Smooth spaces: convenient categories for differential geometry

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Higher Structures in Topology and Geometry II

- Smooth manifolds and convenient categories
- A category of smooth spaces
- Convenient properties of smooth spaces
- Smooth spaces as generalized spaces
- The idea of the proof
- Smooth categories and differential forms

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Outline

In differential geometry, the most popular category, that of finite-dimensional smooth manifolds, fails to be cartesian closed.

If X and Y are finite-dimensional smooth manifolds, the space of smooth maps $C^\infty(X, Y)$ usually is not.

It is some sort of *infinite-dimensional* manifold.

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The category of finite-dimensional smooth manifolds lacks other desirable features such as:

subspaces and quotient spaces,

and limits and colimits.

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Chen, Souriau

Maps out

Smith, Sikorski, Mostow

Maps in & out

Frölicher

- See Andrew Stacey's "Comparitive Smootheology"

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Souriau's notion of 'diffeological space' (Souriau:1980) is very simple:

(but first a preliminary definition)

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An **open set** is an open subset of $\mathbb{R}^n$. A function $f : U \to U'$ between open sets is called **smooth** if it has continuous derivatives of all orders.

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Definition

A **diffeological space** is a set X equipped with, for each open set U, a set of functions

$$\varphi \colon U \to X$$

called **plots in** X, such that:

1. If φ is a plot in X and $f \colon U' \to U$ is a smooth function between open sets, then φf is a plot in X.
2. Suppose the open sets $U_j \subseteq U$ form an open cover of the open set U, with inclusions $i_j \colon U_j \to U$. If φi_j is a plot in X for every j, then φ is a plot in X.
3. Every map from the one point of $\mathbb{R}^0$ to X is a plot in X.

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Given diffeological spaces X and Y, a function $f\colon X \to Y$ is a **smooth map** if, for every plot φ in X, the composite $f\varphi$ is a plot in Y.

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Outline

Some examples

- Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense.
- Every smooth space has a natural topology, and smooth maps between smooth spaces are automatically continuous.
- Any set X has a **discrete** smooth structure such that the plots $\varphi\colon U \to X$ are just the constant functions.
- Any set X has an **indiscrete** smooth structure where every function $\varphi\colon U \to X$ is a plot.

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Subspaces

Any subset $Y \subseteq X$ of a smooth space X becomes a smooth space if we define $\varphi\colon U \to Y$ to be a plot in Y if and only if its composite with the inclusion $i\colon Y \to X$ is a plot in X. We call this the **subspace** smooth structure.

- The inclusion $i\colon Y \to X$ is smooth.
- It is a monomorphism of smooth spaces.
- Not every monomorphism is of this form.
- **Example** The natural map from $\mathbb{R}$ with its discrete smooth structure to $\mathbb{R}$ with its standard smooth structure is also a monomorphism.

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Quotient spaces

If X is a smooth space and $\sim$ is any equivalence relation on X, the quotient space $Y = X/\sim$ becomes a smooth space if we define a plot in Y to be any function of the form

$$U \xrightarrow{\varphi} X \xrightarrow{p} Y$$

where φ is a plot in X.

Motivations

We will see that subspaces are useful in defining smooth categories and quotients are good for defining path groupoids (paths modulo thin homotopy) which are also smooth categories. (John Baez's talk)

Products

Given smooth spaces X and Y, the product $X \times Y$ of their underlying sets becomes a smooth space where $\varphi \colon U \to X \times Y$ is a plot if and only if its composites with the projections

$$p_X \colon X \times Y \to X, \qquad p_Y \colon X \times Y \to Y$$

are plots in X and Y, respectively.

Coproducts

Given smooth spaces X and Y, the disjoint union $X + Y$ of their underlying sets becomes a smooth space where $\varphi\colon U \to X + Y$ is a plot if and only if for each connected component C of U, $\varphi|_C$ is either the composite of a plot in X with the inclusion $i_X\colon X \to X + Y$, or the composite of a plot in Y with the inclusion $i_Y\colon Y \to X + Y$.

Mapping spaces

Given smooth spaces X and Y, the set

$$\mathcal{C}^\infty(X, Y) = \{f \colon X \to Y \colon f \text{ is smooth}\}$$

becomes a smooth space where a function $\tilde{\varphi} \colon U \to \mathcal{C}^\infty(X, Y)$ is a plot if and only if the corresponding function $\varphi \colon U \times X \to Y$ given by

$$\varphi(z, x) = \tilde{\varphi}(z)(x)$$

is smooth.

Mapping spaces cont.

With this smooth structure one can show that the natural map

$$\begin{array}{ccc} \mathcal{C}^\infty(X \times Y, Z) & \to & \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z)) \\ f & \mapsto & \tilde{f} \end{array}$$

$$\tilde{f}(x)(y) = f(x, y)$$

is smooth, with a smooth inverse. So, we say the category of diffeological spaces is cartesian closed.

Outline

Theorem

$\mathcal{C}^\infty$ has a weak subobject classifier, all (small) limits and colimits, and is locally cartesian closed.

$\mathcal{C}^\infty$ is an example of a category of 'concrete sheaves'.

Proof is given by showing that these special categories of sheaves always have these properties.

Other examples include Chen spaces and simplicial complexes.

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We proceed by defining the concept of sheaves on sites, and, ultimately, ‘concrete sheaves’ on ‘concrete sites’.

Along the way we see that diffeological spaces are indeed concrete sheaves on a concrete site.

Presheaves

Definition

Let Diffeological be the category whose objects are open subsets of $\mathbb{R}^n$ and whose morphisms are smooth maps.

Any diffeological space gives a functor.

Axiom 1 - Presheaf condition

If φ is a plot in X and $f\colon U' \to U$ is a smooth function between open sets, then φf is a plot in X.

Definition

A **presheaf** X on a category D is a functor $X\colon \mathrm{D}^{\mathrm{op}} \to \mathrm{Set}$. For any object $D \in \mathrm{D}$, we call the elements of $X(D)$ **plots in** X **with domain** D.

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Covering families

Given an open cover

$$(i_j \colon U_j \to U | j \in J)$$

of an open set U, and a smooth map between open sets

$$\mathbf{f} \colon V \to U,$$

we can always find an open cover $(i_k \colon V_k \to V | k \in K)$ such that each fi_k factors through some U_j.

Given a category D we can define a general notion of coverings of objects in D.

Definition

A **coverage** on a category D is a function assigning to each object $D \in \mathrm{D}$ a collection $\mathcal{J}(D)$ of families $(f_i \colon D_i \to D | i \in I)$ called **covering families**, with the following property:

- Given a covering family $(f_i \colon D_i \to D | i \in I)$ and a morphism $g \colon C \to D$, there exists a covering family $(h_j \colon C_j \to C | j \in J)$ such that each morphism gh_j factors through some f_i.

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Let X be a diffeological space, $(i_j\colon U_j \to U | j \in J)$ an open cover, and $\{\varphi_j \in X(U_j) | j \in J\}$ a family of plots.

If every two plots agree where both are defined then we can define a unique global function $\varphi\colon U \to X$ by gluing together the local plots.

Axiom 2

Suppose the open sets $U_j \subseteq U$ form an open cover of the open set U, with inclusions $i_j\colon U_j \to U$. If φi_j is a plot in X for every j, then φ is a plot in X.

Axiom 2 guarantees φ is a plot.

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Compatible families

Definition

Given a covering family $(f_i\colon D_i \to D | i \in I)$ in D and a presheaf $X\colon \mathrm{D}^{\mathrm{op}} \to \mathrm{Set}$, a collection of plots $\{\varphi_i \in X(D_i) | i \in I\}$ is called **compatible** if whenever $g\colon C \to D_i$ and $h\colon C \to D_j$ make this diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{h} & D_j \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_j} \\ D_i & \xrightarrow[f_i]{} & D \end{array}$$

then $X(g)(\varphi_i) = X(h)(\varphi_j)$.

Sheaves

Definition

Given a site D, a presheaf $X\colon \mathrm{D}^{\mathrm{op}} \to \mathrm{Set}$ is a **sheaf** if it satisfies the following condition:

- Given a covering family $(f_i\colon D_i \to D | i \in I)$ and a compatible collection of plots $\{\varphi_i \in X(D_i) | i \in I\}$, then there exists a unique plot $\varphi \in X(D)$ such that $X(f_i)(\varphi) = \varphi_i$ for each $i \in I$.

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Diffeological spaces are sheaves.

Properties of Diffeological

We notice...

- the objects of our site category Diffeological are sets;
- the functor $\hom(1, -)$: Diffeological $\to \mathrm{Set}$ takes any open set U to its "set of points" $\hom(1, U)$;
- morphisms in Diffeological are functions;
- the coverages defined for Diffeological were honest covers.

- Site categories can in general be more abstract.
- We want to consider sites whose objects are sets with extra structure.

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Properties of Diffeological

We notice...

- the objects of our site category Diffeological are sets;
- the functor $\mathrm{hom}(1, -)$: Diffeological $\to \mathrm{Set}$ takes any open set U to its "set of points" $\mathrm{hom}(1, U)$;
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Concrete sites

Definition

A **concrete site** D is a site with a terminal object 1 satisfying the following conditions:

- The functor $\hom(1, -)\colon \mathrm{D} \to \mathrm{Set}$ is faithful.
- For each covering family $(f_i\colon D_i \to D | i \in I)$, the family of functions $(\hom(1, f_i)\colon \hom(1, D_i) \to \hom(1, D) | i \in I)$ is **jointly surjective**, meaning that

$$\bigcup_{i \in I} \hom(1, f_i)(\hom(1, D_i)) = \hom(1, D).$$

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Given a concrete site D, we say a sheaf $X\colon \mathrm{D}^{\mathrm{op}} \to \mathrm{Set}$ is **concrete** if for every object $D \in \mathrm{D}$, the function sending plots $\varphi \in X(D)$ to functions $\underline{\varphi}\colon \hom(1, D) \to X(1)$ is one-to-one.

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Diffeological spaces are concrete sheaves.

Outline

Generalized spaces

Definition

Given a concrete site D, a **generalized space** or D **space** is a concrete sheaf $X \colon \mathrm{D}^{\mathrm{op}} \to \mathrm{Set}$. A **map** between D spaces $X, Y \colon \mathrm{D}^{\mathrm{op}} \to \mathrm{Set}$ is a natural transformation $F \colon X \Rightarrow Y$. We define DSpace to be the category of D spaces and maps between these.

An equivalence of categories

Axiom 3

Every map from the one point of $\mathbb{R}^0$ to X is a plot in X.

Axiom 3 guarantees that a diffeological space and its corresponding concrete sheaf have the same underlying set.

Theorem

The category of diffeological spaces is equivalent to the category of concrete sheaves on the site Diffeological, *(that is, the category of* Diffeological *spaces).*

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Weak subobject classifier

Definition

In a category $\mathcal{C}$ with finite limits, a **weak subobject classifier** is an object Ω equipped with a morphism $\top\colon 1 \to \Omega$ such that, given any strong monomorphism $i\colon C' \to C$ in $\mathcal{C}$, there is a unique morphism $\chi_i\colon C \to \Omega$ making

$$\begin{array}{ccc} C' & \overset{i}{\rightarrowtail} & C \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_i} \\ 1 & \underset{\top}{\rightarrowtail} & \Omega \end{array}$$

a pullback.

Lemma

For any concrete site D*, the category of* D *spaces has a weak subobject classifier* Ω*.*

In other words there is a 1-1 correspondence between subspaces of a D space X and maps from X to Ω.

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Cartesian closed

Definition

A category $\mathcal{C}$ is **cartesian closed** if it has finite products and for every object $Y \in \mathcal{C}$, the functor

$$- \times Y : \mathcal{C} \to \mathcal{C}$$

has a right adjoint, called the **internal hom** and denoted

$$\mathcal{C}(Y, -) : \mathcal{C} \to \mathcal{C}.$$

The fact that $\mathcal{C}(Y, -)$ is right adjoint to $- \times Y$ means that we have a natural bijection of sets

$$\hom(X \times Y, Z) \cong \hom(X, \mathcal{C}(Y, Z)).$$

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A category $\mathcal{C}$ is called **locally cartesian closed** if for every $B \in \mathcal{C}$, the category $\mathcal{C}_B$ of objects over B is cartesian closed.

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Limits and colimits

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For any concrete site D*, the category of* D *spaces has all (small) limits and colimits.*

Quasitopoi

Definition

A **quasitopos** is a locally cartesian closed category with finite colimits and a weak subobject classifier.

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Outline

Some motivation

In the category of sets, pullbacks are constructed by taking certain subsets of the cartesian product of two sets. For smooth spaces, we construct pullbacks in the same way and then provide the appropriate smooth structure.

So if one wants pullbacks in a category it is convenient to have subspaces in a category.

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Internalization

Let $\mathcal{K}$ be a category.

Definition

A **category** $\mathcal{C}$ **in** $\mathcal{K}$ consists of

- an object $C_0 \in \mathrm{Ob}(\mathcal{K})$ of objects;
- an object $C_1 \in \mathrm{Ob}(\mathcal{K})$ of morphisms;

together with

- source and target morphisms $s, t \colon C_1 \to C_0$ in $\mathcal{K}$;
- an identity assigning morphism $i \colon C_0 \to C_1$ in K;
- a composition morphism $\circ \colon C_{1\,t} \times_s C_1 \to C_1$;

such that the usual category axioms expressed as diagrams in $\mathcal{K}$ commute.

Internalization is good for categorifying algebraic concepts (and eventually differential geometric concepts), for example

- 2-vector spaces (categories in Vect)
- Lie 2-algebras

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Why pullbacks?

Smooth spaces can be used as a tool for defining a notion of 'smooth category'.

A smooth category is a category $\mathcal{C}$ internal to a category of smooth spaces $\mathcal{C}^\infty$.

- $\mathrm{Ob}(\mathcal{C})$ is a smooth space;
- $\mathrm{Mor}(\mathcal{C})$ is a smooth space;
- the structure maps are smooth, in particular, the composition map

$$\circ \colon \mathrm{Mor}(\mathrm{C})_t \times_s \mathrm{Mor}(\mathrm{C}) \to \mathrm{Mor}(\mathrm{C})$$

should be smooth.

Pullbacks guarantee $\mathrm{Mor}(\mathrm{C})_t \times_s \mathrm{Mor}(\mathrm{C})$ to be a smooth space.

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Internal functors

While we are at it we can define smooth functors as functors internal to $\mathcal{C}^\infty$.

Definition

A **functor** $F: A \to B$ **in** K is

- a morphism of objects $\mathrm{Ob}(F): \mathrm{Ob}(A) \to \mathrm{Ob}(B)$ in $\mathcal{K}$;
- a morphism of morphisms $\mathrm{Mor}(F): \mathrm{Mor}(A) \to \mathrm{Mor}(B)$ in $\mathcal{K}$;

such that diagrams expressing respect for the source, target, identity, and composition maps commute.

Differential forms

Definition

A p-**form** on the smooth space X is an assignment of a smooth p-form ω_φ on U to each plot $\varphi \colon U \to X$, satisfying this **pullback compatibility condition** for any smooth map $f \colon U' \to U$.

$$(f^*\omega)_\varphi = \omega_{\varphi \circ f}$$

The space of p-forms on X is denoted by $\Omega^p(X)$.

Proposition

Given a smooth map $f\colon X \to Y$ and $\omega \in \Omega^p(Y)$ there is a p-form $f^*\omega \in \Omega^p(X)$ given by

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Given a differential form ω on X, the forms ω_φ defining it turn out to be just its pullbacks along plots:

Proposition

Given a plot $\varphi \colon U \to X$ and $\omega \in \Omega^p(X)$ we have

$$\varphi^* \omega = \omega_\varphi .$$