Contemporary Mathematics
Volume **613**, 2014
http://dx.doi.org/10.1090/conm/613/12236

Ambidexterity

Gijs Heuts and Jacob Lurie

1. Introduction

These notes were taken from lectures given by Jacob Lurie at the 'Thematic Program on Topology and Field Theories', held at the University of Notre Dame in June 2012, and prepared by Gijs Heuts. Sections 2 until 5 correspond to a mini-course given as part of a graduate workshop, section 6 describes a talk given at the conference which followed the workshop. The contents of the mini-course deal with the concept of *ambidexterity*, a duality phenomenon which is closely related to the existence of variants of Dijkgraaf-Witten topological field theories. The lectures are concerned with reviewing Dijkgraaf-Witten theory, motivating and exploring the concept of ambidexterity, and using the theory to produce "generalized Dijkgraaf-Witten theories" with coefficients in $K(n)$-local spectra. A talk given at the conference, entitled 'Loop spaces, p-divisible groups and character theory' describes a categorification of the generalized character theory developed by Hopkins, Kuhn, and Ravenel. We include this material because it has a close connection with the theory of ambidexterity.

For the reader's convenience a list of references is provided at the end of these notes. These are mostly intended to serve as a guide for further background reading.

2. Dijkgraaf-Witten theory

Our goal in this lecture is to give an overview of Dijkgraaf-Witten theory. We begin with an general review of topological quantum field theory.

2.1. Topological quantum field theories.

Definition 2.1. Let M and N be compact oriented manifolds without boundary. An *oriented cobordism from M to N* is a pair (B, ϕ), where B is an oriented compact manifold with boundary and

$$\phi : \partial B \xrightarrow{\simeq} \overline{M} \coprod N$$

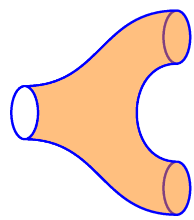
is an orientation-preserving diffeomorphism. Here $\overline{M}$ denotes M with orientation reversed. We will say two such cobordisms (B, ϕ) and (B', ϕ') are equivalent if there exists an orientation-preserving diffeomorphism between B and B' which is compatible with the maps ϕ and ϕ' in the obvious way.

2010 *Mathematics Subject Classification.* Primary 55U35, 55U30.

Definition 2.2. The category $\mathbf{Cob}(n)$ has as objects $(n-1)$-dimensional smooth compact oriented manifolds without boundary. Given two such manifolds M and N, a morphism from M to N is an equivalence class of oriented cobordisms from M to N. Composition is given by gluing cobordisms together. We will regard $\mathbf{Cob}(n)$ as a symmetric monoidal category, with tensor product given by disjoint union of manifolds.

Remark 2.3. The above description of the composition law is imprecise, since it does not provide a smooth structure on the composed cobordism. This can be fixed by introducing auxiliary data such as collars for cobordisms. It turns out the equivalence class of the resulting cobordisms does not depend on these choices.

Example 2.4. The following picture is a representative of a morphism in $\mathbf{Cob}(2)$, where M is a circle and N is a disjoint union of two circles:



The following definition was originally proposed by Atiyah:

Definition 2.5. An *n-dimensional topological quantum field theory* is a symmetric monoidal functor

$$\mathcal{Z} : \mathbf{Cob}(n) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

Here $\mathbf{Vect}_{\mathbb{C}}$ is the category of complex vector spaces endowed with the symmetric monoidal structure coming from the usual tensor product.

Remark 2.6. One can cook up many variants of this definition by replacing oriented manifolds with, for example, framed manifolds, unoriented manifolds or manifolds with spin structure.

Let's unwrap the previous definition a little bit. To specify a topological field theory, we have to provide:

- For each $(n-1)$-dimensional manifold M in $\mathbf{Cob}(n)$, a complex vector space $\mathcal{Z}(M)$
- For each n-dimensional cobordism B from M to N, a complex linear map

$$\mathcal{Z}(B) : \mathcal{Z}(M) \longrightarrow \mathcal{Z}(N)$$

- A collection of isomorphisms

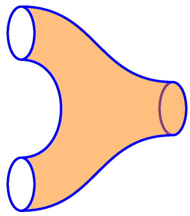
$$\mathcal{Z}(M \coprod N) \simeq \mathcal{Z}(M) \otimes \mathcal{Z}(N) \qquad \mathcal{Z}(\emptyset) \simeq \mathbb{C}$$

These data then have to satisfy various coherence conditions, which we will not spell out explicitly.

To get a better feel for the definition of a field theory it is instructive to analyze the special case $n = 2$. Let $\mathcal{Z}$ be a 2-dimensional topological quantum field theory. Objects of $\mathbf{Cob}(2)$ are compact 1-manifolds without boundary, which are necessarily disjoint unions of circles. Using the fact that $\mathcal{Z}$ is symmetric monoidal, we see that the value of $\mathcal{Z}$ on any object of $\mathbf{Cob}(2)$ is completely determined by $\mathcal{Z}(S^1)$. Set

$$A := \mathcal{Z}(S^1)$$

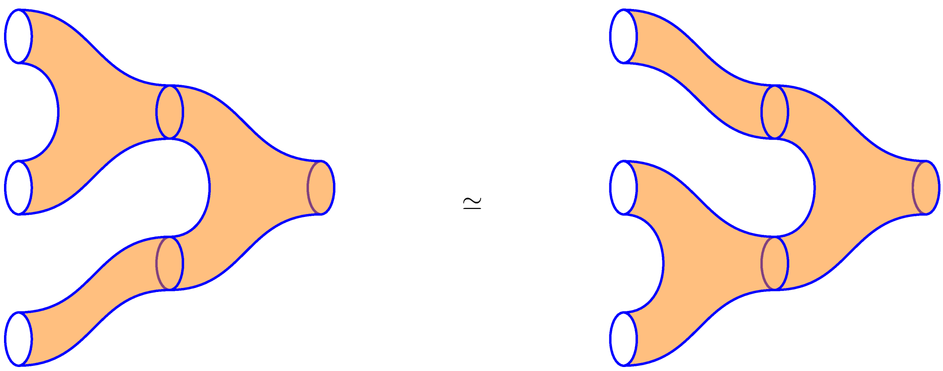
A priori A is just a complex vector space, but we can exploit the fact that $\mathcal{Z}$ is a functor to endow A with a much richer structure. First, we can evaluate $\mathcal{Z}$ on the pair of pants:



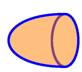
Expressing the domain and codomain of this map in terms of A, this defines a map

$$\mu : A \otimes A \longrightarrow A$$

EXERCISE 2.7. Show that μ makes A into a $\mathbb{C}$-algebra that is both associative and commutative. One of the relevant pictures is the following:



We can also evaluate $\mathcal{Z}$ on the cup, interpreted as a cobordism from the empty set to S^1:



This corresponds to a map

$$\eta : \mathbb{C} \longrightarrow A$$

EXERCISE 2.8. Show that the element $\eta(1)$ provides a unit for the algebra A.

By reversing the cup, we obtain a map

$$\mathrm{tr} : A \longrightarrow \mathbb{C}$$

which, as notation suggests, we will interpret as a trace map.

EXERCISE 2.9. Show that the trace pairing

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\mathrm{tr}} \mathbb{C}$$

is non-degenerate, in the sense that it induces an isomorphism of A with its dual space $A^{\vee}$. Use this to show that A is finite-dimensional.

The above discussion motivates the following definition:

DEFINITION 2.10. A *commutative Frobenius algebra* is a finite-dimensional commutative, associative $\mathbb{C}$-algebra A equipped with a map

$$\mathrm{tr} : A \longrightarrow \mathbb{C}$$

such that the trace pairing $(x, y) \mapsto \mathrm{tr}(xy)$ induces an isomorphism from A to its dual $A^{\vee}$.

Our discussion yields the following:

PROPOSITION 2.11. *Suppose $\mathcal{Z}$ is a 2-dimensional topological quantum field theory. Then $\mathcal{Z}(S^1)$ inherits the structure of a commutative Frobenius algebra.*

The converse assertion is a well-known folk theorem:

THEOREM 2.12. *Suppose A is a commutative Frobenius algebra. Then there exists a 2-dimensional topological quantum field theory $\mathcal{Z}$ such that $\mathcal{Z}(S^1)$, with its canonical Frobenius algebra structure, is isomorphic to A. This field theory is unique up to unique isomorphism.*

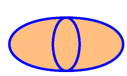
REMARK 2.13. There are many examples of Frobenius algebras. For example:

- Let G be a finite group, and let A be the center of the group ring $\mathbb{C}[G]$. Then A is a commutative Frobenius algebra, with trace given by $\mathrm{tr}(\sum \lambda_g g) = \lambda_e / |G|$, where e denotes the identity element of g and $|G|$ denotes the order of G. (The denominator of $|G|$ is not necessary here, but is convenient: for example, it ensures that the characters of irreducible representations of G form an orthonormal basis for A.)
- The cohomology ring of a compact oriented manifold M is a (graded) commutative Frobenius algebra, with trace map $\mathrm{tr} : H^*(M; \mathbb{C}) \to \mathbb{C}$ given by evaluation on the fundamental homology class of M (the nondegeneracy of tr is equivalent to Poincare duality).
- A complete intersection $A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is a commutative Frobenius algebra provided that it is finite-dimensional as a vector space over $\mathbb{C}$. A nondegenerate trace on A can be given by

$$\mathrm{tr}(p) = \mathrm{Res}\Big(\frac{p\, dx_1 \wedge \cdots \wedge dx_n}{f_1 \cdots f_n}\Big)$$

Suppose we are given a commutative Frobenius algebra A, and let $\mathcal{Z}$ be the associated 2-dimensional topological quantum field theory. We can calculate the value of $\mathcal{Z}$ on an arbitrary surface Σ by cutting Σ into relatively simple pieces (like disks and pairs of pants), and invoking the fact that $\mathcal{Z}$ is a functor.

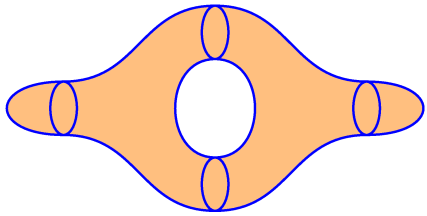
EXAMPLE 2.14. Let $\Sigma = S^2$ be the 2-sphere. Cut this manifold along the equator:



The left hemisphere gives us a map $\mathbb{C} \longrightarrow A$ corresponding to the unit 1 of the algebra A. The right hemisphere corresponds to the trace map $A \longrightarrow \mathbb{C}$. Therefore

$$\mathcal{Z}(\Sigma) = \mathrm{tr}(1)$$

EXAMPLE 2.15. Let Σ be a torus. Then we can decompose Σ as follows:



The left half part of the diagram determines a map

$$\mathbb{C} \longrightarrow A \otimes A^{\vee} \simeq \mathrm{Hom}(A, A)$$

which sends 1 to id_A. The right half gives a map

$$\mathrm{Hom}(A, A) \longrightarrow \mathbb{C}$$

which can be identified with the usual trace on $\mathrm{Hom}(A, A)$. Therefore

$$\mathcal{Z}(\Sigma) = \dim_{\mathbb{C}}(A)$$

2.2. Untwisted Dijkgraaf-Witten theory. In this section we will introduce the untwisted version of Dijkgraaf-Witten theory associated to a finite group G. Let us quickly recall the essential facts about principal G-bundles.

DEFINITION 2.16. Let X be a topological space. A *principal G-bundle on X* is a covering space

$$\pi : \widetilde{X} \longrightarrow X$$

equipped with an action of G on $\widetilde{X}$ which is simply transitive on each fiber. In particular, π induces a homeomorphism $\widetilde{X}/G \simeq X$.

DEFINITION 2.17. We let BG denote any CW-complex satisfying

$$\pi_i(BG) = \begin{cases} G & \text{if } i = 1 \\ * & \text{otherwise} \end{cases}$$

We call such a space a *classifying space* for the group G.

REMARK 2.18. For any group G there exists a classifying space BG. This space is uniquely determined up to homotopy equivalence.

EXAMPLE 2.19. Let G be the group $\mathbb{Z}/2\mathbb{Z}$. Then the infinite-dimensional real projective space $\mathbb{RP}^{\infty}$ is a classifying space for G.

If BG is a classifying space for G, then its universal cover EG is a contractible space equipped with a free action of G. In particular, the covering map $EG \to BG$ is a G-bundle. This G-bundle enjoys the following universal property:

PROPOSITION 2.20. *Let G be a finite group with a classifying space BG. Let X be a CW-complex. Pulling the bundle $\pi : EG \to BG$ back along maps $X \longrightarrow BG$ induces a bijection between $[X, BG]$, the set of homotopy classes of maps from X to BG, and the set of isomorphism classes of principal G-bundles on X.*

In what follows, let us fix a finite group G and an integer $n \geq 0$. We will introduce an n-dimensional topological quantum field theory $\mathcal{Z}$, which we will refer to as (untwisted) Dijkgraaf-Witten theory. For a connected n-manifold M, we set

$$\mathcal{Z}(M) := \frac{|\mathrm{Hom}(\pi_1 M, G)|}{|G|},$$

where $\mathrm{Hom}(\pi_1 M, G)$ denotes the set of group homomorphisms from the fundamental group of M (with respect to an arbitrarily chosen base point) into G. Note that this set is always finite, since the fundamental group of M is finitely generated.

Let us now describe a more conceptual description of the number $\mathcal{Z}(M)$, which makes sense also when M is not connected.

$$\mathcal{Z}(M) := \sum_{[G\text{-bundles } \widetilde{M} \longrightarrow M]} \frac{1}{|\mathrm{Aut}(\widetilde{M})|}$$

where the sum is over isomorphism classes of principal G-bundles $\widetilde{M} \longrightarrow M$ and $\mathrm{Aut}(\widetilde{M})$ denotes the automorphism group of such a bundle. In other words, this formula counts the number of G-bundles on M, but weighted by a 'mass' which depends on the number of automorphisms of each bundle.

We claim that when M is connected, the two definitions we have given for $\mathcal{Z}(M)$ agree. Elementary covering space theory tells us that each homomorphism $\pi_1 M \longrightarrow G$ determines a principal G-bundle. Two G-bundles obtained in this way are isomorphic precisely if the two homomorphisms are conjugate by an element of G. This construction determines a bijection between isomorphism classes of G-bundles and conjugation classes of homomorphisms from $\pi_1 M$ to G. Using this, one sees that

$$\sum_{[G\text{-bundles } \widetilde{M} \longrightarrow M]} \frac{1}{|\mathrm{Aut}(\widetilde{M})|} = \sum_{\{\alpha: \pi_1 M \longrightarrow G\}/\text{conjugacy}} \frac{1}{|Z_G(\alpha)|}$$

where $Z_G(\alpha)$ denotes the centralizer of α in G. But instead of summing over conjugacy classes of homomorphisms, we might as well sum over all homomorphisms and add a factor

$$\frac{1}{|\{\text{conjugates of } \alpha\}|}$$

in each term. But the size of the centralizer of α times the number of its conjugates equals the size of G. This proves

$$\sum_{\{\alpha: \pi_1 M \longrightarrow G\}/\text{conjugacy}} \frac{1}{|Z_G(\alpha)|} = \frac{|\mathrm{Hom}(\pi_1 M, G)|}{|G|}$$

which was our original expression.

We now explain how to extend the construction $M \mapsto \mathcal{Z}(M)$ to $(n-1)$-manifolds, so that it satisfies Atiyah's axioms. For a closed $(n-1)$-manifold M, we set

$$\mathcal{Z}(M) := H^0(\mathrm{Map}(M, BG); \mathbb{C})$$

In words, $\mathcal{Z}(M)$ is the space of locally constant complex functions on the space $\mathrm{Map}(M, BG)$. Recall that the connected components of $\mathrm{Map}(M, BG)$ correspond to isomorphism classes of G-bundles on M, so the dimension of $\mathcal{Z}(M)$ is simply the number of such isomorphism classes.

One can think of the space $\mathrm{Map}(M, BG)$ as a classifying space for G-bundles on M. In fact it is not very hard to describe; the connected component of $\mathrm{Map}(M, BG)$ corresponding to the isomorphism class of a G-bundle $\widetilde{M} \longrightarrow M$ can be identified with a classifying space $B\mathrm{Aut}(\widetilde{M})$ (in the case where M is connected, so that $\widetilde{M}$ is classified by a group homomorphism $\alpha : \pi_1 M \to G$, this automorphism group is isomorphic to the centralizer of α in G). We can therefore write

$$\mathrm{Map}(M, BG) \simeq \coprod_{[\widetilde{M} \longrightarrow M]} B\mathrm{Aut}(\widetilde{M})$$

where the disjoint union is over isomorphism classes of G-bundles.

In order to make $\mathcal{Z}$ a functor we need to specify its behavior on morphisms in $\mathbf{Cob}(n)$. Let B be a cobordism from a closed $(n-1)$-manifold M to another such manifold N. The inclusions of M and N into B determine a diagram

$$\mathrm{Map}(M, BG) \xleftarrow{\;p_M\;} \mathrm{Map}(B, BG) \xrightarrow{\;p_N\;} \mathrm{Map}(N, BG)$$

Suppose we are given $f \in \mathcal{Z}(M)$, which we identify with a locally constant function $\mathbb{C}$-valued on $\mathrm{Map}(M, BG)$. We can pull this function back along p_M to obtain a locally constant function $f \circ p_M$ on the mapping space $\mathrm{Map}(B, BG)$. We can then obtain a locally constant function on $\mathrm{Map}(N, BG)$ by 'integration along the fibers' of p_N. Let us elaborate on what we mean by this. A point $x \in \mathrm{Map}(N, BG)$ determines to a G-bundle $\widetilde{N} \to N$. Its inverse image under p_N is a classifying space for G-bundles on B which restrict to $\widetilde{N}$ on N. This space has finitely many connected components, each of which is the classifying space for a finite group. To 'integrate' the locally constant function $f \circ p_M$ over this space, we sum its values over these connected components, dividing by an auxiliary "mass" factor given by the size of the corresponding finite group. More precisely, we set

$$\mathcal{Z}(B)(f) : \mathrm{Map}(N, BG) \longrightarrow \mathbb{C} : x \longmapsto \sum_{C \in \pi_0(p_N^{-1}(x))} \frac{f(p_M(C))}{|\pi_1 C|}$$

Here $f(p_M(C))$ is meant to denote the value that f takes on any point of $p_M(C)$. This is a well-defined number since f is locally constant.

Example 2.21. Suppose M and N are empty, so that the cobordism B is a closed n-manifold. In this case we are considering the diagram

$$* \longleftarrow \mathrm{Map}(B, BG) \longrightarrow *$$

The map $\mathcal{Z}(B)$ is simply multiplication by a complex number. According to our formula, this number is

$$\sum_{C \in \pi_0(\mathrm{Map}(B, BG))} \frac{1}{|\pi_1 C|} = \sum_{[G-\text{bundles } \widetilde{B} \longrightarrow B]} \frac{1}{|\mathrm{Aut}(\widetilde{B})|}$$

which is exactly the definition of $\mathcal{Z}$ on closed n-manifolds we gave before.

Example 2.22. Consider the special case $n = 2$. According to Theorem 2.12, the topological field theory $\mathcal{Z}$ is completely determined by $A = \mathcal{Z}(S^1)$, regarded as a commutative Frobenius algebra. Unwinding the definitions, we have

$$A = H^0(\mathrm{Map}(S^1, BG); \mathbb{C})$$

For any connected space X we have

$$[S^1, X] = \pi_1(X)/\text{conjugacy}$$

where the left-hand side denotes free homotopy classes of maps. In particular, the connected components of $\mathrm{Map}(S^1, BG)$ can be identified with the set of conjugacy classes of elements of G, so that A can be identified with the set of $\mathbb{C}$-valued class functions on G. This space is familiar from representation theory; the character of a G-representation is a class function on G. Conversely, any class function is a linear combination of irreducible characters in a unique way. Therefore we can write

$$\mathcal{Z}(S^1) = \mathrm{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C}$$

The space of class functions on G can also be identified with the center of the group ring $\mathbb{C}[G]$. Under this identification, the Frobenius algebra structure on A agrees with the one given in Remark 2.13.

By evaluating the topological field theory $\mathcal{Z}$ on closed n-manifolds, we obtain a large number of numerical invariants of the group G. The axiomatics of topological field theory give a nice way of organizing these numbers and expressing the relations between them. Sometimes one can obtain information by computing $\mathcal{Z}$ on a closed manifold in two different ways. For example, consider the torus $T = S^1 \times S^1$. We know that $\mathcal{Z}(T)$ is the number of homomorphisms $\pi_1(T) \longrightarrow G$ divided by the order of G. Since $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$, the number of such homomorphisms is simply the number of pairs of commuting elements of G. On the other hand, by Example 2.14 we know that $\mathcal{Z}(T)$ is the dimension of the space $\mathcal{Z}(S^1)$, i.e. the number of conjugacy classes of G. As a result we get

$$|\{(g, h) \in G \times G \mid gh = hg\}| \quad = \quad |G| \cdot |\mathrm{Conj}(G)|$$

where $\mathrm{Conj}(G)$ denotes the set of conjugacy classes of G.

Exercise 2.23. Verify this directly.

Our goal in the remaining lectures is to describe a general paradigm for producing variants of Dijkgraaf-Witten theory. In particular, we will emphasize two parameters that one might wish to vary:

(1) *The finite group G*: the definition of $\mathcal{Z}$ depends not so much on the group G itself, but on its classifying space BG. We might therefore try to replace BG by some more general space X, and produce a topological field theory whose value on a closed n-manifold M measures the "count" (with appropriate multiplicities) of the number of homotopy classes of maps from M into X. For this to be sensible, we will need to assume that X satisfies some finiteness conditions.
(2) *The complex numbers*: in the above discussion, one could imagine replacing the field $\mathbb{C}$ by an arbitrary commutative ring k. This presents no problem so long as k is an algebra over the field $\mathbb{Q}$ of rational numbers. However, if the order of G is not invertible in k, then we encounter difficulties (many of the expressions we considered above become ill-defined, since

they require us to divide by the order of certain automorphism groups). The ultimate goal of these lectures is to describe a somewhat exotic setting in which these difficulties can be circumvented.

Remark 2.24. Of course one might wonder why we include the denominators in our counting procedures. Let us give some informal justification for this justification. Suppose we are studying spaces with finitely many components, each component having a finite fundamental group and trivial higher homotopy groups (as in the discussion above). Let us try to devise a definition for the 'size' of such a space X. Here are two candidates:

$$\begin{array}{lrcl} \text{Definition 1:} & \text{size}(X) & := & |\pi_0(X)| \\ \text{Definition 2:} & \text{size}(X) & := & \displaystyle\sum_{C \in \pi_0(X)} \frac{1}{|\pi_1(C)|} \end{array}$$

Now suppose we were given an n-fold covering map $\widetilde{X} \longrightarrow X$. Then it is natural to require the equality

$$\text{size}(\widetilde{X}) = n \cdot \text{size}(X)$$

The second definition satisfies this requirement, but the first does not. In fact, this second definition is uniquely characterized by the following three properties:

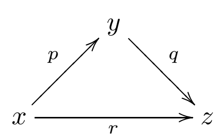
(1) If X is contractible then $\text{size}(X) = 1$.
(2) Size is additive for disjoint unions of spaces.
(3) Size is multiplicative for covering maps, as above.

3. Local systems and twisted Dijkgraaf-Witten theory

3.1. Local systems. We begin this lecture by reviewing the notion of a *local system* on a topological space X. Throughout the following discussion, we fix a category $\mathcal{C}$ (a good example to keep in mind is the case where $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$ is the category of complex vector spaces).

Definition 3.1. A *local system* $\mathcal{L}$ *on* X *with values in* $\mathcal{C}$ is given by the following:

(a) For every point $x \in X$, an object $\mathcal{L}_x \in \mathcal{C}$
(b) For every path $p : [0,1] \longrightarrow X$, an isomorphism $\mathcal{L}_p : \mathcal{L}_{p(0)} \simeq \mathcal{L}_{p(1)}$
(c) For every 2-simplex



in X, the isomorphisms specified in (b) should satisfy $\mathcal{L}_r = \mathcal{L}_q \circ \mathcal{L}_p$

Equivalently, a local system on X is a functor

$$\mathcal{L} : \pi_{\leq 1}(X) \longrightarrow \mathcal{C}$$

where $\pi_{\leq 1}(X)$ denotes the fundamental groupoid of X.

Example 3.2. Let X be a smooth manifold and let $\mathcal{C}$ be the category of real vector spaces. If $\mathcal{L}$ is a vector bundle on X equipped with a flat connection, then $\mathcal{L}$ determines a $\mathcal{C}$-valued local system on X: the value of $\mathcal{L}$ at a point $x \in X$ is the fiber $\mathcal{L}_x$, and the map $\mathcal{L}_p : \mathcal{L}_{p(0)} \to \mathcal{L}_{p(1)}$ associated to a path $p : [0,1] \to X$ is given by parallel transport along p (or, if p is not smooth, along any smooth path homotopic to p).

Remark 3.3. If X is connected and we choose a base point $x \in X$, then specifying a local system $\mathcal{L}$ on X is equivalent to specifying the object $\mathcal{L}_x \in \mathcal{C}$ and an action of the group $\pi_1(X, x)$ on that object. In particular, if $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$ then giving a local system is equivalent to giving a representation of the fundamental group of X.

Example 3.4. Take $\mathcal{C}$ to be the category of 1-dimensional complex vector spaces. When defining a local system on X, we have (up to isomorphism) only one choice of what to assign to every point, namely $\mathbb{C}$. To every path p in X we have to assign an invertible complex number $\mathcal{L}_p \in \mathbb{C}^*$. These numbers have to satisfy the condition $\mathcal{L}_r = \mathcal{L}_q \mathcal{L}_p$ for every 2-simplex as described in the Definition 3.1. This is exactly the data of a 1-cocycle with values in $\mathbb{C}^*$. More precisely, the data of a $\mathbb{C}^*$-valued 1-cocycle on X is equivalent to the data of a $\mathcal{C}$-valued local system $\mathcal{L}$ together with a choice of isomorphism $\mathcal{L}_x \simeq \mathbb{C}$ for each $x \in X$. Different choices of isomorphism will give cocycles which differ by a 1-coboundary. This sets up a bijection

$$\Big\{\text{1-dimensional local systems on } X\Big\}/\text{isomorphism} \quad \simeq \quad H^1(X; \mathbb{C}^*)$$

We will henceforth assume that the category $\mathcal{C}$ admits small limits and colimits.

Definition 3.5. Let X be a topological space and let $\mathcal{L}$ be a $\mathcal{C}$-valued local system on X. We define an object $H^0(X; \mathcal{L}) \in \mathcal{C}$ by the formula

$$H^0(X; \mathcal{L}) := \varprojlim \mathcal{L}.$$

We will refer to $H^0(X; \mathcal{L})$ as the *space of sections* of $\mathcal{L}$.

Concretely, in case $\mathcal{C}$ is the category of complex vector spaces, a section is just a choice of vector $v_x \in \mathcal{L}_x$ for every $x \in X$ which is holonomy invariant in the sense that

$$\mathcal{L}_p(v_{p(0)}) = v_{p(1)}$$

for every path p in X. For a local system coming from a vector bundle with flat connection the space $H^0(X; \mathcal{L})$ is exactly the space of flat sections of that bundle.

We also have the following dual definition:

Definition 3.6. Let X be a topological space and let $\mathcal{L}$ be a $\mathcal{C}$-valued local system on X. We define an object $H_0(X; \mathcal{L}) \in \mathcal{C}$ by the formula

$$H_0(X; \mathcal{L}) := \varinjlim \mathcal{L}.$$

In case $\mathcal{C}$ is the category of complex vector spaces, this admits an explicit description as follows:

$$H_0(X; \mathcal{L}) = \Big(\bigoplus_{x \in X} \mathcal{L}_x\Big)/(v - \mathcal{L}_p(v))_{p \in PX}$$

In words, we take the direct sum of all the vector spaces $\mathcal{L}_x$ and quotient by relations which identify two vectors if one is the image under the other under parallel transport along some path p in X.

3.2. Twisted Dijkgraaf-Witten theory. Throughout this section, let us fix a finite group G, and integer $n \geq 0$, and a cohomology class $\eta \in H^n(BG; \mathbb{C}^*)$. From this data we construct an n-dimensional topological quantum field theory $\mathcal{Z}$, which reduces to our earlier construction in the special case $\eta = 0$.

Let us begin by describing the value of $\mathcal{Z}$ on a closed connected n-manifold B. As in the previous lecture, we consider homomorphisms $\alpha : \pi_1 B \longrightarrow G$. To any such homomorphism we can associate a map $B \longrightarrow BG$ which we will denote by $\overline{\alpha}$. The value of our field theory is given by

$$\mathcal{Z}(B) = \frac{1}{|G|} \sum_{\alpha:\pi_1 B \to G} (\overline{\alpha}^* \eta)[B]$$

Note that this formula reduces to what we had before in case η is the trivial cohomology class.

Let us now describe the vector space $\mathcal{Z}(M)$ associated to a closed $(n-1)$-manifold M. For such a manifold M we again consider the mapping space $\mathrm{Map}(M, BG)$. We have a diagram

$$\begin{array}{ccc} M \times \mathrm{Map}(M, BG) & \xrightarrow{\mathrm{ev}_M} & BG \\ \downarrow{\scriptstyle \pi} & & \\ \mathrm{Map}(M, BG) & & \end{array}$$

where the horizontal map is the evaluation and the vertical map is the projection. Take the n-dimensional cohomology class $\mathrm{ev}_M^* \eta$ and integrate it along the fibers of π. Since these fibers have dimension $n-1$ we obtain a 1-dimensional cohomology class

$$\int_M \mathrm{ev}_M^* \eta \in H^1(\mathrm{Map}(M, BG); \mathbb{C}^*)$$

Invoking the discussion of the previous section we obtain, up to isomorphism, a local system $\mathcal{L}_M$ on $\mathrm{Map}(M, BG)$. Now define

$$\mathcal{Z}(M) := H^0(\mathrm{Map}(M, BG); \mathcal{L}_M)$$

Remark 3.7. We are cheating slightly here, since the local system $\mathcal{L}_M$ is only defined up to isomorphism. To address this, we should specify not only the cohomology class η, but a choice of cocycle that represents η. We will suppress mention of this in our discussion.

Let us now discuss the functoriality of the preceding construction with respect to morphisms in $\mathbf{Cob}(n)$. Suppose we are given a cobordism B from M to N, and consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Map}(B, BG) & & \\ & \swarrow^{p_M} & & \searrow^{p_N} & \\ \mathrm{Map}(M, BG) & & & & \mathrm{Map}(N, BG) \end{array}$$

We have local systems $\mathcal{L}_M$ and $\mathcal{L}_N$ given by thecohomology classes $\int_M \mathrm{ev}_M^* \eta$ and $\int_N \mathrm{ev}_N^* \eta$ on $\mathrm{Map}(M, BG)$ and $\mathrm{Map}(N, BG)$ respectively. It follows that the local systems $p_M^* \mathcal{L}_M$ and $p_N^* \mathcal{L}_N$ are associated to the cohomology classes given by integrating $\mathrm{ev}_B^* \eta$ over M and N respectively, where ev_B is the evaluation map

$$B \times \mathrm{Map}(B, BG) \longrightarrow BG$$

The fundamental classes of M and N are homologous in B; indeed, B is a homology between them. This choice of homology determines an isomorphism between the local systems $p_M^*\mathcal{L}_M$ and $p_N^*\mathcal{L}_N$. Let us denote both of these pullbacks by $\mathcal{L}_B$.

We would like to show that B determines a map from global sections of $\mathcal{L}_M$ to global sections of $\mathcal{L}_N$. Note first that we can pull back sections of $\mathcal{L}_M$ to sections of $\mathcal{L}_B$. Unfortunately, there is *a priori* no canonical map from $H^0(\mathrm{Map}(B, BG); \mathcal{L}_B)$ to $H^0(\mathrm{Map}(N, BG); \mathcal{L}_N)$: the formation of cohomology is functorial, but in the wrong direction. However, in this situation, we happen to get lucky: there are canonical isomorphisms

$$\begin{aligned} H_0(\mathrm{Map}(B, BG); \mathcal{L}_B) &\simeq H^0(\mathrm{Map}(B, BG); \mathcal{L}_B) \\ H_0(\mathrm{Map}(N, BG); \mathcal{L}_N) &\simeq H^0(\mathrm{Map}(N, BG); \mathcal{L}_N). \end{aligned}$$

We can use these isomorphisms to convert cohomology classes into homology classes, which have the desired variance properties.

3.3. Norm maps. Let us now study in detail the special features of our situation which allow us to identify homology with cohomology. Consider a space of the form $X = \mathrm{Map}(K, BG)$, where K is a finite cell complex (for example, any compact manifold with boundary). Let $\mathcal{L}$ be a local system of complex vector bundles on X. As we have seen, there is a homotopy equivalence

$$X \simeq \coprod_{[G\text{-bundles } \widetilde{K} \to K]} B\mathrm{Aut}(\widetilde{K}),$$

where the union is taken over all isomorphism classes of G-bundles on K. Let X_0 be a connected component of X, and write $X_0 \simeq BH$ for some finite group H. It follows that the restriction $\mathcal{L}_M|_{X_0}$ can be identified with a complex vector space V equipped with an action of H. We have canonical isomorphisms

$$H^0(X_0; \mathcal{L}_M|_{X_0}) \simeq V^H = \{v \in V \mid hv = v \quad \forall h \in H\}$$

$$H_0(X_0; \mathcal{L}_M|_{X_0}) \simeq V_H = V/(hv - v)_{h \in H, v \in V}$$

Using the finiteness of the group H, we can produce a canonical map from V_H to V^H, called the *norm map.* Indeed, consider the averaging map

$$\mathrm{av} : V \longrightarrow V : v \longmapsto \sum_{h \in H} hv$$

Clearly any vector in the image of this map is H-invariant, so that av takes values in the subset $V^H \subseteq V$. Moreover, the map av annihilates any vector of the form $hv - v$, and therefore factors through the quotient V_H. It follows that the averaging map factors as a composition

$$V \to V_H \stackrel{\mathrm{Nm}}{\to} V^H \hookrightarrow V.$$

We refer to Nm as the *norm map.* The fact that this map is an isomorphism follows from the following simple observation:

PROPOSITION 3.8. *Suppose A is an abelian group acted on by a finite group H. If multiplication by the order of H induces an isomorphism from A to itself, then the norm map*

$$\mathrm{Nm} : A_H \longrightarrow A^H$$

admits an inverse, which is induced by the map

$$A \longrightarrow A : a \longmapsto \frac{a}{|H|}$$

REMARK 3.9. The hypothesis that multiplication by $|H|$ is an isomorphism from A to itself is automatic when A is a vector space over the complex numbers (or the field of rational numbers).

The invertibility of the norm maps are what make the construction of Dijkgraaf-Witten theory as in the previous section possible: they provide an identification between homology and cohomology, and thereby provide a single invariant which exhibits both covariant and contravariant dependence on the domain space X. Moreover, the auxiliary factors $\frac{1}{|H|}$ which appear in construction of Dijkgraaf-Witten theory arise from the necessity of writing down an inverse to the norm map, as in Proposition 3.8.

Our goal for the remainder of this lectures is to develop a framework that allows for a definition of Dijkgraaf-Witten theory using local systems with values in categories which are more exotic than $\mathbf{Vect}_{\mathbb{C}}$. This motivates our central question:

Question: Let $\mathcal{C}$ be a category and let X be a topological space. Under what conditions can we produce a canonical isomorphism $H^0(X; \mathcal{L}) \simeq H_0(X; \mathcal{L})$, for every $\mathcal{C}$-valued local system on X?

3.4. Ambidexterity. Let us now fix an arbitrary category $\mathcal{C}$ which admits small limits and colimits. For every topological space X, the collection of $\mathcal{C}$-valued local systems on X can be organized into a category which we will denote by $\mathcal{C}^X$. A map of spaces $f : X \longrightarrow Y$ induces a pullback functor

$$f^* : \mathcal{C}^Y \longrightarrow \mathcal{C}^X,$$

given on objects by the formula $(f^*\mathcal{L})_x = \mathcal{L}_{f(x)}$. Using the fact that $\mathcal{C}$ has all limits and colimits one deduces that f^* has a right adjoint

$$f_* : \mathcal{C}^X \longrightarrow \mathcal{C}^Y$$

and a left adjoint

$$f_! : \mathcal{C}^X \longrightarrow \mathcal{C}^Y$$

In the language of category theory, these functors are respectively the right and left Kan extension along the functor of fundamental groupoids $\pi_{\leq 1}X \longrightarrow \pi_{\leq 1}Y$ determined by f. In the special case where f is a fibration (which can always be arranged by a suitable enlargement of X, without changing its homotopy type), these functors are given explicitly by the formulae

$$(f_*\mathcal{L})_y \simeq H^0(f^{-1}(y); \mathcal{L}|_{f^{-1}(y)}) \qquad (f_!\mathcal{L})_y \simeq H_0(f^{-1}(y); \mathcal{L}|_{f^{-1}(y)})$$

More other words, the functors f_* and $f_!$ are given respectively by cohomology and homology along the fibers of f.

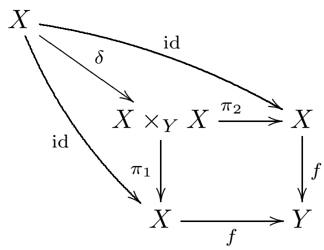
EXAMPLE 3.10. If Y consists of a single point, then $\mathcal{C}^Y = \mathcal{C}$ and we have

$$f_*\mathcal{L} \simeq H^0(X; \mathcal{L}) \qquad f_!\mathcal{L} \simeq H_0(X; \mathcal{L}).$$

We can therefore ask the following relative version of our previous question:

Question: Given a map $f : X \longrightarrow Y$, can we find an isomorphism between the functors $f_!$ and f_*?

We now outline a construction which attempts to give a positive answer to this question, at least in some very special cases. Let us assume for simplicity that the map f is a fibration. Consider the diagram



Assume that we are given an isomorphism $\mu : \delta_* \simeq \delta_!$. We will use μ to produce a natural transformation $\mathrm{Nm} : f_! \to f_*$, which generalizes the usual norm map for representations of finite groups. In the diagram above, we have identities

$$\mathrm{id}_X = \pi_1 \circ \delta = \pi_2 \circ \delta$$

Therefore we can write

$$\mathrm{id}_{\mathcal{C}^X} \simeq (\mathrm{id}_X)_*(\mathrm{id}_X)^* \simeq (\pi_2)_*\delta_*\delta^*(\pi_1)^*$$

Composing with the isomorphism $\mu : \delta_* \simeq \delta_!$, we obtain a natural isomorphism

$$\mathrm{id}_{\mathcal{C}^X} \simeq (\pi_2)_*\delta_!\delta^*(\pi_1)^*$$

Applying the counit of the adjunction $(\delta_!, \delta^*)$, which is a natural transformation from $\delta_!\delta^*$ to the identity on $\mathcal{C}^{X\times_Y X}$, we now have a natural transformation

$$\mathrm{id}_{\mathcal{C}^X} \longrightarrow (\pi_2)_*(\pi_1)^*$$

We claim that the functor on the right is isomorphic to f^*f_*. Indeed, since the square in the previous diagram is a pullback, the fiber of π_2 over a point $x \in X$ is isomorphic to the fiber of f over $f(x) \in Y$. Therefore pulling back along π_1 and taking cohomology along the fibers of π_2 gives the same result as first taking cohomology along the fibers of f and then pulling back to X along f. This gives us a natural transformation

$$\mathrm{id}_{\mathcal{C}^X} \longrightarrow f^*f_*,$$

which is equivalent to the data of a natural transformation

$$\mathrm{Nm} : f_! \longrightarrow f_*,$$

which we will refer to as the *norm map*.

Definition 3.11. The class of $\mathcal{C}$-*ambidextrous maps* is the smallest class of maps containing all homotopy equivalences and satisfying the following closure property: if $f : X \longrightarrow Y$ is a map such that

(1) The map $\delta : X \longrightarrow X \times_Y X$ is $\mathcal{C}$-ambidextrous
(2) The natural transformation $\mathrm{Nm} : f_! \longrightarrow f_*$ is an isomorphism

then f is $\mathcal{C}$-ambidextrous. Here the norm map Nm of f is defined using the natural isomorphism $\mu : \delta_* \simeq \delta_!$, given by the *inverse* of the norm map associated to δ.

Remark 3.12. Although the definition focuses on ambidexterity as a property of the map f, it also depends heavily on the category $\mathcal{C}$. This will become apparent in our examples.

4. Ambidexterity

In this lecture, we let $\mathcal{C}$ denote a category which admits small limits and colimits. Recall that we are trying to answer the following:

Question: Given a map of topological spaces $f : X \longrightarrow Y$, when does there exists a canonical isomorphism of functors $f_! \simeq f_*$ from $\mathcal{C}^X$ to $\mathcal{C}^Y$?

In the previous lecture, we introduced the class of $\mathcal{C}$-*ambidextrous* maps of spaces, for which there exists such an isomorphism $\mathrm{Nm} : f_! \simeq f_*$. We begin by unwrapping the recursion implicit in our definition of Nm. To simplify the discussion, we will use the term *ambidextrous* to refer to the property of being $\mathcal{C}$-ambidextrous (though we should emphasize that this property depends strongly on the choice of category $\mathcal{C}$).

Definition 4.1. A space X is ambidextrous if the map $X \longrightarrow *$ is ambidextrous.

Remark 4.2. If X is ambidextrous and $\mathcal{L}$ is a $\mathcal{C}$-valued local system on X, we get an isomorphism

$$\mathrm{Nm} : H_0(X; \mathcal{L}) \xrightarrow{\simeq} H^0(X; \mathcal{L})$$

If we have a fibration $f : X \longrightarrow Y$ which is ambidextrous, then all the fibers of f are ambidextrous spaces. The converse is almost true; if a fibration $f : X \longrightarrow Y$ has ambidextrous fibers and there exists an $n \geq 0$ such that the homotopy groups of all the fibers vanish in degrees n and higher, then f is ambidextrous. So for many purposes, it suffices to restrict our attention to the case where Y is a point.

Let us consider the situation of a map $f : Z \longrightarrow *$. If Z is ambidextrous there exists an inverse to the norm map, i.e. some natural transformation

$$\mu : f_* \longrightarrow f_!$$

What exactly does such a natural transformation buy us? Suppose we are given objects $C, D \in \mathcal{C}$, which we can think of as local systems on the point, and suppose we are given a continuous map

$$\rho : Z \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, D)$$

where the space on the right just has the discrete topology. Then this determines a map of local systems

$$\rho : f^* C \longrightarrow f^* D$$

Using μ we can write down a sequence of maps

$$C \longrightarrow f_* f^* C \xrightarrow{\mu f^*} f_! f^* C \xrightarrow{f_! \rho} f_! f^* D \longrightarrow D$$

The first map is the unit of the (f^*, f_*)-adjunction, the last one the counit of the $(f_!, f^*)$-adjunction. We will denote this composite map by

$$\int_{z \in Z} \rho(z)\, d\mu$$

As notation suggests, we would like to think of this map as some sort of integral of ρ over the space Z, with respect to a "measure" given by μ. We can summarize the situation informally by saying that for every ambidextrous space Z, the existence of the inverse of the norm map determines an 'integration procedure'

$$\mathrm{Map}(Z, \mathrm{Hom}_{\mathcal{C}}(C, D)) \stackrel{\int d\mu}{\to} \mathrm{Hom}_{\mathcal{C}}(C, D).$$

Now suppose X is a space such that the diagonal map $\delta : X \longrightarrow X \times X$ is ambidextrous. Let $f : X \to *$ denote the constant map. Then we have an isomorphism $\mu : \delta^* \simeq \delta_!$ and this allows us to construct a norm map

$$\mathrm{Nm} : \varinjlim_{x \in X} \mathcal{L}_x = f_! \mathcal{L} \longrightarrow f_* \mathcal{L} = \varprojlim_{y \in X} \mathcal{L}_y$$

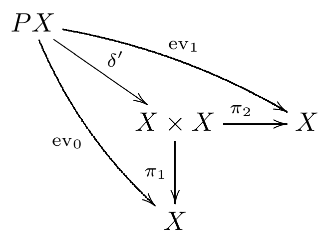
for any $\mathcal{C}$-valued local system $\mathcal{L}$ on X. But by the universal properties of limits and colimits, specifying such a map is the same thing as specifying a compatible family of maps

$$\mathrm{Nm}_{x,y} : \mathcal{L}_x \longrightarrow \mathcal{L}_y$$

Unwinding the definitions, we see that the maps $\mathrm{Nm}_{x,y}$ are given by the formula

$$\mathrm{Nm}_{x,y} = \int_{p \in PX_{(x,y)}} \mathcal{L}_p \, d\mu$$

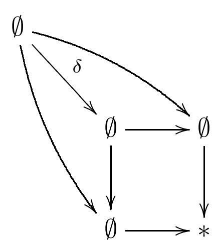
where $PX_{(x,y)}$ denotes the space of paths in X starting at x and ending at y. One obtains this formula by replacing the diagonal map $\delta : X \longrightarrow X \times X$ by the path fibration $\delta' : PX \longrightarrow X \times X$ and analyzing the diagram



The maps ev_0 and ev_1 are given by evaluation and the points $0, 1 \in [0,1]$, and the fibers of δ' are exactly the path spaces $PX_{(x,y)}$.

4.1. Examples of Ambidexterity. Let us now unwind the meaning of ambidexterity for some reasonably simple examples of topological spaces X.

Example 4.3. Let $X = \emptyset$ be the empty space, and consider the unique map $f : \emptyset \longrightarrow *$. The relevant diagram now becomes



The diagonal map δ is an isomorphism, so that the evident isomorphism $\delta_* \simeq \delta_!$ yields a norm map $\mathrm{Nm}_\emptyset : f_! \longrightarrow f_*$. The category of $\mathcal{C}$-valued local systems on a point is equivalent to $\mathcal{C}$. The functor $f_!$ is given by taking the colimit of the empty diagram, which yields the initial object of $\mathcal{C}$. Similarly the functor f_* yields the final object of $\mathcal{C}$. Consequently, the norm map Nm can be identified with the unique morphism in $\mathcal{C}$ from the initial object to the final object. The empty space $\emptyset$ is ambidextrous if and only if this map is an isomorphism. This condition is satisfied if and only $\mathcal{C}$ is a *pointed* category: that is, if it has an object which is both initial and final. In this case, we will denote such an object by 0, and refer to it as a *zero object* of $\mathcal{C}$. The condition that $\mathcal{C}$ be pointed is nontrivial: for example, the category of sets is not pointed. However, it is satisfied in many other categories of interest, such as the category of groups or the category of vector spaces.

Suppose now that $\mathcal{C}$ is pointed, and let $\mu : f_* \to f_!$ be an inverse to the norm map. For every pair of objects $C, D \in \mathcal{C}$, the integration procedure

$$\mathrm{Map}(\emptyset, \mathrm{Hom}_{\mathcal{C}}(C, D)) \xrightarrow{\int d\mu} \mathrm{Hom}_{\mathcal{C}}(C, D).$$

can be identified with a single homomorphism from C to D. Unwinding the definitions, we see that this is the *zero map*, uniquely characterized by the fact that it factors as a composition

$$C \longrightarrow 0 \longrightarrow D$$

Example 4.4. Let $\mathcal{C}$ be a pointed category. Then the empty space $\emptyset$ is ambidextrous (Example 4.3). It follows that any fibration $f : X \to Y$ is ambidextrous, provided that the fibers of f are either empty or contractible. To find more ambidextrous spaces X, we might therefore first try spaces for which the diagonal map $X \longrightarrow X \times X$ has either empty or contractible homotopy fibers. This condition is satisfied whenever X has the discrete topology. In this case, a local system $\mathcal{L}$ on X can be identified with a collection of objects of $\mathcal{C}$ indexed by the points of X. For such a local system $\mathcal{L}$ we have isomorphisms

$$f_!\mathcal{L} \simeq \coprod_{x \in X} \mathcal{L}_x \qquad f_*\mathcal{L} = \prod_{y \in X} \mathcal{L}_y$$

Using Example 4.3, we see that the associated norm map

$$\coprod_{x \in X} \mathcal{L}_x \to \prod_{y \in X} \mathcal{L}_y$$

is given by the 'identity matrix'

$$\mathrm{Nm}_{(x,y)} = \begin{cases} \mathrm{id}_{\mathcal{L}_x} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

The discrete space X is ambidextrous if this map is an isomorphism, for every $\mathcal{C}$-valued local system $\mathcal{L}$ on X. This condition holds for some categories $\mathcal{C}$, and fails for others. For example, suppose that $\mathcal{C}$ is the category of groups. In this case, a two-point space is not ambidextrous: if $\mathcal{L}$ is the local system with value $\mathbb{Z}$, then the norm map

$$\mathbb{Z} * \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$$

has domain given by a free group on two generators, and codomain the free *abelian* group on two generators.

The situation is better if $\mathcal{C}$ is the category of abelian groups. In this case, the norm map

$$\coprod_{x \in X} \mathcal{L}_x \to \prod_{y \in X} \mathcal{L}_y$$

is an isomorphism whenever the set X is finite (both sides can be identified with the *direct sum* $\bigoplus_{x \in X} \mathcal{L}_x$). This generally fails when X is infinite: the canonical map from the direct sum of abelian groups into the direct product is in general only a monomorphism, but not an isomorphism.

Assume now that the category $\mathcal{C}$ has the property that finite sets are ambidextrous. Then for any finite set X, we have an integration procedure

$$\mathrm{Map}(X, \mathrm{Hom}_{\mathcal{C}}(C, D)) \xrightarrow{\int} \mathrm{Hom}_{\mathcal{C}}(C, D).$$

In particular, taking X to have two elements, we obtain an 'addition law'

$$+ : \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{C}}(C, D).$$

One can show that these addition laws endow each mapping space $\mathrm{Hom}_{\mathcal{C}}(C, D)$ with the structure of a commutative monoid, and that the composition law

$$\mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathrm{Hom}_{\mathcal{C}}(D, E) \to \mathrm{Hom}_{\mathcal{C}}(C, E)$$

is bilinear. This motivates the following:

Definition 4.5. The category $\mathcal{C}$ is called *semi-additive* if finite sets are $\mathcal{C}$-ambidextrous.

Remark 4.6. If $\mathcal{C}$ is semiadditive, the integration procedure described above gives a way of adding up different maps from C to D. In this way the sets $\mathrm{Hom}_{\mathcal{C}}(C, D)$ acquire the structure of commutative monoids.

Example 4.7. Any additive category is semi-additive. In particular, for any ring R, the category of (left or right) R-modules is semi-additive. The category of commutative monoids is an example of a semi-additive category which is not additive.

Example 4.8. Assume that the category $\mathcal{C}$ is semi-additive. It follows that any finite-sheeted covering map $\widetilde{Y} \to Y$ is ambidextrous. To find more examples of ambidextrous spaces, we might consider spaces X for which the diagonal map $\delta_X : X \longrightarrow X \times X$ induces a homotopy equivalence from X to a finite-sheeted covering of $X \times X$. This condition is satisfied when $X = BG$ is the classifying space of a finite group G: this case, δ_X factors as a composition

$$EG/G \xrightarrow{\delta'} (EG \times EG)/G \xrightarrow{\delta''} (EG \times EG)/(G \times G),$$

where δ' is a homotopy equivalence and δ'' is a covering space whose fibers can be identified with G.

The category $\mathcal{C}^{BG}$ of local systems $\mathcal{L}$ on BG can be identified with the category of objects V of $\mathcal{C}$ equipped with an action of G. We then have isomorphisms

$$H_0(BG; \mathcal{L}) \simeq V_G \qquad H^0(BG; \mathcal{L}) \simeq V^G.$$

Using the semi-additivity of $\mathcal{C}$, we obtain a norm map $H_0(BG; \mathcal{L}) \to H^0(BG; \mathcal{L})$, which is uniquely characterized by the fact that the composite map

$$V \to V_G \xrightarrow{\mathrm{Nm}} V^G \to V$$

is obtained by 'integrating' the map $G \to \mathrm{Hom}_{\mathcal{C}}(V, V)$ which classifies the action of G on V.

It follows that the classifying space BG is ambidextrous if and only if, for each object $V \in \mathcal{C}$ equipped with an action of G, the norm map $\mathrm{Nm} : V_G \to V^G$ is invertible. This condition is not satisfied when $\mathcal{C}$ is the category of abelian groups. However, it is satisfied if $\mathcal{C}$ is the category $\mathbf{Vect}_{\mathbb{C}}$ of complex vector spaces, or in any situation where it is possible to 'divide by the order of G' (see Proposition 3.8).

The invertibility of the norm map $\mathrm{Nm} : V_G \to V^G$ was the crucial ingredient needed in the construction of Dijkgraaf-Witten style topological field theories (in either the twisted or untwisted case).

Example 4.9. Suppose that the category $\mathcal{C}$ is semiadditive and that the classifying space BG is ambidextrous for some *abelian* group G. Let X denote the Eilenberg-MacLane space $K(G, 2)$, which is determined up to (weak) homotopy equivalence by the existence of isomorphisms

$$\pi_i X \simeq \begin{cases} G & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let us again consider the diagonal map $\delta_X : X \longrightarrow X \times X$. The homotopy fibers of this map are equivalent to BG. Consequently, our assumption that BG is ambidextrous guarantees that δ_X is ambidextrous, so that we obtain a norm map

$$\mathrm{Nm} : H_0(X; \mathcal{L}) \longrightarrow H^0(X; \mathcal{L})$$

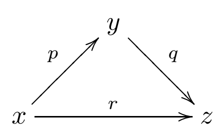
However, this situation is not very interesting. Recall that local systems are functors from the fundamental groupoid of X to $\mathcal{C}$. In this case, the fundamental groupoid of X is trivial, so specifying a local system on X boils down to picking an object $C \in \mathcal{C}$. In this case, we can identify $H_0(X; \mathcal{L})$ and $H^0(X; \mathcal{L})$ with C (under this identification, the norm map is still slightly interesting: for example, if $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$, then it is given by multiplication by $\frac{1}{|G|}$).

It is possible to proceed further with Example 4.9, taking X to be Eilenberg-MacLane spaces of the form $K(G, n)$ for $n > 2$. This does not really get any more interesting: since X is simply connected, all local systems on X are trivial, so the norm maps supplied by the theory of ambidexterity reduce to an elaborate bookkeeping device for keeping track of powers of $|G|$. However, the situation becomes considerably more interesting if pass to the world of *higher* category theory.

Definition 4.10. An ∞-*category* is a simplicial set $\mathcal{C}$ which satisfies the following *weak Kan condition*: for $0 < i < n$, every map of simplicial sets from the inner horn Λ^n_i into $\mathcal{C}$ can be extended to an n-simplex of $\mathcal{C}$.

Let $\mathcal{C}$ be an ∞-category and let X be a topological space. Then a $\mathcal{C}$-*valued local system on* X is a map of simplicial sets $\mathcal{L}$ from the singular complex of X to the ∞-category $\mathcal{C}$. More informally, such a map is given by the following data:

(0) For each point $x \in X$ an object $\mathcal{L}_x \in \mathcal{C}$
(1) For each path $p : x \longrightarrow y$ in X an equivalence $\mathcal{L}_p : \mathcal{L}_x \longrightarrow \mathcal{L}_y$
(2) For each 2-simplex



in X a homotopy between the morphisms $\mathcal{L}_r$ and $\mathcal{L}_q \circ \mathcal{L}_p$

$\vdots$

Remark 4.11. Simplicial sets satisfying the weak Kan condition of Definition 4.10 are also called *quasi-categories* and *weak Kan complexes* in the literature.

REMARK 4.12. For every ordinary category $\mathcal{C}$, the nerve $N(\mathcal{C})$ is an ∞-category. The construction $\mathcal{C} \mapsto N(\mathcal{C})$ is fully faithful, so we can regard the theory of ∞-categories as a generalization of classical category theory. Moreover, it is a robust generalization: most important notions from classical category theory (initial and final objects, limits and colimits, adjoint functors, ...) can be generalized to the setting of ∞-categories. In particular, the theory of ambidexterity can be developed in the ∞-categorical setting. Moreover, it is potentially much more interesting: if $\mathcal{C}$ is an ∞-category, then the notion of a $\mathcal{C}$-valued local system on a topological space X does *not* depend only on the fundamental groupoid of X. In particular, there may exist nontrivial examples of local systems, even when the space X is simply connected.

Roughly speaking, we can think of an ∞-category $\mathcal{C}$ as a kind of higher category where all n-morphisms are invertible for $n > 1$. In particular, $\mathcal{C}$ consists of a collection of objects (given by the vertices of $\mathcal{C}$), morphisms (given by the 1-simplices of $\mathcal{C}$), homotopies between morphisms (encoded by 2-simplices of $\mathcal{C}$), homotopies between homotopies, and so forth. In what follows, we will be content to describe ∞-categories in these informal terms, referring the reader to the references for more details.

Let us list some important examples of ∞-categories.

(a) The archetypical example is the ∞-category of spaces: the objects are topological spaces (which we typically assume to be homotopy equivalent to cell complexes), 1-morphisms are given by continuous maps, 2-morphisms are given by homotopies, 3-morphisms are given by homotopies between homotopies, and so forth.
(b) If R is a ring, we can construct an ∞-category whose objects are chain complexes of modules over R. The 1-morphisms are given by chain maps, 2-morphisms are given by chain homotopies, and so forth.
(c) The collection of all (extraordinary) cohomology theories can be organized into an ∞-category, called the *∞-category of spectra*. We will return to this example in the next lecture.

EXAMPLE 4.13. Let k be a field and let $\mathcal{C}$ be the ∞-category of chain complexes of vector spaces over k. We can consider k as an object of $\mathcal{C}$ (namely, the chain complex which is given by k in degree zero, and vanishes in nonzero degrees). If $f : X \longrightarrow *$ is a map of spaces and $\mathcal{L}$ is the constant local system with value k on X, then we obtain chain complexes $f_!\mathcal{L}, f_*\mathcal{L} \in \mathcal{C}$. These chain complexes are only defined up to equivalence $\mathcal{C}$, which in this case means quasi-isomorphism. Concretely, they are given by

$$f_!\mathcal{L} \simeq C_*(X;k) \qquad f_*\mathcal{L} \simeq C^*(X;k).$$

We can now ask which spaces are $\mathcal{C}$-ambidextrous.

- The empty set is ambidextrous: the ∞-category $\mathcal{C}$ has a zero object, given by the chain complex which is zero in each degree.
- Finite sets are ambidextrous, since finite sums and products agree in the category (or ∞-category) of chain complexes.
- Let G be a finite group. Local systems on BG are now chain complexes with an action of G. Given such a chain complex, we can talk about the homology and cohomology of G with coefficients in that chain complex. Our constructions give a norm map from homology to cohomology. The failure

of that map to be an isomorphism is measured by the *Tate cohomology* of G. The space BG is ambidextrous if and only if the Tate cohomology of G vanishes (with coefficients in any chain complex of representations of G). This condition holds if and only if the characteristic of k does not divide the order of G. In particular, if the field k has characteristic zero, then BG is always ambidextrous.

- Assume the characteristic of k is zero, and let G be a finite abelian group. Then BG is ambidextrous, so it makes sense to consider the ambidexterity of the Eilenberg-MacLane space $K(G,2)$. One can show $K(G,2)$ is also ambidextrous. Unfortunately, this is true for a boring reason: the using the finiteness of G and the assumption $\mathrm{char}(k) = 0$, one can show that all $\mathcal{C}$-valued local systems on $K(G,2)$ are trivial. One can find interesting local systems in the case where $\mathrm{char}(k)$ divides the order of G, but in this case the ambidexterity of BG will fail.

Example 4.13 raises the following question: is it possible for a space like $K(G,2)$ to be $\mathcal{C}$-ambidextrous in a setting where there *are* nontrivial local systems on $K(G,2)$? In the next lecture we will study a more exotic situation (arising from the study of chromatic homotopy theory) where the answer is affirmative.

5. Ambidexterity in the $K(n)$-local category

5.1. Stable homotopy theory. Our goal in this lecture is to give some rather nontrivial example of ∞-categories where ambidexterity holds for a large class of spaces. We begin with a brief review of stable homotopy theory.

Recall that a *cohomology theory* is a sequence of contravariant functors E^n from the category of pairs of spaces to the category of abelian groups. These functors should satisfy several axioms, such as homotopy invariance, excision and existence of certain long exact sequences. The standard example is singular cohomology with coefficients in an abelian group A:

$$(Y \subseteq X) \longmapsto H^n(X,Y;A)$$

A key fact about cohomology theories is the *Brown representability theorem*, which reads:

Theorem 5.1 (Brown representability). *For any cohomology theory E there exist pointed spaces $Z(n)$ such that for any space X we have*

$$E^n(X) = [X, Z(n)]$$

where the brackets indicate homotopy classes of maps.

Example 5.2. If E is singular cohomology with coefficients in an abelian group A, then the spaces $Z(n)$ are Eilenberg-MacLane spaces $K(A,n)$. The weak homotopy type of these spaces is completely determined by the formulae

$$\pi_i K(A,n) = \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

In general, the spaces $Z(n)$ representing a cohomology theory E are related to one another. It follows from the definitions that any cohomology theory E is equipped with *suspension isomorphisms*

$$\widetilde{E}^n(\Sigma X) \simeq \widetilde{E}^{n-1}(X);$$

here ΣX denotes the suspension of a pointed space X, and $\widetilde{E}$ denotes the reduced cohomology theory associated to E (so that $\widetilde{E}(X) = \mathrm{coker}(E(*) \to E(X))$). It follows that for any pointed space X, we have functorial bijections

$$[X, Z(n-1)] \simeq [\Sigma X, Z(n)] \simeq [X, \Omega Z(n)],$$

so that $Z(n-1)$ is homotopy equivalent to the loop space of $Z(n)$.

Therefore there exist homotopy equivalences

$$Z(n-1) \simeq \Omega Z(n)$$

This type of data has a name:

Definition 5.3. A *spectrum* is a sequence of spaces $\{Z(n)\}_{n \geq 0}$ equipped with homotopy equivalences $Z(n-1) \simeq \Omega Z(n)$.

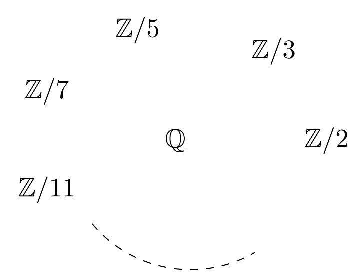
Every cohomology theory gives rise to a spectrum; conversely, any spectrum Z gives rise to a cohomology theory E by defining

$$E^n(X) = [X, Z(n)]$$

The collection of all spectra can be organized into an ∞-category, which we will denote by $\mathcal{S}\mathrm{p}$. Equivalence classes of objects in $\mathcal{S}\mathrm{p}$ are in one-to-one correspondence with cohomology theories.

Example 5.4. For every abelian group A, the sequence of Eilenberg-MacLane spaces $\{K(A, n)\}_{n \geq 0}$ determine a spectrum. This spectrum is typically denoted by HA, and referred to as the *Eilenberg-MacLane spectrum* associated to A. The associated cohomology theory is given by $X \mapsto H^*(X; A)$.

The construction $A \mapsto HA$ determines a fully faithful embedding from the ordinary category of abelian groups to the ∞-category of spectra. Consequently, we can think of the theory of spectra as a generalization of the theory of abelian groups. By definition, an abelian group is a module over the ring $\mathbb{Z}$ of integers. The structure of the category of $\mathbb{Z}$-modules is to some extent 'controlled' by the Zariski spectrum $\mathrm{Spec}\,\mathbb{Z}$, which we indicate impressionistically with the following diagram:



Here we identify the points of $\mathrm{Spec}\,\mathbb{Z}$ with their corresponding residue fields, which we can regard as abelian groups. These residue fields can be thought of as basic building blocks, out of which any other abelian group can be constructed. For example, if M is any abelian group, we have a canonical map

$$\theta : M \to M \otimes \mathbb{Q}.$$

Here $M \otimes \mathbb{Q}$ is a vector space over $\mathbb{Q}$, hence a direct sum of copies of $\mathbb{Q}$. Moreover, the kernel and cokernel of θ are torsion abelian groups and therefore admit filtrations whose successive quotients are vector spaces over $\mathbb{Z}/p$, where p varies over the prime numbers.

Among all abelian groups, the groups $\mathbb{Q}$ and $\mathbb{Z}/p$ can be characterized as those abelian groups M which satisfy the following pair of conditions:

- There exists a multiplication map $M \times M \to M$ which endows M with the structure of a field.
- The abelian group M is indecomposable: that is, we cannot write $M = M_0 \oplus M_1$ for $M_0, M_1 \neq 0$.

Let us now try to translate these hypotheses to conditions on the corresponding cohomology theory. Note that for any commutative ring R, the cohomology theory $X \mapsto H^*(X; R)$ is *multiplicative*: that is, we have multiplication maps

$$\mu_{X,Y} : H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R).$$

Moreover, the following conditions are equivalent:

- The commutative ring R is a field.
- The cohomology theory $X \mapsto H^*(X; R)$ satisfies the Künneth formula. That is, the map $\mu_{X,Y}$ is an isomorphism whenever X and Y are finite cell complexes.

This motivates the following definition:

Definition 5.5. Let E be a multiplicative cohomology theory. We will say that E is a *field* if, whenever X and Y are finite cell complexes, the multiplication map

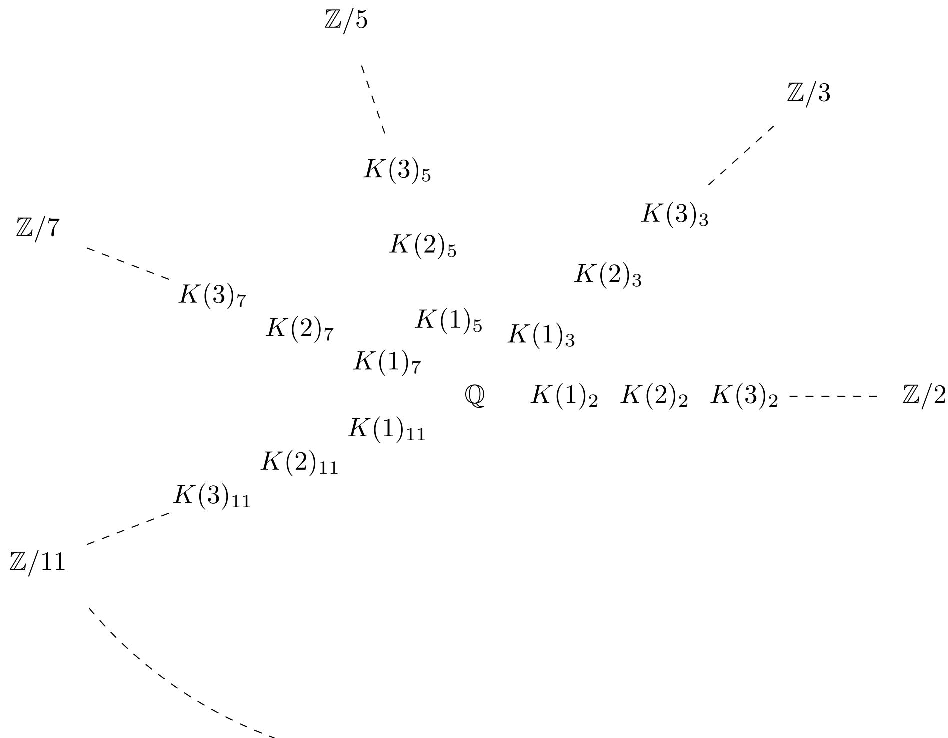
$$E^*(X) \otimes_{E^*(*)} E^*(Y) \to E^*(X \times Y)$$

is an isomorphism.

In this case, we will say that E is a *prime field* if it cannot be written as a direct sum $E_0 \oplus E_1$, where $E_0, E_1 \neq 0$.

Example 5.6. Let R be a commutative ring. Then the Eilenberg-MacLane spectrum HR is a field if and only if R is a field. The Eilenberg-MacLane spectrum HR is a prime field if and only if $R = \mathbb{Q}$ or $R = \mathbb{Z}/p$ for some prime number p.

In addition to the Eilenberg-MacLane spectra $H\mathbb{Q}$ and $H\mathbb{Z}/p$, there are many 'exotic' prime fields in the ∞-category of spectra, called *Morava K-theories.* These can be organized into a 'geometric' picture which controls the organization of the ∞-category of spectra, much as the affine scheme Spec$\mathbb{Z}$ controls the organization of the ordinary category of abelian groups.



Here one refers to the cohomology theory $K(n)_p$ as the *nth Morava K-theory at the prime p*. It is characterized among prime fields by the structure of the coefficient ring $K(n)^*_p(*) \simeq \mathbb{Z}/p[v^{\pm 1}]$, where the generator v has degree $2p^n - 2$. In what follows, we will fix a prime number p and denote the cohomology theory $K(n)_p$ simply by $K(n)$.

EXAMPLE 5.7. If X is a finite cell complex, we let $K^0(X)$ denote the Grothendieck group of complex vector bundles on X. The construction $X \mapsto K^0(X)$ can be extended to a cohomology theory which is represented by a spectrum K, which we refer to as the (periodic) *complex K-theory spectrum*. Let K/p denote the cofiber of the map $p : K \to K$. One can show that that the cohomology theory K/p is a field, which can be written as a direct sum of $p-1$ (shifted) copies of the prime field $K(1)$. In particular, the Morava K-theory $K(1)$ appears as a direct summand of 'complex K-theory modulo p'. However, the higher Morava K-theories $\{K(n)\}_{n\geq 2}$ are much more mysterious from a geometric point of view.

REMARK 5.8. For a fixed prime number p, the Morava K-theories $K(n)$ are defined for $0 < n < \infty$. However, it is often convenient to extend this definition to the case $n = 0$ (by taking $K(0) = H\mathbb{Q}$) and the case $n = \infty$ (by taking $K(\infty) = H\mathbb{Z}/p$). One should think of the sequence of cohomology theories

$$H\mathbb{Q} = K(0), K(1), K(2), \dots, K(\infty) = H\mathbb{Z}/p$$

as increasing in complexity: the cohomology theory $X \mapsto K(n)^*(X)$ becomes increasingly sensitive to 'p-torsion phenomena' as n increases.

To help the reader's intuition we offer the following table of analogies between the category of abelian groups and the category of spectra:

Abelian groups	Spectra
prime fields $\mathbb{Z}/p$, $\mathbb{Q}$	Morava K-theories
the tensor product $\otimes$	the smash product $\wedge$
reduction mod p, i.e. $M \longmapsto M \otimes \mathbb{Z}/p$	$E \longmapsto E \wedge K(n)$

In the category of abelian groups we can choose to work 'at the prime p'. We can do a similar thing in the category of spectra, not only at p but at every Morava K-theory. Let us introduce the relevant definitions.

Definition 5.9. A spectrum X is *$K(n)$-acyclic* if $X \wedge K(n)$ is contractible. A spectrum X is *$K(n)$-local* if, for every $K(n)$-acyclic spectrum Y, every map $f : Y \to X$ is homotopic to zero. The collection of $K(n)$-local spectra comprise a full subcategory of the ∞-category of spectra, which we will denote by $\mathcal{S}\mathrm{p}^{K(n)}$.

Remark 5.10. One can also describe the ∞-category $\mathcal{S}\mathrm{p}^{K(n)}$ as the quotient $\mathcal{S}\mathrm{p}/\big(K(n)\text{-acyclic spectra}\big)$. In other words, it is obtained from the ∞-category of spectra by 'throwing away' all information which cannot be detected by the Morava K-theory $K(n)$.

Example 5.11. Let $n = 0$, so that $K(n) = H\mathbb{Q}$. One can show that the ∞-category $\mathcal{S}\mathrm{p}^{K(n)}$ is equivalent to the ∞-category of chain complexes of $\mathbb{Q}$-vector spaces, studied in the previous lecture. We can informally summarize the situation with the slogan 'rational stable homotopy theory is easy': that is, it can be reduced to linear algebra.

5.2. $K(n)$-local ambidexterity. We have now set the scene for our main result.

Definition 5.12. Let X be a topological space. We will say that X is *π-finite* if it has finitely many connected components, each component has finite homotopy groups, and there exists a number $n > 0$ such that all homotopy groups vanish in degrees higher than n.

Theorem 5.13 (Hopkins, Lurie). *Let $0 \leq n < \infty$ and let $\mathcal{C} = \mathcal{S}\mathrm{p}^{K(n)}$ be the ∞-category of $K(n)$-local spectra. Then every π-finite space is $\mathcal{C}$-ambidextrous.*

Remark 5.14. In case $n = 0$, this is the 'boring' ambidexterity we encountered in the previous lecture: every $\mathcal{C}$-valued local system on a π-finite space X is trivial as soon as X is simply connected, and the invertibility of the relevant norm maps follows from our ability to "divide" by the sizes of certain finite groups. However, for $n > 0$, the situation is much more subtle. In general, the classification of $\mathcal{C}$-valued local systems on X depends on the first $(n + 1)$-homotopy groups of X. Moreover, one generally cannot 'divide by p' in the $K(n)$-local ∞-category: for example, multiplication by p induces the zero map from the spectrum $K(n)$ to itself.

Here is an interesting consequence of the theorem:

Corollary 5.15. *Let $\mathcal{C} = \mathcal{S}\mathrm{p}^{K(n)}$ for $0 \leq n < \infty$, let X be a π-finite space, and let $f : X \longrightarrow *$ be the projection map. Then the functor*

$$f_* : \mathcal{C}^X \longrightarrow \mathcal{C}$$

commutes with colimits. Similarly

$$f_! : \mathcal{C}^X \longrightarrow \mathcal{C}$$

commutes with limits.

This is surprising behavior: f_* is a right adjoint and therefore automatically commutes with limits. Commutation with colimits, however, is special. The proof of this corollary is immediate by using that f_* is isomorphic to $f_!$, which is a left adjoint. In fact, the statements in the corollary are equivalent to the theorem, although it requires a bit of work to prove this.

Another consequence of our theorem brings us back to the considerations at the start of this lecture series:

Corollary 5.16. *Let E be a $K(n)$-local E_∞-ring spectrum. Then we can construct a version of Dijkgraaf-Witten theory with coefficients in E.*

Remark 5.17. An E_∞-ring spectrum is the analog of a commutative ring in the world of ordinary algebra. A commutative ring is the same thing as a commutative monoid in the category of abelian groups. An E_∞-ring should be thought of as a commutative monoid in the category of spectra. However, we do not impose commutativity on the nose, but rather up to given homotopies. These homotopies then, should be related by given higher homotopies, and so forth all the way to infinity. Said differently, an E_∞-ring is a spectrum which is a commutative ring up to *coherent* homotopy.

Let us conclude by giving a very brief indication of the proof of Theorem 5.13. The first step is to use that any π-finite space X admits a finite filtration by Eilenberg-MacLane spaces of the form $K(\mathbb{Z}/\ell, m)$ (some additional complications arise when the fundamental group of X is not nilpotent, which we will ignore). We can therefore reduce to the case $X = K(\mathbb{Z}/\ell, m)$, where we proceed by induction on m. When $\ell \neq p$, ambidexterity is a consequence of the fact that one can divide by ℓ in the group $\mathbb{Z}/p$, as in the characteristic zero case. The interesting part is ambidexterity for spaces of the form $K(\mathbb{Z}/p, m)$. Given any local system $\mathcal{L}$ on such a space, we would have to prove that the norm map

$$\mathrm{Nm} : f_!\mathcal{L} \longrightarrow f_*\mathcal{L}$$

(which exists by the inductive hypothesis) is an equivalence. It turns out one can reduce to proving this only in the case where $\mathcal{L}$ is the trivial local system, i.e. the constant local system with value $K(n)$. In this case, the norm map becomes a map from $K(n)$-homology to $K(n)$-cohomology:

$$\mathrm{Nm} : K(n)_*K(\mathbb{Z}/p, m) \longrightarrow K(n)^*K(\mathbb{Z}/p, m)$$

We wish to show that this map is an isomorphism: in other words, that it gives a nondegenerate "trace" pairing on the homology ring $K(n)_*K(\mathbb{Z}/p, m)$. The proof then proceeds by explicit calculation, using the fundamental work of Ravenel-Wilson (which provides a very convenient description of the ring $K(n)_*K(\mathbb{Z}/p, m)$).

EXAMPLE 5.18. Let $K_p^\wedge$ denote the p-adic completion of the complex K-theory spectrum. Then $K_p^\wedge$ is a $K(1)$-local spectrum. It follows from Theorem 5.13 that for every π-finite space X, the function spectrum $(K_p^\wedge)^X$ is (canonically) self-dual as a $K_p^\wedge$-module.

Let $X = BG$, where G is a finite p-group. It follows from the Atiyah-Segal completion theorem that $K_p^\wedge(X)$ can be identified with the p-adic completion of the representation ring of G. The self-duality of $(K_p^\wedge)^X$ induces an isomorphism of $\mathrm{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with its $\mathbb{Z}_p$-linear dual, which agrees with the usual trace pairing on $\mathrm{Rep}(G)$ (see Remark 2.13).

6. Loop spaces, p-divisible groups and character theory

REMARK 6.1. The notes in this section were taken from a lecture that was not part of the mini-course that is the content of the rest of these notes. This section is therefore not a continuation of the previous one, but contains some closely related ideas.

Let G be a finite group and let $\mathrm{Rep}(G)$ denote its representation ring. A fundamental theorem of the representation theory of finite groups is that a representation is determined up to isomorphism by its character. So there is an injective map

$$\chi : \mathrm{Rep}(G) \longrightarrow \{\text{conjugation invariant functions } G \longrightarrow \mathbb{C}\}$$

which takes a representation V to its character χ_V. This map becomes an isomorphism after tensoring $\mathrm{Rep}(G)$ with $\mathbb{C}$.

Let us now translate the preceding result into the language of homotopy theory. There is a close relationship between the representation theory of G and the group $K(BG)$, where K denotes complex K-theory. A representation of G can be identified with a local system on BG with values in the category of finite-dimensional complex vector spaces, and yields in particular a finite-dimensional complex vector bundle on BG and therefore an element of the K-group $K(BG)$. This procedure defines a homomorphism

$$\mathrm{Rep}(G) \longrightarrow K(BG)$$

This map is almost an isomorphism. More precisely, the Atiyah-Segal completion theorem tells us that this map becomes an isomorphism after completing $\mathrm{Rep}(G)$ with respect to the augmentation ideal $I = \ker(\mathrm{Rep}(G) \xrightarrow{\dim} \mathbb{Z})$.

Now assume G is a finite p-group. In this case, the operation of I-adic completion is subsumed by the operation of p-adic completion (since $I^n \subseteq (p)$ for $n \gg 0$). Using the Atiyah-Segal completion theorem, we obtain an isomorphism

$$\mathrm{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \widehat{K}(BG)$$

where the notation $\widehat{K}$ indicates p-adically completed K-theory. This is a cohomology theory that on finite CW-complexes gives exactly the p-completion of the ordinary K-groups. In particular, if we choose an embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}$, then we obtain an isomorphism

$$\mathrm{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \widehat{K}(BG) \otimes_{\mathbb{Z}_p} \mathbb{C}$$

Let us identify the space of conjugation-invariant functions on G with the space of locally constant functions on the free loop space of BG: that is, with the cohomology group $H^0(\mathrm{Map}(S^1, BG); \mathbb{C})$ We can then rephrase the basic theorem of representation theory as asserting the existence of an isomorphism

$$\widehat{K}(BG) \otimes_{\mathbb{Z}_p} \mathbb{C} \xrightarrow{\simeq} H^0(\mathrm{Map}(S^1, BG); \mathbb{C})$$

As such, it is telling us something about the relation between K-theory and ordinary cohomology. More precisely, it says that after killing the torsion in K-theory, the K-theory of the space BG is the same as the cohomology of its free loop space.

There is a generalization of this theorem by Hopkins-Kuhn-Ravenel, where we replace p-adic K-theory by some more general cohomology theory E. It is a statement of the same flavour; it relates the complexified E-cohomology of a space X to the ordinary complex cohomology of a space which is closely related to X. In order to state this result precisely, we need some preliminaries.

DEFINITION 6.2. A *Morava E-theory* is a cohomology theory E satisfying the following conditions:

(1) E can be represented by an E_∞-ring spectrum
(2) The cohomology of a point is given by

$$E^*(\mathrm{pt}) \simeq \mathbb{Z}_p[\![v_1, \ldots, v_{n-1}]\!][u^{\pm 1}]$$

for some integer $n \geq 1$. The v_i have degree 0 and u has degree 2.
(3) $E^*(B\mathbb{Z}/p)$ is a free module over $E^*(\mathrm{pt})$ of rank p^n

REMARK 6.3. Since E is a ring spectrum, $E^*(B\mathbb{Z}/p)$ is automatically a module over $E^*(\mathrm{pt})$. The requirement is just that it be free of the right dimension.

For any p and $n \geq 1$ there exists a Morava E-theory satisfying the conditions listed above. This theory is usually denoted E_n, suppressing p from the notation. Moreover, this E_n is unique up to 'Galois twisting'. If one modifies the definition by replacing $\mathbb{Z}_p$ by the ring of Witt vectors of the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{Z}/p$, then the corresponding cohomology theory would be uniquely determined.

EXAMPLE 6.4. If $n = 1$, then the p-adically completed K-theory spectrum $\widehat{K}$ is an example of a Morava E-theory. Indeed, $\widehat{K}$ admits an E_∞-structure (arising from the tensor product operation on complex vector bundles), the $\widehat{K}$-cohomology ring of a point is $\mathbb{Z}_p[u^{\pm 1}]$ where u is the inverse of the Bott element, and $\widehat{K}(B\mathbb{Z}/p)$ is the p-completed representation ring of $\mathbb{Z}/p$. Since the group $\mathbb{Z}/p$ has exactly p irreducible representations, this is a free module of rank p over $\mathbb{Z}_p$.

For the remainder of this lecture, we will fix a Morava E-theory E and an embedding of commutative rings $E^0(\mathrm{pt}) \hookrightarrow \mathbb{C}$.

THEOREM 6.5 (Hopkins-Kuhn-Ravenel). *Let G be a finite p-group. Then there is an isomorphism*

$$E^0(BG) \otimes_{E^0(\mathrm{pt})} \mathbb{C} \xrightarrow{\simeq} \{\text{conjugation invariant functions on } \mathrm{Hom}(\mathbb{Z}^n, G)\}$$

Note that in case $n = 1$ this reduces to our earlier reformulation of the fundamental result on the representation theory of G. Let us rewrite the assertion of the theorem in the following form, where X denotes BG:

$$E^0(X) \otimes_{E^0(\mathrm{pt})} \mathbb{C} \simeq H^0(\mathrm{Map}(T^n, X); \mathbb{C})$$

Here T^n denotes the n-torus, i.e. the product of n copies of S^1. In fact, this statement is not only true when $X = BG$, but whenever X is a p-finite space.

Definition 6.6. Let X be a topological space. We will say that X is *p-finite* if it has finitely many connected components, and the homotopy groups $\pi_i(X, x)$ are finite p-groups for $i > 0$ which vanish for $i \gg 0$ and moreover all these groups vanish for $i \gg 0$.

The goal of the rest of this lecture is to obtain a categorification of Theorem 6.5. More precisely, we would like to deduce the Hopkins-Kuhn-Ravenel theorem from the following:

Hypothesis: Let E be a Morava E-theory and X a p-finite space. Then there exists a category $\mathcal{C}$ with a unit object $\mathbf{1}$, there exists some procedure of 'tensoring with $\mathbb{C}$' in this category $\mathcal{C}$, which we will denote by $V \mapsto V_{\mathbb{C}}$, satisfying the following axioms:

(a) $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) \simeq E^0(X)$

(b) There exists an isomorphism

$$\mathrm{Hom}^{\mathbb{C}}_{\mathcal{C}}(\mathbf{1}_{\mathbb{C}}, \mathbf{1}_{\mathbb{C}}) \simeq H^0(\mathrm{Map}(T^n, X); \mathbb{C}),$$

where the left hand side denotes the set of $\mathbb{C}$-linear maps from $\mathbf{1}_{\mathbb{C}}$ to itself in $\mathcal{C}$.

(c) $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) \otimes_{E^0(\mathrm{pt})} \mathbb{C} \simeq \mathrm{Hom}^{\mathbb{C}}_{\mathcal{C}}(\mathbf{1}_{\mathbb{C}}, \mathbf{1}_{\mathbb{C}})$

To prove Theorem 6.5, it will suffice to show that there exists a category $\mathcal{C}$ which satisfies axioms (a), (b), and (c). To produce $\mathcal{C}$, it will be convenient to work at the level of ∞-categories, rather than ordinary categories. Let us begin with a naive attempt.

Non-example 6.7. Since E is represented by an E_∞-ring spectrum, there exists a good theory of modules over E, which is very much analogous to the theory of modules over an ordinary ring. The modules over E can be organized into an ∞-category Mod_E. Now let $\mathcal{C}$ be the homotopy category of the ∞-category of Mod_E-valued local systems on X. In words, an object of $\mathcal{C}$ is a family of cohomology theories parametrized by X, which are acted on by E. Let $\mathbf{1}$ be the constant local system with value E. The embedding $E^0(\mathrm{pt}) \hookrightarrow \mathbb{C}$ determines a complexification operation on the ∞-category $\mathcal{C}$. Let us consider each of our requirements in turn:

(a) The existence of a canonical isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) \simeq E^0(X)$ follows immediately from the definitions.

(b) It follows easily from the definitions that we have an isomorphism

$$\mathrm{Hom}^{\mathbb{C}}_{\mathcal{C}}(\mathbf{1}_{\mathbb{C}}, \mathbf{1}_{\mathbb{C}}) \simeq H^0(X; \mathbb{C}),$$

so that condition (b) fails.

(c) Combining Theorem 6.5 with the failure of (b), we see that condition (c) must also fail.

Remark 6.8. If condition (c) were to hold for our category $\mathcal{C}$, we would obtain a much more version of Theorem 6.5: namely, the existence of an isomorphism

$$E^0(X) \otimes_{E^0(\mathrm{pt})} \mathbb{C} \simeq H^{\mathrm{even}}(X; \mathbb{C})$$

However, Theorem 6.5 is interesting precisely because this naive formula fails: passing from E-cohomology to $\mathbb{C}$-cohomology is somehow related to passage to the n-fold free loop space of X.

We now proceed to modify the ∞-category of local systems introduced in Nonexample 6.7, so that it will satisfy the axioms (a), (b), and (c). Suppose we have a space X and a local system $\mathcal{L}$ of E-modules on X. We will denote by $\Gamma(X;\mathcal{L})$ the global sections of $\mathcal{L}$, which is again an E-module. Actually, the situation is better than that: $\Gamma(X;\mathcal{L})$ is a module over the spectrum E^X, whose ring structure comes from pointwise multiplication in E. Given a map of spaces $f : X \longrightarrow Y$ and a local system $\mathcal{L}$ on Y we get a map

$$\Gamma(Y;\mathcal{L}) \longrightarrow \Gamma(X; f^*\mathcal{L})$$

The domain of this map is an E^Y-module, the codomain an E^X-module, and we get an induced map

$$E^X \otimes_{E^Y} \Gamma(Y;\mathcal{L}) \longrightarrow \Gamma(X; f^*\mathcal{L})$$

This map is not necessarily an isomorphism. For example, suppose that Y is a point and that $\mathcal{L}$ is the local system given by 'complexifying' E. Then, after passing to connected components, we obtain a map

$$E^0(X) \otimes_{E^0(\mathrm{pt})} \mathbb{C} \to H^{\mathrm{even}}(X;\mathbb{C}).$$

This is exactly the map encountered in Nonexample 6.7, and it will generally not be an isomorphism. We would like to modify the category of local systems in such a way that maps obtained by procedure *are* isomorphisms.

Definition 6.9. A *twisted local system on X* consists of the following data:

(1) For every space T which is the classifying space of a finite abelian p-group and every map $\alpha : T \longrightarrow X$, a module $\mathcal{L}(\alpha)$ over the function spectrum spectrum E^T.

(2) For maps $\beta : T' \longrightarrow T$ and $\alpha : T \longrightarrow X$ we should have a map

$$E^{T'} \otimes_{E^T} \mathcal{L}(\alpha) \to \mathcal{L}(\alpha \circ \beta),$$

which is an equivalence provided that β has connected homotopy fibers.

The maps appearing (2) are required to be (coherently) compatible with composition, in a sense which we will not make explicit here.

(3) For every map $\alpha : T \longrightarrow X$ and every covering map $\beta : \widetilde{T} \to T$ where $\widetilde{T}$ is connected, the canonical map

$$\mathcal{L}(\alpha) \to \mathcal{L}(\beta \circ \alpha)^{h\mathrm{Aut}(\widetilde{T}/T)}$$

exhibits the homotopy fixed point spectrum $\mathcal{L}(\beta\circ\alpha)^{h\mathrm{Aut}(\widetilde{T}/T)}$ as a completion of $\mathcal{L}(\alpha)$ with respect to the ideal given by the kernel of the restriction map $E(T) \to E(\widetilde{T})$.

Remark 6.10. Roughly speaking, the intent of Definition 6.9 is to correct the failure of passage to global sections to be compatible with pullback, at least for a small class of morphisms between p-finite spaces.

Example 6.11. For any space X, the construction $(\alpha : T \to X) \mapsto E^T$ determines a twisted local system on X, which we will denote by $\mathbf{1}$.

For every space X, the collection of twisted local systems on X can be organized into an ∞-category, which we will denote by $\widetilde{\mathrm{Loc}}(X)$. We claim that this ∞-category (or rather, its homotopy category) satisfies our axioms (a), (b), and (c), and can therefore be used to supply a proof of Theorem 6.5. Assertions (a) and (b) are immediate consequences of the following facts, which describe the behavior of the ∞-category $\widetilde{\mathrm{Loc}_0}(X)$ after 'localization' at various geometric loci in $\mathrm{Spec}E^0(\mathrm{pt})$.

(a') Let X be any topological space, and let $\widetilde{\mathrm{Loc}}_0(X)$ be the full subcategory of $\widetilde{\mathrm{Loc}}(X)$ spanned by those twisted local systems $\mathcal{L}$ for which the spectrum $\mathcal{L}(\alpha)$ is $K(n)$-local, for every map $\alpha : T \to X$. Then $\widetilde{\mathrm{Loc}}_0(X)$ is equivalent to the ∞-category of local systems of $K(n)$-local E-modules on X.

(b') Let $E_{\mathbb{C}}$ denote the complexification of E, and let $\widetilde{\mathrm{Loc}}_{\mathbb{C}}(X)$ denote the ∞-category of $E_{\mathbb{C}}$-modules in $\widetilde{\mathrm{Loc}}(X)$. Then $\widetilde{\mathrm{Loc}}_{\mathbb{C}}(X)$ is equivalent to the ∞-category of $E_{\mathbb{C}}$-valued local systems on the n-fold free loop space $\mathrm{Map}(T^n, X)$.

The proof of (c) is more interesting. Note that any map of spaces $f : X \to Y$ determines a pullback functor $f^* : \widetilde{\mathrm{Loc}}(Y) \longrightarrow \widetilde{\mathrm{Loc}}(X)$, which admits left and right adjoints $f_!, f_* : \widetilde{\mathrm{Loc}}(X) \longrightarrow \widetilde{\mathrm{Loc}}(Y)$. Using an adaptation of the theory of ambidexterity to the setting of *twisted* local systems, one can prove the following:

THEOREM 6.12. *Let $f : X \to Y$ be a map of p-finite spaces. Then the functors $f_!, f_* : \widetilde{\mathrm{Loc}}(X) \longrightarrow \widetilde{\mathrm{Loc}}(Y)$ are canonically isomorphic.*

REMARK 6.13. Recall that we are supposed to think of $f_!$ as taking homology along the fibers of f and f_* as taking cohomology along the fibers of f.

COROLLARY 6.14. *Let X be a p-finite space, and let $f : X \to \mathrm{pt}$ be the projection map. Then the 'global sections' functor*

$$f_* : \widetilde{\mathrm{Loc}}(X) \to \widetilde{\mathrm{Loc}}(\mathrm{pt}) \simeq \mathrm{Mod}_E$$

preserves colimits.

By writing the complexification $E_{\mathbb{C}}$ as a filtered colimit of *finite* E-module spectra, one can use Corollary 6.14 to show that the global sections functor f_* commutes with the operation of complexification. Applying this observation to the trivial local system $\mathbf{1}$, we obtain a proof of (c), and therefore a proof of Theorem 6.5.

REMARK 6.15. Theorem 6.12 can be regarded as a "transchromatic" analogue of Theorem 5.13: it articulates a relationship between the ambidexterity properties of the $K(m)$-local stable homotopy categories for several different values of m.

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Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: gheuts@math.harvard.edu

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: lurie@math.harvard.edu