Lazard's theorem (characterizing flatness)

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Throughout, A is a commutatitve ring with identity $1 \neq 0$ and we use the notation $A^S = \bigoplus_{s \in S} A$ for the free A-module with basis indexed by the set S. Our goal is to present a proof of the following

Theorem (Lazard)**.** *An A-module M is* **flat** *if and only if M is a* **direct limit** *of finite rank free A-modules.*

Here is a simple example in abelian groups showing that the "directed" assumption is necessary. If G is the colimit of the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"'] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd}$$

then $G \cong \mathbb{Z}^3/((1,0,-2),(1,-2,0))$ has torsion, and therefore is not a flat $\mathbb{Z}$-module. We have $2(1,-1,-1) = (1,0,-2) + (1,-2,0) = 0$ but $(1,-1,-1) \neq 0$ since if

$$(1,-1,-1) = a(1,0,-2) + b(1,-2,0), \ a, b \in \mathbb{Z},$$

then $1 = 2a = 2b$, impossible.

We first define the words in bold.

Definition 1. *An A-module N is* **flat** *if whenever*

$$M' \to M \to M''$$

is an exact sequence of A-modules, then

$$M' \otimes N \to M \otimes N \to M'' \otimes N$$

is also exact.

An condition equivalent to flatness is given by the following

Lemma 1 (Equational criterion for flatness, [E] corollary 6.5)**.** *M is a flat A-module if and only if any relation $\sum_{i=1}^n f_i m_i = 0$, $f_i \in A$, $m_i \in M$, is "trivial": there are $m'_j \in M$ and $a_{ij} \in A$ such that*

$$m_i = \sum_j a_{ij} m'_j, \ \sum_i a_{ij} f_i = 0.$$

[I.e., we can write $0 = \sum_{i=1}^n f_i m_i$ using zero coefficients, $\sum_j (\sum_i a_{ij} f_i) m'_j = \sum_j 0 \cdot m'_j = 0$.]

Proof. Suppose M is flat, $\sum_i f_i m_i = 0$ in M, and let $I = (f_i)_i \subseteq A$ be the ideal generated by the coefficients. The exact sequence $0 \to I \to A$ remains exact after tensoring with M, so that $\sum_i f_i \otimes m_i = 0$ in $I \otimes M$. If $\phi : A^n \to I$ is defined by $\phi(a_1, \dots, a_n) = \sum_i a_i f_i$ and K is its kernel, then the exact sequence $0 \to K \to A^n \xrightarrow{\phi} I \to 0$ remains exact after tensoring with M, so there is $\sum_j k_j \otimes m'_i$ in $K \otimes M$ such that $\sum_j k_j \otimes m'_j = \sum_i e_i \otimes m_i$, (where $\{e_i\}_i$ is the standard basis for A^n), since $(\phi \otimes 1)(\sum_i e_i \otimes m_i) = 0$. Each k_j can be expressed as $k_j = \sum_i a_{ij} e_i$ so that

$$\sum_i e_i \otimes (\sum_j a_{ij} m'_j) = \sum_i e_i \otimes m_i \in A^n \otimes M \cong M^n,$$

and $m_i = \sum_j a_{ij} m'_j$ (the first condition above). Also, $k_j \in K = \ker(\phi)$ so that $\phi(k_j) = 0 = \sum_i a_{ij} f_i$ (the second condition above).

Conversely, suppose every relation in M is trivial. Let $I \xrightarrow{\iota} A$ be the inclusion of a finitely generated ideal. [Flatness of M is equivalent to injectivity of $I \otimes M \to A \otimes M$ for any such I, cf. [AM] chapter 2, exercise 26 using Tor or [R], corollary 6.143 using Baer's criterion and the fact that M is flat iff $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective.] If $(\iota \otimes 1)(\sum_i f_i \otimes m_i) = 0 \in A \otimes M$ then $\sum_i f_i m_i$ is a relation in M, hence trivial. Using the notation of the previous lemma, we have

$$\sum_i f_i \otimes m_i = \sum_i f_i \otimes (\sum_j a_{ij} m'_j) = \sum_j (\sum_i f_i a_{ij}) \otimes m'_j = 0,$$

and $\iota \otimes 1$ is injective. $\square$

A corollary of lemma 1 is the following flatness criterion that we will use below.

Corollary 1. *M is a flat A-module if and only if whenever given a map $A^n \xrightarrow{f} M$ and a finitely generated $N \leq \ker(f)$, there is a factorization*

$$\begin{tikzcd} A^n \arrow[r, "h"] \arrow[dr, "f"'] & F \arrow[d, "g"] \\ & M \end{tikzcd}$$

with F a finite rank free A-module and $N \leq \ker(h)$.

Proof. If N is generated by one element $x = (f_i)_i$, $f(x) = \sum_i f_i m_i = 0$, then M is flat iff this relation is trivial $m_i = \sum_j a_{ij} m'_j$, $\sum_i a_{ij} f_i = 0$ by lemma 1. Take $F = A^m$, $h(a_1, \dots, a_n) = \sum_{i,j} a_{ij} a_i e_j$ ($\{e_j\}_j$ the standard basis), and $g(a_1, \dots, a_m) = \sum_j a_i m'_j$. Then

$$h(f_1, \dots, f_n) = \sum_{i,j} a_{ij} f_i e_j = 0,$$

$$(g \circ h)(a_1, \dots, a_n) = g\left(\sum_{i,j} a_{ij} a_i e_j\right) = \sum_{i,j} a_{ij} a_i m'_j = \sum_i a_i m_i = f(a_1, \dots, a_n).$$

Now if $N = N' + Ax'$ and $A^n \xrightarrow{h'} F' \xrightarrow{g'} M$ factors f with $N' \leq \ker(h')$, then we can find F, h'', and g with $h'(x) \in \ker(h'')$ such that the following commutes

$$\begin{tikzcd} A^n \arrow[r, "h'"] \arrow[dr, "f"'] & F' \arrow[r, "h''"] \arrow[d, "g'"] & F \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

Taking $h = h'' \circ h'$, we have $N \leq \ker(h)$ and $f = g \circ h$. $\square$

Definition 2. *A poset $(I, \leq)$ is* **directed** *if for any $i, j \in I$ there is $k \in I$ such that $i, j \leq k$. A* **directed system** *of A-modules is a functor from I to the category of A-modules (i.e. for each $i \in I$ we associate an A-module M_i and for each $i \leq j$ we associate an A-module homomorphism $f^i_j : M_i \to M_j$ such that whenever $i \leq j \leq k$, $f^j_k \circ f^i_j = f^i_k$). The* **direct limit** *of a directed system of A-modules is an A-module $\varinjlim_{i\in I} M_i$ and maps $p_i : M_i \to \varinjlim_{i\in I} M_i$ that are universal with respect to maps out of the directed system. That is to say, if N is any A-module with maps $q_i : M_i \to N$ commuting with the $\{f^i_j\}_{i,j\in I}$, then there is a unique map $h : \varinjlim_{i\in I} M_i \to N$ commuting with $\{f^i_j\}$, $\{p_i\}$, and $\{q_i\}$. Rephrased yet again, maps out of the directed system factor uniquely through the direct limit.*

More concretely, a construction of the direct limit is $\left(\bigoplus_{i\in I} M_i\right)/N$ where N is the submodule generated by $\{m_i - f^i_j(m_i) : m_i \in M_i, i \leq j \in I\}$.

The following two lemmata are exercises in [AM] and together they show that the direct limit of a directed system of flat A-modules is flat.

Lemma 2. *If*

$$\{M'_i\}_{i\in I} \xrightarrow{\{f_i\}_{i\in I}} \{M_i\}_{i\in I} \xrightarrow{\{g_i\}_{i\in I}} \{M''_i\}_{i\in I}$$

is exact at each i, then

$$\varinjlim_{i\in I} M'_i \xrightarrow{f} \varinjlim_{i\in I} M_i \xrightarrow{g} \varinjlim_{i\in I} M''_i$$

is exact.

Proof. Exercise 19, chapter 2 of [AM]. □

Lemma 3. *Tensor product commutes with direct limits, i.e.*

$$(\varinjlim_{i\in I} M_i) \otimes N \cong \varinjlim_{i\in I}(M_i \otimes N).$$

Proof. Exercise 20, chapter 2 of [AM]. □

We do not necessarily need all of a directed system to determine the direct limit.

Definition 3. *An induced sub-poset $(J, \leq)$ of $(I, \leq)$ (J is a subset of I and $j_1 \leq j_2$ in J if and only if $j_1 \leq j_2$ in I) is* ***cofinal*** *in $(I, \leq)$ if for any $i \in I$ there is a $j \in J$ such that $i \leq j$.*

Lemma 4. *If J is cofinal in the directed poset I and $\{M_i\}_{i\in I}$ is a directed system of A-modules, then $\varinjlim_{i\in I} M_i \cong \varinjlim_{j\in J} M_j$.*

Proof. The maps $M_j \to \varinjlim_{i\in I} M_i$ induce a unique map $\phi : \varinjlim_{j\in J} M_j \to \varinjlim_{i\in I} M_i$. By cofinality, for each $i \in I$, there is a $j(i) \in J$ such that $i \leq j$ (and we take $j(j) = j$ for $j \in J \subseteq I$). This gives maps $M_i \to M_{j(i)} \to \varinjlim_{j\in J} M_j$ commuting with the maps between $\{M_i\}_{i\in I}$ hence induces a unique $\psi : \varinjlim_{i\in I} M_i \to \varinjlim_{j\in J} M_j$. Let α^i_k, $\alpha_i : M_i \to \varinjlim_{i\in I}$ and β^j_k, $\beta_j : M_j \to \varinjlim_{j\in J} M_j$ be the maps associated to the directed systems (the β maps are a subset of the α maps). Then the maps ϕ, ψ just described give

$$\phi \circ \beta_j = \alpha_j,\ \psi \circ \alpha_i = \beta_{j(i)}\alpha^i_{j(i)}.$$

Finally, we have

$$\psi \circ \phi \circ \beta_j = \psi \circ \alpha_j = \beta_{j(j)}\alpha^j_{j(j)} = \beta_j,$$
$$\phi \circ \psi \circ \alpha_i = \phi \circ \beta_{j(i)}\alpha^i_{j(i)} = \alpha_{j(i)}\alpha^i_{j(i)} = \alpha_i,$$

so that ϕ, ψ are inverse isomorphisms between the direct limits. □

Before proving the theorem, we have the following

Lemma 5. *Every A-module M is a direct limit of finitely presented A-modules.*

Proof. Take an exact sequence $0 \to K \to A^I \to M \to 0$ and let

$$\Lambda = \{(J, N) : J \subseteq I \text{ finite}, N \subseteq K \cap A^J \text{ finitely generated}\},$$

partially ordered by $(J', N') \leq (J, N)$ if $J' \subseteq J$ and $N' \subseteq N$ (Λ is clearly directed). For $\lambda = (J, N) \in \Lambda$, define

$$M_\lambda := A^J/N.$$

For $\kappa \leq \lambda$, there are obvious maps $f_\lambda^\kappa : M_\kappa \to M_\lambda$ making $\{M_\lambda\}_{\lambda\in\Lambda}$ a directed system of A-modules. There are also the obvious maps $q_\lambda : M_\lambda \to M$ inducing $h : \varinjlim_{\lambda\in\Lambda} M_\lambda \to M$. The induced map h is surjective since every element of M is in the image of some q_λ. For $x \varinjlim_{\lambda\in\Lambda}$, let $x_\lambda \in M_\lambda$ such that $p_\lambda(x_\lambda) = x$ ($\{p_\lambda\}_{\lambda\in\Lambda}$ being the maps from the directed system into the direct limit), cf. exercise 15, chapter 2 of [AM]. We have $q_\lambda(x_\lambda) = (h \circ p_\lambda)(x_\lambda) = 0$. If $\lambda = (J, N)$, we can enlarge J, N to obtain a $\mu \geq \lambda$ that witnesses the fact that $q_\lambda(x_\lambda) = 0$, i.e. $x_\mu := f_\mu^\lambda(x_\lambda) = 0$. Hence we have

$$x = q_\lambda(x_\lambda) = q_\mu(x_\mu) = 0$$

and h is injective as well. □

Proof of theorem (following [S]). Suppose M is a flat A-module. In the construction of lemma 5, take $I = M \times \mathbb{Z}$ with $A^I \xrightarrow{q} M$ defined by $1_{(m,\nu)} \mapsto m$ (the generator of the ith coordinate gets mapped to the projection of $i = (m, \nu)$ onto M). [The factor of $\mathbb{Z}$ in I guarantees extra space we'll use later.] Let $\lambda = (J, N) \in \Lambda$ ($J \subset I$ finite and $N \leq \ker(f) \cap A^J$ finitely generated). By corollary 1, the map $M_\lambda \xrightarrow{q_\lambda} M$ (where $M_\lambda = A^J/N$) factors through a finite rank free A-module F

$$M_\lambda \xrightarrow{\bar{h}} F \xrightarrow{g} M,\ h : A^J \to F,$$

$q_\lambda = g \circ \bar{h}$. We now realize this F as M_μ for some $\lambda \leq \mu$, giving a cofinal subset of Λ consisting of finite rank free A-modules.

Let $\{b_1, \ldots, b_n\}$ be a basis for F and choose $i_1, \ldots, i_n \in I$ such that $i_l \notin J$ and

$$q(1_{i_l}) = g(b_l),\ q : A^I \to M,\ g : F \to M$$

(the factor of $\mathbb{Z}$ in I guaranteeing this possibility). Let $J' = J \cup \{i_1, \ldots, i_n\}$ and define a map $A^{J'} \xrightarrow{\tilde{h}} F$ extending h and mapping 1_{i_l} to b_l, with kernel N'. The following diagram commutes

$$\begin{tikzcd}
A^{J'} \arrow[r, "\tilde{h}"] \arrow[d] & F \arrow[d, "g"] \\
A^I \arrow[r, "q"'] & M
\end{tikzcd}$$

so that $N' \leq \ker(q)$. The top map $A^{J'} \to F$ is split (since it is a surjection onto a free module) so that $A^{J'} \cong N' \bigoplus F$ and N' is finitely generated as well. Hence $\mu := (J', N') \in \Lambda$ and $\lambda \leq \mu$.

By lemma 5, M is the direct limit of a directed system of finite rank A-modules, proving the theorem. □

There is a proof of a similar statment (M is flat iff M is a filtered colimit of a diagram of free A-modules) in [E], A6.2.

Exercises

To be thorough, one should do the exercises referenced above: [AM] 2.15, 2.19, 2.20, 2.26 (which were all presented in class).

References

[AM] M. F. Atiyah, I. G. MacDonald, *Introduction to Commutative Algebra*, Westview Press, 1969.

[E] David Eisenbud, *Commutative Algebra with a View towards Algebraic Geometry*, Graduate Texts in Mathematics; v. 150, Springer, 2004.

[R] Joseph Rotman, *Advanced Modern Algebra* (second edition), Graduate Studies in Mathematics; v. 114, AMS, 2010.

[S] The Stacks Project, Tag 058C, `http://stacks.math.columbia.edu/tag/058C`