Čech data for holomorphic Chern classes

Work so far, and work for the future

Tim Hosgood

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Topos Institute

The problem

Aim: Categorical construction of Čech representatives of Chern classes in Deligne cohomology of coherent analytic sheaves.

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- holomorphic (not smooth)
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From vector bundles to coherent sheaves

Global resolutions

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Theorem (Toledo–Tong)

These local resolutions do not glue together on the nose, but can be glued up to homotopy, with the homotopies themselves satisfying some higher homotopies, and so on, to get a twisting cochain.

Corollary (O'Brian–Toledo–Tong)

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Theorem (Green)

Twisting cochains can be semi-strictified into strict complexes of "(co)simplicial locally free sheaves". (Conjecture (H): This is a Dold–Kan type result).

Global resolutions

Theorem (H–Zeinalian,H–Glass)

The following three concepts are (give or take some technicalities) sort of equivalent:

1. *coherent sheaf*
2. *homotopy-coherent complex of locally free sheaves (twisting cochain)*
3. *complex of locally free sheaves on the Čech nerve (Green complex)*
4. *homotopy-coherent complex of locally free sheaves on the Čech nerve*

Furthermore, each can be constructed as a homotopy limit (over the Čech nerve) of a simplicial presheaf.

Global resolutions

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Another big technicality is whether we want an equivalence of mere objects, or an equivalence of $(\infty, 1)$-categories.

But this is not today's talk! (Though it is the subject of work-in-progress).

The setup

Conventions

- $(X, \mathscr{O}_X)$ holomorphic manifold
- E locally free sheaf of rank $\mathfrak{r}$ on X
- $\mathcal{U} = \{U_\alpha\}$ "Stein-good" cover of X that trivialises E
- $\varphi_\alpha \colon E|U_\alpha \xrightarrow{\sim} (\mathscr{O}_X|U_\alpha)^{\mathfrak{r}}$ trivialisation maps
- $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ transition maps
- $s_\alpha^{(1)}, \ldots, s_\alpha^{(\mathfrak{r})}$ local sections of E over U_α
  $\rightsquigarrow s_\alpha^{(j)} = \sum_i (g_{\alpha\beta})_j^i s_\beta^{(i)}$
- holomorphic connections *(to be defined)* ∇_α on each $E|U_\alpha$

Atiyah class

Definition

The *Atiyah exact sequence* (or *jet sequence*) of E is the SES

$$0 \to E \otimes_{\mathscr{O}_X} \Omega^1_X \to J^1(E) \to E \to 0$$

of $\mathscr{O}_X$-modules, where $J^1(E) = (E \otimes \Omega^1) \oplus E$ as a $\mathbb{C}_X$-module but with $\mathscr{O}_X$ action given by

$$f(s \otimes \omega, t) = (fs \otimes \omega + t \otimes \mathrm{d}f, ft).$$

Definition

The *Atiyah class* of E is the corresponding Ext class

$$\mathrm{at}_E = [J^1(E)] \in \mathrm{Ext}^1_{\mathscr{O}_X}(E, E \otimes \Omega^1_X).$$

Atiyah class

The Atiyah class is “strongly related” to the Chern class. Briefly, we can recover the Chern classes by taking traces of the Atiyah classes. [Huybrechts, Proposition 4.3.10, Example 4.4.8.i, Exercise 4.4.11].

For the rest of this talk, think “Atiyah class = Chern class”

Holomorphic connections

Definition

A *holomorphic (Koszul) connection* on E is a holomorphic splitting

$$\nabla \colon E \to E \otimes \Omega^1_X$$

of the Atiyah exact sequence of E.

By enforcing the Leibniz rule

$$\nabla(s \otimes \omega) = \nabla s \wedge \omega + s \otimes \mathrm{d}\omega$$

we can extend any connection to higher order morphisms

$$\nabla^k \colon E \otimes \Omega^{k-1} \to E \otimes \Omega^k.$$

Existence of connections

Lemma

Any locally free over an arbitrary holomorphic manifold "rarely" admits a holomorphic connection.

Proof.

The first Chern class. □

Lemma

Any locally free sheaf over a ***Stein*** *manifold admits a holomorphic connection.*

Proof.

Cartan's Theorem B. □

Čech representatives of the Atiyah class

Lemma

The Atiyah class of E is represented by the cocycle

$$\begin{aligned}\{\omega_{\alpha\beta} \coloneqq \nabla_\beta - \nabla_\alpha\}_{\alpha,\beta} &\in \check{\mathscr{C}}^1_{\mathcal{U}}\big(\mathscr{H}om(E, E \otimes \Omega^1_X)\big)\\ &\cong \check{\mathscr{C}}^1_{\mathcal{U}}\big(\mathscr{E}nd(E) \otimes \Omega^1_X\big).\end{aligned}$$

Proof.

Given an SES $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ (in an abelian category "over X") and local sections $\sigma_\alpha \colon \mathcal{C}|U_\alpha \to \mathcal{B}|U_\alpha$, we have the correspondences

$$\begin{array}{ccccc}\mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{C}, \mathcal{A}) & \cong & \mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{C}, \mathcal{A}[1]) & \cong & \mathrm{H}^1(X, \mathscr{H}om(\mathcal{C}, \mathcal{A}))\\ [\mathcal{B}] & \leftrightarrow & \mathcal{C} \xrightarrow{\sim} (\mathcal{A} \to \mathcal{B}) \xrightarrow{\mathrm{id}} \mathcal{A}[1] & \leftrightarrow & [\{\sigma_\beta - \sigma_\alpha\}_{\alpha,\beta}].\end{array}$$

□

Čech representatives of the Atiyah class

If the local sections $s_\alpha^{(k)}$ are ∇_α-*flat* (i.e. lie in the kernel), then

$$\omega_{\alpha\beta}(s_\alpha^{(k)}) = \nabla_\beta(s_\alpha^{(k)})$$

and so

$$\begin{aligned}
\omega_{\alpha\beta}(s_\alpha^{(k)}) &= \nabla_\beta\left[\sum_\ell (g_{\alpha\beta})_k^\ell s_\beta^{(\ell)}\right] \\
&= \sum_\ell \left[\nabla_\beta(s_\beta^{(\ell)}) \wedge (g_{\alpha\beta})_k^\ell + s_\beta^{(\ell)} \otimes \mathrm{d}(g_{\alpha\beta})_k^\ell\right] \\
&= \sum_\ell \left[\left(\sum_m (g_{\alpha\beta}^{-1})_\ell^m s_\alpha^{(m)}\right) \otimes \mathrm{d}(g_{\alpha\beta})_k^\ell\right] \\
&= \sum_m s_\alpha^{(m)} \otimes (g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta})_k^m.
\end{aligned}$$

Čech representatives of the Atiyah class

So, in the U_α trivialisation,

$$\omega_{\alpha\beta} = \mathrm{dlog}\, g_{\alpha\beta}$$

which we might recognise as the *first Chern class*.

Čech representatives of the Atiyah class

Lemma

1. $\mathrm{d}\omega_{\alpha\beta} = -\omega_{\alpha\beta}^2$
2. $\mathrm{d}\,\mathrm{tr}\,\omega_{\alpha\beta} = 0$

Proof.

1. Use the fact that $\mathrm{d}(A^{-1}) = -A^{-1} \cdot \mathrm{d}A \cdot A^{-1}$.
2. $\mathrm{d}\,\mathrm{tr}\,\omega_{\alpha\beta} = \mathrm{tr}\,\mathrm{d}\omega_{\alpha\beta} = -\,\mathrm{tr}\,\omega_{\alpha\beta}^2$, and $\mathrm{tr}(A^{2k}) = 0$ for all $k \in \mathbb{N}$. □

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Corollary

*The **trace** $\mathrm{tr}\,\omega_{\alpha\beta}$ of the Atiyah class defines a class in de Rham cohomology.*

Exponential and standard Chern classes

The trace of the Atiyah class recovers the first Chern class; to recover higher Chern classes, we need to take the trace of powers of the Atiyah class. But there is a choice in how we multiply endomorphism-valued forms:

- *Exponential* — **Compose** the endomorphisms, wedging the forms

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \in \Gamma\left(U, \Omega_X^2 \otimes \mathscr{E}\mathit{nd}(E)\right)$$

- *Standard* — **Wedge** the endomorphisms, wedging the forms

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \wedge \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \det \begin{pmatrix} a & f \\ c & h \end{pmatrix} \in \Gamma\left(U, \Omega_X^2 \otimes \mathscr{E}\mathit{nd}(E \wedge E)\right) \cong \Gamma\left(U, \Omega_X^2\right)$$

Note that if we take the trace, then both of these simply become 2-forms on U. More generally, $\mathrm{tr}(M^k)$ and $\mathrm{tr}(\wedge^k M)$ are both k-forms on U.

Higher exponential Atiyah classes

Definition

The second exponential Atiyah class at_E^2 of E is the image

$$
\begin{aligned}
\left((\mathrm{at}_E \otimes \mathrm{id}_{\Omega^1}) \smile \mathrm{at}_E\right) &\in \mathrm{H}^2\left(X, \mathscr{H}\!om(E\otimes\Omega^1, E\otimes\Omega^1\otimes\Omega^1) \otimes \mathscr{H}\!om(E, E\otimes\Omega^1)\right) \\
&\downarrow_{\mathscr{H}om(\mathcal{F}\otimes\mathcal{G})\otimes\mathscr{H}om(\mathcal{E},\mathcal{F})\to\mathscr{H}om(\mathcal{E},\mathcal{G})} \\
&\in \mathrm{H}^2\left(X, \mathscr{H}\!om(E, E\otimes\Omega^1\otimes\Omega^1)\right) \\
&\downarrow_{\alpha\otimes\beta\mapsto\alpha\wedge\beta} \\
&\in \mathrm{H}^2\left(X, \mathscr{H}\!om(E, E\otimes\Omega^2)\right).
\end{aligned}
$$

Generally, $\mathrm{at}_E^k \in \mathrm{H}^k(X, \mathscr{E}\!nd(E)\otimes\Omega^k)$.

Čech representatives of the second Atiyah class

When we apply the composition map

$$\mathscr{H}\mathit{om}(\mathcal{F},\mathcal{G}) \otimes \mathscr{H}\mathit{om}(\mathcal{E},\mathcal{F}) \to \mathscr{H}\mathit{om}(\mathcal{E},\mathcal{G})$$

to calculate at_E^2 in terms of $\omega_{\alpha\beta}$, we need to account for the change of trivialisation $U_{\beta\gamma} \rightsquigarrow U_{\alpha\beta}$, and so

$$(\mathrm{at}_E^2)_{\alpha\beta\gamma} = \omega_{\alpha\beta} \wedge g_{\alpha\beta}\omega_{\beta\gamma}g_{\alpha\beta}^{-1}.$$

Čech representatives of the third Atiyah class

What about at_E^3? Here we risk a coherence problem: does the order in which we change trivialisations and apply the composition map give us different answers?

$$\omega_{\alpha\beta} \wedge g_{\alpha\beta}(\omega_{\beta\gamma} \wedge g_{\beta\gamma}\omega_{\gamma\delta}g_{\beta\gamma}^{-1})g_{\alpha\beta}^{-1} \stackrel{?}{=} \omega_{\alpha\beta} \wedge g_{\alpha\beta}\omega_{\beta\gamma}g_{\alpha\beta}^{-1} \wedge g_{\alpha\gamma}\omega_{\gamma\delta}g_{\alpha\gamma}^{-1}$$

Čech representatives of the third Atiyah class

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Thankfully these two are equal, due to (1) the cocycle condition on the $g_{\alpha\beta}$, and (2) the fact that $A \cdot MB = AM \cdot B$ whenever M is a matrix of 0-forms.

Truncated de Rham cohomology

Everything is nice enough (paracompact X, Stein $\mathcal{U}$, coherent Ω^k_X) for the Čech–de Rham complex to compute singular cohomology:

$$\mathrm{H}^k \operatorname{Tot}^\bullet \check{\mathscr{C}}^i_{\mathcal{U}}(\Omega^j_X) \cong \mathrm{H}^k(X, \mathbb{C}).$$

Definition

$\mathrm{H}^k_{\mathrm{tDR}}(X) := \mathbb{H}^k(X, \Omega^{\bullet \geqslant k}_X)$

Note that, if we have some *closed* class $c = (c_0, \ldots, c_{2k}) \in \operatorname{Tot}^{2k} \check{\mathscr{C}}^\star_{\mathcal{U}}(\Omega^\bullet_X)$ with $c_i \in \check{\mathscr{C}}^i_{\mathcal{U}}(\Omega^{2k-i}_X)$ *such that* $c_i = 0$ for $i \geqslant k+1$, then we can refine the corresponding singular cohomology class $[c] \in \mathrm{H}^{2k}(X, \mathbb{C})$ to a tDR cohomology class $[c] \in \mathrm{H}^{2k}_{\mathrm{tDR}}(X)$.

The manual construction

The main idea

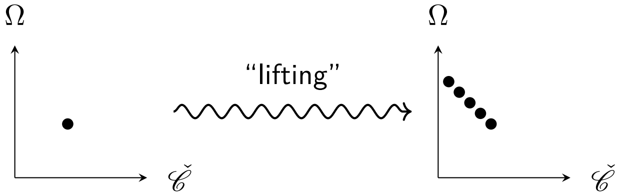
If we are given some $c_k \in \check{\mathscr{C}}^k(\Omega^k)$ that is

- Čech-closed ($\check{\delta} c_k = 0$), but ...
- ...**not** de Rham-closed ($\mathrm{d}c_k \neq 0$)

(main example: at_E^k) then we can try to "lift" it to an element

$$(0, c_1, c_2, \ldots, c_{k-1}, c_k, 0, 0, \ldots, 0) \in \mathrm{Tot}^{2k}\, \check{\mathscr{C}}^\star(\Omega^\bullet)$$

such that $\check{\delta} c_{i-1} = \mathrm{d}c_i$. This then gives a class in singular/tDR cohomology.



The first Atiyah class

We're actually already done: $\operatorname{tr}\omega_{\alpha\beta}$ is both Čech- and de Rham-closed (which is fine because $c_0 = 0$, i.e. our minimal Čech degree is that of $\omega_{\alpha\beta}$), i.e.

$$(0, \operatorname{tr}\omega_{\alpha\beta}, 0) \in \operatorname{Tot}^2 \check{\mathscr{C}}^\star(\Omega^\bullet) \rightsquigarrow [(0, \operatorname{tr}\omega_{\alpha\beta}, 0)] \in \mathrm{H}^2_{\mathrm{tDR}}(X)$$

The second Atiyah class

We have seen that $(\mathrm{at}_E^2)_{\alpha\beta} = \omega_{\alpha\beta} g_{\alpha\beta} \omega_{\beta\gamma} g_{\alpha\beta}^{-1}$.

We're going to make everything live over U_α, so let's introduce some notation:

$$A = \omega_{\alpha\beta} \qquad B = \omega_{\alpha\gamma} \qquad g = g_{\alpha\beta} \qquad (X = g\omega_{\beta\gamma}g^{-1}).$$

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Lemma

1. $\mathrm{d}A = -A^2$ *(and similarly for B and X)*
2. $A + X = B$

Corollary

$\mathrm{at}_E^2 = AX = A(B - A)$

The second Atiyah class

Corollary

$\mathrm{d}\,\mathrm{tr}(\mathrm{at}_E^2) = -\,\mathrm{tr}(A(B - A)B)$

The second Atiyah class

Corollary

$\mathrm{d}\,\mathrm{tr}(\mathrm{at}_E^2) = -\,\mathrm{tr}(A(B-A)B)$

$$
\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle \mathrm{d}} & & & & \\
? & \xrightarrow{\check{\delta}} & \mathrm{tr}(A(B-A)B) & & \\
 & & \uparrow{\scriptstyle \mathrm{d}} & & \\
 & & \mathrm{tr}(A(B-A)) & \xrightarrow{\check{\delta}} & 0
\end{array}
$$

The second Atiyah class

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle \mathrm{d}} & & & & \\
? & \xrightarrow{\check\delta} & \mathrm{tr}(A(B-A)B) & & \\
 & & \uparrow{\scriptstyle \mathrm{d}} & & \\
 & & \mathrm{tr}(A(B-A)) & \xrightarrow{\check\delta} & 0
\end{array}$$

We know that $?$ must be some homogeneous degree-3 polynomial in A, but up to a scalar we have only one choice: $\mathrm{tr}(A^3)$. Computing the Čech coboundary, we see that

$$\check\delta\colon\ \mathrm{tr}(A^3) \mapsto \mathrm{tr}\left((B-A)^3 - B^3 + A^3\right) = \ldots = 3\,\mathrm{tr}(A^2B - AB^2)$$

and

$$\mathrm{d}\,\mathrm{tr}(A^3) = \mathrm{tr}(\mathrm{d}A\cdot A^2 - A\mathrm{d}A^2) = -\,\mathrm{tr}(A^4) = 0.$$

The second Atiyah class

So, equating coefficients, we get Čech representatives for the (trace of the) second Atiyah class:

$$\begin{array}{ccccc} 0 & & & & \\ \uparrow{\scriptstyle \mathrm{d}} & & & & \\ -\frac{1}{3}\operatorname{tr}(A^3) & \xrightarrow{\check{\delta}} & \operatorname{tr}(A(B-A)B) & & \\ & & \uparrow{\scriptstyle \mathrm{d}} & & \\ & & \operatorname{tr}(AX) & \xrightarrow{\check{\delta}} & 0 \end{array}$$

The third Atiyah class

Expanding our previous notation:

$$\begin{array}{lll} A = \omega_{\alpha\beta} & g = g_{\alpha\beta} & X = g\omega_{\beta\gamma}g^{-1} \\ B = \omega_{\alpha\gamma} & h = g_{\alpha\gamma} & Y = h\omega_{\gamma\delta}h^{-1} \\ C = \omega_{\alpha\delta} & & \end{array}$$

so that $\mathrm{at}_E^3 = AXY = A(B-A)(C-B)$.

It's easy to calculate that

$$\mathrm{d}\,\mathrm{tr}\,\mathrm{at}_E^3 = \mathrm{tr}\left(A(B-A)(C-B)C\right)$$

but it's harder to find some $c_2 \in \check{\mathscr{C}}^2(\Omega^4)$ such that $\check{\delta} c_2 = \mathrm{d}\,\mathrm{tr}\,\mathrm{at}_E^3$.

The third Atiyah class

Since we can (up to a sign) cyclically permute under the trace, we can take the following brute-force approach:

1. List all monomials in the **non-commuting** variables A, X of degree 4, modulo equivalence under cyclic permutation (A^2X^2, $AXAX$, A^3X, and AX^3).
2. Calculate the Čech differential applied to each monomial.
3. Equate coefficients with $\mathrm{d}\,\mathrm{tr}\,\mathrm{at}_E^3$.

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2. Calculate the Čech differential applied to each monomial.
3. Equate coefficients with $\mathrm{d}\,\mathrm{tr}\,\mathrm{at}_E^3$.

In fact, we can iterate this: once we have c_2, we can do the same thing to calculate c_1.

The third Atiyah class

$$
\begin{array}{ccccccc}
0 & & & & & & \\
\uparrow{\scriptstyle \mathrm{d}} & & & & & & \\
\frac{1}{10}\operatorname{tr}(A^5) & \xrightarrow{\check\delta} & \frac{1}{10}\operatorname{tr}\big((B-A)^5 - B^5 + A^5\big) & & & & \\
 & & \uparrow{\scriptstyle \mathrm{d}} & & & & \\
 & & \rho(A,X) & \xrightarrow{\check\delta} & -\operatorname{tr}\big(A(B-A)(C-B)C\big) & & \\
 & & & & \uparrow{\scriptstyle \mathrm{d}} & & \\
 & & & & \operatorname{tr}\big(A(B-A)(C-B)\big) & \xrightarrow{\check\delta} & 0
\end{array}
$$

where $\rho(A,X) = -\frac{1}{4}\operatorname{tr}(AXAX) + \frac{1}{2}\operatorname{tr}(A^2X^2) - \frac{1}{2}\operatorname{tr}(A^3X) - \frac{1}{2}\operatorname{tr}(AX^3)$.

The fourth Atiyah class

There is clearly some pattern, and we can reduce this to the computational problem of working with polynomials in non-commuting variables modulo cyclic permutation, but things get very messy very quickly...

The fourth Atiyah class

$$
\begin{aligned}
5c_3 \overset{\text{tr}}{=} {} & \frac{13}{5}A^5 + 13A^4(B-A) + 5A^3(B-A)^2 + 5A^3(B-A)(C-A) \\
& + 3A^3(C-A)(B-A) + 4A^2(B-A)A(B-A) + 4A^2(B-A)A(C-A) \\
& + 3A^2(B-A)^3 - A^2(B-A)^2(C-A) + 5A^2(B-A)(C-A)^2 \\
& + 5A^2(C-A)A(B-A) + 2A^2(C-A)(B-A)^2 + A^2(C-A)(B-A)(C-A) \\
& + 3A^2(C-A)^2(B-A) - A(B-A)A(C-A)(B-A) + 5A(B-A)A(C-A)^2 \\
& - 5A(B-A)^2(C-A)(B-A) + 5A(B-A)(C-A)A(C-A) + 5A(B-A)(C-A)^3 \\
& + 4\big(A(C-A)\big)^2(B-A) - 2A(C-A)(B-A)^3 + 4A(C-A)(B-A)^2(C-A) \\
& + A\big((C-A)(B-A)\big)^2 + 2A(C-A)^2(B-A)^2 + A(C-A)^2(B-A)(C-A) \\
& + 3A(C-A)^3(B-A)
\end{aligned}
$$

$$
\begin{aligned}
5c_2 \overset{\text{tr}}{=} {} & 5A^5(B-A) - 4A^4(B-A)^2 + A^3(B-A)A(B-A) + A^3(B-A)^3 \\
& - 5A^2(B-A)A(B-A)^2 - 4A^2(B-A)^2A(B-A) - 4A^2(B-A)^4 \\
& + \frac{1}{3}\big(A(B-A)\big)^3 + A(B-A)A(B-A)^3 + A(B-A)^5
\end{aligned}
$$

$$
-35c_1 \overset{\text{tr}}{=} A^7
$$

The categorical construction

Sheaves on simplicial spaces

Definition (Detailed)

A *sheaf* $\mathcal{E}^\bullet$ *on a simplicial space* $Y_\bullet$ is a family of sheaves $\{\mathcal{E}^p \in \mathsf{Sh}(Y_p)\}_{p\in\mathbb{N}}$ along with, for all $\varphi\colon [p] \to [q]$ in Δ, morphisms

$$\mathcal{E}^\bullet\varphi\colon (Y_\bullet\varphi)^*\mathcal{E}^p \to \mathcal{E}^q$$

such that $\mathcal{E}^\bullet(\psi \circ \varphi) = \mathcal{E}^\bullet(\psi) \circ \mathcal{E}^\bullet(\varphi)$.

A morphism of such objects is a collection of morphisms that makes the squares commute.

Definition (Brief)

An object in a Grothendieck construction.

Note that we do **not** ask for the $\mathcal{E}^\bullet\varphi$ to be isos/quasi-isos/weak equivalences ("the rank can jump across simplicial levels").

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Example
$Y_\bullet = \check{\mathscr{C}}(\mathcal{U})_\bullet$ and $\mathcal{E}^\bullet = (\check{\mathscr{C}}(\mathcal{U}) \to X)^*E$ (so here the $\mathcal{E}^\bullet\varphi$ **are** isos).

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Theorem (Green)

Any coherent sheaf can be resolved by a complex of locally free sheaves on the Čech nerve.

Note that we do **not** ask for the $\mathcal{E}^\bullet\varphi$ to be isos/quasi-isos/weak equivalences ("the rank can jump across simplicial levels").

Example

$Y_\bullet = \check{\mathscr{C}}(\mathcal{U})_\bullet$ and $\mathcal{E}^\bullet = (\check{\mathscr{C}}(\mathcal{U}) \to X)^* E$ (so here the $\mathcal{E}^\bullet\varphi$ **are** isos).

Theorem (Green)

Any coherent sheaf can be resolved by a complex of locally free sheaves on the Čech nerve. Furthermore, these can be constructed such that the $\mathcal{E}^\bullet\varphi$ *are quasi-isomorphisms.*

Sheaves on simplicial spaces

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Any coherent sheaf can be resolved by a complex of locally free sheaves on the Čech nerve. Furthermore, these can be constructed such that the $\mathcal{E}^\bullet\varphi$ are quasi-isomorphisms. Furthermore, these quasi-isomorphisms are the inclusion into a direct sum with an elementary complement.

Proof.

Really a construction of such objects from the data of a twisting cochain; Toledo–Tong show that any coherent sheaf can be resolved by a twisting cochain. □

Simplicial differential forms

Definition (Brief)

A *simplicial differential r-form $\omega_\bullet$ on a simplicial complex manifold $Y_\bullet$* is a family of differential r-forms ω_p on $Y_p \times \Delta^p$ that are holomorphic on Y_p, smooth on Δ^p, and descend to a differential form on the fat geometric realisation of $Y_\bullet$: for all coface maps $f\colon [p-1] \to p$, we have that

$$(Y_\bullet f \times \mathrm{id})^* \omega_{p-1} = (\mathrm{id} \times f)^* \omega_p \in \Omega^r(Y_p \times \Delta^{p-1}).$$

We get a differential by decomposing into Y_p and Δ^p parts and then enforcing a Koszul sign convention.

Fibre integration

Theorem (Dupont)

For each fixed r, fibre integration

$$\int_{\Delta^p} : \Omega^{r,\Delta}(Y_\bullet) \to \Omega^{r-p,\Delta}(Y_p)$$

*induces a **quasi-isomorphism***

$$\int_{\Delta^\bullet} : \Omega^{r,\Delta}(Y_\bullet) \xrightarrow{\simeq} \bigoplus_{p=0}^{r} \Omega^{r-p}(Y_p)$$

Fibre integration

Example
Taking $Y_\bullet = \check{\mathscr{C}}(\mathcal{U})_\bullet$ gives

$$\int_{\Delta^\bullet} : \Omega^{r,\Delta}(\check{\mathscr{C}}(\mathcal{U})_\bullet) \xrightarrow{\sim} \bigoplus_{p=0}^{r} \Omega^{r-p}(\check{\mathscr{C}}(\mathcal{U})_p) \cong \mathrm{Tot}^r\, \check{\mathscr{C}}^\star(\Omega^\bullet).$$

Caution: if we actually want to **compute** these integrals, then we need to be careful about our choices of orientations and sign conventions.

More notation

- $\pi_p \colon \check{\mathscr{C}}(\mathcal{U})_p \times \Delta^p \twoheadrightarrow \check{\mathscr{C}}(\mathcal{U})_p$
- $E^p \coloneqq (\check{\mathscr{C}}(\mathcal{U})_p \to X)^* E$ (giving a locally free sheaf on the Čech nerve)
- $\bar{E}^p \coloneqq \pi_p^* E^p$ (giving a locally free sheaf on the product of the Čech nerve with the simplex)
- $\xi_p^i \colon [0] \to [p]$

Note that we get isomorphisms $E^\bullet \xi_p^i \colon (\check{\mathscr{C}}(\mathcal{U})_\bullet \xi_p^i)^* E^0 \xrightarrow{\sim} E^p$ and so any connection ∇_α on E gives a connection (also denoted ∇_α) on E^p.

The barycentric connection

Definition

The *barycentric connection* $\nabla^{\mu}_{\bullet}$ *on* $\bar{E}^{\bullet}$ is the map

$$\nabla^{\mu}_{\bullet} \colon \bar{E}^{\bullet} \to \bar{E}^{\bullet} \otimes \Omega^{1}_{\check{\mathscr{C}}(\mathcal{U})_{\bullet} \times \Delta^{\bullet}}$$

defined by

$$\nabla^{\mu}_{p} = \sum_{i=0}^{p} t_i \nabla_{\alpha_i} = \nabla_{\alpha_0} + \sum_{i=1}^{p} t_i \omega_{\alpha_0 \alpha_i}.$$

The curvature of the barycentric connection

Let's try to be more explicit: how does (the curvature of) this connection act on sections σ_{α_0} of $\bar{E}^p$ over U_{α_0}?

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\begin{aligned}
(\nabla_p^\mu)^2(\sigma_{\alpha_0}) &= \left[\nabla_{\alpha_0} + \sum_{i=1}^{p} t_i \omega_{\alpha_0\alpha_i}\right]^2 (\sigma_{\alpha_0}) \\
&\overset{!}{=} \sum_{i=1}^{p} \sigma_{\alpha_0} \otimes \mathrm{d}(t\omega_{\alpha_0\alpha_i}) + \sum_{i,j=1}^{k} \sigma_{\alpha_0} \otimes (t_j t_i \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i})
\end{aligned}
$$

where we again make the assumption (!) that σ_{α_0} is ∇_{α_0}-flat.

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where we again make the assumption (!) that σ_{α_0} is ∇_{α_0}-flat.

In other words,

$$(\nabla_p^\mu)^2 = \mathrm{d}\bar{\omega}_p + \bar{\omega}_p^2$$

where $\bar{\omega}_p = \sum_{i=1}^{p} t_i\omega_{\alpha_0\alpha_i}$. (Note also that $\nabla_p^\mu = \mathrm{d} + \bar{\omega}_p$).

Analogies and differences

Things seem familiar when compared to before, if we just substitute $\bar{\omega}$ for ω, e.g.

- The barycentric connection is of the form $\mathrm{d} + \bar{\omega}_p$.
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However there are some key differences, e.g.

- The $\omega_{\alpha\beta}$ was the **obstruction** towards admitting a global connection; the $\bar{\omega}_p$ **is** a "global" connection.
- $\mathrm{d}\omega_{\alpha\beta} = -\omega_{\alpha\beta}^2$ but $\mathrm{d}\bar{\omega}_p \neq -\bar{\omega}_p^2$ (i.e. the barycentric connection is not "flat").

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As a consequence, there will be an odd mismatch: the Atiyah class is the "connection" $\omega_{\alpha\beta}$, but the *simplicial Atiyah class* is the "**curvature**" of the "connection" $\bar{\omega}_p$.

The simplicial Atiyah class

Definition

$$\bar{\mathrm{at}}_E^k = \left\{(-1)^{k(k-1)/2}(\mathrm{d}\bar{\omega}_p + \bar{\omega}_p^2)^k\right\}_{p\in\mathbb{N}}$$
$$= \left\{(-1)^{k(k-1)/2}\left(-\sum_{i=1}^{p}\omega_{\alpha_0\alpha_i}\otimes \mathrm{d}t_i - \sum_{i=1}^{p} t_i\omega_{\alpha_0\alpha_i}^2 + \sum_{i,j=1}^{p} t_j t_i \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i}\right)^k\right\}_{p\in\mathbb{N}}$$

Theorem (H)

$\bar{\mathrm{at}}_E$ *is an admissible endomorphism-valued simplicial differential form.*

The simplicial Atiyah class

Lemma

$\operatorname{tr} \bar{\mathrm{at}}_E^k$ *is* d*-closed.*

Corollary

$\operatorname{tr} \int_{\Delta^\bullet} \bar{\mathrm{at}}_E^k$ *defines a cohomology class.*

Theorem (H)

$$\varsigma_k \left(\operatorname{tr} \int_{\Delta^\bullet} \bar{\mathrm{at}}_E^k \right)^{(k,k)} = \varsigma_k \operatorname{tr}(\mathrm{at}_E^k) \in \check{\mathscr{C}}^k(\Omega_X^k)$$

where ς_k *denotes the skew-symmetrisation of a cochain (which is the identity in cohomology).*

Proof.

Combinatorics. □

The simplicial Atiyah class

So, in summary, we can recover these complicated manual lifting constructions from before by simply computing some integrals over simplices, which recover the $c_0, \ldots, c_k$ (by looking at the type-(i, j) parts of the simplicial differential form), even including their coefficients (e.g. $\int_{\Delta^2} t_1 t_2 \mathrm{d}t_1 \mathrm{d}t_2 = \frac{1}{24}$).

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So, in summary, we can recover these complicated manual lifting constructions from before by simply computing some integrals over simplices, which recover the $c_0, \ldots, c_k$ (by looking at the type-(i,j) parts of the simplicial differential form), even including their coefficients (e.g. $\int_{\Delta^2} t_1 t_2 \mathrm{d}t_1 \mathrm{d}t_2 = \frac{1}{24}$).

The caveat is that we don't recover exactly the same Čech representatives on the nose: for $k \geqslant 3$ we only get equality in cohomology. However, this is somewhat artificial, in that it is caused by us not using skew-symmetric Čech cohomology initially.

The second simplicial Atiyah class

By definition,

$$\bar{\mathrm{at}}_E^2 = \left\{ -\left(-\sum_{i=1}^{p} \omega_{\alpha_0\alpha_i} \otimes \mathrm{d}t_i - \sum_{i=1}^{p} t_i \omega_{\alpha_0\alpha_i}^2 + \sum_{i,j=1}^{p} t_j t_i \omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \right)^2 \right\}_{p\in\mathbb{N}}$$

but the only parts that will be non-zero after fibre integration are the $(2,2)$ parts on the 2-simplex, and the $(3,1)$ parts on the 1-simplex.

The second simplicial Atiyah class

The only $(2,2)$ part comes from the first half of the $(\mathrm{d}\overline{\omega}_2)^2$ term, which gives us

$$\begin{aligned}
\operatorname{tr}\int_{\Delta^2}\bar{\mathrm{at}}_E^2 &= \operatorname{tr}\int_{\Delta^2} -\left(\sum_{i,j=1}^{2}(\omega_{\alpha_0\alpha_j}\otimes \mathrm{d}t_j)\cdot(\omega_{\alpha_0\alpha_j}\otimes \mathrm{d}t_j)\right)\\
&= \operatorname{tr}\int_{\Delta^2}\sum_{i,j=1}^{2}\omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i}\otimes \mathrm{d}t_j\mathrm{d}t_i\\
&= \operatorname{tr}\int_{\Delta^2}\Big(\omega_{\alpha_0\alpha_1}^2\otimes(\mathrm{d}t_1)^2+\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2}\otimes \mathrm{d}t_1\mathrm{d}t_2\\
&\qquad +\omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1}\otimes \mathrm{d}t_2\mathrm{d}t_1+\omega_{\alpha_0\alpha_2}^2\otimes(\mathrm{d}t_2)^2\Big)\\
&= \operatorname{tr}\int_{\Delta^2}\Big(\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2}-\omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1}\Big)\otimes \mathrm{d}t_1\mathrm{d}t_2\\
&= \operatorname{tr}\int_0^1\int_0^{1-t_2}\Big(\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2}-\omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1}\Big)\otimes \mathrm{d}t_1\mathrm{d}t_2\\
&= \frac{1}{2}\operatorname{tr}\Big(\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2}-\omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1}\Big)\\
&= \frac{1}{2}\cdot 2\cdot\operatorname{tr}(\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2})=\operatorname{tr}\left(\omega_{\alpha_0\alpha_1}(\omega_{\alpha_0\alpha_1}+\omega_{\alpha_1\alpha_2})\right)=\operatorname{tr}(\omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2}).
\end{aligned}$$

The second simplicial Atiyah class

So far, we have

$$\operatorname{tr}\int_{\Delta^\bullet} \bar{\mathrm{at}}_E^2 = \underbrace{?}_{p=1} + \underbrace{\operatorname{tr}(\omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2})}_{p=2}.$$

For the $(3,1)$ part, we work on the 1-simplex and get

$$\begin{aligned}
\operatorname{tr}\int_{\Delta^1} \bar{\mathrm{at}}_E^2 = \operatorname{tr}(-1)^{3\cdot 1}\int_{\Delta^1} -\Bigg(&-\sum_{i,j=1}^{1}(\omega_{\alpha_0\alpha_j}\otimes \mathrm{d}t_j)\cdot(-t_i\omega^2_{\alpha_0\alpha_i}) \\
&-\sum_{i,j=1}^{1}(-t_j\omega^2_{\alpha_0\alpha_j})\cdot(\omega_{\alpha_0\alpha_i}\otimes \mathrm{d}t_i) \\
&-\sum_{i,j,k=1}^{1}(\omega_{\alpha_0\alpha_k}\otimes \mathrm{d}t_k)\cdot(t_jt_i\omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i}) \\
&-\sum_{i,j,k=1}^{1}(t_kt_j\omega_{\alpha_0\alpha_k}\omega_{\alpha_0\alpha_j})\cdot(\omega_{\alpha_0\alpha_i}\otimes \mathrm{d}t_i)\Bigg) \\
= \operatorname{tr}\int_0^1 2\omega^3_{\alpha_0\alpha_1}(t_1-t_1^2)\mathrm{d}t_1 \\
= \frac{1}{3}\operatorname{tr}\omega^3_{\alpha_0\alpha_1}.
\end{aligned}$$

The second simplicial Atiyah class

Finally, we have

$$\operatorname{tr}\int_{\Delta^\bullet} \bar{\mathrm{at}}_E^2 = \underbrace{\frac{1}{3}\operatorname{tr}\omega_{\alpha_0\alpha_1}^3}_{p=1} + \underbrace{\operatorname{tr}(\omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2})}_{p=2}$$

which is exactly our previously calculated lift of $\operatorname{tr}\mathrm{at}_E^2$.

Questions

What about Deligne cohomology?

The first Atiyah class (first Chern class) is “already in” Deligne cohomology, just for degree reasons.

But if we try to manually lift the second Atiyah class to a closed element in the Čech–Deligne bicomplex then we immediately run into issues: there are no obvious elements to take the Čech differential of.

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But if we try to manually lift the second Atiyah class to a closed element in the Čech–Deligne bicomplex then we immediately run into issues: there are no obvious elements to take the Čech differential of.

In fact, I don't even know how to do the very simplest thing at the desired level of detail: *write down* ***Čech representatives*** *for the (non-trivial) second Chern class in Deligne cohomology of any holomorphic vector bundle*.

Does anybody know how to do this?

Fin

Thank you for your time.