HOPF-GALOIS EXTENSIONS

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Let $\mathcal{C}_n$ be a $K(n)$-algebra in p-local stable homotopy theory. $\mathcal{C}_n$ can then be considered as a wedge of the $K(n)$s, i.e., $\mathcal{C}_n = K(n) \vee K(n) \vee \cdots \vee K(n)$. Consider a morphism $K(n) \to \mathcal{C}_n$ - it is easy to see that any map $\mathcal{C}_n \bigwedge_{K(n)} \mathcal{C}_n \xrightarrow{\mathcal{C}_n \bigwedge_{K(n)} \beta} \mathcal{C}_n \bigwedge_{K(n)} \mathcal{C}_n \bigwedge_{K(n)} \mathcal{C}_n \xrightarrow{\mu \bigwedge_{K(n)} \mathcal{C}_n} \mathcal{C}_n \bigwedge_{K(n)} \mathcal{C}_n$ is always a weak equivalence, and the cofixed points of the $K(n)$-coaction on $\mathcal{C}_n$ is always (weakly) equivalent to $K(n)$, so that any map $K(n) \to \mathcal{C}_n$ is a $K(n)$-Hopf-Galois extension. That makes Hopf-Galois extensions of fields in p-local stable homotopy not very interesting.

But can we do something interesting with Morava E-theories, perhaps? Consider any set of maps $K(n) \to \mathcal{C}_n$ for $0 \leq n \leq i$ (where in each $\mathcal{C}_n$ is wedged exactly k times, regardless of the choice of n), and wedge all of them together. We get a pattern (p.9 of my handwritten notes) which allows us to state that any map $E(i) \to E(i) \vee E(i) \vee \cdots \vee E(i)$, where the $E(i)$s are wedged $(i + k)$-times. More generally, we can state that any wedge A_i of $K(n)$-algebras from $0 \leq n \leq i$ (where the $K(n)$s can be wedged any number of times) is a Hopf-Galois extension of $E(i)$, with essentially the same method of proof.

What we'd like to do is extend this to stable ∞-categories, because they're in essence the ∞-categorification of spectra. (This is the same reasoning used in my notes on K-theory and unital ∞-operads ([Dev]).) So we'll first do soemthing simple and define Morava K-theories in the stable ∞-context. Let **DeCat** denote the left adjoint to ∞-categorification, and let $\mathcal{C}_\infty$ be a stable ∞-category. If $\mathbf{DeCat}(\mathcal{C}_\infty)$ is a Morava K-theory, then we call $\mathcal{C}_\infty$ a Morava **K**-theory in "p-local" $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$. The statements that hold for Morava K-theory also hold for this Morava **K**-theory. One can analagously define Morava **E**-theory, I guess, with similar statements holding true.

Maybe one can use Goerss-Hopkins Obstruction Theory to understand HG extensions for general $\mathbf{E}_n$-algebras, because we have HG extensions for $\mathbf{E}_1$ and $\mathbf{E}_\infty$-algebras. I'm not sure, but I think the best person to ask would be Aaron Mazel-Gee. (I'm trying to develop my own kind of GHOsT, like what Aaron is trying to do/has already done, but I've just began.)

References

[Dev] Sanath Devalapurkar. K-theory and morita theory. Available upon request.

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