A CATEGORICAL CONCEPT OF CONNECTEDNESS

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An object C of a category $\mathcal{C}$ is a *Z-object* iff $Hom(C, -)$ preserves coproducts; C is called *Z-generated* iff $C = \coprod_{i \in J} C_i$ for Z-objects C_i; the class of Z-objects is denoted by $B(\mathcal{C})$, the class of Z-generated objects by $Z(\mathcal{C})$ (for the corresponding subcategories the same symbols are used); $\mathcal{C}$ is called *based* iff $\mathcal{C}$ has coproducts and every object of $\mathcal{C}$ is Z-generated.

In *Top* (topological spaces and continuous maps) Z-objects are the non-void connected spaces (non-void, because an initial object in a category $\mathcal{C}$ is not a Z-object); analogously in *Cat* (small categories and functors) and *Graph* (directed graphs) the non-void connected categories, resp. graphs are the Z-objects. For a group G, $[G, Ens]$ will denote the category of G-sets: the transitive (or simple) G-sets are exactly the Z-objects. In the category ${}^{q}Met$ of the quasi-metric spaces (two different points may have infinite distance or null distance) and non-expansive maps Z-objects are those non-void spaces with finite distance only. *Cat*, *Graph*, $[G, Ens]$, ${}^{q}Met$, and specially *Ens*, are based categories (Z-objects in *Ens* are final objects), but *Top* is not. There are negative criteria (existence of a null object or a strictly final object) that exclude the existence of Z-objects in the categories of groups, etc., and of unital rings ($\{0\}$ is strictly final); especially if both $\mathcal{C}$ and $\mathcal{C}^{o}$ are based categories, then $\mathcal{C}$ is equivalent to the final category $\mathbf{1}$.

A based category $\mathcal{C}$ is exactly the universal completion of its bases $B(\mathcal{C})$ with respect to coproducts; the functor $B(\mathcal{C}) \to \mathbf{1}$ obviously induces a coproduct-preserving functor $S: \mathcal{C} \to Ens$ (since $B(Ens) \simeq \mathbf{1}$), and if, in addition, $\mathcal{C}$ has a final object t, one has the following adjunction $S \dashv L$, where L is a left adjoint to $Hom(t, -): \mathcal{C} \to Ens$.

A cone $\{f_i : A_i \to A\}_{i \in J}$ is called a *Z-system* iff every A_i is a Z-object and for every Z-object C, $\{Hom(C, f_i)\}_{i \in J}$ is a coproduct; Z-systems are uniquely determined (up to ...). In *Top* a Z-system is the family of connected components; in a based category it is the (unique!) representation of an object as a coproduct of Z-objects. If $\mathcal{C}$ has coproducts, $Z(\mathcal{C}) \to \mathcal{C}$ is coreflective iff $\mathcal{C}$ has Z-systems (for every object). If $\mathcal{C}$ has colimits, $Z(\mathcal{C}) \to \mathcal{C}$ is closed under colimits. This provides an idea for an existence criterion for Z-systems: if $\mathcal{C}$ has an $(\mathfrak{E}, \mathfrak{M}$-mono)-factorization (diagonal condition), if the class of Z-objects is closed under $\mathfrak{E}$-« quotients », if $\mathcal{C}$ is $\mathfrak{M}$-well-powered, and if $\mathcal{C}$ has colimits for connected diagrams $T : \Sigma \to \mathcal{C}$ with $\mathfrak{M}$-transition morphisms (i.e. $Tp \in \mathfrak{M}$ for $p \in \Sigma$), then every object $A \in \mathcal{C}$ has a Z-system $\{ f_i : A_i \to A \}_{i \in J}$ and every $f_i \in \mathfrak{M}$. -A category $\mathcal{C}$ with coproducts is based iff $\mathcal{C}$ has Z-systems and $B(\mathcal{C}) \to \mathcal{C}$ is dense.

If $(T, \eta, \mu) = \mathbf{T}$ is a monad in a based category $\mathcal{C}$ such that T preserves coproducts, then $\mathcal{C}^{\mathbf{T}}$ is based too (e.g. *Cat* over *Graph*, $[G, Ens]$ over *Ens*); since for a small category $\mathfrak{A}$,

$$[\mathfrak{A}, \mathcal{C}] \to [Ob\,\mathfrak{A}, \mathcal{C}] = \mathcal{C}^{Ob\,\mathfrak{A}}$$

is monadic, and a power of a based category is based too (its basis is a co-power of the original basis), $[\mathfrak{A}, \mathcal{C}]$ is based (because $\mathcal{C}$ is) (e.g. $Graph = [\cdot \rightrightarrows \cdot, Ens]$). In a power of *Ens* one has the following characterization of « coproduct-preserving » triples:

If $V : \mathcal{C} \to Ens^M$ (for a set M) is monadic and preserves coproducts, then there is exactly one (up to ...) category $\mathfrak{A}$ with $Ob\,\mathfrak{A} \cong M$ and an isomorphism $\mathcal{C} \cong [\mathfrak{A}, Ens]$ such that the following square commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \simeq\ } & [\mathfrak{A}, Ens] \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle Evaluation} \\ Ens^M & \xrightarrow[\ \simeq\ ]{} & [Ob\,\mathfrak{A}, Ens] \end{array}$$