CFT correspondence, the first problem is to reconstruct an on-shell field in $\boldsymbol{L}_{d+1}$ from its boundary behavior. The near-boundary, that is, small z, behavior of the classical solution is

$$\phi(z, \boldsymbol{x}) \to z^{d-\Delta} \left[\phi_0(\boldsymbol{x}) + O(z^2) \right] + z^{\Delta} \left[A(\boldsymbol{x}) + O(z^2) \right] \qquad [22]$$

where Δ is one of the roots of

$$\Delta(\Delta - d) = m^2 L^2 \qquad [23]$$

$\phi_0(\boldsymbol{x})$ is regarded as a "source" in [19] that couples to the dual gauge-invariant operator $\mathcal{O}$ of dimension Δ, while $A(\boldsymbol{x})$ is related to the expectation value,

$$A(\boldsymbol{x}) = \frac{1}{2\Delta - d}\langle\mathcal{O}(\boldsymbol{x})\rangle \qquad [24]$$

It is possible to regularize the Euclidean action to obtain the following value as a functional of the source:

$$I[\phi_0(\boldsymbol{x})] = -(\Delta - (d/2))\pi^{-d/2}\frac{\Gamma(\Delta)}{\Gamma(\Delta-(d/2))} \times \int \mathrm{d}^d\boldsymbol{x}\int \mathrm{d}^d\boldsymbol{x}'\frac{\phi_0(\boldsymbol{x})\phi_0(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^{2\Delta}} \qquad [25]$$

Varying twice with respect to ϕ_0, we find that the two-point function of the corresponding operator is

$$\langle\mathcal{O}(\boldsymbol{x})\mathcal{O}(\boldsymbol{x}')\rangle = \frac{(2\Delta-d)\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-(d/2))}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|^{2\Delta}} \qquad [26]$$

Which of the two roots, Δ_+ or Δ_-, of [23]

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2} \qquad [27]$$

should we choose for the operator dimension? For positive m^2, Δ_+ is certainly the right choice: here the other root, Δ_-, is negative. However, it turns out that for

$$-\frac{d^2}{4} < m^2L^2 < -\frac{d^2}{4} + 1 \qquad [28]$$

both roots of [23] may be chosen. Thus, there are two possible CFTs corresponding to the same classical AdS action: in one of them the corresponding operator has dimension Δ_+, while in the other the dimension is Δ_-. We note that Δ_- is bounded from below by $(d-2)/2$, which is precisely the unitarity bound on dimensions of scalar operators in d-dimensional field theory! Thus, the ability to choose dimension Δ_- is crucial for consistency of the AdS/CFT duality.

Whether string theory on $\mathrm{AdS}_5 \times Y_5$ contains fields with m^2 in the range [28] depends on Y_5. The example discussed in the next section, $Y_5 = T^{1,1}$, turns out to contain such fields, and the possibility of having dimension Δ_-, [27], is crucial for consistency of the AdS/CFT duality in that case. However, for $Y_5 = \boldsymbol{S}^5$, which is dual to the $\mathcal{N}=4$ large-N SYM theory, there are no such fields and all scalar dimensions are given by [27].

The operators in the $\mathcal{N}=4$ large-N SYM theory naturally break up into two classes: those that correspond to the Kaluza–Klein states of supergravity and those that correspond to massive string states. Since the radius of the S^5 is L, the masses of the Kaluza–Klein states are proportional to $1/L$. Thus, the dimensions of the corresponding operators are independent of L and therefore also of λ. On the gauge-theory side, this independence is explained by the fact that the supersymmetry protects the dimensions of certain operators from being renormalized: they are completely determined by the representation under the superconformal symmetry. All families of the Kaluza–Klein states, which correspond to such protected operators, were classified long ago. Correlation functions of such operators in the strong 't Hooft coupling limit may be obtained from the dependence of the supergravity action on the boundary values of corresponding Kaluza–Klein fields, as in [19]. A variety of explicit calculations have been performed for two-, three-, and even four-point functions. The four-point functions are particularly interesting because their dependence on operator positions is not determined by the conformal invariance.

On the other hand, the masses of string excitations are $m^2 = 4n/\alpha'$, where n is an integer. For the corresponding operators the formula [27] predicts that the dimensions do depend on the 't Hooft coupling and, in fact, blow up for large $\lambda = g^2_{\mathrm{YM}}N$ as $2\lambda^{1/4}\sqrt{n}$.

Calculation of Wilson Loops

The Wilson loop operator of a nonabelian gauge theory

$$W(\mathcal{C}) = \mathrm{tr}\left[P\exp\left(\mathrm{i}\oint_{\mathcal{C}} A\right)\right] \qquad [29]$$

involves the path-ordered integral of the gauge connection A along a contour $\mathcal{C}$. For $\mathcal{N}=4$ SYM, one typically uses a generalization of this loop operator which incorporates other fields in the $\mathcal{N}=4$ multiplet, the adjoint scalars and fermions. Using a rectangular contour, we can calculate the quark–antiquark potential from the expectation value $\langle W(\mathcal{C})\rangle$. One thinks of the quarks located a distance L apart for a time T, yielding

$$\langle W\rangle \sim \mathrm{e}^{-TV(L)} \qquad [30]$$

where $V(L)$ is the potential.

According to Maldacena, and Rey and Yee, the AdS/CFT correspondence relates the Wilson loop expectation value to a sum over string world sheets ending on the boundary of $L_5(z=0)$ along the contour $\mathcal{C}$:

$$\langle W\rangle \sim \int \mathrm{e}^{-S} \qquad [31]$$

where S is the action functional of the string world sheet. In the large 't Hooft coupling limit $\lambda \to \infty$, this path integral may be evaluated using a saddle-point approximation. The leading answer is $\sim e^{-S_0}$, where S_0 is the action for the classical solution, which is proportional to the minimal area of the string world sheet in L_5 subject to the boundary conditions. The area as currently defined is actually divergent, and to regularize it one must position the contour at $z = \epsilon$ (this is the same type of regulator as used in the definition of correlation functions).

Consider a circular Wilson loop of radius a. The action of the corresponding classical string world sheet is

$$S_0 = \sqrt{\lambda}\left(\frac{a}{\epsilon} - 1\right) \qquad [32]$$

Subtracting the linearly divergent term, which is proportional to the length of the contour, one finds

$$\ln\langle W \rangle = \sqrt{\lambda} + O(\ln \lambda) \qquad [33]$$

a result which has been duplicated in field theory by summing certain classes of rainbow Feynman diagrams in $\mathcal{N} = 4$ SYM. From these sums, one finds

$$\langle W \rangle_{\text{rainbow}} = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) \qquad [34]$$

where I_1 is a Bessel function. This formula is one of the few available proposals for extrapolation of an observable from small to large coupling. At large λ,

$$\langle W \rangle_{\text{rainbow}} \sim \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}} \qquad [35]$$

in agreement with the geometric prediction.

The quark–antiquark potential is extracted from a rectangular Wilson loop of width L and length T. After regularizing the divergent contribution to the energy, one finds the attractive potential

$$V(L) = -\frac{4\pi^2\sqrt{\lambda}}{\Gamma(1/4)^4 L} \qquad [36]$$

The Coulombic $1/L$ dependence is required by the conformal invariance of the theory. The fact that the potential scales as the square root of the 't Hooft coupling indicates some screening of the charges at large coupling.

Conformal Field Theories and Einstein Manifolds

Interesting generalizations of the duality between $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM with less supersymmetry and more complicated gauge groups can be produced by placing D3 branes at the tip of a Ricci-flat six-dimensional cone X (see Figure 1). The cone metric may be cast in the form

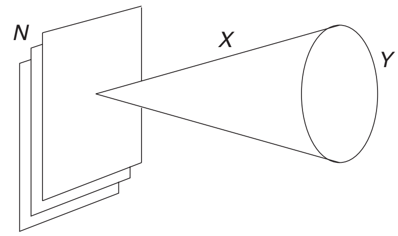

Figure 1 D3 branes placed at the tip of a Ricci-flat cone X.

$$ds_X{}^2 = dr^2 + r^2\, ds_Y{}^2 \qquad [37]$$

where Y is the level surface of X. In particular, Y is a positively curved Einstein manifold, that is, one for which $R_{ij} = 4g_{ij}$. In order to preserve the $\mathcal{N} = 1$ supersymmetry, X must be a Calabi–Yau space; then Y is defined to be Sasaki–Einstein.

The D3 branes appear as a point in X and span the transverse Minkowski space $\mathbb{R}^{3,1}$. The ten-dimensional metric they produce assumes the form [9], but with the sphere metric $d\Omega_5{}^2$ replaced by the metric on Y, ds_Y^2. The equality of tensions [10] now requires that

$$L^4 = \frac{\sqrt{\pi}\kappa N}{2\,\text{vol}(Y)} = 4\pi g_s N \alpha'^2 \frac{\pi^3}{\text{vol}(Y)} \qquad [38]$$

In the near-horizon limit, $r \to 0$, the geometry factors into $AdS_5 \times Y$. Because the D3 branes are located at a singularity, the gauge theory becomes much more complicated, typically involving a product of several SU(N) factors coupled to matter in bifundamental representations, often described using a quiver diagram (see Figure 2 for an example).

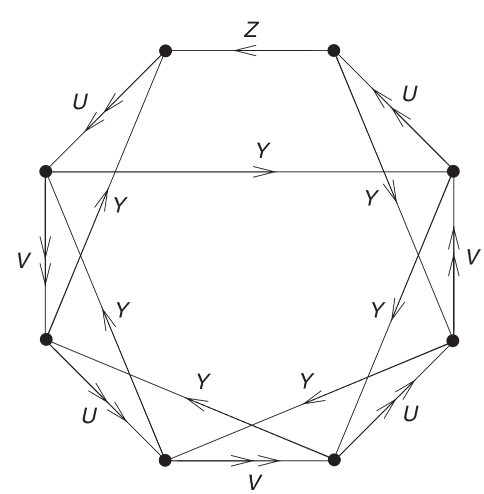

Figure 2 The quiver for $Y^{4,3}$. Each node corresponds to an SU(N) gauge group and each arrow to a bifundamental chiral superfield.

The simplest examples of X are orbifolds $\mathbb{C}^3/\Gamma$, where Γ is a discrete subgroup of SO(6). Indeed, if $\Gamma \subset \mathrm{SU}(3)$, then $\mathcal{N}=1$ supersymmetry is preserved. The level surface of such an X is $Y = S^5/\Gamma$. In this case, the product structure of the gauge theory can be motivated by thinking about image stacks of D3 branes from the action of Γ.

The next simplest example of a Calabi–Yau cone X is the conifold which may be described by the following equation in four complex variables:

$$\sum_{a=1}^{4} {z_a}^2 = 0 \qquad [39]$$

Since this equation is symmetric under an overall rescaling of the coordinates, this space is a cone. The level surface Y of the conifold is a coset manifold $T^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{U}(1)$. This space has the $\mathrm{SO}(4) \sim \mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry which rotates the z's, and also the U(1) R-symmetry under $z_a \to \mathrm{e}^{\mathrm{i}\theta} z_a$. The metric on $T^{1,1}$ is known explicitly; it assumes the form of an S^1 bundle over $S^2 \times S^2$.

The supersymmetric field theory on the D3 branes probing the conifold singularity is $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory coupled to two chiral superfields, A_i, in the $(\boldsymbol{N}, \overline{\boldsymbol{N}})$ representation and two chiral superfields, B_j, in the $(\overline{\boldsymbol{N}}, \boldsymbol{N})$ representation. The A's transform as a doublet under one of the global SU(2)'s, while the B's transform as a doublet under the other SU(2). Cancelation of the anomaly in the U(1) R-symmetry requires that the A's and the B's each have R-charge 1/2. For consistency of the duality, it is necessary that we add an exactly marginal superpotential which preserves the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_{\mathrm{R}}$ symmetry of the theory. Since a marginal superpotential has R-charge equal to 2 it must be quartic, and the symmetries fix it uniquely up to overall normalization:

$$W = \epsilon^{ij}\epsilon^{kl} \operatorname{tr} A_i B_k A_j B_l \qquad [40]$$

There are in fact infinite families of Calabi–Yau cones X, but there are two problems one faces in studying these generalized AdS/CFT correspondences. The first is geometric: the cones X are not all well understood and only for relatively few do we have explicit metrics. However, it is often possible to calculate important quantities such as the vol(Y) without knowing the metric. The second problem is gauge theoretic: although many techniques exist, there is no completely general procedure for constructing the gauge theory on a stack of D-branes at an arbitrary singularity.

Let us mention two important classes of Calabi–Yau cones X. The first class consists of cones over the so-called $Y^{p,q}$ Sasaki–Einstein spaces. Here, p and q are integers with $p \geq q$. Gauntlett *et al.* (2004) discovered metrics on all the $Y^{p,q}$, and the quiver gauge theories that live on the D-branes probing the singularity are now known. Making contact with the simpler examples discussed above, the $Y^{p,0}$ are orbifolds of $T^{1,1}$ while the $Y^{p,p}$ are orbifolds of S^5.

In the second class of cones X, a del Pezzo surface shrinks to zero size at the tip of the cone. A del Pezzo surface is an algebraic surface of complex dimension 2 with positive first Chern class. One simple del Pezzo surface is a complex projective space of dimension 2, $\mathbb{P}^2$, which gives rise to the $\mathcal{N}=1$ preserving $S^5/\mathbb{Z}_3$ orbifold. Another simple case is $\mathbb{P}^1 \times \mathbb{P}^1$, which leads to $T^{1,1}/\mathbb{Z}_2$. The remaining del Pezzos surfaces B_k are $\mathbb{P}^2$ blown up at k points, $1 \leq k \leq 8$. The cone where B_1 shrinks to zero size has level surface $Y^{2,1}$. Gauge theories for all the del Pezzos have been constructed. Except for the three del Pezzos just discussed, and possibly also for B_6, metrics on the cones over these del Pezzos are not known. Nevertheless, it is known that for $3 \leq k \leq 8$, the volume of the Sasaki–Einstein manifold Y associated with B_k is $\pi^3(9-k)/27$.

The Central Charge

The central charge provides one of the most amazing ways to check the generalized AdS/CFT correspondences. The central charge c and conformal anomaly a can be defined as coefficients of certain curvature invariants in the trace of the stress energy tensor of the conformal gauge theory:

$$\langle T^{\alpha}_{\alpha} \rangle = -aE_4 - cI_4 \qquad [41]$$

(The curvature invariants E_4 and I_4 are quadratic in the Riemann tensor and vanish for Minkowski space.) As discussed above, correlators such as $\langle T_{\mu\nu} \rangle$ can be calculated from supergravity, and one finds

$$a = c = \frac{\pi^3 N^2}{4\,\mathrm{vol}(Y)} \qquad [42]$$

On the gauge-theory side of the correspondence, anomalies completely determine a and c:

$$a = \tfrac{3}{32}(3 \operatorname{tr} R^3 - \operatorname{tr} R)$$
$$c = \tfrac{1}{32}(9 \operatorname{tr} R^3 - 5 \operatorname{tr} R) \qquad [43]$$

The trace notation implies a sum over the R-charges of all of the fermions in the gauge theory. (From the geometric knowledge that $a = c$, we can conclude that $\operatorname{tr} R = 0$.)

The R-charges can be determined using the principle of a-maximization. For a superconformal gauge theory, the R-charges of the fermions maximize a subject to the constraints that the

Novikov–Shifman–Vainshtein–Zakharov (NSVZ) beta function of each gauge group vanishes and the R-charge of each superpotential term is 2.

For the $Y^{p,q}$ spaces mentioned above, one finds that

$$\mathrm{vol}(Y^{p,q}) = \frac{q^2\left(2p + \sqrt{4p^2 - 3q^2}\right)}{3p^2\left(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}\right)}\pi^3 \quad [44]$$

The gauge theory consists of $p - q$ fields Z, $p + q$ fields Y, $2p$ fields U, and $2q$ fields V. These fields all transform in the bifundamental representation of a pair of $SU(N)$ gauge groups (the quiver diagram for $Y^{4,3}$ is given in **Figure 2**). The NSVZ beta function and superpotential constraints determine the R-charges up to two free parameters x and y. Let x be the R-charge of Z and y the R-charge of Y. Then the U have R-charge $1 - (1/2)(x + y)$ and the V have R-charge $1 + (1/2)(x - y)$.

The technique of a maximization leads to the result

$$x = \frac{1}{3q^2}\left(-4p^2 + 2pq + 3q^2 + (2p - q)\sqrt{4p^2 - 3q^2}\right)$$

$$y = \frac{1}{3q^2}\left(-4p^2 - 2pq + 3q^2 + (2p + q)\sqrt{4p^2 - 3q^2}\right)$$

Thus, as calculated by Benvenuti *et al.* (2004) and Bertolini *et al.* (2004)

$$a(Y^{p,q}) = \frac{\pi^3 N^2}{4\,\mathrm{vol}(Y^{p,q})} \quad [45]$$

in remarkable agreement with the prediction [42] of the AdS/CFT duality.

A Path to a Confining Theory

There exists an interesting way of breaking the conformal invariance for spaces Y whose topology includes an S^2 factor (examples of such spaces include $T^{1,1}$ and $Y^{p,q}$, which are topologically $S^2 \times S^3$). At the tip of the cone over Y, one may add M wrapped D5 branes to the N D3 branes. The gauge theory on such a combined stack is no longer conformal; it exhibits a novel pattern of quasiperiodic renormalization group flow, called a duality cascade.

To date, the most extensive study of a theory of this type has been carried out for the conifold, where one finds an $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N + M)$ theory coupled to chiral superfields A_1, A_2 in the $(\mathbf{N}, \overline{\mathbf{N+M}})$ representation, and B_1, B_2 in the $(\overline{\mathbf{N}}, \mathbf{N+M})$ representation. D5 branes source RR 3-form flux; hence, the supergravity dual of this theory has to include M units of this flux. Klebanov and Strassler (2000) found an exact nonsingular supergravity solution incorporating the 3-form and the 5-form RR field strengths, and their back-reaction on the geometry. This back-reaction creates a "geometric transition" to the deformed conifold

$$\sum_{a=1}^{4} z_a^2 = \epsilon^2 \quad [46]$$

and introduces a "warp factor" so that the full ten-dimensional geometry has the form

$$ds_{10}{}^2 = h^{-1/2}(\tau)(-(dx^0)^2 + (dx^i)^2) + h^{1/2}(\tau)\,d\tilde{s}_6{}^2 \quad [47]$$

where $d\tilde{s}_6{}^2$ is the Calabi–Yau metric of the deformed conifold, which is known explicitly.

The field-theoretic interpretation of this solution is unconventional. After a finite amount of RG flow, the $SU(N + M)$ group undergoes a Seiberg duality transformation. After this transformation, and an interchange of the two gauge groups, the new gauge theory is $SU(\tilde{N}) \times SU(\tilde{N} + M)$ with the same matter and superpotential, and with $\tilde{N} = N - M$. The self-similar structure of the gauge theory under the Seiberg duality is the crucial fact that allows this pattern to repeat many times. If $N = (k + 1)M$, where k is an integer, then the duality cascade stops after k steps, and we find $SU(M) \times SU(2M)$ gauge theory. This IR gauge theory exhibits a multitude of interesting effects visible in the dual supergravity background. One of them is confinement, which follows from the fact that the warp factor h is finite and nonvanishing at the smallest radial coordinate, $\tau = 0$. The methods presented in the section "Calculation of Wilson loops," then imply that the quark–antiquark potential grows linearly at large distances. Other notable IR effects are chiral symmetry breaking and the Goldstone mechanism. Particularly interesting is the appearance of an entire "baryonic branch" of the moduli space in the gauge theory, whose existence has been demonstrated also in the dual supergravity language.

Conclusions

This article tries to present a logical path from studying gravitational properties of D-branes to the formulation of an exact duality between conformal field theories and string theory in anti-de Sitter backgrounds, and also sketches some methods for breaking the conformal symmetry. Due to space limitations, many aspects and applications of the AdS/CFT correspondence have been omitted. At the moment, practical applications of this duality are limited mainly to very strongly coupled, large-N gauge theories, where the dual string description is well approximated by classical supergravity. To understand the implications of the duality for more general parameters, it is necessary to find better methods for attacking the world sheet approach to string theories in anti-de Sitter backgrounds with RR background fields turned on. When such methods are found, it is likely that the material presented here will have turned out to be just a tiny tip of a monumental iceberg of dualities between fields and strings.

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See also: Brane Construction of Gauge Theories; Branes and Black Hole Statistical Mechanics; Einstein Equations: Exact Solutions; Gauge Theories from Strings; Large-*N* and Topological Strings; Large-*N* Dualities; Mirror Symmetry: A Geometric Survey; Quantum Chromodynamics; Quantum Field Theory in Curved Spacetime; Superstring Theories.

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Affine Quantum Groups

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Affine quantum groups are certain pseudoquasitriangular Hopf algebras that arise in mathematical physics in the context of integrable quantum field theory, integrable quantum spin chains, and solvable lattice models. They provide the algebraic framework behind the spectral parameter dependent Yang–Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \qquad [1]$$

One can distinguish three classes of affine quantum groups, each leading to a different dependence of the R-matrices on the spectral parameter u: Yangians lead to rational R-matrices, quantum affine algebras lead to trigonometric R-matrices, and elliptic quantum groups lead to elliptic R-matrices. We will mostly concentrate on the quantum affine algebras but many results hold similarly for the other classes.

After giving mathematical details about quantum affine algebras and Yangians in the first two sections, we describe how these algebras arise in different areas of mathematical physics in the three following sections. We end with a description of boundary quantum groups which extend the formalism to the boundary Yang–Baxter (reflection) equation.